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RATE OF CONVERGENCE IN SINGULAR PERTURBATIONS

by W. M. GREENLEE

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Introduction.

In a singular perturbation problem one is concerned with a differential equation of the form

$$L(\varepsilon)u_\varepsilon = f_\varepsilon, \tag{1}$$

with initial or boundary conditions

$$B(\varepsilon)u_\varepsilon = g_\varepsilon, \tag{2}$$

where ε is a small parameter. The distinguishing feature of this problem is that the orders of $L(\varepsilon)$ and $B(\varepsilon)$ for $\varepsilon \neq 0$ are higher than the orders of $L(0)$ and $B(0)$ respectively. The differential problem (1), (2) is referred to as a perturbed problem when $\varepsilon \neq 0$ and a degenerate problem when $\varepsilon = 0$. The singular perturbation problem consists of studying the behavior of solutions or eigenvalues of (1), (2) as $\varepsilon \to 0$. Such problems can also be considered with more than one parameter.

Singular perturbation problems arise frequently in applied mathematics and have been considered at least as far back in history as Lord Rayleigh's treatise, *Theory of Sound* [5'] (1), first published in 1877. Rayleigh considered the effect of a small amount of stiffness on the modes of vibration of a violin string. A discussion of the role of singular perturbation phenomena in mathematical physics can be found in Friedrichs [1'].

Some difficulties are inherent in singular perturbation problems. Solutions of the degenerate problem will not in general be as smooth as solutions of the perturbed problem. Moreover, solutions of the degenerate problem usually will not satisfy as many initial or boundary conditions as do solutions of the perturbed problem. Hence, if solutions of the perturbed problem are to converge to solutions of the degenerate problem, the notion of convergence will probably have to be rather weak. Due to the « loss » of initial or boundary data it may also happen that solutions of the perturbed problem converge in a stronger sense in the interior of the underlying domain, than in the vicinity of the boundary. This as known as the boundary layer phenomenon.

There is by now a vast amount of literature on singular perturbation problems for ordinary differential equations, both linear and non-linear. An extensive bibliography of this literature is contained in Wasow [9']. In Chapter 10 of [9'], Wasow also presents a lucid discussion of the boundary layer phenomenon. Moser [4'] has obtained asymptotic expansions for eigenvalues and eigenfunctions of the perturbed problem in the case of linear equations of even order.

There is also a considerable amount of literature on singular perturbation problems for partial differential equations. Višik and Lyusternik [8'] have obtained asymptotic expressions for solutions of the perturbed problem for linear equations

(1) Numbers in brackets refer to the bibliography; primed numbers in brackets denote references mentioned only in the introduction. The references mentioned only in the introduction are listed separately as supplementary references after the bibliography.

using boundary layer techniques. [8′] also contains a sizable bibliography.

In 1960, Huet [8] published several theorems on convergence in singular perturbation problems for linear elliptic and parabolic partial differential equations. One particular feature distinguishes this paper from those previously mentioned. This is that convergence theorems are first proven in a Hilbert space setting and then applied to the differential problems as opposed to starting directly with the differential equations. In the elliptic case, theorems on local convergence and convergence of tangential derivatives at the boundary are also proven. In [6′], Ton has extended some of the results of [8] to nonlinear elliptic and parabolic boundary value problems. Ton has also obtained results on singular perturbation with application to non-linear parabolic boundary value problems in [7′].

The work of Huet [8] is fundamental to the considerations in this paper even though the results of [8] are not specifically used here.

The principal aim of the present paper is to obtain rate of convergence estimates for solutions of singular perturbations of linear elliptic boundary value problems. The problem can be described as follows. Let D be a domain in R^n and let ε be a positive real parameter. Consider two boundary value problems on D,

$$(\varepsilon\mathfrak{A} + \mathfrak{B})w_\varepsilon = f, \qquad \mathfrak{B}u = f, \tag{3}$$

where $\mathfrak{A}$ and $\mathfrak{B}$ are elliptic differential operators with the order of $\mathfrak{A}$ greater than the order of $\mathfrak{B}$. The problem is to determine in what sense w_ε converges to u on D as $\varepsilon \downarrow 0$ and to estimate the rate of convergence. This problem is investigated in the present work within the L^2 theory of elliptic partial differential problems.

To approach this problem, rate of convergence theorems are first proven in an abstract Hilbert space setting. A brief sketch of the method will now be given.

Consider two Hilbert spaces $V \subset V_0$ with V dense in V_0. Let $a(v, w)$, $b(v, w)$ be Hermitian bilinear (sesquilinear) forms on V and V_0 respectively such that $b(v, w)$ and $\varepsilon a(v, w) + b(v, w)$ are coercive. Define the operator $\mathcal{A}$ by

$a(v, w) = b(\mathcal{A}v, w)$ and consider the spaces obtained by quadratic interpolation between the domain of $\mathcal{A}$, provided with the graph norm, and V_0. Denoting the domain of $\mathcal{A}$ by V_1 the basic rate of convergence criterion is the following. $(\varepsilon\mathcal{A} + I)^{-1}u$ converges to u in V_0 as $\varepsilon \downarrow 0$ with rate $o(\varepsilon^\tau)$ if $u \in V_\tau$, $0 \leqslant \tau < 1$, and rate $O(\varepsilon)$ if $u \in V_1$. Moreover, if the domain of $\mathcal{A}$ contains the domain of the adjoint of $\mathcal{A}$ in V_0, and $u \in V_\tau$ where $0 < \tau \leqslant 1$, then $(\varepsilon\mathcal{A} + I)^{-1}u$ converges to u in V_γ with rate $o(\varepsilon^{\tau-\gamma})$ for all $\gamma \in (0, \tau]$.

This estimates the rate of convergence in Theorem 1.4, p. 76, Huet [8], and also provides conditions for convergence in a stronger norm. The use of fractional powers of positive self adjoint operators (quadratic interpolation) to estimate the rate of convergence has some relation to work of Kato [2′], [12], [3′]. Kato uses a square root condition in his work on asymptotic perturbation theory for eigenvalues.

In the case of the differential problem (3) observe that, formally, $w_\varepsilon = (\varepsilon\mathcal{B}^{-1}\,\mathcal{U} + I)^{-1}u$. Let $a(v, w)$ and $b(v, w)$ be the Hermitian bilinear forms corresponding to $\mathcal{U}$ and $\mathcal{B}$ respectively. The rate of convergence criterion is then to be applied by noting that $\varepsilon\mathcal{A} + I$ is an extension of the Hilbert space realization of the formal operator $\varepsilon\mathcal{B}^{-1}\mathcal{U} + I$ and proving that the solution u of $\mathcal{B}u = f$ is in V_τ for some $\tau > 0$.

This investigation has been divided into six chapters.

Chapter 1 consists of preliminary material. The operator $\mathcal{A}$ is defined by a simple variant of a standard Hilbert space framework for boundary value problems (cf. Lions [16], Chap. II, § 1).

The main rate of convergence theorems (theorems 2.1 and 2.3) are proven in Chapter 2. The basic setting is similar to that used by Huet [8], Chapter 1, n. 2. The chapter concludes with some simple examples which show that the rate of convergence theorems are sharp.

In Chapter 3 it is proven that if $\mathcal{A}$ is positive self adjoint, then a classical asymptotic expansion can be obtained by use of the so-called « negative norms ».

Chapter 4 deals with reformulating the results of Chapter 2 in the framework to be used in the applications to differential problems in Chapter 6. In particular, the aforementioned relation between the operator $\mathcal{A}$ and the Hilbert space realizations of the differential operators is established.

The terminology and several of the results from the theory of Bessel potentials are used in Chapter 5 (cf. Adams, Aronszajn and Smith [2], Aronszajn [5]). An outline of the relevant facts about Bessel potentials is included. The interpolation spaces by quadratic interpolation between $\check{P}_0^m(D)$ and $L^2(D)$ are characterized for D a Lipschitzian graph domain. In the case of a bounded domain for which one has regularity at the boundary for weak solutions of elliptic boundary value problems, the spaces obtained by quadratic interpolation between $\check{P}^{2m}(D) \cap \check{P}_0^m(D)$ and $\check{P}_0^m(D)$ are also characterized. Some of the methods of proof are related to methods used by Lions and Magenes [19].

In Chapter 6 the results of the preceding chapters are applied to singular perturbations of the Dirichlet problem with homogeneous boundary conditions. In particular, if ∂D, u, and u_ε are smooth, the problems considered are of the form

$$\begin{gathered}(\varepsilon\mathfrak{U} + \mathfrak{B})u_\varepsilon = f_\varepsilon \in L^2(D),\\ u_\varepsilon = \frac{\partial u_\varepsilon}{\partial n} = \cdots = \frac{\partial^{m'-1} u_\varepsilon}{\partial n^{m'-1}} = 0 \text{ on } \partial D,\\ \mathfrak{B}u = f \in L^2(D),\\ u = \frac{\partial u}{\partial n} = \cdots = \frac{\partial^{m-1} u}{\partial n^{m-1}} = 0 \text{ on } \partial D,\end{gathered}$$

where $\mathfrak{U}$ and $\mathfrak{B}$ are elliptic partial differential operators, $\mathfrak{U}$ is of order $2m'$, $\mathfrak{B}$ is of order $2m$, $m' > m$, and $\dfrac{\partial}{\partial n}$ denotes differentiation in the direction normal to ∂D. It is proven that if D is bounded, ∂D is smooth enough that u is regular at the boundary, and $|f_\varepsilon - f|_{0,D} = o(\varepsilon^\tau)$ as $\varepsilon \downarrow 0$ for all $\tau < 1/4(m' - m)$, then $|u_\varepsilon - u|_{m,D} = o(\varepsilon^\tau)$ as $\varepsilon \downarrow 0$ for all $\tau < 1/4(m' - m)$ (Theorem 6.1). If, in addition, ∂D is smooth enough that the solution w_ε of $(\varepsilon\mathfrak{U} + \mathfrak{B})w_\varepsilon = f$ is regular at the boundary and $\mathfrak{B}$ is positive self adjoint, then $w_\varepsilon \to u$ in $\check{P}^\alpha(D)$ for all α such that $m \leqslant \alpha < m + 1/2$. A theorem is then given in which the perturbed operator is of the form $\varepsilon\mathfrak{B} + I$. This theorem supplements rate of convergence results of Huet [9] and Ton [22]. The chapter concludes with some elementary examples and a brief comment on singular perturbations of Neumann problems.

As this paper was in the final stages of preparation, recent results of Grisvard [7] came to the attention of the author. Grisvard has characterized the interpolation spaces associated with spaces of potentials satisfying quite general homogeneous boundary conditions. These results make the methods of this paper applicable to a much larger class of elliptic boundary value problems.

The results of Chapter 5 were obtained independently and characterize these spaces, in the case of homogeneous Dirichlet boundary conditions, for a larger class of domains. In particular, the results of Chapter 5 enable one to estimate the rate of convergence when the domain is bounded convex and the degenerate problem is of second order via the results of Kadlec [11].

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1. Preliminaries.

Notation.

In this section some notations and results are given which are used in the sequel. The results are minor variants of those given in Lions [16], pp. 9-13.

Let V_0 be a complex Hilbert space with norm denoted by $|u|_0$ and scalar product by $(u, v)_0$. Let V_1 be a complex Hilbert space which is continuously contained in V_0, written

$$V_1 \underset{c}{\subset} V_0$$

i.e., V_1 is a vector subspace of V_0 and the injection of V_1 into V_0 is continuous. Further assume that V_1 is dense in V_0 and let $|u|_1$, $(u, v)_1$ denote the norm and scalar product in V_1, respectively.

Let $b(u, v)$ be a continuous Hermitian bilinear (sesquilinear) form on V_0, i.e., $b(u, v)$ is a mapping of $V_0 \times V_0$ into C (the complex field) which is linear in the left hand variable and anti-(semi, conjugate, skew) linear in the right hand variable, with

$$|b(u, v)| \leqslant c|u|_0|v|_0, \quad c = \text{constant},$$

for all $u, v \in V_0$. Prescribing $b(u, v)$ is equivalent to giving an operator $B \in \mathcal{L}(V_0, V_0)$ (the space of continuous linear operators of V_0 into V_0) with

$$b(u, v) = (Bu, v)_0. \tag{1.1}$$

Proposition 1.1. — *Suppose in addition that*

$$|b(u, u)| \geqslant \beta|u|_0^2, \quad \beta > 0, \quad \textit{for all } u \in V_0. \tag{1.2}$$

Then:

i) *the operator* B *is a linear homeomorphism of* V_0 *onto* V_0,

ii) *for every continuous linear functional* F *on* V_0 *there exists a unique element* $f \in V_0$ *such that*

$$F(u) = b(u, f), \quad u \in V_0,$$

iii) *for every continuous anti-linear functional* G *on* V_0 *there exists a unique element* $g \in V_0$ *such that*

$$G(u) = b(g, u), \quad u \in V_0.$$

This proposition is essentially the Lax-Milgram Lemma, Lax and Milgram [14], p. 169. It follows from (1.1) and (1.2) that $|Bu|_0 \geq \beta|u|_0$. When (1.2) is satisfied, $b(u, v)$ will be said to be V_0-*coercive.*

The adjoint form to $b(u, v)$, $b^*(u, v)$, is defined by

$$b^*(u, v) = \overline{b(v, u)} \text{ for all } u, v \in V_0$$

and is likewise a continuous Hermitian bilinear form on V_0. One has

$$b^*(u, v) = (B^*u, v)_0$$

where B^* is the (Hilbert space) adjoint of B. Under hypothesis (1.2), $|B^*u|_0 \geqslant \beta|u|_0$.

Now let $a(u, v)$ be a continuous Hermitian bilinear form on V_1 and assume that (1.2) holds for the remainder of this section. Denote by N the set of all $u \in V_1$ such that the anti-linear functional

$$v \to a(u, v) \tag{1.3}$$

is continuous on V_1 in the topology induced by V_0. Then N is a linear set and since V_1 is dense in V_0, the functional (1.3) may be extended by continuity to a continuous anti-linear functional on V_0. Hence by Proposition 1.1, iii),

$$a(u, v) = b(\mathcal{C}u, v), \ \mathcal{C}u \in V_0, \ v \in V_1, \ u \in N.$$

This defines a linear operator $\mathcal{C}$, in general unbounded, with domain $D(\mathcal{C}) = N$ and range $R(\mathcal{C}) \subset V_0$. The operator $\mathcal{C}$ will be referred to as the *operator in* V_0 *associated with* $a(u, v)$ *relative to* $b(u, v)$. The operator in V_0 associated with $a(u, v)$ relative to $(u, v)_0$ will be denoted by A and referred to simply as the *operator in* V_0 *associated with* $a(u, v)$. Note that $B\mathcal{C} = A$.

Consider the following two problems.

Problem 1.1. — *Given* $f \in V_0$, *does there exist* $u \in D(\mathcal{C})$ *such that* $\mathcal{C}u = f$?

Problem 1.2. — *Given* $f \in V_0$, *does there exist* $u \in V_1$ *such that*

$$a(u, v) = b(f, v) \text{ for all } v \in V_1? \tag{1.4}$$

Proposition 1.2. — *Problems* 1.1 *and* 1.2 *are equivalent.*

Proof. — If u satisfies $\mathcal{C}u = f$, then for any $v \in V_1$,

$$a(u, v) = b(\mathcal{C}u, v) = b(f, v).$$

Conversely, let $u \in V_1$ satisfy (1.4). Then the functional $v \to a(u, v)$ is continuous on V_1 in the topology induced by V_0. Hence, by the definition of $\mathcal{C}$, $u \in D(\mathcal{C})$ and $a(u, v) = b(\mathcal{C}u, v)$. Consequently, $b(\mathcal{C}u, v) = b(f, v)$ for all $v \in V_1$. Since V_1 is dense in V_0, it follows from (1.2) that $\mathcal{C}u = f$.

Proposition 1.3. — *Let* $a(u, v)$ *be* V_1-coercive, *i.e.*

$$|a(v, v)| \geqslant \alpha |v|_1^2, \quad \alpha > 0, \quad \text{for all} \quad v \in V_1. \tag{1.5}$$

Then for every $f \in V_0$ *there exists a unique* $u \in D(\mathcal{A})$ *satisfying* $\mathcal{A}u = f$.

Proof. — Given $f \in V_0$, let $Bf = g$. The equation $Au = g$, where A is the operator in V_0 associated with $a(u, v)$, is uniquely solvable (cf. [16], pp. 11-12). Hence u is the unique solution of $\mathcal{A}u = B^{-1}Au = B^{-1}g = f$. (Recall that B is a linear homeomorphism of V_0 onto V_0).

Now assume for the remainder of this section that (1.5) holds. Then the following facts about the operator A are given in Lions [16], p. 12. A is closed and D(A) is dense in V_0. D(A) is also dense in V_1. The operator A^* associated with $a^*(u, v) = \overline{a(v, u)}$ in V_0 is the (Hilbert space) adjoint of A and has all the properties mentioned above for A including the unique solvability of $A^*u = f$, $f \in V_0$.

PROPOSITION 1.4. — *The operator* $\mathcal{A}$ *in* V_0 *associated with a* V_1*-coercive form* $a(u, v)$ *relative to a* V_0*-coercive form* $b(u, v)$ *has the following properties:*

i) $\mathcal{A}$ *is closed,*

ii) $D(\mathcal{A})$ *is dense in* V_0,

iii) $D(\mathcal{A})$, *is dense in* V_1,

iv) $D(\mathcal{A})$, *provided with the graph norm*

$$|u|_{D(\mathcal{A})} = (|u|_0^2 + |\mathcal{A}u|_0^2)^{1/2},$$

is a Hilbert space and $\mathcal{A}$ *is a linear homeomorphism of* $D(\mathcal{A})$ *(provided with this topology) onto* V_0.

Proof. — i), ii), and iii) are obvious since $B\mathcal{A} = A$. iv) follows from i), Proposition 1.3, and the closed graph theorem.

Quadratic Interpolation.

Let Λ be the operator in V_0 associated with the V_1-coercive form $(u, v)_1$, i.e.,

$$(u, v)_1 = (\Lambda u, v)_0, \qquad \Lambda u \in V_0, \qquad v \in V_1.$$

It follows from the preceding that Λ is self adjoint, and that $(\Lambda v, v)_0 \geqslant \gamma |v|_0^2$, $v \in D(\Lambda)$, where $\gamma > 0$ is such that

$$|v|_1^2 \geqslant \gamma |v|_0^2, \qquad v \in V_1.$$

For ρ real, denote by Λ^ρ the positive ρ^{th} power of Λ as defined by use of the spectral theorem; Λ^ρ is a positive definite self adjoint operator in V_0. Furthermore, $D(\Lambda^{1/2}) = V_1$ and

$$(u, v)_1 = (\Lambda^{1/2}u, \Lambda^{1/2}v)_0 \qquad \text{for all} \qquad u, v \in V_1.$$

(cf. Kato [12], pp. 26-27).

DÉFINITION 1.1. — Let $S = \Lambda^{1/2}$. For $0 \leqslant \tau \leqslant 1$, *the* τ^{th} *interpolation space by quadratic interpolation* between V_1 and V_0, V_τ, is defined as the Hilbert space

$$V_\tau = D(S^\tau)$$

with inner product

$$(u, v)_\tau = (S^\tau u, S^\tau v)_0.$$

Now let H_1 and H_0 be another couple of Hilbert spaces with the same properties as V_1 and V_0, i.e., $H_1 \underset{c}{\subset} H_0$ with H_1 dense in H_0, and consider the corresponding quadratic interpolation spaces. Then the following theorem of quadratic interpolation holds (cf. Lions [15], pp. 431-432 and Adams, Aronszajn and Hanna [1], App. I).

PROPOSITION 1.5. — *Let* T *be a continuous linear mapping of* V_0 *into* H_0 *with bound* M_0 *which is also continuous from* V_1 *into* H_1 *with bound* M_1. *Then for each* $\tau \in (0,1)$, T *is a continuous linear mapping of* V_τ *into* H_τ *with bound* $M_\tau \leqslant M_1^\tau M_0^{1-\tau}$.

2. Singular Perturbation. Rate of Convergence Theorems.

Let V and V_0 be complex Hilbert spaces with

$$V \underset{c}{\subset} V_0 \text{ and } V \text{ dense in } V_0. \tag{2.1}$$

Denote by $|v|_V$, $(u, v)_V$, $|v|_0$, and $(u, v)_0$ the norms and inner products in V and V_0 respectively. Let $a(u, v)$ be a continuous Hermitian bilinear form on V and let $b(u, v)$ be a continuous Hermitian bilinear form on V_0 with upper bound *c*. Further assume that:

(2.2) $b(u, v)$ is V_0-coercive, i.e. there exists $\beta > 0$ such that

$$|b(v, v)| \geqslant \beta |v|_0^2 \qquad \text{for all} \qquad v \in V_0;$$

and

(2.3) for $0 < \varepsilon \leqslant \varepsilon_0$, there exist $\alpha(\varepsilon) > 0$, $\alpha(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$, and $\delta > 0$ such that

$$|\varepsilon a(v, v) + b(v, v)| \geqslant \alpha(\varepsilon)|v|_V^2 + \delta |v|_0^2 \qquad \text{for all} \qquad v \in V.$$

In particular, $\varepsilon a(u, v) + b(u, v)$ is V-coercive for $0 < \varepsilon \leqslant \varepsilon_0$.

PROPOSITION 2.1. — *Assume hypotheses* (2.1), (2.2), *and* (2.3), *and let* $\mathcal{A}$ *be the operator in* V_0 *associated with* $a(u, v)$ *relative to* $b(u, v)$. *Then:*

i) $\mathcal{A}$ *is closed,*

ii) $D(\mathcal{A})$ *is dense in* V_0,

iii) $D(\mathcal{A})$ *is dense in* V,

iv) *for* $0 < \varepsilon \leqslant \varepsilon_0$, $\varepsilon\mathcal{A} + I$ (I *is the identity map on* V_0) *is a linear homeomorphism of* $D(\mathcal{A})$, *provided with the graph norm* $|v|_{D(\mathcal{A})} = (|v|_0^2 + |\mathcal{A}v|_0^2)^{1/2}$, *onto* V_0.

Proof. — Since $b(u, v)$ is continuous on V_0 it is easily seen that $\varepsilon\mathcal{A} + I$ is the operator in V_0 associated with $\varepsilon a(u, v) + b(u, v)$ relative to $b(u, v)$. Then by Proposition 1.4, i), ii) and iii) hold with $\mathcal{A}$ replaced by $\varepsilon\mathcal{A} + I$. Since $D(\mathcal{A}) = D(\varepsilon\mathcal{A} + I)$ and the identity map is bounded, i), ii) and iii) follow.

Now for $v \in D(\mathcal{A})$,

$$\begin{aligned}(\varepsilon\mathcal{A}v + v, \varepsilon\mathcal{A}v + v)_0 &\leqslant 2(\varepsilon^2|\mathcal{A}v|_0^2 + |v|_0^2) \\ &\leqslant 2 \max(\varepsilon^2, 1)\ |v|_{D(\mathcal{A})}^2, \qquad 0 < \varepsilon \leqslant \varepsilon_0.\end{aligned}$$

Thus $\varepsilon\mathcal{A} + I$ is a continuous linear mapping of $D(\mathcal{A})$ into V_0. By Proposition 1.3, $\varepsilon\mathcal{A} + I$ is also one-to-one and onto V_0. Hence by the closed graph theorem $\varepsilon\mathcal{A} + I$ has a continuous linear inverse and iv) holds.

Let V_0^* be the anti-dual of V_0, i.e. the Hilbert space of continuous anti-linear functionals on V_0, with the usual norm, $\|L\| = \sup\{|L(v)| : v \in V_0 \text{ and } |v|_0 \leqslant 1\}$. Let L, L_ε be given in V_0^*, $0 < \varepsilon \leqslant \varepsilon_0$. Denote by u the unique solution in V_0 of

$$(2.4) \qquad b(u, v) = L(v) \qquad \text{for all} \qquad v \in V_0.$$

For each ε such that $0 < \varepsilon \leqslant \varepsilon_0$, let w_ε be the unique solution in V of

$$(2.5) \qquad \varepsilon a(w_\varepsilon, v) + b(w_\varepsilon, v) = L(v) \qquad \text{for all} \qquad v \in V,$$

and let u_ε be the unique solution in V of

$$(2.6) \qquad \varepsilon a(u_\varepsilon, v) + b(u_\varepsilon, v) = L_\varepsilon(v) \qquad \text{for all} \qquad v \in V.$$

Recall that (2.4) is uniquely solvable by Proposition 1.1, iii) (Lax-Milgram), while (2.5) and (2.6) are uniquely solvable by Proposition 1.1, iii), Propositions 1.2, 1.3, and the density of V in V_0.

Denote $D(\mathcal{A})$ by V_1, $|v|_{D(\mathcal{A})}$ by $|v|_1$, and $(w, v)_{D(\mathcal{A})}$ by $(w, v)_1$. Then the following rate of convergence theorem describes the behavior of u_ε (and w_ε) as $\varepsilon \downarrow 0$.

THEOREM 2.1. — *Assume hypotheses* (2.1) *through* (2.6). *Let* $\mathcal{A}$ *be the operator in* V_0 *associated with* $a(u, v)$ *relative to* $b(u, v)$. *Consider the interpolation spaces* V_τ, $0 \leqslant \tau \leqslant 1$, *obtained by quadratic interpolation between* $V_1 = D(\mathcal{A})$ *and* V_0. *Then one has:*

i) *if* $u \in D(\mathcal{A})$ *and* $\|L_\varepsilon - L\| = 0(\varepsilon)$ *as* $\varepsilon \downarrow 0$, *then*

$$|u_\varepsilon - u|_0 = 0(\varepsilon) \qquad \textit{as} \qquad \varepsilon \downarrow 0;$$

ii) *if, for fixed* $\tau \in [0,1)$, $u \in V_\tau$ *and* $\|L_\varepsilon - L\| = o(\varepsilon^\tau)$ *as* $\varepsilon \downarrow 0$, *then*

$$|u_\varepsilon - u|_0 = o(\varepsilon^\tau) \qquad \textit{as} \qquad \varepsilon \downarrow 0.$$

Remark. — The proof of Theorem 2.1 will be carried out in three steps. In the first step an easy reduction is performed which estimates $|u_\varepsilon - u|_0$ in terms of $|w_\varepsilon - u|_0$. In the second section of the proof, $|w_\varepsilon - u|_0$ is estimated ((2.17) below) by $|x_\varepsilon - u|_0$ where x_ε is the solution of a problem with a self adjoint operator ((2.8) below). The proof is then completed in the third section where conclusion ii) is obtained by estimating $|x_\varepsilon - u|_0$.

Proof. — *a) Elimination of* L_ε. Subtracting (2.5) from (2.6) yields

$$\varepsilon a(u_\varepsilon - w_\varepsilon, v) + b(u_\varepsilon - w_\varepsilon, v) = (L_\varepsilon - L)(v) \quad \text{for all} \quad v \in V.$$

In particular, letting $v = u_\varepsilon - w_\varepsilon$,

$$|\varepsilon a(u_\varepsilon - w_\varepsilon, u_\varepsilon - w_\varepsilon) + b(u_\varepsilon - w_\varepsilon, u_\varepsilon - w_\varepsilon)| \leqslant \|L_\varepsilon - L\| . |u_\varepsilon - w_\varepsilon|_0.$$

Thus by (2.3),

$$\alpha(\varepsilon)|u_\varepsilon - w_\varepsilon|_V^2 + \delta|u_\varepsilon - w_\varepsilon|_0^2 \leqslant \|L_\varepsilon - L\| . |u_\varepsilon - w_\varepsilon|_0,$$

and so,

$$|u_\varepsilon - w_\varepsilon|_0 \leqslant (1/\delta)\|L_\varepsilon - L\|.$$

Hence,

$$|u_\varepsilon - u|_0 \leqslant (1/\delta)\|L_\varepsilon - L\| + |w_\varepsilon - u|_0. \tag{2.7}$$

From the hypotheses it is now sufficient to prove that i) and ii) hold with u_ε replaced by w_ε.

b) Reduction to an associated problem with a self adjoint operator. Let Λ be the operator in V_0 associated with $(w, v)_1$ and let $S = \Lambda^{1/2}$. Then it is easily seen that $\Lambda = \mathcal{A}^*\mathcal{A} + I$, where $\mathcal{A}^*$ is the adjoint of $\mathcal{A}$ in V_0, and that $D(S) = D(\mathcal{A})$. Consider the Hermitian symmetric bilinear form $(S^{1/2}w, S^{1/2}v)_0$ defined on $D(S^{1/2})$. Let x_ε be the unique solution in $D(S^{1/2})$ of

$$\varepsilon(S^{1/2}x_\varepsilon, S^{1/2}v)_0 + (x_\varepsilon, v)_0 = (u, v)_0 \quad \text{for all} \quad v \in D(S^{1/2}).$$

Since $(w, v)_0$ is bounded on V_0, x_ε is the unique solution in $D(S)$ of

$$\varepsilon S x_\varepsilon + x_\varepsilon = u. \tag{2.8}$$

According to (2.4) and (2.5),

$$\varepsilon a(w_\varepsilon, v) + b(w_\varepsilon, v) = b(u, v) \qquad \text{for all} \qquad v \in V.$$

Then since $b(w, v)$ is bounded on V_0, w_ε is the unique solution in $D(\mathcal{A})$ of

$$\varepsilon \mathcal{A} w_\varepsilon + w_\varepsilon = u. \tag{2.9}$$

$|w_\varepsilon - u|_0$ will now be estimated in terms of $|x_\varepsilon - u|_0$.

From (2.8),

$$(2.10)\quad x_\varepsilon = (\varepsilon S + I)^{-1}u, \qquad x_\varepsilon - u = -\varepsilon S(\varepsilon S + I)^{-1}u,$$

while from (2.9),

$$(2.11)\quad w_\varepsilon = (\varepsilon\mathcal{A} + I)^{-1}u, \quad w_\varepsilon - u = -\varepsilon\mathcal{A}(\varepsilon\mathcal{A} + I)^{-1}u.$$

Furthermore, (2.10) and (2.11) yield,

$$(2.12)\quad w_\varepsilon - u = \mathcal{A}(\varepsilon\mathcal{A} + I)^{-1}(\varepsilon S + I)S^{-1}(x_\varepsilon - u).$$

It will now be proven that $\mathcal{A}(\varepsilon\mathcal{A} + I)^{-1}(\varepsilon S + I)S^{-1}$ is a bounded operator on V_0 with bound independant of ε.

Let $y \in V_0$. Then

$$(2.13)\quad \mathcal{A}(\varepsilon\mathcal{A} + 1)^{-1}(\varepsilon S + I)S^{-1}y = \varepsilon\mathcal{A}(\varepsilon\mathcal{A} + I)^{-1}y + \mathcal{A}(\varepsilon\mathcal{A} + I)^{-1}S^{-1}y = y - y_\varepsilon + (\varepsilon\mathcal{A} + I)^{-1}\mathcal{A}S^{-1}y$$

since $S^{-1}y \in D(\mathcal{A})$ and where y_ε is the unique solution in $D(\mathcal{A})$ of $\varepsilon\mathcal{A}y_\varepsilon + y_\varepsilon = y$ (cf. (2.9) and (2.11)). Now

$$\varepsilon a(y_\varepsilon, v) + b(y_\varepsilon, v) = b(y, v) \qquad \text{for all} \qquad v \in V$$

and letting $v = y_\varepsilon$,

$$\alpha(\varepsilon)|y_\varepsilon|_V^2 + \delta|y_\varepsilon|_0^2 \leqslant |\varepsilon a(y_\varepsilon, y_\varepsilon) + b(y_\varepsilon, y_\varepsilon)| = |b(y, y_\varepsilon)| \leqslant c|y|_0|y_\varepsilon|_0.$$

So,

$$(2.14)\quad |(\varepsilon\mathcal{A} + I)^{-1}y|_0 = |y_\varepsilon|_0 \leqslant (c/\delta)|y|_0 \quad \text{for all} \quad y \in V_0.$$

Similarly, letting $\mathcal{A}S^{-1}y = z \in V_0$ and letting z_ε be the unique solution in $D(\mathcal{A})$ of $\varepsilon\mathcal{A}z_\varepsilon + z_\varepsilon = z$, one has

$$(2.15)\quad |(\varepsilon\mathcal{A} + I)^{-1}\mathcal{A}S^{-1}y|_0 = |z_\varepsilon|_0 \leqslant (c/\delta)|z|_0 = (c/\delta)|\mathcal{A}S^{-1}y|_0 \leqslant (c/\delta)|\mathcal{A}||S^{-1}y|_1 \leqslant (c/\delta)|S^{-1}||y|_0 = (c/\delta)|y|_0$$

where $|\mathcal{A}|$ is the norm of $\mathcal{A}$ in $\mathcal{L}(V_1, V_0)$ and $|S^{-1}|$ is the norm of S^{-1} in $\mathcal{L}(V_0, V_1)$. (2.13), (2.14), (2.15) and the triangle inequality yield

$$(2.16)\quad |\mathcal{A}(\varepsilon\mathcal{A} + I)^{-1}(\varepsilon S + I)S^{-1}| \leqslant 1 + 2(c/\delta)$$

where the operator norm is that of $\mathcal{L}(V_0, V_0)$. (2.12) and (2.16) give

$$(2.17)\quad |w_\varepsilon - u|_0 \leqslant [1 + 2(c/\delta)]|x_\varepsilon - u|_0,$$

the desired estimate.

c) Completion of the proof. When $u \in D(\mathcal{A})$, (2.11) gives $w_\varepsilon - u = -\varepsilon(\varepsilon\mathcal{A} + I)^{-1}\mathcal{A}u$ and so,

$$(2.18) \quad |w_\varepsilon - u|_0 \leqslant \varepsilon(c/\delta)|\mathcal{A}u|_0 = 0(\varepsilon) \quad \text{as} \quad \varepsilon \downarrow 0$$

as in (2.14). Thus i) follows from (2.7) and (2.18)

Now let $u \in V_\tau$ for fixed $\tau \in [0, 1)$. Then letting E be the resolution of the identity for the self adjoint operator S, the spectral theorem for functions of a self adjoint operator gives,

$$\begin{aligned}(2.19) \quad |x_\varepsilon - u|_0^2 &= |[(\varepsilon S + I)^{-1} - I]u|_0^2 \\ &= \int_0^\infty \left|\frac{1}{\varepsilon\lambda + 1} - 1\right|^2 (E(d\lambda)u, u)_0 \\ &= \int_0^\infty \frac{\varepsilon^2\lambda^2}{(\varepsilon\lambda + 1)^2}(E(d\lambda)u, u)_0 \\ &= \varepsilon^{2\tau}\int_0^\infty \lambda^{2\tau}\cdot\frac{(\varepsilon\lambda)^{2-2\tau}}{(\varepsilon\lambda + 1)^{2-2\tau}}\cdot\frac{1}{(\varepsilon\lambda + 1)^{2\tau}}(E(d\lambda)u, u)_0 \\ &\leqslant \varepsilon^{2\tau}\int_0^\infty \lambda^{2\tau}\frac{(\varepsilon\lambda)^{2-2\tau}}{(\varepsilon\lambda + 1)^{2-2\tau}}(E(d\lambda)u, u)_0.\end{aligned}$$

But $u \in D(S^\tau) = V_\tau$ if and only if $\int_0^\infty \lambda^{2\tau}(E(d\lambda)u, u)_0 < \infty$, and since for each fixed λ, $\dfrac{\varepsilon\lambda}{\varepsilon\lambda + 1}$ converges monotonically to zero as $\varepsilon \downarrow 0$, one has $\left|\lambda^{2\tau}\cdot\dfrac{(\varepsilon\lambda)^{2-2\tau}}{(\varepsilon\lambda + 1)^{2-2\tau}}\right| \leqslant \lambda^{2\tau}$ and $\lambda^{2\tau}\cdot\dfrac{(\varepsilon\lambda)^{2-2\tau}}{(\varepsilon\lambda + 1)^{2-2\tau}}$ converges in $(E(d\lambda)u, u)_0$ measure to zero as $\varepsilon \downarrow 0$. Thus by the dominated convergence theorem,

$$(2.20) \quad \int_0^\infty \lambda^{2\tau}\frac{(\varepsilon\lambda)^{2-2\tau}}{(\varepsilon\lambda + 1)^{2-2\tau}}(E(d\lambda)u, u)_0 \to 0 \quad \text{as} \quad \varepsilon \downarrow 0.$$

(2.19) and (2.20) give

$$(2.21) \qquad |x_\varepsilon - u|_0^2 = o(\varepsilon^{2\tau}) \qquad \text{as} \qquad \varepsilon \downarrow 0$$

and ii) now follows from (2.7), (2.17) and (2.21).

COROLLARY 2.1. — *Assume the hypotheses of Theorem 2.1. Let* $\mathcal{A} \geqslant 0$, *i.e.* $(\mathcal{A}\varphi, \varphi)_0 \geqslant 0$ *for all* $\varphi \in D(\mathcal{A})$. *Then if for fixed* $\tau \in [0, 1)$, $u \in D(\mathcal{A}^\tau)$ *and* $\|L_\varepsilon - L\| = o(\varepsilon^\tau)$ *as* $\varepsilon \downarrow 0$, *one has*

$$|u_\varepsilon - u|_0 = o(\varepsilon^\tau) \qquad as \qquad \varepsilon \downarrow 0.$$

Proof. — $(\varepsilon\mathcal{A} + I)^{-1}$ is a bounded self adjoint operator on V_0, hence $\varepsilon\mathcal{A} + I$ and $\mathcal{A}$ are self adjoint operators in V_0. The proof now consists of applying the spectral theorem to $\mathcal{A}$ as applied to S in part *c*) of the proof of Theorem 2.1.

Corollary 2.2. — *Assume hypotheses* (2.1) *through* (2.3). *Let* $\mathcal{A}$ *be one-to-one and onto* V_0. *Let* w_ε, t_ε *be the unique solutions of* $\varepsilon\mathcal{A}w_\varepsilon + w_\varepsilon = u$ *and* $\varepsilon(\mathcal{A}^*\mathcal{A})^{1/2}t_\varepsilon + t_\varepsilon = u$ *respectively where* $u \in V_0$. *Then the ratio* $|w_\varepsilon - u|_0 : |t_\varepsilon - u|_0$ *is bounded both from above and below by positive constants independent of* ε *and* u.

Proof. — Both $\mathcal{A}$ and $(\mathcal{A}^*\mathcal{A})^{1/2}$ have bounded inverses on V_0. So apply the methods used in part *b*) of the proof of Theorem 2.1 twice, using $(\mathcal{A}^*\mathcal{A})^{1/2}$ instead of S.

The last corollary shows that when $\mathcal{A}$ is one-to-one and onto V_0, the rate of convergence of w_ε to u in V_0 is the same as the rate of convergence of t_ε to u in V_0. In turn, $|t_\varepsilon - u|_0$ may be estimated in terms of powers of ε by applying the spectral theorem to $(\mathcal{A}^*\mathcal{A})^{1/2}$ as applied to S in part *c*) of the proof of theorem 2.1.

Corollary 2.3. — *Assume hypotheses* (2.1) *through* (2.3). *Let* $\mathcal{A} \geqslant 0$ *and let* w_ε *be the unique solution of* $\varepsilon\mathcal{A}w_\varepsilon + w_\varepsilon = u$ *where* $u \in V_0$. *Then one has:*

i) *if* $u \in D(\mathcal{A})$, *then*

$$|w_\varepsilon - u|_0 = \varepsilon|\mathcal{A}u|_0 . \sigma(\varepsilon, u)$$

where $0 \leqslant \sigma(\varepsilon, u) \leqslant 1$ *and* $\sigma(\varepsilon, u) \to 1$ *as* $\varepsilon \downarrow 0$;

ii) *if for fixed* $\tau \in [0,1)$, $u \in D(\mathcal{A}^\tau)$, *then*

$$|w_\varepsilon - u|_0 = \varepsilon^\tau|\mathcal{A}^\tau u|_0 . \sigma(\varepsilon, \tau, u)$$

where $0 \leqslant \sigma(\varepsilon, \tau, u) \leqslant 1$ *and* $\sigma(\varepsilon, \tau, u) \to 0$ *as* $\varepsilon \downarrow 0$.

Proof. — $\mathcal{A}$ is self adjoint (cf. the proof of Corollary (2.1)). So apply the spectral theorem to $\mathcal{A}$ as applied to S in part *c*) of the proof of Theorem 2.1. Note in particular that the estimate corresponding to (2.19) gives $|w_\varepsilon - u|_0 \leqslant \varepsilon^\tau|\mathcal{A}^\tau u|_0$ when $u \in D(\mathcal{A}^\tau)$, $0 \leqslant \tau \leqslant 1$.

The next theorem gives an improvement of the estimates obtained in the proof of case ii) of Theorem 2.1.

THEOREM 2.2. — *Assume the hypotheses of Theorem* 2.1. *Let* $C = c/\delta$. *Then one has:*

i) *if* $u \in D(\mathcal{A})$ *and* $\|L_\varepsilon - L\| \leqslant K\varepsilon$ *for* $0 < \varepsilon \leqslant \varepsilon_0$, *then*

$$|u_\varepsilon - u|_0 \leqslant [(K/\delta) + C|\mathcal{A}u|_0]\varepsilon \qquad \textit{for} \qquad 0 < \varepsilon \leqslant \varepsilon_0;$$

ii) *if for fixed* $\tau \in [0,1)$, $u \in V_\tau$, *and for* $0 < \varepsilon \leqslant \varepsilon_0$, $\|L_\varepsilon - L\| \leqslant K\varepsilon^\tau\eta(\varepsilon)$, $0 \leqslant \eta(\varepsilon) \leqslant 1$, $\eta(\varepsilon) \to 0$ *as* $\varepsilon \downarrow 0$, *then for* $0 < \varepsilon \leqslant \varepsilon_0$

$$|u_\varepsilon - u|_0 \leqslant [(K/\delta)\eta(\varepsilon) + C^\tau(C+1)^{1-\tau}|u|_\tau\nu(\varepsilon, \tau, u)]\varepsilon^\tau$$

where $0 \leqslant \nu(\varepsilon, \tau, u) \leqslant 1$ *and* $\nu(\varepsilon, \tau, u) \to 0$ *as* $\varepsilon \downarrow 0$.

Remark. — An examination of the proof of part ii) of Theorem 2.1. gives

$$|u_\varepsilon - u|_0 \leqslant [(K/\delta)\eta(\varepsilon) + (2C+1)|S^\tau u|_0\nu(\varepsilon, \tau, u)]\varepsilon^\tau.$$

However, it is easily verified that $2C + 1 > C^\tau(C+1)^{1-\tau}$. So the bound obtained in Theorem 2.2 is sharper than that obtained from the proof of Theorem 2.1 alone.

Proof. — i) follows from (2.7) and (2.18) of the proof of Theorem 2.1. Under the hypotheses of ii), it follows from (2.7) that

$$(2.22) \qquad |u_\varepsilon - u|_0 \leqslant (K/\delta)\varepsilon^\tau\eta(\varepsilon) + |w_\varepsilon - u|_0.$$

The appropriate bound for $|w_\varepsilon - u|_0$ will now be obtained by use of Proposition 1.5, the quadratic interpolation theorem.

First consider $(\varepsilon\mathcal{A} + I)^{-1} - I$ as a mapping of $V_1 = D(\mathcal{A})$ into V_0. Then

$$|[(\varepsilon\mathcal{A} + I)^{-1} - I]u|_0 = |\varepsilon(\varepsilon\mathcal{A} + I)^{-1}\mathcal{A}u|_0 \leqslant \varepsilon C|\mathcal{A}u|_0 \leqslant \varepsilon C|u|_1$$

as in the derivation of (2.14). So $(\varepsilon\mathcal{A} + I)^{-1} - I$ is continuous from V_1 into V_0 with bound $\leqslant \varepsilon C$.

Now consider $(\varepsilon\mathcal{A} + I)^{-1} - I$ as a mapping of V_0 into V_0. Then

$$|[(\varepsilon\mathcal{A} + I)^{-1} - I]u|_0 \leqslant |(\varepsilon\mathcal{A} + I)^{-1}u|_0 + |u|_0 \leqslant (C+1)|u|_0$$

again as in the derivation of (2.14). Thus the bound of $(\varepsilon\mathcal{A} + I)^{-1} - I$ as an operator in V_0 is $\leqslant C + 1$. So by quadratic interpolation $(\varepsilon\mathcal{A} + I)^{-1} - I$ is a continuous mapping of V_τ into V_0 with bound $\leqslant (\varepsilon C)^\tau (C + I)^{1-\tau}$. Hence, for $u \in V_\tau$,

$$(2.23)\quad |w_\varepsilon - u|_0 = |[(\varepsilon\mathcal{A} + I)^{-1} - I]u|_0 \leqslant \varepsilon^\tau C^\tau (C + 1)^{1-\tau} |u|_\tau;$$

ii) now follows from (2.22), (2.23) and the fact that by Theorem 2.1, $|w_\varepsilon - u|_0 = o(\varepsilon^\tau)$.

THEOREM 2.3. — *Assume hypotheses* (2.1) *through* (2.5.). *Let* $D(\mathcal{A}) \supset D(\mathcal{A}^*)$ *where* $\mathcal{A}^*$ *is the adjoint of* $\mathcal{A}$ *in* V_0. *Then one has : if for fixed* $\tau \in (0,1]$, $u \in V_\tau$, *then for any* γ *such that* $0 < \gamma \leqslant \tau$,

$$|w_\varepsilon - u|_\gamma = o(\varepsilon^{\tau-\gamma}) \qquad \text{as} \qquad \varepsilon \downarrow 0.$$

Remark. — Theorem 2.3 will be proven by a technique similar to that used in proving Theorem 2.1, i.e. by looking at an associated problem with a self adjoint operator and then employing the spectral theorem. Theorem 4.3 below gives conditions under which the conclusion of Theorem 2.3 holds for $|u_\varepsilon - u|_\gamma$.

Proof. — As in part *b*) of the proof of Theorem 2.1, let Λ be the operator in V_0 associated with $(w, v)_1$, i.e. $(w, v)_1 = (\Lambda w, v)_0$ for all $v \in V_1$ and let $S = \Lambda^{1/2}$. Letting x_ε be the unique solution of $\varepsilon S x_\varepsilon + x_\varepsilon = u$, one has,

$$x_\varepsilon = (\varepsilon S + I)^{-1}u, \; x_\varepsilon - u = [(\varepsilon S + I)^{-1} - I]u = - \varepsilon S(\varepsilon S + I)^{-1}u,$$

and since w_ε satisfies $\varepsilon\mathcal{A}w_\varepsilon + w_\varepsilon = u$,

$$w_\varepsilon = (\varepsilon\mathcal{A} + I)^{-1}u, \; w_\varepsilon - u = [(\varepsilon\mathcal{A} + I)^{-1} - I]u$$
$$= - \varepsilon\mathcal{A}(\varepsilon\mathcal{A} + I)^{-1}u.$$

Then since S has a bounded inverse on V_0,

$$w_\varepsilon - u = \mathcal{A}(\varepsilon\mathcal{A} + I)^{-1}(\varepsilon S + I)S^{-1}(x_\varepsilon - u).$$

In part *b*) of the proof of Theorem 2.1 it was proven that

$$(2.24)\qquad \mathcal{A}(\varepsilon\mathcal{A} + I)^{-1}(\varepsilon S + I)S^{-1} \in \mathcal{L}(V_0, V_0)$$

with bound $\leqslant 1 + 2(c/\delta)$ (cf. (2.16)). In order to estimate

$|w_\varepsilon - u|_\gamma$ in terms of $|x_\varepsilon - u|_\gamma$ it is sufficient to prove that

$$\mathcal{A}(\varepsilon\mathcal{A} + I)^{-1}(\varepsilon S + I)S^{-1} \in \mathcal{L}(V_1, V_1) \tag{2.25}$$

with bound independent of ε. For, if (2.25) is proven, then it follows from (2.24) and the quadratic interpolation theorem (Proposition 1.5) that for any $\gamma \in (0,1)$

$$\mathcal{A}(\varepsilon\mathcal{A} + I)^{-1}(\varepsilon S + I)S^{-1} \in \mathcal{L}(V_\gamma, V_\gamma)$$

with bound independent of ε. Hence for $u \in V_\gamma$, $0 \leqslant \gamma \leqslant 1$,

$$|w_\varepsilon - u|_\gamma \leqslant M_\gamma |x_\varepsilon - u|_\gamma. \tag{2.26}$$

For this purpose, let $y \in V_1 = D(\mathcal{A})$ and $Sy = v$. Then, $S^{-1}y = S^{-2}v = (\mathcal{A}^*\mathcal{A} + I)^{-1}v$. Hence $S^{-1}y \in D(\mathcal{A}^*\mathcal{A})$ which implies that $\mathcal{A}S^{-1}y \in D(\mathcal{A}^*)$. By hypothesis $D(\mathcal{A}) \supset D(\mathcal{A}^*)$, so $\mathcal{A}S^{-1}y \in D(\mathcal{A})$. Therefore,

$$\begin{aligned}\mathcal{A}^2(\varepsilon\mathcal{A} + I)^{-1}&(\varepsilon S + I)S^{-1}y \\ &= \varepsilon\mathcal{A}(\varepsilon\mathcal{A} + I)^{-1}\mathcal{A}y + (\varepsilon\mathcal{A} + I)^{-1}\mathcal{A}^2S^{-1}y \\ &= [I - (\varepsilon\mathcal{A} + I)^{-1}]\mathcal{A}y + (\varepsilon\mathcal{A} + I)^{-1}\mathcal{A}^2S^{-1}y\end{aligned}$$

and hence (2.14) yields,

$$\begin{aligned}|\mathcal{A}^2(\varepsilon\mathcal{A} + I)^{-1}&(\varepsilon S + I)S^{-1}y|_0 \\ &\leqslant |\mathcal{A}y|_0 + (c/\delta)|\mathcal{A}y|_0 + (c/\delta)|\mathcal{A}^2S^{-1}y|_0 \\ &\leqslant [1 + (c/\delta)]|y|_1 + (c/\delta)|\mathcal{A}S^{-1}y|_1.\end{aligned} \tag{2.27}$$

Now, $\mathcal{A}S^{-1}$ is a closed operator on V_1, for, suppose $\{y_n\}$ is a sequence of elements of V_1 such that $y_n \to y$ in V_1 and $\mathcal{A}S^{-1}y_n \to z$ in V_1. Then $y \in D(\mathcal{A}S^{-1})$ and since $\mathcal{A} \in \mathcal{L}(V_1, V_0)$ and $S^{-1} \in \mathcal{L}(V_0, V_1)$, $\mathcal{A}S^{-1}y_n \to \mathcal{A}S^{-1}y$ in V_0. Necessarily $\mathcal{A}S^{-1}y = z$ since $V_1 \underset{c}{\subseteq} V_0$. So $\mathcal{A}S^{-1}$ is a closed operator on V_1 and the closed graph theorem gives $\mathcal{A}S^{-1} \in \mathcal{L}(V_1, V_1)$. Hence,

$$|\mathcal{A}^2(\varepsilon\mathcal{A} + I)^{-1}(\varepsilon S + I)S^{-1}y|_0 \leqslant M|y|_1 \tag{2.28}$$

for all $y \in V_1$.

Since $|v|_1^2 = |v|_0^2 + |\mathcal{A}v|_0^2$, (2.24) and (2.27) give

$$|\mathcal{A}(\varepsilon\mathcal{A} + I)^{-1}(\varepsilon S + I)S^{-1}y|_1 \leqslant M'|y|_1 \qquad \text{for all} \qquad y \in V_1,$$

which proves (2.25) and thus (2.26).

Now let $\tau \in (0,1]$, $u \in V_\tau = D(S^\tau)$, and $\gamma \in (0, \tau]$. By (2.26) it remains to show that $|x_\varepsilon - u|_\gamma = o(\varepsilon^{(\tau-\gamma)})$ as $\varepsilon \downarrow 0$. Let E be the resolution of the identity for the self adjoint operator S. Then

$$
\begin{aligned}
(2.29)\quad & |x_\varepsilon - u|_\gamma^2 \\
&= |S^\gamma[(\varepsilon S + I)^{-1} - I]u|_0^2 \\
&= \int_0^\infty \lambda^{2\gamma}\left[\frac{1}{\varepsilon\lambda + 1} - 1\right]^2 (E(d\lambda)u, u)_0 \\
&= \int_0^\infty \frac{\varepsilon^2\lambda^{2+2\gamma}}{(\varepsilon\lambda + 1)^2}(E(d\lambda)u, u)_0 \\
&= \varepsilon^{2(\tau-\gamma)}\int_0^\infty \lambda^{2\tau}.\frac{(\varepsilon\lambda)^{2+2(\gamma-\tau)}}{(\varepsilon\lambda + 1)^{2+2(\gamma-\tau)}}.\frac{1}{(\varepsilon\lambda + 1)^{2(\tau-\gamma)}}(E(d\lambda)u, u)_0.
\end{aligned}
$$

Since $0 < \gamma \leqslant \tau \leqslant 1$ implies that $0 \geqslant \gamma - \tau > -1$, (2.29) and an application of the dominated convergence theorem as in part *c*) of the proof of Theorem 2.1 yield

$$(2.30)\qquad |x_\varepsilon - u|_\gamma^2 = o(\varepsilon^{2(\tau-\gamma)}) \qquad \text{as} \qquad \varepsilon \downarrow 0.$$

The theorem now follows from (2.26) and (2.30).

THEOREM 2.4. — *Assume hypotheses* (2.1) *through* (2.5). *Let* $\mathcal{A}$ *be a normal operator in* V_0 *and let* $C = c/\delta$. *Then one has: if for fixed* $\tau \in (0,1]$, $u \in V_\tau$, *then for* $0 < \varepsilon \leqslant \varepsilon_0$ *and* $0 < \gamma \leqslant \tau$,

$$|w_\varepsilon - u|_\gamma \leqslant 2^{(\gamma/2)}(1 + 2C)|u|_\tau \varepsilon^{(\tau-\gamma)}\omega(\varepsilon, \tau - \gamma, u)$$

where $0 \leqslant \omega(\varepsilon, \tau - \gamma, u) \leqslant 1$ *and* $\omega(\varepsilon, \tau - \gamma, u) \to 0$ *as* $\varepsilon \downarrow 0$.

Proof. — Referring to the proof of Theorem 2.3,

$$w_\varepsilon - u = \mathcal{A}(\varepsilon\mathcal{A} + I)^{-1}(\varepsilon S + I)S^{-1}(x_\varepsilon - u)$$

where $S = (\mathcal{A}^*\mathcal{A} + I)^{1/2}$ and x_ε satisfies $\varepsilon S x_\varepsilon + x_\varepsilon = u$. Recall (2.24), i.e. that

$$\mathcal{A}(\varepsilon\mathcal{A} + I)^{-1}(\varepsilon S + I)S^{-1} \in \mathcal{L}(V_0, V_0) \quad \text{with bound} \ \leqslant 1 + 2C.$$

Since $\mathcal{A}$ is normal, $D(\mathcal{A}) = D(\mathcal{A}^*)$, and so (2.27) of the proof of Theorem 2.3 holds, i.e. for $y \in V_1 = D(\mathcal{A})$,

$$|\mathcal{A}^2(\varepsilon\mathcal{A} + I)^{-1}(\varepsilon S + I)S^{-1}y|_0 \leqslant (1 + C)|y|_1 + C|\mathcal{A}S^{-1}y|_1.$$

Furthermore, $\mathcal{A}\mathcal{A}^*\mathcal{A} = \mathcal{A}^*\mathcal{A}^2$, which implies that

$$\mathcal{A}S^{-1}y = S^{-1}\mathcal{A}y,$$

and since $|y|_1 = |Sy|_0$,

$$|\mathcal{A}^2(\varepsilon\mathcal{A} + I)^{-1}(\varepsilon S + I)S^{-1}y| \leqslant (1 + C)|y|_1 + C|\mathcal{A}y|_0 \leqslant (1 + 2C)|y|_1.$$

But $|v|_1^2 = |v|_0^2 + |\mathcal{A}v|_0^2$ and so,

$$\mathcal{A}(\varepsilon\mathcal{A} + I)^{-1}(\varepsilon S + I)S^{-1} \in \mathcal{L}(V_1, V_1)$$

with bound $\leqslant 2^{1/2}(1 + 2C)$.
Thus by the quadratic interpolation theorem (Proposition 1.5) it follows that for any $\gamma \in (0,1)$,

$$\mathcal{A}(\varepsilon\mathcal{A} + I)^{-1}(\varepsilon S + I)S^{-1} \in \mathcal{L}(V_\gamma, V_\gamma)$$

with bound $\leqslant 2^{\gamma/2}(1 + 2C)$.
Hence for $u \in V_\gamma$, $0 \leqslant \gamma \leqslant 1$,

$$|w_\varepsilon - u|_\gamma \leqslant 2^{\gamma/2}(1 + 2C)|x_\varepsilon - u|_\gamma.$$

The theorem now follows from (2.29) of the proof of Theorem 2.3.

Example 2.1. — Let $V_0 = l^2$ and let V be the Hilbert space of all sequences $v = \{\nu_n\}$ of complex numbers such that $\sum_{n=1}^{\infty} n^n|\nu_n|^2 < \infty$ with $(w, v)_V = \sum_{n=1}^{\infty} n^n\omega_n\bar{\nu}_n$ where $w = \{\omega_n\}$. Let $a(w, v) = (w, v)_V$ and $b(w, v) = (w, v)_0 = \sum_{n=1}^{\infty} \omega_n\bar{\nu}_n$. Then $\mathcal{A} = A$ is the operator given by $\mathcal{A}v = \{n^n\nu_n\}$ on

$$D(\mathcal{A}) = \{v = \{\nu_n\} : \{n^n\nu_n\} \in l^2\},$$

i.e. $v \in D(\mathcal{A})$ if and only if $\sum_{n=1}^{\infty} n^{2n}|\nu_n|^2 < \infty$. Now, for fixed $\tau \in [0,1]$, let $u = \{\xi_n\}$ be such that $u \in D(\mathcal{A}^\tau)$ but $u \notin D(\mathcal{A}^\beta)$ for any $\beta > \tau$ (e.g. $u = \{1/n^{\tau n+1}\}$). The solution w_ε of $\varepsilon\mathcal{A}w_\varepsilon + w_\varepsilon = u$ is given by $w_\varepsilon = \{\xi_n/(\varepsilon n^n + 1)\}$. Let $\gamma \in [0, \tau]$, $\alpha > \tau - \gamma$, and consider

$$\frac{1}{\varepsilon^{2\alpha}}|\mathcal{A}^\gamma(w_\varepsilon - u)|_0^2 = \frac{1}{\varepsilon^{2\alpha}}\sum_{n=1}^{\infty}|\xi_n|^2\frac{\varepsilon^2n^{2n}n^{2\gamma n}}{(\varepsilon n^n + 1)^2}.$$

Let $\varepsilon_m = 1/m^m$ and assume that this series, with ε replaced by ε_m, is bounded, by M say, as $m \to \infty$. Then from the m^{th} term one has

$$|\xi_m|^2m^{2(\alpha+\gamma)m}/4 \leqslant M.$$

Multiplying by $m^{(\tau-\alpha-\gamma)m}$, one gets

$$|\xi_m|^2 m^{(\tau+\alpha+\gamma)m} \leqslant 4\mathrm{M} m^{(\tau-\alpha-\gamma)m}$$

so

$$\sum_{m=1}^{\infty} m^{(\tau+\alpha+\gamma)m}|\xi_m|^2 \leqslant 4\mathrm{M} \sum_{m=1}^{\infty} m^{(\tau-\alpha-\gamma)m} < \infty .$$

But this is a contradiction since $\tau + \alpha + \gamma > 2\tau$ and by hypothesis $u \notin \mathrm{D}(\mathcal{A}^{(\tau+\alpha+\gamma)/2})$, i.e. $\sum_{m=1}^{\infty} m^{(\tau+\alpha+\gamma)m}|\xi_m|^2 = \infty$. Thus for every $\alpha > \tau - \gamma$, $(1/\varepsilon^{\alpha})|\mathcal{A}^{\gamma}(w_\varepsilon - u)|_0$ is unbounded as $\varepsilon \downarrow 0$. Since $|\mathcal{A}^{\gamma} v|_0$ and $|\mathrm{S}^{\gamma} v|_0 = |v|_{\gamma}$ are equivalent norms on $\mathrm{D}(\mathcal{A}^{\gamma}) = \mathrm{D}(\mathrm{S}^{\gamma}) = \mathrm{V}_{\gamma}$ this shows that, under the assumed hypotheses, the powers of ε cannot be improved on in the preceding theorems and corollaries.

The next two examples show that if $0 < \beta \leqslant 1$, $u \in \mathrm{V}_{\tau}$ for all $\tau < \beta$, but $u \notin \mathrm{V}_{\beta}$, then it may or may not be true that $|w_\varepsilon - u|_{\gamma} = 0(\varepsilon^{\beta-\gamma})$ where $0 \leqslant \gamma \leqslant \tau$ and $\varepsilon\mathcal{A}w_\varepsilon + w_\varepsilon = u$.

Example 2.2. — Let V, V_0 and $\mathcal{A}$ be the same as in Example 2.1. Choices of u and β, $0 < \beta \leqslant 1$, will be made such that $u \in \mathrm{D}(\mathcal{A}^{\tau})$ for all $\tau < \beta$, $u \notin \mathrm{D}(\mathcal{A}^{\beta})$, and for all $\gamma \in [0, \tau]$, $(1/\varepsilon^{\beta-\gamma})|w_\varepsilon - u|_{\gamma}$ is unbounded as $\varepsilon \downarrow 0$ where $\varepsilon\mathcal{A}w_\varepsilon + w_\varepsilon = u$. Let $u = \{n/n^{n/2}\}$. Then

$$|\mathcal{A}^{\tau}u|_0^2 = \sum_{n=1}^{\infty} \frac{n^{2\tau n} n^2}{n^n} \quad \begin{cases} < \infty, \ \tau < 1/2 \\ = \infty, \ \tau \geqslant 1/2. \end{cases}$$

So let $\beta = 1/2$ and $0 \leqslant \gamma < 1/2$. Then

$$\frac{1}{\varepsilon^{1-2\gamma}}|\mathcal{A}^{\gamma}(w_\varepsilon - u)|_0^2 = \frac{1}{\varepsilon^{1-2\gamma}} \sum_{n=1}^{\infty} \frac{n^2}{n^n} \frac{\varepsilon^2 n^{2n} n^{2\gamma n}}{(\varepsilon n^n + 1)^2} = \sum_{n=1}^{\infty} n^2 \frac{\varepsilon^{1+2\gamma} n^{(1+2\gamma)n}}{(\varepsilon n^n + 1)^2}.$$

For $\varepsilon_m = 1/m^m$, the m^{th} term of this series is $m^2/4$. Thus $(1/\varepsilon^{(1-2\gamma)/2})|w_\varepsilon - u|_{\gamma}$ is unbounded as $\varepsilon \downarrow 0$.

Example 2.3. — Let $\mathrm{V}_0 = l^2$ and let V be the Hilbert space of all sequences $v = \{\nu_n\}$ of complex numbers such that $\sum_{n=1}^{\infty} n|\nu_n|^2 < \infty$ with $(w, v)_{\mathrm{V}} = \sum_{n=1}^{\infty} n\omega_n\bar{\nu}_n$ where $w = \{\omega_n\}$. Let $a(w, v) = (w, v)_{\mathrm{V}}$ and $b(w, v) = (w, v)_0$. Then $\mathcal{A}$ is the operator given by $\mathcal{A}v = \{n\nu_n\}$ on $\mathrm{D}(\mathcal{A}) = \{v = \{\nu_n\} :$

$\{n\nu_n\} \in l^2\}$, i.e. $v \in D(\mathcal{A})$ if and only if $\sum_{n=1}^{\infty} n^2|\nu_n|^2 < \infty$. Choices of u and β, $0 < \beta \leqslant 1$, will be made such that $u \in D(\mathcal{A}^\tau)$ for all $\tau < \beta$, $u \notin D(\mathcal{A}^\beta)$, and for all $\gamma \in [0, \tau]$,

$$|w_\varepsilon - u|_\gamma = O(\varepsilon^{\beta-\gamma})$$

where $\varepsilon\mathcal{A}w_\varepsilon + w_\varepsilon = u$. Let $u = \{1/n\}$. Then

$$|\mathcal{A}^\tau u|_0^2 = \sum_{n=1}^{\infty} \frac{n^{2\tau}}{n^2} \quad \begin{cases} < \infty, & \tau < 1/2 \\ = \infty, & \tau \leqslant 1/2. \end{cases}$$

So let $\beta = 1/2$ and $0 \leqslant \gamma < 1/2$. Then

$$|\mathcal{A}^\gamma(w_\varepsilon - u)|_0^2 = \sum_{n=1}^{\infty} \frac{\varepsilon^2 n^{2\gamma}}{(\varepsilon n + 1)^2}.$$

Now, for $\gamma = 0$,

$$\sum_{n=1}^{\infty} \frac{\varepsilon^2}{(\varepsilon n + 1)^2} \leqslant \int_0^\infty \frac{\varepsilon^2}{(\varepsilon x + 1)^2}\, dx$$

while for $0 < \gamma < 1/2$ an elementary calculus argument shows that for $0 < \varepsilon \leqslant \gamma(1-\gamma)^{-1}$,

$$\sum_{n=1}^{\infty} \frac{\varepsilon^2 n^{2\gamma}}{(\varepsilon n + 1)^2} \leqslant 2\int_1^\infty \frac{\varepsilon^2 x^{2\gamma}}{(\varepsilon x + 1)^2}\, dx.$$

Thus for $0 \leqslant \gamma < 1/2$ and $0 < \varepsilon \leqslant \gamma(1-\gamma)^{-1}$,

$$|\mathcal{A}^\gamma(w_\varepsilon - u)|_0^2 \leqslant 2\int_0^\infty \frac{\varepsilon^2 x^{2\gamma}}{(\varepsilon x + 1)^2}\, dx.$$

Under the transformation $\varepsilon x = y$ the right hand side becomes

$$2\varepsilon^{1-2\gamma} \int_0^\infty \frac{y^{2\gamma}}{(y+1)^2}\, dy.$$

Hence $|w_\varepsilon - u|_\gamma = O(\varepsilon^{(1-2\gamma)/2})$ as $\varepsilon \downarrow 0$.

3. An Asymptotic Expansion.

Assume hypotheses (2.1) through (2.3). Let $\mathcal{A}$ be the operator in V_0 associated with $a(u, v)$ relative to $b(u, v)$ and denote the Hilbert space $D(\mathcal{A})$, provided with the graph norm $(|v|_0^2 + |\mathcal{A}v|_0^2)^{1/2}$, by V_1. Then $(u, v)_1 = (Su, Sv)_0$

where $S = (\mathcal{A}^*\mathcal{A} + I)^{1/2}$. For any $\tau \in [0, \infty)$ let V_τ be the Hilbert space $D(S^\tau)$ with inner product $(u, v)_\tau = (S^\tau u, S^\tau v)_0$. For $v \in V_0$ and $\tau \in [0, \infty)$ let

$$|v|_{-\tau} = \sup \{|(v, w)_0| : w \in V_\tau \quad \text{and} \quad |w|_\tau \leqslant 1\}.$$

Then $|v|_{-\tau} = |S^{-\tau}v|_0$. Let $V_{-\tau}$ be the completion of V_0 in the norm $|v|_{-\tau}$. This defines the Hilbert space V_τ for all real τ and for $\sigma < \tau$, $V_\tau \underset{c}{\subset} V_\sigma$ with V_τ dense in V_σ. Furthermore, employing extension by continuity, for all real γ, τ, S^γ is an isometric isomorphism of V_τ onto $V_{\tau-\gamma}$.

Now assume

$$(3.1) \qquad (\mathcal{A}v, v)_0 \geqslant 0 \qquad \text{for all} \qquad v \in V_1 = D(\mathcal{A}).$$

Then as noted in the proof of Corollary 2.1, $\mathcal{A}$ is a self adjoint operator in V_0 and so $S = (\mathcal{A}^2 + I)^{1/2}$. It now follows that $\mathcal{A}$ can also be extended by continuity so that for any real τ, $\mathcal{A}$ is a continuous linear mapping of V_τ into $V_{\tau-1}$. Furthermore, for any $\sigma \geqslant 0$ and any real γ, τ,

$$(3.2) \qquad S^\gamma \mathcal{A}^\sigma = \mathcal{A}^\sigma S^\gamma$$

is a continuous linear mapping of V_τ into $V_{\tau-\sigma-\gamma}$.

The « negative norms » defined above will now be used to obtain an asymptotic expansion for w_ε, the solution of (2.5), in terms of u, the solution of (2.4). Extension by continuity will be understood wherever necessary in the statement and proof of Theorem 3.1.

THEOREM 3.1. — *Assume hypotheses* (2.1) *through* (2.5) *and* (3.1). *Let n be a non-negative integer and let $u \in V_\tau$ where $0 \leqslant \tau \leqslant n$. Then for $0 < \varepsilon \leqslant \varepsilon_0$,*

$$\left| w_\varepsilon - \sum_{k=0}^{n} (-1)^k \varepsilon^k \mathcal{A}^k u \right|_{\tau-n} = o(\varepsilon^n) \qquad as \qquad \varepsilon \downarrow 0.$$

Proof. — For $n = 0$, (2.11) gives

$$w_\varepsilon = u - \varepsilon\mathcal{A}(\varepsilon\mathcal{A} + I)^{-1}u.$$

If for $n = m$,

$$w_\varepsilon = \sum_{k=0}^{m} (-1)^k \varepsilon^k \mathcal{A}^k u + (-1)^{m+1}\varepsilon^{m+1}\mathcal{A}(\varepsilon\mathcal{A} + 1)^{-1}\mathcal{A}^m u,$$

then

$$\begin{aligned} w_\varepsilon - \sum_{k=0}^{m} (-1)^k \varepsilon^k \mathcal{A}^k u &= (-1)^{m+1}\varepsilon^{m+1}(\varepsilon\mathcal{A} + \mathrm{I})^{-1}\mathcal{A}^{m+1}u \\ &= (-1)^{m+1}\varepsilon^{m+1}(\varepsilon\mathcal{A} + \mathrm{I} - \varepsilon\mathcal{A})(\varepsilon\mathcal{A} + \mathrm{I})^{-1}\mathcal{A}^{m+1}u \\ &= (-1)^{m+1}\varepsilon^{m+1}\mathcal{A}^{m+1}u + (-1)^{m+2}\varepsilon^{m+2}\mathcal{A}(\varepsilon\mathcal{A} + \mathrm{I})^{-1}\mathcal{A}^{m+1}u. \end{aligned}$$

Thus for any non-negative integer n,

$$w_\varepsilon = \sum_{k=0}^{n} (-1)^k \varepsilon^k \mathcal{A}^k u + (-1)^{n+1}\varepsilon^{n+1}\mathcal{A}(\varepsilon\mathcal{A} + \mathrm{I})^{-1}\mathcal{A}^n u,$$

Since $u \in \mathrm{V}_\tau$, (3.2) gives $\mathrm{S}^{\tau-n}\mathcal{A}^n u \in \mathrm{V}_0$. So, letting $z = \mathrm{S}^{\tau-n}\mathcal{A}^n u$ and letting z_ε be the unique solution in $\mathrm{V}_1 = \mathrm{D}(\mathcal{A})$ of $\varepsilon\mathcal{A}z_\varepsilon + z_\varepsilon = z$, conclusion ii) of Theorem 2.1 yields

$$|z_\varepsilon - z|_0 = |\varepsilon\mathcal{A}(\varepsilon\mathcal{A} + \mathrm{I})^{-1}z|_0 = o(1) \qquad \text{as} \qquad \varepsilon \downarrow 0.$$

Thus

$$\begin{aligned} \Big|w_\varepsilon - \sum_{k=0}^{n} (-1)^k \varepsilon^k \mathcal{A}^k u\Big|_{\tau-n} &= \varepsilon^{n+1}|\mathrm{S}^{\tau-n}\mathcal{A}(\varepsilon\mathcal{A} + \mathrm{I})^{-1}\mathcal{A}^n u|_0 \\ &= \varepsilon^n|\varepsilon\mathcal{A}(\varepsilon\mathcal{A} + \mathrm{I})^{-1}\mathrm{S}^{\tau-n}\mathcal{A}^n u|_0 \\ &= o(\varepsilon^n) \text{ as } \varepsilon \downarrow 0. \end{aligned}$$

Example 3.1. — Let V, V_0 and $\mathcal{A}$ be as in Example 2.1. Then for any real τ, $\mathrm{V}_\tau = \{v = \{v_n\} : \{n^{\tau n}v_n\} \in l^2\}$, i.e. $v \in \mathrm{V}_\tau$ if and only if $\sum_{n=1}^{\infty} n^{2\tau n}|v_n|^2 < \infty$. The same method as employed in Example 2.1 shows that the estimate obtained in Theorem 3.1 is sharp.

4. Hilbert Space Framework for Singular Perturbation of Elliptic Boundary Value Problems.

Let V, V_0 and H be Hilbert spaces with

(4.1) $\mathrm{V} \underset{c}{\subset} \mathrm{V}_0 \underset{c}{\subset} \mathrm{H}$, V dense in V_0, and V dense in H.

Denote the norms and inner products in V and V_0 as before and let $|v|_\mathrm{H}$, $(v, w)_\mathrm{H}$ be the norm and inner product in H

respectively. As previously let $a(v, w)$ be a continuous Hermitian bilinear form on V and let $b(v, w)$ be a continuous Hermitian bilinear form on V_0 with upper bound c. Further assume the coerciveness inequalities (2.2) and (2.3), i.e. there exists $\beta > 0$ such that $|b(v, v)| \geqslant \beta |v|_0^2$ for all $v \in V_0$, and for $0 < \varepsilon \leqslant \varepsilon_0$ there exist $\alpha(\varepsilon) > 0$, $\alpha(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$, and $\delta > 0$ such that

$$|\varepsilon a(v, v) + b(v, v)| \geqslant \alpha(\varepsilon)|v|_V^2 + \delta |v|_0^2$$

for all $v \in V$, respectively.

For $f \in H$, the anti-linear functional $v \to (f, v)_H$, $v \in H$, is continuous from H to C and thus its restriction to V_0 is in V_0^*. So let f, f_ε be given in H, $0 < \varepsilon \leqslant \varepsilon_0$. Let u be the unique solution in V_0 of

$$(4.2) \qquad b(u, v) = (f, v)_H \qquad \text{for all} \qquad v \in V_0$$

and for each $\varepsilon \in (0, \varepsilon_0]$, let u_ε be the unique solution in V of

$$(4.3) \qquad \varepsilon a(u_\varepsilon, v) + b(u_\varepsilon, v) = (f_\varepsilon, v)_H \quad \text{for all} \quad v \in V.$$

Theorem 2.1 will now be reformulated in the present context.

THEOREM 4.1. — *Assume hypotheses* (4.1) *through* (4.3), (2.2) *and* (2.3). *Let* $\mathfrak{A}$ *be the operator in* V_0 *associated with* $a(v, w)$ *relative to* $b(v, w)$. *Consider the interpolation spaces* V_τ, $0 \leqslant \tau \leqslant 1$, *obtained by quadratic interpolation between* $V_1 = D(\mathfrak{A})$ *and* V_0. *Then one has:*

i) *if* $u \in D(\mathfrak{A})$ *and* $|f_\varepsilon - f|_H = 0(\varepsilon)$ *as* $\varepsilon \downarrow 0$, *then*

$$|u_\varepsilon - u|_0 = 0(\varepsilon) \quad as \quad \varepsilon \downarrow 0;$$

ii) *if, for fixed* $\tau \in [0,1)$, $u \in V_\tau$ *and* $|f_\varepsilon - f|_H = o(\varepsilon^\tau)$ *as* $\varepsilon \downarrow 0$, *then*

$$|u_\varepsilon - u|_0 = o(\varepsilon^\tau) \quad as \quad \varepsilon \downarrow 0.$$

Proof. — It is sufficient to carry out the reduction corresponding to part *a*) of the proof of Theorem 2.1. The rest of the proof then follows word for word as in parts *b*) and *c*) of the proof of Theorem 2.1. So let $L(v) = (f, v)_H$ for all $v \in H$ and $L_\varepsilon(v) = (f_\varepsilon, v)_H$ for all $v \in H$. Then the restrictions of L and L_ε to V_0 are in V_0^*. By (4.1) there exists

$K > 0$ such that $|v|_H \leqslant K|v|_0$ for all $v \in V_0$, and V_0 is dense in H. Thus for the norm of $L_\varepsilon - L$ in V_0^* one has

$$\begin{aligned}||L_\varepsilon - L|| &= \sup\{|(L_\varepsilon - L)(v)| : v \in V_0 \quad \text{and} \quad |v|_0 \leqslant 1\}\\ &\leqq K \sup\{|(L_\varepsilon - L)(v)| : v \in V_0 \quad \text{and} \quad |v|_H \leqslant 1\}\\ &= K|f_\varepsilon - f|_H.\end{aligned}$$

Therefore, letting w_ε be the unique solution in V of $\varepsilon a(w_\varepsilon, v) + b(w_\varepsilon, v) = (f, v)_H = L(v)$ for all $v \in V$, (2.7) gives,

$$|u_\varepsilon - u|_0 \leqslant (K/\delta)|f_\varepsilon - f|_H + |w_\varepsilon - u|_0.$$

The theorem follows.

It is now obvious that Theorem 2.2 can also be reformulated in the present context. One merely replaces $||L_\varepsilon - L||$ by $|f_\varepsilon - f|_H$ and $1/\delta$ by K/δ in the statement of Theorem 2.2.

The elliptic boundary value problems to be considered in Chapter 6 will be of the form given by (4.2) and (4.3). To apply the results of Chapters 2 and 3 to these problems it is necessary to relate the operator equation in H corresponding to (4.3) to the operator equation $\varepsilon\mathcal{A}w_\varepsilon + w_\varepsilon = u$ in V_0. The operator forms of equations (4.2) and (4.3) will now be considered.

So assume hypotheses (4.1), (2.2) and (2.3). Let A_ε be the operator in H associated with $\varepsilon a(v, w) + b(v, w)$ and let $\mathcal{B}$ be the operator in H associated with $b(v, w)$. Then, given $f \in H$, there exists a unique $u \in V_0$ such that

$$(4.4) \qquad b(u, v) = (\mathcal{B}u, v)_H = (f, v)_H \quad \text{for all} \quad v \in V_0,$$

and for $0 < \varepsilon \leqslant \varepsilon_0$ there exists a unique $w_\varepsilon \in V$ such that

$$(4.5) \qquad \varepsilon a(w_\varepsilon, v) + b(w_\varepsilon, v) = (A_\varepsilon w_\varepsilon, v)_H = (f, v)_H$$

for all $v \in V$. Clearly, if the anti-linear functional $v \to \varepsilon a(w, v) + b(w, v)$ is continuous on V in the topology induced by H it is also continuous on V in the topology induced by V_0. Then since by Proposition 1.4, iv), $\mathcal{B}$ has an inverse on H,

$$\begin{aligned}\varepsilon a(w_\varepsilon, v) + b(w_\varepsilon, v) = (A_\varepsilon w_\varepsilon, v)_H = (f, v)_H &= b(\mathcal{B}^{-1}A_\varepsilon w_\varepsilon, v)\\ &= b(u, v) = b((\varepsilon\mathcal{A} + I)w_\varepsilon, v)\end{aligned}$$

for all $v \in V$, where $\mathcal{A}$ is the operator in V_0 associated with

$a(v, w)$ relative to $b(v, w)$. Hence for each $\varepsilon \in (0, \varepsilon_0]$,

$$\mathcal{B}^{-1}A_\varepsilon \subset \varepsilon\mathcal{A} + I, \tag{4.6}$$

i.e., $\mathcal{B}^{-1}A_\varepsilon$ is a restriction of $\varepsilon\mathcal{A} + I$.

The relation (4.6) holds even though $D(\mathcal{B}^{-1}A_\varepsilon) = D(A_\varepsilon)$ may depend on ε while $D(\varepsilon\mathcal{A} + I) = D(\mathcal{A})$ is independent of ε. In the applications to differential problems to be considered in Chapter 6, it will be easier to determine A_ε and $\mathcal{B}$ than to determine $\mathcal{A}$. The following lemma will be helpful in the determination of $D(\mathcal{A})$.

Lemma 4.1. — *For any* $\varepsilon \in (0, \varepsilon_0]$, $D(A_\varepsilon)$ *is dense in* $D(\mathcal{A})$, *where* $D(\mathcal{A})$ *is provided with the graph norm,* $|v|_1 = (|v|_0^2 + |\mathcal{A}v|_0^2)^{1/2}$.

Proof. — Let $\varepsilon \in (0, \varepsilon_0]$. By the proof of Proposition 2.1, iv) and a consequence of the closed graph theorem, $(|v|_0^2 + |\mathcal{A}v|_0^2)^{1/2}$ and $(|v|_0^2 + |(\varepsilon\mathcal{A} + I)v|_0^2)^{1/2}$ are equivalent norms on $D(\mathcal{A})$. The proof will be carried out with the latter norm.

Let $v \in D(\mathcal{A})$ and let g be the unique element of V_0 such that $(\varepsilon\mathcal{A} + I)v = g$. Let $\{g_n\}$ be a sequence of elements of $D(\mathcal{B})$ for which $|g_n - g|_0 \to 0$ as $n \to \infty$. Such a sequence exists according to Proposition 1.4, iii). For each n let v_n be the unique element of $D(A_\varepsilon)$ such that $\mathcal{B}^{-1}A_\varepsilon v_n = g_n$ (cf. Proposition 1.4, iv)). $(\varepsilon\mathcal{A} + I)^{-1}$ is a continuous operator on V_0 and so by (4.6), $(\mathcal{B}^{-1}A_\varepsilon)^{-1} = A_\varepsilon^{-1}\mathcal{B}$ is continuous on its domain in the norm of V_0. Therefore,

$$v_n = A_\varepsilon^{-1}\mathcal{B}g_n = (\varepsilon\mathcal{A} + I)^{-1}g_n \to (\varepsilon\mathcal{A} + I)^{-1}g = v$$

in V_0 as $n \to \infty$ and

$$\mathcal{B}^{-1}A_\varepsilon v_n = (\varepsilon\mathcal{A} + I)v_n = g_n \to g = (\varepsilon\mathcal{A} + I)v$$

in V_0 as $n \to \infty$.

Hence,

$$\begin{aligned}(|v_n - v|_0^2 &+ |\mathcal{B}^{-1}A_\varepsilon v_n - (\varepsilon\mathcal{A} + I)v|_0^2)^{1/2} \\ &= (|v_n - v|_0^2 + |(\varepsilon\mathcal{A} + I)(v_n - v)|_0^2)^{1/2} \to 0 \quad \text{as} \quad n \to \infty,\end{aligned}$$

the desired conclusion.

In order to apply Theorem 2.3 to differential problems, criteria will be needed to establish the hypothesis $D(\mathcal{A}) \supset D(\mathcal{A}^*)$. In the present work this will be accomplished by examining the adjoint problems to (4.4) and (4.5) (cf. (4.10) and (4.11) below).

So consider the adjoint forms to $a(v, w)$ and $b(v, w)$, i.e., the Hermitian bilinear forms $a^*(v, w) = \overline{a(w, v)}$ and $b^*(v, w) = \overline{b(w, v)}$. Then $|b^*(v, w)| = |b(v, w)|$ for all $v, w \in V_0$, so $b^*(v, w)$ has upper bound c and $b^*(v, v)$ satisfies (2.2). Thus by Proposition 1.1, iii) one can define $\mathcal{A}'$ as the operator in V_0 associated with $a^*(v, w)$ relative to $b^*(v, w)$, i.e.,

$$a^*(v, w) = b^*(\mathcal{A}'v, w), \qquad w \in V,$$

with $D(\mathcal{A}') = \{v \in V : w \to a^*(v, w)$ is continuous on V in the topology induced by $V_0\}$. Then since $a^*(v, w)$ is continuous on V and by (2.3), for $0 < \varepsilon \leqslant \varepsilon_0$,

$$|\varepsilon a^*(v, v) + b^*(v, v)| \geqslant \alpha(\varepsilon)|v|_V^2 + \delta|v|_0^2$$

for all $v \in V$, one obtains the following proposition by the same proof as used for Proposition 2.1.

PROPOSITION 4.1. — *Assume hypotheses* (2.1), (2.2) *and* (2.3), *and let* $\mathcal{A}'$ *be the operator in* V_0 *associated with* $a^*(v, w)$ *relative to* $b^*(v, w)$. *Then:*

i) $\mathcal{A}'$ *is closed,*

ii) $D(\mathcal{A}')$ *is dense in* V_0,

iii) $D(\mathcal{A}')$ *is dense in* V,

iv) *for* $0 < \varepsilon \leqslant \varepsilon_0$, $\varepsilon\mathcal{A}' + I$ *is a linear homeomorphism of* $D(\mathcal{A}')$, *provided with the graph norm* $|v|_{D(\mathcal{A}')} = (|v|_0^2 + |\mathcal{A}'v|_0^2)^{1/2}$, *onto* V_0.

Now let B be the operator in V_0 associated with $b(v, w)$. Then by Proposition 1.1, i) B is a linear homeormorphism of V_0 onto V_0. Furthermore the operator B^* in V_0 associated with $b^*(v, w)$ is the adjoint of B in V_0 and is also a linear homeomorphism of V_0 onto V_0. Then for $v \in D(\mathcal{A})$ and $w \in D(\mathcal{A}')$,

$$(\mathcal{A}v, B^*w)_0 = (B\mathcal{A}v, w)_0 = b(\mathcal{A}v, w) = a(v, w),$$

and,

$$(B^*\mathcal{A}'w, v)_0 = b^*(\mathcal{A}'w, v)_0 = a^*(w, v) = \overline{a(v, w)}.$$

Thus $(\mathcal{A}v, B^*w)_0 = (v, B^*\mathcal{A}'w)_0$ and letting $w = B^{*-1}z$ one has that for $v \in D(\mathcal{A})$ and $z \in D(\mathcal{A}'B^{*-1})$,

$$(4.7) \qquad (\mathcal{A}v, z)_0 = (v, B^*\mathcal{A}'B^{*-1}z)_0.$$

PROPOSITION 4.2. — $\mathcal{A}^* = B^*\mathcal{A}'B^{*-1}$.

Proof. — Let $w \in D(\mathcal{A}^*)$. Then the functional $v \to (\mathcal{A}v, w)_0$ is continuous on $D(\mathcal{A})$ in the topology induced by V_0, and $(\mathcal{A}v, w)_0 = (v, \mathcal{A}^*w)_0$. Let $\varepsilon \in (0, \varepsilon_0]$ and let z be the unique solution in $D(\mathcal{A}'B^{*-1})$ of

$$B^*(\varepsilon\mathcal{A}' + I)B^{*-1}z = (\varepsilon B^*\mathcal{A}'B^{*-1} + I)z = (\varepsilon\mathcal{A}^* + I)w.$$

Then by (4.7), for $v \in D(\mathcal{A})$,

$$(v, (\varepsilon\mathcal{A}^* + I)w)_0 = (v, (\varepsilon B^*\mathcal{A}'B^{*-1} + I)z)_0 = ((\varepsilon\mathcal{A} + I)v, z)_0.$$

Since $\varepsilon\mathcal{A} + I$ maps $D(\mathcal{A})$ onto V_0, $z = w$, and since $D(\mathcal{A})$ is dense in V_0, $(\varepsilon\mathcal{A}^* + I)w = (\varepsilon B^*\mathcal{A}'B^{*-1} + I)w$, which implies that $\mathcal{A}^*w = B^*\mathcal{A}'B^{*-1}\, w$. Thus $B^*\mathcal{A}'B^{*-1} \supset \mathcal{A}^*$ and since (4.7) clearly implies the reverse inclusion, the proposition follows.

From Proposition 4.2. it follows that

$$(4.8) \qquad \mathcal{A}' = B^{*-1}\mathcal{A}^*B^*.$$

Note that if (2.2) is strengthened to

$$(4.9) \qquad b(v, v) \geqslant \beta|v|_0^2 \quad \text{for all} \quad v \in V_0, \quad \beta > 0,$$

then $b(v, w)$ is an equivalent inner product to $(v, w)_0$ on V_0 and $\mathcal{A}'$ is the adjoint of $\mathcal{A}$ as an operator on V_0 with $b(v, w)$ as inner product. It will now be proven that $\mathcal{A}'$ satisfies relations corresponding to (4.6) and Lemma 4.1.

Assume hypotheses (4.1), (2.2) and (2.3). Let A_ε^* be the operator in H associated with $\varepsilon a^*(v, w) + b^*(v, w)$ and let $\mathcal{B}^*$ be the operator in H associated with $b^*(v, w)$. Then, as noted in Chapter 1, $\mathcal{B}^*$ is the adjoint of $\mathcal{B}$ in H and for $0 < \varepsilon \leqslant \varepsilon_0$, A_ε^* is the adjoint of A_ε in H. Furthermore, given $f \in \mathrm{H}$, there exists a unique $z \in \mathrm{V}_0$ such that

$$(4.10) \quad b^*(z, v) = (\mathcal{B}^* z, v)_{\mathrm{H}} = (f, v)_{\mathrm{H}} \quad \text{for all } v \in \mathrm{V}_0,$$

and for $0 < \varepsilon \leqslant \varepsilon_0$ there exists a unique $z_\varepsilon \in \mathrm{V}$ such that

$$(4.11) \quad \varepsilon a^*(z_\varepsilon, v) + b^*(z_\varepsilon, v) = (\mathrm{A}_\varepsilon^* z_\varepsilon, v)_{\mathrm{H}} = (f, v)_{\mathrm{H}}$$

for all $v \in \mathrm{V}$. Then by the method used to derive (4.6), one has for $\varepsilon \in (0, \varepsilon_0]$

$$(4.12) \quad \mathcal{B}^{*-1}\mathrm{A}_\varepsilon^* \subset \varepsilon\mathcal{A}' + \mathrm{I}.$$

Also, the argument used to obtain Lemma 4.1 yields the following lemma.

Lemma 4.2. *For any* $\varepsilon \in (0, \varepsilon_0]$, $\mathrm{D}(\mathrm{A}_\varepsilon^*)$ *is dense in* $\mathrm{D}(\mathcal{A}')$ *where* $\mathrm{D}(\mathcal{A}')$ *is provided with the graph norm,* $|v|_{\mathrm{D}(\mathcal{A}')} = (|v|_0^2 + |\mathcal{A}'v|_0^2)^{1/2}$.

In view of (4.12) and Lemma 4.2 it would be convenient to replace the hypothesis $\mathrm{D}(\mathcal{A}) \supset \mathrm{D}(\mathcal{A}^*)$ in Theorem 2.3 by a hypothesis relating $\mathrm{D}(\mathcal{A})$ and $\mathrm{D}(\mathcal{A}')$ in order to apply the theorem to differential problems. This is readily accomplished if one assumes (4.9) in place of (2.2) for then $\mathcal{A}'$ is the adjoint of $\mathcal{A}$ in V_0 with inner product $b(v, w)$.

So assume (4.9) for the remainder of this chapter and let

$$(4.13) \quad [v, w]_0 = b(v, w),\ [v]_0 = \sqrt{b(v, v)} \quad \text{for} \quad v, w \in \mathrm{V}_0.$$

Then $\beta|v|_0^2 \leqslant [v]_0^2 \leqslant c|v|_0^2$ for all $v \in \mathrm{V}_0$ and, letting $[v]_1 = ([v]_0^2 + [\mathcal{A}v]_0^2)^{1/2}$ for $v \in \mathrm{V}_1 = \mathrm{D}(\mathcal{A})$ an application of Proposition 1.5 to the identity mapping yields

$$\beta|v|_\tau^2 \leqslant [v]_\tau^2 \leqq c|v|_\tau^2$$

for the corresponding interpolation norms on V_τ, $0 \leqslant \tau \leqslant 1$. Furthermore (2.3) implies

(4.14) for $0 < \varepsilon \leqq \varepsilon_0$, there exist $\alpha(\varepsilon) > 0$, $\alpha(\varepsilon) \to 0$, as $\varepsilon \downarrow 0$, and $\mu > 0$ such that

$$|\varepsilon a(v, v) + [v, v]_0| \geqslant \alpha(\varepsilon)|v|_{\mathrm{V}}^2 + \mu[v]_0^2 \quad \text{for all} \quad v \in \mathrm{V}.$$

Now the method of proof used to obtain Theorem 2.3 gives the following theorem.

THEOREM 4.2. — *Assume hypotheses* (2.1), (4.9), (4.14), (2.4) *and* (2.5). *Let* $D(\mathcal{A}) \supset D(\mathcal{A}')$ *and suppose that for some* $\tau \in (0,1]$, $u \in V_\tau$. *Then for any* $\gamma \in (0, \tau]$,

$$[w_\varepsilon - u]_\gamma = o(\varepsilon^{\tau-\gamma}) \qquad as \qquad \varepsilon \downarrow 0.$$

It is also obvious that if one assumes that $\mathcal{A}\mathcal{A}' = \mathcal{A}'\mathcal{A}$, then an estimate corresponding to Theorem 2.4 is obtained. A theorem corresponding to Theorems 2.3 and 2.4 will now be proven for the rate of convergence of u_ε, the solution of (2.6), to u, the solution of (2.4).

THEOREM 4.3. — *Assume hypotheses* (2.1), (4.9), (4.14) *and* (2.4) *through* (2.6). *Assume that* $a(v, w)$ *is Hermitian symmetric so that* $\mathcal{A} = \mathcal{A}'$. *Further suppose that for fixed* $\tau \in (0,1]$ *and* $\gamma \in (0, \tau]$, L *and* L_ε *for* $\varepsilon \in (0, \varepsilon_0]$ *are extendable by continuity to* $V_{-\tau}$ *and* $\|L_\varepsilon - L\|_{-\gamma} = o(\varepsilon^{\tau-\gamma})$ *as* $\varepsilon \downarrow 0$ *where the norm is that of* $V^*_{-\gamma}$). *Then* $u \in V_\tau$ *and*

$$[u_\varepsilon - u]_\gamma = o(\varepsilon^{\tau-\gamma}) \qquad as \qquad \varepsilon \downarrow 0.$$

Proof. — Since L is extendable by continuity to $V_{-\tau}$, it follows from (2.4), (4.9), and the procedure used by Lax [13], p. 623, that $u \in V_\tau$. Similarly, letting g_ε be the unique solution in V_0 of

$$(4.15) \qquad b(g_\varepsilon, v) = L_\varepsilon(v) \qquad \text{for all} \qquad v \in V_0,$$

$g_\varepsilon \in V_\tau$. Now (2.5), (2.6) and (4.15) yield

$$(4.16) \quad \varepsilon a(u_\varepsilon - w_\varepsilon, v) + b(u_\varepsilon, - w_\varepsilon, v) = b(g_\varepsilon - u, v)$$

for all $v \in V$, and so

$$(4.17) \qquad (\varepsilon\mathcal{A} + I)(u_\varepsilon - w_\varepsilon) = g_\varepsilon - u.$$

Note that since g_ε, $u \in V_\tau$, u_ε, $w_\varepsilon \in V_{1+\tau}$.

Now let Γ be the operator in V_0 associated with $[v, w]_1$ relative to $[v, w]_0$, i.e. $[\Gamma v, w]_0 = [v, w]_1$, $w \in V_1$. Then $\Gamma = \mathcal{A}\mathcal{A}' + I = \mathcal{A}^2 + I$ and, letting T be the positive square root of Γ (relative to $[v, w]_0$), one has $[v]_\gamma = [T^\gamma v]_0$ for $v \in V_\gamma$. Furthermore, since g_ε, $u \in V_\tau$, u_ε, $w_\varepsilon \in V_{1+\tau}$, and $\gamma \leqslant \tau$, (4.17) implies

$$(4.18) \qquad (\varepsilon\mathcal{A} + I)T^\gamma(u_\varepsilon - w_\varepsilon) = T^\gamma(g_\varepsilon - u).$$

Hence, by (4.16) and (4.18),

$$(4.19)\quad \varepsilon a(T^\gamma(u_\varepsilon - w_\varepsilon),\ v) + b(T^\gamma(u_\varepsilon - w_\varepsilon),\ v) \\ = b(T^\gamma(g_\varepsilon - u,)\ v) \qquad \text{for all} \qquad v \in V.$$

Letting $v = T^\gamma(u_\varepsilon - w_\varepsilon)$, (4.14) and (4.19) give,

$$\alpha(\varepsilon)|T^\gamma(u_\varepsilon - w_\varepsilon)|_V^2 + \mu[T^\gamma(u_\varepsilon - w_\varepsilon)]_0^2 \\ \leqslant [T^\gamma(g_\varepsilon - u)]_0 \cdot [T^\gamma(u_\varepsilon - w_\varepsilon)]_0.$$

Thus, $[u_\varepsilon - w_\varepsilon]_\gamma \leqslant (1/\mu)[g_\varepsilon - u]_\gamma$, and so,

$$[u_\varepsilon - u]_\gamma \leqslant (1/\mu)[g_\varepsilon - u]_\gamma + [w_\varepsilon - u]_\gamma.$$

Now (cf. [13], p. 623),

$$\begin{aligned}[g_\varepsilon - u]_\gamma &= \sup\{|b(g_\varepsilon - u,\ v)| : v \in V_0 \quad \text{and} \quad [v]_{-\gamma} \leqslant 1\} \\ &= \sup\{|(L_\varepsilon - L)(v)| : v \in V_0 \quad \text{and} \quad [v]_{-\gamma} \leqslant 1\} \\ &= o(\varepsilon^{\tau-\gamma}) \quad \text{as} \quad \varepsilon \downarrow 0\end{aligned}$$

by hypothesis. The theorem now follows by estimating $[w_\varepsilon - u]_\gamma$ in the same fashion as $|w_\varepsilon - u|_\gamma$ was estimated in the proof of Theorem 2.3, using T in place of S.

It is now apparent that under the hypotheses of Theorem 4.3 one may obtain an explicit estimate for $[u_\varepsilon - u]_\gamma$ analogous to the evaluations obtained in Theorems 2.2 and 2.4. It is also obvious that if one assumes (4.9) and that $a(v, v) \geqslant 0$ for $v \in V$, Theorem 3.1 follows by using the norms $[v]_\tau$. A reformulation of Theorem 4.3 will now be proven in the context of hypotheses (4.1) through (4.3). Again, it is possible to give an explicit estimate in this theorem.

THEOREM 4.4. — *Assume hypotheses* (4.1) *through* (4.3), (4.9), and (4.14). *Further assume that* $a(v, w)$ *is Hermitian symmetric and that for fixed* $\tau \in (0,1]$ *and* $\gamma \in (0, \tau]$, $V_{-\tau} \underset{c}{\subset} H$ *and* $|f_\varepsilon - f|_H = o(\varepsilon^{\tau-\gamma})$ *as* $\varepsilon \downarrow 0$. *Then* $u \in V_\tau$ *and*

$$[u_\varepsilon - u]_\gamma = o(\varepsilon^{\tau-\gamma}) \quad \textit{as} \quad \varepsilon \downarrow 0.$$

Proof. — Let $L(v) = (f, v)_H$ for all $v \in H$ and $L_\varepsilon(v) = (f_\varepsilon, v)_H$ for all $v \in H$. Then the restrictions of L and L_ε to $V_{-\tau}$ are in $V^*_{-\tau}$. Moreover there exists $M > 0$ such that $|v|_H \leqslant M|v|_{-\gamma}$ for all $v \in V_{-\gamma}$ and $V_{-\gamma}$ is dense in H. Thus for the norm of $L_\varepsilon - L$ in $V^*_{-\gamma}$ one has

$$\begin{aligned}\|L_\varepsilon - L\|_{-\gamma} &= \sup\{|(L_\varepsilon - L)(v)| : v \in V_{-\gamma} \text{ and } |v|_{-\gamma} \leqslant 1\}\\ &\leqslant M \sup\{|(L_\varepsilon - L)(v)| : v \in V_{-\gamma} \text{ and } |v|_H \leqslant 1\}\\ &= M|f_\varepsilon - f|_H.\end{aligned}$$

The theorem now follows from Theorem 4.3.

5. Quadratic Interpolation Theorems.

In order to apply the results of the preceding chapters to singular perturbations of elliptic boundary value problems, it is necessary to know when the solution of degenerate problem is in a space obtained by quadratic interpolation from the perturbed problem. It is thus essential to have concrete characterizations of the interpolation spaces by quadratic interpolation between spaces of Bessel potentials satisfying homogeneous boundary conditions. Such a characterization will be obtained in the present chapter for the spaces appropriate to the Dirichlet problem with homogeneous boundary data.

The results of this chapter supplement the $p=2$ case of some theorems of Lions and Magenes [19], [20]. The terminology and a number of the results of Aronszajn and Smith [5], [6], Adams, Aronszajn and Smith [2], and Adams, Aronszajn and Hanna [1] will be used. For the sake of completeness some of the relevant definitions and theorems will now be recalled.

The Bessel kernel of order $\alpha > 0$ on R^n is the function given by

$$G_\alpha(x) \equiv G_\alpha^{(n)}(x) = \frac{1}{2^{(n+\alpha-2)/2}\pi^{n/2}\Gamma(\alpha/2)} K_{(n-\alpha)/2}(|x|)|x|^{(\alpha-n)/2}$$

where K_ν is the modified Bessel function of the third kind. For $0 < \alpha < 1$, let

$$C(n, \alpha) = \frac{2^{-2\alpha+1}\pi^{(n+2)/2}}{\Gamma(\alpha+1)\Gamma(\alpha+n/2)\sin \pi\alpha}.$$

Now let D be a domain in R^n and let $u : R^n \to \mathbf{C}$ be in

$C^\infty(D)$. The standard α-norm over D, $|u|_{\alpha,D}$, is defined as follows,

$$|u|^2_{0,D} = \int_D |u(x)|^2 \, dx,$$

and for $0 < \alpha < 1$,

$$|u|^2_{\alpha,D} = |u|^2_{0,D} + \frac{1}{C(n,\alpha)G_{2n+2\alpha}(0)} \int_D \int_D \frac{G_{2n+2\alpha}(x-y)}{|x-y|^{n+2\alpha}} |u(x) - u(y)|^2 \, dx \, dy.$$

For arbitrary $\alpha \geqslant 0$, let $m = [\alpha]$ be the greatest integer $\leqslant \alpha$ and $\beta = \alpha - m$. Then

$$|u|^2_{\alpha,D} = \sum_{k=0}^{m} \binom{m}{k} \sum_{|i| \leqslant k} |D_i u|^2_{\beta,D}.$$

The space $\check{P}^\alpha(D)$ is the perfect functional completion in the sense of Aronszajn and Smith [5] of the functions in $C^\infty(D)$ for which $|u|_{\alpha,D} < \infty$. For $D = R^n$, $\check{P}^\alpha(D)$ is denoted simply by P^α and $|u|_{\alpha,R^n}$ by $\|u\|_\alpha$. Henceforth, $\int$ stands for $\int_{R^n}$, L^2 for $L^2(R^n)$, etc. $P^\alpha(D)$ is defined as the space of all restrictions to D of functions in P^α with norm

$$\|u\|_{\alpha,D} = \inf \|\tilde{u}\|_\alpha,$$

the infimum being taken over all $\tilde{u} \in P^\alpha$ such that $\tilde{u} = u$ except on a subset of D of 2α-capacity 0. $P^\alpha(D)$ is the perfect functional completion of the class of restrictions to D of functions in C_0^∞.

Throughout the rest of this paper it will (at least) be assumed, unless explicit mention is made to the contrary, that

(5.1) D is a Lipschitzian graph (LG) domain in R^n

(cf. [2], §11). For $n = 1$ it is understood that a LG domain is simply an open interval. For LG domains $\check{P}^\alpha(D) = P^\alpha(D)$ with equivalent norms (cf. [2], § 7 and § 11). It should be noted that $\check{P}^\alpha(D)$ is the class of corrections (cf. [2], § 0) of functions in the more familiar class $W^{\alpha,2}(D)$ (cf. Lions and Magenes [18], *n*. 2).

For $\alpha > 0$ and $u \in \check{P}^\alpha(D)$, let

$$J_{\alpha,D}(u) = \sum_{|i| \leqslant \alpha^*} \int_D |D_i u(x)|^2 r(x)^{-2\alpha+2|i|} \, dx$$

where α^* is the greatest integer $< \alpha$ and, denoting the exterior of D by Ext D, $r(x) = \text{dist}\,(x, \text{Ext D})$. For $\alpha = 0$, let $J_{0,D}(u) = 0$. The present $J_{\alpha,D}(u)$ was denoted by $J_{\alpha,D,\text{Ext D}}(u)$ in [2], § 9. The more explicit notation will not be needed in the present work.

Recall now that a LG domain admits a simultaneous extension mapping (cf. [2], § 11 or Aronszajn [4], § 5). From this result it follows that (cf. [1], App. I) for $0 \leqslant \tau \leqslant 1$, the τ^{th} interpolation space by quadratic interpolation between $\check{P}^{\alpha}(D)$ and $\check{P}^{\beta}(D)$ is $\check{P}^{\alpha(1-\tau)+\beta\tau}(D)$ with an equivalent norm. Since the spaces $\check{P}^{\alpha}(D)$ are not exactly subspaces of $L^2(D)$, one must apply the procedure given in [1], App. I to recover the proper class of exceptional sets.

Now, denote the closure of $C_0^{\infty}(D)$ in $\check{P}^{\alpha}(D)$ by $\check{P}_0^{\alpha}(D)$. Since the identity mapping is bounded from $\check{P}_0^{\alpha}(D)$ into $\check{P}^{\alpha}(D)$, it is apparent from the above considerations and Proposition 1.5 that the τ^{th} interpolation space by quadratic interpolation between $\check{P}_0^{\alpha}(D)$ and $\check{P}_0^{\beta}(D)$ can be realized as a (not necessarily closed) subspace of $\check{P}^{\alpha(1-\tau)+\beta\tau}(D)$. A theorem will now be proven characterizing the interpolation spaces V_{τ}, $0 \leqslant \tau \leqslant 1$, obtained by quadratic interpolation between $V_1 = \check{P}_0^{m}(D)$, m a positive integer, and $V_0 = \check{P}_0^{0}(D) = L^2(D)$. The theorem refines the $p = 2$ case of a theorem of Lions and Magenes [19], p. 322.

THEOREM 5.1. — *Assume that* D *satisfies* (5.1). *Let* m *be a positive integer,* $V_1 = \check{P}_0^{m}(D)$, *and* $V_0 = L^2(D)$. *Furthermore, let* $E_0 : L^2(D) \to L^2$ *be extension by* 0, i.e.

$$E_0 u(x) = \begin{cases} u(x), & x \in D \\ 0, & x \in R^n \setminus D. \end{cases}$$

Denote the extension constant $\Gamma[0, m]$ *of* D (cf. [2], § 7 and § 11) *by* K. *Then:*

i) *for* $0 \leqslant \tau \leqslant 1$, $u \in V_{\tau}$ *if and only if* $u \in \check{P}^{m\tau}(D)$ *and* $E_0 u \in P^{m\tau}$;

ii) *for* $0 \leqslant \tau \leqslant 1$, $u \in V_{\tau}$ *if and only if* $u \in \check{P}_{m\tau}(D)$ *and* $J_{m\tau,D}(u) < \infty$;

iii) $(1/2)\ [K^{-2}|u|^2_{m\tau,D} + CJ_{m\tau,D}(u)] \leqslant |u|^2_\tau$
$\leqslant (1+K)^2\{[1+2n(1-\beta)]|u|^2_{m\tau,D} + 2n(1-\beta)J_{m\tau,D}(u)\}$
where $\beta = m\tau - [m\tau]$ *and* C *depends only on* n, $[m\tau]$ *and* D, *and* $u \in V_\tau$.

Proof. — i) Necessity. $E_0 : L^2(D) \to L^2$ is continuous with bound 1. Since m is a positive integer, it is apparent, from the definition of the standard m-norm and the density of $C_0^\infty(D)$ in $\check{P}_0^m(D)$, that E_0 maps $\check{P}_0^m(D)$ into P^m continuously with bound 1. Moreover, the τ^{th} interpolation space by quadratic interpolation between P^m and L^2 is $P^{m\tau}$ with the same norm (cf. [1], App. I or Lions and Magenes [17], pp. 300-301). Thus by quadratic interpolation (Proposition 1.5), E_0 is continuous from V_τ into $P^{m\tau}$ with bound $\leqslant 1$, i.e. $u \in V_\tau$, implies that $E_0 u \in P^{m\tau}$ and

$$(5.2) \qquad \|E_0 u\|_{m\tau} \leqslant |u|_\tau.$$

Now let I be the identity mapping on $L^2(D)$. Then $I : L^2(D) \to L^2(D)$ is continuous with bound 1 and $I : \check{P}_0^m(D) \to \check{P}^m(D)$ is continuous with bound 1. As noted previously the τ^{th} interpolation space by quadratic interpolation between $\check{P}^m(D)$ and $L^2(D)$ is $\check{P}^{m\tau}(D)$ with an equivalent norm. Furthermore (cf. [1], App. I), $K^{-1}|u|_{m\tau,D} \leqslant$ the τ^{th} interpolated norm of u between $\check{P}^m(D)$ and $L^2(D) \leqslant K|u|_{m\tau,D}$. By quadratic interpolation I is continuous from V_τ into $\check{P}^{m\tau}(D)$ with bound $\leqslant 1$. So, $u \in V_\tau$ implies that $u \in \check{P}^{m\tau}(D)$ and

$$(5.3) \qquad K^{-1}|u|_{m\tau,D} \leqslant |u|_\tau.$$

i) Sufficiency. Since D is LG, D is the interior of its closure and so $\partial D = \partial(\text{Ext } D)$. Furthermore, Ext D is LG and the extension constant $K = \Gamma[0, m]$ is the same for Ext D as for D. So let E be the associated simultaneous extension mapping for Ext D. In particular

$$E(C^\infty(\text{Ext } D) \cap \check{P}^m(\text{Ext } D)) \subset C^\infty$$

and for every $\alpha \in [0, m]$, E is a continuous linear mapping of $\check{P}^\alpha(\text{Ext } D)$ into P^α with bound $\leqslant K$.

Let $R : L^2 \to L^2(D)$ be restriction to D and let

$S : L^2 \to L^2(\text{Ext } D)$ be restriction to Ext D. For $\varphi \in L^2$, define

$$Q'\varphi = \varphi - ES\varphi$$

and

$$Q\varphi = RQ'\varphi = (R - RES)\varphi.$$

Then since $R : L^2 \to L^2(D)$ is continuous with bound 1, $S : L^2 \to L^2(\text{Ext } D)$ is continuous with bound 1, and $E : L^2(\text{Ext } D) \to L^2$ is continuous with bound $\leqslant K$, it follows that Q maps L^2 into $L^2(D)$ with bound $\leqslant 1 + K$. Moreover, Q maps L^2 onto $L^2(D)$ and $QE_0 = I$ on $L^2(D)$. For, if $u \in L^2(D)$, then $E_0 u \in L^2$ and, since $SE_0 u = 0$ and $RE_0 u = u$, $QE_0 u = u$.

Similarly, Q maps P^m into $\check{P}^m(D)$ continuously with bound $\leqslant 1 + K$. Suppose that it has been proven that

$$Q(P^m) \subset \check{P}_0^m(D). \tag{5.4}$$

Sufficiency follows then readily from (5.4). For now Q maps P^m into $\check{P}_0^m(D)$ continuously with bound $\leqslant 1 + K$. Furthermore, Q maps P^m onto $\check{P}_0^m(D)$ and $QE_0 = I$ on $\check{P}_0^m(D)$. For if $u \in \check{P}_0^m(D)$, then $E_0 u \in P^m$ and $QE_0 u = u$ as in the L^2 case. Thus by quadratic interpolation, if $u \in \check{P}^{m\tau}(D)$ is such that $E_0 u \in P^{m\tau}$, then $QE_0 u = u \in V_\tau$. Also the bound of $Q : P^{m\tau} \to V_\tau$ is $\leqslant 1 + K$. Thus

$$|Q\varphi|_\tau \leqslant (1 + K)\|\varphi\|_{m\tau}. \tag{5.5}$$

To complete the proof of i) it remains to verify (5.4). For this purpose, let $\varphi \in C^\infty \cap P^m$. Then since E is a simultaneous extension mapping, $Q'\varphi \in C^\infty \cap P^m$ and so $Q\varphi \in C^\infty(D) \cap \check{P}^m(D)$. Moreover, since $ES\varphi(x) = \varphi(x)$ for all $x \in \text{Ext } D$, $Q\varphi$ and all partial derivatives of $Q\varphi$ vanish at every point of $\partial D = \partial(\text{Ext } D)$. Since $Q : P^m \to \check{P}^m(D)$ is continuous and $C^\infty \cap P^m$ is dense in P^m, the proof will be finished by showing that if $\psi \in C^\infty(D) \cap \check{P}^m(D)$ is such that ψ and all partial derivatives of ψ vanish on ∂D, then $\psi \in \check{P}_0^m(D)$.

So let ψ be such a function. Since D is LG there exists a δ-loose open cover of ∂D and a C^∞ partition of unity subordinate to the cover. Furthermore the functions in $\check{P}_0^m(D)$ with bounded support are dense in $\check{P}_0^m(D)$. It is therefore sufficient to consider D to be of the following form. Letting

B be a bounded rectangle in R^{n-1}, $D = \{(x', x_n) : x' \in B$ and $0 < x_n < f(x')\}$ where f is Lipschitzian on B with a positive lower bound. Now, for sufficiently small positive ε, define

$$w_\varepsilon(x) = \begin{cases} w(x', (1-\varepsilon)^{-1}x_n), & 0 < x_n < (1-\varepsilon)f(x') \\ 0, & (1-\varepsilon)f(x') \leqslant x_n < f(x'). \end{cases}$$

Then $w_\varepsilon \in C^\infty(D)$. $w_\varepsilon \to w$ in $\check{P}^m(D)$ as $\varepsilon \downarrow 0$, and the support of w_ε is bounded away from $x_n = f(x')$. The proof of i) is now complete.

ii) R^n is obviously L-convex and since Ext D is LG, Ext D is a (C)-domain (cf. [2], § 5, § 9 and § 11, and [4], § 5). Thus by Theorem 1, § 9, [2], if $u \in \check{P}^{m\tau}(D)$, then $E_0 u \in P^{m\tau}$ if and only if $J_{m\tau,D}(u) < \infty$. So ii) follows from i).

iii) In the proof of part i) it has been shown that (cf. (5.2), (5.3) and (5.5)) if $u \in V_\tau$ and $v \in P^{m\tau}$, then

$$\text{(5.6)} \quad \|E_0 u\|_{m\tau} \leqslant |u|_\tau, \ |Qv|_\tau \leqslant (1+K)\|v\|_{m\tau}, \quad QE_0 u = u, \ \text{and} \ K^{-1}|u|_{m\tau,D} \leqslant |u|_\tau.$$

Thus

$$\text{(5.7)} \quad \|E_0 u\|_{m\tau} \leqslant |u|_\tau = |QE_0 u|_\tau \leqslant (1+K)\|E_0 u\|_{m\tau}.$$

Now by Theorem 1, § 9, [2].

$$\text{(5.8)} \quad \|E_0 u\|^2_{m\tau} \leqslant |u|^2_{m\tau,D} + 2n(1-\beta)\{J_{m\tau,D}(u) + |u|^2_{[m\tau],D}\} \leqslant [1 + 2n(1-\beta)]|u|^2_{m\tau,D} + 2n(1-\beta)J_{m\tau,D}(u)$$

and

$$\text{(5.9)} \qquad J_{m\tau,D}(u) \leqslant c\|E_0 u\|^2_{m\tau}$$

where c depends only on n, $[m\tau]$, and D. Therefore, letting $C = (1/c)$, (5.6) through (5.9) give

$$(1/2)\ [K^{-2}|u|^2_{m\tau,D} + CJ_{m\tau,D}(u)] \leqslant |u|^2_\tau \leqslant (1+K)^2\{[1 + 2n(1-\beta)]|u|^2_{m\tau,D} + 2n(1-\beta)J_{m\tau,D}(u)\},$$

and ii) is proven.

Corollary 5.1. — *Assume that* D *is a* LG *domain. Let* m *be a positive integer,* $V_1 = \check{P}^m_0(D)$, *and* $V_0 = L^2(D)$. *Then for* $0 \leqslant \tau \leqslant 1$, $u \in V_\tau$ *implies* $u \in \check{P}^{m\tau}_0(D)$ *and* V_τ *is dense in* $\check{P}^{m\tau}_0(D)$.

Proof. — By definition, $C_0^\infty(D)$ is dense in $\check{P}_0^\alpha(D)$ and since for any $\tau \in [0,1)$, $V_1 = \check{P}_0^m(D)$ is dense in V_τ, $C_0^\infty(D)$ is also dense in V_τ. By Theorem 5.1, $|u|_\tau^2$ is equivalent to $|u|_{m\tau,D}^2 + J_{m\tau,D}(u)$. Since the norm in $\check{P}_0^{m\tau}(D)$ is just $|u|_{m\tau,D}$, the conclusion follows.

Though Theorem 5.1 characterizes the functional spaces V_τ obtained by quadratic interpolation between $\check{P}_0^m(D)$ and $L^2(D)$ some more information about these spaces will be required to apply the Hilbert space perturbation theory to the Dirichlet problem. The additional information which will be needed is to know that in « most » cases the inclusion relation in Corollary 5.1 is an equality, i.e. $V_\tau = \check{P}_0^{m\tau}(D)$ with an equivalent norm. For $m = 1$ and for bounded domains D with smooth boundary this result was obtained for the spaces $W_0^{\alpha,p}(D)$ in the aforementioned theorem of Lions and Magenes [19], p. 322. The functional Hilbert space case of the corresponding theorem for general m and Lipschitzian graph domains will be derived here by virtually the same techniques as those used in Lions and Magenes [19]. A few more preliminaries are in order before the statement of the theorem.

For $u \in \check{P}^\beta(D)$, $0 \leqslant \beta \leqslant 1$, the Dirichlet integral of order β, $d_{\beta,D}(u)$, is defined by

$$d_{0,D}(u) = |u|_{0,D}^2,$$

$$d_{1,D}(u) = |u|_{1,D}^2 - |u|_{0,D}^2 = \sum_{l=1}^{n} \int_D \left|\frac{\partial u}{\partial x_l}\right|^2 dx,$$

and for $0 < \beta < 1$,

$$d_{\beta,D}(u) = \frac{1}{C(n, \beta)} \int_D \int_D \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\beta}} \, dx \, dy.$$

The approximate α-norm, $|u\rceil_{\alpha,D}$, for $u \in \check{P}^\alpha(D)$ is

$$|u\rceil_{0,D}^2 = d_{0,D}(u) = |u|_{0,D}^2,$$

and for $0 < \alpha < 1$,

$$|u\rceil_{\alpha,D}^2 = |u|_{0,D}^2 + d_{\alpha,D}(u).$$

For arbitrary $\alpha > 0$, let $m = [\alpha]$, $\beta = \alpha - m$, and

$$|u\rceil_{\alpha,D}^2 = \sum_{k=0}^{m} \binom{m}{k} \sum_{|i|=k} |D_i u\rceil_{\beta,D}^2.$$

Then for an arbitrary domain D

$$2^{-1/2}|u\rceil_{\alpha,\mathrm{D}} \leqslant |u|_{\alpha,\mathrm{D}} \leqslant |u\rceil_{\alpha,\mathrm{D}}$$

(*cf.* [2], § 2).

The following lemma is a particular case of the results of [19], N. 1. Its proof is included here to accomodate the reader.

LEMMA 5.1. — *For* $0 \leqslant \alpha \leqslant 1/2$, $\mathrm{C}_0^\infty(0, \infty)$ *is dense in* $\check{\mathrm{P}}^\alpha(0, \infty)$, *i.e.*

$$\check{\mathrm{P}}_0^\alpha(0, \infty) = \check{\mathrm{P}}^\alpha(0, \infty) \qquad \textit{for} \qquad 0 \leqslant \alpha \leqslant 1/2.$$

Proof. — By density it is sufficient to prove that if u is the restriction of a function in $\mathrm{C}_0^\infty(\mathrm{R}^1)$ to $(0, \infty)$, then $u \in \check{\mathrm{P}}_0^\alpha(0, \infty)$. Let

$$\varphi_n(x) = \begin{cases} 0, & 0 < x < 1/n \\ nx - 1, & 1/n \leqslant x \leqslant 2/n \\ 1, & 2/n < x. \end{cases}$$

Assume for the present that there exists $\mathrm{M} > 0$ such that for all n and all $\alpha \in [0,1/2]$,

$$d_{\alpha,(0,\infty)}(\varphi_n u) \leqslant \mathrm{M}. \tag{5.10}$$

The lemma follows readily from (5.10). For, obviously $\varphi_n u \to u$ in $\mathrm{L}^2(0, \infty)$ as $n \to \infty$. Thus (5.10) implies that $|\varphi_n u|_{\alpha,(0,\infty)}$ is bounded uniformly in n for $0 \leqslant \alpha \leqslant 1/2$. Hence there exists a subsequence which converges weakly in $\check{\mathrm{P}}^\alpha(0, \infty)$, for which, the corresponding sequence of arithmetic means, $\{\psi_n\}$, converges strongly in $\check{\mathrm{P}}^\alpha(0, \infty)$. Since $\psi_n \to u$ in $\mathrm{L}^2(0, \infty)$, $\psi_n \to u$ in $\check{\mathrm{P}}^\alpha(0, \infty)$. By regularization of the ψ_n'*s* one obtains a sequence of functions in $\mathrm{C}_0^\infty(0, \infty)$ converging to u in $\check{\mathrm{P}}^\alpha(0, \infty)$.

To prove (5.10), write

$$\varphi_n(x)u(x) - \varphi_n(y)u(y) = \varphi_n(x)[u(x) - u(y)] + [\varphi_n(x) - \varphi_n(y)]u(y).$$

Then it is sufficient to show that

$$\int_0^\infty \int_0^\infty \frac{|\varphi_n(x) - \varphi_n(y)|^2 |u(y)|^2}{|x - y|^{1+2\alpha}}\, dx\, dy$$

is bounded uniformly in n. But since u is the restriction of a function in $C_0^\infty(R^1)$ to $(0, \infty)$, u is bounded. Thus a direct computation of

$$\int_0^\infty \int_0^\infty \frac{|\varphi_n(x) - \varphi_n(y)|^2}{|x - y|^{1+2\alpha}} \, dx \, dy$$

proves the lemma.

The following lemma gives explicit bounds in the $p = 2$ case of a proposition of [19], N. 2.

LEMMA 5.2. — i) *For* $1/2 < \alpha < 1$ *and* $u \in \check{P}_0^\alpha(0, \infty)$,

$$J_{\alpha,(0,\infty)}(u) = \int_0^\infty |u(x)|^2 x^{-2\alpha} \, dx \leqslant \frac{\pi}{(\alpha - 1/2)^2} d_{\alpha,(0,\infty)}(u).$$

ii) For $0 < \alpha < 1/2$ *and* $u \in \check{P}^\alpha(0, \infty) = \check{P}_0^\alpha(0, \infty)$,

$$J_{\alpha,(0,\infty)}(u) = \int_0^\infty |u(x)|^2 x^{-2\alpha} \, dx \leqslant \frac{5\pi}{(\alpha - 1/2)^2} d_{\alpha,(0,\infty)}(u).$$

Proof. — For $\alpha > 1/2$ the functions in $\check{P}_0^\alpha(0, \infty)$ are continuous with limit 0 at $x = 0$. Thus i) is a special case of Lemma 3, § 9, [2] and ii) will now be proven by a simple modification of the proof of this lemma.

Since for $0 < \alpha < 1/2$, $\check{P}^\alpha(0, \infty) = \check{P}_0^\alpha(0, \infty)$, it is sufficient to prove ii) for $u \in C_0^\infty(0, \infty)$. For $t > 1$,

$$\begin{aligned}\left[\int_0^\infty |u(x) - u(t^n x)|^2 x^{-2\alpha} \, dx\right]^{1/2} &\leqslant \sum_{k=0}^{n-1} \left[\int_0^\infty |u(t^k x) - u(t^{k+1} x)|^2 x^{-2\alpha} \, dx\right]^{1/2} \\ &= \sum_{k=0}^{n-1} t^{k(\alpha-1/2)} \left[\int_0^\infty |u(x) - u(tx)|^2 x^{-2\alpha} \, dx\right]^{1/2},\end{aligned}$$

and letting $n \to \infty$, the dominated convergence theorem gives

$$(1 - t^{\alpha-1/2})^2 \int_0^\infty |u(x)|^2 x^{-2\alpha} \, dx \leqslant \int_0^\infty |u(x) - u(tx)|^2 x^{-2\alpha} \, dx.$$

Since

$$d_{\alpha,(0,\infty)}(u) = \frac{1}{C(1, \alpha)} \int_0^\infty \int_0^\infty \frac{|u(x) - u(tx)|^2}{x^{2\alpha}|1 - t|^{1+2\alpha}} \, dx \, dt,$$

$$\frac{1}{C(1, \alpha)} \int_1^\infty \frac{(1 - t^{\alpha-1/2})^2}{(t - 1)^{1+2\alpha}} \, dt \cdot \int_0^1 |u(x)|^2 x^{-2\alpha} \, dx \leqslant d_{\alpha,(0,\infty)}(u).$$

Now,

$$\int_1^\infty \frac{(1 - t^{\alpha-1/2})^2}{(t-1)^{1+2\alpha}}\, dt = \int_0^1 \frac{s^{2\alpha-1}(1 - s^{1/2-\alpha})^2}{(1-s)^{1+2\alpha}}\, ds$$
$$\geqslant (\alpha - 1/2)^2 \int_0^1 s^{2\alpha-1}(1-s)^{1-2\alpha}\, ds$$
$$= (\alpha - 1/2)^2\Gamma(2\alpha)\Gamma(2-2\alpha).$$

By use of $\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha}$, Legendre's duplication formula, and tabulated values of the Γ function (cf. Jahnke and Emde [10]), one obtains

$$\frac{(\alpha - 1/2)^2\Gamma(2\alpha)\Gamma(2-2\alpha)}{C(1, \alpha)} > \frac{(\alpha - 1/2)^2}{5\pi}$$

which yields the inequality.

Theorem 5.2. — *Assume that* D *is a* LG *domain. Let* m *be a positive integer,* $V_1 = \check{P}_0^m(D)$, *and* $V_0 = L^2(D)$. *Let* $\tau \in [0,1]$ *be such that* $m\tau - [m\tau] \neq 1/2$, *i.e.* $m\tau \neq l + 1/2$, $l = 0,1, \ldots, m-1$. *Then* $u \in V_\tau$ *if and only if* $u \in \check{P}_0^{m\tau}(D)$. *Moreover if* $0 < m\tau - [m\tau] < 1/2$, $\check{P}_0^{m\tau}(D) = \check{P}_0^{[m\tau]}(D) \cap \check{P}^{m\tau}(D)$.

Proof. — By Theorem 5.1 and Corollary 5.1 it must be proven that: if $0 \leqslant m\tau - [m\tau] < 1/2$ and

$$u \in \check{P}_0^{[m\tau]}(D) \cap \check{P}^{m\tau}(D)$$

then $E_0 u \in P^{m\tau}$ ($E_0 : L^2(D) \to L^2$ is extension by 0); and, if $1/2 < m\tau - [m\tau] < 1$ and $u \in \check{P}_0^{m\tau}(D)$ then $E_0 u \in P^{m\tau}$. Since u and $E_0 u$ have the same exceptional set, $E_0 u \in P^{m\tau}$ if and only if $\|E_0 u\|_{m\tau} < \infty$.

Now for any multi-index i it follows from the density of $C_0^\infty(D)$ in $\check{P}_0^{|i|}(D)$ that $D_i E_0 u = E_0 D_i u$ for all $u \in \check{P}_0^{|i|}(D)$. Thus if $m\tau$ is an integer, i.e. $m\tau = [m\tau]$, it follows trivially from the density of $C_0^\infty(D)$ in $\check{P}_0^{m\tau}(D)$ and the definition of the approximate norm that $|u|_{m\tau,D} = \|E_0 u\|_{m\tau} < \infty$. Hence it is sufficient to prove that if

$$|i| < m\tau \qquad \text{and} \qquad 0 < \beta = m\tau - [m\tau] < 1$$

with $\beta \neq 1/2$, then

$$d_{\beta,\mathrm{R}^n}(\mathrm{E}_0\mathrm{D}_i u) = \frac{1}{\mathrm{C}(n,\beta)}\int_{\mathrm{R}^n}\int_{\mathrm{R}^n}\frac{|\mathrm{E}_0\mathrm{D}_i u(x) - \mathrm{E}_0\mathrm{D}_i u(y)|^2}{|x-y|^{n+2\beta}}\,dx\,dy < \infty,$$

for $u \in \check{\mathrm{P}}_0^{[m\tau]}(\mathrm{D}) \cap \check{\mathrm{P}}^{m\tau}(\mathrm{D})$ if $0 < \beta < 1/2$ and for $u \in \check{\mathrm{P}}_0^{m\tau}(\mathrm{D})$ if $1/2 < \beta < 1$. Since a $\mathrm{C}^{(0,1)}$ homeomorphism preserves potential classes of order $\leqslant 1$ (cf. [2], § 2), by use of partition of a unity it is sufficient to prove that $d_{\beta,\mathrm{R}^n}(\mathrm{E}_0\mathrm{D}_i u) < \infty$ with $\mathrm{D} = \mathrm{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathrm{R}^n : x_n > 0\}$. Moreover Theorem 5.1 yields $d_{\beta,\mathrm{R}^n}(\mathrm{E}_0\mathrm{D}_i u) < \infty$ if $\mathrm{J}_{\beta,\mathrm{R}^n_+}(\mathrm{D}_i u) < \infty$.

Letting $\varphi = \mathrm{D}_i u$ it now suffices to prove that $\mathrm{J}_{\beta,\mathrm{R}^n_+}(\varphi) < \infty$ if $0 < \beta < 1/2$ and $\varphi \in \check{\mathrm{P}}^{\beta}(\mathrm{R}^n_+)$ and that $\mathrm{J}_{\beta,\mathrm{R}^n_+}(\varphi) < \infty$ if $1/2 < \beta < 1$ and $\varphi \in \check{\mathrm{P}}_0^{\beta}(\mathrm{R}^n_+)$. For this purpose let φ be the restriction of a C_0^∞ function to R^n_+ if $0 < \beta < 1/2$ and let $\varphi \in \mathrm{C}_0^\infty(\mathrm{R}^n_+)$ if $1/2 < \beta < 1$. Then Lemma 5.2 yields

$$\begin{aligned}\mathrm{J}_{\beta,\mathrm{R}^n_+}(\varphi) &= \int_{\mathrm{R}^n_+}|\varphi(x)^2|x_n^{-2\beta}\,dx\\ &\leqslant \mathrm{K}(\beta)\int_{\mathrm{R}^n_+}\int_0^\infty\frac{|\varphi(x)-\varphi(x_1,\ldots,x_{n-1},y_n)|^2}{|x_n-y_n|^{1+2\beta}}\,dy_n\,dx\end{aligned}$$

where $\mathrm{K}(\beta) = 5\pi(\beta - 1/2)^{-2}$ if $0 < \beta < 1/2$ and

$$\mathrm{K}(\beta) = \pi(\beta - 1/2)^{-2} \quad \text{if} \quad 1/2 < \beta < 1.$$

Now let E be a simultaneous extension mapping for R^n_+ and $\mathrm{E}\varphi = \psi$. Then

$$\begin{aligned}&\int_{\mathrm{R}^n_+}\int_0^\infty\frac{|\varphi(x)-\varphi(x_1,\ldots,x_{n-1},y_n)|^2}{|x_n-y_n|^{1+2\beta}}\,dy_n\,dx\\ &\qquad\leqslant \int_{\mathrm{R}^n}\int_{-\infty}^\infty\frac{|\psi(x)-\psi(x_1,\ldots,x_{n-1},y_n)|^2}{|x_n-y_n|^{1+2\beta}}\,dy_n\,dx\\ &\qquad= \int_{\mathrm{R}^n}\int_{-\infty}^\infty\frac{|\psi(x)-\psi(x_1,\ldots,x_{n-1},x_n+t_n)|^2}{|t_n|^{1+2\beta}}\,dt_n\,dx\\ &\qquad= \mathrm{M}(n,\beta)\int_{\mathrm{R}^n}\int_{\mathrm{R}^n}\frac{|\psi(x)-\psi(x_1,\ldots,x_{n-1},x_n+t_n)|^2}{|t|^{n+2\beta}}\,dt\,dx\end{aligned}$$

where

$$\begin{aligned}[\mathrm{M}(n,\beta)]^{-1} &= \int_{\mathrm{R}^{n-1}}|1+z_1^2+\cdots+z_{n-1}^2|^{-(n+2\beta)/2}\,dz\\ &= \frac{\pi^{(n-1)/2}\Gamma(\beta+1/2)}{\Gamma(\beta+n/2)}.\end{aligned}$$

Now

$$\left[\int_{\mathrm{R}^n}\int_{\mathrm{R}^n}\frac{|w(x)-w(x_1,\ldots,x_{n-1},x_n+t_n)|^2}{|t|^{n+2\beta}}\,dt\,dx\right]^{1/2}$$
$$\leqslant\left[\int_{\mathrm{R}^n}\int_{\mathrm{R}^n}\frac{|w(x)-w(x_1+t_1/2,x_2+t_2/2,\ldots,x_n+t_n/2)|^2}{|t|^{n+2\beta}}\,dt\,dx\right]^{1/2}$$
$$+\left[\int_{\mathrm{R}^n}\int_{\mathrm{R}^n}\frac{\begin{array}{l}|w(x_1+t_1/2,x_2+t_2/2,\ldots,x_n+t_n/2)\\ \qquad -w(x_1,\ldots,x_n,x_n+t_n)|^2\end{array}}{|t|^{n+2\beta}}\,dt\,dx\right]^{1/2}$$
$$=2^{1-\beta}\left[\int_{\mathrm{R}^n}\int_{\mathrm{R}^n}\frac{|w(x)-w(y)|^2}{|x-y|^{n+2\beta}}\,dx\,dy\right]^{1/2}$$
$$\leqslant \mathrm{L}|v|_{\beta,\mathrm{R}^n_+}$$

since E is a simultaneous extension mapping. By density of the class of $v's$ considered, the proof is complete.

Remark. — Corollary 5.1 shows that for Lipschitzian graph domains $\mathrm{V}_\tau \underset{c}{\subset} \check{\mathrm{P}}_0^{m\tau}(\mathrm{D})$ and Theorem 5.2 gives the opposite inclusion for $m\tau - [m\tau] \neq 1/2$. The $p = 2$ case of Theorem 5.2. p. 322, [19] states that for $m = 1$, $\tau = 1/2$, and D a bounded domain with smooth boundary, $\mathrm{V}_{1/2}$ is strictly contained in $\check{\mathrm{P}}_0^{1/2}(\mathrm{D})$ with a stronger topology. An example will now be given which shows that for $\mathrm{D} = (0,1)$, m any positive integer, and $m\tau = l + 1/2$ where $l = 0,1,\ldots,m-1$, V_τ is strictly contained in $\check{\mathrm{P}}_0^{l+1/2}(0,1)$ with a stronger topology.

Example 5.1. — Let $\psi \in \mathrm{C}^\infty(0,1)$ be such that $0 \leqslant \psi(x) \leqslant 1$, $\psi(x) = 1$ for $x < 1/3$, and $\psi(x) = 0$ for $x > 2/3$. Let $u(x) = x^l\psi(x)$ where l is a non-negative integer. Then

$$\mathrm{J}_{l+1/2,(0,1)}(u) \geqslant l\,!\int_0^{1/3} x^{-1}\,dx = \infty.$$

It will now be verified that $u \in \check{\mathrm{P}}_0^{l+1/2}(0,1)$.

For $\varepsilon \in (0,1/4]$ let $\chi_\varepsilon \equiv \chi \in \mathrm{C}^\infty(\mathrm{R}^1)$ be such that $0 \leqslant \chi(x) \leqslant 1$, $\chi(x) = 0$ for $x < 0$, $\chi(x) = 1$ for $x > \varepsilon$, and $|\chi^{(k)}(x)| \leqslant \mathrm{C}_k\varepsilon^{-k}$ where C_k is a constant depending only on k (cf. [2], § 1, Lemma 1). Denote by φ the restriction of χ to $(0,1)$. Then by a simple version of the argument used to verify (5.4) in the proof of Theorem 5.1, $\varphi u \in \check{\mathrm{P}}_0^m(0,1)$ for any positive integer m and so $\varphi u \in \check{\mathrm{P}}_0^{l+1/2}(0,1)$. Then, by the same method as employed in the proof of Lemma 5.1, it is sufficient to prove that $|\varphi u|_{l+1/2,(0,1)}$ is bounded uniformly in ε in order to conclude that $u \in \check{\mathrm{P}}_0^{l+1/2}(0,1)$. Moreover, since for $\alpha < \beta$, $\check{\mathrm{P}}^{\beta}(0,1) \underset{c}{\subset} \check{\mathrm{P}}^{\alpha}(0,1)$, $|\varphi u|_{l+1/2,(0,1)}$ is bounded uniformly in ε if

$$(5.11) \qquad \int_0^1 \int_0^1 \frac{|\mathrm{D}^l(\varphi u)(x) - \mathrm{D}^l(\varphi u)(y)|^2}{|x-y|^2}\, dx\, dy \leqslant \mathrm{M}$$

and

$$(5.12) \qquad \int_0^1 |\mathrm{D}^k(\varphi u)(x)|^2\, dx \leqslant \mathrm{M}_k, \qquad 0 \leqslant k \leqslant l,$$

where M, M_k are independent of ε.

To obtain (5.12) it is sufficient to note that

$$\int_0^{\varepsilon} |\mathrm{D}^k(\varphi u)(x)|^2\, dx = \int_0^{\varepsilon} \left| \sum_{i=0}^{k} \binom{k}{i} \frac{l\,!}{(l-k+i)\,!} \varphi^{(i)}(x) x^{l-k+i} \right|^2 dx$$
$$\leqslant 2 \sum_{i=0}^{k} \left[\binom{k}{i} \frac{l\,!}{(l-k+i)\,!} \right]^2 \mathrm{C}_i^2 \varepsilon^{2l-2k}$$

and $l \geqslant k$. To verify (5.11), first observe that

$$\int_0^{\varepsilon} \int_0^{\varepsilon} \frac{|\mathrm{D}^l(\varphi u)(x) - \mathrm{D}^l(\varphi u)(y)|^2}{|x-y|^2}\, dx\, dy$$
$$\leqslant 2 \sum_{k=0}^{l} \left[\binom{l}{k} \frac{l!}{k!} \right]^2 \int_0^{\varepsilon} \int_0^{\varepsilon} \frac{|x^k \varphi^{(k)}(x) - y^k \varphi^{(k)}(y)|^2}{|x-y|^2}\, dx\, dy$$

and that for each k the integral on the right does not exceed

$$2\, \mathrm{C}_k^2 \varepsilon^{-2k} \int \int_0^{\varepsilon} \frac{|x^k - y^k|^2}{|x-y|^2}\, dx\, dy + 2\mathrm{C}_{k+1}^2 \leqslant 2k^2 \mathrm{C}_k^2 + 2\mathrm{C}_{k+1}^2.$$

It remains only to observe that

$$\int_0^{\varepsilon} \int_{\varepsilon}^{1} \frac{|\mathrm{D}^l(\varphi u)(x) - \mathrm{D}^l(\varphi u)(y)|^2}{|x-y|^2}\, dx\, dy$$
$$\leqslant \int_0^{\varepsilon} \int_{\varepsilon}^{1/3} \frac{\left| l\,! - \sum_{k=0}^{l} \binom{l}{k} \frac{l!}{k\,!} y^k \varphi^{(k)}(y) \right|^2}{|x-y|^2}\, dx\, dy$$
$$+ \mathrm{K} \int_0^{\varepsilon} \int_{1/3}^{1} [|\mathrm{D}^l(\varphi u)(x)|^2 + |u^{(l)}(y)|^2]\, dx\, dy$$

and the last integral is appropriately bounded by (5.12). Finally,

$$\int_0^\varepsilon \int_\varepsilon^{1/3} \frac{\left|1 - \sum_{k=0}^{l} \binom{l}{k} \frac{y^k}{k!} \varphi^{(k)}(y)\right|^2}{|x-y|^2} \, dx \, dy$$
$$\leqslant 2\left\{\int_0^\varepsilon |1-\varphi(y)|^2(\varepsilon - y)^{-1} \, dy + \sum_{k=1}^{l} \left[\binom{l}{k}\frac{1}{k!}\right]^2 \int_0^\varepsilon [y^k \varphi^{(k)}(y)]^2 (\varepsilon - y)^{-1} \, dy\right\}$$
$$\leqslant 2\left\{C_1^2 \varepsilon^{-2} \int_0^\varepsilon (\varepsilon - y) \, dy + \sum_{k=1}^{l} \left[\binom{l}{k}\frac{1}{k!}\right]^2 C_{k+1}^2 \varepsilon^{-2l-2} \int_0^\varepsilon y^{2k}(\varepsilon - y) \, dy\right\}$$

which is obviously bounded and the example is complete.

A domain $D \subset R^n$ will be said to be $C^{1,1}$-*bounded convex* if:

(5.13) to each point $x \in \overline{D}$ there corresponds a neighborhood $\mathfrak{N}$ of x and a homeomorphism T of class $C^{1,1}$ of $\mathfrak{N}$ onto a neighborhood $\mathfrak{M} = T(\mathfrak{N}) \subset R^n$ of the point $y = Tx$ such that $\mathfrak{N} \cap D = T^{-1}(K \cap \mathfrak{M})$ where K is a fixed bounded convex domain.

Note in particular that if D is $C^{1,1}$-bounded convex, then D is a bounded Lipschitzian graph domain.

THEOREM 5.3. — *Let m be a positive integer and if $m = 1$, assume that D is a $C^{1,1}$-bounded convex domain while if $m > 1$, assume that D is a bounded domain of class C^{2m}. Let $V_1 = \check{P}^{2m}(D) \cap \check{P}_0^m(D)$ and $V_0 = \check{P}_0^m(D)$. Then for $0 \leqslant \tau \leqslant 1$,*

$$V_\tau = \check{P}^{m+m\tau}(D) \cap \check{P}_0^m(D)$$

with an equivalent norm.

Proof. — Letting I be the identity mapping on $\check{P}^m(D)$ one has that $I : \check{P}_0^m(D) \to \check{P}^m(D)$ is continuous and $I : \check{P}^{2m}(D) \cap \check{P}_0^m(D) \to \check{P}^{2m}(D)$ is continuous. Thus by quadratic interpolation,

$$V_\tau \underset{c}{\subset} \check{P}^{m+m\tau}(D) \cap \check{P}_0^m(D).$$

Let P be the orthogonal projection of $\check{P}^m(D)$ onto $\check{P}_0^m(D)$.

Then $P : \check{P}^m(D) \to \check{P}_0^m(D)$ is continuous and onto. Now let $u \in \check{P}^{2m}(D)$ and $Pu = u_0$. Then

$$(u - u_0, v)_{m,D} = 0 \qquad \text{for all} \qquad v \in \check{P}_0^m(D).$$

Thus u_0 is a weak solution of the equation

$$\sum_{k=0}^{m} (-1)^m \binom{m}{k} \Delta^m u_0 = \sum_{k=0}^{m} (-1)^m \binom{m}{k} \Delta^m u \in L^2(D),$$
$$\Delta = \text{Laplacian}.$$

According to the regularity results of Nirenberg [21], Agmon [3], Ch. 9, for $m > 1$, and Kadlec [11] for $m = 1$, $u_0 \in \check{P}^{2m}(D)$. Hence P maps $\check{P}^{2m}(D)$ onto $\check{P}^{2m}(D) \cap \check{P}_0^m(D)$ continuously (by the closed graph theorem). So by quadratic interpolation, $P : \check{P}^{m+m\tau}(D) \to V_\tau$ is continuous. Since $PI = I$ on $\check{P}_0^m(D)$, $u \in \check{P}^{m+m\tau}(D) \cap \check{P}_0^m(D)$ implies $PIu = u \in V_\tau$.

6. Singular Perturbation of Dirichlet Problems.

Let D be a domain in R^n and let m', m be positive integers with $m' \geqslant m$. If $m = 1$, D is assumed to be $C^{1,1}$–bounded convex (cf. 5.13) while if $m > 1$, D is assumed to be a bounded domain of class C^{2m}. For $v, w \in \check{P}^{m'}(D)$, let

$$a(v, w) = \sum_{|i|,|j| \leqslant m'} \int_D a_{ij}(x) D_j v \overline{D_i w}\, dx$$

where $a_{ij} \in C^{|i|}(\overline{D})$ and for $v, w \in \check{P}^m(D)$, let

$$b(v, w) = \sum_{|i|,|j| \leqslant m} \int_D b_{ij}(x) D_j v \overline{D_i w}\, dx$$

where $b_{ij} \in C^{|i|}(\overline{D})$. Further assume

(6.1) there exists $\beta > 0$ such that

$$|b(v, v)| \geqslant \beta |v|_{m,D}^2 \qquad \text{for all} \qquad v \in \check{P}_0^m(D);$$

and

(6.2) for $0 < \varepsilon \leqslant \varepsilon_0$, there exist $\alpha(\varepsilon) > 0$, $\alpha(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$, and $\delta > 0$ such that

$$|\varepsilon a(v, v) + b(v, v)| \geqslant \alpha(\varepsilon) |v|_{m',D}^2 + \delta |v|_{m,D}^2 \quad \text{for all} \quad v \in \check{P}_0^{m'}(D).$$

Consider the formal differential operators

$$\mathcal{U} = \sum_{|i|,|j|\leqslant m'} (-1)^{|i|} D_i(a_{ij} D_j \cdot)$$

and

$$\mathcal{B} = \sum_{|i|,|j|\leqslant m} (-1)^{|i|} D_i(b_{ij} D_j \cdot).$$

Let A_ε be the operator in $L^2(D)$ associated with $\varepsilon a(v, w) + b(v, w)$. i.e. $\varepsilon a(v, w) + b(v, w) = (A_\varepsilon v, w)_{0,D}$. Then for $w \in C_0^\infty(D)$,

$$\text{(6.3)} \qquad \varepsilon a(v, w) + b(v, w) = \langle (\varepsilon\mathcal{U} + \mathcal{B})v, \overline{w} \rangle$$

where $\langle (\varepsilon\mathcal{U} + \mathcal{B})v, \overline{w} \rangle$ denotes the value of the distribution $(\varepsilon\mathcal{U} + \mathcal{B})v$ at $\overline{w}$. Now, if $v \in D(A_\varepsilon)$, the functional $w \to \langle (\varepsilon\mathcal{U} + \mathcal{B})v, \overline{w} \rangle$ is continuous on $C_0^\infty(D)$ in the topology of $L^2(D)$. So $(\varepsilon\mathcal{U} + \mathcal{B})v \in L^2(D)$. Moreover, $v \in D(A_\varepsilon)$ implies that

$$(A_\varepsilon v, w)_{0,D} = \varepsilon a(v, w) + b(v, w) = ((\varepsilon\mathcal{U} + \mathcal{B})v, w)_{0,D}$$

for all $w \in C_0^\infty(D)$ and hence for all $w \in \check{P}_0^{m'}(D)$. Conversely, if $v \in \check{P}_0^{m'}(D)$ is such that $(\varepsilon\mathcal{U} + \mathcal{B})v \in L^2(D)$, then (6.3) gives $v \in D(A_\varepsilon)$. Thus $D(A_\varepsilon) = \{v \in \check{P}_0^{m'}(D) : (\varepsilon\mathcal{U} + \mathcal{B})v \in L^2(D)\}$ and for $v \in D(A_\varepsilon)$,

$$A_\varepsilon v = (\varepsilon\mathcal{U} + \mathcal{B})v.$$

Now $|A_\varepsilon v|_{0,D}$ is equivalent to $(|v|_{0,D}^2 + |A_\varepsilon v|_{0,D}^2)^{1/2}$ on $D(A_\varepsilon)$, so let $|v|_{D(\widehat{A_\varepsilon})} = |A_\varepsilon v|_{0,D}$. Then since $a_{ij}, b_{ij} \in C^{|i|}(\overline{D})$, it is apparent that $\check{P}^{2m'}(D) \cap \check{P}_0^{m'}(D) \underset{c}{\subset} D(A_\varepsilon)$. Thus $|A_\varepsilon v|_{0,D}$ is not stronger than $|v|_{2m',D}$ on $\check{P}^{2m'}(D) \cap \check{P}_0^{m'}(D)$ which will be written

$$\text{(6.4)} \qquad |A_\varepsilon v|_{0,D} \prec |v|_{2m',D} \qquad \text{on} \qquad \check{P}^{2m'}(D) \cap \check{P}_0^{m'}(D).$$

Similarly, the operator in $L^2(D)$ associated with $b(v, w)$ is given by $\mathcal{B}$ with domain equal to $\{v \in \check{P}_0^{m}(D) : \mathcal{B}v \in L^2(D)\}$. Henceforth $\mathcal{B}$ will denote the operator in $L^2(D)$ associated with $b(v, w)$ rather than the corresponding formal differential operator. Furthermore the regularity results of [21], [3], Chapter 9, for $m > 1$, and [11] for $m = 1$ state that

$D(\mathcal{B}) = \check{P}^{2m}(D) \cap \check{P}_0^m(D)$ and

$$(6.5) \qquad |\mathcal{B}v|_{0,D} \sim |v|_{2m,D} \qquad \text{on} \qquad \check{P}^{2m}(D) \cap \check{P}_0^m(D)$$

where $\sim$ is read « is equivalent to ».

Now for l a positive integer, let $\check{P}^{-l}(D)$ be the completion of $L^2(D)$ in

$$|v|_{-l,D} = \sup\{|(v, w)_{0,D}| : w \in \check{P}_0^l(D) \quad \text{and} \quad |w|_{l,D} \leqslant 1\}.$$

$\check{P}^{-l}(D)$ can be realized as a space of distributions on D and is topologically isomorphic to the dual of $\check{P}_0^l(D)$. Then, using (6.2,) it is easy to verify that for $\varepsilon \in (0, \varepsilon_0]$ there exists $K(\varepsilon) > 0$ such that

$$\alpha(\varepsilon)|v|_{m',D} \leqslant |A_\varepsilon v|_{-m',D} \leqslant K(\varepsilon)|v|_{m',D} \qquad \text{for all} \qquad v \in D(A_\varepsilon).$$

Thus A_ε may be extended by continuity to a topological isomorphism of $\check{P}_0^{m'}(D)$ onto $\check{P}^{-m'}(D)$. Denoting this extension by A_ε,

$$(6.6) \qquad |A_\varepsilon v|_{-m',D} \sim |v|_{m',D} \qquad \text{on} \qquad \check{P}_0^{m'}(D).$$

Similarly $\mathcal{B}$ may be extended by continuity to a topological isomorphism of $\check{P}_0^m(D)$ onto $\check{P}^{-m'}(D)$. Denoting this extension by $\mathcal{B}$,

$$(6.7) \qquad |\mathcal{B}v|_{-m,D} \sim |v|_{m,D} \qquad \text{on} \qquad \check{P}_0^m(D).$$

Now let $\mathcal{A}$ be the operator in $\check{P}_0^m(D)$ associated with $a(v, w)$ relative to $b(v, w)$, i.e. $a(v, w) = b(\mathcal{A}v, w)$.

Proposition 6.1. — *Assume hypotheses* (6.1) *and* (6.2.). *Then*

$$\check{P}_0^{2m'-m}(D) \underset{c}{\subset} D(\mathcal{A}).$$

Proof: (6.4), (6.6), Theorem 5.2 and duality give,

$$(6.8) \qquad |A_\varepsilon v|_{-m,D} \prec |v|_{2m'-m,D} \qquad \text{on} \qquad \check{P}_0^{2m'-m}(D),$$

by quadratic interpolation.

Now, (6.7) yields

$$|\mathcal{B}^{-1}w|_{m,D} \sim |w|_{-m,D} \qquad \text{on} \qquad \check{P}^{-m}(D)$$

and for $w \in \check{P}^{-m}(D)$, $\mathcal{B}^{-1}w \in \check{P}^m_0(D)$. Hence, letting $w = A_\varepsilon v$ in (6.8),

$$(6.9)\quad |\mathcal{B}^{-1}A_\varepsilon v|_{m,D} \sim |A_\varepsilon v|_{-m,D} \prec |v|_{2m'-m,D} \quad \text{on} \quad \check{P}^{2m'-m}_0(D).$$

By (4.6), $\mathcal{B}^{-1}A_\varepsilon \subset \varepsilon\mathcal{A} + I$ and by Lemma 4.1,

$$D(A_\varepsilon) = D(\mathcal{B}^{-1}A_\varepsilon)$$

is dense in $D(\mathcal{A})$. Moreover the norm on $D(\mathcal{A})$ is equivalent to $(|v|^2_{m,D} + |(\varepsilon\mathcal{A} + I)v|^2_{m,D})^{1/2}$. The proposition follows.

Let f, f_ε be given in $L^2(D)$, $0 < \varepsilon \leqslant \varepsilon_0$. Let u be the unique solution in $\check{P}^m_0(D)$ of

$$(6.10)\quad b(u, v) = (f, v)_{0,D} \qquad \text{for all} \qquad v \in \check{P}^m_0(D).$$

Equivalently, u is the unique solution in

$$D(\mathcal{B}) = \check{P}^{2m}(D) \cap \check{P}^m_0(D)$$

of

$$\mathcal{B}u = f.$$

For each $\varepsilon \in (0, \varepsilon_0]$, denote by w_ε the unique solution in $\check{P}^{m'}_0(D)$ of

$$(6.11)\quad \varepsilon a(w_\varepsilon, v) + b(w_\varepsilon, v) = (f, v)_{0,D} \qquad \text{for all} \qquad v \in P^{m'}_0(D).$$

Then w_ε is the unique solution in $D(A_\varepsilon)$ of

$$A_\varepsilon w_\varepsilon = (\varepsilon\mathcal{U} + \mathcal{B})w_\varepsilon = f.$$

Further, for each $\varepsilon \in (0, \varepsilon_0]$, let u_ε be the unique solution in $\check{P}^{m'}_0(D)$ of

$$(6.12)\quad \varepsilon a(u_\varepsilon, v) + b(u_\varepsilon, v) = (f_\varepsilon, v)_{0,D} \quad \text{for all} \quad v \in \check{P}^{m'}_0(D),$$

i.e., u_ε is the unique solution in $D(A_\varepsilon)$ of

$$A_\varepsilon u_\varepsilon = (\varepsilon\mathcal{U} + \mathcal{B})u_\varepsilon = f_\varepsilon.$$

THEOREM 6.1. *Assume hypotheses* (6.1), (6.2), (6.10), *and* (6.12). *Then one has:*

i) *if* $m' = m$ *and* $|f_\varepsilon - f|_{0,D} = 0(\varepsilon)$ *as* $\varepsilon \downarrow 0$, *then*

$$|u_\varepsilon - u|_{m,D} = 0(\varepsilon) \qquad \text{as} \qquad \varepsilon \downarrow 0;$$

ii) *if* $m' > m$ *and* $|f_\varepsilon - f|_{0,D} = o(\varepsilon^\tau)$ *for all* $\tau < 1/4(m' - m)$ *as* $\varepsilon \downarrow 0$, *then*

$$|u_\varepsilon - u|_{m,D} = o(\varepsilon^\tau) \quad \textit{for all} \quad \tau < 1/4(m' - m) \quad \textit{as} \quad \varepsilon \downarrow 0.$$

Proof. — Let $\mathcal{A}$ be the operator in $\check{P}_0^m(D)$ associated with $a(v, w)$ relative to $b(v, w)$. Then if $m' = m$, $\mathcal{A}$ is a bounded operator on $\check{P}_0^m(D)$, and i) follows from Theorem 4.1, i).

So let $m' > m$, $H_1 = \check{P}_0^{2m'-m}(D)$, $V_1 = D(\mathcal{A})$, and

$$H_0 = V_0 = \check{P}_0^m(D).$$

Then by Proposition 6.1, $H_1 \underset{c}{\subset} V_1$, and so by quadratic interpolation, $H_\tau \underset{c}{\subset} V_\tau$ for all $\tau \in [0,1]$. Since u, the solution of (6.10), is in $\check{P}^{2m}(D) \cap \check{P}_0^m(D)$, Theorem 5.2 gives both that $u \in \check{P}_0^\alpha(D)$ and $\check{P}_0^\alpha(D)$ is an interpolation space by quadratic interpolation between H_1 and H_0 for all α such that $m \leqslant \alpha < m + 1/2$. Thus $u \in H_\tau \subset V_\tau$ for all τ such that $2(m' - m)\tau < 1/2$, i.e., $\tau < 1/4(m' - m)$. ii) now follows from Theorem 4.1, ii).

THEOREM 6.2. — *Let* $m' > m$ *and assume that* D *is a bounded domain of class* $C^{2m'}$. *Let* $b(v, w)$ *be Hermitian symmetric with* $b(v, v) \geqslant \beta |v|^2_{m,D}$, $\beta > 0$, *for all* $v \in \check{P}_0^m(D)$, *and let* a_{ij}, $b_{ij} \in C^{\max(|i|,|j|)}(\overline{D})$. *Further assume hypotheses* (6.2), (6.10), *and* (6.11). *Then for any* γ, τ *such that* $0 < \gamma \leqslant \tau < 1/4(m' - m)$,

$$|w_\varepsilon - u|_{m+2\gamma(m'-m),D} = o(\varepsilon^{\tau-\gamma}) \qquad \textit{as} \qquad \varepsilon \downarrow 0.$$

In particular, $w_\varepsilon \to u$ *in* $\check{P}^\alpha(D)$ *for all* α *such that* $m \leqslant \alpha < m + 1/2$.

Proof. — Since D is a bounded domain of class $C^{2m'}$, the regularity results of [21], [3], Chapter 9, state that for each $\varepsilon \in (0, \varepsilon_0]$, $D(A_\varepsilon) = D(A_\varepsilon^*) = \check{P}^{2m'}(D) \cap \check{P}_0^{m'}(D)$ and

$$(6.13) \quad |A_\varepsilon v|_{0,D} \sim |A_\varepsilon^* v|_{0,D} \sim |v|_{2m',D} \quad \text{on} \quad \check{P}^{2m'}(D) \cap \check{P}_0^{m'}(D).$$

Then (6.6), (6.13), Theorems 5.2, 5.3, and duality yield,

$$|A_\varepsilon v|_{-m,D} \sim |v|_{2m'-m,D} \qquad \text{on} \qquad \check{P}^{2m'-m}(D) \cap \check{P}_0^{m'}(D).$$

Hence the argument used in the proof of Proposition 6.1 gives $D(\mathcal{A}) = \check{P}^{2m'-m}(D) \cap \check{P}_0^{m'}(D)$.

Now let $\mathcal{A}'$ be the operator in $\check{P}_0^m(D)$ associated with $a^*(v, w)$ relative to $b^*(v, w) = b(v, w)$, i.e., $a^*(v, w) = b(\mathcal{A}'v, w)$. Then by using (4.12), Lemma 4.2 and the method of the preceding paragraph, one obtains $D(\mathcal{A}') = \check{P}^{2m'-m}(D) \cap \check{P}_0^{m'}(D)$. Thus Theorem 4.2 is applicable.

It now follows from Theorems 5.2 and 5.3 that for $0 \leqslant \tau < 1/4(m' - m)$, the interpolation spaces by quadratic interpolation between $V_1 = D(\mathcal{A})$ and $V_0 = \check{P}_0^m(D)$ are the spaces $\check{P}^{m+2\tau(m'-m)}(D) \cap \check{P}_0^m(D)$. Since $u \in \check{P}^{2m}(D) \cap P_0^m(D)$, $u \in V_\tau$ for all $\tau < 1/4(m' - m)$. The theorem now follows from Theorem 4.2.

The following theorem is readily proven by the methods of this chapter. It supplements the results of Huet [9] and Ton [22] with $p = 2$ and homogeneous Dirichlet boundary conditions. The assumptions on D and the right hand side of the equation are, for the most part, weaker here.

THEOREM 6.3. — *Let* D *be a bounded Lipschitzian graph domain and assume that for* $0 < \varepsilon \leqslant \varepsilon_0$, *there exist* $\eta(\varepsilon) > 0$, $\eta(\varepsilon) \to 0$ *as* $\varepsilon \downarrow 0$ *and* $\mu > 0$ *such that*

$$|\varepsilon b(v, v) + (v, v)_{0,D}| \geqslant \eta(\varepsilon)|v|^2_{m,D} + \mu|v|^2_{0,D} \qquad \textit{for all} \qquad v \in \check{P}_0^m(D).$$

Let $f, f_\varepsilon \in L^2(D)$, $0 < \varepsilon \leqslant \varepsilon_0$. *Let* u_ε *be the unique solution in* $D(\mathcal{B}) = \{v \in \check{P}_0^m(D) : \mathcal{B}u \in L^2(D)\}$ *of*

$$\varepsilon \mathcal{B} u_\varepsilon + u_\varepsilon = f_\varepsilon$$

and let w_ε *be the unique solution in* $D(\mathcal{B})$ *of*

$$\varepsilon \mathcal{B} w_\varepsilon + w_\varepsilon = f.$$

Then one has:

i) *if for some* $\alpha \in [0, 2m)$, $f \in \check{P}^\alpha(D)$, $J_{\alpha,D}(f) < \infty$, *and* $|f_\varepsilon - f|_{0,D} = o(\varepsilon^{\alpha/2m})$ *as* $\varepsilon \downarrow 0$, *then*

$$|u_\varepsilon - f|_{0,D} = o(\varepsilon^{\alpha/2m}) \qquad \textit{as} \qquad \varepsilon \downarrow 0;$$

ii) *if* $f \in \check{P}_0^{2m}(D)$ *and* $|f_\varepsilon - f|_{0,D} = O(\varepsilon)$ *as* $\varepsilon \downarrow 0$, *then*

$$|u_\varepsilon - f|_{0,D} = O(\varepsilon) \qquad \textit{as} \qquad \varepsilon \downarrow 0;$$

iii) *if* $b(v, w)$ *is Hermitian symmetric and for some* $\alpha \in (0,2m]$, $f \in \check{P}^{\alpha}(D)$ *and* $J_{\alpha,D}(f) < \infty$, *then for any* $\gamma \in (0, \alpha]$,

$$|w_\varepsilon - f|_{\gamma,D} = o(\varepsilon^\beta) \quad \textit{as} \quad \varepsilon \downarrow 0 \quad \textit{where} \quad \beta = (\alpha - \gamma)/2m;$$

iv) *if* D *is of class* C^{2m}, $b_{ij} \in C^{\max(|i|,|j|)}(\overline{D})$ *and for some* $\alpha \in (0, m]$, $f \in \check{P}^{\alpha}(D)$ *and* $J_{\alpha,D}(f) < \infty$, *then for any* $\gamma \in (0, \alpha]$,

$$|w_\varepsilon - f|_{\gamma,D} = o(\varepsilon^\beta) \quad \textit{as} \quad \varepsilon \downarrow 0 \quad \textit{where} \quad \beta = (\alpha - \gamma)/2m;$$

v) if D is of class C^{2m}, $b_{ij} \in C^{\max(|i|,|j|)}(\overline{D})$ and for some $\alpha \in [m, 2m]$, $f \in \check{P}^{\alpha}(D) \cap \check{P}^{m}_{0}(D)$, then for any $\gamma \in (0, \alpha]$,

$$|w_\varepsilon - f|_{\gamma,D} = o(\varepsilon^\beta) \quad \text{as} \quad \varepsilon \downarrow 0 \quad \text{where} \quad \beta = (\alpha - \gamma)/2m.$$

If $m = 1$, iv) *and* v) *remain true for* D *a* $C^{1,1}$*-bounded convex domain.*

Some examples will now be given to indicate the degree of precision of the methods of this paper. The calculations are elementary but tedious and are omitted.

Example 6.1. — Let $D = (0,1)$, let w_ε be the unique solution in $\check{P}^4(D) \cap \check{P}^2_0(D)$ of

$$\left(\varepsilon \frac{d^4}{dx^4} - \frac{d^2}{dx^2}\right) w_\varepsilon = 1, \qquad w_\varepsilon(0) = w'_\varepsilon(0) = w_\varepsilon(1) = w'_\varepsilon(1) = 0$$

and let u be the unique solution in $\check{P}^2(D) \cap \check{P}^1_0(D)$ of

$$-u'' = 1, \qquad u(0) = u(1) = 0.$$

Direct calculation gives

$$\varepsilon^{-1/4}|w_\varepsilon - u|_{1,D} = 1/2 + o(1) \qquad \text{as} \qquad \varepsilon \downarrow 0$$

while Theorem 6.1 gives

$$|w_\varepsilon - u|_{1,D} = o(\varepsilon^\tau) \qquad \text{for all} \qquad \tau < 1/4 \qquad \text{as} \qquad \varepsilon \downarrow 0.$$

Example 6.2. — Let $D = (0,1)$ and let w_ε be the unique solution in $\check{P}^2(D) \cap \check{P}^1_0(D)$ of

$$-\varepsilon w''_\varepsilon + w_\varepsilon = 1, \qquad w_\varepsilon(0) = w_\varepsilon(1) = 0.$$

Theorem 6.3. gives $w_\varepsilon \to 1$ in $\check{P}^\alpha(D)$ for all $\alpha < 1/2$. By calculating the solution and using the inequality, $x \leqslant \sinh x$

for $x \geqslant 0$, one obtains,

$C(1,1/2)d_{1/2}(w_\varepsilon - 1) \geqslant K(\varepsilon)$ where $K(\varepsilon) \to 2$ as $\varepsilon \downarrow 0$.

So w_ε does not converge to 1 in $\check{P}^{1/2}(D)$ even though the norm of $\check{P}^{1/2}(D)$ is strictly weaker than the interpolation norm of $V_{1/4}$. Thus Theorem 6.3 in fact gives the strongest potential norm in which one can expect convergence of w_ε to f for f an arbitrary element of $C^\infty(\overline{D})$.

Example 6.3. — Let $D = (0,1)$ and consider the following « intermediate » problem.

$$\left(\varepsilon \frac{d^4}{dx^4} - \frac{d^2}{dx^2}\right) w_\varepsilon = 1, \qquad w_\varepsilon(0) = w''_\varepsilon(0) = w_\varepsilon(1) = w''_\varepsilon(1) = 0,$$
$$-u'' = 1, \qquad u(0) = u(1) = 0.$$

Direct calculation gives

$$\varepsilon^{-3/4}|w_\varepsilon - u|_{1,D} = 2^{-1/2} + o(1) \quad \text{as} \quad \varepsilon \downarrow 0,$$

and

$$\varepsilon^{-1/4}|w_\varepsilon - u|_{2,D} = 1 + o(1) \quad \text{as} \quad \varepsilon \downarrow 0.$$

It is not difficult to work the interpolation problem directly in this case to obtain $V_0 = \check{P}^1_0(D)$, $V_{1/2} = \check{P}^2(D) \cap \check{P}^1_0(D)$, and that Theorems 4.1 and 4.2 give,

$$|w_\varepsilon - u|_{1,D} = o(\varepsilon^\tau) \quad \text{for all} \quad \tau < 3/4,$$

and

$$|w_\varepsilon - u|_{2,D} = o(\varepsilon^{\tau-1/2}) \quad \text{for all} \quad \tau < 3/4.$$

In conclusion, a comment on singular perturbation of Neumann problems (i.e. the coerciveness inequalities (6.1) and (6.2) are assumed over $\check{P}^m(D)$ and $\check{P}^{m'}(D)$ respectively) is in order. If D is a bounded smooth domain, $a(v, v) \geqslant 0$ for all $v \in \check{P}^{m'}(D)$, and there exists $\beta > 0$ such that $b(v, v) \geqslant \beta|v|^2_{m,D}$ for all $v \in \check{P}^m(D)$, then the rates of convergence in the problems corresponding to theorems 6.1 and 6.2 are faster than for the Dirichlet problems. This is easy to see, even without having a characterization of the higher order interpolation spaces. For, in this case, $D(A_\varepsilon^{1/2}) = V_{1/2} = \check{P}^{m'}(D)$ for any $\varepsilon \in (0, \varepsilon_0]$. Assuming that f and the coefficients of

$a(v, w)$ and $b(v, w)$ are smooth enough, the results of Grisvard [7] and the methods of this paper yield, $|w_\varepsilon - u|_{m,D} = o(\varepsilon^\tau)$ for all $\tau < (2m' - 2m + 1)/4(m' - m)$, and $w_\varepsilon \to u$ in $\check{P}^\alpha(D)$ for all α such that $m \leqslant \alpha < m' + 1/2$. Similar results follow for problems in which V and V_0 are obtained by other homogeneous boundary conditions as in [3], Chapter 10.

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