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CLASSIFYING COMPLEMENTS FOR GROUPS. APPLICATIONS

by Ana-Loredana AGORE & Gigel MILITARU (*)

Dedicated to Professor Constantin Năstăsescu
on the occasion of his 70th birthday

Abstract. — Let $A \leqslant G$ be a subgroup of a group G. An A–complement of G is a subgroup H of G such that $G = AH$ and $A \cap H = \{1\}$. The *classifying complements problem* asks for the description and classification of all A–complements of G. We shall give the answer to this problem in three steps. Let H be a given A–complement of G and $(\triangleright, \triangleleft)$ the canonical left/right actions associated to the factorization $G = AH$. First, H is deformed to a new A–complement of G, denoted by H_r, using a deformation map $r : H \to A$ of the matched pair $(A, H, \triangleright, \triangleleft)$. Then the description of all complements is given: $\mathbb{H}$ is an A–complement of G if and only if $\mathbb{H}$ is isomorphic to H_r, for some deformation map $r : H \to A$. Finally, the classification of complements proves that there exists a bijection between the isomorphism classes of all A–complements of G and a cohomological object $\mathcal{D}\,(H, A \,|\, (\triangleright, \triangleleft))$. As an application we show that the theoretical formula for computing the number of isomorphism types of all groups of order n arises only from the factorization $S_n = S_{n-1}C_n$.

Keywords: Matched pairs, bicrossed products, the classification of finite groups.
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Résumé. — Soit G un groupe et $A \leqslant G$ un sous-groupe de G. Un A–complément de G est un sous-groupe H de G tel que $G = AH$ et $A \cap H = \{1\}$. Le problème auquel on s'intéresse est de classifier et décrire tous les A–compléments de G. Nous donnons la réponse à ce problème en trois étapes. Fixons H un A–complément de G et soient $(\triangleright, \triangleleft)$ les actions canoniques associées à la factorisation $G = AH$. On commence par déformer H en un nouveau A–complément H_r à l'aide d'une certaine fonction $r : H \to A$ appelée fonction de déformation de $(A, H, \triangleright, \triangleleft)$. Ensuite on donne la description de tous les A–compléments : $\mathbb{H} \leqslant G$ est un A–complément de G si et seulement si $\mathbb{H}$ est isomorphe à H_r pour une certaine fonction de déformation $r : H \to A$. Enfin, la classification des A–compléments prouve qu'il existe une bijection entre les classes d'isomorphisme de tous les A–compléments de G et un objet cohomologique $\mathcal{D}\,(H, A \,|\, (\triangleright, \triangleleft))$. Comme application, on démontre que la formule qui calcule le nombre de classes d'isomorphisme des groupes d'ordre n peut être retrouvée à partir de la factorisation $S_n = S_{n-1} C_n$.

Introduction

Group factorizations have been intensively studied starting with the classical papers by Szép [20, 22, 21], Douglas [6] and Ito [13] but the problem goes back to Maillet [17] and the 1900 Minkowski conjecture on tiling (another name for factorizations) proved 40 years later by Hajós [12]. Let $A \leqslant G$ be a subgroup of G. An A–complement of G is a subgroup $H \leqslant G$ such that G factorizes through A and H, that is $G = AH$ and $A \cap H = \{1\}$. $\mathcal{F}(A, G)$ will denote the (possibly empty) set of isomorphism types of all A–complements of G. We define the factorization index of A in G to be the cardinal of $\mathcal{F}(A, G)$ and it will be denoted by $[G : A]^f := |\,\mathcal{F}(A, G)\,|$.

The problem of existence of complements has to be treated "case by case" for every given subgroup A of G, a computational part of it can not be avoided. It was studied in its global form: *find all factorizations of a given group G*. Particular attention was given to finding all factorizations of simple groups. Starting with the 1970's a very rich literature on the subject was developed: see for instance [4, 5], [7, 8, 9, 10, 11], [15], [18], [24, 25]. For more details on this problem we refer to the two fundamental monographs [14], [16] and the references therein. The present paper deals with the following question:

Classifying complements problem (CCP)*: Let A be a subgroup of G. If an A–complement of G exists, describe explicitly, classify all A–complements of G and compute the factorization index $[G : A]^f$.*

We shall give the answer to the CCP in three steps called: deformation of complements, description of complements and classification of complements. First of all, in Section 1 we recall briefly the definition of a matched pair of groups and the construction of the bicrossed product of two groups as defined by Takeuchi [23]. Let H be a given A–complement of G and $(\triangleright, \triangleleft)$

the canonical left/right actions associated to the factorization $G = AH$ such that $(A, H, \triangleright, \triangleleft)$ is a matched pair of groups and $G = A \bowtie H$. Theorem 2.4 is called the *deformation of complements*: if $r : H \to A$ is a deformation map of the matched pair $(A, H, \triangleright, \triangleleft)$, then the group H is deformed to a new group H_r, called the r-deformation of H, such that H_r remains an A–complement of $G = A \bowtie H$. The key point is Theorem 2.5 called the *description of complements*: $\mathbb{H}$ is an A–complement of G if and only if $\mathbb{H}$ is isomorphic to H_r, for some deformation map $r : H \to A$ of the canonical matched pair $(A, H, \triangleright, \triangleleft)$. Finally, the *classification of complements* is proven in Theorem 2.9: there exists a bijection between the set of isomorphism types of all A–complements of G and a cohomological type object $\mathcal{D}\,(H, A \,|\, (\triangleright, \triangleleft))$ which is explicitly constructed. In particular, the factorization index is computed by the formula $[G : A]^f = |\, \mathcal{D}(H, A \,|\, (\triangleright, \triangleleft))\,|$. In Section 3 we provide some explicit examples. Let S_n be the symmetric group and C_n the cyclic group of order n. By applying our results to the factorization $S_n = S_{n-1} C_n$ we obtain the following: (1) any group H of order n is isomorphic to $(C_n)_r$, the r-deformation of the cyclic group C_n for some deformation map $r : C_n \to S_{n-1}$ of the canonical matched pair $(S_{n-1}, C_n, \triangleright, \triangleleft)$ and (2) the number of isomorphism types of all groups of order n is equal to $|\, \mathcal{D}(C_n, S_{n-1} \,|\, (\triangleright, \triangleleft))\,|$. Therefore, we obtain a combinatorial formula for computing the number of isomorphism types of all groups of order n which arises from a minimal set of data: the factorization $S_n = S_{n-1} C_n$.

The factorization problem as well as the bicrossed product were introduced and studied in other fields such as topological groups, local compact groups, Hopf algebras, groups and Lie algebras etc. The results presented here for groups can be used as a model for developing similar theories in the fields listed above. For Hopf algebras and Lie algebras we refer to [2] and for associative algebras to [1].

1. Preliminaries

Let G, G' be two groups containing A as a subgroup. We say that a morphism of groups $\psi : G \to G'$ *stabilizes* A if $\psi(a) = a$, for all $a \in A$. Let A and H be two groups and $\triangleright : H \times A \to A$ and $\triangleleft : H \times A \to H$ two maps. The map $\triangleright$ (resp. $\triangleleft$) is called trivial if $h \triangleright a = a$ (resp. $h \triangleleft a = h$), for all $a \in A$ and $h \in H$. A *matched pair* [23] of groups is a quadruple $(A, H, \triangleright, \triangleleft)$, where A and H are groups, $\triangleright : H \times A \to A$ is a left action of the group H on the set A, $\triangleleft : H \times A \to H$ is a right action of the group A on the set H

satisfying the following compatibilities for any $a, b \in A$, $h, g \in H$:

$$h \triangleright (ab) = (h \triangleright a)((h \triangleleft a) \triangleright b) \tag{1.1}$$

$$(hg) \triangleleft a = (h \triangleleft (g \triangleright a))(g \triangleleft a) \tag{1.2}$$

If $(A, H, \triangleright, \triangleleft)$ is a matched pair then the following normalizing conditions hold:

$$1 \triangleright a = a, \quad h \triangleleft 1 = h, \quad h \triangleright 1 = 1, \quad 1 \triangleleft a = 1 \tag{1.3}$$

for all $a \in A$ and $h \in H$. Let $\triangleright : H \times A \to A$, $\triangleleft : H \times A \to H$ be two maps and $A \bowtie H := A \times H$ with the binary operation defined by the formula:

$$(a,\, h) \cdot (b,\, g) := \big(a(h \triangleright b),\, (h \triangleleft b)g\big) \tag{1.4}$$

for all $a, b \in A$, $h, g \in H$. The following is [23, Proposition 2.2.]:

Proposition 1.1. — *Let A and H be groups and $\triangleright : H \times A \to A$, $\triangleleft : H \times A \to H$ two maps. Then $A \bowtie H$ is a group with unit $(1, 1)$ if and only if $(A, H, \triangleright, \triangleleft)$ is a matched pair of groups. In this case $A \bowtie H$ is called the bicrossed product of A and H.*

If $A \bowtie H$ is a bicrossed product then $i_A : A \to A \bowtie H$, $i_A(a) = (a, 1)$ and $i_H : H \to A \bowtie H$, $i_H(h) = (1, h)$ are morphisms of groups. A and H will be viewed as subgroups of $A \bowtie H$ via the identifications $A \cong A \times \{1\}$, $H \cong \{1\} \times H$. If the right action $\triangleleft$ of a matched pair $(A, H, \triangleright, \triangleleft)$ is the trivial action then the bicrossed product $A \bowtie H$ is just the semidirect product $A \ltimes H$ of A and H. Thus, the bicrossed product is a generalization of the semidirect product to the case when none of the factors is required to be normal.

We recall that a group G *factorizes* through two subgroups A and H if $G = AH$ and $A \cap H = \{1\}$. The bicrossed product $A \bowtie H$ factorizes through $A \cong A \times \{1\}$ and $H \cong \{1\} \times H$ as for any $a \in A$ and $h \in H$ we have that $(a, h) = (a, 1) \cdot (1, h)$. Conversely, the main motivation for defining the bicrossed product of groups is the following:

Proposition 1.2. — *A group G factorizes through two subgroups A and H if and only if there exists a matched pair of groups $(A, H, \triangleright, \triangleleft)$ such that the multiplication map*

$$m_G : A \bowtie H \to G, \qquad m_G(a,\, h) = ah$$

for all $a \in A$ and $h \in H$ is an isomorphism of groups that stabilizes A.

Proof. — The detailed proof is given in [23, Proposition 2.4]. We only indicate the construction of the matched pair $(A, H, \triangleright, \triangleleft)$ associated to the factorization $G = AH$. Indeed, if G factorizes through A and H then for any $g \in G$ there exists a unique pair $(a, h) \in A \times H$ such that $g = ah$. This allows us to attach to any $(a, h) \in A \times H$ a unique pair of elements $(h \triangleright a,\ h \triangleleft a) \in A \times H$ such that

$$h\,a = (h \triangleright a)(h \triangleleft a) \in AH \tag{1.5}$$

Then $(A, H, \triangleright, \triangleleft)$ is a matched pair of groups and $m_G : A \bowtie H \to G$ is an isomorphism of groups that stabilizes A. □

Remark 1.3. — Let $A \leqslant G$ be a given subgroup of G. We will see that a factorization $G = AH$ is not necessarily unique as there may exist other subgroups $H' \leqslant G$, not isomorphic to H, such that $G = AH'$. Such an example is presented below. Let k be a positive integer. In what follows we view A_{4k-1} as a subgroup of A_{4k} by letting $4k$ to be a fixed point in the alternating group A_{4k}. Then we have two factorizations: $A_{4k} = A_{4k-1}D_{4k} = A_{4k-1}(C_2 \times C_{2k})$, where D_{4k} is the dihedral group and C_m is the cyclic group of order m. Indeed, let σ, $\tau \in A_{4k}$ be the even permutations

$$\sigma = (1, 3, 5, \cdots, 4k-1)(2, 4, 6, \cdots, 4k)$$
$$\tau = (1, 2k+2)(2, 2k+1)(3, 2k+4)(4, 2k+3)\cdots(2k-1, 4k)(2k, 4k-1)$$

It is straightforward to check that σ and τ generate a subgroup of A_{4k} isomorphic to the dihedral group D_{4k} of order $4k$ and $A_{4k} = A_{4k-1}D_{4k}$. On the other hand, let σ', $\tau' \in A_{4k}$ given by

$$\sigma' = (1, 2, \cdots, 2k)(2k+1, 2k+2, \cdots, 4k)$$
$$\tau' = (1, 2k+1)(2, 2k+2)\cdots(2k, 4k)$$

Then $\sigma'\tau' = \tau'\sigma'$ and the subgroup of A_{4k} generated by σ and τ is $C_2 \times C_{2k}$. Moreover, we have $A_{4k} = A_{4k-1}(C_2 \times C_{2k})$. This example reveals yet another important fact: a possible attempt to generalize the Krull-Schmidt decomposition of groups into direct products ([19, Theorem 6.36]) fails for bicrossed products since $A_{4k} = A_{4k-1} \bowtie D_{4k} \cong A_{4k-1} \bowtie (C_2 \times C_{2k})$, and of course the direct product $C_2 \times C_{2k}$ is not isomorphic to the dihedral group D_{4k}.

From now on, the matched pair constructed in (1.5) will be called the *canonical matched pair* associated to the factorization $G = AH$. We use the above terminology in order to distinguish this matched pair among other possible matched pairs $(A, H, \triangleright', \triangleleft')$ such that $A \bowtie' H \cong G$ (isomorphism of groups that stabilizes A), where $A \bowtie' H$ is the bicrossed product associated to the matched pair $(A, H, \triangleright', \triangleleft')$. The following result provides more details: it can be obtained from [3, Proposition 2.1] for $\sigma = Id_H$. However, we state the result below for the sake of completeness as it will be used in the sequel.

Proposition 1.4. — *Let $(A, H, \triangleright, \triangleleft)$ and $(A, H', \triangleright', \triangleleft')$ be two matched pairs of groups. There exists a bijection between the set of all morphisms of groups $\psi : A \bowtie' H' \to A \bowtie H$ that stabilize A and the set of all pairs (r, v), where $r : H' \to A$, $v : H' \to H$ are two unit preserving maps satisfying the following compatibilities for any h', $g' \in H'$, $a \in A$:*

$$h' \triangleright' a = r(h') \big(v(h') \triangleright a\big) r(h' \triangleleft' a)^{-1} \tag{1.6}$$

$$v(h' \triangleleft' a) = v(h') \triangleleft a \tag{1.7}$$

$$r(h'g') = r(h') \big(v(h') \triangleright r(g')\big) \tag{1.8}$$

$$v(h'g') = \big(v(h') \triangleleft r(g')\big) v(g') \tag{1.9}$$

Under the above correspondence the morphism of groups $\psi : A \bowtie' H' \to A \bowtie H$ corresponding to (r, v) is given by:

$$\psi(a,\, h') = \big(a\, r(h'),\, v(h')\big) \tag{1.10}$$

for all $a \in A$, $h' \in H'$ and $\psi : A \bowtie' H' \to A \bowtie H$ is an isomorphism of groups if and only if the map $v : H' \to H$ is bijective.

2. Classifying complements

This section contains the main results of the paper. First we need to introduce the following:

Definition 2.1. — *Let $A \leqslant G$ be a subgroup of G. An A–*complement *of G is a subgroup $H \leqslant G$ such that G factorizes through A and H. We denote by $\mathcal{F}(A, G)$ the set of isomorphism types of all A–complements of G. We define the* factorization index *of A in G as the cardinal of $\mathcal{F}(A, G)$ and it will be denoted by $[G : A]^f := |\,\mathcal{F}(A, G)\,|$. We shall write $[G : A]^f = 0$, if $\mathcal{F}(A, G)$ is empty.*

Let H be a given A–complement of G and $(A, H, \triangleright, \triangleleft)$ the canonical matched pair associated to it as in (1.5) of Proposition 1.2. We shall describe all A–complements of G in terms of $(H, \triangleleft, \triangleright)$ and certain maps $r : H \to A$, called deformation maps. The classification of all A–complements of G is also given by proving that $\mathcal{F}(A, G)$ is in bijection with a cohomological object.

Examples 2.2. — 1. Many group extensions $A \leqslant G$ have the factorization index $[G : A]^f$ equal to 0 (that is there exists no factorization $G = AH$) or 1. For instance, if G is an abelian group, then $[G : A]^f \in \{0, 1\}$, for any subgroup A of G ($[G : A]^f = 1$ if and only if A is a direct summand of G).

Group extensions $A \leqslant G$ of factorization index 1 are exactly those for which the factorization is unique. In other words, for these extensions the Krull-Schmidt theorem [19, Theorem 6.36] for bicrossed products holds: if $G \cong A \bowtie H \cong A \bowtie H'$, then $H \cong H'$. A generic example of an extension of factorization index 1 is provided in Corollary 2.7 below: if $A \ltimes H$ is an arbitrary semidirect product of A and H, then $[A \ltimes H : A]^f = 1$.

2. Examples of extensions $A \leqslant G$ for which $[G : A]^f \geqslant 2$ are quite rare, which makes them tempting to identify. Remark 1.3 proves in fact that $[A_{4k} : A_{4k-1}]^f \geqslant 2$. We provide below an example of an extension of factorization index 2.

The extension $S_3 \leqslant S_4$ has factorization index 2. Indeed, let $C_4 =< (1234) >$ be the cyclic group of order 4 and $C_2 \times C_2$ the Klein's group viewed as a subgroup of S_4 being generated by $(12)(34)$ and $(13)(24)$. Then S_4 has two factorizations: $S_4 = S_3 C_4 = S_3(C_2 \times C_2)$. Since there are no other groups of order four we obtain that $[S_4 : S_3]^f = 2$.

3. Example (2) above can be generalized as follows: the factorization index $[S_n : S_{n-1}]^f = g(n)$, the number of isomorphism types of groups of order n. Indeed, let H be a group of order n. We see H as a subgroup of S_n through the regular representation, i.e. $T : H \to S_n$ given by $T(h) = \sigma_h$, where $\sigma_h(x) = hx$, for all h, $x \in H$. It is now obvious that through this representation n is not fixed by any other element in H besides 1. Since we consider S_{n-1} as a subgroup in S_n by letting n to be a fixed point we have $H \cap S_{n-1} = 1$ and therefore $S_n = S_{n-1} H$.

Definition 2.3. — *Let $(A, H, \triangleright, \triangleleft)$ be a matched pair of groups. A* deformation map *of the matched pair $(A, H, \triangleright, \triangleleft)$ is a function $r : H \to A$ such that $r(1) = 1$ and for all g, $h \in H$ we have:*

$$r\big(\big(h \triangleleft r(g)\big)\, g\,\big) = r(h)\,\big(h \triangleright r(g)\big) \tag{2.1}$$

Let $\mathcal{DM}\,(H, A \,|\, (\triangleright, \triangleleft))$ be the set of all deformation maps of the matched pair $(A, H, \triangleright, \triangleleft)$. The trivial map $H \to A$, $h \mapsto 1$, for any $h \in H$ is a deformation map. If both actions $(\triangleright, \triangleleft)$ of the matched pair are trivial then a deformation map is just a morphism of groups $r : H \to A$. The following result is called the deformation of complements: it shows that any A–complement can be deformed to a new A–complement using a deformation map $r : H \to A$.

Theorem 2.4. — *Let $(A, H, \triangleright, \triangleleft)$ be a matched pair of groups and $r : H \to A$ a deformation map. The following hold:*

(1) Let $H_r := H$, as a set, with the new multiplication $\bullet$ on H defined for any h, $g \in H$ as follows:

$$h \bullet g := \big(h \triangleleft r(g)\big)\, g \tag{2.2}$$

Then $(H_r, \bullet)$ is a group called the r-deformation of H.

(2) The map

$$\triangleright^r : H_r \times A \to A, \quad h \triangleright^r a := r(h)\,\big(h \triangleright a\big)\, r(h \triangleleft a)^{-1} \tag{2.3}$$

for all $h \in H_r$, $a \in A$ is a left action of the group H_r on the set A and $(A, H_r, \triangleright^r, \triangleleft)$ is a matched pair of groups. Furthermore, the map

$$\psi : A \bowtie^r H_r \to A \bowtie H, \quad \psi(a, h) = (a\, r(h), h) \tag{2.4}$$

for all $a \in A$ and $h \in H$ is an isomorphism of groups, where $A \bowtie^r H_r$ is the bicrossed product associated to the matched pair $(A, H_r, \triangleright^r, \triangleleft)$.

(3) H_r is an A–complement of $A \bowtie H$.

Proof. — (1) Using the normalizing conditions (1.3) and the fact that $r : H \to A$ is a unitary map, 1 remains the unit for the new multiplication $\bullet$ given by (2.2). On the other hand for any h, g, $t \in H$ we have:

$$\begin{aligned}
(h \bullet g) \bullet t &= \big[\big(h \triangleleft r(g)\big) g\big] \bullet t = \Big(\underline{\big((h \triangleleft r(g)) g\big) \triangleleft r(t)}\Big)\, t \\
&\overset{(1.2)}{=} \Big(\underline{\big(h \triangleleft r(g)\big) \triangleleft \big(g \triangleright r(t)\big)}\Big) \big(g \triangleleft r(t)\big)\, t \\
&= \Big(h \triangleleft \big(\underline{r(g)(g \triangleright r(t))}\big)\Big) \big(g \triangleleft r(t)\big)\, t \\
&\overset{(2.1)}{=} \Big(h \triangleleft r\big((g \triangleleft r(t))\, t\big)\Big)\, \big(g \triangleleft r(t)\big)\, t \\
&= h \bullet \big[\big(g \triangleleft r(t)\big)\, t\big] = h \bullet (g \bullet t)
\end{aligned}$$

Thus, the multiplication $\bullet$ is associative and has 1 as a unit. We prove now that the inverse of an element $h \in H_r$ is given by $h^{-1} = h^{-1} \triangleleft r(h)^{-1}$, for all $h \in H$. Indeed, for any $h \in H$ we have:

$$\begin{aligned}
h^{-1} \bullet h &= \big(h^{-1} \triangleleft r(h)^{-1}\big) \bullet h = \Big(\big(h^{-1} \triangleleft r(h)^{-1}\big) \triangleleft r(h)\Big)\, h \\
&= \Big(h^{-1} \triangleleft \big(r(h)^{-1} r(h)\big)\Big)\, h = h^{-1}\, h = 1
\end{aligned}$$

Thus we proved that $(H_r, \bullet)$ is a monoid in which every element has a left inverse. Hence $(H_r, \bullet)$ is a group.

(2) Instead of using a rather long computation to prove that $(A, H_r, \triangleright^r, \triangleleft)$ satisfies the axioms (1.1)-(1.2) of a matched pair we proceed as follows: first, observe that the map $\psi : A \times H_r \to A \bowtie H$, $\psi(a, h) = (a\, r(h), h)$ is

a bijection between the set $A \times H_r$ and the group $A \bowtie H$ with the inverse given by

$$\psi^{-1} : A \bowtie H \to A \times H_r, \quad \psi^{-1}(a,\, h) = (a\, r(h)^{-1},\, h)$$

for all $a \in A$ and $h \in H$. Thus, there exists a unique group structure $\diamond$ on the set $A \times H_r$ such that ψ becomes an isomorphism of groups and this unique group structure $\diamond$ is obtained by transferring the group structure from the group $A \bowtie H$ via the bijection of sets ψ, i.e. is given by:

$$(a,\, h) \diamond (b,\, g) := \psi^{-1}\big(\psi(a,\, h) \cdot \psi(b,\, g)\big)$$

for all $a,\, b \in A$ and $h,\, g \in H_r = H$. If we prove that this group structure $\diamond$ on the direct product of sets $A \times H_r$ is exactly the one given by (1.4) associated to the pair of maps $(\triangleright^r, \triangleleft)$ the proof is finished by using Proposition 1.1. Indeed, for any $a,\, b \in A$ and $g,\, h \in H$ we have:

$$\begin{aligned}
(a,\, h) \diamond (b,\, g) &= \psi^{-1}\big(\psi(a,\, h) \cdot \psi(b,\, g)\big)\\
&= \psi^{-1}\Big(\big(a\, r(h),\, h\big) \cdot \big(b\, r(g),\, g\big)\Big)\\
&= \psi^{-1}\Big(a\, r(h)\big(h \triangleright br(g)\big),\, \big(h \triangleleft br(g)\big)\, g\Big)\\
&= \Big(a\, r(h)\big(h \triangleright br(g)\big) r\Big(\big(h \triangleleft br(g)\big)\, g\Big)^{-1},\, \big(h \triangleleft br(g)\big)\, g\Big)\\
&= \Big(a\, r(h)\big(h \triangleright br(g)\big) \underline{r\Big(\big((h \triangleleft b) \triangleleft r(g)\big)\, g\Big)^{-1}},\, \big(h \triangleleft br(g)\big)\, g\Big)\\
&\overset{(2.1)}{=} \Big(a\, r(h)\big(\underline{h \triangleright br(g)}\big)\Big[r\big(h \triangleleft b\big)\Big(\big(h \triangleleft b\big) \triangleright r(g)\Big)\Big]^{-1},\\
&\qquad \big(h \triangleleft br(g)\big)\, g\Big)\\
&\overset{(1.1)}{=} \Big(a\, r(h)\big(h \triangleright b\big)\underline{\Big(\big(h \triangleleft b\big) \triangleright r(g)\Big)\Big(\big(h \triangleleft b\big) \triangleright r(g)\Big)^{-1}} r\big(h \triangleleft b\big)^{-1},\\
&\qquad \big(h \triangleleft br(g)\big)\, g\Big)\\
&= \Big(a\, r(h)\big(h \triangleright b\big) r\big(h \triangleleft b\big)^{-1},\, \underline{\big(h \triangleleft br(g)\big)\, g}\Big)\\
&\overset{(2.2)}{=} \Big(\underline{a\, r(h)\big(h \triangleright b\big) r\big(h \triangleleft b\big)^{-1}},\, \big(h \triangleleft b\big) \bullet g\Big)\\
&\overset{(2.3)}{=} \Big(a\, (h \triangleright^r b),\, (h \triangleleft b) \bullet g\Big)\\
&= (a,\, h) \cdot^r (b,\, g)
\end{aligned}$$

where $\cdot^r$ is the multiplication given by (1.4) associated to the new pair of maps $(\triangleright^r, \triangleleft)$. Now we apply Proposition 1.1.

(3) First we remark that the isomorphism of groups $\psi : A \bowtie^r H_r \to A \bowtie H$ given by (2.4) stabilizes A. Hence $A \cong \psi(A) = A \times \{1\} \leqslant A \bowtie H$ and $H_r \cong \psi(\{1\} \times H_r) = \{(r(h), h) \,|\, h \in H\}$ is a subgroup of $A \bowtie H$. Now, $A \bowtie H$ factorizes through A and H_r since in $A \bowtie H$ we have:

$$(a, h) = (ar(h)^{-1}, 1) \cdot (r(h), h)$$

for all $a \in A$ and $h \in H$. Of course, $A \times \{1\}$ and $\{(r(h), h) \,|\, h \in H\} \cong H_r$ have trivial intersection in $A \bowtie H$ as r is a unitary map. The proof is now completely finished. □

Now we prove the converse of Theorem 2.4 which gives the description of all A–complements of G in terms of a fixed one H.

Theorem 2.5. — *Let $A \leqslant G$ be a subgroup of G and H a given A–complement of G. Then $\mathbb{H}$ is an A–complement of G if and only if there exists an isomorphism of groups $\mathbb{H} \cong H_r$, for some deformation map $r : H \to A$ of the canonical matched pair $(A, H, \triangleright, \triangleleft)$ associated to the factorization $G = AH$.*

Proof. — Let $A \bowtie H$ be the bicrossed product of the canonical matched pair $(A, H, \triangleright, \triangleleft)$. Then the multiplication map $m_G : A \bowtie H \to G$ is an isomorphism of groups that stabilizes A. Consider $(A, \mathbb{H}, \triangleright', \triangleleft')$ to be the canonical matched pair associated to the factorization $G = A\mathbb{H}$; hence the multiplication map $m'_G : A \bowtie' \mathbb{H} \to G$ is also an isomorphism of groups that stabilizes A. Then $\psi := m_G^{-1} \circ m'_G : A \bowtie' \mathbb{H} \to A \bowtie H$ is a group isomorphism that stabilizes A as a composition of such morphisms. Now by applying Proposition 1.4 it follows that ψ is uniquely determined by a pair of maps $(\overline{r}, \overline{v})$ consisting of a unitary map $\overline{r} : \mathbb{H} \to A$ and a unitary bijective map $\overline{v} : \mathbb{H} \to H$ satisfying the compatibility conditions

$$h' \triangleright' a = \overline{r}(h') \big(\overline{v}(h') \triangleright a\big) \overline{r}(h' \triangleleft' a)^{-1} \tag{2.5}$$

$$\overline{v}(h' \triangleleft' a) = \overline{v}(h') \triangleleft a \tag{2.6}$$

$$\overline{r}(h'g') = \overline{r}(h') \big(\overline{v}(h') \triangleright \overline{r}(g')\big) \tag{2.7}$$

$$\overline{v}(h'g') = \big(\overline{v}(h') \triangleleft \overline{r}(g')\big) \overline{v}(g') \tag{2.8}$$

for all h', $g' \in \mathbb{H}$ and $a \in A$. Moreover, $\psi : A \bowtie' \mathbb{H} \to A \bowtie H$ is given by:

$$\psi(a, h') = (a\,\overline{r}(h'), \overline{v}(h'))$$

for all $a \in A$ and $h' \in \mathbb{H}$. We define

$$r : H \to A, \quad r := \overline{r} \circ \overline{v}^{-1}$$

and we will prove that r is a deformation map of the matched pair $(A, H, \triangleright, \triangleleft)$ and $\overline{v} : \mathbb{H} \to H_r$ is an isomorphism of groups. First, notice that r is unitary as $\overline{r}$, $\overline{v}$ are both unitary. We have to show that the compatibility condition (2.1) holds for r. Indeed, from (1.8) and (1.9) we obtain:

$$\overline{r} \circ \overline{v}^{-1}\big[\, \big(\overline{v}(h') \triangleleft \overline{r}(g')\big)\, \overline{v}(g')\, \big] = \overline{r}(h')\big(\overline{v}(h') \triangleright \overline{r}(g')\big) \tag{2.9}$$

for all h', $g' \in \mathbb{H}$. Let h, $g \in H$ and write the compatibility condition (2.9) for $h' = \overline{v}^{-1}(h)$ and $g' = \overline{v}^{-1}(g)$. We obtain

$$r\Big(\big(h \triangleleft r(g)\big)\, g\,\Big) = r(h)\,\big(h \triangleright r(g)\big)$$

that is (2.1) holds and hence $r : H \to A$ is a deformation map. Finally, $\overline{v} : \mathbb{H} \to H_r$ is a bijective map as $H = H_r$ as sets. Hence, we are left to prove that $\overline{v}$ is also a morphism of groups. Indeed, for any h', $g' \in \mathbb{H}$ we have:

$$\overline{v}(h'g') \overset{(1.9)}{=} \big(\overline{v}(h') \triangleleft \overline{r}(g')\big)\, \overline{v}(g') \overset{(2.2)}{=} \overline{v}(h') \bullet \overline{v}(g')$$

where $\bullet$ is the multiplication on H_r as defined by (2.2). Hence $\overline{v} : \mathbb{H} \to H_r$ is an isomorphism of groups and the proof is finished. □

Remark 2.6. — Assume that in Theorem 2.4 the deformation map $r : H \to A$ is the trivial one or the right action $\triangleleft$ is the trivial action of A on H. Then $H_r = H$ as groups. In general, the new group H_r may not be isomorphic to H as groups. Example 3.3 shows how the Klein's group $C_2 \times C_2$ can be constructed as an r-deformation of the cyclic group C_4, for some deformation map $r : C_4 \to S_3$. On the other hand, there are also examples of non-trivial deformation maps, with a non-trivial action $\triangleleft$, such that H_r is a group isomorphic to H. Such an example is provided in Example 3.5.

Corollary 2.7. — *Let A and H be two groups, $A \ltimes H$ an arbitrary semidirect product of A and H. Then the factorization index $[A \ltimes H : A]^f = 1$.*

In particular, the following Krull-Schmidt type theorem for bicrossed product holds: if $A \ltimes H \cong A \bowtie H'$ (isomorphism of groups that stabilizes A), then the groups H' and H are isomorphic, where $A \bowtie H'$ is an arbitrary bicrossed product.

Proof. — Indeed, $H \cong \{1\} \times H$ is an A–complement of the semidirect product $A \ltimes H$. Moreover, the right action $\triangleleft$ of the canonical matched pair $(A, H, \triangleright, \triangleleft)$ constructed in (1.5) for the factorization $A \ltimes H = (A \times \{1\})(\{1\} \times H)$ is the trivial action. Thus, using Remark 2.6, any r-deformation of $H \cong \{1\} \times H$ coincides with H.

The rest follows from Theorem 2.5. □

In order to provide the classification of complements we need one more definition:

Definition 2.8. — *Let* $(A, H, \triangleright, \triangleleft)$ *be a matched pair of groups. Two deformation maps* r, $R : H \to A$ *are called* equivalent *and we denote this by* $r \sim R$ *if there exists* $\sigma : H \to H$ *a permutation on the set* H *such that* $\sigma(1_H) = 1_H$ *and for all* g, $h \in H$ *we have:*

$$\sigma\big((h \triangleleft r(g))\, g\big) = \big(\sigma(h) \triangleleft R(\sigma(g))\big)\, \sigma(g) \tag{2.10}$$

As a conclusion of all the above results, our main theorem which gives the classification of all A–complements of a group G now follows.

Theorem 2.9. — *Let* $A \leqslant G$ *be a subgroup of* G, H *a given* A*–complement of* G *and* $(A, H, \triangleright, \triangleleft)$ *the associated canonical matched pair. Then:*

(1) $\sim$ *is an equivalence relation on* $\mathcal{DM}(H, A \,|\, (\triangleright, \triangleleft))$ *and the map*

$$\mathcal{D}\,(H, A \,|\, (\triangleright, \triangleleft)) \to \mathcal{F}\,(A, G), \quad \overline{r} \mapsto H_r$$

is a bijection between sets, where $\mathcal{D}\,(H, A \,|\, (\triangleright, \triangleleft)) := \mathcal{DM}\,(H, A \,|\, (\triangleright, \triangleleft))/ \sim$ *is the quotient set through the relation* $\sim$ *and* $\overline{r}$ *is the equivalence class of* r *via* $\sim$.

(2) *The factorization index* $[G : A]^f$ *is computed by the formula:*

$$[G : A]^f = |\mathcal{D}\,(H, A \,|\, (\triangleright, \triangleleft))|$$

Proof. — It follows from Theorem 2.5 that if $\mathbb{H}$ is an arbitrary A–complement of G, then there exists an isomorphism of groups $\mathbb{H} \cong H_r$, for some deformation map $r : H \to A$ of the matched pair $(A, H, \triangleright, \triangleleft)$. Thus, in order to classify all A–complements on G we can consider only r-deformations of H, for various deformation maps $r : H \to A$. Now let r, $R : H \to A$ be two deformation maps of the matched pair $(A, H, \triangleright, \triangleleft)$. As H_r and H_R coincide as sets, the groups H_r and H_R are isomorphic if and only if there exists $\sigma : H \to H$ a unitary bijective map such that $\sigma : H_r \to H_R$ is a morphism of groups. Taking into account the definition of the multiplication on H_r given by (2.2) it follows that σ is a group morphism if and only if the compatibility condition (2.10) of Definition 2.8 holds, i.e. $r \sim R$. Hence, $r \sim R$ if and only if there exists a map σ such that $\sigma : H_r \to H_R$ is an isomorphism of groups. Therefore $\sim$ is an equivalence relation on $\mathcal{DM}(H, A \,|\, (\triangleright, \triangleleft))$ and the map

$$\mathcal{D}\,(H, A \,|\, (\triangleright, \triangleleft)) \to \mathcal{F}\,(A, E), \qquad \overline{r} \mapsto H_r$$

is well defined and a bijection between sets, where $\overline{r}$ is the equivalence class of r via the relation $\sim$. (2) follows from (1) and the proof is now finished. □

3. Examples

Let n be a positive integer. In this section we apply the results obtained in Section 2 to the factorization $S_n = S_{n-1}C_n$. As a consequence, we derive a combinatorial formula for computing the number of types of groups of order n as well as an explicit description for the multiplication on any group of order n. In what follows we consider the usual presentation of the symmetric group S_n:

$$S_n = \langle s_1,\, s_2,\, \ldots,\, s_{n-1} \mid s_i^2 = 1,\, s_i\, s_{i+1}\, s_i = s_{i+1}\, s_i\, s_{i+1},\, s_i\, s_j = s_j\, s_i,\, |i-j| > 1\rangle$$

We shall see the cyclic group C_n as a subgroup of S_n generated by $x := s_1\, s_2\, \ldots\, s_{n-1}$ while S_{n-1} will be seen as the subgroup of S_n generated by $s_1,\, s_2,\, \ldots\, s_{n-2}$. To start with, we describe the canonical matched pair associated to the factorization $S_n = S_{n-1}C_n$. It is enough to define the two actions $\triangleright : C_n \times S_{n-1} \to S_{n-1}$ and $\triangleleft : C_n \times S_{n-1} \to C_n$ on the generators of S_{n-1} and C_n as they can be extended to the entire group by using the compatibilities (1.1) and (1.2).

Proposition 3.1. — *The canonical matched pair $(S_{n-1}, C_n, \triangleright, \triangleleft)$ associated to the factorization $S_n = S_{n-1}C_n$ is given as follows:*

$$x \triangleright s_i = \begin{cases} s_{i+1}, & \text{if} \quad i < n-2 \\ s_{n-2}\, s_{n-3}\, \ldots\, s_1, & \text{if} \quad i = n-2 \end{cases}$$

$$x \triangleleft s_i = \begin{cases} x, & \text{if} \quad i < n-2 \\ x^2, & \text{if} \quad i = n-2 \end{cases}$$

where $x := s_1\, s_2\, \ldots\, s_{n-1}$.

Proof. — We compute the canonical matched pair by using the approach highlighted in the proof of Proposition 1.2. We start by computing the xs_i's, for al $i \in 1, 2, \ldots, n-2$. If $i < n-2$ we have:

$$\begin{aligned} xs_i &= s_1 \ldots s_{i-1}\, s_i\, s_{i+1}\, s_{i+2} \ldots s_{n-1}\, s_i \\ &= s_1 \ldots s_{i-1}\, \big(s_i\, s_{i+1}\, s_i\big)\, s_{i+2} \ldots s_{n-1} \\ &= s_1 \ldots s_{i-1}\, \big(s_{i+1}\, s_i\, s_{i+1}\big)\, s_{i+2} \ldots s_{n-1} \\ &= s_{i+1}\, s_1\, \ldots\, s_{n-1} = s_{i+1}x \end{aligned}$$

If $i = n-2$ we obtain:

$$\begin{aligned} x s_{n-2} &= s_1 \dots s_{n-3} \left(s_{n-2}\, s_{n-1}\, s_{n-2}\right) \\ &= s_1 \dots s_{n-3} \left(s_{n-1}\, s_{n-2}\, s_{n-1}\right) \\ &= s_{n-1}\, s_1 \dots s_{n-1} \\ &= s_{n-2}\, s_{n-3} \dots s_1 (s_1\, s_2 \dots s_{n-1})^2 = x' x^2 \end{aligned}$$

where $x' := s_{n-2}\, s_{n-3}\, \dots\, s_1$ and the conclusion follows easily. □

By applying Theorem 2.5 and Theorem 2.9 for the factorization $S_n = S_{n-1} C_n$ we obtain the following result concerning the structure and the number of types of groups of finite order.

Corollary 3.2. — *Let n be a positive integer and $(S_{n-1}, C_n, \triangleright, \triangleleft)$ the canonical matched pair associated to the factorization $S_n = S_{n-1} C_n$. Then:*

(1) *Any group of order n is isomorphic to an r-deformation of the cyclic group C_n, for some deformation map $r : C_n \to S_{n-1}$ of the canonical matched pair $(S_{n-1}, C_n, \triangleright, \triangleleft)$. The multiplication $\bullet$ on $(C_n)_r$ is given by: $x \bullet y = \big(x \triangleleft r(y)\big)\, y$, for all $x, y \in (C_n)_r$, where we denoted by juxtaposition the multiplication in the cyclic group C_n.*

(2) *The number of isomorphism types of all groups of order n is equal to*

$$|\, \mathcal{D}(C_n, S_{n-1} \,|\, (\triangleright, \triangleleft))\, |$$

Proof. — It follows from Theorem 2.5 and Theorem 2.9 taking into account that any group H of order n is an S_{n-1}–complement of S_n according to (3) of Example 2.2. □

Now we provide some explicit examples in order to see how Corollary 3.2 works.

Example 3.3. — Consider the extension $S_3 \leqslant S_4$ of factorization index 2. Then the canonical matched pair $(S_3, C_4, \triangleright, \triangleleft)$ associated to the factorization $S_4 = S_3 C_4$ from Proposition 3.1 takes the following form:

$\triangleright$	1	s_1	$s_1\, s_2$	$s_2\, s_1$	s_2	$s_1\, s_2\, s_1$
1	1	s_1	$s_1\, s_2$	$s_2\, s_1$	s_2	$s_1\, s_2\, s_1$
x	1	s_2	s_1	$s_1\, s_2$	$s_2\, s_1$	$s_1\, s_2\, s_1$
x^2	1	$s_2\, s_1$	s_2	s_1	$s_1\, s_2$	$s_1\, s_2\, s_1$
x^3	1	$s_1\, s_2$	$s_2\, s_1$	s_2	s_1	$s_1\, s_2\, s_1$

$\triangleleft$	1	s_1	$s_1\, s_2$	$s_2\, s_1$	s_2	$s_1\, s_2\, s_1$
1	1	1	1	1	1	1
x	x	x	x^2	x^3	x^2	x^3
x^2	x^2	x^3	x^3	x	x	x^2
x^3	x^3	x^2	x	x^2	x^3	x

By a straightforward computation one can prove that there are two deformation maps for the canonical matched pair $(S_3, C_4, \triangleright, \triangleleft)$: namely the trivial one $r' : C_4 \to S_3$, $r'(c) = 1$, for any $c \in C_4$ and the map given by

$$r : C_4 \to S_3, \qquad r(1) = r(x^2) = 1, \qquad r(x) = r(x^3) = s_1\, s_2\, s_1$$

We consider the following presentation of the Klein's group: $C_2 \times C_2 = \langle a = (12)(34),\, b = (13)(24) \mid a^2 = b^2 = 1,\, ab = ba\rangle$. Then we can easily prove that the map:

$$\varphi : C_2 \times C_2 \to (C_4)_r, \quad \varphi(1) = 1, \quad \varphi(a) = x, \quad \varphi(b) = x^2, \quad \varphi(ab) = x^3$$

is an isomorphism of groups, that is $C_2 \times C_2 \cong (C_4)_r$.

Corollary 3.2 proves that any finite group of order n is isomorphic to an r-deformation of the cyclic group C_n, for some deformation map $r : C_n \to S_{n-1}$ of the canonical matched pair associated to the factorization $S_n = S_{n-1}C_n$. The next example shows how the symmetric group S_3 appears as an r-deformation of the cyclic group C_6 arising from a given matched pair $(C_3, C_6, \triangleright, \triangleleft)$.

Example 3.4. — Let $C_3 = \langle a \mid a^3 = 1\rangle$ and $C_6 = \langle b \mid b^6 = 1\rangle$ be the cyclic groups of order 3 respectively 6. As a special case of [3, Proposition 4.2] we have a matched pair of groups $(C_3, C_6, \triangleright, \triangleleft)$, where the actions $(\triangleright, \triangleleft)$ on generators are defined by:

$$b \triangleright a := a^2 \qquad b \triangleleft a := b^3$$

By a rather long but straightforward computation it can be seen that the map:

$$r : C_6 \to C_3, \quad r(1) = r(b^3) = 1, \quad r(b) = r(b^4) = a^2, \quad r(b^2) = r(b^5) = a$$

is a deformation map of the matched pair $(C_3, C_6, \triangleright, \triangleleft)$ and $\varphi : S_3 \to (C_6)_r$ given by:

$$\varphi(1) = 1, \quad \varphi(s_1) = b, \quad \varphi(s_1\, s_2) = b^2, \quad \varphi(s_2\, s_1) = b^4, \\ \varphi(s_2) = b^5, \quad \varphi(s_1\, s_2\, s_1) = b^3$$

is an isomorphism of groups. Hence S_3 is an r-deformation of the cyclic group C_6.

Our last example provides a non-trivial deformation map $r : H \to A$ such that $H_r \cong H$.

Example 3.5. — Let $(C_3, C_6, \triangleright, \triangleleft)$ be the matched pair of Example 3.4. Then the map

$$R : C_6 \to C_3, \quad R(1) = R(b^2) = R(b^4) = 1, \quad R(b) = R(b^3) = R(b^5) = a$$

is also a deformation map of $(C_3, C_6, \triangleright, \triangleleft)$. Then, one can easily check that $(C_6)_R$ is a group isomorphic to C_6.

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