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Kottwitz-Rapoport and p-rank strata in the reduction of Shimura varieties of PEL type

KOTTWITZ-RAPOPORT AND p-RANK STRATA IN THE REDUCTION OF SHIMURA VARIETIES OF PEL TYPE

by Philipp HARTWIG

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ABSTRACT. — We study the reduction of certain integral models of Shimura varieties of PEL type with Iwahori level structure. On these spaces we have the Kottwitz-Rapoport and the p-rank stratification. We show that the p-rank is constant on a KR stratum, generalizing a result of Ngô and Genestier. We prove an abstract, uniform formula for the p-rank on a KR stratum. In the symplectic and in the unitary case we derive explicit formulas for its value. We apply these formulas to the question of the density of the ordinary locus and to the question of the dimension of the p-rank 0 locus.

RÉSUMÉ. — Nous étudions la réduction de certains modèles entiers des variétés de Shimura de type PEL à structure de niveau Iwahori. Sur ces espaces on a la stratification de Kottwitz-Rapoport et la stratification de p-rang. Nous montrons que le p-rang est constant sur un strate de Kottwitz-Rapoport, généralisant un résultat de Ngô et Genestier. Nous montrons une formule abstraite, uniforme pour le p-rang sur un strate de Kottwitz-Rapoport. Dans les cas symplectique et unitaire nous trouvons des formules explicites pour sa valeur. Nous appliquons ces formules à la question de la densité du lieu ordinaire et à la question de la dimension du lieu de p-rang 0.

1. Introduction

In [24] Rapoport and Zink construct integral models for certain Shimura varieties of PEL type. These integral models have since been an object of intense study and our paper is concerned with their geometric special fiber. One strategy for studying the geometry of these special fibers is to make use of different natural stratifications on them. So far this strategy

has been particularly successful in the case of the Siegel modular variety
with Iwahori level structure $\mathcal{A}_I$: For example it has been used by Ngô
and Genestier to prove the density of the ordinary locus in $\mathcal{A}_I$, see [16],
and it has been used by Görtz and Yu to obtain various results about the
geometry of the supersingular locus in $\mathcal{A}_I$, see [5] and [6]. In both cases the
Kottwitz-Rapoport stratification, first introduced by Ngô and Genestier in
loc. cit., plays a crucial role. However beyond the Siegel case not much
is known about this stratification. The aim of this paper is to study the
Kottwitz-Rapoport stratification in the general PEL setup and to use it to
obtain results about the geometry of the special fibers of the Rapoport-Zink
integral models in other cases than the Siegel case.

Let us now give a more detailed overview of the content of this paper.
Fix a rational prime $p \neq 2$ and a PEL datum $\mathcal{B} = (B, *, V, (\cdot, \cdot), J)$ with
auxiliary data $\mathcal{B}_p = (\mathcal{O}_B, \mathcal{L})$, see Section 2.1. The datum $\mathcal{B}$ gives rise to
a reductive group G over $\mathbb{Q}$ and a conjugacy class h of homomorphisms
$\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_{\mathbb{R}}$. Fix a compact open subgroup $C^p \subset G(\mathbb{A}_f^p)$. From C^p
and $\mathcal{B}_p$ one obtains a compact open subgroup $C \subset G(\mathbb{A}_f)$ and thus a
Shimura datum (G, h, C). In [24, Section 6], Rapoport and Zink construct
from $\mathcal{B}$, $\mathcal{B}_p$ and C^p an integral model $\mathcal{A} = \mathcal{A}_{C^p}$ of the Shimura variety
associated with (G, h, C). Concretely $\mathcal{A}$ is defined as a moduli space of
abelian schemes with additional structure.

In order to study properties of the scheme $\mathcal{A}$, Rapoport and Zink in-
troduce the so-called local model[1] M^{loc}. It is defined purely in terms of
linear algebra and therefore easier to investigate than $\mathcal{A}$. The schemes $\mathcal{A}$
and M^{loc} are related via an intermediate object $\widetilde{\mathcal{A}}$ fitting into the so called
local model diagram

$$
\begin{array}{ccc}
 & \widetilde{\mathcal{A}} & \\
{}^{\widetilde{\varphi}}\swarrow & & \searrow{}^{\widetilde{\psi}} \\
\mathcal{A} & & M^{\mathrm{loc}}.
\end{array}
$$

Étale locally on $\mathcal{A}$, there is a section $s : \mathcal{A} \to \widetilde{\mathcal{A}}$ of $\widetilde{\varphi}$ such that the compo-
sition $\mathcal{A} \xrightarrow{s} \widetilde{\mathcal{A}} \xrightarrow{\widetilde{\psi}} M^{\mathrm{loc}}$ is étale. Consequently $\mathcal{A}$ inherits any property from
M^{loc} which is local for the étale topology. In particular, questions about
singularities of $\mathcal{A}$ or the flatness of $\mathcal{A}$ can equivalently be studied for M^{loc}.

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[1] The local models introduced by Rapoport and Zink have come to be called "naive
local models" because they fail to be flat in general. We have chosen to omit the word
"naive" from their name since the question of flatness is not relevant for the purposes of
this paper. But the reader should be warned that this is not completely consistent with
the terminology used in the more recent literature.

The PEL datum also gives rise to an affine smooth group scheme $\mathrm{Aut}(\mathcal{L})$, and $\mathrm{Aut}(\mathcal{L})$ acts on both $\tilde{\mathcal{A}}$ and M^{loc}. The map $\widetilde{\varphi}$ is an $\mathrm{Aut}(\mathcal{L})$-torsor, while the map $\widetilde{\psi}$ is $\mathrm{Aut}(\mathcal{L})$-equivariant. Denote by $\mathbb{F}$ an algebraic closure of $\mathbb{F}_p$. Via the local model diagram, the decomposition of $M^{\mathrm{loc}}(\mathbb{F})$ into $\mathrm{Aut}(\mathcal{L})(\mathbb{F})$-orbits induces the *Kottwitz-Rapoport* (or *KR*) *stratification*

$$\mathcal{A}(\mathbb{F}) = \coprod_{x \in \mathrm{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F})} \mathcal{A}_x,$$

which was first introduced by Ngô and Genestier in [16].

Assume for the rest of this introduction that we are in the **Iwahori case** (i.e. that $\mathcal{L}$ is complete in the sense of Definition 2.4.2). In all explicit cases studied thus far (cf. the discussion in [22, §3.3]), the $\mathrm{Aut}(\mathcal{L})(\mathbb{F})$-orbits in $M^{\mathrm{loc}}(\mathbb{F})$ are intimately related to Schubert cells in a suitable affine flag variety. In order to observe this relationship one first constructs an embedding of the geometric special fiber $M^{\mathrm{loc}}_{\mathbb{F}}$ of M^{loc} into an affine flag variety $\mathcal{F}$. This embedding is constructed on a case-by-case basis, using a suitable realization of the affine flag variety as a moduli space of lattice chains. (Let us mention though that recently a purely group-theoretic definition of a local model was given in [23] by Pappas and Zhu; for this local model the special fiber is canonically contained in an affine flag variety.) In analogy with the Bruhat decomposition of the classical flag variety, indexed by the finite Weyl group W, the affine flag variety admits the *Iwahori decomposition* $\mathcal{F}(\mathbb{F}) = \coprod_{x \in \widetilde{W}} \mathcal{C}_x$ into Schubert cells $\mathcal{C}_x$, indexed by the extended affine Weyl group $\widetilde{W}$. It then turns out in each of these explicit cases that $M^{\mathrm{loc}}_{\mathbb{F}} \subset \mathcal{F}$ is a disjoint union of Schubert cells and that the decomposition $M^{\mathrm{loc}}(\mathbb{F}) = \coprod_{\mathcal{C}_x \subset M^{\mathrm{loc}}(\mathbb{F})} \mathcal{C}_x$ coincides with the decomposition of $M^{\mathrm{loc}}(\mathbb{F})$ into $\mathrm{Aut}(\mathcal{L})(\mathbb{F})$-orbits. As in the case of the Bruhat decomposition, many properties of the Iwahori decomposition are easily expressed by combinatorial properties of the corresponding index element in $\widetilde{W}$. Notably, the dimension of $\mathcal{C}_x$ is given by the length $\ell(x)$ of x in $\widetilde{W}$, and the closure relation between Schubert cells is expressed by the Bruhat order on $\widetilde{W}$. We conclude that the same statements hold for the KR stratification on $\mathcal{A}(\mathbb{F})$.

Let us explain in detail one case in which this convenient combinatorial behavior of the KR stratification was fruitfully exploited. For $B = \mathbb{Q}$, the moduli problem $\mathcal{A}$ specializes to the Siegel moduli space $\mathcal{A}_I$ of principally polarized abelian varieties with Iwahori level structure. In [6], Görtz and Yu compute the dimension of the p-rank 0 locus in $\mathcal{A}_I$, and this computation was later generalized in [9] by Hamacher to the case of all p-rank strata. The method is the same in both cases: Determine all KR strata contained in a given p-rank stratum and compute the maximum of their dimensions. For

this method to work one of course needs to know that a p-rank stratum is indeed the union of the KR strata contained in it. Thus both papers depend crucially on the result [16, Théorème 4.1] of Ngô and Genestier, which states that indeed the p-rank is constant on a KR stratum in $\mathcal{A}_I$, and also provides an explicit formula for the p-rank on a given KR stratum.

The subject of this paper is to generalize the result of Ngô and Genestier on the relationship between the KR and the p-rank stratification to more general PEL data. Let us give an outline of the structure of this paper and of the results that we have obtained.

In Sections 2.1 through 2.3, we recall the construction of the local model diagram. We then show the following result.

THEOREM 1.0.1. — *Let $\mathcal{B}$ be an arbitrary PEL datum. If $\mathcal{L}$ is complete (in the sense of Definition 2.4.2), then the p-rank is constant on a KR stratum.*

Before being able to state our next result, we need some more notation. Denote by $\mathcal{O}_K = W(\mathbb{F})$ the Witt ring of $\mathbb{F}$, by K the fraction field of $\mathcal{O}_K$ and by σ the Frobenius on K. Denote by $\mathbb{D}$ the diagonalizable affine group with character group $\mathbb{Q}$ over K. For $b \in G(K)$ denote by $\nu_b : \mathbb{D} \to G_K$ the corresponding Newton map. By definition, the group G_K acts on V_K and thus ν_b gives rise to a representation of $\mathbb{D}$ on V_K. Consider the corresponding weight decomposition $V_K = \oplus_{\chi \in \mathbb{Q}} V_\chi$ and define

$$\nu_{b,0} := \dim_K V_0.$$

Denote by I the stabilizer of $\mathcal{L} \otimes \mathcal{O}_K$ in $G(K)$. For $b \in G(K)$ and $x \in I \backslash G(K)/I$ we denote by $X_x(b) = \{g \in G(K)/I \mid g^{-1} b \sigma(g) \in IxI\}$ the affine Deligne-Lusztig variety associated with b and x.

By interpreting the KR stratification in terms of the relative position of $\mathcal{L} \otimes \mathcal{O}_K$ to its image under Frobenius, we show that the KR strata are precisely the fibers of a canonical map $\gamma : \mathcal{A}(\mathbb{F}) \to I \backslash G(K)/I$. Denote by Perm $\subset I \backslash G(K)/I$ the image of γ, and for $x \in$ Perm by $\mathcal{A}_x = \gamma^{-1}(x)$ the corresponding KR stratum.

THEOREM 1.0.2. — *Let $x \in$ Perm and let $b \in G(K)$. Assume that $X_x(b) \neq \emptyset$. Then the p-rank on $\mathcal{A}_x$ is constant with value $\nu_{b,0}$.*

It is conjectured that the non-emptiness of $X_x(b)$ implies the non-emptiness of the intersection of $\mathcal{A}_x$ with the Newton stratum $\mathcal{N}_b$ associated with b, compare Remark 2.5.17. Our result can be seen as providing further evidence for this conjecture.

In Sections 3 through 7 we turn to the aforementioned interpretation of the KR stratification in terms of the affine flag variety. Section 3 deals with the case of the symplectic group. Section 5 (resp. 6, resp. 7) deals with the case of a unitary group associated with a ramified (resp. inert, resp. split) quadratic extension. Let us note that the embedding of $M_{\mathbb{F}}^{\mathrm{loc}}$ into an affine flag variety has a long history. In particular we want to emphasize that we have greatly profited from the expositions by Pappas and Rapoport in [19], [20], [21], and by Smithling in [26], [28]. We have decided to repeat part of their discussions, on the one hand for the convenience of the reader, and on the other to provide several proofs and details that have been omitted in loc. cit.

Our discussion is quite similar in all cases. We begin with describing in detail the PEL datum at hand, including the Hodge structure and the resulting determinant morphism. We proceed by making explicit the definition of the local model and investigate its base-change to $\mathbb{F}$. We then recall the definition of the affine flag variety as a suitable quotient of loop groups and prove in detail that it can be realized as a moduli space of lattice chains. We conclude the discussion of the local model by embedding it into the affine flag variety and proving that the $\mathrm{Aut}(\mathcal{L})$-orbits on $M^{\mathrm{loc}}(\mathbb{F})$ are precisely the Schubert cells contained in $M^{\mathrm{loc}}(\mathbb{F})$. We then prove an explicit formula for the p-rank on a KR stratum.

Using these explicit formulas and the aforementioned combinatorial structure of the KR stratification, we prove the following geometric results.

1.1. Density of the ordinary locus

It is an interesting question whether the ordinary locus lies dense in $\mathcal{A}_{\mathbb{F}}$. In the case of hyperspecial level structure, this question has been studied in detail by Wedhorn in [31], who shows that one should work with the μ-ordinary locus instead of the (classical) ordinary locus, and that the former is open and dense.

We focus on the case of Iwahori level structure. In the corresponding Siegel case $\mathcal{A}_I$, the result [16, Corollaire 4.3] of Ngô and Genestier answers this question affirmatively. On the other hand, Stamm obtains in [29] a negative answer in the two-dimensional Hilbert-Blumenthal case. The following result generalizes these two results and explains the general pattern, thereby answering a question by M. Rapoport.

THEOREM 1.1.1 (Corollary 3.11.2). — *Assume that $\mathcal{B}$ is the symplectic PEL datum associated with a totally real extension $F/\mathbb{Q}$ (see Section 3.2*

for a detailed description of $\mathcal{B}$*). Assume (without loss of generality) that there is only a single prime of* $\mathcal{O}_F$ *dividing* p*. Then the ordinary locus lies dense in* $\mathcal{A}_{\mathbb{F}}$ *if and only if* p *is totally ramified in* $\mathcal{O}_F$*.*

Unlike in the case of hyperspecial level structure mentioned above, the non-density of the ordinary locus occurring in Theorem 1.1.1 does not stem from using the wrong notion of ordinariness, but rather should be seen as a natural phenomenon.

1.2. Dimension of the p-rank 0 locus

As mentioned above, Görtz and Yu use [16, Théorème 4.1] to compute the dimension of the p-rank 0 locus in $\mathcal{A}_I$, see [6, Theorem 8.8]. By copying their approach and using our formula for the p-rank on a KR stratum in the split unitary case, we obtain the following result.

THEOREM 1.2.1 (Theorem 7.5.7). — *Assume that* $\mathcal{B}$ *is the unitary PEL datum of signature* $(r, n - r)$ *associated with an imaginary quadratic extension of* $\mathbb{Q}$ *in which* p *splits (see Section 7.1 for a detailed description of* $\mathcal{B}$*). Denote by* $\mathcal{A}^{(0)} \subset \mathcal{A}(\mathbb{F})$ *the subset where the* p*-rank of the underlying abelian variety is equal to* 0*. Then*

$$\dim \mathcal{A}^{(0)} = \min\big((r-1)(n-r), r(n-r-1)\big).$$

1.3. The Hilbert-Blumenthal case

As an illustrative example, we look in Section 3.12 at the case of the Hilbert-Blumenthal modular varieties. Without any additional work, we obtain the following result.

THEOREM 1.3.1 (Theorem 3.12.3). — *Let* $g \geqslant 2$ *and let* $\mathcal{A}$ *be the Hilbert-Blumenthal modular variety with* $\Gamma_0(p)$*-level structure associated with a totally real extension of degree* g *of* $\mathbb{Q}$*. Denote by* $\mathcal{A}^{(0)} \subset \mathcal{A}_{\mathbb{F}}$ *and* $\mathcal{A}^{(g)} \subset \mathcal{A}_{\mathbb{F}}$ *the subsets where the* p*-rank of the underlying abelian variety is equal to* 0 *and* g*, respectively. Then*

$$\mathcal{A}_{\mathbb{F}} = \mathcal{A}^{(0)} \amalg \mathcal{A}^{(g)}.$$

The ordinary locus $\mathcal{A}^{(g)}$ *is the union of only two KR strata* $\mathcal{A}_{x_1}$ *and* $\mathcal{A}_{x_2}$*. Consequently we have*

$$\mathcal{A}_{\mathbb{F}} = \overline{\mathcal{A}}_{x_1} \cup \overline{\mathcal{A}}_{x_2} \cup \mathcal{A}^{(0)}.$$

Here $\overline{\mathcal{A}}_x$ denotes the closure of the KR stratum $\mathcal{A}_x$ in $\mathcal{A}_{\mathbb{F}}$.

Each of $\mathcal{A}_{\mathbb{F}}, \overline{\mathcal{A}}_{x_1}, \overline{\mathcal{A}}_{x_2}$ and $\mathcal{A}^{(0)}$ is equidimensional of dimension 2^g. Furthermore, we have

$$\overline{\mathcal{A}}_{x_1} \cap \overline{\mathcal{A}}_{x_2} \subset \mathcal{A}^{(0)}.$$

Taking $g = 2$, we recover the result [29, Theorem 2 (p. 408)] of Stamm.

Acknowledgments

Notation

We fix once and for all a rational prime $p \neq 2$ and an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$.

Let $n \in \mathbb{N}_{\geqslant 1}$.

- For elements $x_1, \ldots, x_n$ of some set and $k_1, \ldots k_n \in \mathbb{N}$, we denote by $(x_1^{(k_1)}, \ldots, x_n^{(k_n)})$ the tuple

$$\big(\underbrace{x_1, \ldots, x_1,}_{k_1\text{-times}}, \ldots, \underbrace{x_n, \ldots, x_n}_{k_n\text{-times}}\big).$$

  For a tuple $x \in \mathbb{Z}^n$, we denote by $x(i)$ its i-th entry.

- For an element w of S_n, the symmetric group on n letters, we denote by $A_w = (\delta_{iw(j)})_{ij}$ the corresponding permutation matrix.

- We write

$$\widetilde{J}_{2n} = \begin{pmatrix} 0 & \widetilde{I}_n \\ -\widetilde{I}_n & 0 \end{pmatrix}, \quad \text{where} \quad \widetilde{I}_n = \text{anti-diag}(1, \ldots, 1).$$

Let R be a ring and let $R \to R'$ be an R-algebra.

- We denote the dual of various objects over R by a superscript $\cdot^\vee = \cdot^{\vee,R}$.
- We often denote the base-change from R to R' by a subscript $\cdot_{R'}$.
- If G is a functor on the category of R'-algebras, we denote by $\operatorname{Res}_{R'/R} G$ the functor on the category of R-algebras with $(\operatorname{Res}_{R'/R} G)(S) = G(S \otimes_R R')$.
- If F is a functor on the category of $R((u))$-algebras (resp. $R[\![u]\!]$-algebras), we denote by $\operatorname{L} F = \operatorname{L}_u F$ (resp. $\operatorname{L}^+ F = \operatorname{L}_u^+ F$) the functor on the category of R-algebras with $\operatorname{L} F(S) = F(S((u)))$ (resp. $\operatorname{L}^+ F(S) = F(S[\![u]\!])$).
- For $\lambda \in \mathbb{Z}^n$, we write $u^\lambda = \operatorname{diag}(u^{\lambda(1)}, \ldots, u^{\lambda(n)}) \in \operatorname{GL}_n(R((u)))$.

2. The general case

We assume that the reader is familiar with at least the definitions of [24, 3.1-3.27] and [24, 6.1-6.9]. The required results on orders in semisimple algebras can all be found in Reiner's excellent [25]. In Sections 2.1 through 2.3 we recall from [24] the general setup of integral models of PEL-type Shimura varieties and their local models.

2.1. PEL data

A *PEL datum* consists of the following objects.

(1) A finite-dimensional semisimple $\mathbb{Q}$-algebra B.
(2) A positive[2] involution $*$ on B.
(3) A finitely generated left B-module V. We assume that $V \neq 0$.
(4) A symplectic form $(\cdot, \cdot) : V \times V \to \mathbb{Q}$ on the underlying $\mathbb{Q}$-vector space of V, such that for all $v, w \in V$ and all $b \in B$ the relation

$$(bv, w) = (v, b^* w)$$

is satisfied.
(5) An element $J \in \operatorname{End}_{B \otimes \mathbb{R}}(V \otimes \mathbb{R})$ with $J^2 + 1 = 0$ such that the bilinear form $(\cdot, J\cdot)_\mathbb{R} : V_\mathbb{R} \times V_\mathbb{R} \to \mathbb{R}$ is symmetric and positive definite.

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[2] By this we mean that the involution on $B \otimes \mathbb{R}$ arising from $*$ via base-change is a positive involution in the sense of [13, §2].

We also fix the following data.

(a) A $\mathbb{Z}$-order $\mathcal{O}_B$ in B such that $\mathcal{O}_B \otimes \mathbb{Z}_p$ is a maximal $\mathbb{Z}_p$-order in $B \otimes \mathbb{Q}_p$. We assume that $\mathcal{O}_B \otimes \mathbb{Z}_p$ is stable under $*$.

(b) A self-dual multichain $\mathcal{L}$ of $\mathcal{O}_B \otimes \mathbb{Z}_p$-lattices in $V \otimes \mathbb{Q}_p$.

Denote by G the group on the category of $\mathbb{Q}$-algebras with

$$G(R) = \left\{ g \in \mathrm{GL}_{B \otimes R}(V \otimes R) \mid \exists c = c(g) \in R^\times \left(\begin{matrix} \forall x, y \in V \otimes R \\ (gx, gy)_R = c\, (x, y)_R \end{matrix} \right) \right\}$$

Note that for $g \in G(R)$, the unit $c(g) \in R^\times$ is indeed uniquely determined in view of the assumption $V \neq 0$ and the perfectness of $(\cdot, \cdot)$, justifying the notation. We also denote by $c : G \to \mathbb{G}_{m,\mathbb{Q}}$ the resulting morphism.

Let $\Lambda \in \mathcal{L}$. We deviate slightly from the notation of [24] in writing $\Lambda^\vee = \{ x \in V_{\mathbb{Q}_p} \mid (x, \Lambda)_{\mathbb{Q}_p} \subset \mathbb{Z}_p \}$ (in loc. cit. the notation Λ^* is used instead). We denote by $(\cdot, \cdot)_\Lambda : \Lambda \times \Lambda^\vee \to \mathbb{Z}_p$ the restriction of $(\cdot, \cdot)_{\mathbb{Q}_p}$. It is a perfect pairing and induces an isomorphism $\Lambda^\vee \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}(\Lambda, \mathbb{Z}_p) = \Lambda^{\vee, \mathbb{Z}_p}$ of $\mathcal{O}_B \otimes \mathbb{Z}_p$-modules, justifying the notation. For $\Lambda \subset \Lambda'$ in $\mathcal{L}$ we denote by $\rho_{\Lambda', \Lambda} : \Lambda \to \Lambda'$ the inclusion. For $b \in (B \otimes \mathbb{Q}_p)^\times$ in the normalizer of $\mathcal{O}_B \otimes \mathbb{Z}_p$, denote by Λ^b the $\mathcal{O}_B \otimes \mathbb{Z}_p$-module obtained from Λ by restriction of scalars with respect to the morphism $\mathcal{O}_B \otimes \mathbb{Z}_p \to \mathcal{O}_B \otimes \mathbb{Z}_p$, $x \mapsto b^{-1}xb$, and let $\vartheta_{\Lambda, b} : \Lambda^b \to b\Lambda$ be the isomorphism given by multiplication with b. Then $(\Lambda, \rho_{\Lambda', \Lambda}, \vartheta_{\Lambda, b}, (\cdot, \cdot)_\Lambda)$ is a polarized multichain of $\mathcal{O}_B \otimes \mathbb{Z}_p$-modules of type $(\mathcal{L})$ which, by abuse of notation, we also denote by $\mathcal{L}$.

Let $B \otimes \mathbb{Q}_p = B_1 \times \cdots \times B_m$ be the decomposition into simple factors. It induces a decomposition

$$(2.1.1) \qquad \mathcal{O}_B \otimes \mathbb{Z}_p = \mathcal{O}_{B_1} \times \cdots \times \mathcal{O}_{B_m}$$

and each O_{B_i} is a maximal $\mathbb{Z}_p$-order in B_i.

We also get a decomposition $V \otimes \mathbb{Q}_p = V_1 \times \cdots \times V_m$ into left B_i-modules V_i. Denote by $\mathcal{L}_i$ the projection of $\mathcal{L}$ to V_i. It is a chain of $\mathcal{O}_{B_i}$-lattices in V_i. For $\Lambda \in \mathcal{L}$ we denote by $\Lambda = \Lambda_1 \times \cdots \times \Lambda_m$, $\Lambda_i \in \mathcal{L}_i$ the corresponding decomposition.

Denote by $V_{\mathbb{C}, \pm i}$ the $(\pm i)$-eigenspace of $J_\mathbb{C}$. Complex conjugation induces an isomorphism $V_{\mathbb{C}, i} \to V_{\mathbb{C}, -i}$ and consequently

$$(2.1.2) \qquad \dim_\mathbb{C} V_{\mathbb{C}, i} = \dim_\mathbb{C} V_{\mathbb{C}, -i} = \frac{1}{2} \dim_\mathbb{Q} V.$$

Let us quickly recall from [24, 3.23] the determinant morphism. See [10, 2.3] for a more detailed discussion. Let R be a ring, A a (not necessarily commutative) R-algebra and let M be a left A-module which is finite locally

free as an R-module. Denote by $V = V_A$ the functor on the category of R-algebras with $V(S) = A \otimes_R S$. We define a morphism $\det_{M,A} = \det_M :$ $V \to \mathbb{A}^1_R$ on S-valued points by

$$\det_M(S) : A \otimes_R S \to S, \quad x \mapsto \det_S(M_S \xrightarrow{x\cdot} M_S).$$

For $x \in A$ denote by $\chi_R(x|M)$ the characteristic polynomial of $M \xrightarrow{x\cdot} M$ over R. Below we will phrase the determinant condition using characteristic polynomials. This is warranted by the following statement.

PROPOSITION 2.1.1. — *Let A be a (not necessarily commutative) R-algebra and let M and N be A-modules which are finite locally free over R. Let $A_0 \subset A$ be a generating set of A as an R-module. Then $\det_M = \det_N$ if and only if for all $a \in A_0$ we have $\chi_R(a|M) = \chi_R(a|N)$.*

Proof. — Clear by the existence of Amitsur's formula [1, Theorem A], which, in a suitable sense, expresses the characteristic polynomial of a linear combination of endomorphisms in terms of the characteristic polynomials of the summands. $\qquad\square$

As $V_{\mathbb{C},-i}$ is a $B \otimes \mathbb{C}$-module we get a morphism $\det_{V_{\mathbb{C},-i}} : V_{B \otimes \mathbb{C}} \to \mathbb{A}^1_\mathbb{C}$. Consider the reflex field $E = \mathbb{Q}(\mathrm{tr}_\mathbb{C}(b \otimes 1|V_{\mathbb{C},-i}); \ b \in B)$. The morphism $\det_{V_{\mathbb{C},-i}}$ is defined over $\mathcal{O}_E$. Fix a place $\mathcal{Q}$ of $\mathcal{O}_E$ lying over p.

2.2. Polarized $\mathcal{L}$-sets of abelian varieties

DEFINITION 2.2.1. — *Let R be an $\mathcal{O}_{E_\mathcal{Q}}$-algebra. A polarized $\mathcal{L}$-set of abelian varieties over R is a pair (A, λ), where $A = (A_\Lambda, \varrho_{\Lambda',\Lambda})$ is an $\mathcal{L}$-set of abelian varieties over R in the sense of [24, Definition 6.5], and where $\lambda : A \to A^\vee$ is a principal polarization in the sense of [24, Definition 6.6]. We say that (A, λ) is of determinant $\det_{V_{\mathbb{C},-i}}$ if for all $\Lambda \in \mathcal{L}$ we have an equality*

$$\det_{\mathrm{Lie}\, A_\Lambda} = \det_{V_{\mathbb{C},-i}} \otimes_{\mathcal{O}_E} R$$

of morphisms $V_{\mathcal{O}_B \otimes R} \to \mathbb{A}^1_R$.

We denote by $\mathcal{A}$ the functor on the category of $\mathcal{O}_{E_\mathcal{Q}}$-algebras with $\mathcal{A}(R)$ the set of isomorphism classes of polarized $\mathcal{L}$-sets of abelian varieties of determinant $\det_{V_{\mathbb{C},-i}}$ over R.

Remark 2.2.2. — After additionally imposing a suitable level structure away from p in the definition of $\mathcal{A}$, we may (and will) assume that $\mathcal{A}$ is representable by a quasi-projective scheme over $\mathcal{O}_{E_\mathcal{Q}}$, see [24, Definition 6.9] and the discussion following it. We have decided not to include this

level structure in our notation as it is of no importance for the question of the p-rank on a KR stratum.

Let R be a ring. For an abelian scheme A/R, we denote by $H_1^{dR}(A/R)$ the first de Rham homology of A. It is part of a canonical short exact sequence

$$(2.2.1) \qquad 0 \to \omega_{A^\vee} \to H_1^{dR}(A/R) \to \mathrm{Lie}(A) \to 0,$$

where $\omega_{A^\vee} \subset H_1^{dR}(A/R)$ denotes the Hodge filtration. All terms of (2.2.1) are finite locally free R-modules. We have $\mathrm{rk}_R H_1^{dR}(A/R) = 2\dim_R A$ and $\mathrm{rk}_R \mathrm{Lie}(A) = \mathrm{rk}_R \omega_{A^\vee} = \dim_R A$.

DEFINITION 2.2.3. — *We denote by $\widetilde{\mathcal{A}}$ the functor on the category of $\mathcal{O}_{E_\mathcal{Q}}$-algebras with $\widetilde{\mathcal{A}}(R)$ the set of isomorphism classes of pairs (A, γ), where A is a polarized $\mathcal{L}$-set of abelian varieties of determinant $\det_{V_\mathbb{C}, -i}$ over R and*

$$\gamma : H_1^{dR}(A) \xrightarrow{\sim} \mathcal{L} \otimes R$$

is an isomorphism of polarized multichains of $\mathcal{O}_B \otimes R$-modules of type $(\mathcal{L})$.

Denote by $\widetilde{\varphi} : \widetilde{\mathcal{A}} \to \mathcal{A}$ the morphism given on R-valued points by $\widetilde{\mathcal{A}}(R) \to \mathcal{A}(R)$, $(A, \gamma) \mapsto A$.

$\mathrm{Aut}(\mathcal{L})$ acts from the left on $\widetilde{\mathcal{A}}$ via $g \cdot (A, \gamma) = (A, g \circ \gamma)$ and $\widetilde{\varphi}$ is invariant for this action.

PROPOSITION 2.2.4 ([18, Theorem 2.2]). — *The morphism $\widetilde{\varphi} : \widetilde{\mathcal{A}} \to \mathcal{A}$ is an $\mathrm{Aut}(\mathcal{L})$-torsor for the étale topology. In particular $\widetilde{\varphi}(\mathbb{F})$ is an $\mathrm{Aut}(\mathcal{L})(\mathbb{F})$-torsor in the set-theoretic sense.*

2.3. The local model diagram and the KR stratification

We will use the following obvious variant of [24, Definition 3.27].

DEFINITION 2.3.1. — *The local model M^{loc} is the functor on the category of $\mathcal{O}_{E_\mathcal{Q}}$-algebras with $M^{\mathrm{loc}}(R)$ the set of tuples $(t_\Lambda)_{\Lambda \in \mathcal{L}}$ of $\mathcal{O}_B \otimes R$-submodules $t_\Lambda \subset \Lambda_R$ satisfying the following conditions for all $\Lambda \subset \Lambda'$ in $\mathcal{L}$.*

(1) We have $\rho_{\Lambda', \Lambda, R}(t_\Lambda) \subset t_{\Lambda'}$, so that we get a commutative diagram

$$\begin{array}{ccc} t_\Lambda & \longrightarrow & t_{\Lambda'} \\ \downarrow & & \downarrow \\ \Lambda_R & \xrightarrow{\rho_{\Lambda', \Lambda, R}} & \Lambda'_R. \end{array}$$

(2) *The quotient Λ_R/t_Λ is a finite locally free R-module.*
(3) *We have an equality*

$$\det{}_{\Lambda_R/t_\Lambda} = \det{}_{V_{\mathbb{C},-i}} \otimes_{\mathcal{O}_E} R$$

   of morphisms $V_{\mathcal{O}_B \otimes R} \to \mathbb{A}^1_R$.
(4) *Under the pairing $(\cdot,\cdot)_{\Lambda,R} : \Lambda_R \times \Lambda_R^\vee \to R$, the submodules t_Λ and $t_{\Lambda^\vee}$ pair to zero.*
(5) *We have $\vartheta_{\Lambda,b,R}(t_\Lambda) = t_{b\Lambda}$ for all $b \in (B \otimes \mathbb{Q}_p)^\times$ that normalize $\mathcal{O}_B \otimes \mathbb{Z}_p$.*

Remark 2.3.2. — We have added the natural condition 2.3.1(5), which seems to be missing from [24, Definition 3.27].

Remark 2.3.3. — By definition, M^{loc} is a closed subscheme of a finite product of Grassmannians. In particular M^{loc} is a projective scheme over $\operatorname{Spec} \mathcal{O}_{E_{\mathcal{Q}}}$.

Remark 2.3.4. — Let R be an $\mathcal{O}_{E_{\mathcal{Q}}}$-algebra and $(t_\Lambda)_\Lambda \in M^{\mathrm{loc}}(R)$. For $\Lambda \in \mathcal{L}$ the decomposition $\Lambda = \Lambda_1 \times \cdots \times \Lambda_m$ induces a decomposition $t_\Lambda = t_{\Lambda,1} \times \cdots \times t_{\Lambda,m}$ into $\mathcal{O}_{B_i} \otimes R$-submodules $t_{\Lambda,i} \subset \Lambda_{i,R}$. Let $i \in \{1,\ldots,m\}$ and let $\Lambda \subset \Lambda'$ in $\mathcal{L}$ with $\Lambda_i = \Lambda'_i$. From condition 2.3.1(1) we conclude that $t_{\Lambda,i} \subset t_{\Lambda',i}$. From condition 2.3.1(3) we conclude that $t_{\Lambda,i}$ and $t_{\Lambda',i}$ both have the same rank over R. Thus $t_{\Lambda,i} = t_{\Lambda',i}$ in view of 2.3.1(2). Consequently we may unambiguously write $t_{\Lambda_i} = t_{\Lambda,i}$.

We conclude that the family $(t_\Lambda)_{\Lambda \in \mathcal{L}}$ is determined by the tuple of families

$$\left((t_{\Lambda_1})_{\Lambda_1 \in \mathcal{L}_1}, \ldots, (t_{\Lambda_m})_{\Lambda_m \in \mathcal{L}_m} \right).$$

All conditions of Definition 2.3.1 with the exception of condition (4) translate into independent conditions on the individual (t_{Λ_i}).

DEFINITION 2.3.5. — *Denote by $\widetilde{\psi} : \widetilde{\mathcal{A}} \to M^{\mathrm{loc}}$ the morphism given on R-valued points by*

$$\widetilde{\mathcal{A}}(R) \to M^{\mathrm{loc}}(R),$$
$$((A_\Lambda),(\gamma_\Lambda)) \mapsto (\gamma_\Lambda(\omega_{A_\Lambda^\vee}))_\Lambda.$$

$\operatorname{Aut}(\mathcal{L})$ acts from the left on M^{loc} via $(\varphi_\Lambda) \cdot (t_\Lambda) = (\varphi_\Lambda(t_\Lambda))$ and $\widetilde{\psi}$ is equivariant for this action.

Definition 2.3.6. — *The diagram*

$$
\begin{array}{ccc}
 & \widetilde{\mathcal{A}} & \\
{}^{\widetilde{\varphi}}\swarrow & & \searrow{}^{\widetilde{\psi}} \\
\mathcal{A} & & M^{\mathrm{loc}}
\end{array}
$$
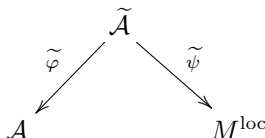
of $\mathcal{O}_{E_{\mathcal{Q}}}$-schemes is called the local model diagram.

Remark 2.3.7 ([24, Chapter 3], *cf.* [18, Theorem 2.2]). — The morphisms $\widetilde{\varphi}$ and $\widetilde{\psi}$ are smooth of the same relative dimension. There is, étale locally on $\mathcal{A}$, a section $s : \mathcal{A} \to \widetilde{\mathcal{A}}$ of $\widetilde{\varphi}$, such that the composition $\widetilde{\psi} \circ s : \mathcal{A} \to M^{\mathrm{loc}}$ is étale.

Consider the decomposition

$$
M^{\mathrm{loc}}(\mathbb{F}) = \coprod_{x \in \mathrm{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F})} M_x^{\mathrm{loc}}
$$

into $\mathrm{Aut}(\mathcal{L})(\mathbb{F})$-orbits.

Remark 2.3.8. — Let $x \in \mathrm{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F})$. The subset $M_x^{\mathrm{loc}} \subset M^{\mathrm{loc}}(\mathbb{F})$ is locally closed, and we equip it with the reduced scheme structure. By [24, Theorem 3.16] the $\mathbb{F}$-group $\mathrm{Aut}(\mathcal{L})_{\mathbb{F}}$ is smooth and affine. Thus M_x^{loc} is a smooth quasi-projective variety over $\mathbb{F}$.

For $x \in \mathrm{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F})$, we define $\widetilde{\mathcal{A}}_x = \widetilde{\psi}(\mathbb{F})^{-1}(M_x^{\mathrm{loc}})$ and $\mathcal{A}_x = \widetilde{\varphi}(\mathbb{F})(\widetilde{\mathcal{A}}_x)$. It follows from Proposition 2.2.4 that the $\mathcal{A}_x$ are pairwise disjoint and cover $\mathcal{A}(\mathbb{F})$ as x runs through $\mathrm{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F})$.

Definition 2.3.9. — *The decomposition*

$$
\mathcal{A}(\mathbb{F}) = \coprod_{x \in \mathrm{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F})} \mathcal{A}_x
$$

is called the Kottwitz-Rapoport *(or KR) stratification on $\mathcal{A}$.*

Remark 2.3.10. — By Remark 2.3.7 there is, étale locally on $\mathcal{A}_{\mathbb{F}}$, an étale morphism $\beta : \mathcal{A}_{\mathbb{F}} \to M_{\mathbb{F}}^{\mathrm{loc}}$ with $\mathcal{A}_x = \beta^{-1}(M_x^{\mathrm{loc}})$ for $x \in \mathrm{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F})$. Hence the subset $\mathcal{A}_x \subset \mathcal{A}(\mathbb{F})$ is locally closed, and after equipping it with the reduced scheme structure, $\mathcal{A}_x$ is a smooth variety over $\mathbb{F}$.

2.4. The p-rank on a KR stratum

Lemma 2.4.1. — *Let $A/\mathbb{Q}_p$ be a finite simple algebra and let $\mathcal{O}_A \subset A$ be a maximal $\mathbb{Z}_p$-order. Then all simple left $\mathcal{O}_A$-modules and all simple right $\mathcal{O}_A$-modules have the same finite cardinality.*

Proof. — By [25, Theorem 17.3] there exist a finite division algebra $D/\mathbb{Q}_p$, an integer $n \in \mathbb{N}$ and an isomorphism $A \simeq M^{n \times n}(D)$ inducing an isomorphism $\mathcal{O}_A \simeq M^{n \times n}(\mathcal{O}_D)$. Here $\mathcal{O}_D \subset D$ denotes the unique maximal $\mathbb{Z}_p$-order, see [25, Theorem 12.8].

Denote by $\mathfrak{p} \subset \mathcal{O}_D$ the unique maximal ideal and by $k = \mathcal{O}_D/\mathfrak{p}$ the corresponding residue field, see [25, Theorem 13.2]. By loc. cit. every simple left (resp. right) $\mathcal{O}_D$-module is isomorphic to k. Hence by Morita equivalence (see [25, §§16]) every simple left (resp. right) $M^{n \times n}(\mathcal{O}_D)$-module is isomorphic to $k^n = M^{n \times 1}(k)$ (resp. $k^n = M^{1 \times n}(k)$). $\square$

DEFINITION 2.4.2. — *The multichain $\mathcal{L}$ is called* complete *if for any two neighbors $\Lambda \subset \Lambda'$ in $\mathcal{L}$, the quotient Λ'/Λ is a simple $\mathcal{O}_B \otimes \mathbb{Z}_p$-module.*

For a finite commutative group scheme $G/\mathbb{F}$, we denote by $G^{e,u}$ the étale unipotent and by $G^{i,m}$ the infinitesimal multiplicative part of G. Let $\mathrm{rk}_{e,u}(G) := \mathrm{rk}(G^{e,u})$ and $\mathrm{rk}_{i,m}(G) := \mathrm{rk}(G^{i,m})$.

LEMMA 2.4.3. — *Assume that $\mathcal{L}$ is complete. Let $(A_\Lambda, \varrho_{\Lambda',\Lambda})$ be an $\mathcal{L}$-set of abelian varieties over $\mathbb{F}$ and let $\Lambda \subset \Lambda'$ be neighbors in $\mathcal{L}$. Then $K = \ker \varrho_{\Lambda',\Lambda}$ is either étale unipotent or infinitesimal multiplicative or infinitesimal unipotent.*

Proof. — The decomposition (2.1.1) induces a decomposition $K = K_1 \times \cdots \times K_m$ into finite locally free group schemes K_i with actions $\mathcal{O}_{B_i} \to \mathrm{End}(K_i)$.

As Λ and Λ' are neighbors, there is a unique $i_0 \in \{1, \ldots, m\}$ with $\Lambda_{i_0} \subsetneqq \Lambda'_{i_0}$, and as $\mathcal{L}$ is complete we know that $\Lambda'_{i_0}/\Lambda_{i_0}$ is a simple left $\mathcal{O}_{B_{i_0}}$-module. Let $N = |\Lambda'_{i_0}/\Lambda_{i_0}|$. By the definition of an $\mathcal{L}$-set of abelian varieties we know that $K_i = 0$ for $i \neq i_0$ and that $G := K_{i_0}$ has rank N over $\mathbb{F}$.

The action $\mathcal{O}_{B_{i_0}} \to \mathrm{End}\, G$ induces on $G(\mathbb{F})$ the structure of a left $O_{B_{i_0}}$-module, and as $|G(\mathbb{F})| \leqslant \mathrm{rk}\, G = N$, Lemma 2.4.1 implies $|G(\mathbb{F})| \in \{0, N\}$. As $|G(\mathbb{F})| = \mathrm{rk}(G^{e,u})$, we conclude that $G^{e,u} \in \{0, G\}$.

Denote by $D(G)$ the Cartier dual of G. We also obtain on $D(G)(\mathbb{F})$ the structure of a right $O_{B_{i_0}}$-module and we analogously obtain that $D(G)^{e,u} \in \{0, D(G)\}$. As $D(G)^{e,u} = D(G^{i,m})$, it follows that $G^{i,m} \in \{0, G\}$. $\square$

DEFINITION 2.4.4. — *Let $A/\mathbb{F}$ be an abelian variety. Denote by $[p]_A : A \to A$ the multiplication by p and by $A[p]$ the kernel of $[p]_A$. The integer $\log_p \mathrm{rk}_{e,u} A[p]$ is called the p-rank of A.*

PROPOSITION 2.4.5. — *Assume that $\mathcal{L}$ is complete. Let $(A_\Lambda, \varrho_{\Lambda',\Lambda})$ be an $\mathcal{L}$-set of abelian varieties over $\mathbb{F}$. Let $\Lambda \in \mathcal{L}$ and choose a sequence*

$p^{-1}\Lambda = \Lambda^{(0)} \supsetneqq \Lambda^{(1)} \supsetneqq \cdots \supsetneqq \Lambda^{(k)} = \Lambda$ of neighbors $\Lambda^{(j-1)} \supsetneqq \Lambda^{(j)}$ in $\mathcal{L}$. Define

$$J_{e,u} = \{j \in \{1, \ldots, k\} \mid \ker \varrho_{\Lambda^{(j-1)}, \Lambda^{(j)}} \text{ is étale}\}.$$

Then the p-rank of A_Λ is equal to

$$\sum_{j \in J_{e,u}} \log_p |\Lambda^{(j-1)}/\Lambda^{(j)}|.$$

Proof. — By the definition of an $\mathcal{L}$-set of abelian varieties, there is a periodicity isomorphism $\theta_{p^{-1}\Lambda, p} : A_{p^{-1}\Lambda} \xrightarrow{\sim} A_\Lambda$ such that

$$[p]_{A_\Lambda} = \theta_{p^{-1}\Lambda, p} \circ \prod_{j=1}^{k} \varrho_{\Lambda^{(j-1)}, \Lambda^{(j)}}.$$

This implies

$$\mathrm{rk}_{e,u} A_\Lambda[p] = \prod_{j=1}^{k} \mathrm{rk}_{e,u} \ker \varrho_{\Lambda^{(j-1)}, \Lambda^{(j)}}.$$

Lemma 2.4.3 and the definition of an $\mathcal{L}$-set of abelian varieties yield

$$\mathrm{rk}_{e,u} \ker \varrho_{\Lambda^{(j-1)}, \Lambda^{(j)}} = \begin{cases} |\Lambda^{(j-1)}/\Lambda^{(j)}| & \text{if } j \in J_{e,u}, \\ 1 & \text{otherwise.} \end{cases}$$

$\square$

PROPOSITION 2.4.6. — *Let $A = (A_\Lambda, \varrho_{\Lambda', \Lambda}) \in \mathcal{A}(\mathbb{F})$, choose a lift $\widetilde{A} \in \widetilde{\mathcal{A}}(\mathbb{F})$ of A under $\widetilde{\varphi}(\mathbb{F})$ and let $(t_\Lambda) = \widetilde{\psi}(\mathbb{F})(\widetilde{A}) \in M_x^{\mathrm{loc}}$. Let $\Lambda \subset \Lambda'$ in $\mathcal{L}$. Then*

(2.4.1)
$$\ker \varrho_{\Lambda', \Lambda} \text{ is multiplicative}$$
$$\Leftrightarrow \rho_{\Lambda', \Lambda, \mathbb{F}}(t_\Lambda) = t_{\Lambda'}$$

and

(2.4.2)
$$\ker \varrho_{\Lambda', \Lambda} \text{ is étale}$$
$$\Leftrightarrow \Lambda'_{\mathbb{F}} = \mathrm{im}\, \rho_{\Lambda', \Lambda, \mathbb{F}} + t_{\Lambda'}.$$

Proof. — In view of the definition of $\widetilde{\psi}$, the stated equivalences amount to well-known characterizations of the respective conditions on $\ker \varrho_{\Lambda', \Lambda}$ in terms of the Hodge filtration inside the de Rham homology. $\square$

COROLLARY 2.4.7. — *Let $x \in \mathrm{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F})$ and $(A_\Lambda, \varrho_{\Lambda', \Lambda})$, $(A'_\Lambda, \varrho'_{\Lambda', \Lambda}) \in \mathcal{A}_x$. Let $\Lambda \subset \Lambda'$ in $\mathcal{L}$. Then $\ker \varrho_{\Lambda', \Lambda}$ is étale if and only if $\ker \varrho'_{\Lambda', \Lambda}$ is étale.*

Proof. — For $(t_\Lambda) \in M^{\mathrm{loc}}(\mathbb{F})$, the condition $\Lambda'_{\overline{\mathbb{F}}} = \operatorname{im} \rho_{\Lambda',\Lambda,\mathbb{F}} + t_{\Lambda'}$ is clearly invariant under the $\operatorname{Aut}(\mathcal{L})(\mathbb{F})$-action on $M^{\mathrm{loc}}(\mathbb{F})$. The claim therefore follows from (2.4.2). $\qquad\square$

THEOREM 2.4.8. — *Assume that $\mathcal{L}$ is complete.*
Let $x \in \operatorname{Aut}(\mathcal{L})(\mathbb{F})\backslash M^{\mathrm{loc}}(\mathbb{F})$ and $(A_\Lambda, \varrho_{\Lambda',\Lambda}), (A'_\Lambda, \varrho'_{\Lambda',\Lambda}) \in \mathcal{A}_x$. Let $\Lambda, \Lambda' \in \mathcal{L}$. Then the p-ranks of A_Λ and $A'_{\Lambda'}$ coincide. In other words, the p-rank is constant on a KR stratum.

Proof. — The p-rank of an abelian variety is an isogeny invariant by [15, p. 147], so that it suffices to treat the case $\Lambda = \Lambda'$. The statement then follows from Proposition 2.4.5 and Corollary 2.4.7. $\qquad\square$

2.5. A formula for the p-rank on a KR stratum

Denote by K the completion of the maximal unramified extension of $\mathbb{Q}_p$ and by $\mathcal{O}_K$ the valuation ring of K. We identify the residue field of K with $\mathbb{F}$. We denote by σ the Frobenius automorphism on K, inducing the usual Frobenius $\mathbb{F} \to \mathbb{F}, x \mapsto x^p$ on the residue field. By abuse of notation we also denote by σ the morphism $G(\sigma) : G(K) \to G(K)$.

2.5.1. A p-divisible group analogue

Let $Y/\mathbb{F}$ be a p-divisible group. We denote by $\mathbb{D}(Y)$ the covariant Dieudonné module of Y, see for example [2]. It is a free $\mathcal{O}_K$-module, equipped with a σ-linear endomorphism $F_\mathbb{D}$ and a σ^{-1}-linear endomorphism $V_\mathbb{D}$, satisfying $F_\mathbb{D} \circ V_\mathbb{D} = F_\mathbb{D} \circ V_\mathbb{D} = p$. Let Y, Y' be p-divisible groups over $\mathbb{F}$ and let $\lambda : Y \to (Y')^\vee$ be a morphism. It induces a pairing $(\cdot, \cdot)_\lambda : \mathbb{D}(Y) \times \mathbb{D}(Y') \to \mathcal{O}_K$ satisfying

$$(2.5.1) \qquad (F_\mathbb{D} x, y)_\lambda = \sigma\, (x, V_\mathbb{D} y)_\lambda\,, \;\; x \in \mathbb{D}(Y), y \in \mathbb{D}(Y').$$

We denote by $(\overline{\mathbb{D}}(Y), F_{\overline{\mathbb{D}}}, V_{\overline{\mathbb{D}}})$ the reduction of $\mathbb{D}(Y), F_\mathbb{D}, V_\mathbb{D})$ modulo p.
If $A/\mathbb{F}$ is an abelian variety, let $\mathbb{D}(A) = \mathbb{D}(A[p^\infty])$.

PROPOSITION 2.5.1 ([17, Corollary 5.11]). —

(1) *Let $A/\mathbb{F}$ be an abelian variety. There is a canonical isomorphism*

$$(2.5.2) \qquad\qquad \iota = \iota_A : \overline{\mathbb{D}}(A) \xrightarrow{\sim} H_1^{dR}(A),$$
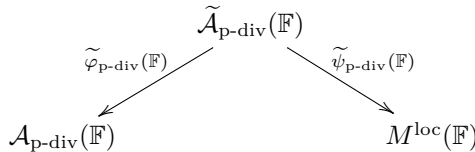
inducing an isomorphism of short exact sequences

(2.5.3)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{im} V_{\overline{\mathbb{D}}} & \longrightarrow & \overline{\mathbb{D}}(A) & \longrightarrow & \operatorname{Lie}(A[p^\infty]) & \longrightarrow & 0 \\
& & \downarrow{\simeq} & & \iota\downarrow{\simeq} & & \downarrow{\simeq} & & \\
0 & \longrightarrow & \omega_{A^\vee} & \longrightarrow & H_1^{dR}(A) & \longrightarrow & \operatorname{Lie}(A) & \longrightarrow & 0.
\end{array}
$$

(2) Let $A = (A_\Lambda) \in \mathcal{A}(\mathbb{F})$. For $\Lambda \in \mathcal{L}$ let $\iota_\Lambda = \iota_{A_\Lambda} : \overline{\mathbb{D}}(A_\Lambda) \to H_1^{dR}(A_\Lambda)$ be the isomorphism from (2.5.2). The resulting morphism

(2.5.4)
$$\iota = (\iota_\Lambda)_\Lambda : (\overline{\mathbb{D}}(A_\Lambda))_\Lambda \xrightarrow{\sim} (H_1^{dR}(A_\Lambda))_\Lambda$$

is an isomorphism of polarized multichains of $\mathcal{O}_B \otimes \mathbb{F}$-modules of type $(\mathcal{L})$.

In complete analogy with [24, Definition 6.5] and Definition 2.2.1 we have the notion of a polarized $\mathcal{L}$-set $Y = (Y_\Lambda, \varrho_{\Lambda',\Lambda}, \lambda_\Lambda)$ of p-divisible groups of determinant $\det_{V_{\mathbb{C},-i}}$ over $\mathbb{F}$. We further obtain a set $\mathcal{A}_{\text{p-div}}(\mathbb{F})$ of isomorphism classes of such Y. In analogy with Definition 2.2.3, denote by $\widetilde{\mathcal{A}}_{\text{p-div}}(\mathbb{F})$ the set of isomorphism classes of pairs (Y, γ), where Y is as above and $\gamma : \overline{\mathbb{D}}(Y) \xrightarrow{\sim} \mathcal{L} \otimes \mathbb{F}$ is an isomorphism of polarized multichains of $\mathcal{O}_B \otimes \mathbb{F}$-modules of type $(\mathcal{L})$. We have the canonical map $\widetilde{\varphi}_{\text{p-div}}(\mathbb{F}) : \widetilde{\mathcal{A}}_{\text{p-div}}(\mathbb{F}) \to \mathcal{A}_{\text{p-div}}(\mathbb{F})$, $(Y, \gamma) \mapsto Y$, and the morphism $\widetilde{\psi}_{\text{p-div}}(\mathbb{F}) : \widetilde{\mathcal{A}}_{\text{p-div}}(\mathbb{F}) \to M^{\text{loc}}(\mathbb{F})$, $(Y, \gamma) \mapsto \gamma(\operatorname{im} V_{\overline{\mathbb{D}}})$.

$$
\begin{array}{ccc}
 & \widetilde{\mathcal{A}}_{\text{p-div}}(\mathbb{F}) & \\
{\scriptstyle \widetilde{\varphi}_{\text{p-div}}(\mathbb{F})}\swarrow & & \searrow{\scriptstyle \widetilde{\psi}_{\text{p-div}}(\mathbb{F})} \\
\mathcal{A}_{\text{p-div}}(\mathbb{F}) & & M^{\text{loc}}(\mathbb{F})
\end{array}
$$

By Proposition 2.5.1 we obtain maps $\delta : \mathcal{A}(\mathbb{F}) \to \mathcal{A}_{\text{p-div}}(\mathbb{F})$, $A \mapsto A[p^\infty]$ and $\widetilde{\delta} : \widetilde{\mathcal{A}}(\mathbb{F}) \to \widetilde{\mathcal{A}}_{\text{p-div}}(\mathbb{F})$, $(A, \gamma) \mapsto (A[p^\infty], \gamma \circ \iota)$, and the following diagrams commute.

$$
\begin{array}{ccc}
\widetilde{\mathcal{A}}(\mathbb{F}) & \xrightarrow{\widetilde{\delta}} & \widetilde{\mathcal{A}}_{\text{p-div}}(\mathbb{F}) \\
{\scriptstyle \widetilde{\varphi}(\mathbb{F})}\downarrow & & \downarrow{\scriptstyle \widetilde{\varphi}_{\text{p-div}}(\mathbb{F})} \\
\mathcal{A}(\mathbb{F}) & \xrightarrow{\delta} & \mathcal{A}_{\text{p-div}}(\mathbb{F}),
\end{array}
\qquad
\begin{array}{ccc}
\widetilde{\mathcal{A}}(\mathbb{F}) & \xrightarrow{\widetilde{\delta}} & \widetilde{\mathcal{A}}_{\text{p-div}}(\mathbb{F}) \\
{\scriptstyle \widetilde{\psi}(\mathbb{F})}\downarrow & & \downarrow{\scriptstyle \widetilde{\psi}_{\text{p-div}}(\mathbb{F})} \\
M^{\text{loc}}(\mathbb{F}) & = = = & M^{\text{loc}}(\mathbb{F}).
\end{array}
$$

In absolute analogy with Definition 2.3.9, we obtain a decomposition

$$\mathcal{A}_{\text{p-div}}(\mathbb{F}) = \coprod_{x \in \operatorname{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\text{loc}}(\mathbb{F})} \mathcal{A}_{\text{p-div},x},$$

which we call the *KR stratification* on $\mathcal{A}_{\text{p-div}}(\mathbb{F})$. For $x \in \operatorname{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\text{loc}}(\mathbb{F})$ we have $\mathcal{A}_x = \delta^{-1}(\mathcal{A}_{\text{p-div},x})$. We define as in Definition 2.4.4 the p-rank of a p-divisible group over $\mathbb{F}$. The proof of Theorem 2.4.8 then carries over without any changes to show the following statement.

THEOREM 2.5.2. — *Assume that $\mathcal{L}$ is complete. Then the p-rank is constant on a KR stratum in $\mathcal{A}_{\text{p-div}}(\mathbb{F})$.*

2.5.2. The map $\alpha : \mathcal{A}_{\text{p-div}}(\mathbb{F}) \to B_I(G)$

LEMMA 2.5.3. — *Let $\mathcal{M}$ and $\mathcal{M}'$ be polarized multichains of $\mathcal{O}_B \otimes \mathcal{O}_K$-modules of type $(\mathcal{L})$. Then the canonical map $\operatorname{Isom}(\mathcal{M}, \mathcal{M}')(\mathcal{O}_K) \to \operatorname{Isom}(\mathcal{M}, \mathcal{M}')(\mathbb{F})$ is surjective. In particular $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic.*

Proof. — Let $\mathcal{I} = \operatorname{Isom}(\mathcal{M}, \mathcal{M}')$. Clearly $\mathcal{I}$ is representable by an affine scheme over $\mathcal{O}_K$. By [24, Theorem 3.16] we know that the base-change $\mathcal{I} \otimes_{\mathcal{O}_K} \mathcal{O}_K/p^n$ is in particular formally smooth over $\mathcal{O}_K/p^n$ for every $n \in \mathbb{N}$. This easily implies the surjectivity of $\mathcal{I}(\mathcal{O}_K) \to \mathcal{I}(\mathbb{F})$ in view of $\mathcal{I}(\mathcal{O}_K) = \lim_{n \in \mathbb{N}} \mathcal{I}(\mathcal{O}_K/p^n)$. The second claim follows, as $\mathcal{I}(\mathbb{F}) \neq \emptyset$ by loc. cit. $\quad\square$

For $g \in G(K)$, we denote by $g \cdot (\mathcal{L} \otimes \mathcal{O}_K)$ the tuple $(g(\Lambda \otimes \mathcal{O}_K))_{\Lambda \in \mathcal{L}}$ of $\mathcal{O}_B \otimes \mathcal{O}_K$-submodules of $V \otimes K$. Let

$$I := \{ g \in G(K) \mid g \cdot (\mathcal{L} \otimes \mathcal{O}_K) = \mathcal{L} \otimes \mathcal{O}_K \}$$
$$= \{ g \in G(K) \mid \forall \Lambda \in \mathcal{L} : \; g(\Lambda \otimes \mathcal{O}_K) = \Lambda \otimes \mathcal{O}_K \}$$

and

$$I_0 := \{ g \in I \mid c(g) = 1 \}.$$

Our notation for these groups is motivated by the Iwahori case (i.e. the case where $\mathcal{L}$ is complete), cf. Remark 2.5.19. But note that we have not yet specialized to this case.

LEMMA 2.5.4. — *We have $I = \mathcal{O}_K^\times I_0$. In particular, any $g \in I$ satisfies $c(g) \in \mathcal{O}_K^\times$.*

Proof. — Let $g \in I$ and $\Lambda \in \mathcal{L}$. Then Λ is in particular an $\mathcal{O}_K$-lattice in the K-vector space V_K and the fact that g restricts to an automorphism of Λ implies that $\det(g) \in \mathcal{O}_K^\times$. The equation $\det(g)^2 = c(g)^{\dim_\mathbb{Q} V}$ then yields $c(g) \in \mathcal{O}_K^\times$. As $\mathcal{O}_K$ is strictly Henselian of residue characteristic different from 2, there is an $x \in \mathcal{O}_K^\times$ with $x^2 = c(g)$. Then $x^{-1}g \in I_0$, as desired. $\quad\square$

LEMMA 2.5.5. — *Let $g \in I_0$. Then g restricts to an automorphism $g_\Lambda : \Lambda \otimes \mathcal{O}_K \to \Lambda \otimes \mathcal{O}_K$ for each $\Lambda \in \mathcal{L}$. The assignment $g \mapsto (g_\Lambda)_\Lambda$ defines an isomorphism $I_0 \xrightarrow{\sim} \operatorname{Aut}(\mathcal{L})(\mathcal{O}_K)$.*

Proof. — If $(\varphi_\Lambda) \in \mathrm{Aut}(\mathcal{L})(\mathcal{O}_K)$, then $\varphi_\Lambda \otimes_{\mathcal{O}_K} K$ is an element of $G(K)$ which is independent of Λ. This provides an inverse to the map in question.
□

PROPOSITION 2.5.6 ([24, 3.23]). — *Let* $Y = (Y_\Lambda, \varrho_{\Lambda',\Lambda}, \lambda_\Lambda) \in \mathcal{A}_{\text{p-div}}(\mathbb{F})$. *For* $\Lambda \in \mathcal{L}$ *let* $\mathcal{E}_\Lambda : \mathbb{D}(Y_\Lambda) \times \mathbb{D}(Y_{\Lambda^\vee}) \to \mathcal{O}_K$ *be the pairing induced by* λ_Λ. *Then* $(\mathbb{D}(Y_\Lambda))_\Lambda$, *equipped with the pairings* $(\mathcal{E}_\Lambda)_\Lambda$, *is a polarized multichain of* $\mathcal{O}_B \otimes \mathcal{O}_K$-*modules of type* $(\mathcal{L})$.

Define equivalence relations $\sim$ and $\sim_I$ on $G(K)$ by

$$x \sim y :\Leftrightarrow \exists g \in G(K) : y = g x \sigma(g)^{-1},$$
$$x \sim_I y :\Leftrightarrow \exists i \in I : y = i x \sigma(i)^{-1}$$

and denote by $B(G) = G(K)/\sim$ and $B_I(G) = G(K)/\sim_I$ the corresponding quotients. For an element $b \in G(K)$, we denote by $[b]$ its equivalence class in $B(G)$.

Let $(Y_\Lambda) \in \mathcal{A}_{\text{p-div}}(\mathbb{F})$. By Lemma 2.5.3 there is an isomorphism $\varphi = (\varphi_\Lambda)_\Lambda : (\mathbb{D}(Y_\Lambda))_\Lambda \xrightarrow{\sim} \mathcal{L} \otimes \mathcal{O}_K$ of polarized multichains of $\mathcal{O}_B \otimes \mathcal{O}_K$-modules of type $(\mathcal{L})$. Let $\Lambda \in \mathcal{L}$. Then $\mathfrak{F}_\Lambda = \varphi_\Lambda \circ F_\mathbb{D} \circ \varphi_\Lambda^{-1}$ is a σ-linear endomorphism of $\Lambda \otimes \mathcal{O}_K$. By functoriality of the Dieudonné module, the base-change $\mathfrak{F}_\Lambda \otimes_{\mathcal{O}_K} K : V \otimes K \to V \otimes K$ is independent of Λ and we simply denote it by $\mathfrak{F}$. In the same way, we obtain from the morphisms $V_\mathbb{D}$ on the Dieudonné modules a σ^{-1}-linear endomorphism $\mathfrak{V}$ of $V \otimes K$.

Let $b = \mathfrak{F} \circ (\mathrm{id}_V \otimes \sigma)^{-1}$. Then b is a $B \otimes K$-linear endomorphism of $V \otimes K$, and in view of (2.5.1) we have $b \in G(K)$, with $c(b) = p$. If $\varphi' : (\mathbb{D}(Y_\Lambda))_\Lambda \xrightarrow{\sim} \mathcal{L} \otimes \mathcal{O}_K$ is another isomorphism, we have $\varphi' = i \circ \varphi$ for some $i \in I_0$, and the resulting element $b' \in G(K)$ will satisfy $b' = i b \sigma(i)^{-1}$. In this way we obtain a well-defined map $\alpha : \mathcal{A}_{\text{p-div}}(\mathbb{F}) \to B_I(G)$.

The canonical projection $G(K) \to I \backslash G(K)/I$ factors through $B_I(G)$, so that we obtain a map $B_I(G) \xrightarrow{\mathrm{can.}} I \backslash G(K)/I$. Denote by γ the composition $\mathcal{A}_{\text{p-div}}(\mathbb{F}) \xrightarrow{\alpha} B_I(G) \xrightarrow{\mathrm{can.}} I \backslash G(K)/I$. Denote by Perm $\subset I \backslash G(K)/I$ the image of γ. By abuse of notation, we also denote by $\gamma : \mathcal{A}_{\text{p-div}}(\mathbb{F}) \to \mathrm{Perm}$ the induced map.

Denote by $B_I(G)_{\mathrm{Perm}} \subset B_I(G)$ the preimage of Perm under the canonical map $B_I(G) \xrightarrow{\mathrm{can.}} I \backslash G(K)/I$. By abuse of notation, we also denote by $\alpha : \mathcal{A}_{\text{p-div}}(\mathbb{F}) \to B_I(G)_{\mathrm{Perm}}$ the induced map. The situation is visualized by

the following commutative diagram, in which the square is cartesian.

$$
\begin{array}{ccc}
B_I(G) & \xrightarrow{\ \text{can.}\ } & I\backslash G(K)/I \\[4pt]
\cup & & \cup \\[6pt]
\mathcal{A}_{\text{p-div}}(\mathbb{F}) \xrightarrow{\ \alpha\ } B_I(G)_{\text{Perm}} & \xrightarrow{\ \text{can.}\ } & \text{Perm}
\end{array}
$$

$$\gamma$$

The following two results show that the map $\alpha : \mathcal{A}_{\text{p-div}}(\mathbb{F}) \to B_I(G)_{\text{Perm}}$ is very close to being a bijection. Our proofs are based on the discussion of the Siegel case by Hoeve in [11, Chapter 7].

PROPOSITION 2.5.7. — *The map* $\alpha : \mathcal{A}_{\text{p-div}}(\mathbb{F}) \to B_I(G)_{\text{Perm}}$ *is surjective.*

Proof. — Let $\bar{b} \in B_I(G)_{\text{Perm}}$ and pick any representative $b \in G(K)$ of $\bar{b}$. By (2.5.1) and Lemma 2.5.4 there is a unit $v \in \mathcal{O}_K^\times$ with $c(b) = vp$. Using Lang's Lemma in combination with an approximation argument, we find a $u \in \mathcal{O}_K^\times$ with $v = (u\sigma(u)^{-1})^2$. After replacing b by $u^{-1}b\sigma(u)$, we may assume that $c(b) = p$.

Define $\mathfrak{F} = b \circ (\text{id}_V \otimes \sigma)$ and $\mathfrak{V} = p\mathfrak{F}^{-1}$. Let $\Lambda \in \mathcal{L}$. Then $\Lambda \otimes \mathcal{O}_K$ is stable under $\mathfrak{F}$ and $\mathfrak{V}$, and we denote by $\mathfrak{F}_\Lambda$ and $\mathfrak{V}_\Lambda$ the induced endomorphisms of $\Lambda \otimes \mathcal{O}_K$. Dieudonné theory implies that the chain $((\Lambda \otimes \mathcal{O}_K, \mathfrak{F}_\Lambda, \mathfrak{V}_\Lambda), \rho_{\Lambda', \Lambda})$ is of the form $\mathbb{D}(Y)$ for an $\mathcal{L}$-set $Y = (Y_\Lambda, \varrho_{\Lambda', \Lambda})$ of p-divisible groups of determinant $\det_{V_{\mathbb{C}, -i}}$ over $\mathbb{F}$.

From $c(b) = p$, we obtain

$$
(\mathfrak{F}_\Lambda x, y)_{\Lambda, \mathcal{O}_K} = \sigma\, (x, \mathfrak{V}_{\Lambda^\vee} y)_{\Lambda, \mathcal{O}_K}\,, \ \ x \in \Lambda \otimes \mathcal{O}_K, y \in \Lambda^\vee \otimes \mathcal{O}_K,
$$

compare (2.5.1). Dieudonné theory then implies that $(\cdot, \cdot)_{\Lambda, \mathcal{O}_K}$ is induced by an isomorphism $\lambda_\Lambda : Y_\Lambda \to Y_{\Lambda^\vee}^\vee$, and the tuple $\lambda = (\lambda_\Lambda)$ provides us with a polarization of Y. The isomorphism class of (Y, λ) is the desired preimage of $\bar{b}$ under α. $\qquad\square$

PROPOSITION 2.5.8. — *Let* $Y = (Y_\Lambda, \varrho_{\Lambda', \Lambda}, \lambda_\Lambda)$ *and* $Y' = (Y'_\Lambda, \varrho'_{\Lambda', \Lambda}, \lambda'_\Lambda)$ *be two points of* $\mathcal{A}_{\text{p-div}}(\mathbb{F})$. *Then* $\alpha(Y) = \alpha(Y')$ *if and only if there exist both an isomorphism* $\phi = (\phi_\Lambda) : (Y_\Lambda, \varrho_{\Lambda', \Lambda}) \to (Y'_\Lambda, \varrho'_{\Lambda', \Lambda})$ *of* $\mathcal{L}$*-sets of* p-*divisible groups over* $\mathbb{F}$ *and a unit* $u \in \mathbb{Z}_p^\times$ *such that the following diagram*

commutes for all $\Lambda \in \mathcal{L}$.

$$
\begin{array}{ccc}
Y_\Lambda & \xrightarrow{\phi_\Lambda} & Y'_\Lambda \\
{\scriptstyle u\lambda_\Lambda}\downarrow & & \downarrow{\scriptstyle \lambda'_\Lambda} \\
Y^\vee_{\Lambda^\vee} & \xleftarrow{\phi^\vee_{\Lambda^\vee}} & Y'^\vee_{\Lambda^\vee}.
\end{array}
$$

Proof. — Choose isomorphisms $\varphi : \mathbb{D}(Y) \xrightarrow{\sim} \mathcal{L} \otimes \mathcal{O}_K$ and $\varphi' : \mathbb{D}(Y') \xrightarrow{\sim} \mathcal{L} \otimes \mathcal{O}_K$, and denote by b and b' the resulting elements of $G(K)$ as above. If $\alpha(Y) = \alpha(Y')$, there is an $i \in I$ with $b' = ib\sigma(i)^{-1}$. We have seen above that $c(b) = p = c(b')$, so that $c(i) = \sigma(c(i))$. By Lemma 2.5.4 and [12, Lemma 1.2], the element $u := c(i)$ lies in $\mathbb{Z}_p^\times$. As in Lemma 2.5.5 we consider i as a tuple of endomorphisms $(\Lambda \otimes \mathcal{O}_K \to \Lambda \otimes \mathcal{O}_K)_{\Lambda \in \mathcal{L}}$. The composition $\varphi'^{-1} \circ i \circ \varphi$ corresponds under Dieudonné theory to the desired morphism ϕ. Similarly for the converse. $\qquad\square$

2.5.3. The map γ and the KR stratification

PROPOSITION 2.5.9. — *Two points* $Y, Y' \in \mathcal{A}_{\text{p-div}}(\mathbb{F})$ *lie in the same KR stratum if and only if* $\gamma(Y) = \gamma(Y')$.

Proof. — The map $G(K) \to G(K)$, $g \mapsto p\sigma^{-1}(g^{-1})$ descends to a well-defined bijection $\tau : I\backslash G(K)/I \to I\backslash G(K)/I$, and it suffices to show the corresponding statement for the composition $\mathcal{A}_{\text{p-div}}(\mathbb{F}) \xrightarrow{\gamma} I\backslash G(K)/I \xrightarrow{\tau} I\backslash G(K)/I$ instead of γ.

Let $Y = (Y_\Lambda)_\Lambda \in \mathcal{A}_{\text{p-div}}(\mathbb{F})$. Choose an isomorphism $\varphi : (\mathbb{D}(Y_\Lambda))_\Lambda \xrightarrow{\sim} \mathcal{L} \otimes \mathcal{O}_K$ and denote by $\mathfrak{F}$ and $\mathfrak{V}$ the resulting endomorphisms of $V \otimes K$, as above. Let $b \in G(K)$ with $\mathfrak{F} = b \circ (\text{id}_V \otimes \sigma)$. We have $\mathfrak{V} = p\sigma^{-1}(b^{-1}) \circ (\text{id}_V \otimes \sigma^{-1})$, so that $\mathfrak{V}(\Lambda \otimes \mathcal{O}_K) = p\sigma^{-1}(b^{-1})(\Lambda \otimes \mathcal{O}_K)$, $\Lambda \in \mathcal{L}$.

For $\Lambda \in \mathcal{L}$, denote by $\pi_\Lambda : \Lambda \otimes \mathcal{O}_K \to \Lambda \otimes \mathbb{F}$ the canonical projection. For a tuple $M = (M_\Lambda)_{\Lambda \in \mathcal{L}}$ of $\mathcal{O}_B \otimes \mathcal{O}_K$-submodules $p\Lambda \otimes \mathcal{O}_K \subset M_\Lambda \subset \Lambda \otimes \mathcal{O}_K$ further write $\pi(M) = (\pi_\Lambda(M_\Lambda))_\Lambda$. Note that M and $\pi(M)$ mutually determine each other.

By definition the KR stratum that $Y \in \mathcal{A}_{\text{p-div}}(\mathbb{F})$ lies in is given by the $\text{Aut}(\mathcal{L})(\mathbb{F})$-orbit of the point $\pi\big((\mathfrak{V}(\Lambda \otimes \mathcal{O}_K))_\Lambda\big) \in M^{\text{loc}}(\mathbb{F})$. By Lemma 2.5.3, Lemma 2.5.4 and Lemma 2.5.5, this orbit is equal to

$$
\pi\big(\{ip\sigma^{-1}(b^{-1}) \cdot (\mathcal{L} \otimes \mathcal{O}_K) \mid i \in I\}\big).
$$

We conclude by noting that for an element $g \in G(K)$, the set $\{ig \cdot (\mathcal{L} \otimes \mathcal{O}_K) \mid i \in I\}$ and the image of g in $I\backslash G(K)/I$ mutually determine each other. $\qquad\square$

DEFINITION 2.5.10. — *For $x \in \mathrm{Perm}$, we denote by $\mathcal{A}_{\mathrm{p\text{-}div},x} = \gamma^{-1}(x)$ and $\mathcal{A}_x = \delta^{-1}(\mathcal{A}_{\mathrm{p\text{-}div},x})$ the corresponding KR stratum in $\mathcal{A}_{\mathrm{p\text{-}div}}(\mathbb{F})$ and $\mathcal{A}(\mathbb{F})$, respectively.*

Remark 2.5.11 (Compare [11, 11.3]). — The normalization of Definition 2.5.10 amounts to indexing the KR stratification by the relative position of $\mathcal{L} \otimes \mathcal{O}_K$ to its image under *Frobenius*. This seems to be the natural normalization in the current context, see in particular Remark 2.5.17 below. In the context of affine flag varieties however, to be explained in the following sections, the natural normalization seems to be by the relative position of $\mathcal{L} \otimes \mathcal{O}_K$ to its image under *Verschiebung*, see for example Remark 3.9.4. This amounts to replacing the map γ by the composition $\mathcal{A}(\mathbb{F}) \xrightarrow{\gamma} I\backslash G(K)/I \xrightarrow{g \mapsto p\sigma^{-1}(g^{-1})} I\backslash G(K)/I$, as we have for example done in the proof of Proposition 2.5.9.

Let us note that for the question of the p-rank on a KR stratum both normalizations yield the same results.

Denote by $r_p : \mathcal{A}_{\mathrm{p\text{-}div}}(\mathbb{F}) \to \mathbb{N}$ the map with $r_p((Y_\Lambda)_\Lambda)$ equal to the common p-rank of the Y_Λ. In view Proposition 2.5.9, Theorem 2.5.2 amounts to the following statement.

THEOREM 2.5.12. — *Assume that $\mathcal{L}$ is complete.*
The map $r_p : \mathcal{A}_{\mathrm{p\text{-}div}}(\mathbb{F}) \to \mathbb{N}$ factors through $\mathcal{A}_{\mathrm{p\text{-}div}}(\mathbb{F}) \xrightarrow{\gamma} \mathrm{Perm}$.

2.5.4. The Newton stratification

The canonical projection $G(K) \to B(G)$ factors through $B_I(G)$, so that we get a map $B_I(G) \xrightarrow{\mathrm{can.}} B(G)$. Denote by β the composition $\mathcal{A}_{\mathrm{p\text{-}div}}(\mathbb{F}) \xrightarrow{\alpha} B_I(G) \xrightarrow{\mathrm{can.}} B(G)$.

DEFINITION 2.5.13. — *Let $b \in B(G)$. We define $\mathcal{N}_{\mathrm{p\text{-}div},b} := \beta^{-1}(b) \subset \mathcal{A}_{\mathrm{p\text{-}div}}(\mathbb{F})$ and $\mathcal{N}_b := (\beta \circ \delta)^{-1}(b) \subset \mathcal{A}(\mathbb{F})$, and call it the* Newton stratum *associated with b in $\mathcal{A}_{\mathrm{p\text{-}div}}(\mathbb{F})$ and $\mathcal{A}(\mathbb{F})$, respectively.*

Denote by $\mathbb{D}$ the diagonalizable affine group with character group $\mathbb{Q}$ over K. Let $b \in G(K)$. We denote by $\nu_b : \mathbb{D} \to G_K$ the corresponding Newton map, defined in [12, 4.2].[3] The morphism ν_b makes V_K into a representation of $\mathbb{D}$ and we consider the corresponding weight decomposition $V_K = \oplus_{\chi \in \mathbb{Q}} V_\chi$. We define

$$\nu_{b,0} := \dim_K V_0.$$

[3] Note that the discussion in loc. cit. still remains valid for not necessarily connected reductive groups over $\mathbb{Q}_p$.

If $g \in G(K)$, we know that $\nu_{gb\sigma(g)^{-1}} = \operatorname{Int}(g) \circ \nu_b$, where $\operatorname{Int}(g) : G(K) \to G(K)$, $h \mapsto ghg^{-1}$, and consequently $\nu_{b,0} = \nu_{gb\sigma(g)^{-1},0}$. Thus the map $G(K) \to \mathbb{N}$, $b \mapsto \nu_{b,0}$ factors through $B(G)$, and we also denote by $B(G) \to \mathbb{N}$, $b \mapsto \nu_{b,0}$ the resulting map.

PROPOSITION 2.5.14. — The map $r_p : \mathcal{A}_{\text{p-div}}(\mathbb{F}) \to \mathbb{N}$ factors as

$$\mathcal{A}_{\text{p-div}}(\mathbb{F}) \xrightarrow{\beta} B(G) \xrightarrow{b \mapsto \nu_{b,0}} \mathbb{N}.$$

In other words, for $b \in B(G)$ the p-rank on $\mathcal{N}_b$ is constant with value $\nu_{b,0}$.

Proof. — This follows from the fact that for a p-divisible group $Y/\mathbb{F}$ the isotypical component of slope 0 in $\mathbb{D}(Y) \otimes_{\mathcal{O}_K} K$ comes precisely from the étale part of Y, see for instance [2, IV]. $\square$

Assume from now on that $\mathcal{L}$ is complete. We can summarize the above discussion in the following commutative diagram, with the dotted arrow coming from Theorem 2.5.12.

$$(2.5.5)$$

DEFINITION 2.5.15. — Let $b \in G(K)$ and $x \in I \backslash G(K)/I$. The affine Deligne-Lusztig variety associated with b and x is defined by

$$X_x(b) = \{g \in G(K)/I \mid g^{-1}b\sigma(g) \in IxI\}.$$

PROPOSITION 2.5.16. — Let $x \in \operatorname{Perm}$ and $b \in G(K)$. Then the following equivalence holds.

$$X_x(b) \neq \emptyset \Leftrightarrow \mathcal{A}_{\text{p-div},x} \cap \mathcal{N}_{\text{p-div},[b]} \neq \emptyset.$$

Proof. — Follows from the definitions and Proposition 2.5.7. $\square$

Remark 2.5.17. — Let $x \in \operatorname{Perm}$ and let $b \in G(K)$. Although not established in full generality, it is expected that the following equivalence holds, see [7, Proposition 12.6].

$$(2.5.6) \qquad X_x(b) \neq \emptyset \Leftrightarrow \mathcal{A}_x \cap \mathcal{N}_{[b]} \neq \emptyset.$$

The difficulty in proving this equivalence lies in the construction of a suitable $\mathcal{L}$-set of abelian varieties with prescribed $\mathcal{L}$-set of p-divisible groups.

For recent progress in the *unramified* case due to Viehmann and Wedhorn see [30].

THEOREM 2.5.18. — *Let $x \in$ Perm and let $b \in G(K)$. Assume that $X_x(b) \neq \emptyset$. Then the p-rank on $\mathcal{A}_{\text{p-div},x}$ (and a fortiori on $\mathcal{A}_x$) is constant with value $\nu_{[b],0}$.*

Proof. — Clear from Proposition 2.5.16, Theorem 2.5.12 and Proposition 2.5.14. □

Remark 2.5.19. — We expect that $I \subset G(K)$ is an Iwahori subgroup. This is true in the situations to be studied in the following sections, and would provide a more natural view on the set $I \backslash G(K)/I$ occurring above as we could then identify it with a suitable Iwahori-Weyl group, see [20, Appendix]. Proving this statement seems to require a case-by-case analysis. The case of a ramified unitary group has been studied in [21, §1.2], and the case of an even, split orthogonal group has been investigated in [27, §4.3].

2.6. A combinatorial lemma

The following combinatorial result explains the relationship between the abstract formula of Theorem 2.5.18 and the more concrete formulas of the following sections.

Let $n \in \mathbb{N}$ and consider the canonical semidirect product $\widetilde{W} := S_n \ltimes \mathbb{Z}^n$. To avoid any confusion of the product inside $\widetilde{W}$ and the canonical action of S_n on $\mathbb{Z}^n$, we will always denote the element of $\widetilde{W}$ corresponding to $\lambda \in \mathbb{Z}^n$ by u^λ.

Let Ξ be a finite cyclic group of order f with generator σ. We have the shift $\prod_{\xi \in \Xi} \widetilde{W} \to \prod_{\xi \in \Xi} \widetilde{W}$, $(x_\xi)_\xi \mapsto (x_{\sigma^{-1}\xi})_\xi$. By abuse of notation, we simply denote it by σ.

LEMMA 2.6.1. — *Let $(w_\xi)_\xi \in \prod_{\xi \in \Xi} S_n$ and $(\lambda_\xi)_\xi \in \prod_{\xi \in \Xi} \mathbb{Z}^n$. Assume that for all $\xi \in \Xi$ and all $1 \leqslant i \leqslant n$, the following statement holds.*

$$(2.6.1) \qquad \lambda_\xi(i) \geqslant 0 \quad \text{and} \quad (\lambda_\xi(i) = 0 \Rightarrow w_\xi(i) \leqslant i).$$

Let $x = (w_\xi u^{\lambda_\xi})_\xi \in \prod_{\xi \in \Xi} \widetilde{W}$. Choose $N \in \mathbb{N}_{\geqslant 1}$ such that $\prod_{k=0}^{Nf-1} \sigma^k(x) \in \prod_{\xi \in \Xi} \mathbb{Z}^n$. Consider the element

$$\nu = (\nu_\xi)_\xi := \frac{1}{Nf} \prod_{k=0}^{Nf-1} \sigma^k(x)$$

of $\prod_{\xi \in \Xi} \mathbb{Q}_{\geqslant 0}$. Then for each $1 \leqslant i \leqslant n$, the following statements are equivalent.

(1) $\exists \xi \in \Xi : \ \nu_\xi(i) = 0.$
(2) $\forall \xi \in \Xi : \ \nu_\xi(i) = 0.$
(3) $\forall \xi \in \Xi : \ (w_\xi(i) = i$ and $\lambda_\xi(i) = 0).$ □

3. The symplectic case

3.1. Number fields

We first fix some notation concerning number fields. Let $K/\mathbb{Q}$ be a number field; note that this use of K differs from Section 2.5. We will always denote by $\mathcal{O}_K$ the ring of integers of K. If $\mathcal{P}$ is a nonzero prime of $\mathcal{O}_K$, we will always denote by $k_\mathcal{P} = \mathcal{O}_K/\mathcal{P}$ its residue field and by $\rho_\mathcal{P} : \mathcal{O}_K \to k_\mathcal{P}$ the corresponding residue morphism. We further denote by $K_\mathcal{P}$ the completion of K with respect to $\mathcal{P}$ and by $\mathcal{O}_{K_\mathcal{P}}$ the valuation ring of $K_\mathcal{P}$.

Let $K_0/\mathbb{Q}$ be a number field and assume that $p\mathcal{O}_{K_0} = \mathcal{P}_0^{e_0}$ for a single prime $\mathcal{P}_0$ of $\mathcal{O}_{K_0}$ and some $e_0 \in \mathbb{N}$. Denote by Σ_0 the set of all embeddings $K_0 \hookrightarrow \mathbb{C}$. Fix a finite Galois extension $L/\mathbb{Q}$ with $K_0 \subset L$ and write $G = \mathrm{Gal}(L/\mathbb{Q})$ and $H_0 = \mathrm{Gal}(L/K_0)$. Fix a prime $\mathcal{Q}$ of $\mathcal{O}_L$ lying over $\mathcal{P}_0$ and denote by $G_\mathcal{Q} \subset G$ the corresponding decomposition group. Our assumption that $\mathcal{P}_0$ is the only prime of $\mathcal{O}_{K_0}$ lying over p implies that $G = G_\mathcal{Q} H_0$.

LEMMA 3.1.1. — *There is a unique map* $\gamma_0 = \gamma_{\mathcal{P}_0} : \Sigma_0 \to \mathrm{Gal}(k_{\mathcal{P}_0}/\mathbb{F}_p)$ *satisfying*

(3.1.1) $$\forall \sigma \in \Sigma_0 \forall a \in \mathcal{O}_{K_0} : \quad \rho_\mathcal{Q}(\sigma(a)) = \gamma_0(\sigma)(\rho_{\mathcal{P}_0}(a)).$$

It is surjective and all its fibers have cardinality e_0.

Proof. — Left to the reader. □

Let K/K_0 be a quadratic extension with $K \subset L$. Denote by Σ the set of all embeddings $K \hookrightarrow \mathbb{C}$ and write $H = \mathrm{Gal}(L/K)$. Denote by $*$ the non-trivial element of $\mathrm{Gal}(K/K_0)$. Assume that $\mathcal{P}_0\mathcal{O}_K = \mathcal{P}_+\mathcal{P}_-$ for two distinct primes $\mathcal{P}_+, \mathcal{P}_-$ of $\mathcal{O}_K$, say $\mathcal{Q} \cap \mathcal{O}_K = \mathcal{P}_+$. Consequently $\mathcal{P}_- = \mathcal{P}_+^*$. Denote by $\alpha : G \to \Sigma$ the restriction map. Fix a lift $\tau_* \in G$ of $*$ under α. Define subsets $\Sigma_\pm \subset \Sigma$ by $\Sigma_+ = \alpha(G_\mathcal{Q} H)$ and $\Sigma_- = \alpha(G_\mathcal{Q} \tau_* H)$. Then $\Sigma = \Sigma_+ \amalg \Sigma_-$. We identify $k_{\mathcal{P}_\pm}$ with $k_{\mathcal{P}_0}$ via the isomorphism induced by the inclusion $\mathcal{O}_{K_0} \subset \mathcal{O}_K$

LEMMA 3.1.2. — *There are unique maps* $\gamma_\pm : \Sigma_\pm \to \mathrm{Gal}(k_{\mathcal{P}_0}/\mathbb{F}_p)$ *satisfying*

(3.1.2) $$\forall \sigma \in \Sigma_\pm \forall a \in \mathcal{O}_K : \ \rho_\mathcal{Q}(\sigma(a)) = \gamma_0(\sigma|_{K_0})(\rho_{\mathcal{P}_\pm}(a)).$$

Proof. — Left to the reader. □

3.2. The PEL datum

Let $g, n \in \mathbb{N}_{\geqslant 1}$. We start with the PEL datum consisting of the following objects.

(1) A totally real field extension $F/\mathbb{Q}$ of degree g.
(2) The identity involution id_F on F.
(3) A $2n$-dimensional F-vector space V.
(4) The symplectic form $(\cdot, \cdot) : V \times V \to \mathbb{Q}$ on the underlying $\mathbb{Q}$-vector space of V constructed as follows: Fix once and for all a symplectic form $(\cdot, \cdot)' : V \times V \to F$ and a basis $\mathfrak{E}' = (e_1', \dots, e_{2n}')$ of V such that $(\cdot, \cdot)'$ is described by the matrix $\widetilde{J}_{2n}$ with respect to $\mathfrak{E}'$. Define $(\cdot, \cdot) = \mathrm{tr}_{F/\mathbb{Q}} \circ (\cdot, \cdot)'$.
(5) The $F \otimes \mathbb{R}$-endomorphism J of $V \otimes \mathbb{R}$ described by the matrix $-\widetilde{J}_{2n}$ with respect to $\mathfrak{E}'$.

Remark 3.2.1. — Denote by $\mathrm{GSp}_{(\cdot, \cdot)'}$ the F-group of symplectic similitudes with respect to $(\cdot, \cdot)'$, and by $c : \mathrm{GSp}_{(\cdot, \cdot)'} \to \mathbb{G}_m$ the factor of similitude. Then the reductive $\mathbb{Q}$-group G associated with the above PEL datum fits into the following cartesian diagram.

$$
\begin{array}{ccc}
G & \lhook\joinrel\longrightarrow & \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GSp}_{(\cdot, \cdot)'} \\
{\scriptstyle c}\big\downarrow & & \big\downarrow {\scriptstyle c} \\
\mathbb{G}_{m,\mathbb{Q}} & \lhook\joinrel\longrightarrow & \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F}.
\end{array}
$$

We assume that $p\mathcal{O}_F = \mathcal{P}^e$ for a single prime $\mathcal{P}$ of $\mathcal{O}_F$. Denote by $f = [k_\mathcal{P} : \mathbb{F}_p]$ the corresponding inertia degree, so that $g = ef$. We have $F \otimes \mathbb{Q}_p = F_\mathcal{P}$ and $\mathcal{O}_F \otimes \mathbb{Z}_p = \mathcal{O}_{F_\mathcal{P}}$. Fix once and for all a uniformizer π of $\mathcal{O}_F \otimes \mathbb{Z}_{(p)}$.

Denote by $\mathfrak{C} = \mathfrak{C}_{\mathcal{O}_{F_\mathcal{P}} | \mathbb{Z}_p}$ the inverse different of the extension $F_\mathcal{P}/\mathbb{Q}_p$. Fix a generator δ of $\mathfrak{C}$ over $\mathcal{O}_{F_\mathcal{P}}$ and define a basis $(e_1, \dots, e_{2n})$ of $V_{\mathbb{Q}_p}$ over $F_\mathcal{P}$ by $e_i = e_i'$, $e_{n+i} = \delta e_{n+i}'$, $1 \leqslant i \leqslant n$.

Let $0 \leqslant i < 2n$. We denote by Λ_i the $\mathcal{O}_{F_\mathcal{P}}$-lattice in $V_{\mathbb{Q}_p}$ with basis

$$\mathfrak{E}_i = (\pi^{-1}e_1, \dots, \pi^{-1}e_i, e_{i+1}, \dots, e_{2n}).$$

For $k \in \mathbb{Z}$ we further define $\Lambda_{2nk+i} = \pi^{-k}\Lambda_i$ and we denote by $\mathfrak{E}_{2nk+i}$ the corresponding basis obtained from $\mathfrak{E}_i$. Then $\mathcal{L} = (\Lambda_i)_i$ is a complete chain of $\mathcal{O}_{F_\mathcal{P}}$-lattices in V. For $i \in \mathbb{Z}$, the dual lattice $\Lambda_i^\vee := \{x \in V_{\mathbb{Q}_p} \mid (x, \Lambda_i)_{\mathbb{Q}_p} \subset \mathbb{Z}_p\}$ of Λ_i is given by Λ_{-i}. Consequently the chain $\mathcal{L}$ is self-dual.

Let $i \in \mathbb{Z}$. We denote by $\rho_i : \Lambda_i \to \Lambda_{i+1}$ the inclusion, by $\vartheta_i : \Lambda_{2n+i} \to \Lambda_i$ the isomorphism given by multiplication with π and by $(\cdot, \cdot)_i : \Lambda_i \times \Lambda_{-i} \to \mathbb{Z}_p$ the restriction of $(\cdot, \cdot)_{\mathbb{Q}_p}$. Then $(\Lambda_i, \rho_i, \vartheta_i, (\cdot, \cdot)_i)_i$ is a polarized chain of

$\mathcal{O}_{F_\mathcal{P}}$-modules of type $(\mathcal{L})$, which, by abuse of notation, we also denote by $\mathcal{L}$.

Denote by $\langle \cdot, \cdot \rangle_i : \Lambda_i \times \Lambda_{-i} \to \mathcal{O}_{F_\mathcal{P}}$ the restriction of the pairing $\delta^{-1}(\cdot, \cdot)'_{\mathbb{Q}_p}$. It is the perfect pairing described by the matrix $\widetilde{J}_{2n}$ with respect to the bases $\mathfrak{E}_i$ and $\mathfrak{E}_{-i}$.

3.3. The determinant morphism

Denote by Σ the set of all embeddings $F \hookrightarrow \mathbb{C}$. The canonical isomorphism

$$(3.3.1) \qquad\qquad F \otimes \mathbb{C} = \prod_{\sigma \in \Sigma} \mathbb{C}$$

induces a decomposition $V \otimes \mathbb{C} = \prod_{\sigma \in \Sigma} V_\sigma$ into $\mathbb{C}$-vector spaces V_σ, and the morphism $J_\mathbb{C}$ decomposes into the product of $\mathbb{C}$-linear maps $J_\sigma : V_\sigma \to V_\sigma$. Each J_σ induces a decomposition $V_\sigma = V_{\sigma,i} \oplus V_{\sigma,-i}$, where $V_{\sigma,\pm i}$ denotes the $\pm i$-eigenspace of J_σ. From the explicit description of J in terms of $\mathcal{B}$ above one sees that both $V_{\sigma,i}$ and $V_{\sigma,-i}$ have dimension n over $\mathbb{C}$.

The $(-i)$-eigenspace V_{-i} of $J_\mathbb{C}$ is given by $V_{-i} = \prod_{\sigma \in \Sigma} V_{-i,\sigma}$. As $\dim_\mathbb{C} V_{-i,\sigma} = n$ for all σ, there is an isomorphism $V_{-i} \simeq (\prod_\sigma \mathbb{C})^n$ of $\prod_\sigma \mathbb{C}$-modules and hence the $\mathcal{O}_F \otimes \mathbb{C}$-module corresponding to V_{-i} under (3.3.1) is isomorphic to $\mathcal{O}_F^n \otimes \mathbb{C}$. In particular, the morphism $\det_{V_{-i}} : V_{\mathcal{O}_F \otimes \mathbb{C}} \to \mathbb{A}^1_\mathbb{C}$ is defined over $\mathbb{Z}$.

3.4. The local model

For the chosen PEL datum, Definition 2.3.1 amounts to the following.

DEFINITION 3.4.1. — *The local model M^{loc} is the functor on the category of $\mathbb{Z}_p$-algebras with $M^{\mathrm{loc}}(R)$ the set of tuples $(t_i)_{i \in \mathbb{Z}}$ of $\mathcal{O}_F \otimes R$-submodules $t_i \subset \Lambda_{i,R}$ satisfying the following conditions for all $i \in \mathbb{Z}$.*

    (a) *$\rho_{i,R}(t_i) \subset t_{i+1}$.*
    (b) *The quotient $\Lambda_{i,R}/t_i$ is a finite locally free R-module.*
    (c) *We have an equality*

$$\det\nolimits_{\Lambda_{i,R}/t_i} = \det\nolimits_{V_{-i}} \otimes R$$

    of morphisms $V_{\mathcal{O}_F \otimes R} \to \mathbb{A}^1_R$.
    (d) *Under the pairing $(\cdot, \cdot)_{i,R} : \Lambda_{i,R} \times \Lambda_{-i,R} \to R$, the submodules t_i and t_{-i} pair to zero.*
    (e) *$\vartheta_i(t_{2n+i}) = t_i$.*

3.5. The geometric special fiber of the local model

For $i \in \mathbb{Z}$, denote by $\overline{\Lambda}_i$ the $\mathbb{F}[u]/u^e$-module $(\mathbb{F}[u]/u^e)^{2n}$ and by $\overline{\mathfrak{E}}_i$ its canonical basis. Denote by $\overline{\langle \cdot, \cdot \rangle}_i : \overline{\Lambda}_i \times \overline{\Lambda}_{-i} \to \mathbb{F}[u]/u^e$ the pairing described by the matrix $\widetilde{J}_{2n}$ with respect to $\overline{\mathfrak{E}}_i$ and $\overline{\mathfrak{E}}_{-i}$. Denote by $\overline{\vartheta}_i : \overline{\Lambda}_{2n+i} \to \overline{\Lambda}_i$ the identity morphism. For $k \in \mathbb{Z}$ and $0 \leqslant i < 2n$, let $\overline{\rho}_{2n+i} : \overline{\Lambda}_{2n+i} \to \overline{\Lambda}_{2n+i+1}$ be the morphism described by the matrix $\mathrm{diag}(1^{(i)}, u, 1^{(2n-i-1)})$ with respect to $\overline{\mathfrak{E}}_{2nk+i}$ and $\overline{\mathfrak{E}}_{2nk+i+1}$.

DEFINITION 3.5.1. — *Let $M^{e,n}$ be the functor on the category of $\mathbb{F}$-algebras with $M^{e,n}(R)$ the set of tuples $(t_i)_{i \in \mathbb{Z}}$ of $R[u]/u^e$-submodules $t_i \subset \overline{\Lambda}_{i,R}$ satisfying the following conditions for all $i \in \mathbb{Z}$.*

    (a) *$\overline{\rho}_{i,R}(t_i) \subset t_{i+1}$.*
    (b) *The quotient $\overline{\Lambda}_{i,R}/t_i$ is finite locally free over R.*
    (c) *For all $P \in R[u]/u^e$, we have*

$$\chi_R(P|\overline{\Lambda}_{i,R}/t_i) = \big(T - P(0)\big)^{ne}$$

    in $R[T]$.
    (d) *$t_i^{\perp, \overline{\langle \cdot, \cdot \rangle}_{i,R}} = t_{-i}$.*
    (e) *$\overline{\vartheta}_i(t_{2n+i}) = t_i$.*

Denote by $\mathfrak{S}$ the set of all embeddings $k_{\mathcal{P}} \hookrightarrow \mathbb{F}$. Our choice of uniformizer π induces a canonical isomorphism

$$(3.5.1) \qquad\qquad \mathcal{O}_F \otimes \mathbb{F} = \prod_{\sigma \in \mathfrak{S}} \mathbb{F}[u]/(u^e).$$

Let $i \in \mathbb{Z}$. From (3.5.1) we obtain an isomorphism

$$(3.5.2) \qquad\qquad \Lambda_{i,\mathbb{F}} = \prod_{\sigma \in \mathfrak{S}} \overline{\Lambda}_i$$

by identifying the basis $\mathfrak{E}_{i,\mathbb{F}}$ with the product of the bases $\overline{\mathfrak{E}}_i$. Under this identification, the morphism $\rho_{i,\mathbb{F}}$ decomposes into the morphisms $\overline{\rho}_i$, the pairing $\langle \cdot, \cdot \rangle_{i,\mathbb{F}}$ decomposes into the pairings $\overline{\langle \cdot, \cdot \rangle}_i$ and the morphism $\vartheta_{i,\mathbb{F}}$ decomposes into the morphisms $\overline{\vartheta}_i$.

Let R be an $\mathbb{F}$-algebra and let $(t_i)_{i \in \mathbb{Z}}$ be a tuple of $\mathcal{O}_F \otimes R$-submodules $t_i \subset \Lambda_{i,R}$. Then (3.5.2) induces decompositions $t_i = \prod_{\sigma \in \mathfrak{S}} t_{i,\sigma}$ into $R[u]/u^e$-submodules $t_{i,\sigma} \subset \overline{\Lambda}_{i,R}$.

PROPOSITION 3.5.2. — *The morphism $M_{\mathbb{F}}^{\mathrm{loc}} \to \prod_{\sigma \in \mathfrak{S}} M^{e,n}$ given on R-valued points by*

(3.5.3)
$$M_{\mathbb{F}}^{\mathrm{loc}}(R) \to \prod_{\sigma \in \mathfrak{S}} M^{e,n}(R),$$
$$(t_i) \mapsto ((t_{i,\sigma})_i)_\sigma$$

is an isomorphism of functors on the category of $\mathbb{F}$-algebras.

Proof. — The only point requiring an argument is the transition from $(\cdot, \cdot)_i$ to $\langle \cdot, \cdot \rangle_i$. It is warranted by the perfectness of the pairing $\mathcal{O}_{F_\mathcal{P}} \times \mathcal{O}_{F_\mathcal{P}} \to \mathbb{Z}_p$, $(x, y) \mapsto \mathrm{tr}_{F_\mathcal{P}/\mathbb{Q}_p}(\delta xy)$. $\qquad\square$

3.6. The affine Grassmannian and the affine flag variety for GL_n

Let R be an $\mathbb{F}$-algebra and let $n \in \mathbb{N}$.

DEFINITION 3.6.1. — *A lattice in $R((u))^n$ is an $R[\![u]\!]$-submodule $L \subset R((u))^n$ satisfying the following conditions for some $N \in \mathbb{N}$.*
 (1) $u^N R[\![u]\!]^n \subset L \subset u^{-N} R[\![u]\!]^n$.
 (2) $u^{-N} R[\![u]\!]^n / L$ *is a finite locally free R-module.*

The following statement is well-known. See for example [10, Proposition 4.5.5] for a proof.

PROPOSITION 3.6.2. — *Let L be a lattice in $R((u))^n$. Then L is a finite locally free $R[\![u]\!]$-module of rank n.*

DEFINITION 3.6.3. — *Denote by $\mathcal{G}$ the functor on the category of $\mathbb{F}$-algebras with $\mathcal{G}(R)$ the set of lattices in $R((u))^n$.*

Denote by $\widetilde{\Lambda}_0 = R[\![u]\!]^n$ the *standard lattice*. Clearly $\mathrm{L}\,\mathrm{GL}_n(R)$ acts on $\mathcal{G}(R)$ by multiplication from the left, and the stabilizer of $\widetilde{\Lambda}_0$ for this action is given by $\mathrm{L}^+\,\mathrm{GL}_n(R)$. Consequently we get an injective map

$$\phi(R) : \mathrm{L}\,\mathrm{GL}_n(R)/\mathrm{L}^+\,\mathrm{GL}_n(R) \to \mathcal{G}(R)$$
$$g \mapsto g\widetilde{\Lambda}_0.$$

It is equivariant for the left action by $\mathrm{L}\,\mathrm{GL}_n$.

PROPOSITION 3.6.4. — *The map ϕ identifies $\mathcal{G}$ with both the Zariski and the fpqc sheafification of the presheaf $\mathrm{L}\,\mathrm{GL}_n/\mathrm{L}^+\,\mathrm{GL}_n$.*

Proof. — By Proposition 3.6.2 it is clear that any lattice lies in the image of ϕ Zariski locally on R. It follows that ϕ is the Zariski sheafification of the presheaf $\mathrm{L}\,\mathrm{GL}_n/\mathrm{L}^+\,\mathrm{GL}_n$. The fact that $\mathcal{G}$ is already an fpqc sheaf implies formally that ϕ is also the fpqc sheafification of the presheaf $\mathrm{L}\,\mathrm{GL}_n/\mathrm{L}^+\,\mathrm{GL}_n$. $\qquad\square$

DEFINITION 3.6.5. — *The Zariski sheafification of the presheaf* $\mathrm{L}\,\mathrm{GL}_n/\mathrm{L}^+\,\mathrm{GL}_n$ *is called the* affine Grassmannian *for* GL_n.

By Proposition 3.6.4 the functor $\mathcal{G}$ provides a realization of the affine Grassmannian for GL_n.

DEFINITION 3.6.6. — *A (complete, periodic)* lattice chain *in* $R((u))^n$ *is a tuple* $(L_i)_{i\in\mathbb{Z}}$ *of lattices* L_i *in* $R((u))^n$ *satisfying the following conditions for each* $i \in \mathbb{Z}$.

    (1) $L_i \subset L_{i+1}$.
    (2) *(completeness)* L_{i+1}/L_i *is a locally free* R-*module of rank* 1.
    (3) *(periodicity)* $L_{n+i} = u^{-1}L_i$.

DEFINITION 3.6.7. — *Denote by* $\mathcal{F}$ *the functor on the category of* $\mathbb{F}$-*algebras with* $\mathcal{F}(R)$ *the set of (complete, periodic) lattice chains in* $R((u))^n$.

Denote by $(e_1,\ldots,e_n)$ the standard basis of $R((u))^n$ over $R((u))$. For $0 \leqslant i < n$ we denote by $\widetilde{\Lambda}_i$ the lattice in $R((u))^n$ with basis

$$\widetilde{\mathfrak{E}}_i = \left\langle u^{-1}e_1,\ldots,u^{-1}e_i, e_{i+1},\ldots,e_n \right\rangle.$$

For $k \in \mathbb{Z}$ we further define $\widetilde{\Lambda}_{nk+i} = u^{-k}\widetilde{\Lambda}_i$ and we denote by $\widetilde{\mathfrak{E}}_{nk+i}$ the corresponding basis obtained from $\widetilde{\mathfrak{E}}_i$. Then $\widetilde{\mathcal{L}} = (\widetilde{\Lambda}_i)_i$ is a (complete, periodic) lattice chain in $R((u))^n$, called the *standard lattice chain*.

In complete analogy with [24, p. 131], we have for an $\mathbb{F}[\![u]\!]$-algebra R the notion of a chain $\mathcal{M} = (M_i, \varrho_i : M_i \to M_{i+1}, \theta_i : M_{n+i} \xrightarrow{\sim} M_i)_{i\in\mathbb{Z}}$ of R-modules of type $(\widetilde{\mathcal{L}})$ (cf. [10, Definition 7.5.1]). The proof of [24, Proposition A.4] then carries over without any changes to show the following result.

PROPOSITION 3.6.8. — *Let* R *be an* $\mathbb{F}[\![u]\!]$-*algebra such that the image of* u *in* R *is nilpotent. Then any two chains* $\mathcal{M},\mathcal{N}$ *of* R-*modules of type* $(\widetilde{\mathcal{L}})$ *are isomorphic locally for the Zariski topology on* R. *Furthermore the functor* $\mathrm{Isom}(\mathcal{M},\mathcal{N})$ *is representable by a smooth affine scheme over* R.

PROPOSITION 3.6.9. — *Let* R *be an* $\mathbb{F}$-*algebra and let* $\mathcal{M},\mathcal{N}$ *be chains of* $R[\![u]\!]$-*modules of type* $(\widetilde{\mathcal{L}})$. *Then the canonical map* $\mathrm{Isom}(\mathcal{M},\mathcal{N})(R[\![u]\!]) \to \mathrm{Isom}(\mathcal{M},\mathcal{N})(R[\![u]\!]/u^m)$ *is surjective for all* $m \in \mathbb{N}_{\geqslant 1}$. *In particular* $\mathcal{M}$ *and* $\mathcal{N}$ *are isomorphic locally for the Zariski topology on* R.

Proof. — Analogous to the proof of Lemma 2.5.3. □

Remark 3.6.10. — Let R be an $\mathbb{F}$-algebra and let $(L_i)_i \in \mathcal{F}(R)$. For $i \in \mathbb{Z}$ denote by $\varrho_i : L_i \to L_{i+1}$ the inclusion and by $\theta_i : L_{n+i} \to L_i$ the isomorphism given by multiplication with u. Then $(L_i, \varrho_i, \theta_i)$ is a chain of $R[\![u]\!]$-modules of type $(\widetilde{\mathcal{L}})$.

Remark 3.6.11. — The group $\operatorname{L} \operatorname{GL}_n(R)$ acts on $\mathcal{F}(R)$ via $g \cdot (L_i)_i = (gL_i)_i$. Denote by $I(R)$ the stabilizer of $\widetilde{\mathcal{L}}$ for this action. One checks that $I(R) \subset \operatorname{GL}_n(R[\![u]\!])$ is equal to the preimage of $B(R)$ under the reduction map $\operatorname{GL}_n(R[\![u]\!]) \to \operatorname{GL}_n(R)$, $u \mapsto 0$. Here $B(R) \subset \operatorname{GL}_n(R)$ denotes the subgroup of upper triangular matrices.

We obtain for each $\mathbb{F}$-algebra R an injective map

$$\operatorname{L} \operatorname{GL}_n(R)/I(R) \xrightarrow{\phi(R)} \mathcal{F}(R),$$

$$g \longmapsto g \cdot \widetilde{\mathcal{L}}.$$

PROPOSITION 3.6.12. — *The morphism ϕ identifies $\mathcal{F}$ with both the Zariski and the fpqc sheafification of the presheaf $\operatorname{L} \operatorname{GL}_n/I$.*

Proof. — Let R be an $\mathbb{F}$-algebra and let $\mathcal{M} \in \mathcal{F}(R)$. We consider $\mathcal{M}$ as a chain of $R[\![u]\!]$-modules of type $(\widetilde{\mathcal{L}})$ as in Remark 3.6.10. By Proposition 3.6.9, the chains $\widetilde{\mathcal{L}}$ and $\mathcal{M}$ are isomorphic locally for the Zariski topology on R. Such an isomorphism $\widetilde{\mathcal{L}} \to \mathcal{M}$ is given by multiplication with a single $g \in \operatorname{GL}_n(R(\!(u)\!))$. Consequently $\mathcal{M}$ lies in the image of ϕ Zariski locally on R. The fact that $\mathcal{F}$ is already an fpqc sheaf implies formally that ϕ is also the fpqc sheafification of the presheaf $\operatorname{L} \operatorname{GL}_n/I$. □

DEFINITION 3.6.13. — *The Zariski sheafification of the presheaf $\operatorname{L} \operatorname{GL}_n/I$ is called the* affine flag variety *for* GL_n.

By Proposition 3.6.12 the functor $\mathcal{F}$ provides a realization of the affine flag variety for GL_n.

3.7. The affine flag variety

This section deals with the affine flag variety for the symplectic group. Our discussion loosely follows the one in [19, §10-11]. Note though that in loc. cit. there is a minor problem with the definition of the notion of self-duality for lattice chains, see Remark 3.7.8 below. We have learned the correct formulation of this definition from [26, §4.2], which deals with the case of a ramified unitary group.

Let R be an $\mathbb{F}$-algebra. Let $\widetilde{\langle \cdot, \cdot \rangle}$ be the symplectic form on $R((u))^{2n}$ described by the matrix $\widetilde{J}_{2n}$ with respect to the standard basis of $R((u))^{2n}$ over $R((u))$. We denote by $\mathrm{Sp} = \mathrm{Sp}_{2n}$ the symplectic group and by $\mathrm{GSp} = \mathrm{GSp}_{2n}$ the group of symplectic similitudes with respect to $\widetilde{\langle \cdot, \cdot \rangle}$.

For a lattice Λ in $R((u))^{2n}$ we define $\Lambda^\vee := \{x \in R((u))^{2n} \mid \widetilde{\langle x, \Lambda \rangle} \subset R[\![u]\!]\}$. Recall from Section 3.6 the standard lattice chain $\widetilde{\mathcal{L}} = (\widetilde{\Lambda}_i)_i$ in $R((u))^{2n}$. Note that $(\widetilde{\Lambda}_i)^\vee = \widetilde{\Lambda}_{-i}$ for all $i \in \mathbb{Z}$. We denote by $\widetilde{\langle \cdot, \cdot \rangle}_i : \widetilde{\Lambda}_i \times \widetilde{\Lambda}_{-i} \to R[\![u]\!]$ the restriction of $\widetilde{\langle \cdot, \cdot \rangle}$.

In complete analogy with [24, Definition 3.14], we have for an $\mathbb{F}[\![u]\!]$-algebra R the notion of a polarized chain $\mathcal{M} = (M_i, \varrho_i : M_i \to M_{i+1}, \theta_i : M_{2n+i} \xrightarrow{\sim} M_i, \mathcal{E}_i : M_i \times M_{-i} \to R)_{i \in \mathbb{Z}}$ of R-modules of type $(\widetilde{\mathcal{L}})$ (cf. [10, Definition 5.5.1]). The proof of [24, Proposition A.21] then carries over without any changes to show the following result.

PROPOSITION 3.7.1. — *Let R be an $\mathbb{F}[\![u]\!]$-algebra such that the image of u in R is nilpotent. Then any two polarized chains $\mathcal{M}, \mathcal{N}$ of R-modules of type $(\widetilde{\mathcal{L}})$ are isomorphic locally for the Zariski topology on R. Furthermore the functor $\mathrm{Isom}(\mathcal{M}, \mathcal{N})$ is representable by a smooth affine scheme over R.*

PROPOSITION 3.7.2. — *Let R be an $\mathbb{F}$-algebra and let $\mathcal{M}, \mathcal{N}$ be polarized chains of $R[\![u]\!]$-modules of type $(\widetilde{\mathcal{L}})$. Then the canonical map $\mathrm{Isom}(\mathcal{M}, \mathcal{N})(R[\![u]\!]) \to \mathrm{Isom}(\mathcal{M}, \mathcal{N})(R[\![u]\!]/u^m)$ is surjective for all $m \in \mathbb{N}_{\geqslant 1}$. In particular $\mathcal{M}$ and $\mathcal{N}$ are isomorphic locally for the Zariski topology on R.*

Proof. — Analogous to the proof of Lemma 2.5.3. $\qquad\qquad\square$

The following definition is a straightforward variant of [26, §4.2].

DEFINITION 3.7.3. — *Let R be an $\mathbb{F}$-algebra and let $(L_i)_i$ be a lattice chain in $R((u))^{2n}$.*

(1) *Let $r \in \mathbb{Z}$. The chain $(L_i)_i$ is called r-self-dual if*

$$\forall i \in \mathbb{Z} : \quad L_i^\vee = u^r L_{-i}.$$

Denote by $\mathcal{F}_{\mathrm{Sp}}^{(r)}$ the functor on the category of $\mathbb{F}$-algebras with $\mathcal{F}_{\mathrm{Sp}}^{(r)}(R)$ the set of r-self-dual lattice chains in $R((u))^{2n}$.

(2) *The chain $(L_i)_i$ is called self-dual if Zariski locally on R there is an $a \in R((u))^\times$ such that*

(3.7.1) $$\forall i \in \mathbb{Z} : \quad L_i^\vee = a L_{-i}.$$

We denote by $\mathcal{F}_{\mathrm{GSp}}$ the functor on the category of $\mathbb{F}$-algebras with $\mathcal{F}_{\mathrm{GSp}}(R)$ the set of self-dual lattice chains in $R((u))^{2n}$.

Note that $\widetilde{\mathcal{L}} \in \mathcal{F}_{\mathrm{Sp}}^{(0)}(R)$.

LEMMA 3.7.4. — *Let R be a ring and let $a \in R((u))^{\times}$. Then Zariski locally on R, there are integers $n \leqslant n_0$, nilpotent elements $a_n, a_{n+1}, \ldots,$ $\ldots, a_{n_0-1} \in R$, a unit $a_{n_0} \in R^{\times}$ and elements $a_{n_0+1}, a_{n_0+2}, \ldots \in R$ such that $a = \sum_{i=n}^{\infty} a_i u^i$.*

If $\operatorname{Spec} R$ is connected, such integers and elements exist globally on R.

Remark 3.7.5. — Let R be a reduced $\mathbb{F}$-algebra such that $\operatorname{Spec} R$ connected. Then

$$\mathcal{F}_{\mathrm{GSp}}(R) = \bigcup_{r \in \mathbb{Z}} \mathcal{F}_{\mathrm{Sp}}^{(r)}(R).$$

Proof. — This follows immediately from Lemma 3.7.4. $\qquad \square$

Remark 3.7.6. — Let R be an $\mathbb{F}$-algebra and let $(L_i)_i \in \mathcal{F}_{\mathrm{Sp}}^{(0)}(R)$. For $i \in \mathbb{Z}$ denote by $\varrho_i : L_i \to L_{i+1}$ the inclusion, by $\theta_i : L_{2n+i} \to L_i$ the isomorphism given by multiplication with u and by $\mathcal{E}_i : L_i \times L_{-i} \to R[u]$ the restriction of $\widetilde{\langle \cdot, \cdot \rangle}$. Then $(L_i, \varrho_i, \theta_i, \mathcal{E}_i)$ is a polarized chain of $R[\![u]\!]$-modules of type $(\widetilde{\mathcal{L}})$.

Recall from Remark 3.6.11 the subfunctor $I \subset \mathrm{L\,GL}_{2n}$. We define a subfunctor $I_{\mathrm{GSp}} = I_{\mathrm{GSp}_{2n}}$ of $\mathrm{L\,GSp} = \mathrm{L\,GSp}_{2n}$ by $I_{\mathrm{GSp}} = \mathrm{L\,GSp}_{2n} \cap I$. We consider all of these functors as functors on the category of $\mathbb{F}$-algebras.

The proof of the following result is similar to and therefore based on the proof of [20, Theorem 4.1].

PROPOSITION 3.7.7. — *The natural action of $\mathrm{L\,GL}_{2n}$ on $\mathcal{F}$ (cf. Remark 3.6.11) restricts to an action of $\mathrm{L\,GSp}$ on $\mathcal{F}_{\mathrm{GSp}}$. Consequently we obtain an injective map*

$$\mathrm{L\,GSp}(R)/I_{\mathrm{GSp}}(R) \xrightarrow{\phi(R)} \mathcal{F}_{\mathrm{GSp}}(R),$$
$$g \longmapsto g \cdot \widetilde{\mathcal{L}}$$

for each $\mathbb{F}$-algebra R. The morphism ϕ identifies $\mathcal{F}_{\mathrm{GSp}}$ with both the Zariski and the fpqc sheafification of the presheaf $\mathrm{L\,GSp}/I_{\mathrm{GSp}}$.

Proof. — Let R be an $\mathbb{F}$-algebra and let $\mathcal{M} = (L_i)_i \in \mathcal{F}_{\mathrm{GSp}}(R)$. Working Zariski locally on R we may assume that there is an $a \in R((u))^{\times}$ such that (3.7.1) holds. Choose any $h \in \mathrm{GSp}(R((u)))$ with factor of similitude a, e.g. $h = \operatorname{diag}(a^{(n)}, 1^{(n)})$. An easy computation shows that $h\mathcal{M} \in \mathcal{F}_{\mathrm{Sp}}^{(0)}(R)$. We see as in the proof of Proposition 3.6.12 that Zariski locally on R, there is a $g \in \mathrm{Sp}(R((u)))$ with $h\mathcal{M} = g\widetilde{\mathcal{L}}$. Consequently $\mathcal{M} = h^{-1}g\widetilde{\mathcal{L}}$ lies in the image

of ϕ Zariski locally on R. As $\mathcal{F}_{\mathrm{GSp}}$ is clearly a Zariski sheaf, it follows that $\mathcal{F}_{\mathrm{GSp}}$ is indeed the Zariski sheafification of the presheaf $\mathrm{L}\,\mathrm{GSp}/I_{\mathrm{GSp}}$.

To see that $\mathcal{F}_{\mathrm{GSp}}$ is also the fpqc sheafification of $\mathrm{L}\,\mathrm{GSp}/I_{\mathrm{GSp}}$, it suffices to show that $\mathcal{F}_{\mathrm{GSp}}$ is an fpqc sheaf. Let $(L_i)_i$ be a lattice chain in $R(\!(u)\!)^{2n}$. Assume that fpqc locally on R there is an $a \in R(\!(u)\!)^\times$ such that (3.7.1) holds. The scalar a gives rise to a well-defined element of $(\mathrm{L}\,\mathbb{G}_m/\mathrm{L}^+\,\mathbb{G}_m)_{\mathrm{fpqc}}(R)$, where $(\mathrm{L}\,\mathbb{G}_m/\mathrm{L}^+\,\mathbb{G}_m)_{\mathrm{fpqc}}$ denotes the fpqc sheafification of the presheaf $\mathrm{L}\,\mathbb{G}_m/\mathrm{L}^+\,\mathbb{G}_m$. By Proposition 3.6.4 any element of $(\mathrm{L}\,\mathbb{G}_m/\mathrm{L}^+\,\mathbb{G}_m)_{\mathrm{fpqc}}(R)$ can be represented in $\mathrm{L}\,\mathbb{G}_m(R)$ Zariski locally on R, so that the scalar a exists in fact Zariski locally on R. $\qquad\square$

Remark 3.7.8. — Let us note that there seems to exist a misconception surrounding the notion of self-duality for lattice chains. In the literature one finds the following definition: Let R be an $\mathbb{F}$-algebra. A lattice chain $(L_i)_i \in \mathcal{F}(R)$ is called *(naively) self-dual* if for each $i \in \mathbb{Z}$ there is a $j \in \mathbb{Z}$ such that $L_i^\vee = L_j$. It is then claimed that the fpqc local $\mathrm{L}\,\mathrm{GSp}$-orbit of $\widetilde{\mathcal{L}}$ (in the sense of Proposition 3.7.7) is precisely the set of (naively) self-dual lattice chains. This is wrong in both directions, as shown by the following easy examples.

- Let $n = 1$ and $a \in R(\!(u)\!)^\times$. The chain $(L_i)_i = a\widetilde{\mathcal{L}}$ satisfies $L_i^\vee = a^{-2}L_{-i}$, $i \in \mathbb{Z}$. Assume there is a $j \in \mathbb{Z}$ with $L_0^\vee = L_j$. Then $a^{-2}L_0 = L_j$ and hence $a^{-2}\widetilde{\Lambda}_0 = \widetilde{\Lambda}_j$. Projecting this equality inside $R(\!(u)\!)^2$ to its first components yields the existence of a $k \in \mathbb{Z}$ with $a^{-2}R[\![u]\!] = u^k R[\![u]\!]$, so that $u^k a^2 \in R[\![u]\!]^\times$. If for example $R = \mathbb{F}[x]/x^2$ and $a = 1 + xu^{-1}$, such a k does not exist.
- Conversely one easily sees that for $n \geqslant 2$, the (naively) self-dual chain $(\widetilde{\Lambda}_{i+1})_{i\in\mathbb{Z}}$ does not lie in the $\mathrm{L}\,\mathrm{GSp}(R)$-orbit of $\widetilde{\mathcal{L}}$ (unless $R = \{0\}$).

DEFINITION 3.7.9. — *The Zariski sheafification of the presheaf* $\mathrm{L}\,\mathrm{GSp}/I_{\mathrm{GSp}}$ *is called the* affine flag variety for GSp.

By Proposition 3.7.7 the functor $\mathcal{F}_{\mathrm{GSp}}$ provides a realization of the affine flag variety for GSp.

3.8. Embedding the local model into the affine flag variety

Let R be an $\mathbb{F}$-algebra. We consider an $R[u]/u^e$-module as an $R[\![u]\!]$-module via the canonical projection $R[\![u]\!] \to R[u]/u^e$. For $i \in \mathbb{Z}$ denote by $\alpha_i : \widetilde{\Lambda}_i \to \overline{\Lambda}_{i,R}$ the morphism described by the identity matrix with

respect to $\widetilde{\mathfrak{E}}_i$ and $\overline{\mathfrak{E}}_i$. It induces an isomorphism $\widetilde{\Lambda}_i/u^e\widetilde{\Lambda}_i \xrightarrow{\sim} \overline{\Lambda}_{i,R}$. Clearly the following diagrams commute.

$$
\begin{array}{ccc}
\widetilde{\Lambda}_i & \subset & \widetilde{\Lambda}_{i+1} \\
\downarrow{\scriptstyle\alpha_i} & & \downarrow{\scriptstyle\alpha_{i+1}} \\
\overline{\Lambda}_{i,R} & \xrightarrow{\overline{\rho}_{i,R}} & \overline{\Lambda}_{i+1,R}
\end{array}
\qquad
\begin{array}{ccc}
\widetilde{\Lambda}_i \times \widetilde{\Lambda}_{-i} & \xrightarrow{\langle\cdot,\cdot\rangle_i} & R[\![u]\!] \\
\downarrow{\scriptstyle\alpha_i\times\alpha_{-i}} & & \downarrow \\
\overline{\Lambda}_{i,R} \times \overline{\Lambda}_{-i,R} & \xrightarrow{\overline{\langle\cdot,\cdot\rangle}_{i,R}} & R[u]/u^e
\end{array}
\qquad
\begin{array}{ccc}
\widetilde{\Lambda}_i & \xleftarrow{u\cdot} & \widetilde{\Lambda}_{2n+i} \\
\downarrow{\scriptstyle\alpha_i} & & \downarrow{\scriptstyle\alpha_{2n+i}} \\
\overline{\Lambda}_{i,R} & \xleftarrow{\overline{\vartheta}_{i,R}} & \overline{\Lambda}_{2n+i,R}
\end{array}
$$

The following proposition allows us to consider $M^{e,n}$ as a subfunctor of $\mathcal{F}_{\mathrm{Sp}}^{(-e)}$.

PROPOSITION 3.8.1 ([19, §11]). — *There is an embedding $\alpha : M^{e,n} \hookrightarrow \mathcal{F}_{\mathrm{Sp}}^{(-e)}$ given on R-valued points by*

$$M^{e,n}(R) \to \mathcal{F}_{\mathrm{Sp}}^{(-e)}(R),$$
$$(t_i)_i \mapsto (\alpha_i^{-1}(t_i))_i.$$

It induces a bijection from $M^{e,n}(R)$ onto the set of those $(L_i)_i \in \mathcal{F}_{\mathrm{Sp}}^{(-e)}(R)$ satisfying the following conditions for all $i \in \mathbb{Z}$.

(1) *$u^e\widetilde{\Lambda}_i \subset L_i \subset \widetilde{\Lambda}_i$.*
(2) *For all $P \in R[u]/u^e$, we have*

$$\chi_R(P|\widetilde{\Lambda}_i/L_i) = (T - P(0))^{ne}$$

in $R[T]$. Here $\widetilde{\Lambda}_i/L_i$ is considered as an $R[u]/u^e$-module using (1).

Proof. — Let $(t_i)_i \in M^{e,n}(R)$ and set $(L_i)_i = (\alpha_i^{-1}(t_i))_i$. It is clear that this defines a periodic lattice chain in $R(\!(u)\!)^{2n}$. Let $i \in \mathbb{Z}$. We have

$$(3.8.1) \quad \mathrm{rk}_R(\widetilde{\Lambda}_{i+1}/\widetilde{\Lambda}_i) + \mathrm{rk}_R(\widetilde{\Lambda}_i/L_i) = \mathrm{rk}_R(\widetilde{\Lambda}_{i+1}/L_{i+1}) + \mathrm{rk}_R(L_{i+1}/L_i),$$

as both sides are equal to $\mathrm{rk}_R(\widetilde{\Lambda}_{i+1}/L_i)$. We conclude from condition 3.5.1(c) that $\mathrm{rk}_R(\widetilde{\Lambda}_i/L_i) = ne = \mathrm{rk}_R(\widetilde{\Lambda}_{i+1}/L_{i+1})$. Thus (3.8.1) amounts to the equation $\mathrm{rk}_R(\widetilde{\Lambda}_{i+1}/\widetilde{\Lambda}_i) = \mathrm{rk}_R(L_{i+1}/L_i)$, so that the chain $(L_i)_i$ is complete.

From $\overline{\langle t_i, t_{-i}\rangle}_{i,R} = 0$ we deduce that $\langle\widetilde{L_i, L_{-i}}\rangle \subset u^e R[\![u]\!]$ and hence that $u^{-e}L_{-i} \subset L_i^\vee$.

From $u^e\widetilde{\Lambda}_i \subset L_i$ on the other hand we deduce $u^e L_i^\vee \subset \widetilde{\Lambda}_{-i}$. By definition, we know that $\langle\widetilde{L_i, u^e L_i^\vee}\rangle \subset u^e R[\![u]\!]$, which implies $\overline{\langle t_i, \alpha_{-i}(u^e L_i^\vee)\rangle}_i = 0$. Consequently $\alpha_{-i}(u^e L_i^\vee) \subset t_{-i}$, which shows that $u^e L_i^\vee \subset L_{-i}$. Hence also $L_i^\vee \subset u^{-e}L_{-i}$.

This proves the existence of the map α. Its injectivity as well as the characterization of its image are immediate. $\qquad\square$

Note that $\overline{\mathcal{L}} = (\overline{\Lambda}_i, \overline{\rho}_i, \overline{\vartheta}_i, \overline{\langle \cdot, \cdot \rangle}_i)_i$ is a polarized chain of $\mathbb{F}[u]/u^e$-modules of type $(\widetilde{\mathcal{L}})$. In fact $\overline{\mathcal{L}} = \widetilde{\mathcal{L}} \otimes_{\mathbb{F}[\![u]\!]} \mathbb{F}[u]/u^e$. Let R be an $\mathbb{F}$-algebra. There is an obvious action of $\mathrm{Aut}(\overline{\mathcal{L}})(R[u]/u^e)$ on $M^{e,n}(R)$, given by $(\varphi_i) \cdot (t_i) = (\varphi_i(t_i))$. The canonical morphism $R[\![u]\!] \to R[u]/u^e$ induces a morphism $\mathrm{Aut}(\widetilde{\mathcal{L}})(R[\![u]\!]) \to \mathrm{Aut}(\overline{\mathcal{L}})(R[u]/u^e)$ and we thereby extend this $\mathrm{Aut}(\overline{\mathcal{L}})(R[u]/u^e)$-action on $M^{e,n}(R)$ to an $\mathrm{Aut}(\widetilde{\mathcal{L}})(R[\![u]\!])$-action.

LEMMA 3.8.2. — *Let R be an $\mathbb{F}$-algebra and let $t \in M^{e,n}(R)$. We have $\mathrm{Aut}(\widetilde{\mathcal{L}})(R[\![u]\!]) \cdot t = \mathrm{Aut}(\overline{\mathcal{L}})(R[u]/u^e) \cdot t$.*

Proof. — The map $\mathrm{Aut}(\widetilde{\mathcal{L}})(R[\![u]\!]) \to \mathrm{Aut}(\overline{\mathcal{L}})(R[u]/u^e)$ is surjective by Proposition 3.7.2. □

Define a subfunctor $I_{\mathrm{Sp}} = I_{\mathrm{Sp}_{2n}}$ of $\mathrm{L}\,\mathrm{Sp}_{2n}$ by $I_{\mathrm{Sp}} = \mathrm{L}\,\mathrm{Sp}_{2n} \cap I_{\mathrm{GSp}}$.

LEMMA 3.8.3. — *We have $I_{\mathrm{GSp}}(\mathbb{F}) = \mathbb{F}[\![u]\!]^\times I_{\mathrm{Sp}}(\mathbb{F})$.*

Proof. — Let $g \in I_{\mathrm{GSp}}(\mathbb{F})$. Clearly $c(g) \in \mathbb{F}[\![u]\!]^\times$. As $\mathrm{char}\,\mathbb{F} \neq 2$, there is an $x \in \mathbb{F}[\![u]\!]^\times$ with $x^2 = c(g)$. Then $x^{-1}g \in I_{\mathrm{Sp}}(\mathbb{F})$. □

LEMMA 3.8.4. — *Let $g \in I_{\mathrm{Sp}}(\mathbb{F})$. Then g restricts to an automorphism $g_i : \widetilde{\Lambda}_i \to \widetilde{\Lambda}_i$ for each $i \in \mathbb{Z}$. The assignment $g \mapsto (g_i)_i$ defines an isomorphism $I_{\mathrm{Sp}}(\mathbb{F}) \xrightarrow{\sim} \mathrm{Aut}(\widetilde{\mathcal{L}})(\mathbb{F}[\![u]\!])$.*

Proof. — Analogous to the proof of Lemma 2.5.5. □

PROPOSITION 3.8.5. — *Let $t \in M^{e,n}(\mathbb{F})$. Then α induces a bijection*
$$\mathrm{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u]/u^e) \cdot t \xrightarrow{\sim} I_{\mathrm{GSp}}(\mathbb{F}) \cdot \alpha(t).$$
Consequently we obtain an embedding
$$\mathrm{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u]/u^e) \backslash M^{e,n}(\mathbb{F}) \hookrightarrow I_{\mathrm{GSp}}(\mathbb{F}) \backslash \mathcal{F}_{\mathrm{GSp}}(\mathbb{F}).$$

Proof. — The composition $M^{e,n}(\mathbb{F}) \xrightarrow{\alpha} \mathcal{F}_{\mathrm{Sp}}^{(-e)}(\mathbb{F}) \subset \mathcal{F}_{\mathrm{GSp}}(\mathbb{F})$ is equivariant for the $\mathrm{Aut}(\widetilde{\mathcal{L}})(\mathbb{F}[\![u]\!])$-action on $M^{e,n}(\mathbb{F})$, the $I_{\mathrm{Sp}}(\mathbb{F})$-action on $\mathcal{F}_{\mathrm{GSp}}(\mathbb{F})$ and the isomorphism $I_{\mathrm{Sp}}(\mathbb{F}) \xrightarrow{\sim} \mathrm{Aut}(\widetilde{\mathcal{L}})(\mathbb{F}[\![u]\!])$ of Lemma 3.8.4. It therefore induces a bijection $\mathrm{Aut}(\widetilde{\mathcal{L}})(\mathbb{F}[\![u]\!]) \cdot t \xrightarrow{\sim} I_{\mathrm{Sp}}(\mathbb{F}) \cdot \alpha(t)$. We conclude by applying Lemmata 3.8.2 and 3.8.3. □

Consider $\alpha' : M^{e,n}(\mathbb{F}) \hookrightarrow \mathcal{F}_{\mathrm{GSp}}(\mathbb{F}) \xrightarrow{\phi(\mathbb{F})^{-1}} \mathrm{L}\,\mathrm{GSp}(\mathbb{F})/I_{\mathrm{GSp}}(\mathbb{F})$.

PROPOSITION 3.8.6. — *Let $t \in M^{e,n}(\mathbb{F})$. Then α' induces a bijection*
$$\mathrm{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u]/u^e) \cdot t \xrightarrow{\sim} I_{\mathrm{GSp}}(\mathbb{F}) \cdot \alpha'(t).$$
Consequently we obtain an embedding
$$\mathrm{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u]/u^e) \backslash M^{e,n}(\mathbb{F}) \hookrightarrow I_{\mathrm{GSp}}(\mathbb{F}) \backslash \mathrm{GSp}(\mathbb{F}(\!(u)\!)) / I_{\mathrm{GSp}}(\mathbb{F}).$$

Proof. — Clear from Proposition 3.8.5, as the isomorphism $\phi(\mathbb{F})$ is in particular $I_{\mathrm{GSp}}(\mathbb{F})$-equivariant. $\qquad\square$

Let R be an $\mathbb{F}$-algebra and $(\varphi_i)_i \in \mathrm{Aut}(\mathcal{L})(R)$. The decomposition (3.5.2) induces for each i a decomposition of $\varphi_i : \Lambda_{i,R} \xrightarrow{\sim} \Lambda_{i,R}$ into the product of $R[u]/u^e$-linear automorphisms $\varphi_{i,\sigma} : \overline{\Lambda}_{i,R} \xrightarrow{\sim} \overline{\Lambda}_{i,R}$. The following statement is then clear (cf. the proof of Proposition 3.5.2).

PROPOSITION 3.8.7. — *Let R be an $\mathbb{F}$-algebra. The following map is an isomorphism, functorial in R.*

$$\mathrm{Aut}(\mathcal{L})(R) \to \prod_{\sigma \in \mathfrak{S}} \mathrm{Aut}(\overline{\mathcal{L}})(R[u]/u^e),$$

$$(\varphi_i)_i \mapsto ((\varphi_{i,\sigma})_i)_\sigma.$$

Consider the composition

$$\tilde{\alpha} : M^{\mathrm{loc}}(\mathbb{F}) \xrightarrow{(3.5.3)} \prod_{\sigma \in \mathfrak{S}} M^{e,n}(\mathbb{F}) \xrightarrow{\prod_\sigma \alpha'} \prod_{\sigma \in \mathfrak{S}} \mathrm{L\,GSp}(\mathbb{F})/I_{\mathrm{GSp}}(\mathbb{F}).$$

For $\sigma \in \mathfrak{S}$ denote by $\tilde{\alpha}_\sigma : M^{\mathrm{loc}}(\mathbb{F}) \to \mathrm{L\,GSp}(\mathbb{F})/I_{\mathrm{GSp}}(\mathbb{F})$ the corresponding component of $\tilde{\alpha}$.

THEOREM 3.8.8. — *Let $t \in M^{\mathrm{loc}}(\mathbb{F})$. Then $\tilde{\alpha}$ induces a bijection*

$$\mathrm{Aut}(\mathcal{L})(\mathbb{F}) \cdot t \xrightarrow{\sim} \prod_{\sigma \in \mathfrak{S}} I_{\mathrm{GSp}}(\mathbb{F}) \cdot \tilde{\alpha}_\sigma(t).$$

Consequently we obtain an embedding

$$\mathrm{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F}) \hookrightarrow \prod_{\sigma \in \mathfrak{S}} I_{\mathrm{GSp}}(\mathbb{F}) \backslash \mathrm{GSp}(F((u)))/I_{\mathrm{GSp}}(\mathbb{F}).$$

Proof. — The isomorphism $M^{\mathrm{loc}}(\mathbb{F}) \xrightarrow{(3.5.3)} \prod_{\sigma \in \mathfrak{S}} M^{e,n}(\mathbb{F})$ is equivariant for the $\mathrm{Aut}(\mathcal{L})(\mathbb{F})$ action on $M^{\mathrm{loc}}(\mathbb{F})$, the $\prod_{\sigma \in \mathfrak{S}} \mathrm{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u]/u^e)$ action on $\prod_{\sigma \in \mathfrak{S}} M^{e,n}(\mathbb{F})$ and the isomorphism of Lemma 3.8.7. The statement thus follows from Proposition 3.8.6. $\qquad\square$

3.9. The extended affine Weyl group

Let T be the maximal torus of diagonal matrices in GSp_{2n} and let N be its normalizer. We denote by $\widetilde{W} = N(\mathbb{F}((u)))/T(\mathbb{F}[\![u]\!])$ the extended affine Weyl group of GSp with respect to T. Setting

$$W = \{w \in S_{2n} \mid \forall i \in \{1, \ldots, 2n\} : \; w(i) + w(2n+1-i) = 2n+1\}$$

and

$$X = \{(a_1, \ldots, a_{2n}) \in \mathbb{Z}^{2n} \mid a_1 + a_{2n} = a_2 + a_{2n-1} = \cdots = a_n + a_{n+1}\},$$

the group homomorphism $v : W \ltimes X \to N(\mathbb{F}((u)))$, $(w, \lambda) \mapsto A_w u^\lambda$ induces an isomorphism $W \ltimes X \xrightarrow{\sim} \widetilde{W}$. We use it to identify $\widetilde{W}$ with $W \ltimes X$ and consider $\widetilde{W}$ as a subgroup of $\mathrm{GSp}(\mathbb{F}((u)))$ via v.

To avoid any confusion of the product inside $\widetilde{W}$ and the canonical action of S_{2n} on $\mathbb{Z}^{2n}$, we will always denote the element of $\widetilde{W}$ corresponding to $\lambda \in X$ by u^λ.

Recall from [5, §2.5-2.6] the notion of an extended alcove $(x_i)_{i=0}^{2n-1}$ for GSp_{2n}. Also recall the standard alcove $(\omega_i)_{i=0}^{2n-1}$. As in loc. cit. we identify $\widetilde{W}$ with the set of extended alcoves by using the standard alcove as a base point.

Write $\mathbf{e} = (e^{(2n)})$.

DEFINITION 3.9.1 (Cf. [14], [5, Definition 2.4]). — *An extended alcove* $(x_i)_{i=0}^{2n-1}$ *is called* permissible *if it satisfies the following conditions for all* $i \in \{0, \ldots, 2n-1\}$.

(1) $\omega_i \leqslant x_i \leqslant \omega_i + \mathbf{e}$, *where* $\leqslant$ *is to be understood componentwise.*
(2) $\sum_{j=1}^{2n} x_i(j) = ne - i$.

Denote by Perm *the set of all permissible extended alcoves.*

PROPOSITION 3.9.2. — *The inclusion* $N(\mathbb{F}((u))) \subset \mathrm{GSp}(\mathbb{F}((u)))$ *induces a bijection* $\widetilde{W} \xrightarrow{\sim} I_{\mathrm{GSp}}(\mathbb{F}) \backslash \mathrm{GSp}(\mathbb{F}((u))) / I_{\mathrm{GSp}}(\mathbb{F})$. *In other words,*

$$\mathrm{GSp}(\mathbb{F}((u))) = \coprod_{x \in \widetilde{W}} I_{\mathrm{GSp}}(\mathbb{F}) x I_{\mathrm{GSp}}(\mathbb{F}).$$

Under this bijection, the subset

$$\mathrm{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u]/u^e) \backslash M^{e,n}(\mathbb{F}) \subset I_{\mathrm{GSp}}(\mathbb{F}) \backslash \mathrm{GSp}(\mathbb{F}((u))) / I_{\mathrm{GSp}}(\mathbb{F})$$

of Proposition 3.8.6 corresponds to the subset Perm $\subset \widetilde{W}$.

Proof. — The first statement is the well-known Iwahori decomposition. The second statement follows easily from the explicit description of the image of α in Proposition 3.8.1. $\qquad\square$

COROLLARY 3.9.3. — *Under the identifications of Theorem 3.8.8, the set* $\prod_{\sigma \in \mathfrak{S}}$ Perm *constitutes a set of representatives of* $\mathrm{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F})$.

Remark 3.9.4. — In the normalization of Corollary 3.9.3 we have indexed the KR stratification by the relative position of $\mathcal{L} \otimes \mathbb{F}$ to its image

under *Verschiebung*, compare Remark 2.5.11. The normalization of Corollary 3.9.3 therefore differs from the one of Definition 2.5.10 by the automorphism $\prod_{\sigma \in \mathfrak{S}} \mathrm{Perm} \to \prod_{\sigma \in \mathfrak{S}} \mathrm{Perm},\ (x_\sigma) \mapsto (u^e x_\sigma^{-1})$.

3.10. The p-rank on a KR stratum

Recall from Section 2.3 the scheme $\mathcal{A}/\mathbb{Z}_p$ associated with our choice of PEL datum, and the KR stratification

$$\mathcal{A}(\mathbb{F}) = \coprod_{x \in \mathrm{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F})} \mathcal{A}_x.$$

We have identified the occurring index set with $\prod_{\sigma \in \mathfrak{S}} \mathrm{Perm}$ in Corollary 3.9.3. We can then state the following result.

THEOREM 3.10.1. — *Let $x = (x_\sigma)_\sigma \in \prod_{\sigma \in \mathfrak{S}} \mathrm{Perm}$. Write $x_\sigma = w_\sigma u^{\lambda_\sigma}$ with $w_\sigma \in W,\ \lambda_\sigma \in X$. Then the p-rank on $\mathcal{A}_x$ is constant with value*

$$g \cdot |\{1 \leqslant i \leqslant 2n \mid \forall \sigma \in \mathfrak{S}(w_\sigma(i) = i \text{ and } \lambda_\sigma(i) = 0)\}|.$$

Remark 3.10.2. — For $F = \mathbb{Q}$, we recover the result [16, Théorème 4.1] of Ngô and Genestier.

Proof of Theorem 3.10.1. — Let $1 \leqslant i \leqslant 2n$ and $x \in \mathrm{Perm}$. Write $x = wu^\lambda$ with $w \in W, \lambda \in X$. By Propositions 2.4.5 and 2.4.6 it suffices to show the following equivalence.

$$\widetilde{\Lambda}_i = x(\widetilde{\Lambda}_i) + \widetilde{\Lambda}_{i-1} \Leftrightarrow (w(i) = i \text{ and } \lambda(i) = 0).$$

Consider the subset $\mathcal{S} = \{u^k e_j \mid k \in \mathbb{Z},\ 1 \leqslant j \leqslant 2n\}$ of $\mathbb{F}((u))^{2n}$. Then x induces a permutation of $\mathcal{S}$, namely $x(u^k e_j) = u^{\lambda(j)+k} e_{w(j)}$. We have $\widetilde{\Lambda}_i \cap \mathcal{S} = \widetilde{\Lambda}_{i-1} \cap \mathcal{S} \amalg \{u^{-1} e_i\}$, and $x \in \mathrm{Perm}$ implies $x(\widetilde{\Lambda}_{i-1} \cap \mathcal{S}) \subset \widetilde{\Lambda}_{i-1} \cap \mathcal{S}$. Consequently $u^{-1} e_i \in x(\widetilde{\Lambda}_i \cap \mathcal{S})$ if and only if $x(u^{-1} e_i) = u^{-1} e_i$, which in turn is equivalent to $(w(i) = i \text{ and } \lambda(i) = 0)$, as desired. $\square$

3.11. The density of the ordinary locus

Denote by $\leqslant$ and ℓ the partial order and the length function on $\widetilde{W}$ defined in [5, §2.1], respectively. We extend $\leqslant$ and ℓ to $\prod_{\sigma \in \mathfrak{S}} \widetilde{W}$ by setting $(x_\sigma)_\sigma \leqslant (x'_\sigma)_\sigma :\Leftrightarrow (\forall \sigma \in \mathfrak{S} : x_\sigma \leqslant x'_\sigma)$ and $\ell((x_\sigma)_\sigma) = \sum_\sigma \ell(x_\sigma)$.

LEMMA 3.11.1. — *Let $x \in \prod_{\sigma \in \mathfrak{S}} \mathrm{Perm}$. The smooth $\mathbb{F}$-variety $\mathcal{A}_x$ is equidimensional of dimension $\ell(x)$. Furthermore the closure $\overline{\mathcal{A}}_x$ of $\mathcal{A}_x$ in $\mathcal{A}_{\mathbb{F}}$ is given by*

$$(3.11.1) \qquad\qquad \overline{\mathcal{A}}_x = \coprod_{y \leqslant x} \mathcal{A}_y.$$

Proof. — As in Remark 2.3.10 there is, étale locally on $\mathcal{A}_{\mathbb{F}}$, an étale morphism $\beta : \mathcal{A}_{\mathbb{F}} \to M^{\mathrm{loc}}_{\mathbb{F}}$ with $\mathcal{A}_x = \beta^{-1}(M^{\mathrm{loc}}_x)$. In Theorem 3.8.8 we have identified M^{loc}_x with the Schubert cell $\mathcal{C}_x \subset \prod_{\sigma \in \mathfrak{S}} \mathrm{GSp}_{2n}(\mathbb{F}((u)))/I_{\mathrm{GSp}}(\mathbb{F})$ corresponding to x. The statements therefore follow from well-known properties of Schubert cells once we know that all KR strata are non-empty. This is true in the Siegel case by Genestier's result [3, Proposition 1.3.2], in the case that p is unramified in F by the result [4, Theorem 2.5.2(1)] of Goren and Kassaei, and in the ramified case by a yet to be published result of Yu [32]. □

Our next goal is to generalize the result [16, Corollaire 4.3] of Ngô and Genestier on the density of the ordinary locus. Denote by $\mathcal{A}^{(ng)} \subset \mathcal{A}(\mathbb{F})$ the subset where the p-rank of the underlying abelian variety is equal to ng. By the determinant condition imposed in the definition of $\mathcal{A}$ this is precisely the ordinary locus in $\mathcal{A}(\mathbb{F})$.

COROLLARY 3.11.2. — *The ordinary locus $\mathcal{A}^{(ng)}$ is dense in $\mathcal{A}(\mathbb{F})$ if and only if p is totally ramified in F.*

Proof. — Let $\mu = (e^{(g)}, 0^{(g)}) \in X$. Our subset $\mathrm{Perm} \subset \widetilde{W}$ is precisely the set denoted by $\mathrm{Perm}(\mu)$ in [8]. By [8, Theorem 10.1], we have

$$(3.11.2) \qquad\qquad \mathrm{Perm}(\mu) = \mathrm{Adm}(\mu),$$

where $\mathrm{Adm}(\mu) := \{x \in \widetilde{W} \mid \exists w \in W : x \leqslant u^{w(\mu)}\}$.

Write $\mathfrak{M} = \{x \in \widetilde{W} \mid \exists w \in W : x = u^{w(\mu)}\}$. Then (3.11.2) implies that $\prod_{\sigma \in \mathfrak{S}} \mathfrak{M}$ is precisely the subset of maximal elements for $\leqslant$ in $\prod_{\sigma \in \mathfrak{S}} \mathrm{Perm}$. Denote by $\Delta_{\mathfrak{M}} \subset \prod_{\sigma \in \mathfrak{S}} \mathfrak{M}$ the diagonal. By Theorem 3.10.1 we have $\mathcal{A}^{(ng)} = \coprod_{x \in \Delta_{\mathfrak{M}}} \mathcal{A}_x$. The statement therefore follows from (3.11.1) by noting that $\Delta_{\mathfrak{M}} = \prod_{\sigma \in \mathfrak{S}} \mathfrak{M}$ if and only if p is totally ramified in F. □

3.12. An explicit example: Hilbert-Blumenthal modular varieties

In this section we use the explicit case of the Hilbert-Blumenthal modular varieties to illustrate how Theorem 3.10.1 and the KR stratification in

general yield results about the geometry of the moduli spaces $\mathcal{A}$. We also compare these results to some of those obtained by Stamm in [29].

Assume from now on that p is *inert* in $\mathcal{O}_F$, so that we have $e = 1$ and $f = g$. Assume also that $\dim_F V = 2$, so that $n = 1$.

Let us start with a discussion of the index set $\prod_{\sigma \in \mathfrak{S}}$ Perm of the KR stratification. From Definition 3.9.1 one immediately obtains that the subset Perm $\subset \widetilde{W}$ is given by Perm $= \{u^{(1,0)}, u^{(0,1)}, (1,2)u^{(1,0)}\}$. To put this set into a group theoretic perspective, we recall the setup described in [5, §2.1] in this easy special case. Consider the elements $\tau = (1,2)u^{(1,0)}$, $s_1 = (1,2)$ and $s_0 = (1,2)u^{(1,-1)}$ of $\widetilde{W}$. The subgroup W_a of $\widetilde{W}$ generated by s_0 and s_1 is a Coxeter group on the generators s_0 and s_1, and we denote by $\leqslant$ and ℓ the corresponding Bruhat order and length function on W_a, respectively. Denoting by Ω the cyclic subgroup of $\widetilde{W}$ generated by τ, we have $\widetilde{W} = W_a \rtimes \Omega$. The extension of $\leqslant$ and ℓ to $\widetilde{W}$ is given by $w'\tau' \leqslant w''\tau'' \Leftrightarrow (w' \leqslant w''$ and $\tau' = \tau'')$ and $\ell(w'\tau') = \ell(w')$, for $w', w'' \in W_a$ and $\tau', \tau'' \in \Omega$. We extend $\leqslant$ and ℓ to $\prod_{\sigma \in \mathfrak{S}} \widetilde{W}$ as in Section 3.11.

We see that
$$\text{Perm} = \{s_1\tau, s_0\tau, \tau\} \subset W_a\tau.$$
The Bruhat order on Perm is determined by the non-trivial relations $\tau \leqslant s_1\tau$ and $\tau \leqslant s_0\tau$, while the length function on Perm is given by $\ell(\tau) = 0$ and $\ell(s_1\tau) = \ell(s_0\tau) = 1$.

Let us state Theorem 3.10.1 in this special case. Denote by $\mathcal{A}^{(0)} \subset \mathcal{A}(\mathbb{F})$ and $\mathcal{A}^{(g)} \subset \mathcal{A}(\mathbb{F})$ the subsets where the p-rank of the underlying abelian variety is equal to 0 and g, respectively.

PROPOSITION 3.12.1. — *We have*
$$\mathcal{A}(\mathbb{F}) = \mathcal{A}^{(0)} \amalg \mathcal{A}^{(g)}.$$

The ordinary locus $\mathcal{A}^{(g)}$ is the union of only two KR strata, namely those corresponding to the elements $((s_1\tau)^{(g)}) = (s_1\tau, s_1\tau, \ldots, s_1\tau)$ and $((s_0\tau)^{(g)}) = (s_0\tau, s_0\tau, \ldots, s_0\tau)$ of $\prod_{\sigma \in \mathfrak{S}}$ Perm. The p-rank on all other KR strata is equal to 0.

LEMMA 3.12.2. — *The maximal elements in $\prod_{\sigma \in \mathfrak{S}}$ Perm for the Bruhat order are precisely the elements of length 2^g in $\prod_{\sigma \in \mathfrak{S}}$ Perm. The set of these maximal elements is given by $\prod_{\sigma \in \mathfrak{S}} \{s_1\tau, s_0\tau\}$.* □

From the preceding results, we obtain without any additional work the following theorem.

THEOREM 3.12.3. — *Let $g \geqslant 2$. Then*
$$\mathcal{A}_\mathbb{F} = \overline{\mathcal{A}}_{((s_1\tau)^{(g)})} \cup \overline{\mathcal{A}}_{((s_0\tau)^{(g)})} \cup \mathcal{A}^{(0)}.$$

Each of $\overline{\mathcal{A}}_{((s_1\tau)^{(g)})}, \overline{\mathcal{A}}_{((s_0\tau)^{(g)})}$ and $\mathcal{A}^{(0)}$ is equidimensional of dimension 2^g, and hence so is $\mathcal{A}_{\mathbb{F}}$.

More precisely, $\mathcal{A}^{(0)}$ is the union

$$\mathcal{A}^{(0)} = \bigcup_{\substack{x \in \prod_{\sigma \in \mathfrak{S}} \{s_1\tau, s_0\tau\} \\ x \neq ((s_1\tau)^{(g)}), ((s_0\tau)^{(g)})}} \overline{\mathcal{A}}_x$$

of $2^g - 2$ closed subsets, all equidimensional of dimension 2^g.

Furthermore, we have

$$\overline{\mathcal{A}}_{((s_1\tau)^{(g)})} \cap \overline{\mathcal{A}}_{((s_0\tau)^{(g)})} \subset \mathcal{A}^{(0)}.$$

Taking $g = 2$, we recover [29, Theorem 2 (p. 408)]. Note that for $g = 2$, the set $\mathcal{A}^{(0)}$ is precisely the supersingular locus in $\mathcal{A}_{\mathbb{F}}$, because a 2-dimensional abelian variety is supersingular if and only if its p-rank is equal to zero.

4. The unitary PEL datum

Let $n \in \mathbb{N}_{\geqslant 1}$. In Sections 5 through 7 we will be concerned with the PEL datum consisting of the following objects.

(1) An imaginary quadratic extension F/F_0 of a totally real extension $F_0/\mathbb{Q}$. Let $g_0 = [F_0 : \mathbb{Q}]$ and $g = [F : \mathbb{Q}]$, so that $g = 2g_0$.

(2) The non-trivial element $*$ of $\mathrm{Gal}(F/F_0)$.

(3) An n-dimensional F-vector space V.

(4) The symplectic form $(\cdot, \cdot) : V \times V \to \mathbb{Q}$ on the underlying $\mathbb{Q}$-vector space of V constructed as follows: Fix once and for all a $*$-skew-hermitian form $(\cdot, \cdot)' : V \times V \to F$ (i.e. $(av, bw)' = ab^*(v, w)'$ and $(v, w)' = -(w, v)'^*$ for $v, w \in V$, $a, b \in F$). Define $(\cdot, \cdot) = \mathrm{tr}_{F/\mathbb{Q}} \circ (\cdot, \cdot)'$.

(5) The element $J \in \mathrm{End}_{B \otimes \mathbb{R}}(V \otimes \mathbb{R})$ to be defined separately in each case, see Sections 5.1, 6.1 and 7.1.

Remark 4.0.1. — Denote by $\mathrm{GU}_{(\cdot,\cdot)'}$ the F_0-group given on R-valued points by $\mathrm{GU}_{(\cdot,\cdot)'}(R) = \{g \in \mathrm{GL}_{F \otimes_{F_0} R}(V \otimes_{F_0} R) \mid \exists c = c(g) \in R^\times \forall x, y \in V \otimes_{F_0} R : (gx, gy)'_R = c(x, y)'_R\}$. Then the reductive $\mathbb{Q}$-group G associated with the above PEL datum fits into the following cartesian diagram.

$$\begin{array}{ccc} G & \lhook\joinrel\longrightarrow & \mathrm{Res}_{F_0/\mathbb{Q}} \mathrm{GU}_{(\cdot,\cdot)'} \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c} \\ \mathbb{G}_{m,\mathbb{Q}} & \lhook\joinrel\longrightarrow & \mathrm{Res}_{F_0/\mathbb{Q}} \mathbb{G}_{m,F_0}. \end{array}$$

5. The ramified unitary case

5.1. The PEL datum

We start with the PEL datum defined in Section 4. We assume that $p\mathcal{O}_{F_0} = (\mathcal{P}_0)^{e_0}$ for a single prime $\mathcal{P}_0$ of $\mathcal{O}_{F_0}$ and that $\mathcal{P}_0\mathcal{O}_F = \mathcal{P}^2$ for a prime $\mathcal{P}$ of $\mathcal{O}_F$. Write $e = 2e_0$, so that $p\mathcal{O}_F = \mathcal{P}^e$. Denote by $f = [k_{\mathcal{P}_0} : \mathbb{F}_p]$ the corresponding inertia degree, so that $g = ef$ and $g_0 = fe_0$. Fix once and for all uniformizers π_0 of $\mathcal{O}_{F_0} \otimes \mathbb{Z}_{(p)}$ and π of $\mathcal{O}_F \otimes \mathbb{Z}_{(p)}$, satisfying $\pi^2 = \pi_0$. We have $\pi^* = -\pi$.

For typographical reasons, we denote the ring of integers in $F_{\mathcal{P}}$ by $\mathcal{O}_{\mathcal{P}}$ and the ring of integers in $(F_0)_{\mathcal{P}_0}$ by $\mathcal{O}_{\mathcal{P}_0}$. Denote by $\mathfrak{C} = \mathfrak{C}_{\mathcal{O}_{\mathcal{P}}|\mathbb{Z}_p}$, $\mathfrak{C}_0 = \mathfrak{C}_{\mathcal{O}_{\mathcal{P}_0}|\mathbb{Z}_p}$ and $\mathfrak{C}' = \mathfrak{C}_{\mathcal{O}_{\mathcal{P}}|\mathcal{O}_{\mathcal{P}_0}}$ the corresponding inverse differents. Then $\mathfrak{C}_0 = (\pi_0^{-k})$ for some $k \in \mathbb{N}$. The extension $F_{\mathcal{P}}/(F_0)_{\mathcal{P}_0}$ is tamely ramified, so that $\mathfrak{C}' = (\pi^{-1})$. The equality $\mathfrak{C} = \mathfrak{C}' \cdot \mathfrak{C}_0$ then implies that $\mathfrak{C} = (\pi^{-2k-1})$ and we denote by $\delta = \pi^{-2k-1}$ the corresponding generator of $\mathfrak{C}$. It satisfies $\delta^* = -\delta$. Consequently the form $\delta^{-1} (\cdot, \cdot)'_{\mathbb{Q}_p} : V_{\mathbb{Q}_p} \times V_{\mathbb{Q}_p} \to F_{\mathcal{P}}$ is $*$-hermitian and we assume that it *splits*, i.e. that there is a basis $(e_1, \ldots, e_n)$ of $V_{\mathbb{Q}_p}$ over $F_{\mathcal{P}}$ such that $(e_i, e_{n+1-j})'_{\mathbb{Q}_p} = \delta \delta_{ij}$ for $1 \leqslant i, j \leqslant n$. Here we denote by δ_{ij} the Kronecker delta.

Let $0 \leqslant i < n$. We denote by Λ_i the $\mathcal{O}_{\mathcal{P}}$-lattice in $V_{\mathbb{Q}_p}$ with basis

$$\mathfrak{E}_i = (\pi^{-1}e_1, \ldots, \pi^{-1}e_i, e_{i+1}, \ldots, e_n).$$

For $k \in \mathbb{Z}$ we further define $\Lambda_{nk+i} = \pi^{-k}\Lambda_i$ and we denote by $\mathfrak{E}_{nk+i}$ the corresponding basis obtained from $\mathfrak{E}_i$. Then $\mathcal{L} = (\Lambda_i)_i$ is a complete chain of $\mathcal{O}_{\mathcal{P}}$-lattices in $V_{\mathbb{Q}_p}$. For $i \in \mathbb{Z}$, the dual lattice $\Lambda_i^{\vee} := \{x \in V_{\mathbb{Q}_p} \mid (x, \Lambda_i)_{\mathbb{Q}_p} \subset \mathbb{Z}_p\}$ of Λ_i is given by Λ_{-i}. Consequently the chain $\mathcal{L}$ is self-dual.

Let $i \in \mathbb{Z}$. We denote by $\rho_i : \Lambda_i \to \Lambda_{i+1}$ the inclusion, by $\vartheta_i : \Lambda_{n+i} \to \Lambda_i$ the isomorphism given by multiplication with π and by $(\cdot, \cdot)_i : \Lambda_i \times \Lambda_{-i} \to \mathbb{Z}_p$ the restriction of $(\cdot, \cdot)_{\mathbb{Q}_p}$. Then $(\Lambda_i, \rho_i, \vartheta_i, (\cdot, \cdot)_i)_i$ is a polarized chain of $\mathcal{O}_{F_{\mathcal{P}}}$-modules of type $(\mathcal{L})$, which, by abuse of notation, we also denote by $\mathcal{L}$.

Denote by $\langle \cdot, \cdot \rangle_i : \Lambda_i \times \Lambda_{-i} \to \mathcal{O}_{\mathcal{P}}$ the restriction of the $*$-hermitian form $\delta^{-1} (\cdot, \cdot)'_{\mathbb{Q}_p}$, and by H_i the matrix describing $\langle \cdot, \cdot \rangle_i$ with respect to $\mathfrak{E}_i$ and $\mathfrak{E}_{-i}$. We have

(5.1.1) $$H_i = \text{anti-diag}((-1)^{a_{i,1}}, \ldots, (-1)^{a_{i,n}})$$

for some $a_{i,1}, \ldots, a_{i,n} \in \mathbb{Z}/2\mathbb{Z}$.

Denote by Σ_0 the set of all embeddings $F_0 \hookrightarrow \mathbb{R}$ and by Σ the set of all embeddings $F \hookrightarrow \mathbb{C}$. For each $\sigma \in \Sigma_0$, we denote by $\tau_{\sigma,1}, \tau_{\sigma,2} \in \Sigma$ the two embeddings with $\tau_{\sigma,j}\big|_{F_0} = \sigma$. Of course we have $\tau_{\sigma,2} = \tau_{\sigma,1} \circ *$.

We obtain isomorphisms

$$(5.1.2) \qquad F \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{\sigma \in \Sigma_0} \mathbb{C}, \qquad\qquad F \ni x \mapsto (\tau_{\sigma,1}(x))_\sigma,$$

$$(5.1.3) \qquad F \otimes_{\mathbb{Q}} \mathbb{C} = \prod_{\sigma \in \Sigma_0} \mathbb{C} \times \mathbb{C}, \qquad F \ni x \mapsto (\tau_{\sigma,1}(x), \tau_{\sigma,2}(x))_\sigma$$

of $\mathbb{R}$- and $\mathbb{C}$-algebras, respectively.

The isomorphism (5.1.2) induces a decomposition $V \otimes \mathbb{R} = \prod_{\sigma \in \Sigma_0} V_\sigma$ into $\mathbb{C}$-vector spaces V_σ and $(\cdot, \cdot)'_{\mathbb{R}}$ decomposes into the product of skew-hermitian forms $(\cdot, \cdot)'_\sigma : V_\sigma \times V_\sigma \to \mathbb{C}$, $\sigma \in \Sigma_0$. For each $\sigma \in \Sigma_0$, there are $r_\sigma, s_\sigma \in \mathbb{N}$ with $r_\sigma + s_\sigma = n$ and a basis $\mathfrak{B}_\sigma$ of V_σ over $\mathbb{C}$ such that $(\cdot, \cdot)'_\sigma$ is described by the matrix $D_\sigma = \mathrm{diag}(i^{(r_\sigma)}, (-i)^{(s_\sigma)})$ with respect to $\mathfrak{B}_\sigma$. Denote by J_σ the endomorphism of V_σ described by the matrix D_σ with respect to $\mathfrak{B}_\sigma$. We complete the description of the PEL datum by defining $J := \prod_{\sigma \in \Sigma_0} J_\sigma \in \mathrm{End}_{B \otimes \mathbb{R}}(V \otimes \mathbb{R})$.

5.2. The determinant morphism

The isomorphism (5.1.3) induces a decomposition $V \otimes \mathbb{C} = \prod_{\sigma \in \Sigma_0} (V_{\tau_{\sigma,1}} \times V_{\tau_{\sigma,2}})$ into $\mathbb{C}$-vector spaces $V_{\tau_{\sigma,j}}$. The basis $\mathfrak{B}_\sigma$ of V_σ induces bases $\mathfrak{B}_{\tau_{\sigma,j}}$ of $V_{\tau_{\sigma,j}}$ over $\mathbb{C}$, and the endomorphism $J_{\sigma,\mathbb{C}}$ decomposes into the product of endomorphisms $J_{\tau_{\sigma,j}}$ of $V_{\tau_{\sigma,j}}$. We find that $J_{\tau_{\sigma,1}}$ is described by the matrix D_σ with respect to $\mathfrak{B}_{\tau_{\sigma,1}}$, while $J_{\tau_{\sigma,2}}$ is described by the matrix $-D_\sigma$ with respect to $\mathfrak{B}_{\tau_{\sigma,2}}$.

Denote by V_{-i} the $(-i)$-eigenspace of $J_{\mathbb{C}}$. From the explicit description of the $J_{\tau_{\sigma,j}}$ with respect to the $\mathfrak{B}_{\tau_{\sigma,j}}$, one concludes that V_{-i} is the $\mathcal{O}_F \otimes \mathbb{C}$-module corresponding to the $\prod_{\sigma \in \Sigma_0} \mathbb{C} \times \mathbb{C}$-module $\prod_{\sigma \in \Sigma_0} \mathbb{C}^{s_\sigma} \times \mathbb{C}^{r_\sigma}$ under (5.1.3).

Let E' be the Galois closure of F inside $\mathbb{C}$ and choose a prime $\mathcal{Q}'$ of E' over $\mathcal{P}$. In absolute analogy to (5.1.3), we have a decomposition

$$(5.2.1) \qquad\qquad F \otimes_{\mathbb{Q}} E' = \prod_{\sigma \in \Sigma_0} E' \times E'.$$

Let M be the $\mathcal{O}_F \otimes E'$-module corresponding to the $\prod_{\sigma \in \Sigma_0} E' \times E'$-module $\prod_{\sigma \in \Sigma_0} (E')^{s_\sigma} \times (E')^{r_\sigma}$ under (5.2.1). From the present discussion we obtain an identification $M \otimes_{E'} \mathbb{C} = V_{-i}$ of $\mathcal{O}_F \otimes \mathbb{C}$-modules. Let $\mathfrak{B}$ be a basis of M over E' and denote by M_0 the $\mathcal{O}_F \otimes \mathcal{O}_{E'}$-module generated by $\mathfrak{B}$. Then M_0 is an $\mathcal{O}_F \otimes \mathcal{O}_{E'}$-stable $\mathcal{O}_{E'}$-lattice M_0 in M. In particular, the morphism $\det_{V_{-i}} : V_{\mathcal{O}_F \otimes \mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ descends to the morphism $\det_{M_0} : V_{\mathcal{O}_F \otimes \mathcal{O}_{E'}} \to \mathbb{A}^1_{\mathcal{O}_{E'}}$.

5.3. The geometric special fiber of the determinant morphism

We write $\mathfrak{S} = \mathrm{Gal}(k_{\mathcal{P}}/\mathbb{F}_p) = \mathrm{Gal}(k_{\mathcal{P}_0}/\mathbb{F}_p)$. We fix once and for all an embedding $\iota_{\mathcal{Q}'} : k_{\mathcal{Q}'} \hookrightarrow \mathbb{F}$. We consider $\mathbb{F}$ as an $\mathcal{O}_{E'}$-algebra via the composition $\mathcal{O}_{E'} \xrightarrow{\rho_{\mathcal{Q}'}} k_{\mathcal{Q}'} \xrightarrow{\iota_{\mathcal{Q}'}} \mathbb{F}$. Also $\iota_{\mathcal{Q}'}$ induces an embedding $\iota_{\mathcal{P}} : k_P \hookrightarrow \mathbb{F}$ and thereby an identification of the set of all embeddings $k_{\mathcal{P}} \hookrightarrow \mathbb{F}$ with $\mathfrak{S}$. Our choice of uniformizer π induces a canonical isomorphism

$$(5.3.1) \qquad \mathcal{O}_F \otimes \mathbb{F} = \prod_{\sigma \in \mathfrak{S}} \mathbb{F}[u]/(u^e).$$

PROPOSITION 5.3.1. — Let $x \in \mathcal{O}_F$ and let $(p_\sigma)_\sigma \in \prod_{\sigma \in \mathfrak{S}} \mathbb{F}[u]/(u^e)$ be the element corresponding to $x \otimes 1$ under (5.3.1). Then

$$\chi_{\mathbb{F}}(x|M_0 \otimes_{\mathcal{O}_{E'}} \mathbb{F}) = \prod_{\sigma \in \mathfrak{S}} \left(T - p_\sigma(0)\right)^{ne_0}$$

in $\mathbb{F}[T]$.

Proof. — Reduce $\chi_{\mathcal{O}_{E'}}(x|M_0)$ modulo $\mathcal{Q}'$, using Lemma 3.1.1. □

Denote by $E = \mathbb{Q}(\mathrm{tr}_{\mathbb{C}}(x \otimes 1|V_{-i}); \ x \in F)$ the reflex field and define $\mathcal{Q} = \mathcal{Q}' \cap \mathcal{O}_E$. The morphism $\det_{V_{-i}}$ is defined over $\mathcal{O}_E$.

5.4. The local model

For the chosen PEL datum, Definition 2.3.1 amounts to the following.

DEFINITION 5.4.1. — The local model M^{loc} is the functor on the category of $\mathcal{O}_{E_{\mathcal{Q}}}$-algebras with $M^{\mathrm{loc}}(R)$ the set of tuples $(t_i)_{i\in\mathbb{Z}}$ of $\mathcal{O}_F \otimes R$-submodules $t_i \subset \Lambda_{i,R}$ satisfying the following conditions for all $i \in \mathbb{Z}$.

(a) $\rho_{i,R}(t_i) \subset t_{i+1}$.
(b) The quotient $\Lambda_{i,R}/t_i$ is a finite locally free R-module.
(c) We have an equality

$$\det_{\Lambda_{i,R}/t_i} = \det_{V_{-i}} \otimes_{\mathcal{O}_E} R$$

of morphisms $V_{\mathcal{O}_F \otimes R} \to \mathbb{A}^1_R$.
(d) Under the pairing $(\cdot, \cdot)_{i,R} : \Lambda_{i,R} \times \Lambda_{-i,R} \to R$, the submodules t_i and t_{-i} pair to zero.
(e) $\vartheta_i(t_{n+i}) = t_i$.

5.5. The geometric special fiber of the local model

For $i \in \mathbb{Z}$, denote by $\overline{\Lambda}_i$ the free $\mathbb{F}[u]/u^e$-module $(\mathbb{F}[u]/u^e)^n$ and by $\overline{\mathfrak{E}}_i$ its canonical basis. Consider the $\mathbb{F}$-automorphism $\overline{\ast} : \mathbb{F}[u]/u^e \to \mathbb{F}[u]/u^e$, $u \mapsto -u$. Denote by $\overline{\langle \cdot, \cdot \rangle}_i : \overline{\Lambda}_i \times \overline{\Lambda}_{-i} \to \mathbb{F}[u]/u^e$ the $\overline{\ast}$-sesquilinear form described by the matrix H_i of (5.1.1) with respect to $\overline{\mathfrak{E}}_i$ and $\overline{\mathfrak{E}}_{-i}$. Denote by $\overline{\vartheta}_i : \overline{\Lambda}_{n+i} \to \overline{\Lambda}_i$ the identity morphism. For $k \in \mathbb{Z}$ and $0 \leqslant i < n$, let $\overline{\rho}_{nk+i} : \overline{\Lambda}_{nk+i} \to \overline{\Lambda}_{nk+i+1}$ be the morphism described by the matrix $\mathrm{diag}(1^{(i)}, u, 1^{(n-i-1)})$ with respect to $\overline{\mathfrak{E}}_{nk+i}$ and $\overline{\mathfrak{E}}_{nk+i+1}$.

DEFINITION 5.5.1. — *Define a functor $M^{e,n}$ on the category of $\mathbb{F}$-algebras with $M^{e,n}(R)$ the set of tuples $(t_i)_{i \in \mathbb{Z}}$ of $R[u]/u^e$-submodules $t_i \subset \overline{\Lambda}_{i,R}$ satisfying the following conditions for all $i \in \mathbb{Z}$.*

(a) *$\overline{\rho}_{i,R}(t_i) \subset t_{i+1}$.*
(b) *The quotient $\overline{\Lambda}_{i,R}/t_i$ is finite locally free over R.*
(c) *For all $P \in R[u]/u^e$, we have*

$$\chi_R(P | \overline{\Lambda}_{i,R}/t_i) = (T - P(0))^{ne_0}$$

in $R[T]$.

(d) *$t_i^{\perp, \overline{\langle \cdot, \cdot \rangle}_{i,R}} = t_{-i}$.*
(e) *$\overline{\vartheta}_i(t_{n+i}) = t_i$.*

Let $i \in \mathbb{Z}$. From (5.3.1) we obtain an isomorphism

$$\tag{5.5.1} \Lambda_{i,\mathbb{F}} = \prod_{\sigma \in \mathfrak{S}} \overline{\Lambda}_i$$

by identifying the basis $\mathfrak{E}_{i,\mathbb{F}}$ with the product of the bases $\overline{\mathfrak{E}}_i$. Under this identification, the morphism $\rho_{i,\mathbb{F}}$ decomposes into the morphisms $\overline{\rho}_i$, the pairing $\langle \cdot, \cdot \rangle_{i,\mathbb{F}}$ decomposes into the pairings $\overline{\langle \cdot, \cdot \rangle}_i$ and the morphism $\vartheta_{i,\mathbb{F}}$ decomposes into the morphisms $\overline{\vartheta}_i$.

Let R be an $\mathbb{F}$-algebra and let $(t_i)_{i \in \mathbb{Z}}$ be a tuple of $\mathcal{O}_F \otimes R$-submodules $t_i \subset \Lambda_{i,R}$. Then (5.5.1) induces decompositions $t_i = \prod_{\sigma \in \mathfrak{S}} t_{i,\sigma}$ into $R[u]/u^e$-submodules $t_{i,\sigma} \subset \overline{\Lambda}_{i,R}$. The following statement is then clear (cf. the proof of Proposition 3.5.2).

PROPOSITION 5.5.2. — *The morphism $M_{\mathbb{F}}^{\mathrm{loc}} \to \prod_{\sigma \in \mathfrak{S}} M^{e,n}$ given on R-valued points by*

$$\tag{5.5.2} M_{\mathbb{F}}^{\mathrm{loc}}(R) \to \prod_{\sigma \in \mathfrak{S}} M^{e,n}(R),$$

$$(t_i) \mapsto ((t_{i,\sigma})_i)_\sigma$$

is an isomorphism of functors on the category of $\mathbb{F}$-algebras.

5.6. The affine flag variety

This section deals with the affine flag variety for the ramified unitary group. Our discussion is based on and has greatly profited from [20], [21] and [26], [28].

Let R be an $\mathbb{F}$-algebra. Consider the extension $R[\![u]\!]/R[\![u_0]\!]$ with $u_0 = u^2$. Also consider the $R(\!(u_0)\!)$-automorphism $\widetilde{*} : R(\!(u)\!) \to R(\!(u)\!)$, $u \mapsto -u$. Let $\widetilde{\langle\cdot,\cdot\rangle}$ be the $\widetilde{*}$-hermitian form on $R(\!(u)\!)^n$ described by the matrix $\widetilde{I}_n$ with respect to the standard basis of $R(\!(u)\!)^n$ over $R(\!(u)\!)$. For a lattice Λ in $R(\!(u)\!)^n$ we define $\Lambda^\vee := \{x \in R(\!(u)\!)^n \mid \widetilde{\langle x, \Lambda \rangle} \subset R[\![u]\!]\}$. Recall from Section 3.6 the standard lattice chain $\widetilde{\mathcal{L}} = (\widetilde{\Lambda}_i)_i$ in $R(\!(u)\!)^n$. Note that $(\widetilde{\Lambda}_i)^\vee = \widetilde{\Lambda}_{-i}$ for all $i \in \mathbb{Z}$. We denote by $\widetilde{\langle\cdot,\cdot\rangle}_i : \widetilde{\Lambda}_i \times \widetilde{\Lambda}_{-i} \to R[\![u]\!]$ the restriction of $\widetilde{\langle\cdot,\cdot\rangle}$. It is the perfect $\widetilde{*}$-sesquilinear pairing described by the matrix H_i of (5.1.1) with respect to $\widetilde{\mathfrak{E}}_i$ and $\widetilde{\mathfrak{E}}_{-i}$.

In complete analogy with [24, Definition A.41], we have for an $\mathbb{F}[\![u_0]\!]$-algebra R the notion of a polarized chain $\mathcal{M} = (M_i, \varrho_i : M_i \to M_{i+1}, \theta_i : M_{n+i} \xrightarrow{\sim} M_i, \mathcal{E}_i : M_i \times M_{-i} \to \mathbb{F}[\![u]\!] \otimes_{\mathbb{F}[\![u_0]\!]} R)_{i \in \mathbb{Z}}$ of $\mathbb{F}[\![u]\!] \otimes_{\mathbb{F}[\![u_0]\!]} R$-modules of type $(\widetilde{\mathcal{L}})$ (cf. [10, Definition 6.6.1]). The proof of [24, Proposition A.43] then carries over without any changes to show the following result.

PROPOSITION 5.6.1. — *Let R be an $\mathbb{F}[\![u_0]\!]$-algebra such that the image of u_0 in R is nilpotent. Then any two polarized chains $\mathcal{M}, \mathcal{N}$ of $\mathbb{F}[\![u]\!] \otimes_{\mathbb{F}[\![u_0]\!]} R$-modules of type $(\widetilde{\mathcal{L}})$ are isomorphic locally for the étale topology on R. Furthermore the functor $\mathrm{Isom}(\mathcal{M}, \mathcal{N})$ is representable by a smooth affine scheme over R.*

PROPOSITION 5.6.2. — *Let R be an $\mathbb{F}$-algebra and let $\mathcal{M}, \mathcal{N}$ be polarized chains of $\mathbb{F}[\![u]\!] \otimes_{\mathbb{F}[\![u_0]\!]} R[\![u_0]\!]$-modules of type $(\widetilde{\mathcal{L}})$. Then the canonical map $\mathrm{Isom}(\mathcal{M}, \mathcal{N})(R[\![u_0]\!]) \to \mathrm{Isom}(\mathcal{M}, \mathcal{N})(R[\![u_0]\!]/u_0^m)$ is surjective for all $m \in \mathbb{N}_{\geqslant 1}$. In particular $\mathcal{M}$ and $\mathcal{N}$ are isomorphic locally for the étale topology on R.*

Proof. — Analogous to the proof of Proposition 3.7.2. $\qquad\square$

Denote by $\mathrm{U} = \mathrm{U}_n$ and $\mathrm{GU} = \mathrm{GU}_n$ the $\mathbb{F}(\!(u_0)\!)$-groups given on R-valued points by

$$\mathrm{U}(R) = \{g \in \mathrm{GL}_n(\mathbb{F}(\!(u)\!) \otimes_{\mathbb{F}(\!(u_0)\!)} R) \mid g^t \widetilde{I}_n \widetilde{g^*} = \widetilde{I}_n\}$$

and

$$\mathrm{GU}(R) = \left\{g \in \mathrm{GL}_n(\mathbb{F}(\!(u)\!) \otimes_{\mathbb{F}(\!(u_0)\!)} R) \mid \exists c = c(g) \in R^\times : \ g^t \widetilde{I}_n \widetilde{g^*} = c \widetilde{I}_n\right\}.$$

DEFINITION 5.6.3 ([26, §4.2], [28, §6.2]). — *Let R be an $\mathbb{F}$-algebra and let $(L_i)_i$ be a lattice chain in $R((u))^n$.*

(1) *Let $r \in \mathbb{Z}$. The chain $(L_i)_i$ is called r-self-dual if*

$$\forall i \in \mathbb{Z}: \ L_i^\vee = u_0^r L_{-i}.$$

Denote by $\mathcal{F}_{\mathrm{U}}^{(r)}$ the functor on the category of $\mathbb{F}$-algebras with $\mathcal{F}_{\mathrm{U}}^{(r)}(R)$ the set of r-self-dual lattice chains in $R((u))^n$.

(2) *The chain $(L_i)_i$ is called self-dual if Zariski locally on R there is an $a \in R((u_0))^\times$ such that*

(5.6.1) $$\forall i \in \mathbb{Z}: \ L_i^\vee = a L_{-i}.$$

Denote by $\mathcal{F}_{\mathrm{GU}}$ the functor on the category of $\mathbb{F}$-algebras with $\mathcal{F}_{\mathrm{GU}}(R)$ the set of self-dual lattice chains in $R((u))^n$.

Note that $\widetilde{\mathcal{L}} \in \mathcal{F}_{\mathrm{U}}^{(0)}(R)$.

Remark 5.6.4. — Let R be a reduced $\mathbb{F}$-algebra such that $\operatorname{Spec} R$ is connected. Then

$$\mathcal{F}_{\mathrm{GU}}(R) = \bigcup_{r \in \mathbb{Z}} \mathcal{F}_{\mathrm{U}}^{(r)}(R).$$

Proof. — This follows directly from Lemma 3.7.4. $\qquad\square$

Remark 5.6.5. — Let R be an $\mathbb{F}$-algebra and let $(L_i)_i \in \mathcal{F}_{\mathrm{U}}^{(0)}(R)$. For $i \in \mathbb{Z}$ denote by $\varrho_i : L_i \to L_{i+1}$ the inclusion, by $\theta_i : L_{n+i} \to L_i$ the isomorphism given by multiplication with u and by $\mathcal{E}_i : L_i \times L_{-i} \to R[\![u]\!]$ the restriction of $\widetilde{\langle \cdot, \cdot \rangle}$. Then $(L_i, \varrho_i, \theta_i, \mathcal{E}_i)$ is a polarized chain of $\mathbb{F}[\![u]\!] \otimes_{\mathbb{F}[\![u_0]\!]} R[\![u_0]\!]$-modules of type $(\widetilde{\mathcal{L}})$.

Note that for an $\mathbb{F}$-algebra R, the canonical maps

$$\mathbb{F}[\![u]\!] \otimes_{\mathbb{F}[\![u_0]\!]} R[\![u_0]\!] \to R[\![u]\!],$$
$$\mathbb{F}((u)) \otimes_{\mathbb{F}((u_0))} R((u_0)) \to R((u))$$

are isomorphisms. Consequently we can consider $\mathrm{L}_{u_0} \mathrm{GU}$ and $\mathrm{L}_{u_0} \mathrm{U}$ as subfunctors of $\mathrm{L}_u \mathrm{GL}_n$. Recall from Remark 3.6.11 the subfunctor $I \subset \mathrm{L}\,\mathrm{GL}_n$. We define a subfunctor I_{GU} of $\mathrm{L}_{u_0} \mathrm{GU}$ by $I_{\mathrm{GU}} = \mathrm{L}_{u_0} \mathrm{GU} \cap I$.

PROPOSITION 5.6.6. — *The natural action of $\mathrm{L}_u \mathrm{GL}_n$ on $\mathcal{F}$ (cf. Remark 3.6.11) restricts to an action of $\mathrm{L}_{u_0} \mathrm{GU}$ on $\mathcal{F}_{\mathrm{GU}}$. Consequently we obtain an injective map*

$$\mathrm{L}_{u_0} \mathrm{GU}(R)/I_{\mathrm{GU}}(R) \xrightarrow{\ \phi(R)\ } \mathcal{F}_{\mathrm{GU}}(R),$$
$$g \longmapsto g \cdot \widetilde{\mathcal{L}}$$

for each $\mathbb{F}$-algebra R. The morphism ϕ identifies $\mathcal{F}_{\mathrm{GU}}$ with both the étale and the fpqc sheafification of the presheaf $\mathrm{L}_{u_0} \mathrm{GU}/I_{\mathrm{GU}}$.

Proof. — Let R be an $\mathbb{F}$-algebra. We claim that every $a \in R((u_0))^{\times}$ lies in the image of the map $c : \mathrm{GU}(R((u_0))) \to R((u_0))^{\times}$ étale locally on R. Assuming this, one can proceed exactly as in the proof of Proposition 3.7.7.

To prove the claim, first note that for $b \in R((u))$, the matrix $bI_n \in \mathrm{GU}(R((u_0)))$ satisfies $c(bI_n) = b\tilde{b}^*$. Lemma 3.7.4 implies that Zariski locally on R, the element a is of the form $a = u_0^k v(1+n)$ for some $k \in \mathbb{Z}$, a unit $v \in R[\![u_0]\!]^{\times}$ and a nilpotent element $n \in R((u_0))$. Consequently it suffices to show that each of u_0^k, v and $1+n$ is of the form $b\tilde{b}^*$ for some $b \in R((u))$ étale locally on R.

As $2 \in R^{\times}$, one easily sees that v is a square in $R[\![u_0]\!]^{\times}$ whenever $v(0)$ is a square in $R^{\times}$, which is the case étale locally on R. For $b = \sqrt{-1}u^k$, one has $b\tilde{b}^* = u_0^k$. Finally, $1+n$ is a square in $R((u_0))^{\times}$; this follows from the Taylor expansion of $\sqrt{1+x}$ if one notes that $\binom{1/2}{l} \in \mathbb{Z}[\frac{1}{2}]$ for all $l \in \mathbb{N}$. $\square$

DEFINITION 5.6.7. — *The étale sheafification of the presheaf $\mathrm{L}_{u_0} \mathrm{GU}/I_{\mathrm{GU}}$ is called the* affine flag variety for GU.

By Proposition 5.6.6 the functor $\mathcal{F}_{\mathrm{GU}}$ provides a realization of the affine flag variety for GU.

5.7. Embedding the local model into the affine flag variety

Let R be an $\mathbb{F}$-algebra. We consider an $R[u]/u^e$-module as an $R[\![u]\!]$-module via the canonical projection $R[\![u]\!] \to R[u]/u^e$. For $i \in \mathbb{Z}$ denote by $\alpha_i : \widetilde{\Lambda}_i \to \overline{\Lambda}_{i,R}$ the morphism described by the identity matrix with respect to $\widetilde{\mathfrak{E}}_i$ and $\overline{\mathfrak{E}}_i$. It induces an isomorphism $\widetilde{\Lambda}_i/u^e\widetilde{\Lambda}_i \xrightarrow{\sim} \overline{\Lambda}_{i,R}$. Clearly the following diagrams commute.

$$
\begin{array}{ccc}
\widetilde{\Lambda}_i & \subset & \widetilde{\Lambda}_{i+1} \\
\alpha_i \downarrow & & \downarrow \alpha_{i+1} \\
\overline{\Lambda}_{i,R} & \xrightarrow{\overline{\rho}_{i,R}} & \overline{\Lambda}_{i+1,R}
\end{array}
\qquad
\begin{array}{ccc}
\widetilde{\Lambda}_i \times \widetilde{\Lambda}_{-i} & \xrightarrow{\widetilde{\langle\cdot,\cdot\rangle}_i} & R[\![u]\!] \\
\alpha_i \times \alpha_{-i} \downarrow & & \downarrow \\
\overline{\Lambda}_{i,R} \times \overline{\Lambda}_{-i,R} & \xrightarrow{\overline{\langle\cdot,\cdot\rangle}_{i,R}} & R[u]/u^e
\end{array}
\qquad
\begin{array}{ccc}
\widetilde{\Lambda}_i & \xleftarrow{u\cdot} & \widetilde{\Lambda}_{n+i} \\
\alpha_i \downarrow & & \downarrow \alpha_{n+i} \\
\overline{\Lambda}_{i,R} & \xleftarrow{\overline{\vartheta}_{i,R}} & \overline{\Lambda}_{n+i,R}.
\end{array}
$$

The following proposition allows us to consider $M^{e,n}$ as a subfunctor of $\mathcal{F}_{\mathrm{U}}^{(-e_0)}$.

PROPOSITION 5.7.1 ([21, §3.3],[26, §4.4-5.1], [28, §6.4-7.1]). — *There is an embedding $\alpha : M^{e,n} \hookrightarrow \mathcal{F}_{\mathrm{U}}^{(-e_0)}$ given on R-valued points by*

$$M^{e,n}(R) \to \mathcal{F}_{\mathrm{U}}^{(-e_0)}(R),$$

$$(t_i)_i \mapsto (\alpha_i^{-1}(t_i))_i.$$

It induces a bijection from $M^{e,n}(R)$ onto the set of those $(L_i)_i \in \mathcal{F}_{\mathrm{U}}^{(-e_0)}(R)$ satisfying the following conditions for all $i \in \mathbb{Z}$.

(1) $u^e \widetilde{\Lambda}_i \subset L_i \subset \widetilde{\Lambda}_i$.

(2) *For all $P \in R[u]/u^e$, we have*

$$\chi_R(P|\widetilde{\Lambda}_i/L_i) = (T - P(0))^{ne_0}$$

in $R[T]$. Here $\widetilde{\Lambda}_i/L_i$ is considered as an $R[u]/u^e$-module using (1).

Proof. — Identical to the proof of Proposition 3.8.1. □

Note that $\overline{\mathcal{L}} = (\overline{\Lambda}_i, \overline{\rho}_i, \overline{\vartheta}_i, \overline{\langle \cdot, \cdot \rangle}_i)_i$ is a polarized chain of $\mathbb{F}[\![u]\!] \otimes_{\mathbb{F}[\![u_0]\!]} \mathbb{F}[u_0]/u_0^{e_0}$-modules of type $(\widetilde{\mathcal{L}})$. In fact $\overline{\mathcal{L}} = \widetilde{\mathcal{L}} \otimes_{\mathbb{F}[\![u_0]\!]} \mathbb{F}[u_0]/u_0^{e_0}$. Let R be an $\mathbb{F}$-algebra. There is an obvious action of $\mathrm{Aut}(\overline{\mathcal{L}})(R[u_0]/u_0^{e_0})$ on $M^{e,n}(R)$, given by $(\varphi_i) \cdot (t_i) = (\varphi_i(t_i))$. The canonical morphism $R[\![u_0]\!] \to R[u_0]/u_0^{e_0}$ induces a morphism $\mathrm{Aut}(\widetilde{\mathcal{L}})(R[\![u_0]\!]) \to \mathrm{Aut}(\overline{\mathcal{L}})(R[u_0]/u_0^{e_0})$ and we thereby extend this $\mathrm{Aut}(\overline{\mathcal{L}})(R[u_0]/u_0^{e_0})$-action on $M^{e,n}(R)$ to an $\mathrm{Aut}(\widetilde{\mathcal{L}})(R[\![u_0]\!])$-action.

LEMMA 5.7.2. — *Let R be an $\mathbb{F}$-algebra and let $t \in M^{e,n}(R)$. We have* $\mathrm{Aut}(\widetilde{\mathcal{L}})(R[\![u_0]\!]) \cdot t = \mathrm{Aut}(\overline{\mathcal{L}})(R[u_0]/u_0^e) \cdot t$.

Proof. — The map $\mathrm{Aut}(\widetilde{\mathcal{L}})(R[\![u_0]\!]) \to \mathrm{Aut}(\overline{\mathcal{L}})(R[u_0]/u_0^{e_0})$ is surjective by Proposition 5.6.2. □

Define a subfunctor I_{U} of $\mathrm{L}_{u_0} \mathrm{U}$ by $I_{\mathrm{U}} = \mathrm{L}_{u_0} \mathrm{U} \cap I_{\mathrm{GU}}$.

LEMMA 5.7.3. — *We have $I_{\mathrm{GU}}(\mathbb{F}) = \mathbb{F}[\![u_0]\!]^\times I_{\mathrm{U}}(\mathbb{F})$.*

Proof. — Analogous to the proof of Lemma 3.8.3, noting that for $g \in I_{\mathrm{GU}}(\mathbb{F})$ one has $c(g) \in \mathbb{F}[\![u_0]\!]^\times$. □

LEMMA 5.7.4. — *Let $g \in I_{\mathrm{U}}(\mathbb{F})$. Then g restricts to an automorphism $g_i : \widetilde{\Lambda}_i \xrightarrow{\sim} \widetilde{\Lambda}_i$ for each $i \in \mathbb{Z}$. The assignment $g \mapsto (g_i)_i$ defines an isomorphism $I_{\mathrm{U}}(\mathbb{F}) \xrightarrow{\sim} \mathrm{Aut}(\widetilde{\mathcal{L}})(\mathbb{F}[\![u_0]\!])$.*

Proof. — Analogous to the proof of Lemma 2.5.5. □

PROPOSITION 5.7.5. — *Let $t \in M^{e,n}(\mathbb{F})$. Then α induces a bijection*

$$\mathrm{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u_0]/u_0^{e_0}) \cdot t \xrightarrow{\sim} I_{\mathrm{GU}}(\mathbb{F}) \cdot \alpha(t).$$

Consequently we obtain an embedding

$$\operatorname{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u_0]/u_0^{e_0})\backslash M^{e,n}(\mathbb{F}) \hookrightarrow I_{\mathrm{GU}}(\mathbb{F})\backslash \mathcal{F}_{\mathrm{GU}}(\mathbb{F}).$$

Proof. — Analogous to the proof of Proposition 3.8.5. $\qquad\square$

Consider $\alpha' : M^{e,n}(\mathbb{F}) \hookrightarrow \mathcal{F}_{\mathrm{GU}}(\mathbb{F}) \xrightarrow{\phi(\mathbb{F})^{-1}} \mathrm{L}_{u_0}\,\mathrm{GU}(\mathbb{F})/I_{\mathrm{GU}}(\mathbb{F}).$

PROPOSITION 5.7.6. — *Let* $t \in M^{e,n}(\mathbb{F})$. *Then* α' *induces a bijection*

$$\operatorname{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u_0]/u_0^{e_0}) \cdot t \xrightarrow{\sim} I_{\mathrm{GU}}(\mathbb{F}) \cdot \alpha'(t).$$

Consequently we obtain an embedding

$$\operatorname{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u_0]/u_0^{e_0})\backslash M^{e,n}(\mathbb{F}) \hookrightarrow I_{\mathrm{GU}}(\mathbb{F})\backslash \mathrm{GU}(\mathbb{F}((u_0)))/I_{\mathrm{GU}}(\mathbb{F}).$$

Proof. — Clear from Proposition 5.7.5, as the isomorphism $\phi(\mathbb{F})$ is in particular $I_{\mathrm{GU}}(\mathbb{F})$-equivariant. $\qquad\square$

Let R be an $\mathbb{F}$-algebra and $(\varphi_i)_i \in \operatorname{Aut}(\mathcal{L})(R)$. The decomposition (5.5.1) induces for each i a decomposition of $\varphi_i : \Lambda_{i,R} \xrightarrow{\sim} \Lambda_{i,R}$ into the product of $R[u]/u^e$-linear automorphisms $\varphi_{i,\sigma} : \overline{\Lambda}_{i,R} \xrightarrow{\sim} \overline{\Lambda}_{i,R}$. The following statement is then clear (cf. the proof of Proposition 3.5.2).

PROPOSITION 5.7.7. — *Let* R *be an* $\mathbb{F}$-algebra. *The following map is an isomorphism, functorial in* R.

$$\operatorname{Aut}(\mathcal{L})(R) \to \prod_{\sigma\in\mathfrak{S}} \operatorname{Aut}(\overline{\mathcal{L}})(R[u_0]/u_0^{e_0}),$$

$$(\varphi_i)_i \mapsto ((\varphi_{i,\sigma})_i)_{\sigma\in\mathfrak{S}}.$$

Consider the composition

$$\tilde{\alpha} : M^{\mathrm{loc}}(\mathbb{F}) \xrightarrow{(5.5.2)} \prod_{\sigma\in\mathfrak{S}} M^{e,n}(\mathbb{F}) \xrightarrow{\prod_\sigma \alpha'} \prod_{\sigma\in\mathfrak{S}} \mathrm{L}_{u_0}\,\mathrm{GU}(\mathbb{F})/I_{\mathrm{GU}}(\mathbb{F}).$$

For $\sigma \in \mathfrak{S}$ denote by $\tilde{\alpha}_\sigma : M^{\mathrm{loc}}(\mathbb{F}) \to \mathrm{L}_{u_0}\,\mathrm{GU}(\mathbb{F})/I_{\mathrm{GU}}(\mathbb{F})$ the corresponding component of $\tilde{\alpha}$.

THEOREM 5.7.8. — *Let* $t \in M^{\mathrm{loc}}(\mathbb{F})$. *Then* $\tilde{\alpha}$ *induces a bijection*

$$\operatorname{Aut}(\mathcal{L})(\mathbb{F}) \cdot t \xrightarrow{\sim} \prod_{\sigma\in\mathfrak{S}} I_{\mathrm{GU}}(\mathbb{F}) \cdot \tilde{\alpha}_\sigma(t).$$

Consequently we obtain an embedding

$$\operatorname{Aut}(\mathcal{L})(\mathbb{F})\backslash M^{\mathrm{loc}}(\mathbb{F}) \hookrightarrow \prod_{\sigma\in\mathfrak{S}} I_{\mathrm{GU}}(\mathbb{F})\backslash \mathrm{GU}(F((u_0)))/I_{\mathrm{GU}}(\mathbb{F}).$$

Proof. — Identical to the proof of Theorem 3.8.8. $\qquad\square$

5.8. The Iwahori-Weyl group

As in [26, 3.2],[28, 3.2], we denote by S the standard diagonal maximal split torus in GU. Denote by T the centralizer and by N the normalizer of S in GU. By the discussion in [26, 3.4], [28, 5.4], the Kottwitz homomorphism for T is given by

$$\kappa_T : T(\mathbb{F}((u_0))) \to \mathbb{Z}^n, \quad \operatorname{diag}(x_1, \ldots, x_n) \mapsto (\operatorname{val}_u(x_1), \ldots, \operatorname{val}_u(x_n)).$$

Consequently the kernel $T\big(\mathbb{F}((u_0))\big)_1$ of κ_T is equal to $T(\mathbb{F}((u_0))) \cap D_n(\mathbb{F}[\![u]\!])$, with the intersection taking place in $\operatorname{GL}_n(\mathbb{F}((u)))$. Here $D_n \subset \operatorname{GL}_n$ denotes the subgroup of diagonal matrices. By definition, the Iwahori-Weyl group[4] of GU with respect to S is given by $\widetilde{W} := N(\mathbb{F}((u_0)))/T(\mathbb{F}((u_0)))_1$.

Set

$$W = \{w \in S_n \mid \forall i \in \{1, \ldots, n\} : \; w(i) + w(n+1-i) = n+1\}$$

and

$$X = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \mid \exists r \in \mathbb{Z} \forall i \in \{1, \ldots, n\} : \; x_i + x_{n+1-i} = 2r\}.$$

We identify W with a subgroup of $\mathrm{U}(\mathbb{F}((u_0)))$ via $W \ni w \mapsto A_w$. One easily sees that $N(\mathbb{F}((u_0))) = W \ltimes T(\mathbb{F}((u_0)))$. The Kottwitz homomorphism κ_T induces an isomorphism $T(\mathbb{F}((u_0)))/T(\mathbb{F}((u_0)))_1 \xrightarrow{\sim} X$ and we thereby identify $\widetilde{W}$ with $W \ltimes X$.

To avoid any confusion of the product inside $\widetilde{W}$ and the canonical action of S_n on $\mathbb{Z}^n$, we will always denote the element of $\widetilde{W}$ corresponding to $\lambda \in X$ by u^λ.

Recall from [5, §2.5] the notion of an extended alcove $(x_i)_{i=0}^{n-1}$ for GL_n. An *extended alcove for* GU is an extended alcove $(x_i)_{i=0}^{n-1}$ for GL_n such that

$$\exists r \in \mathbb{Z} \forall i \in \{0, \ldots, n\} \forall j \in \{1, \ldots, n\} : \; x_i(j) + x_{n-i}(n+1-j) = 2r-1.$$

Here $x_n = x_0 + (1^{(n)})$.

Also recall the standard alcove $(\omega_i)_{i=0}^{n-1}$. As in the linear case treated in loc. cit., we identify $\widetilde{W}$ with the set of extended alcoves for GU by using the standard alcove as a base point.

Write $\mathbf{e} = (e^{(n)})$.

DEFINITION 5.8.1 (Cf. [14]). — *An extended alcove $(x_i)_{i=0}^{n-1}$ for GU is called* permissible *if it satisfies the following conditions for all $i \in \{0, \ldots, n-1\}$.*

---

[4] Following Rapoport, we use the term "Iwahori-Weyl group" instead of "extended affine Weyl group" in the non-split case.

(1) $\omega_i \leqslant x_i \leqslant \omega_i + \mathbf{e}$, where $\leqslant$ is to be understood componentwise.

(2) $\sum_{j=1}^n x_i(j) = ne_0 - i$.

Denote by Perm the set of all permissible extended alcoves for GU.

PROPOSITION 5.8.2. — The inclusion $N(\mathbb{F}((u_0))) \subset \mathrm{GU}(\mathbb{F}((u_0)))$ induces a bijection $\widetilde{W} \xrightarrow{\sim} I_{\mathrm{GU}}(\mathbb{F})\backslash\mathrm{GU}(\mathbb{F}((u_0)))/I_{\mathrm{GU}}(\mathbb{F})$. In other words,

$$\mathrm{GU}(\mathbb{F}((u_0))) = \coprod_{x \in \widetilde{W}} I_{\mathrm{GU}}(\mathbb{F}) x I_{\mathrm{GU}}(\mathbb{F}).$$

Under this bijection, the subset

$$\mathrm{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u_0]/u_0^{e_0})\backslash M^{e,n}(\mathbb{F}) \subset I_{\mathrm{GU}}(\mathbb{F})\backslash\mathrm{GU}(\mathbb{F}((u_0)))/I_{\mathrm{GU}}(\mathbb{F})$$

of Proposition 5.7.6 corresponds to the subset $\mathrm{Perm} \subset \widetilde{W}$.

Proof. — The first statement is discussed in [26, 4.4], [28, 6.4]. The second statement follows easily from the explicit description of the image of α in Proposition 5.7.1. $\square$

COROLLARY 5.8.3. — Under the identifications of Theorem 5.7.8, the set $\prod_{\sigma \in \mathfrak{S}} \mathrm{Perm}$ constitutes a set of representatives of $\mathrm{Aut}(\mathcal{L})(\mathbb{F})\backslash M^{\mathrm{loc}}(\mathbb{F})$.

Remark 5.8.4. — As explained in Remark 3.9.4, the normalization of Corollary 5.8.3 differs from the one of Definition 2.5.10 by the automorphism $\prod_{\sigma \in \mathfrak{S}} \mathrm{Perm} \to \prod_{\sigma \in \mathfrak{S}} \mathrm{Perm}$, $(x_\sigma) \mapsto (u^e x_\sigma^{-1})$

5.9. The p-rank on a KR stratum

Recall from Section 2.3 the scheme $\mathcal{A}/\mathcal{O}_{E_Q}$ associated with our choice of PEL datum, and the KR stratification

$$\mathcal{A}(\mathbb{F}) = \coprod_{x \in \mathrm{Aut}(\mathcal{L})(\mathbb{F})\backslash M^{\mathrm{loc}}(\mathbb{F})} \mathcal{A}_x.$$

We have identified the occurring index set with $\prod_{\sigma \in \mathfrak{S}} \mathrm{Perm}$ in Corollary 5.8.3 . We can then state the following result.

THEOREM 5.9.1. — Let $x = (x_\sigma)_\sigma \in \prod_{\sigma \in \mathfrak{S}} \mathrm{Perm}$. Write $x_\sigma = w_\sigma u^{\lambda_\sigma}$ with $w_\sigma \in W$, $\lambda_\sigma \in X$. Then the p-rank on $\mathcal{A}_x$ is constant with value

$$g \cdot |\{1 \leqslant i \leqslant n \mid \forall \sigma \in \mathfrak{S}(w_\sigma(i) = i \text{ and } \lambda_\sigma(i) = 0)\}|.$$

Proof. — The proof is identical to the one of Theorem 3.10.1. $\square$

6. The inert unitary case

6.1. The PEL datum

We start with the PEL datum defined in Section 4. We assume that $p\mathcal{O}_{F_0} = (\mathcal{P}_0)^e$ for a single prime $\mathcal{P}_0$ of $\mathcal{O}_{F_0}$ and that $\mathcal{P}_0\mathcal{O}_F = \mathcal{P}$ for a single prime $\mathcal{P}$ of $\mathcal{O}_F$. Denote by $f_0 = [k_{\mathcal{P}_0} : \mathbb{F}_p]$ and $f = [k_{\mathcal{P}} : \mathbb{F}_p]$ the corresponding inertia degrees, so that $f = 2f_0$. We fix once and for all a uniformizer π of $\mathcal{O}_{F_0} \otimes \mathbb{Z}_{(p)}$. Then π is also a uniformizer of $\mathcal{O}_F \otimes \mathbb{Z}_{(p)}$.

Denote by $\mathfrak{C} = \mathfrak{C}_{\mathcal{O}_{F_{\mathcal{P}}}|\mathbb{Z}_p}$ the corresponding inverse different. Choose a generator δ of $\mathfrak{C}$ satisfying $\delta^* = -\delta$. Consequently the form $\delta^{-1}(\cdot,\cdot)'_{\mathbb{Q}_p} : V_{\mathbb{Q}_p} \times V_{\mathbb{Q}_p} \to F_{\mathcal{P}}$ is $*$-hermitian and we assume that it *splits*, i.e. that there is a basis $(e_1,\ldots,e_n)$ of $V_{\mathbb{Q}_p}$ over $F_{\mathcal{P}}$ such that $(e_i, e_{n+1-j})'_{\mathbb{Q}_p} = \delta\delta_{ij}$ for $1 \leqslant i,j \leqslant n$. Here we denote by δ_{ij} the Kronecker delta.

Let $0 \leqslant i < n$. We denote by Λ_i the $\mathcal{O}_{F_{\mathcal{P}}}$-lattice in $V_{\mathbb{Q}_p}$ with basis

$$\mathfrak{E}_i = (\pi^{-1}e_1,\ldots,\pi^{-1}e_i, e_{i+1},\ldots,e_n).$$

For $k \in \mathbb{Z}$ we further define $\Lambda_{nk+i} = \pi^{-k}\Lambda_i$ and we denote by $\mathfrak{E}_{nk+i}$ the corresponding basis obtained from $\mathfrak{E}_i$. Then $\mathcal{L} = (\Lambda_i)_i$ is a complete chain of $\mathcal{O}_{F_{\mathcal{P}}}$-lattices in $V_{\mathbb{Q}_p}$. For $i \in \mathbb{Z}$, the dual lattice $\Lambda_i^\vee := \{x \in V_{\mathbb{Q}_p} \mid (x,\Lambda_i)_{\mathbb{Q}_p} \subset \mathbb{Z}_p\}$ of Λ_i is given by Λ_{-i}. Consequently the chain $\mathcal{L}$ is self-dual.

Let $i \in \mathbb{Z}$. We denote by $\rho_i : \Lambda_i \to \Lambda_{i+1}$ the inclusion, by $\vartheta_i : \Lambda_{n+i} \to \Lambda_i$ the isomorphism given by multiplication with π and by $(\cdot,\cdot)_i : \Lambda_i \times \Lambda_{-i} \to \mathbb{Z}_p$ the restriction of $(\cdot,\cdot)_{\mathbb{Q}_p}$. Then $(\Lambda_i, \rho_i, \vartheta_i, (\cdot,\cdot)_i)_i$ is a polarized chain of $\mathcal{O}_{F_{\mathcal{P}}}$-modules of type $(\mathcal{L})$, which, by abuse of notation, we also denote by $\mathcal{L} = \mathcal{L}^{\text{inert}}$.

Denote by $\langle\cdot,\cdot\rangle_i : \Lambda_i \times \Lambda_{-i} \to \mathcal{O}_{F_{\mathcal{P}}}$ the restriction of the $*$-hermitian form $\delta^{-1}(\cdot,\cdot)'_{\mathbb{Q}_p}$. It is the $*$-sesquilinear form described by the matrix $\widetilde{I}_n$ with respect to $\mathfrak{E}_i$ and $\mathfrak{E}_{-i}$.

Denote by Σ_0 the set of all embeddings $F_0 \hookrightarrow \mathbb{R}$ and by Σ the set of all embeddings $F \hookrightarrow \mathbb{C}$. Also write $\mathfrak{S} = \text{Gal}(k_{\mathcal{P}}/\mathbb{F}_p)$ and $\mathfrak{S}_0 = \text{Gal}(k_{\mathcal{P}_0}/\mathbb{F}_p)$.

Let E' be the Galois closure of F inside $\mathbb{C}$ and choose a prime $\mathcal{Q}'$ of E' over $\mathcal{P}$. Consider the maps $\gamma : \Sigma \to \mathfrak{S}$ and $\gamma_0 : \Sigma_0 \to \mathfrak{S}_0$ of Lemma 3.1.1. For each $\sigma \in \mathfrak{S}_0$ we denote by $\tau_{\sigma,1}, \tau_{\sigma,2} \in \mathfrak{S}$ the two elements with $\tau_{\sigma,j}\big|_{k_{\mathcal{P}_0}} = \sigma$.

Let $\sigma \in \Sigma_0$ and $j \in \{1,2\}$. There is a unique $\tau_{\sigma,j} \in \Sigma$ with $\tau_{\sigma,j}\big|_{F_0} = \sigma$ satisfying

(6.1.1) $$\gamma(\tau_{\sigma,j}) = \tau_{\gamma_0(\sigma),j}.$$

Exactly as in Section 5, we define for each $\sigma \in \Sigma_0$ integers r_σ, s_σ with $r_\sigma + s_\sigma = n$, and using these the element $J \in \mathrm{End}_{B \otimes \mathbb{R}}(V \otimes \mathbb{R})$. Denote by V_{-i} the $(-i)$-eigenspace of $J_\mathbb{C}$. As before, we construct an $\mathcal{O}_F \otimes \mathcal{O}_{E'}$-module M_0 which is finite locally free over $\mathcal{O}_{E'}$, such that $M_0 \otimes_{\mathcal{O}_{E'}} \mathbb{C} = V_{-i}$ as $\mathcal{O}_F \otimes \mathbb{C}$-modules.

6.2. The geometric special fiber of the determinant morphism

Let $\sigma \in \mathfrak{S}_0$. We define

$$\bar{r}_\sigma = \sum_{\sigma' \in \gamma_0^{-1}(\sigma)} r_{\sigma'} \quad \text{and} \quad \bar{s}_\sigma = \sum_{\sigma' \in \gamma_0^{-1}(\sigma)} s_{\sigma'}.$$

As the fibers of γ_0 have cardinality e, it follows that $\bar{r}_\sigma + \bar{s}_\sigma = ne$.

We fix once and for all an embedding $\iota_{\mathcal{Q}'} : k_{\mathcal{Q}'} \hookrightarrow \mathbb{F}$. We consider $\mathbb{F}$ as an $\mathcal{O}_{E'}$-algebra with respect to the composition $\mathcal{O}_{E'} \xrightarrow{\rho_{\mathcal{Q}'}} k_{\mathcal{Q}'} \xrightarrow{\iota_{\mathcal{Q}'}} \mathbb{F}$. Also $\iota_{\mathcal{Q}'}$ induces an embedding $\iota_{\mathcal{P}} : k_P \hookrightarrow \mathbb{F}$ and thereby an identification of the set of all embeddings $k_{\mathcal{P}} \hookrightarrow \mathbb{F}$ with $\mathfrak{S}$.

Our choice of uniformizer π induces a canonical isomorphism

$$(6.2.1) \qquad \mathcal{O}_F \otimes \mathbb{F} = \prod_{\sigma \in \mathfrak{S}_0} \mathbb{F}[u]/(u^e) \times \mathbb{F}[u]/(u^e).$$

Here in the component $\mathbb{F}[u]/(u^e) \times \mathbb{F}[u]/(u^e)$ corresponding to $\sigma \in \mathfrak{S}_0$, the first factor is supposed to correspond to $\tau_{\sigma,1}$ and the second factor is supposed to correspond to $\tau_{\sigma,2}$.

PROPOSITION 6.2.1. — Let $x \in \mathcal{O}_F$ and let $\left((q_{\tau_{\sigma,1}}, q_{\tau_{\sigma,2}})\right)_\sigma \in \prod_{\sigma \in \mathfrak{S}_0} \mathbb{F}[u]/(u^e) \times \mathbb{F}[u]/(u^e)$ be the element corresponding to $x \otimes 1$ under (6.2.1). Then

$$\chi_\mathbb{F}(x|M_0 \otimes_{\mathcal{O}_{E'}} \mathbb{F}) = \prod_{\sigma \in \mathfrak{S}_0} \left(T - q_{\tau_{\sigma,1}}(0)\right)^{\bar{s}_\sigma} \left(T - q_{\tau_{\sigma,2}}(0)\right)^{\bar{r}_\sigma}$$

in $\mathbb{F}[T]$.

Proof. — Reduce $\chi_{\mathcal{O}_{E'}}(x|M_0)$ modulo $\mathcal{Q}'$, using (6.1.1). □

Denote by $E = \mathbb{Q}(\mathrm{tr}_\mathbb{C}(x \otimes 1|V_{-i}); \ x \in F)$ the reflex field and define $\mathcal{Q} = \mathcal{Q}' \cap \mathcal{O}_E$. The morphism $\det_{V_{-i}}$ is defined over $\mathcal{O}_E$.

6.3. The local model

For the chosen PEL datum, Definition 2.3.1 amounts to the following.

DEFINITION 6.3.1. — *The local model $M^{\mathrm{loc}} = M^{\mathrm{loc,inert}}$ is the functor on the category of $\mathcal{O}_{E_{\mathcal{Q}}}$-algebras with $M^{\mathrm{loc}}(R)$ the set of tuples $(t_i)_{i \in \mathbb{Z}}$ of $\mathcal{O}_F \otimes R$-submodules $t_i \subset \Lambda_{i,R}$, satisfying the following conditions for all $i \in \mathbb{Z}$.*

(a) *$\rho_{i,R}(t_i) \subset t_{i+1}$.*
(b) *The quotient $\Lambda_{i,R}/t_i$ is a finite locally free R-module.*
(c) *We have an equality*

$$\det\nolimits_{\Lambda_{i,R}/t_i} = \det\nolimits_{V_{-i}} \otimes_{\mathcal{O}_E} R$$

of morphisms $V_{\mathcal{O}_F \otimes R} \to \mathbb{A}^1_R$.

(d) *Under the pairing $(\cdot, \cdot)_{i,R} : \Lambda_{i,R} \times \Lambda_{-i,R} \to R$, the submodules t_i and t_{-i} pair to zero.*
(e) *$\vartheta_i(t_{n+i}) = t_i$.*

6.4. The geometric special fiber of the local model

For $i \in \mathbb{Z}$, denote by $\overline{\Lambda}_i$ the free $\mathbb{F}[u]/u^e$-module $(\mathbb{F}[u]/u^e)^n$ and by $\overline{\mathfrak{E}}_i$ its canonical basis. Denote by $\overline{\vartheta}_i : \overline{\Lambda}_{n+i} \to \overline{\Lambda}_i$ the identity morphism. Consider the map $\overline{*} : \mathbb{F}[u]/u^e \times \mathbb{F}[u]/u^e \to \mathbb{F}[u]/u^e \times \mathbb{F}[u]/u^e$, $(a,b) \mapsto (b,a)$. Let $\overline{\Lambda}_{i,1}$ and $\overline{\Lambda}_{i,2}$ be two copies of $\overline{\Lambda}_i$ and denote by $\langle \cdot, \cdot \rangle_{i,1} : \overline{\Lambda}_{i,1} \times \overline{\Lambda}_{-i,2} \to \mathbb{F}[u]/u^e$ (resp. $\overline{\langle \cdot, \cdot \rangle}_{i,2} : \overline{\Lambda}_{i,2} \times \overline{\Lambda}_{-i,1} \to \mathbb{F}[u]/u^e$) the perfect bilinear map described by the matrix $\widetilde{I}_n$ with respect to $\overline{\mathfrak{E}}_{i,1}$ and $\overline{\mathfrak{E}}_{-i,2}$ (resp. $\overline{\mathfrak{E}}_{i,2}$ and $\overline{\mathfrak{E}}_{-i,1}$). Consider the pairing

$$\overline{\langle \cdot, \cdot \rangle}_i : (\overline{\Lambda}_{i,1} \times \overline{\Lambda}_{i,2}) \times (\overline{\Lambda}_{-i,1} \times \overline{\Lambda}_{-i,2}) \to \mathbb{F}[u]/u^e \times \mathbb{F}[u]/u^e,$$

$$((x_1, x_2), (y_1, y_2)) \mapsto (\langle x_1, y_2 \rangle_{i,1}, \langle x_2, y_1 \rangle_{i,2}).$$

It is a perfect $\overline{*}$-sesquilinear pairing.

For $k \in \mathbb{Z}$ and $0 \leqslant i < n$, let $\overline{\rho}_{nk+i} : \overline{\Lambda}_{nk+i} \to \overline{\Lambda}_{nk+i+1}$ be the morphism described by the matrix $\mathrm{diag}(1^{(i)}, u, 1^{(n-i-1)})$ with respect to $\overline{\mathfrak{E}}_{nk+i}$ and $\overline{\mathfrak{E}}_{nk+i+1}$.

DEFINITION 6.4.1. — *Let $r, s \in \mathbb{N}$ with $r + s = ne$. Define a functor $M^{e,n,r}$ on the category of $\mathbb{F}$-algebras with $M^{e,n,r}(R)$ the set of tuples $(t_i)_{i \in \mathbb{Z}}$ of $R[u]/u^e$-submodules $t_i \subset \overline{\Lambda}_{i,R}$ satisfying the following conditions for all $i \in \mathbb{Z}$.*

(a) $\overline{\rho}_{i,R}(t_i) \subset t_{i+1}$.
(b) The quotient $\overline{\Lambda}_{i,R}/t_i$ is a finite locally free R-module.
(c) For all $P \in R[u]/u^e$, we have

$$\chi_R(P|\overline{\Lambda}_{i,R}/t_i) = \left(T - P(0)\right)^s$$

in $R[T]$.
(d) $\overline{\vartheta}_i(t_{n+i}) = t_i$.

Let $i \in \mathbb{Z}$. From (6.2.1) we obtain an isomorphism

$$(6.4.1) \qquad \Lambda_{i,\mathbb{F}} = \prod_{\sigma \in \mathfrak{S}_0} \overline{\Lambda}_{i,1} \times \overline{\Lambda}_{i,2}$$

by identifying the basis $\mathfrak{E}_{i,\mathbb{F}}$ with the product of the bases $\overline{\mathfrak{E}}_i$. Under this identification, the morphism $\rho_{i,\mathbb{F}}$ decomposes into the morphisms $\overline{\rho}_i$, the pairing $\langle \cdot, \cdot \rangle_{i,\mathbb{F}}$ decomposes into the pairings $\overline{\langle \cdot, \cdot \rangle}_i$ and the morphism $\vartheta_{i,\mathbb{F}}$ decomposes into the morphisms $\overline{\vartheta}_i$.

Let R be an $\mathbb{F}$-algebra and let $(t_i)_{i \in \mathbb{Z}}$ be a tuple of $\mathcal{O}_F \otimes R$-submodules $t_i \subset \Lambda_{i,R}$. Then (6.4.1) induces decompositions $t_i = \prod_{\sigma \in \mathfrak{S}_0} t_{i,\tau_{\sigma},1} \times t_{i,\tau_{\sigma},2}$ into $R[u]/u^e$-submodules $t_{i,\tau_{\sigma},j} \subset \overline{\Lambda}_{i,j,R}$. The following statement is then clear (cf. the proof of Proposition 3.5.2).

PROPOSITION 6.4.2. — The morphism $\Phi_1 : M_{\mathbb{F}}^{\mathrm{loc}} \to \prod_{\sigma \in \mathfrak{S}_0} M^{e,n,\overline{r}_\sigma}$ given on R-valued points by

$$M_{\mathbb{F}}^{\mathrm{loc}}(R) \to \prod_{\sigma \in \mathfrak{S}_0} M^{e,n,\overline{r}_\sigma}(R),$$

$$(t_i) \mapsto \left((t_{i,\tau_{\sigma},1})_i\right)_\sigma$$

is an isomorphism of functors on the category of $\mathbb{F}$-algebras.

Remark 6.4.3. — For symmetry reasons, also the morphism $\Phi_2 : M_{\mathbb{F}}^{\mathrm{loc}} \to \prod_{\sigma \in \mathfrak{S}_0} M^{e,n,\overline{s}_\sigma}$ given on R-valued points by

$$M_{\mathbb{F}}^{\mathrm{loc}}(R) \to \prod_{\sigma \in \mathfrak{S}_0} M^{e,n,\overline{s}_\sigma}(R),$$

$$(t_i) \mapsto \left((t_{i,\tau_{\sigma},2})_i\right)_\sigma$$

is an isomorphism of functors on the category of $\mathbb{F}$-algebras.

The morphism $\prod_{\sigma \in \mathfrak{S}_0} M^{e,n,\overline{r}_\sigma} \to \prod_{\sigma \in \mathfrak{S}_0} M^{e,n,\overline{s}_\sigma}$ making commutative the diagram

$$\prod_{\sigma \in \mathfrak{S}_0} M^{e,n,\overline{r}_\sigma}$$

$$M_{\mathbb{F}}^{\mathrm{loc}} \xrightarrow{\Phi_1}$$

$$M_{\mathbb{F}}^{\mathrm{loc}} \xrightarrow{\Phi_2} \prod_{\sigma \in \mathfrak{S}_0} M^{e,n,\overline{s}_\sigma}$$

is given by on R-valued points by

$$\prod_{\sigma \in \mathfrak{S}_0} M^{e,n,\overline{r}_\sigma}(R) \to \prod_{\sigma \in \mathfrak{S}_0} M^{e,n,\overline{s}_\sigma}(R),$$

(6.4.2)

$$((t_{i,\sigma})_i)_\sigma \mapsto ((t_{-i,\sigma}^{\perp,\overline{\langle\cdot,\cdot\rangle}-i,1,R})_i)_\sigma.$$

6.5. Embedding the local model into the affine flag variety

Recall from Section 3.6 the realization $\mathcal{F}$ of the affine flag variety for GL_n. Let R be an $\mathbb{F}$-algebra. We consider an $R[u]/u^e$-module as an $R\llbracket u \rrbracket$-module via the canonical projection $R\llbracket u \rrbracket \to R[u]/u^e$. For $i \in \mathbb{Z}$, denote by $\alpha_i : \widetilde{\Lambda}_i \to \overline{\Lambda}_{i,R}$ the morphism described by the identity matrix with respect to $\widetilde{\mathfrak{E}}_i$ and $\overline{\mathfrak{E}}_i$. It induces an isomorphism $\widetilde{\Lambda}_i/u^e\widetilde{\Lambda}_i \xrightarrow{\sim} \overline{\Lambda}_{i,R}$. Clearly the following diagrams commute.

$$
\begin{array}{ccc}
\widetilde{\Lambda}_i & \subset & \widetilde{\Lambda}_{i+1} \\
\alpha_i \downarrow & & \downarrow \alpha_{i+1} \\
\overline{\Lambda}_{i,R} & \xrightarrow{\overline{\rho}_{i,R}} & \overline{\Lambda}_{i+1,R},
\end{array}
\qquad
\begin{array}{ccc}
\widetilde{\Lambda}_i & \xleftarrow{u\cdot} & \widetilde{\Lambda}_{n+i} \\
\alpha_i \downarrow & & \downarrow \alpha_{n+i} \\
\overline{\Lambda}_{i,R} & \xleftarrow{\overline{\vartheta}_{i,R}} & \overline{\Lambda}_{n+i,R}.
\end{array}
$$

Let $r, s \in \mathbb{N}$ with $r+s = ne$. The following proposition allows us to consider $M^{e,n,r}$ as a subfunctor of $\mathcal{F}$.

PROPOSITION 6.5.1 ([19, §4]). — *There is an embedding* $\alpha : M^{e,n,r} \hookrightarrow \mathcal{F}$ *given on R-valued points by*

$$M^{e,n,r}(R) \to \mathcal{F}(R),$$

$$(t_i)_i \mapsto (\alpha_i^{-1}(t_i))_i.$$

It induces a bijection from $M^{e,n,r}(R)$ onto the set of those $(L_i)_i \in \mathcal{F}(R)$ satisfying the following conditions for all $i \in \mathbb{Z}$.

(1) $u^e \widetilde{\Lambda}_i \subset L_i \subset \widetilde{\Lambda}_i$.

(2) For all $P \in R[u]/u^e$, we have

$$\chi_R(P | \widetilde{\Lambda}_i / L_i) = \big(T - P(0)\big)^s$$

in $R[T]$. Here $\widetilde{\Lambda}_i / L_i$ is considered as an $R[u]/u^e$-module using (1).

Proof. — Analogous to the proof of Proposition 3.8.1. □

Let R be an $\mathbb{F}$-algebra. Denote by $\widetilde{\langle \cdot, \cdot \rangle} : R((u))^n \times R((u))^n \to R((u))$ the bilinear form described by the matrix $\widetilde{I_n}$ with respect to the standard basis of $R((u))^n$ over $R((u))$. Further denote by $\widetilde{\langle \cdot, \cdot \rangle}_i : \widetilde{\Lambda}_i \times \widetilde{\Lambda}_{-i} \to R[\![u]\!]$ the restriction of $\widetilde{\langle \cdot, \cdot \rangle}$. Note that the diagram

$$
\begin{array}{ccc}
\widetilde{\Lambda}_i \times \widetilde{\Lambda}_{-i} & \xrightarrow{\widetilde{\langle \cdot, \cdot \rangle}_i} & R[\![u]\!] \\
\downarrow{\scriptstyle \alpha_i \times \alpha_{-i}} & & \downarrow \\
\overline{\Lambda}_{i,1,R} \times \overline{\Lambda}_{-i,2,R} & \xrightarrow{\langle \cdot, \cdot \rangle_{i,1,R}} & R[u]/u^e
\end{array}
$$

commutes. For a lattice Λ in $R((u))^n$ we define $\Lambda^\vee := \{x \in R((u))^n \mid \widetilde{\langle x, \Lambda \rangle} \subset R[\![u]\!]\}$.

As in Remark 6.5.10, the morphism $\Psi : M^{e,n,r} \to M^{e,n,s}$ given on R-valued points by

$$M^{e,n,r}(R) \to M^{e,n,s}(R),$$

$$(t_i)_i \mapsto (t_{-i}^{\perp, \overline{\langle \cdot, \cdot \rangle}_{-i,1,R}})_i$$

is an isomorphism.

PROPOSITION 6.5.2. — *The following diagram commutes.*

$$
\begin{array}{ccc}
M^{e,n,r} & \xhookrightarrow{\alpha} & \mathcal{F} \\
\downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle (L_i)_i \mapsto (u^e L_{-i}^\vee)_i} \\
M^{e,n,s} & \xhookrightarrow{\alpha} & \mathcal{F}.
\end{array}
$$

Proof. — Similar to the proof of the duality statement in the proof of Proposition 3.8.1. □

Note that $\overline{\mathcal{L}} = (\overline{\Lambda}_i, \overline{\rho}_i, \overline{\vartheta}_i)$ is a chain of $\mathbb{F}[u]/u^e$-modules of type $(\widetilde{\mathcal{L}})$. In fact $\overline{\mathcal{L}} = \widetilde{\mathcal{L}} \otimes_{\mathbb{F}[\![u]\!]} \mathbb{F}[u]/u^e$. Let R be an $\mathbb{F}$-algebra. There is an obvious action of $\mathrm{Aut}(\overline{\mathcal{L}})(R[u]/u^e)$ on $M^{e,n,r}(R)$, given by $(\varphi_i) \cdot (t_i) = (\varphi_i(t_i))$. The canonical morphism $R[\![u]\!] \to R[u]/u^e$ induces a map $\mathrm{Aut}(\widetilde{\mathcal{L}})(R[\![u]\!]) \to \mathrm{Aut}(\overline{\mathcal{L}})(R[u]/u^e)$ and we thereby extend this $\mathrm{Aut}(\overline{\mathcal{L}})(R[u]/u^e)$-action on $M^{e,n,r}(R)$ to an $\mathrm{Aut}(\widetilde{\mathcal{L}})(R[\![u]\!])$-action.

LEMMA 6.5.3. — *Let R be an $\mathbb{F}$-algebra and let $t \in M^{e,n,r}(R)$. We have*
$\operatorname{Aut}(\widetilde{\mathcal{L}})(R[\![u]\!]) \cdot t = \operatorname{Aut}(\overline{\mathcal{L}})(R[u]/u^e) \cdot t.$

Proof. — The map $\operatorname{Aut}(\widetilde{\mathcal{L}})(R[\![u]\!]) \to \operatorname{Aut}(\overline{\mathcal{L}})(R[u]/u^e)$ is surjective by Proposition 3.6.9. $\qquad\square$

LEMMA 6.5.4. — *Let $g \in I(\mathbb{F})$. Then g restricts to an automorphism $g_i :$ $\widetilde{\Lambda}_i \xrightarrow{\sim} \widetilde{\Lambda}_i$ for each $i \in \mathbb{Z}$. The assignment $g \mapsto (g_i)_i$ defines an isomorphism $I(\mathbb{F}) \xrightarrow{\sim} \operatorname{Aut}(\widetilde{\mathcal{L}})(\mathbb{F}[\![u]\!]).$*

Proof. — Clear (cf. the proof of Lemma 2.5.5). $\qquad\square$

PROPOSITION 6.5.5. — *Let $t \in M^{e,n,r}(\mathbb{F})$. Then α induces a bijection*
$$\operatorname{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u]/u^e) \cdot t \xrightarrow{\sim} I(\mathbb{F}) \cdot \alpha(t).$$

Consequently we obtain an embedding
$$\operatorname{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u]/u^e) \backslash M^{e,n,r}(\mathbb{F}) \hookrightarrow I(\mathbb{F}) \backslash \mathcal{F}(\mathbb{F}).$$

Proof. — Analogous to the proof of Proposition 3.8.6. $\qquad\square$

Consider $\alpha' : M^{e,n,r}(\mathbb{F}) \hookrightarrow \mathcal{F}(\mathbb{F}) \xrightarrow{\phi(\mathbb{F})^{-1}} \operatorname{L}\operatorname{GL}_n(\mathbb{F})/I(\mathbb{F})$.

PROPOSITION 6.5.6. — *Let $t \in M^{e,n,r}(\mathbb{F})$. Then α' induces a bijection*
$$\operatorname{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u]/u^e) \cdot t \xrightarrow{\sim} I(\mathbb{F}) \cdot \alpha'(t).$$

Consequently we obtain an embedding

$$(6.5.1) \qquad \operatorname{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u]/u^e) \backslash M^{e,n,r}(\mathbb{F}) \hookrightarrow I(\mathbb{F}) \backslash \operatorname{GL}_n(\mathbb{F}(\!(u)\!))/I(\mathbb{F}).$$

Proof. — Clear from Proposition 6.5.5, as the isomorphism $\phi(\mathbb{F})$ is in particular $I(\mathbb{F})$-equivariant. $\qquad\square$

Denote by τ the adjoint involution for $\widetilde{\langle \cdot, \cdot \rangle}$ on $\operatorname{GL}_n(\mathbb{F}(\!(u)\!))$, so that for $g \in \operatorname{GL}_n(\mathbb{F}(\!(u)\!))$ we have $\widetilde{\langle gx, y \rangle} = \widetilde{\langle x, g^\tau \rangle}$, $x, y \in \mathbb{F}(\!(u)\!)^n$.

PROPOSITION 6.5.7. — *The vertical maps in the following diagram are well-defined bijections and the diagram commutes.*

$$
\begin{array}{ccc}
\operatorname{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u]/u^e) \backslash M^{e,n,r}(\mathbb{F}) & \xhookrightarrow{(6.5.1)} & I(\mathbb{F}) \backslash \operatorname{GL}_n(\mathbb{F}(\!(u)\!))/I(\mathbb{F}) \\
\Psi \downarrow & & \downarrow g \mapsto u^e (g^\tau)^{-1} \\
\operatorname{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u]/u^e) \backslash M^{e,n,s}(\mathbb{F}) & \xhookrightarrow{(6.5.1)} & I(\mathbb{F}) \backslash \operatorname{GL}_n(\mathbb{F}(\!(u)\!))/I(\mathbb{F}).
\end{array}
$$

Proof. — In view of Proposition 6.5.2 it suffices to note the following statement, which follows from a short computation: Let Λ be a lattice in $\mathbb{F}(\!(u)\!)^n$ and let $g \in \operatorname{GL}_n(\mathbb{F}(\!(u)\!))$. Then $(g\Lambda)^\vee = (g^\tau)^{-1}(\Lambda^\vee)$. $\qquad\square$

Let R be an $\mathbb{F}$-algebra and $\varphi = (\varphi_i)_i \in \operatorname{Aut}(\mathcal{L})(R)$. The decomposition (6.4.1) induces for each i a decomposition of $\varphi_i : \Lambda_{i,R} \xrightarrow{\sim} \Lambda_{i,R}$ into the product of $R[u]/u^e$-linear automorphisms $\varphi_{i,\tau_{\sigma,j}} : \overline{\Lambda}_{i,j,R} \xrightarrow{\sim} \overline{\Lambda}_{i,j,R}$. The following statement is then clear (cf. the proof of Proposition 3.5.2).

PROPOSITION 6.5.8. — *Let R be an $\mathbb{F}$-algebra. The following map is an isomorphism, functorial in R.*

$$\operatorname{Aut}(\mathcal{L})(R) \to \prod_{\sigma \in \mathfrak{S}_0} \operatorname{Aut}(\overline{\mathcal{L}})(R[u]/u^e),$$

$$(\varphi_i)_i \mapsto ((\varphi_{i,\tau_{\sigma,1}})_i)_\sigma.$$

Consider the composition

$$\tilde{\alpha}_1 : M^{\mathrm{loc}}(\mathbb{F}) \xrightarrow{\Phi_1} \prod_{\sigma \in \mathfrak{S}_0} M^{e,n,\overline{r}_\sigma}(\mathbb{F}) \xrightarrow{\prod_\sigma \alpha'} \prod_{\sigma \in \mathfrak{S}_0} \operatorname{L}\operatorname{GL}_n(\mathbb{F})/I(\mathbb{F}).$$

For $\sigma \in \mathfrak{S}_0$ denote by $\tilde{\alpha}_{1,\sigma} : M^{\mathrm{loc}}(\mathbb{F}) \to \operatorname{L}\operatorname{GL}_n(\mathbb{F})/I(\mathbb{F})$ the corresponding component of $\tilde{\alpha}_1$.

THEOREM 6.5.9. — *Let $t \in M^{\mathrm{loc}}(\mathbb{F})$. Then $\tilde{\alpha}_1$ induces a bijection*

$$\operatorname{Aut}(\mathcal{L})(\mathbb{F}) \cdot t \xrightarrow{\sim} \prod_{\sigma \in \mathfrak{S}_0} I(\mathbb{F}) \cdot \tilde{\alpha}_{1,\sigma}(t).$$

Consequently we obtain an embedding

$$\iota_1 : \operatorname{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F}) \hookrightarrow \prod_{\sigma \in \mathfrak{S}_0} I(\mathbb{F}) \backslash \operatorname{GL}_n(\mathbb{F}((u)))/I(\mathbb{F}).$$

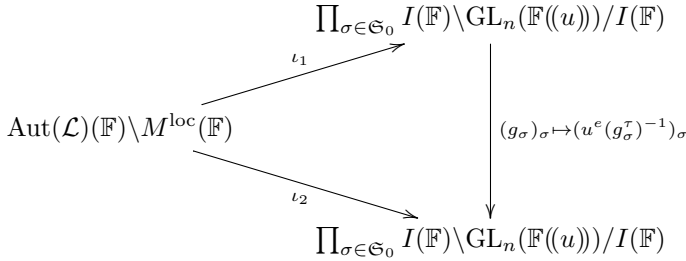*Proof.* — Identical to the proof of Theorem 3.8.8. □

Remark 6.5.10. — In the same way, the composition

$$\tilde{\alpha}_2 : M^{\mathrm{loc}}(\mathbb{F}) \xrightarrow{\Phi_2} \prod_{\sigma \in \mathfrak{S}_0} M^{e,n,\overline{s}_\sigma}(\mathbb{F}) \xrightarrow{\prod_\sigma \alpha'} \prod_{\sigma \in \mathfrak{S}_0} \operatorname{L}\operatorname{GL}_n(\mathbb{F})/I(\mathbb{F})$$

induces an embedding

$$\iota_2 : \operatorname{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F}) \hookrightarrow \prod_{\sigma \in \mathfrak{S}_0} I(\mathbb{F}) \backslash \operatorname{GL}_n(\mathbb{F}((u)))/I(\mathbb{F}).$$

By Proposition 6.5.7 the following diagram commutes.

$$\prod_{\sigma\in\mathfrak{S}_0} I(\mathbb{F})\backslash\mathrm{GL}_n(\mathbb{F}((u)))/I(\mathbb{F})$$

$$\mathrm{Aut}(\mathcal{L})(\mathbb{F})\backslash M^{\mathrm{loc}}(\mathbb{F})$$

ι_1

ι_2

$(g_\sigma)_\sigma \mapsto (u^e(g_\sigma^\tau)^{-1})_\sigma$

$$\prod_{\sigma\in\mathfrak{S}_0} I(\mathbb{F})\backslash\mathrm{GL}_n(\mathbb{F}((u)))/I(\mathbb{F})$$

6.6. The extended affine Weyl group

Let T be the maximal torus of diagonal matrices in GL_n and let N be its normalizer. We denote by $\widetilde{W} = N(\mathbb{F}((u)))/T(\mathbb{F}[\![u]\!])$ the extended affine Weyl group of GL_n with respect to T. Setting $W = S_n$ and $X = \mathbb{Z}^n$, the group homomorphism $\upsilon : W \ltimes X \to N(\mathbb{F}((u)))$, $(w,\lambda) \mapsto A_w u^\lambda$ induces an isomorphism $W \ltimes X \xrightarrow{\sim} \widetilde{W}$. We use it to identify $\widetilde{W}$ with $W \ltimes X$ and consider $\widetilde{W}$ as a subgroup of $\mathrm{GL}_n(\mathbb{F}((u)))$ via υ.

To avoid any confusion of the product inside $\widetilde{W}$ and the canonical action of S_n on $\mathbb{Z}^n$, we will always denote the element of $\widetilde{W}$ corresponding to $\lambda \in X$ by u^λ.

Recall from [5, §2.5] the notion of an extended alcove $(x_i)_{i=0}^{n-1}$ for GL_n. Also recall the standard alcove $(\omega_i)_i$. As in loc. cit. we identify $\widetilde{W}$ with the set of extended alcoves by using the standard alcove as a base point.

Let $r, s \in \mathbb{N}$ with $r + s = ne$ and write $\mathbf{e} = (e^{(n)})$.

DEFINITION 6.6.1 (Cf. [14], [5, Definition 2.4]). — *An extended alcove $(x_i)_{i=0}^{n-1}$ is called r-permissible if it satisfies the following conditions for all $i \in \{0, \ldots, n-1\}$.*

  (1) $\omega_i \leqslant x_i \leqslant \omega_i + \mathbf{e}$, where $\leqslant$ is to be understood componentwise.
  (2) $\sum_{j=1}^n x_i(j) = s - i$.
Denote by Perm_r the set of all r-permissible extended alcoves.

PROPOSITION 6.6.2. — *The inclusion $N(\mathbb{F}((u))) \subset \mathrm{GL}_n(\mathbb{F}((u)))$ induces a bijection $\widetilde{W} \xrightarrow{\sim} I(\mathbb{F})\backslash\mathrm{GL}_n(\mathbb{F}((u)))/I(\mathbb{F})$. In other words,*

$$\mathrm{GL}_n(\mathbb{F}((u))) = \coprod_{x\in\widetilde{W}} I(\mathbb{F})xI(\mathbb{F}).$$

Under this bijection, the subset

$$\mathrm{Aut}(\overline{\mathcal{L}})(\mathbb{F}[u]/u^e)\backslash M^{e,n,r}(\mathbb{F}) \subset I(\mathbb{F})\backslash\mathrm{GL}_n(\mathbb{F}((u)))/I(\mathbb{F})$$

of $(6.5.1)$ corresponds to the subset $\mathrm{Perm}_r \subset \widetilde{W}$.

Proof. — The first statement is the well-known Iwahori decomposition. The second statement follows easily from the explicit description of the image of α in Proposition 6.5.1. $\qquad\square$

COROLLARY 6.6.3. — *With respect to the embedding ι_1 of Theorem 6.5.9, the set $\prod_{\sigma \in \mathfrak{S}_0} \mathrm{Perm}_{\bar{r}_\sigma}$ constitutes a set of representatives of $\mathrm{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F})$.*

The following lemma will be used below.

LEMMA 6.6.4. — *Let $x \in \widetilde{W}$. Write $x = wu^\lambda$ with $w \in W$, $\lambda \in X$. Define $w' \in W$ and $\lambda' \in X$ by*

$$w'(i) = n + 1 - w(n + 1 - i), \quad 1 \leqslant i \leqslant n$$

and

$$\lambda'(i) = e - \lambda(n + 1 - i), \quad 1 \leqslant i \leqslant n.$$

Let $x' = w'u^{\lambda'}$. Then $x' = u^e(x^\tau)^{-1}$.

Proof. — This is an easy computation. $\qquad\square$

6.7. The p-rank on a KR stratum

Recall from Section 2.3 the scheme $\mathcal{A}/\mathcal{O}_{E_\mathcal{Q}}$ associated with our choice of PEL datum, and the KR stratification

$$\mathcal{A}(\mathbb{F}) = \coprod_{x \in \mathrm{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F})} \mathcal{A}_x.$$

We have identified the occurring index set with $\prod_{\sigma \in \mathfrak{S}_0} \mathrm{Perm}_{\bar{r}_\sigma}$ in Corollary 6.6.3. We can then state the following result.

THEOREM 6.7.1. — *Let $x = (x_\sigma)_\sigma \in \prod_{\sigma \in \mathfrak{S}_0} \mathrm{Perm}_{\bar{r}_\sigma}$. Write $x_\sigma = w_\sigma u^{\lambda_\sigma}$ with $w_\sigma \in W$, $\lambda_\sigma \in X$ and define elements $w'_\sigma \in W$ and $\lambda'_\sigma \in X$ as in Lemma 6.6.4. Then the p-rank on $\mathcal{A}_x$ is constant with value*

$$g \cdot \left| \left\{ 1 \leqslant i \leqslant n \, \middle| \, \forall \sigma \in \mathfrak{S}_0 \left(\begin{array}{c} w_\sigma(i) = w'_\sigma(i) = i \text{ and} \\ \lambda_\sigma(i) = \lambda'_\sigma(i) = 0 \end{array} \right) \right\} \right|.$$

Proof. — Follows from Proposition 6.5.7 and Lemma 6.6.4 by the arguments of the proof of Theorem 3.10.1. $\qquad\square$

7. The split unitary case

7.1. The PEL datum

We start with the PEL datum defined in Section 4. We assume that $pO_{F_0} = (\mathcal{P}_0)^e$ for a single prime $\mathcal{P}_0$ of O_{F_0} and that $\mathcal{P}_0 O_F = \mathcal{P}_+ \mathcal{P}_-$ for two distinct primes $\mathcal{P}_\pm$ of O_F. Consequently $\mathcal{P}_- = (\mathcal{P}_+)^*$. Denote by $f_0 = [k_{\mathcal{P}_0} : \mathbb{F}_p]$ the corresponding inertia degree. We fix once and for all a uniformizer π_0 of $O_{F_0} \otimes \mathbb{Z}_{(p)}$.

For typographical reasons, we denote the ring of integers in $(F_0)_{\mathcal{P}_0}$ by $O_{\mathcal{P}_0}$. The inclusion $O_{F_0} \hookrightarrow O_F$ induces identifications

(7.1.1)
$$O_F \otimes \mathbb{Z}_p = O_{\mathcal{P}_0} \times O_{\mathcal{P}_0},$$
$$F \otimes \mathbb{Q}_p = (F_0)_{\mathcal{P}_0} \times (F_0)_{\mathcal{P}_0}.$$

Here the first (resp. second) factor is always supposed to correspond to $\mathcal{P}_+$ (resp. $\mathcal{P}_-$). Under (7.1.1), the base-change $F \otimes \mathbb{Q}_p \to F \otimes \mathbb{Q}_p$ of $*$ takes the simple form $(F_0)_{\mathcal{P}_0} \times (F_0)_{\mathcal{P}_0} \to (F_0)_{\mathcal{P}_0} \times (F_0)_{\mathcal{P}_0}$, $(a, b) \mapsto (b, a)$.

The identification (7.1.1) further induces a decomposition $V \otimes \mathbb{Q}_p = V_+ \times V_-$ into $(F_0)_{\mathcal{P}_0}$-vector spaces $V_\pm$. The pairing $(\cdot, \cdot)'_{\mathbb{Q}_p}$ decomposes into its restrictions $(\cdot, \cdot)_\pm : V_\pm \times V_\mp \to (F_0)_{\mathcal{P}_0}$. Both $(\cdot, \cdot)_+$ and $(\cdot, \cdot)_-$ are perfect $(F_0)_{\mathcal{P}_0}$-bilinear pairings and they are related by the equation $(v, w)_+ = -(w, v)_-$, $v \in V_+, w \in V_-$.

Denote by $\mathfrak{C}_0 = \mathfrak{C}_{O_{\mathcal{P}_0}|\mathbb{Z}_p}$ the corresponding inverse different and fix a generator δ_0 of $\mathfrak{C}_0$. We fix bases $(e_{1,\pm}, \ldots, e_{n,\pm})$ of $V_\pm$ over $(F_0)_{\mathcal{P}_0}$ such that $(e_{i,+}, e_{n+1-j,-})_+ = \delta_0 \delta_{ij}$ for $1 \leqslant i, j \leqslant n$. Here we denote by δ_{ij} the Kronecker delta.

Let $0 \leqslant i < n$. We denote by $\Lambda_{i,\pm}$ the $O_{\mathcal{P}_0}$-lattice in $V_\pm$ with basis

(7.1.2) $\mathfrak{E}_{i,\pm} = (\pi_0^{-1} e_{1,\pm}, \ldots, \pi_0^{-1} e_{i,\pm}, e_{i+1,\pm}, \ldots, e_{n,\pm}).$

For $k \in \mathbb{Z}$ we further define $\Lambda_{nk+i,\pm} = \pi_0^{-k} \Lambda_{i,\pm}$ and we denote by $\mathfrak{E}_{nk+i,\pm}$ the corresponding basis obtained from $\mathfrak{E}_{i,\pm}$. Then $\mathcal{L}_\pm = (\Lambda_{i,\pm})_i$ is a complete chain of $O_{\mathcal{P}_0}$-lattices in $V_\pm$.

Let $i \in \mathbb{Z}$. We denote by $\rho_{i,\pm} : \Lambda_{i,\pm} \to \Lambda_{i+1,\pm}$ the inclusion and by $\vartheta_{i,\pm} : \Lambda_{n+i,\pm} \to \Lambda_{i,\pm}$ the isomorphism given by multiplication with π_0. Then $(\Lambda_{i,\pm}, \rho_{i,\pm}, \vartheta_{i,\pm})$ is a chain of $O_{\mathcal{P}_0}$-modules of type $(\mathcal{L}_\pm)$ which, by abuse of notation, we also denote by $\mathcal{L}_\pm$.

For $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ we define $\Lambda_{(i,j)} := \Lambda_{i,+} \times \Lambda_{j,-}$. Then $\Lambda_{(i,j)}$ is an $O_F \otimes \mathbb{Z}_p$-lattice in $V_{\mathbb{Q}_p}$. A basis $\mathfrak{E}_{(i,j)}$ of $\Lambda_{(i,j)}$ over $O_F \otimes \mathbb{Z}_p$ is given by the diagonal in $\mathfrak{E}_{i,+} \times \mathfrak{E}_{j,-}$. Then $\mathcal{L} = (\Lambda_{(i,j)})_{(i,j)}$ is a complete multichain of $O_F \otimes \mathbb{Z}_p$-lattices in $V_{\mathbb{Q}_p}$. For $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ the dual lattice $\Lambda_{(i,j)}^\vee := \{x \in$

$V_{\mathbb{Q}_p} \mid (x, \Lambda_{(i,j)})_{\mathbb{Q}_p} \subset \mathbb{Z}_p\}$ of $\Lambda_{(i,j)}$ is given by $\Lambda_{(-j,-i)}$. Consequently the multichain $\mathcal{L}$ is a self-dual.

Let $(i,j) \in \mathbb{Z} \times \mathbb{Z}$. We denote by $\rho_{(i,j),+} : \Lambda_{(i,j)} \to \Lambda_{(i+1,j)}$, $\rho_{(i,j),-} : \Lambda_{(i,j)} \to \Lambda_{(i,j+1)}$ and $\rho_{(i,j)} : \Lambda_{(i,j)} \to \Lambda_{(i+1,j+1)}$ the inclusions. We denote by $\vartheta_{(i,j),+} : \Lambda_{(n+i,j)} \to \Lambda_{(i,j)}$ (resp. $\vartheta_{(i,j),-} : \Lambda_{(i,n+j)} \to \Lambda_{(i,j)}$, resp. $\vartheta_{(i,j)} : \Lambda_{(n+i,n+j)} \to \Lambda_{(i,j)}$) the isomorphism given by multiplication with π_0 in the first (resp. second, resp. first and second) component. We further denote by $(\cdot,\cdot)_{(i,j)} : \Lambda_{(i,j)} \times \Lambda_{(-j,-i)} \to \mathbb{Z}_p$ the restriction of $(\cdot,\cdot)_{\mathbb{Q}_p}$.

We find that $(\mathcal{L}_+, \mathcal{L}_-)$, equipped with $((\cdot,\cdot)_{(i,j)})_{(i,j)}$, is a polarized multichain of $\mathcal{O}_F \otimes \mathbb{Z}_p$-modules of type $(\mathcal{L})$, which, by abuse of notation, we also denote by $\mathcal{L} = \mathcal{L}^{\mathrm{split}}$.

Denote by $\langle\cdot,\cdot\rangle_{(i,j)} : \Lambda_{(i,j)} \times \Lambda_{(-j,-i)} \to \mathcal{O}_{\mathcal{P}_0} \times \mathcal{O}_{\mathcal{P}_0}$ the restriction of the $*$-hermitian form $(\delta_0^{-1}, -\delta_0^{-1})(\cdot,\cdot)'_{\mathbb{Q}_p}$. It is the $*$-sesquilinear form described by the matrix $\widetilde{I}_n$ with respect to $\mathfrak{E}_{(i,j)}$ and $\mathfrak{E}_{(-j,-i)}$.

Denote by Σ_0 the set of all embeddings $F_0 \hookrightarrow \mathbb{R}$ and by Σ the set of all embedding $F \hookrightarrow \mathbb{C}$. The inclusion $\mathcal{O}_{F_0} \hookrightarrow \mathcal{O}_F$ induces an identification of $k_{\mathcal{P}_\pm}/\mathbb{F}_p$ with $k_{\mathcal{P}_0}/\mathbb{F}_p$. We write $\mathfrak{S}_0 = \mathrm{Gal}(k_{\mathcal{P}_0}/\mathbb{F}_p)$ and also identify $\mathrm{Gal}(k_{\mathcal{P}_\pm}/\mathbb{F}_p)$ with $\mathfrak{S}_0$. Let E' be the Galois closure of F inside $\mathbb{C}$ and choose a prime $\mathcal{Q}'$ of E' over $\mathcal{P}_+$. Consider the decomposition $\Sigma = \Sigma_+ \amalg \Sigma_-$ and the maps $\gamma_0 : \Sigma_0 \to \mathfrak{S}_0$, $\gamma_\pm : \Sigma_\pm \to \mathfrak{S}_0$ of Lemma 3.1.2. For $\sigma \in \Sigma_0$ we denote by $\tau_{\sigma,\pm}$ the unique lift of σ to $\Sigma_\pm$. Exactly as in Section 5, we define for each $\sigma \in \Sigma_0$ integers r_σ, s_σ with $r_\sigma + s_\sigma = n$,[5] and using these the element $J \in \mathrm{End}_{B \otimes \mathbb{R}}(V \otimes \mathbb{R})$. Denote by $V_{\mathbb{C},-i}$ the $(-i)$-eigenspace of $J_{\mathbb{C}}$. As before, we construct an $\mathcal{O}_F \otimes \mathcal{O}_{E'}$-module M_0 which is finite locally free over $\mathcal{O}_{E'}$, such that $M_0 \otimes_{\mathcal{O}_{E'}} \mathbb{C} = V_{\mathbb{C},-i}$ as $\mathcal{O}_F \otimes \mathbb{C}$-modules.

7.2. The geometric special fiber of the determinant morphism

For $\sigma \in \mathfrak{S}_0$ we write

$$\overline{r}_\sigma = \sum_{\sigma' \in \gamma_0^{-1}(\sigma)} r_{\sigma'} \quad \text{and} \quad \overline{s}_\sigma = \sum_{\sigma' \in \gamma_0^{-1}(\sigma)} s_{\sigma'}.$$

As the fibers of γ_0 have cardinality e, it follows that $\overline{r}_\sigma + \overline{s}_\sigma = ne$.

We fix once and for all an embedding $\iota_{\mathcal{Q}'} : k_{\mathcal{Q}'} \hookrightarrow \mathbb{F}$. We consider $\mathbb{F}$ as an $\mathcal{O}_{E'}$-algebra with respect to the composition $\mathcal{O}_{E'} \xrightarrow{\rho_{\mathcal{Q}'}} k_{\mathcal{Q}'} \xrightarrow{\iota_{\mathcal{Q}'}} \mathbb{F}$. Also $\iota_{\mathcal{Q}'}$ induces an embedding $\iota_{\mathcal{P}_0} : k_{\mathcal{P}_0} \hookrightarrow \mathbb{F}$ and thereby an identification of the set of all embeddings $k_{\mathcal{P}_0} \hookrightarrow \mathbb{F}$ with $\mathfrak{S}_0$.

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[5] In Section 5 we have written $\tau_{\sigma,1}$ and $\tau_{\sigma,2}$ instead of $\tau_{\sigma,+}$ and $\tau_{\sigma,-}$, respectively.

Consider the isomorphism

$$(7.2.1) \qquad \mathcal{O}_F \otimes \mathbb{F} = \prod_{\sigma \in \mathfrak{S}_0} \mathbb{F}[u]/(u^e) \times \mathbb{F}[u]/(u^e)$$

obtained from (7.1.1) and our choice of uniformizer π_0.

PROPOSITION 7.2.1. — *Let* $x \in \mathcal{O}_F$ *and let* $\left((q_{\sigma,+}, q_{\sigma,-})\right)_\sigma \in \prod_{\sigma \in \mathfrak{S}_0} \mathbb{F}[u]/(u^e) \times \mathbb{F}[u]/(u^e)$ *be the element corresponding to* $x \otimes 1$ *under* (7.2.1). *Then*

$$\chi_{\mathbb{F}}(x|M_0 \otimes_{\mathcal{O}_{E'}} \mathbb{F}) = \prod_{\sigma \in \mathfrak{S}_0} \left(T - q_{\sigma,+}(0)\right)^{\bar{s}_\sigma} \left(T - q_{\sigma,-}(0)\right)^{\bar{r}_\sigma}$$

in $\mathbb{F}[T]$.

Proof. — Reduce $\chi_{\mathcal{O}_{E'}}(x|M_0)$ modulo $\mathcal{Q}'$, using (3.1.2). $\qquad \square$

Denote by $E = \mathbb{Q}(\mathrm{tr}_{\mathbb{C}}(x \otimes 1|V_{-i}); \ x \in F)$ the reflex field and define $\mathcal{Q} = \mathcal{Q}' \cap \mathcal{O}_E$. The morphism $\det_{V_{-i}}$ is defined over $\mathcal{O}_E$.

7.3. The local model

For the chosen PEL datum, Definition 2.3.1 amounts to the following.

DEFINITION 7.3.1. — *The local model* $M^{\mathrm{loc}} = M^{\mathrm{loc,split}}$ *is the functor on the category of* $\mathcal{O}_{E_{\mathcal{Q}}}$*-algebras with* $M^{\mathrm{loc}}(R)$ *the set of tuples* $(t_{(i,j)})_{(i,j) \in \mathbb{Z} \times \mathbb{Z}}$ *of* $\mathcal{O}_F \otimes R$*-submodules* $t_{(i,j)} \subset \Lambda_{(i,j),R}$ *satisfying the following conditions for all* $(i,j) \in \mathbb{Z} \times \mathbb{Z}$.

(a) $\rho_{(i,j),+,R}(t_{(i,j)}) \subset t_{(i+1,j)}$ *and* $\rho_{(i,j),-,R}(t_{(i,j)}) \subset t_{(i,j+1)}$.

(b) *The quotient* $\Lambda_{(i,j),R}/t_{(i,j)}$ *is a finite locally free* R*-module.*

(c) *We have an equality*

$$\det_{\Lambda_{(i,j),R}/t_{(i,j)}} = \det_{V_{-i}} \otimes_{\mathcal{O}_E} R$$

of morphisms $V_{\mathcal{O}_F \otimes R} \to \mathbb{A}_R^1$.

(d) *Under the pairing* $(\cdot, \cdot)_{(i,j),R} : \Lambda_{(i,j),R} \times \Lambda_{(-j,-i),R} \to R$, *the submodules* $t_{(i,j)}$ *and* $t_{(-j,-i)}$ *pair to zero.*

(e) $\vartheta_{(i,j),+,R}(t_{(n+i,j)}) = t_{(i,j)}$ *and* $\vartheta_{(i,j),-,R}(t_{(i,n+j)}) = t_{(i,j)}$.

Remark 7.3.2. — Let R be an $\mathcal{O}_{E_{\mathcal{Q}}}$-algebra and let $(t_{(i,j)})_{(i,j)} \in M^{\mathrm{loc}}(R)$. For $(i,j) \in \mathbb{Z} \times \mathbb{Z}$, the decomposition (7.1.1) induces a decomposition $t_{(i,j)} = t_{(i,j),+} \times t_{(i,j),-}$ into $\mathcal{O}_{\mathcal{P}_0} \otimes_{\mathbb{Z}_p} R$-submodules $t_{(i,j),+} \subset \Lambda_{i,+,R}$ and $t_{(i,j),-} \subset \Lambda_{j,-,R}$. As in Remark 2.3.4 one sees that $t_{(i,j),+}$ (resp. $t_{(i,j),-}$) is independent of j (resp. i). Writing $t_{i,+} = t_{(i,j),+}$ and $t_{j,-} = t_{(i,j),-}$, the tuple $(t_{(i,j)})_{(i,j)}$ is determined by the pair of tuples $((t_{i,+})_i, (t_{j,-})_j)$.

Recall from Section 6 the chain $\mathcal{L}^{\text{inert}}$ and the functor $M^{\text{loc,inert}}$. The identifications (6.2.1) and (7.2.1), together with our choices of bases, give rise to a canonical identification of the tuple $(\Lambda_{(i,i),\mathbb{F}}, \rho_{(i,i),\mathbb{F}}, \vartheta_{(i,i),\mathbb{F}}, (\cdot,\cdot)_{(i,i),\mathbb{F}})_i$ with the chain $\mathcal{L}^{\text{inert}} \otimes_{\mathbb{Z}_p} \mathbb{F}$.

We can then state the following result.

PROPOSITION 7.3.3. — (1) The morphism $M_{\mathbb{F}}^{\text{loc,split}} \to M_{\mathbb{F}}^{\text{loc,inert}}$ given on R-valued points by

$$M_{\mathbb{F}}^{\text{loc,split}}(R) \to M_{\mathbb{F}}^{\text{loc,inert}}(R),$$
$$(t_{(i,j)})_{(i,j)} \mapsto (t_{(i,i)})_i$$

is an isomorphism.

(2) The morphism $\operatorname{Aut}(\mathcal{L}^{\text{split}})_{\mathbb{F}} \to \operatorname{Aut}(\mathcal{L}^{\text{inert}})_{\mathbb{F}}$ given on R-valued points by

$$\operatorname{Aut}(\mathcal{L}^{\text{split}})_{\mathbb{F}}(R) \to \operatorname{Aut}(\mathcal{L}^{\text{inert}})_{\mathbb{F}}(R),$$
$$(\varphi_{(i,j)})_{(i,j)} \mapsto (\varphi_{(i,i)})_i$$

is an isomorphism.

Proof. — Clear in view of Remark 7.3.2 and Propositions 6.2.1, 7.2.1. □

Consequently all the statements about $M_{\mathbb{F}}^{\text{loc,inert}}$ from Section 6 are also valid for $M_{\mathbb{F}}^{\text{loc,split}}$.

7.4. The p-rank on a KR stratum

Recall from Section 2.3 the scheme $\mathcal{A}/\mathcal{O}_{E_Q}$ associated with our choice of PEL datum, and the KR stratification

$$\mathcal{A}(\mathbb{F}) = \coprod_{x \in \operatorname{Aut}(\mathcal{L})(\mathbb{F}) \backslash M^{\text{loc}}(\mathbb{F})} \mathcal{A}_x.$$

We have identified the occurring index set with $\prod_{\sigma \in \mathfrak{S}_0} \operatorname{Perm}_{\bar{r}_\sigma}$ in Corollary 6.6.3. We can then state the following result.

THEOREM 7.4.1. — Let $x = (x_\sigma)_\sigma \in \prod_{\sigma \in \mathfrak{S}_0} \operatorname{Perm}_{\bar{r}_\sigma}$. Write $x_\sigma = w_\sigma u^{\lambda_\sigma}$ with $w_\sigma \in W$, $\lambda_\sigma \in X$. Then the p-rank on $\mathcal{A}_x$ is constant with value

$$g_0 \cdot |\{1 \leqslant i \leqslant n \mid \forall \sigma \in \mathfrak{S}_0 (w_\sigma(i) = i \text{ and } \lambda_\sigma(i) = 0)\}|$$
$$+ g_0 \cdot |\{1 \leqslant i \leqslant n \mid \forall \sigma \in \mathfrak{S}_0 (w_\sigma(i) = i \text{ and } \lambda_\sigma(i) = e)\}|.$$

Proof. — Define elements $w'_\sigma \in W$ and $\lambda'_\sigma \in X$ as in Lemma 6.6.4. Let $t = (t_{(i,j)})_{(i,j)} \in M^{\mathrm{loc}}(\mathbb{F})$ and let $(t_{i,\pm})_i$ be the two associated tuples of Remark 7.3.2. Let $(i,j) \in \mathbb{Z} \times \mathbb{Z}$. We have the following equivalences.

$$\Lambda_{(i,j),\mathbb{F}} = \operatorname{im}\rho_{(i-1,j),+,\mathbb{F}} + t_{(i,j)} \Leftrightarrow \Lambda_{i,+,\mathbb{F}} = \operatorname{im}\rho_{i-1,+,\mathbb{F}} + t_{i,+},$$
$$\Lambda_{(i,j),\mathbb{F}} = \operatorname{im}\rho_{(i,j-1),-,\mathbb{F}} + t_{(i,j)} \Leftrightarrow \Lambda_{j,-,\mathbb{F}} = \operatorname{im}\rho_{j-1,-,\mathbb{F}} + t_{j,-}.$$

Assume now that t lies in the $\operatorname{Aut}(\mathcal{L}^{\mathrm{inert}})(\mathbb{F})$-orbit corresponding to x under the identifications of Corollary 6.6.3. Consider the chain of neighbors

$$\Lambda_{(0,0)} \subset \Lambda_{(1,0)} \subset \cdots \subset \Lambda_{(n,0)} \subset \Lambda_{(n,1)} \subset \cdots \subset \Lambda_{(n,n)} = \pi_0^{-1}\Lambda_{(0,0)},$$

and let $1 \leqslant i \leqslant n$. By Propositions 2.4.5 and 2.4.6, the claim of the theorem follows once we can show the following equivalences.

$$\Lambda_{i,+,\mathbb{F}} = \operatorname{im}\rho_{i-1,+,\mathbb{F}} + t_{i,+} \Leftrightarrow \forall \sigma \in \mathfrak{S}_0 (w_\sigma(i) = i \text{ and } \lambda_\sigma(i) = 0),$$
$$\Lambda_{i,-,\mathbb{F}} = \operatorname{im}\rho_{i-1,-,\mathbb{F}} + t_{i,-} \Leftrightarrow \forall \sigma \in \mathfrak{S}_0 (w'_\sigma(i) = i \text{ and } \lambda'_\sigma(i) = 0).$$

These equivalences follow from Proposition 6.5.7 and Lemma 6.6.4 by the arguments of the proof of Theorem 3.10.1. $\qquad\square$

7.5. An application to the dimension of the p-rank 0 locus

Assume from now on that $F_0 = \mathbb{Q}$, so that $F/\mathbb{Q}$ is an imaginary quadratic extension in which p splits. We write $r = r_{\mathrm{id}_\mathbb{Q}}$ and $s = s_{\mathrm{id}_\mathbb{Q}}$, so that $n = r+s$. Also write $I_n = \{1, \ldots, n\}$.

Note that the moduli problem $\mathcal{A}$ is a special case of the "fake unitary case" considered in [7]. Concretely, the moduli problem defined in [7, §5.2] specializes to $\mathcal{A}$ for $D = F$.

Denote by $\ell : \widetilde{W} \to \mathbb{N}$ the length function defined in [5, §2.1].

LEMMA 7.5.1. — *Let $x \in \operatorname{Perm}_r$. The smooth $\mathbb{F}$-variety $\mathcal{A}_x$ is equidimensional of dimension $\ell(x)$.*

Proof. — We know from [7, Lemma 13.1] that $\mathcal{A}_x$ is non-empty. The rest of the proof is identical to the one of Lemma 3.11.1. $\qquad\square$

Let us state Theorem 7.4.1 in this special case.

THEOREM 7.5.2. — *Let $x \in \operatorname{Perm}_r$. Write $x = wu^\lambda$ with $w \in W, \lambda \in X$. Then the p-rank on $\mathcal{A}_x$ is constant with value $|\operatorname{Fix}(w)|$, where $\operatorname{Fix}(w) = \{i \in I_n \mid w(i) = i\}$.*

We want to use this result to compute the dimension of the p-rank 0 locus in $\mathcal{A}_{\mathbb{F}}$. We do this by copying the approach of [6, §8].

Denote by $\mathrm{Perm}_r^{(0)}$ the subset of those $x \in \mathrm{Perm}_r$ such that the p-rank on $\mathcal{A}_x$ is equal to 0. Denote by $W_{n,r}$ the subset of those $w \in W$ satisfying $\mathrm{Fix}(w) = \emptyset$ and

(7.5.1) $$|\{i \in I_n \mid w(i) < i\}| = r.$$

LEMMA 7.5.3 (Cf. [6, Lemma 8.1]). — *The canonical projection $\widetilde{W} \to W$ induces a bijection $\mathrm{Perm}_r^{(0)} \to W_{n,r}$. Its inverse is given by $w \mapsto u^{\lambda(w)}w$ with*

$$\lambda(w)(i) = \begin{cases} 0, & \text{if } w^{-1}(i) > i \\ 1, & \text{if } w^{-1}(i) < i \end{cases}, \quad i \in I_n.$$

Proof. — This is an easy combinatorial consequence of Theorem 7.5.2 and the interpretation of Perm_r in terms of extended alcoves, see Section 6.6. $\qquad\square$

Define for $\sigma \in S_n$ the following sets and natural numbers.

$$\begin{aligned} A_\sigma &= \{(i,j) \in (I_n)^2 \mid i < j < \sigma(j) < \sigma(i)\}, & a_\sigma &= |A_\sigma|, \\ \tilde{A}_\sigma &= \{(i,j) \in (I_n)^2 \mid \sigma(j) < \sigma(i) < i < j\}, & \tilde{a}_\sigma &= |\tilde{A}_\sigma|, \\ B_\sigma &= \{(i,j) \in (I_n)^2 \mid \sigma(i) < i < j < \sigma(j)\}, & b_\sigma &= |B_\sigma|, \\ \tilde{B}_\sigma &= \{(i,j) \in (I_n)^2 \mid i < \sigma(i) < \sigma(j) < j\}, & \tilde{b}_\sigma &= |\tilde{B}_\sigma|, \\ N_\sigma &= a_\sigma + \tilde{a}_\sigma + b_\sigma + \tilde{b}_\sigma. \end{aligned}$$

Note that $N_\sigma = N_{\sigma^{-1}}$ in view of the obvious identities $a_\sigma = \tilde{a}_{\sigma^{-1}}$ and $b_\sigma = \tilde{b}_{\sigma^{-1}}$.

PROPOSITION 7.5.4. — *Let $u^\lambda w \in \mathrm{Perm}_r^{(0)}$, $w \in W, \lambda \in X$. Then $\ell(u^\lambda w) = N_w$.*

Proof. — Denote by e_i the i-th standard basis vector of $\mathbb{Z}^n$. The positive roots $\beta > 0$ of GL_n are given by $\beta_{ij} = e_i - e_j$, $1 \leqslant i < j \leqslant n$. Denote by $\langle \cdot, \cdot \rangle$ the standard symmetric pairing on $\mathbb{Z}^n$, determined by $\langle e_i, e_j \rangle = \delta_{ij}$. By [6, (8.1)] the following Iwahori-Matsumoto formula holds.

(7.5.2) $$\ell(u^\lambda w) = \sum_{\substack{\beta > 0 \\ w^{-1}\beta > 0}} |\langle \beta, \lambda \rangle| + \sum_{\substack{\beta > 0 \\ w^{-1}\beta < 0}} |\langle \beta, \lambda \rangle + 1|.$$

Using Lemma 7.5.3, the equality $\ell(u^\lambda w) = N_{w^{-1}}$ readily follows. $\qquad\square$

Define

$$N_{n,r} := \min\big((r-1)(n-r), r(n-r-1)\big) = \begin{cases} (r-1)(n-r), & \text{if } r \leqslant n/2, \\ r(n-r-1), & \text{if } r \geqslant n/2. \end{cases}$$

PROPOSITION 7.5.5. — *Let* $\sigma \in W_{n,r}$. *Then* $N_\sigma \leqslant N_{n,r}$.

Proof. — Consider the set $M = \{(n,r,i_0) \in \mathbb{N}^3 \mid 1 \leqslant r \leqslant n-1,\ 2 \leqslant i_0 \leqslant n\}$ and equip it with the lexicographical ordering $<$, which is a well-ordering on M. For $(n,r,i_0) \in M$ we define $W_{n,r,i_0} = \{\sigma \in W_{n,r} \mid \min\{2 \leqslant i \leqslant n \mid \sigma(i) < i\} = i_0\}$. Denote by $\mathcal{P}(n,r,i_0)$ the following statement.

$$\forall \sigma \in W_{n,r,i_0} :\ N_\sigma \leqslant N_{n,r}.$$

We will prove it by induction on (n,r,i_0).

Let $(n,r,i_0) \in M$, $\sigma \in W_{(n,r,i_0)}$ and assume that $\mathcal{P}(n',r',i_0')$ is true for all $(n',r',i_0') \in M$ with $(n',r',i_0') < (n,r,i_0)$. Set $\sigma' = \sigma \circ (i_0-1, i_0)$. We distinguish the following four cases.

Case 1: $\sigma(i_0) < i_0 - 1$ and $\sigma(i_0 - 1) > i_0$.
Case 2: $\sigma(i_0) = i_0 - 1$ and $\sigma(i_0 - 1) > i_0$.
Case 3: $\sigma(i_0) < i_0 - 1$ and $\sigma(i_0 - 1) = i_0$.
Case 4: $\sigma(i_0) = i_0 - 1$ and $\sigma(i_0 - 1) = i_0$.

We use the example of **Case 2** to illustrate how to proceed. So assume that $\sigma(i_0) = i_0 - 1$ and $\sigma(i_0 - 1) > i_0$.

We read off the following identities.

$$\begin{aligned} a_\sigma &= a_{\sigma'}, \\ \tilde{a}_\sigma &= \tilde{a}_{\sigma'} + |\{j \in I_n \mid \sigma(j) < i_0 - 1 < i_0 < j\}|, \\ b_\sigma &= b_{\sigma'} + |\{j \in I_n \mid i_0 - 1 < i_0 < j < \sigma(j)\}|, \\ \tilde{b}_\sigma &= \tilde{b}_{\sigma'} + |\{i \in I_n \mid i < \sigma(i) < i_0 - 1 < i_0\}|. \end{aligned}$$

Identifying $\{1, \ldots, \widehat{i_0 - 1}, \ldots, n\}$ with $\{1, \ldots, n-1\}$, we consider the restriction $\sigma'|_{\{1,\ldots,\widehat{i_0-1},\ldots,n\}}$ as an element of $W_{n-1,r-1,j_0}$ for some j_0. By induction hypothesis we know that $N_{\sigma'} = N_{\sigma'|_{\{1,\ldots,\widehat{i_0-1},\ldots,n\}}} \leqslant N_{n-1,r-1}$. In view of $N_{n,r} - N_{n-1,r-1} \geqslant n - r - 1$ it therefore suffices to show the following two inequalities.

$$i_0 - 2 \geqslant |\{j \in I_n \mid \sigma(j) < i_0 - 1 < i_0 < j\}| + |\{i \in I_n \mid i < \sigma(i) < i_0 - 1\}|,$$
$$n - r - (i_0 - 1) \geqslant |\{j \in I_n \mid i_0 < j < \sigma(j)\}|.$$

For the first inequality, it suffices to note that σ maps both sets in question into I_{i_0-2}. On the other hand, by the definition of i_0 we have $I_{i_0-1} \subset \{i \in I_n \mid i < \sigma(i)\}$, so that (7.5.1) implies the second inequality. $\qquad\square$

PROPOSITION 7.5.6. — *We have*

$$\max_{\sigma \in W_{n,r}} N_\sigma = N_{n,r}.$$

Proof. — It suffices to show that there is a $\sigma \in W_{n,r}$ satisfying $N_\sigma = N_{n,r}$. As $W_{n,r} \to W_{n,n-r}$, $\sigma \mapsto \sigma^{-1}$ is a bijection and as $N_\sigma = N_{\sigma^{-1}}$, we may assume that $r \leqslant n/2$. One easily checks that

$$\sigma = (1,2)(3,4)\cdots(2(r-1)-1, 2(r-1))(2r-1, 2r, 2r+1, \ldots, n) \in W_{n,r}$$

satisfies $N_\sigma = (r-1)(n-r) = N_{n,r}$. $\qquad\square$

Denote by $\mathcal{A}^{(0)} \subset \mathcal{A}(\mathbb{F})$ the subset where the p-rank of the underlying abelian variety is equal to 0. It is a closed subset and we equip it with the reduced scheme structure. From the discussion above we obtain the following result.

THEOREM 7.5.7. — $\dim \mathcal{A}^{(0)} = \min\big((r-1)(n-r), r(n-r-1)\big).$

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