ANNALES

DE

L'INSTITUT FOURIER

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Tome 65, nº 3 (2015), p. 1387-1422.

<http://aif.cedram.org/item?id=AIF_2015__65_3_1387_0>

Ann. Inst. Fourier, Grenoble
65, 3 (2015) 1387-1422

ON MOEBIUS AND CONFORMAL MAPS BETWEEN BOUNDARIES OF CAT(−1) SPACES

by Kingshook BISWAS (*)

Abstract. — We study Moebius and conformal maps between boundaries of CAT(−1) spaces equipped with visual metrics. We show that any Moebius map between boundaries of proper, geodesically complete CAT(−1) spaces extends to a (1, log 2)-quasi-isometry between the spaces. For a conformal map f between boundaries of spaces X, Y we define a function $S(f)$ on the space of geodesics of X, called the integrated Schwarzian of f, which measures the deviation of the conjugacy of geodesic flows induced by f from being flip equivariant. The integrated Schwarzian $S(f)$ vanishes identically if f is Moebius. Conversely, when X is a simply connected manifold with pinched negative sectional curvatures, we obtain a formula for the cross-ratio distortion of f in terms of $S(f)$ which shows that if $S(f)$ vanishes then f is Moebius.

Résumé. — Nous étudions les applications conformes de Moebius entre les bords des espaces CAT(−1) équipés avec leurs métriques visuelles. Nous démontrons qu'une application de Moebius entre les bords des espaces CAT(−1) propres et géodésiquement complets s'étend à une (1, log 2)-quasi-isométrie. Nous définissons pour une application conforme f entre les bords des espaces X, Y une fonction $S(f)$, appelée le Schwarzian integrée de f, qui quantifie la déviation de conjugaison des flots géodésiques induits par f d'être équivariant par rapport aux flips. Le Schwarzian integré s'annule si f est de Moebius. Réciproquement, si X est une variété riemannienne simplement connexe à courbure $-b^2 \leqslant K \leqslant -1$, nous obtenons une formule pour la distortion du birapport par f, qui montre que f est de Moebius si $S(f)$ s'annule.

1. Introduction

The problems we consider in this article are motivated by rigidity results for negatively curved manifolds. The Mostow Rigidity Theorem asserts that an isomorphism between fundamental groups of closed hyperbolic

Keywords: CAT(-1) space, cross-ratio, Moebius, conformal.
Math. classification: 53C24.
(*) The author is grateful to Marc Bourdon and Mahan Mj for helpful discussions. The author was supported by CEFIPRA grant no. 4301-1: "Kleinian groups: geometric and analytic aspects".

n-manifolds (where $n \geqslant 3$) is induced by an isometry between the manifolds. Thus hyperbolic manifolds are determined upto isometry by their fundamental groups. It is natural to ask for closed manifolds with variable negative curvature what extra information over and above the fundamental group is required to determine the metric. Recall that each free homotopy class of closed curves in a closed negatively curved manifold contains a unique closed geodesic. Thus a closed negatively curved manifold X comes equipped with a length function $l_X : \pi_1(X) \to \mathbb{R}^+$ (which is constant on conjugacy classes). The marked length spectrum rigidity problem asks whether the pair $(\pi_1(X), l_X)$ (the *marked length spectrum* of X) determines the manifold X upto isometry. More precisely, if X, Y are closed negatively curved n-manifolds and $\Phi : \pi_1(X) \to \pi_1(Y)$ is an isomorphism such that $l_X = l_Y \circ \Phi$, then is Φ induced by an isometry $F : X \to Y$?

Otal proved that this is indeed the case if the dimension $n = 2$ [7]. The problem remains open in higher dimensions. It is known however to be equivalent to two related problems, which we briefly describe. The geodesic conjugacy problem asks whether the existence of a homeomorphism between the unit tangent bundles $\phi : T^1X \to T^1Y$ conjugating the geodesic flows implies isometry of the manifolds. Hamenstadt proved that equality of marked length spectra is equivalent to existence of a geodesic conjugacy [5]. Thus the problems of marked length spectrum rigidity and geodesic conjugacy are equivalent.

We recall that the boundary at infinity ∂X of a CAT(-1) space carries a natural class of metrics $\rho_x, x \in X$ called visual metrics, which are Moebius equivalent, in the sense that metric cross-ratios are the same for all metrics ρ_x. For background on visual metrics and cross-ratios we refer to Bourdon [1], [2]. Recall that a continuous embedding $f : \partial X \to \partial Y$ between boundaries of CAT(-1) spaces X, Y is Moebius if it preserves cross-ratios. Any isometric embedding $F : X \to Y$ extends to a Moebius embedding $\partial F : \partial X \to \partial Y$. Bourdon showed in [1], that for a Gromov-hyperbolic group Γ with two quasi-convex actions on CAT(-1) spaces X, Y, the natural Γ-equivariant homeomorphism f between the limit sets $\Lambda X, \Lambda Y$ is Moebius if and only if there is a Γ-equivariant conjugacy of the abstract geodesic flows $\mathcal{G}\Lambda X$ and $\mathcal{G}\Lambda Y$ compatible with f. In particular for X, Y the universal covers of two closed negatively curved manifolds $\overline{X}, \overline{Y}$, it follows that the geodesic flows of $\overline{X}, \overline{Y}$ are conjugate if and only if the induced equivariant boundary map $f : \partial X \to \partial Y$ is Moebius.

Otal showed that equality of the marked length spectrum for two negatively curved metrics on the same closed manifold is equivalent to the existence of an equivariant Moebius map between the boundaries at infinity of the universal covers [8]. We remark that the same conclusion holds when the marked length spectra of two closed negatively curved manifolds coincide (the manifolds are not necessarily assumed to be diffeomorphic), using the results of Hamenstadt (equality of the marked length spectrum being equivalent to conjugacy of geodesic flows) and Bourdon (conjugacy of geodesic flows being equivalent to the boundary map being Moebius).

It follows that the marked length spectrum, the geodesic flow, and the Moebius structure on the boundary at infinity of the universal cover are all equivalent data for a closed negatively curved manifold, and the question becomes whether any one of these is enough to determine the metric. We discuss in section 5 the proofs of these equivalences. In the case of simply connected, complete Riemannian manifolds of sectional curvature bounded above by -1, the marked length spectrum no longer makes sense, but one may still consider the correspondence between Moebius maps and geodesic conjugacies. We define a certain uniform continuity property for geodesic conjugacies, *uniform continuity along geodesics* (which is satisfied in particular by uniformly continuous maps). Recall that a CAT(-1) space X is *geodesically complete* if every geodesic segment in X can be extended (not necessarily uniquely) to a bi-infinite geodesic. We show in Section 4:

Theorem 1.1. — *Let X be a simply connected complete Riemannian manifold with sectional curvatures bounded above by -1, and let Y be a proper geodesically complete CAT(-1) space. If there is a homeomorphism $\phi : T^1X \to \mathcal{G}Y$ conjugating the geodesic flows of X and Y which is uniformly continuous along geodesics then ϕ induces a map $f : \partial X \to \partial Y$ which is Moebius.*

Recall that there is a notion of a conformal homeomorphism between metric spaces, in particular between boundaries of CAT(-1) spaces equipped with visual metrics. We consider C^1 *conformal* maps, i.e. those for which the pointwise derivative is a continuous function. A C^1 conformal map $f : \partial X \to \partial Y$ between boundaries of CAT(-1) spaces induces a topological conjugacy $\phi : \mathcal{G}X \to \mathcal{G}Y$ between the abstract geodesic flows of X and Y (following Bourdon [1]), where $\mathcal{G}X, \mathcal{G}Y$ are the spaces of bi-infinite geodesics in X and Y. The conjugacy is equivariant with respect to the flips if f is Moebius. We define a function $S(f) : \partial^2 X \to \mathbb{R}$, the *integrated Schwarzian* of f, which measures the deviation of the conjugacy from being flip-equivariant, vanishing in particular if f is Moebius. Coversely, if the domain X is a simply connected negatively curved manifold also satisfying a lower curvature bound $-b^2 \leqslant K \leqslant -1$, then, as in the classical case, bounds on the integrated Schwarzian imply bounds on cross-ratio distortion. Indeed we have an exact formula for the cross-ratio distortion:

Theorem 1.2. — *Let X be a simply connected complete Riemannian manifold with sectional curvatures satisfying $-b^2 \leqslant K \leqslant -1$ for some $b \geqslant 1$, and let Y be a proper geodesically complete CAT(-1) space. Let $f : U \subset \partial X \to V \subset \partial Y$ be a C^1 conformal map between open subsets U, V. Then*

$$\log \frac{[f(\xi), f(\xi'), f(\eta), f(\eta')]}{[\xi, \xi', \eta, \eta']} = \frac{1}{2}\left(S(f)(\xi,\eta) + S(f)(\xi',\eta') - S(f)(\xi,\eta') - S(f)(\xi',\eta)\right)$$

for all $(\xi, \xi', \eta, \eta') \in \partial^4 U$. In particular f is Moebius if and only if $S(f) \equiv 0$.

The integrated Schwarzian also satisfies a cocycle identity, thus two C^1 conformal maps differ by post-composition with a Moebius map if and only if their integrated Schwarzians are equal, as in the classical case.

In the case of a lower curvature bound we have a converse to Theorem 1.1 above:

Theorem 1.3. — *Let X, Y be as in the previous theorem and $f : \partial X \to \partial Y$ a C^1 conformal map. Then the induced topological conjugacy of geodesic flows $\phi : T^1 X \to \mathcal{G}Y$ is uniformly continuous along geodesics if and only if f is Moebius.*

We then consider in the more general context of CAT(-1) spaces, the question of whether a Moebius embedding $f : \partial X \to \partial Y$ between the boundaries of two CAT(-1) spaces extends to an isometric embedding $F : X \to Y$. In [2], Bourdon proved the following Theorem:

Theorem 1.4 (Bourdon). — *If X is a rank one symmetric space of noncompact type with maximum of sectional curvatures equal to -1 and Y a CAT(-1) space then any Moebius embedding $f : \partial X \to \partial Y$ extends to an isometric embedding $F : X \to Y$.*

We consider the general case where the domain X is an arbitrary CAT(-1) space. We prove the following in section 6:

Theorem 1.5. — *Let X, Y be proper geodesically complete CAT(-1) spaces such that ∂X has at least four points, and let $f : \partial X \to \partial Y$ be a Moebius homeomorphism. Then f extends to a $(1, \log 2)$-quasi-isometry $F : X \to Y$, with image $\log 2$-dense in Y.*

The proof of the above Theorem involves a study of the space $\mathcal{M}(\partial X)$ of metrics on the boundary ∂X of a proper geodesically complete CAT(-1) space X which are Moebius equivalent to a visual metric. The key point is that there is a natural metric $d_{\mathcal{M}}$ on $\mathcal{M}(\partial X)$ such that the map $i_X : X \to \mathcal{M}(\partial X)$ sending a point $x \in X$ to the visual metric ρ_x based at x is an isometric embedding. The space $(\mathcal{M}(\partial X), d_{\mathcal{M}})$ is itself isometric to a closed, locally compact subspace of the Banach space $C(\partial X)$ of continuous functions on ∂X. By studying the derivative of the embedding i_X along geodesics in X, we show that it has image $\frac{1}{2}\log 2$-dense in $\mathcal{M}(\partial X)$. Thus we may define a nearest-point projection map (not unique) $\pi_X : \mathcal{M}(\partial X) \to X$ which is a $(1, \log 2)$ quasi-isometry.

A Moebius map $f : \partial X \to \partial Y$ induces a natural map $\hat{f} : \mathcal{M}(\partial X) \to \mathcal{M}(\partial Y)$ (by push-forward of metrics) which is a surjective isometry. The extension $F : X \to Y$ of f is then defined by $F = \pi_Y \circ \hat{f} \circ i_X$.

In the case of a metric tree X we can show that the embedding i_X surjects onto $\mathcal{M}(\partial X)$. We thus obtain a new proof of a result of Coornaert [4]:

Theorem 1.6. — *Let X, Y be proper geodesically complete metric trees such that ∂X has at least four points and let $f : \partial X \to \partial Y$ be a Moebius homeomorphism. Then f extends to a surjective isometry $F : X \to Y$.*

For C^1 conformal maps with bounded integrated Schwarzian, similar arguments lead to the following:

Theorem 1.7. — *Let X be a simply connected complete Riemannian manifold with sectional curvatures satisfying $-b^2 \leqslant K \leqslant -1$ for some $b \geqslant 1$, and let Y be a proper geodesically complete CAT(-1) space. Let $f : \partial X \to \partial Y$ be a C^1 conformal map such that $S(f)$ is bounded. Then f extends to a $(1, \log 2 + 12||S(f)||_\infty)$-quasi-isometry $F : X \to Y$. If Y is also a simply connected complete Riemannian manifold with sectional curvatures satisfying $-b^2 \leqslant K \leqslant -1$ for some $b \geqslant 1$, then the image is $(\log 2 + 12||S(f)||_\infty)$-dense in Y.*

We have as corollaries of the above theorems the following:

Theorem 1.8. — *Let X be a simply connected complete Riemannian manifold with sectional curvatures bounded above by -1 and Y a proper geodesically complete CAT(-1) space. Suppose that there is a conjugacy $\phi : T^1 X \to \mathcal{G}Y$ of geodesic flows which is uniformly continuous along geodesics. Then:*

(1) *There is a $(1, \log 2)$-quasi-isometry $F : X \to Y$ with image $\frac{1}{2}\log 2$-dense in Y.*

(2) *If X is a rank one symmetric space of noncompact type with maximum of sectional curvatures equal to -1, then F can be taken to be a surjective isometry.*

Finally, using the almost-isometric extension of Moebius maps, we obtain in section 7 a dynamical classification of Moebius self-maps into three types, elliptic, parabolic and hyperbolic:

THEOREM 1.9. — *Let X be a proper geodesically complete CAT(-1) space and $f : \partial X \to \partial X$ a Moebius self-map of its boundary. Then one of the following three mutually exclusive cases holds:*

(1) *For all $x \in X$, the iterates $f^n : (\partial X, \rho_x) \to (\partial X, \rho_x)$ are uniformly bi-Lipschitz (we say f is* elliptic*).*
(2) *There is a unique fixed point $\xi_0 \in \partial X$ of f such that $f^n(\xi) \to \xi_0$ for all ξ as $n \to \pm\infty$ (we say f is* parabolic*).*
(3) *There is a pair of distinct fixed points ξ_+, ξ_- of f such that for all $\xi \in \partial X - \{\xi_+, \xi_-\}$, $f^n(\xi) \to \xi_+$ as $n \to +\infty$ and $f^n(\xi) \to \xi_-$ as $n \to -\infty$ (we say f is* hyperbolic*).*

2. Spaces of Moebius equivalent metrics

Let (Z, ρ_0) be a compact metric space with at least four points. For a metric ρ on Z we define the metric cross-ratio with respect to ρ of a quadruple of distinct points (ξ, ξ', η, η') of Z by

$$[\xi\xi'\eta\eta']_\rho := \frac{\rho(\xi,\eta)\rho(\xi',\eta')}{\rho(\xi,\eta')\rho(\xi',\eta)}$$

We say that a diameter one metric ρ on Z is *antipodal* if for any $\xi \in Z$ there exists $\eta \in Z$ such that $\rho(\xi, \eta) = 1$. We assume that ρ_0 is diameter one and antipodal. We say two metrics ρ_1, ρ_2 on Z are *Moebius equivalent* if their metric cross-ratios agree:

$$[\xi\xi'\eta\eta']_{\rho_1} = [\xi\xi'\eta\eta']_{\rho_2}$$

for all (ξ, ξ', η, η'). We define

$$\mathcal{M}(Z, \rho_0) := \{\rho : \rho \text{ is an antipodal, diameter one metric on } Z \text{ Moebius equivalent to } \rho_0\}$$

We will write $\mathcal{M}(Z, \rho_0) = \mathcal{M}$. Note we do not assume that the metrics $\rho \in \mathcal{M}$ induce the same topology on Z as ρ_0, but we will show that they are indeed all bi-Lipschitz equivalent to each other. For $\rho_1, \rho_2 \in \mathcal{M}$ we define a positive function on Z called the derivative of ρ_2 with respect to ρ_1 by

$$\frac{d\rho_2}{d\rho_1}(\xi) := \frac{\rho_2(\xi,\eta)\rho_2(\xi,\eta')\rho_1(\eta,\eta')}{\rho_1(\xi,\eta)\rho_1(\xi,\eta')\rho_2(\eta,\eta')}$$

where $\eta, \eta' \in Z$ are distinct points not equal to ξ.

LEMMA 2.1. — *The function $\frac{d\rho_2}{d\rho_1}$ is well-defined.*

Proof. — Given two pairs of distinct points η, η' and β, β' not equal to x, the desired equality

$$\frac{\rho_2(\xi,\eta)\rho_2(\xi,\eta')\rho_1(\eta,\eta')}{\rho_1(\xi,\eta)\rho_1(\xi,\eta')\rho_2(\eta,\eta')} = \frac{\rho_2(\xi,\beta)\rho_2(\xi,\beta')\rho_1(\beta,\beta')}{\rho_1(\xi,\beta)\rho_1(\xi,\beta')\rho_2(\beta,\beta')}$$

follows from the equality

$$[\xi\beta\eta\eta']_{\rho_2}[\xi\eta\eta'\beta']_{\rho_2} = [\xi\beta\eta\eta']_{\rho_1}[\xi\eta\eta'\beta']_{\rho_1}$$

□

The next Lemma follows from a straightforward computation using the definition of the derivative, we omit the proof:

LEMMA 2.2 (Chain Rule). — *For $\rho_1, \rho_2, \rho_3 \in \mathcal{M}$ we have*

$$\frac{d\rho_3}{d\rho_1} = \frac{d\rho_3}{d\rho_2}\frac{d\rho_2}{d\rho_1}$$

and

$$\frac{d\rho_2}{d\rho_1} = 1/\left(\frac{d\rho_1}{d\rho_2}\right)$$

LEMMA 2.3. — *For $\rho \in \mathcal{M}$ the function $f = \frac{d\rho}{d\rho_0}$ is bounded.*

Proof. — Suppose not, let $\xi_n \in Z$ be a sequence such that $f(\xi_n) \to \infty$. Passing to a subsequence we may assume $\xi_n \to \xi$, choose η, η' distinct points in Z not equal to ξ, then we have

$$\begin{aligned}\limsup f(\xi_n) &= \limsup \frac{\rho(\xi_n,\eta)\rho(\xi_n,\eta')\rho_0(\eta,\eta')}{\rho_0(\xi_n,\eta)\rho_0(\xi_n,\eta')\rho(\eta,\eta')} \\ &\leqslant \frac{1}{\rho_0(\xi,\eta)\rho_0(\xi,\eta')\rho(\eta,\eta')}\ ,\end{aligned}$$

which is a contradiction. □

LEMMA 2.4 (Geometric Mean-Value Theorem).

$$\rho_2(\xi,\eta)^2 = \rho_1(\xi,\eta)^2\frac{d\rho_2}{d\rho_1}(\xi)\frac{d\rho_2}{d\rho_1}(\eta)$$

for all $\xi, \eta \in Z$.

Proof. — Given $\xi \neq \eta$ choose a point β distinct from ξ, η, then by definition we may write

$$\frac{d\rho_2}{d\rho_1}(\xi) = \frac{\rho_2(\xi,\eta)\rho_2(\xi,\beta)\rho_1(\eta,\beta)}{\rho_1(\xi,\eta)\rho_1(\xi,\beta)\rho_2(\eta,\beta)} \ , \ \frac{d\rho_2}{d\rho_1}(\eta) = \frac{\rho_2(\eta,\xi)\rho_2(\eta,\beta)\rho_1(\xi,\beta)}{\rho_1(\eta,\xi)\rho_1(\eta,\beta)\rho_2(\xi,\beta)}$$

from which it follows that

$$\frac{d\rho_2}{d\rho_1}(\xi)\frac{d\rho_2}{d\rho_1}(\eta) = \left(\frac{\rho_2(\xi,\eta)}{\rho_1(\xi,\eta)}\right)^2$$

□

For $\rho \in \mathcal{M}$ since $\frac{d\rho}{d\rho_0}$ is bounded it follows from the above Lemma that $\rho \leqslant K\rho_0$, hence the functions $\xi \mapsto \rho(\xi,\eta)$ are continuous for all $\eta \in Z$, therefore the functions $\frac{d\rho}{d\rho_0}$ are continuous. Since $\frac{d\rho_2}{d\rho_1} = \frac{d\rho_2}{d\rho_0} / \frac{d\rho_1}{d\rho_0}$ it follows that all functions $\frac{d\rho_2}{d\rho_1}$ are continuous, so bounded above and below by positive constants, hence by the above Lemma all metrics $\rho \in \mathcal{M}$ are bi-Lipschitz to each other and induce the same topology on Z as ρ_0. The following Lemma justifies the use of the term 'derivative':

Lemma 2.5. — *If $\xi \in Z$ is not an isolated point then*

$$\frac{d\rho_2}{d\rho_1} = \lim_{\eta \to \xi} \frac{\rho_2(\xi,\eta)}{\rho_1(\xi,\eta)}$$

Proof. — We have

$$\begin{aligned} \frac{\rho_2(\xi,\eta)}{\rho_1(\xi,\eta)} &= \frac{d\rho_2}{d\rho_1}(\xi)^{1/2}\frac{d\rho_2}{d\rho_1}(\eta)^{1/2} \\ &\to \frac{d\rho_2}{d\rho_1}(\xi) \end{aligned}$$

as $\eta \to \xi$. □

Lemma 2.6.

$$\max_{\xi \in Z} \frac{d\rho_2}{d\rho_1}(\xi) \cdot \min_{\xi \in Z} \frac{d\rho_2}{d\rho_1}(\xi) = 1$$

Proof. — Let λ, μ denote the maximum and minimum values of $\frac{d\rho_2}{d\rho_1}$ respectively, and let $\xi, \eta \in Z$ denote points where the maximum and minimum values are attained respectively. Choosing $\eta' \in Z$ such that $\rho_1(\xi,\eta') = 1$ gives

$$1 \geqslant \rho_2(\xi,\eta') = \frac{d\rho_2}{d\rho_1}(\xi)^{1/2}\frac{d\rho_2}{d\rho_1}(\eta')^{1/2} \geqslant \lambda^{1/2} \cdot \mu^{1/2}$$

while choosing $\xi' \in Z$ such that $\rho_2(\xi',\eta) = 1$ gives

$$1 \geqslant \rho_1(\xi',\eta) = 1/\left(\frac{d\rho_2}{d\rho_1}(\xi')^{1/2}\frac{d\rho_2}{d\rho_1}(\eta)^{1/2}\right) \geqslant 1/(\lambda^{1/2}\mu^{1/2})$$

hence $\lambda \cdot \mu = 1$. $\square$

We now define for $\rho_1, \rho_2 \in \mathcal{M}$,

$$d_{\mathcal{M}}(\rho_1, \rho_2) := \max_{\xi \in Z} \log \frac{d\rho_2}{d\rho_1}(\xi)$$

Lemma 2.7. — *The function $d_{\mathcal{M}}$ is a metric on $\mathcal{M}$.*

Proof. — For $\rho_1, \rho_2 \in \mathcal{M}$, $(\max_{\xi \in Z} \frac{d\rho_2}{d\rho_1}(\xi))^2 \geqslant (\max_{\xi \in Z} \frac{d\rho_2}{d\rho_1}(\xi)) \cdot (\min_{\xi \in Z} \frac{d\rho_2}{d\rho_1}(\xi)) = 1$, hence $d_{\mathcal{M}}(\rho_1, \rho_2) \geqslant 0$. Moreover $d_{\mathcal{M}}(\rho_1, \rho_2) = 0$ implies $\max_{\xi \in Z} \frac{d\rho_2}{d\rho_1}(\xi) = 1$ hence $\min_{\xi \in Z} \frac{d\rho_2}{d\rho_1}(\xi) = 1$ by the previous Lemma, hence $\frac{d\rho_2}{d\rho_1} \equiv 1$, and it then follows from the Geometric Mean-Value Theorem that $\rho_1 \equiv \rho_2$.

Symmetry of $d_{\mathcal{M}}$ follows from $\frac{d\rho_1}{d\rho_2} = 1/\frac{d\rho_2}{d\rho_1}$ and the previous Lemma, while the triangle inequality follows easily from the Chain Rule $\frac{d\rho_3}{d\rho_1} = \frac{d\rho_3}{d\rho_2}\frac{d\rho_2}{d\rho_1}$. $\square$

Let $(C(Z), ||\cdot||_\infty)$ denote the Banach space of continuous functions on Z equipped with the supremum norm.

Lemma 2.8. — *The map*

$$\begin{aligned} \mathcal{M} &\to C(Z) \\ \rho &\mapsto \log \frac{d\rho}{d\rho_0} \end{aligned}$$

is an isometric embedding.

Proof. — It follows from Lemma 2.6 that we have $\max_{\xi \in Z} \log \frac{d\rho_2}{d\rho_1}(\xi) = ||\log \frac{d\rho_2}{d\rho_1}||_\infty$, hence

$$d_{\mathcal{M}}(\rho_1, \rho_2) = \left|\left|\log \frac{d\rho_2}{d\rho_1}\right|\right|_\infty = \left|\left|\log \frac{d\rho_2}{d\rho_0} - \log \frac{d\rho_1}{d\rho_0}\right|\right|_\infty$$

(where the second equality uses the Chain Rule). $\square$

Lemma 2.9. — *The image of the above embedding is closed in $C(Z)$.*

Proof. — Let $\rho_n \in \mathcal{M}$ such that $g_n = \log \frac{d\rho_n}{d\rho_0}$ converges in $C(Z)$ to g. Define $f = e^g$ and $\rho(\xi, \eta) := \rho_0(\xi, \eta) f(\xi)^{1/2} f(\eta)^{1/2}, \xi, \eta \in Z$, then it follows from the Geometric Mean Value Theorem that $\rho(\xi, \eta) = \lim \rho_n(\xi, \eta)$. Passing to the limit in the triangle inequality for ρ_n gives the triangle inequality for ρ, while symmetry and positivity of ρ are clear, hence ρ is a metric. Moreover it follows easily from the definition of ρ that ρ is Moebius equivalent to ρ_0, and moreover $\frac{d\rho}{d\rho_0} = f$. Since the ρ_n's have diameter one it follows that ρ has diameter less than or equal to one. Given $\xi \in Z$ let

$\eta_n \in Z$ such that $\rho_n(\xi, \eta_n) = 1$, passing to a subsequence we may assume η_n converges to some η, then

$$\begin{aligned}|\rho(\xi,\eta) - \rho_n(\xi,\eta_n)| &\leqslant |\rho(\xi,\eta) - \rho_n(\xi,\eta)| + |\rho_n(\xi,\eta) - \rho_n(\xi,\eta_n)| \\ &\leqslant |\rho(\xi,\eta) - \rho_n(\xi,\eta)| + \rho_n(\eta,\eta_n) \\ &\to 0\end{aligned}$$

since $\rho_0(\eta, \eta_n) \to 0$ and the ρ_n's are uniformly bi-Lipschitz equivalent to ρ_0 (being a bounded sequence in $\mathcal{M}$), hence $\rho(\xi, \eta) = 1$. Thus ρ is of diameter one and is antipodal, hence $\rho \in \mathcal{M}$ and g is the image of ρ under the isometric embedding. □

Lemma 2.10. — *The function $f = \frac{d\rho_2}{d\rho_1} : (Z, \rho_1) \to \mathbb{R}$ is K-Lipschitz where $K = 2(\max_{\xi \in Z} f(\xi))^2$.*

Proof. — Let $\lambda = \max_{\xi \in Z} f(\xi)$. Let $\xi_1, \xi_2 \in Z$. We may assume $f(\xi_1) \geqslant f(\xi_2)$. Choose $\xi \in Z$ such that $\rho_1(\xi_1, \xi) = 1$, then the inequality $|\rho_2(\xi, \xi_1) - \rho(\xi, \xi_2)| \leqslant \rho_2(\xi_1, \xi_2)$ gives, using the Geometric Mean-Value Theorem,

$$f(\xi)^{1/2} \left| f(\xi_1)^{1/2} - \rho_1(\xi,\xi_2) f(\xi_2)^{1/2} \right| \leqslant \rho_1(\xi_1,\xi_2) f(\xi_1)^{1/2} f(\xi_2)^{1/2}$$

and we have

$$\begin{aligned}|f(\xi_1)^{1/2} - \rho_1(\xi,\xi_2) f(\xi_2)^{1/2}| &= f(\xi_1)^{1/2} - \rho_1(\xi,\xi_2) f(\xi_2)^{1/2} \\ &\geqslant f(\xi_1)^{1/2} - f(\xi_2)^{1/2}\end{aligned}$$

which, combined with the previous inequality, gives

$$(1/\lambda^{1/2})(f(\xi_1)^{1/2} - f(\xi_2)^{1/2}) \leqslant \rho_1(\xi_1,\xi_2)\lambda$$

hence

$$\begin{aligned}|f(\xi_1) - f(\xi_2)| &= |(f(\xi_1)^{1/2} - f(\xi_2)^{1/2})(f(\xi_1)^{1/2} + f(\xi_2)^{1/2})| \\ &\leqslant \lambda^{3/2} \rho_1(\xi_1,\xi_2) 2\lambda^{1/2} \\ &= 2\lambda^2 \rho_1(\xi_1,\xi_2)\end{aligned}$$

□

Lemma 2.11. — *The space $(\mathcal{M}, d_{\mathcal{M}})$ is proper, i.e. closed balls are compact. Hence $(\mathcal{M}, d_{\mathcal{M}})$ is also complete.*

Proof. — It follows from the previous Lemma that for a sequence $\rho_n \in \mathcal{M}$ with $d_{\mathcal{M}}(\rho_n, \rho_0)$ bounded, the functions $f_n = \frac{d\rho_n}{d\rho_0}$ are uniformly Lipschitz, and uniformly bounded away from 0 and ∞, hence the functions $g_n = \log f_n$ are uniformly Lipschitz and uniformly bounded. Therefore g_n has a subsequence g_{n_k} converging uniformly to a continuous function g, which by Lemma 2.9 is equal to $\log \frac{d\rho}{d\rho_0}$ for some $\rho \in \mathcal{M}$. It follows from Lemma 2.8 that $\rho_{n_k} \to \rho$ in $\mathcal{M}$. □

3. Visual metrics on the boundary of a CAT(-1) space

Let (X, d_X) be a proper CAT(-1) space such that ∂X has at least four points.

3.1. Definitions

We recall below the definitions and some elementary properties of visual metrics and Busemann functions; for proofs we refer to [1]:

Let $x \in X$ be a basepoint. The *Gromov product* of two points $\xi, \xi' \in \partial X$ with respect to x is defined by

$$(\xi|\xi')_x = \lim_{(a,a')\to(\xi,\xi')} \frac{1}{2}(d_X(x,a) + d_X(x,a') - d_X(a,a'))$$

where a, a' are points of X which converge radially towards ξ and ξ' respectively. The *visual metric* on ∂X based at the point x is defined by

$$\rho_x(\xi, \xi') := e^{-(\xi|\xi')_x}$$

The distance $\rho_x(\xi, \xi')$ is less than or equal to one, with equality iff x belongs to the geodesic $(\xi\xi')$.

Lemma 3.1. — *If X is geodesically complete then ρ_x is a diameter one antipodal metric.*

Proof. — Let $\xi \in \partial X$, then the geodesic ray $[x, \xi)$ extends to a bi-infinite geodesic $(\xi'\xi)$ for some $\xi' \in \partial X$, hence $\rho_x(\xi, \xi') = 1$, hence ρ_x is diameter one and antipodal. □

The Busemann function $B : \partial X \times X \times X \to \mathbb{R}$ is defined by

$$B(\xi, x, y) := \lim_{a\to\xi} d_X(x,a) - d_X(y,a)$$

where $a \in X$ converges radially towards ξ.

It will be convenient to consider the functions on ∂X, $f_{x,y}(\xi) := e^{B(\xi,x,y)}$, $g_{x,y}(\xi) = B(\xi, x, y), \xi \in \partial X, x, y \in X$. The following Lemma is elementary:

Lemma 3.2. — *We have $|g_{x,y}(\xi)| \leqslant d_X(x,y)$ for all $\xi \in \partial x, x, y \in X$. Moreover $g_{x,y}(\xi) = d_X(x,y)$ iff y lies on the geodesic ray $[x, \xi)$ while $g_{x,y}(\xi) = -d_X(x,y)$ iff x lies on the geodesic ray $[y, \xi)$.*

We recall the following Lemma from [1]:

Lemma 3.3. — *For $x, y \in X, \xi, \xi' \in \partial X$ we have*

$$\rho_y(\xi, \xi') = \rho_x(\xi, \xi') f_{x,y}(\xi)^{1/2} f_{x,y}(\xi')^{1/2}$$

An immediate corollary of the above Lemma is the following:

Lemma 3.4. — *The visual metrics $\rho_x, x \in X$ are Moebius equivalent to each other and*

$$\frac{d\rho_y}{d\rho_x} = f_{x,y}$$

Hence the functions $f_{x,y}, g_{x,y}$ are continuous.

It follows that the metric cross-ratio $[\xi\xi'\eta\eta']_{\rho_x}$ of a quadruple (ξ, ξ', η, η') is independent of the choice of $x \in X$. Denoting this common value by $[\xi\xi'\eta\eta']$, it is shown in [2] that the cross-ratio is given by

$$[\xi\xi'\eta\eta'] = \lim_{(a,a',b,b') \to (\xi,\xi',\eta,\eta')} \exp(\frac{1}{2}(d(a,b) + d(a',b') - d(a,b') - d(a',b)))$$

where the points $a, a', b, b' \in X$ converge radially towards $\xi, \xi', \eta, \eta' \in \partial X$.

We assume henceforth that X is a proper, geodesically complete CAT(-1) space. We let $\mathcal{M} = \mathcal{M}(\partial X, \rho_x)$ (this space is independent of the choice of $x \in X$).

Lemma 3.5. — *The map*

$$i_X : X \to \mathcal{M}$$
$$x \mapsto \rho_x$$

is an isometric embedding and the image is closed in $\mathcal{M}$.

Proof. — Given $x, y \in X$, extend $[x, y]$ to a geodesic ray $[x, \xi)$ where $\xi \in \partial X$, then $g_{x,y}(\xi) = d_X(x, y)$ hence $d_{\mathcal{M}}(\rho_x, \rho_y) = \max_{\eta \in \partial X} g_{x,y}(\eta) = d_X(x, y)$, so i_X is an isometric embedding. Given $x_n \in X$ such that $\rho_{x_n} \to \rho \in \mathcal{M}$, since i_X is an isometry and the sequence ρ_{x_n} is bounded in $\mathcal{M}$, so is the sequence x_n in X. Passing to a subsequence we may assume $x_n \to a$ in X, then $d_{\mathcal{M}}(\rho_{x_n}, \rho_a) = d_X(x_n, a) \to 0$ hence $\rho_a = \rho$. $\square$

3.2. Limiting comparison angles and derivatives of visual metrics

For points $a, x, a' \in X$ we denote by $\angle^{(-1)} axa' \in [0, \pi]$ the angle at the vertex corresponding to x in a comparison triangle in $\mathbb{H}^2$ corresponding to the triangle axa' in X. It is easy to show (see [1]) that the map $X \times X \times X \to [0,\pi], (a,x,a') \mapsto \angle^{(-1)} axa'$ extends to a continuous map $\overline{X} \times X \times \overline{X} \to [0,\pi]$, so for $\xi,\xi' \in \partial X$ and $x \in X$ the limiting comparison angle $\angle^{(-1)}\xi x \xi'$ is defined, and moreover

$$\rho_x(\xi,\xi') = \sin(\angle^{(-1)}\xi x \xi'/2)$$

For any point y on the geodesic ray $[x,\xi)$ it follows easily from the CAT(-1) inequality that

$$\angle^{(-1)} yx\xi' \leqslant \angle^{(-1)}\xi x\xi'$$

We note also that if a geodesic segment $[x,y]$ of length δ is common to both rays $[x,\xi)$ and $[x,\xi')$ then $\angle^{(-1)}yx\xi' = 0$ for $d(x,y) \leqslant \delta$.

Lemma 3.6. — *For $x,y \in X$ and $\xi \in \partial X$, we have*

$$f_{x,y}(\xi) = \frac{d\rho_y}{d\rho_x}(\xi) = \frac{1}{(e^t - e^{-t})\sin^2(\angle^{(-1)}yx\xi/2) + e^{-t}}$$

Proof. — Let a tend to ξ radially, let $r = d_X(x,a), s = d_X(a,y)$ and let θ be the comparison angle $\angle^{(-1)}yxa$. By the hyperbolic law of cosine we have

$$\cosh s = \cosh r \cosh t - \sinh r \sinh t \cos\theta$$

which gives

$$e^{s-r} + e^{-s-r} = (1+e^{-2r})\frac{1}{2}(e^t + e^{-t}) - \frac{1}{2}(1-e^{-2r})(e^t - e^{-t})\cos\theta$$

Now as $r \to \infty$ we have $s \to \infty$, and by definition $r - s \to B(\xi,x,y)$, also $\theta \to \angle^{(-1)}yx\xi$, hence letting $r \to \infty$ above gives

$$\begin{aligned}\frac{1}{f_{x,y}(\xi')} &= \frac{1}{2}(e^t + e^{-t}) - \frac{1}{2}(e^t - e^{-t})\cos(\angle^{(-1)}yx\xi)\\ &= (e^t - e^{-t})\sin^2(\angle^{(-1)}yx\xi)/2) + e^{-t}\end{aligned}$$

□

We now consider the behaviour of the derivatives $f_{x,y}$ as $t = d(x,y) \to 0$ and the point y converges radially towards x along a geodesic. For functions F_t on ∂X we write $F_t = o(t)$ if $||F_t||_\infty = o(t)$. We have the following formula, which may be thought of as a formula for the derivative of the map i_X along a geodesic:

Lemma 3.7. — *As $t \to 0$ we have*

$$g_{x,y}(\xi) = \log\frac{d\rho_y}{d\rho_x}(\xi) = t\cos(\angle^{(-1)}yx\xi) + o(t)$$

Proof. — As $t \to 0$ we have

$$\begin{aligned} g_{x,y}(\xi) &= -\log((e^t - e^{-t})\sin^2(\angle^{(-1)} yx\xi/2) + e^{-t}) \\ &= -\log(2t\sin^2(\angle^{(-1)} yx\xi/2) + 1 - t + o(t)) \\ &= -(2t\sin^2(\angle^{(-1)} yx\xi/2) - t) + o(t) \\ &= t\cos(\angle^{(-1)} yx\xi) + o(t) \end{aligned}$$

□

4. Geodesic conjugacies, Moebius maps, conformal maps, and the integrated Schwarzian

We start by recalling the definitions of conformal maps, Moebius maps, and the abstract geodesic flow of a CAT(-1) space.

Definition 4.1. — *A homeomorphism between metric spaces $f : (Z_1, \rho_1) \to (Z_2, \rho_2)$ with no isolated points is said to be* conformal *if for all $\xi \in Z_1$, the limit*

$$df_{\rho_1,\rho_2}(\xi) := \lim_{\eta \to \xi} \frac{\rho_2(f(\xi), f(\eta))}{\rho_1(\xi, \eta)}$$

exists and is positive. The positive function df_{ρ_1,ρ_2} is called the derivative of f with respect to ρ_1, ρ_2. We say f is C^1 conformal *if its derivative is continuous.*

Two metrics ρ_1, ρ_2 inducing the same topology on a set Z, such that Z has no isolated points, are said to be conformal (respectively C^1 conformal) if the map $id_Z : (Z, \rho_1) \to (Z, \rho_2)$ is conformal (respectively C^1 conformal). In this case we denote the derivative of the identity map by $\frac{d\rho_2}{d\rho_1}$.

Definition 4.2. — *A homeomorphism between metric spaces $f : (Z_1, \rho_1) \to (Z_2, \rho_2)$ (where Z_1 has at least four points) is said to be Moebius if it preserves metric cross-ratios with respect to ρ_1, ρ_2. The derivative of f is defined to be the derivative $\frac{df^*\rho_2}{d\rho_1}$ of the Moebius equivalent metrics $f^*\rho_2, \rho_1$ as defined in section 2 (where $f^*\rho_2$ is the pull-back of ρ_2 under f).*

From the results of section 2 it follows that any Moebius map between compact metric spaces with no isolated points is C^1 conformal, and the two definitions of the derivative of f given above coincide. Moreover any Moebius map f satisfies the geometric mean-value theorem,

$$\rho_2(f(\xi), f(\eta))^2 = \rho_1(\xi, \eta)^2 df_{\rho_1,\rho_2}(\xi) df_{\rho_1,\rho_2}(\xi)$$

DEFINITION 4.3. — *Let (X, d) be a CAT(-1) space. The abstract geodesic flow space of X is defined to be the space of bi-infinite geodesics in X,*

$$\mathcal{G}X := \{\gamma : (-\infty, +\infty) \to X | \gamma \text{ is an isometric embedding}\}$$

endowed with the topology of uniform convergence on compact subsets. This topology is metrizable with a distance defined by

$$d_{\mathcal{G}X}(\gamma_1, \gamma_2) := \int_{-\infty}^{\infty} d(\gamma_1(t), \gamma_2(t)) \frac{e^{-|t|}}{2}\, dt$$

We define also a projection

$$\begin{aligned} \pi_X : \mathcal{G}X &\to X \\ \gamma &\mapsto \gamma(0) \end{aligned}$$

It is shown in Bourdon [1] that π_X is 1-Lipschitz.

The abstract geodesic flow of X is defined to be the one-parameter group of homeomorphisms

$$\begin{aligned} \phi_t^X : \mathcal{G}X &\to \mathcal{G}X \\ \gamma &\mapsto \gamma_t \end{aligned}$$

for $t \in \mathbb{R}$, where γ_t is the geodesic $s \mapsto \gamma(s + t)$.

The flip is defined to be the map

$$\begin{aligned} \mathcal{F}_X : \mathcal{G}X &\to \mathcal{G}X \\ \gamma &\mapsto \overline{\gamma} \end{aligned}$$

where $\overline{\gamma}$ is the geodesic $s \mapsto \gamma(-s)$.

We observe that for a simply connected complete Riemannian manifold X with sectional curvatures bounded above by -1, the map

$$\begin{aligned} \mathcal{G}X &\to T^1X \\ \gamma &\mapsto \gamma'(0) \end{aligned}$$

is a homeomorphism conjugating the abstract geodesic flow of X to the usual geodesic flow of X and the flip $\mathcal{F}$ to the usual flip on T^1X.

We note that that for any CAT(-1) space X there is a continuous surjection

$$\begin{aligned} \mathcal{E}_X : \mathcal{G}X &\to \partial^2 X \\ \gamma &\mapsto (\gamma(-\infty), \gamma(+\infty)) \end{aligned}$$

which induces a homeomorphism $\mathcal{G}X/(\phi_t)_{t\in\mathbb{R}} \to \partial^2 X$. Following Bourdon [1], we have the following:

Proposition 4.4. — *Let $f : \partial X \to \partial Y$ be a conformal map between the boundaries of CAT(-1) spaces X, Y equipped with visual metrics. Then f induces a bijection $\phi_f : \mathcal{G}X \to \mathcal{G}Y$ conjugating the geodesic flows, which is a homeomorphism if f is C^1 conformal. If f is Moebius then ϕ_f is flip-equivariant.*

Proof. — Given $\gamma \in \mathcal{G}X$, let $\mathcal{E}_X(\gamma) = (\xi, \eta), x = \gamma(0)$, then there is a unique point $y \in (f(\xi), f(\eta))$ such that $df_{\rho_x,\rho_y}(\eta) = 1$. Define $\phi_f(\gamma) = \gamma^*$ where γ^* is the unique geodesic in Y satisfying $\mathcal{E}_Y(\gamma^*) = (f(\xi), f(\eta))$, $\gamma^*(0) = y$. Then $\phi_f : \mathcal{G}X \to \mathcal{G}Y$ is a bijection conjugating the geodesic flows.

Claim. — *The map ϕ_f is continuous if f is C^1 conformal.*

Proof of Claim. — Let $\gamma_n \to \gamma$ in $\mathcal{G}X$. Let $x = \gamma(0), x_n = \gamma_n(0), \mathcal{E}_X(\gamma) = (\xi, \eta), \mathcal{E}_X(\gamma_n) = (\xi_n, \eta_n)$. Then $x_n \to x, (\xi_n, \eta_n) \to (\xi, \eta)$, hence

$$\begin{aligned}
\rho_x(\xi_n, \eta_n) &= \rho_{x_n}(\xi_n, \eta_n) \frac{d\rho_x}{d\rho_{x_n}}(\xi_n)^{1/2} \frac{d\rho_x}{d\rho_{x_n}}(\eta_n)^{1/2} \\
&= \frac{d\rho_x}{d\rho_{x_n}}(\xi_n)^{1/2} \frac{d\rho_x}{d\rho_{x_n}}(\eta_n)^{1/2} \\
&\to 1
\end{aligned}$$

since $|g_{x_n,x}| \leqslant d(x, x_n) \to 0$. Letting $y = \pi_Y \circ \phi_f(\gamma)$, this implies $\rho_y(f(\xi_n), f(\eta_n)) \to \rho_y(f(\xi), f(\eta)) = 1$ since f is continuous.

Fix $\epsilon > 0$ small and n large such that $\rho_y(f(\xi_n), f(\eta_n)) \geqslant 1 - \epsilon$. If a_t, b_t are points converging radially towards $f(\xi_n), f(\eta_n)$, then as $t \to +\infty$ there are points $\overline{z_t}$ in the comparison triangle $\overline{a_t y b_t}$ on the side $\overline{a_t b_t}$ such that $d(\overline{z_t}, \overline{y}) \leqslant C(\epsilon)$ for some constant $C(\epsilon)$ which tends to 0 as ϵ tends to 0. Hence we obtain a point $z_n \in (f(\xi_n), f(\eta_n))$ such that $d(z_n, y) \leqslant C(\epsilon)$. Therefore $d(z_n, y) \to 0$ as $n \to \infty$.

Let $z_n^* = \pi_Y \circ \phi_f(\gamma_n)$. Then since z_n, z_n^* both lie on the geodesic $\phi_f(\gamma_n)$ and $df_{\rho_{x_n},\rho_{z_n^*}}(\eta_n) = 1$, we have

$$\begin{aligned}
d(z_n^*, z_n) &= |\log df_{\rho_{x_n},\rho_{z_n}}(\eta_n)| \\
&= \left| \log \left(df_{\rho_x,\rho_y}(\eta_n) \frac{d\rho_x}{d\rho_{x_n}}(\eta_n) \frac{d\rho_{z_n}}{d\rho_y}(f(\eta_n)) \right) \right| \\
&\to |\log(1 \cdot 1 \cdot 1)| = 0
\end{aligned}$$

since f is C^1 conformal with $df_{\rho_x,\rho_y}(\eta) = 1$ and $\eta_n \to \eta$, $d(x_n, x) + d(z_n, y) \to 0$. Hence the basepoints z_n^* of the geodesics $\phi_f(\gamma_n)$ converge to the basepoint y of the geodesic $\phi_f(\gamma)$, and the endpoints $(f(\xi_n), f(\eta_n))$ of $\phi_f(\gamma_n)$ converge to the endpoints $(f(\xi), f(\eta))$ of $\phi_f(\gamma)$, from which it follows easily that $\phi_f(\gamma_n) \to \phi_f(\gamma)$ in $\mathcal{G}Y$. This finishes the proof of the Claim. $\square$

Since the inverse of a C^1 conformal map is clearly C^1 conformal, f^{-1} also induces a continuous conjugacy $\psi_f : \mathcal{G}Y \to \mathcal{G}X$ which is clearly inverse to ϕ_f, hence ϕ_f is a homeomorphism if f is C^1 conformal.

If f is Moebius, then with the same notation as above, by the geometric mean-value theorem we have $df_{\rho_x,\rho_y}(\xi) df_{\rho_x,\rho_y}(\eta) = 1$, hence $df_{\rho_x,\rho_y}(\xi) = 1$, and it follows that ϕ_f is flip-equivariant. $\square$

The proof of flip-equivariance of the conjugacy for a Moebius map above motivates the following definition:

Definition 4.5. — *Let $f : \partial X \to \partial Y$ be a conformal map between boundaries of CAT(-1) spaces equipped with visual metrics. The integrated Schwarzian of f is the function $S(f) : \partial^2 X \to \mathbb{R}$ defined by*

$$S(f)(\xi, \eta) := -\log(df_{\rho_x,\rho_y}(\xi) df_{\rho_x,\rho_y}(\eta)) \quad (\xi, \eta) \in \partial^2 X$$

where x, y are any two points $x \in (\xi, \eta), y \in (f(\xi), f(\eta))$ (it is easy to see that the quantity defined above is independent of the choices of x and y).

We note that $S(f)$ is continuous if f is C^1 conformal, and for any $\gamma \in \mathcal{G}X$ with $\mathcal{E}_X(\gamma) = (\xi, \eta)$, we have

$$\phi_f(\mathcal{F}_X(\gamma)) = \mathcal{F}_Y(\phi^Y_{-t}(\phi_f(\gamma)))$$

where $t = S(f)(\xi, \eta)$, hence the integrated Schwarzian of f measures the deviation of the induced conjugacy ϕ_f from being flip-equivariant.

We consider now the relation between the integrated Schwarzian and the continuity of the conjugacy ϕ_f near infinity. In particular we consider the continuity properties of ϕ_f along geodesics.

Definition 4.6. — *Let X be a simply connected complete Riemannian manifold with sectional curvatures bounded above and below, $-b^2 \leqslant K \leqslant -1$. A sequence of pairs of unit tangent vectors $(v_n, w_n) \in T^1X \times T^1X$ is said to be forward asymptotic along a geodesic $\gamma \in \mathcal{G}X$ if:*

1. *There are times $t_n \to +\infty$ such that $v_n = \gamma'(t_n)$ and $d_{T^1X}(v_n, w_n) \to 0$ (the distance on T^1X being the Sasaki metric).*
2. *Let $\gamma_n \in \mathcal{G}X$ such that $\gamma_n'(0) = w_n$, let $\mathcal{E}_X(\gamma) = (\xi, \eta), \mathcal{E}_X(\gamma_n) = (\xi_n, \eta_n)$. Then we require $\xi_n \to \xi_0 \neq \eta$ as $n \to \infty$.*

We have:

Proposition 4.7. — *Let X be a simply connected complete Riemannian manifold with sectional curvatures bounded above and below, $-b^2 \leqslant$*

$K \leqslant -1$, and let Y be a $CAT(-1)$ space. Let $f : \partial X \to \partial Y$ be a C^1 conformal map and $\phi = \phi_f : T^1X \to \mathcal{G}Y$ the associated geodesic conjugacy. Then for any sequence (v_n, w_n) forward asymptotic along a geodesic γ, we have

$$d_Y(\pi_Y \circ \phi(v_n), \pi_Y \circ \phi(w_n)) \to 0$$

Proof. — Let (v_n, w_n) be a forward asymptotic sequence along a geodesic γ, so there are times $t_n \to +\infty$ such that $v_n = \gamma'(t_n)$ and $d_{T^1X}(v_n, w_n) \to 0$. Let $x = \gamma(0)$, $x_n = \gamma(t_n) \in X$, $y = \pi_Y \circ \phi(\gamma'(0))$, $y_n = \pi_Y \circ \phi(v_n) \in Y$. Let $\gamma_n \in \mathcal{G}X$ with $\gamma_n'(0) = w_n$, let $\mathcal{E}_X(\gamma_n) = (\xi_n, \eta_n), \mathcal{E}_X(\gamma) = (\xi, \eta)$, then by hypothesis $\xi_n \to \xi_0 \neq \eta$. Since the curvature of X is bounded below by $-b^2$, for any $T \in \mathbb{R}$ the time-T-map of the geodesic flow $\phi^X_T : T^1X \to T^1X$ is Lipschitz. This follows from the fact that the differential of the map ϕ^X_T is given in terms of Jacobi fields and their derivatives, and by well known comparison arguments, Jacobi fields in X grow at most as fast as Jacobi fields in the hyperbolic space of constant curvature $-b^2$, hence $||d\phi^X_T||$ is bounded on T^1X. It follows that for any fixed large T, $d_{T^1X}(\phi^X_T(v_n), \phi^X_T(w_n)) \to 0$, hence the visual distance $\rho_{x_n}(\eta, \eta_n) \to 0$. It is easy to see that this also implies $\rho_x(\eta, \eta_n) \to 0$.

Claim. — *We have*

$$\lim_{n\to\infty} \frac{d\rho_{x_n}}{d\rho_x}(\eta_n)e^{-t_n} = \lim_{n\to\infty} \frac{d\rho_{y_n}}{d\rho_y}(f(\eta_n))e^{-t_n} = 1$$

Proof of Claim. — Fix $\epsilon > 0$ small. Let $\alpha_n \in \mathcal{G}X$ be a geodesic with $\alpha_n(0) = x_n, \alpha_n(+\infty) = \eta_n$. Then the Riemannian angle between $\alpha_n'(0), v_n$ tends to 0 (since the comparison angle $\angle^{(-1)}\eta_n x_n \eta$ tends to 0), so the Riemannian angle between $\alpha_n'(0), -v_n$ tends to π. Hence the limit of comparison angles ($\lim_{t\to+\infty} \angle\overline{\alpha_n(t)}\overline{x_n}\overline{x}$) tends to π as $n \to \infty$ (where $\overline{\alpha_n(t)}\overline{x_n}\overline{x}$ is a comparison triangle in $\mathbb{H}^2$). Fix n large such that this limiting angle is larger than $\pi - \epsilon$. For $t > 0$ large the comparison triangles $\overline{\alpha_n(t)}\overline{x_n}\overline{x}$ in $\mathbb{H}^2$ have an angle at the vertex $\overline{x_n}$ greater than $\pi - \epsilon$, hence the sides satisfy

$$d(\alpha_n(t), x) - d(\alpha_n(t), x_n) \geqslant d(x_n, x) - C(\epsilon)$$

for some constant $C(\epsilon)$ which tends to 0 as ϵ tends to 0. Letting $t \to +\infty$, we have $B(\eta_n, x, x_n) \geqslant t_n - C(\epsilon)$, hence

$$e^{t_n} = e^{d(x,x_n)} \geqslant \frac{d\rho_{x_n}}{d\rho_x}(\eta_n) \geqslant e^{-C(\epsilon)}e^{t_n}$$

therefore $\frac{d\rho_{x_n}}{d\rho_x}(\eta_n)e^{-t_n} \to 1$.

Now using the geometric mean value theorem for visual metrics we have

$$\begin{aligned}
\rho_{y_n}(f(\eta_n), f(\eta)) &= \frac{\rho_{y_n}(f(\eta_n), f(\eta))}{\rho_y(f(\eta_n), f(\eta))} \frac{\rho_y(f(\eta_n), f(\eta))}{\rho_x(\eta_n, \eta)} \frac{\rho_x(\eta_n, \eta)}{\rho_{x_n}(\eta_n, \eta)} \rho_{x_n}(\eta_n, \eta) \\
&= \left(e^{t_n} \frac{d\rho_{y_n}}{d\rho_y}(\eta_n) \right)^{1/2} \frac{\rho_y(f(\eta_n), f(\eta))}{\rho_x(\eta_n, \eta)} \\
&\quad \times \left(e^{-t_n} \left(\frac{d\rho_{x_n}}{d\rho_x}(\eta_n) \right)^{-1} \right)^{1/2} \rho_{x_n}(\eta_n, \eta) \\
&\leqslant \frac{\rho_y(f(\eta_n), f(\eta))}{\rho_x(\eta_n, \eta)} \left(e^{t_n} \left(\frac{d\rho_{x_n}}{d\rho_x}(\eta_n) \right)^{-1} \right)^{1/2} \rho_{x_n}(\eta_n, \eta) \\
&\to 1 \cdot 1 \cdot 0 = 0
\end{aligned}$$

Now $\rho_{y_n}(f(\eta_n), f(\eta)) \to 0$ and $d(y_n, y) = t_n$ implies that

$$\lim_{n\to\infty} \frac{d\rho_{y_n}}{d\rho_y}(f(\eta_n)) e^{-t_n} = 1$$

by the same argument used above to show that $\frac{d\rho_{x_n}}{d\rho_x}(\eta_n)e^{-t_n} \to 1$. This finishes the proof of the Claim. □

Now note that since $f(\xi_n) \to f(\xi_0) \neq f(\eta)$ and $y_n \to \eta$ radially, we have $\rho_{y_n}(f(\xi), f(\xi_n)) \to 0$. Hence

$$\begin{aligned}
\rho_{y_n}(f(\xi_n), f(\eta_n)) &\geqslant \rho_{y_n}(f(\xi), f(\eta)) - \rho_{y_n}(f(\xi), f(\xi_n)) - \rho_{y_n}(f(\eta), f(\eta_n)) \\
&= 1 - \rho_{y_n}(f(\xi), f(\xi_n)) - \rho_{y_n}(f(\eta), f(\eta_n)) \\
&\to 1
\end{aligned}$$

Fix $\epsilon > 0$ small. Fix n large such that $\rho_{y_n}(f(\xi_n), f(\eta_n)) \geqslant 1 - \epsilon$. If a_t, b_t are points converging radially towards $f(\xi_n), f(\eta_n)$, then as $t \to +\infty$ there are points $\overline{z_t}$ in the comparison triangle $\overline{a_t y_n b_t}$ on the side $\overline{a_t b_t}$ such that $d(\overline{z_t}, \overline{y_n}) \leqslant C(\epsilon)$ for some constant $C(\epsilon)$ which tends to 0 as ϵ tends to 0. Hence we obtain a point $z_n \in (f(\xi_n), f(\eta_n))$ such that $d(z_n, y_n) \leqslant C(\epsilon)$. Therefore $d(z_n, y_n) \to 0$ as $n \to \infty$.

Let $x_n^* = \pi_X(w_n), z_n^* = \pi_Y \circ \phi(w_n)$. Note $d(x_n^*, x_n) \to 0$. Since z_n, z_n^* lie on the geodesic $(f(\xi_n), f(\eta_n))$ and $df_{\rho_{x_n^*}, \rho_{z_n^*}}(\eta_n) = 1$, we have

$$\begin{aligned} d(z_n^*, z_n) &= \left|\log df_{\rho_{x_n^*}, \rho_{z_n}}(\eta_n)\right| \\ &= \left|\log\left(df_{\rho_x, \rho_y}(\eta_n)\left(\frac{d\rho_x}{d\rho_{x_n}}(\eta_n)\frac{d\rho_{y_n}}{d\rho_y}(f(\eta_n))\right)\right.\right. \\ &\qquad\qquad \left.\left.\times\left(\frac{d\rho_{x_n}}{d\rho_{x_n^*}}(\eta_n)\frac{d\rho_{z_n}}{d\rho_{y_n}}(f(\eta_n))\right)\right)\right| \\ &\to |\log(1\cdot 1\cdot 1)| = 0 \end{aligned}$$

since f is C^1 conformal with $df_{\rho_x, \rho_y}(\eta) = 1$ and $\eta_n \to \eta$, $d(x_n^*, x_n) + d(z_n, y_n) \to 0$, and the term in the middle of the product tends to 1 by the Claim proved earlier.

Hence $d(\pi_Y \circ \phi(v_n), \pi_Y \circ \phi(w_n)) = d(y_n, z_n^*) \to 0$. □

Proposition 4.8. — *Let X, Y, f, ϕ be as in the previous Proposition. Let $x \in X$ and $(\xi, \eta) \in \partial^2 X$. Let $\alpha, \beta : [0, \infty) \to X$ be geodesic rays joining x to ξ, η respectively. Let $x_t = \alpha(t), y_t = \beta(t), v_t = \alpha'(t), w_t = \beta'(t)$, then*

$$d_Y(\pi_Y \circ \phi(v_t), \pi_Y \circ \phi(w_t)) - d_X(x_t, y_t) \to S(f)(\xi, \eta)$$

as $t \to +\infty$.

Proof. — Let γ_t be the bi-infinite geodesic passing through x_t, y_t, with endpoints $(\xi_t, \eta_t) \in \partial^2 X$, so that $(\xi_t, \eta_t) \to (\xi, \eta)$ as $t \to +\infty$. Let v'_t, w'_t be the tangent vectors to γ_t at the points x_t, y_t pointing respectively towards ξ_t, η_t. Then it is a standard fact that for any sequence $t_n \to +\infty$, the sequences of pairs $\{(v_{t_n}, v'_{t_n})\}, \{(w_{t_n}, w'_{t_n})\}$ are forward asymptotic along α, β respectively. Letting $p_t = \pi_Y \circ \phi(v_t), q_t = \pi_Y \circ \phi(w_t), p'_t = \pi_Y \circ \phi(v'_t), q'_t = \pi_Y \circ \phi(w'_t)$, then by Proposition 4.7 we have $d_Y(p_{t_n}, p'_{t_n}) \to 0, d_Y(q_{t_n}, q'_{t_n}) \to 0$ as $n \to \infty$. By definition of the integrated Schwarzian, we have $d_Y(p'_{t_n}, q'_{t_n}) = d_X(x_{t_n}, y_{t_n}) + S(f)(\xi_{t_n}, \eta_{t_n})$, since $S(f)$ is continuous it follows that $d_Y(p'_{t_n}, q'_{t_n}) - d_X(x_{t_n}, y_{t_n}) = S(f)(\xi_{t_n}, \eta_{t_n}) \to S(f)(\xi, \eta)$ as $n \to \infty$. The result follows. □

We can now prove Theorem 1.2:

Proof of Theorem 1.2. — We first note that $f : U \to V$ induces a geodesic conjugacy between the flow invariant subsets of $\mathcal{G}X, \mathcal{G}Y$ with endpoints in U, V respectively, for which the same arguments as above show that the conclusion of Proposition 4.8 above holds. Fix a basepoint $x \in X$. Now given $(\xi, \xi', \eta, \eta') \in \partial^4 U$, let $\alpha, \beta, \gamma, \delta$ be geodesic rays joining x to ξ, η, ξ', η' respectively. Let $x_t = \alpha(t), y_t = \beta(t), a_t = \gamma(t), b_t = \delta(t)$, let $v_t = \alpha'(t), w_t = \beta'(t), v'_t = \gamma'(t), w'_t = \delta'(t)$ and let $p_t = \pi_Y \circ \phi(v_t), q_t =$

$\pi_Y \circ \phi(w_t), r_t = \pi_Y \circ \phi(v_t'), s_t = \pi_Y \circ \phi(w_t')$. Then the points p_t, q_t, r_t, s_t converge radially towards $f(\xi), f(\eta), f(\xi'), f(\eta')$, hence

$$\begin{aligned}\log \frac{[f(\xi), f(\xi'), f(\eta), f(\eta')]}{[\xi, \xi', \eta, \eta']} \\ &= \frac{1}{2}(\lim_{t\to\infty}(d_Y(p_t, q_t) - d_X(x_t, y_t)) + (d_Y(r_t, s_t) - d_X(a_t, b_t)) \\ &\quad - (d_Y(p_t, s_t) - d_X(x_t, b_t)) - (d_Y(q_t, r_t) - d_X(y_t, a_t))) \\ &= \frac{1}{2}(S(f)(\xi, \eta) + S(f)(\xi', \eta') - S(f)(\xi, \eta') - S(f)(\xi', \eta))\end{aligned}$$

(using Proposition 4.8 in the last line above) ☐

Definition 4.9. — *Let X be a simply connected complete Riemannian manifold with sectional curvatures bounded above and below, $-b^2 \leqslant K \leqslant -1$, and let Y be a CAT(-1) space. A homeomorphism $\phi : T^1X \to \mathcal{G}Y$ is said to be uniformly continuous along geodesics, if, given $\gamma \in \mathcal{G}X$, and a sequence $(v_n, w_n) \in T^1X \times T^1X$ which is forward asymptotic along γ, we have*

$$d(\pi_Y \circ \phi(v_n), \pi_Y \circ \phi(w_n)) + d(\pi_Y \circ \phi(-v_n), \pi_Y \circ \phi(-w_n)) \to 0$$

We note that any uniformly continuous homeomorphism $\phi : T^1X \to \mathcal{G}Y$ is uniformly continuous along geodesics. We can now prove Theorem 1.1:

Proof of Theorem 1.1. — We first note that if $\gamma_1, \gamma_2 \in \mathcal{G}X$ are geodesics with $\gamma_1(+\infty) = \gamma_2(+\infty)$, then it follows easily from the definition of uniform continuity along geodesics that $\phi(\gamma_1'(0))(+\infty) = \phi(\gamma_2'(0))(+\infty)$. Hence there is a map $f : \partial X \to \partial Y$ such that $\mathcal{E}_Y(\phi(v)) = (f(\xi), f(\eta))$ where $(\xi, \eta) = \mathcal{E}_X(\gamma)$, $\gamma \in \mathcal{G}X$ being such that $\gamma'(0) = v$, and it is not hard to show that f is continuous. Moreover f is surjective since Y is geodesically complete and ϕ is surjective. Also given $(\xi, \eta) \in \partial^2 X$, choosing γ with $\mathcal{E}_X(\gamma) = (\xi, \eta)$, we have $(f(\xi), f(\eta)) = \mathcal{E}_Y(\phi(\gamma'(0))) \in \partial^2 Y$, in particular $f(\xi) \neq f(\eta)$. Thus f is injective, and since $\partial X, \partial Y$ are compact Hausdorff spaces, f is a homeomorphism.

Given a quadruple of distinct points $(\xi, \xi', \eta, \eta') \in \partial^4 X$, let γ_1, γ_2 be geodesics with $\mathcal{E}_X(\gamma_1) = (\xi, \eta), \mathcal{E}_X(\gamma_2) = (\xi', \eta')$, and let $t_n \to +\infty$. Let $a_n = \gamma_1(-t_n), a_n' = \gamma_2(-t_n), b_n = \gamma_1(t_n), b_n' = \gamma_2(t_n)$ so

$$[\xi\xi'\eta\eta'] = \lim_{n\to\infty} \exp(\frac{1}{2}(d(a_n, b_n) + d(a_n', b_n') - d(a_n, b_n') - d(a_n', b_n)))$$

Let $\alpha_n = \pi_Y \circ \phi(\gamma_1'(-t_n)), \alpha_n' = \pi_Y \circ \phi(\gamma_2'(-t_n)), \beta_n = \pi_Y \circ \phi(\gamma_1'(t_n)), \beta_n' = \pi_Y \circ \phi(\gamma_2'(t_n))$, so that

$$\begin{aligned}&[f(\xi)f(\xi')f(\eta)f(\eta')]\\ &\qquad = \lim_{n\to\infty} \exp(\frac{1}{2}(d(\alpha_n,\beta_n) + d(\alpha_n',\beta_n') - d(\alpha_n,\beta_n') - d(\alpha_n',\beta_n)))\end{aligned}$$

Note that $d(a_n, b_n) = d(\alpha_n, \beta_n), d(a_n', b_n') = d(\alpha_n', \beta_n')$ since ϕ conjugates the geodesic flows. Clearly the Theorem follows from the following claim:

Claim. — *We have $d(a_n, b_n') - d(\alpha_n, \beta_n') \to 0, d(a_n', b_n) - d(\alpha_n', \beta_n) \to 0$ as $n \to \infty$.*

Proof of Claim. — Let $\gamma_n : [0, l_n] \to X$ be the geodesic segment with $\gamma_n(0) = a_n, \gamma_n(l_n) = b_n'$, where $l_n = d(a_n, b_n')$. Then it is a standard fact that the Riemannian angle between the vectors $\gamma_1'(-t_n), v_n = \gamma_n'(0)$ tends to 0, as does the angle between the vectors $\gamma_2'(t_n), w_n = \gamma_n'(l_n)$. Letting $p_n = \pi_Y \circ \phi(v_n), q_n = \pi_Y \circ \phi(w_n)$, we have $d(p_n, q_n) = d(a_n, b_n')$ since ϕ is a geodesic conjugacy. Moreover since ϕ is uniformly continuous along geodesics, it follows that $d(p_n, \alpha_n) \to 0, d(q_n, \beta_n') \to 0$. Hence $d(a_n, b_n') - d(\alpha_n, \beta_n') \to 0$ and a similar argument shows $d(a_n', b_n) - d(\alpha_n', \beta_n) \to 0$. ☐

Proof. — Proof of Theorem 1.3 The forward implication follows from Theorem 1.1. For the backward implication, given $f : \partial X \to \partial Y$ a Moebius map, let $\phi : T^1 X \to \mathcal{G}Y$ denote the induced conjugacy of geodesic flows given by Proposition 4.4. We show that ϕ is uniformly continuous along geodesics.

Let (v_n, w_n) be a forward asymptotic sequence. By Proposition 4.7, we have $d(\pi_Y \circ \phi(v_n), \pi_Y \circ \phi(w_n)) \to 0$. Since f is Moebius, the conjugacy ϕ is flip-equivariant, hence $\pi_Y \circ \phi(-v_n) = \pi_Y \circ \phi(v_n), \pi_Y \circ \phi(-w_n) = \pi_Y \circ \phi(w_n)$, thus $d(\pi_Y \circ \phi(-v_n), \pi_Y \circ \phi(-w_n)) = d(\pi_Y \circ \phi(v_n), \pi_Y \circ \phi(w_n)) \to 0$. ☐

It follows from the chain rule that the integrated Schwarzian satisfies the following transformation rule: given conformal maps $f : \partial X \to \partial Y, g : \partial Y \to \partial Z$, where X, Y, Z are CAT(-1) spaces, we have

$$S(g \circ f) = S(g) \circ f + S(f)$$

For the group G of C^1 conformal self-maps of the boundary ∂X of a CAT(-1) space, the map

$$\begin{aligned} c : G &\to C(\partial^2 X)\\ f &\mapsto S(f)\end{aligned}$$

is therefore a G-cocycle with values in the vector space $C(\partial^2 X)$ of continuous functions on $\partial^2 X$ endowed with its natural G-action.

For the group G of C^1 conformal self-maps of $V \subset \partial Y$, it follows that the subgroup $ker\ c := \{g \in G | S(g) = 0\} < G$ coincides with the group of Moebius self-maps of V. Hence for conformal maps $f, g : U \to V$, $g \circ f^{-1}$ is Moebius if and only if $S(g \circ f^{-1}) = 0$. Using the identities

$$S(g \circ f^{-1}) = S(g) \circ f^{-1} + S(f^{-1})$$
$$0 = S(f \circ f^{-1}) = S(f) \circ f^{-1} + S(f^{-1})$$

it follows that $S(g \circ f^{-1}) = (S(g) - S(f)) \circ f^{-1}$, hence f, g differ by post-composition with a Moebius map if and only if $S(g) = S(f)$.

For C^1 conformal maps $f : \partial X \to \partial Y$ such that the integrated Schwarzian $S(f)$ is bounded, and X is a simply connected manifold with pinched negative sectional curvatures, we have the following version of the geometric mean value theorem:

Theorem 4.10. — *Let X be a simply connected complete Riemannian manifold with sectional curvatures bounded above and below, $-b^2 \leqslant K \leqslant -1$, and let Y be a CAT(-1) space. Let $f : \partial X \to \partial Y$ be a C^1 conformal map such that $S(f)$ is bounded. Then for all $(\xi, \eta) \in \partial^2 X$ and $x \in X, y \in Y$, we have*

$$\begin{aligned} e^{-4||S(f)||_\infty} df_{\rho_x,\rho_y}(\xi) df_{\rho_x,\rho_y}(\eta) &\leqslant \left(\frac{\rho_y(f(\xi), f(\eta))}{\rho_x(\xi, \eta)} \right)^2 \\ &\leqslant e^{4||S(f)||_\infty} df_{\rho_x,\rho_y}(\xi) df_{\rho_x,\rho_y}(\eta) \end{aligned}$$

Proof. — Fix $x \in X, y \in Y$. For a triple $(\xi, \xi', \eta') \in \partial^3 X$, we define

$$\delta(\xi, \xi', \eta') := \frac{\rho_y(f(\xi), f(\xi'))\rho_y(f(\xi), f(\eta'))\rho_x(\xi', \eta')}{\rho_x(\xi, \xi')\rho_x(\xi, \eta')\rho_y(f(\xi'), f(\eta'))}$$

For a quadruple $(\xi, \xi', \eta, \eta') \in \partial^4 X$, by Theorem 1.2 we have

$$e^{-2||S(f)||_\infty} \leqslant \frac{[f(\xi), f(\xi'), f(\eta), f(\eta')]}{[\xi, \xi', \eta, \eta']} \leqslant e^{2||S(f)||_\infty}$$

Passing to the limit above as $\eta \to \xi$, the term in the middle converges to $df_{\rho_x,\rho_y}(\xi)/\delta(\xi, \xi', \eta')$, thus we may write $df_{\rho_x,\rho_y}(\xi) = \delta(\xi, \xi', \eta') \cdot E(\xi, \xi', \eta')$ where $e^{-2||S(f)||_\infty} \leqslant E(\xi, \xi', \eta') \leqslant e^{2||S(f)||_\infty}$.

Now given $(\xi, \eta) \in \partial^2 X$, choose $\beta \in \partial X$ distinct from ξ, η. Then we have:

$$\begin{aligned} df_{\rho_x,\rho_y}(\xi) df_{\rho_x,\rho_y}(\eta) &= \delta(\xi, \eta, \beta)\delta(\eta, \xi, \beta)E(\xi, \eta, \beta)E(\eta, \xi, \beta) \\ &= \left(\frac{\rho_y(f(\xi), f(\eta))}{\rho_x(\xi, \eta)} \right)^2 E(\xi, \eta, \beta)E(\eta, \xi, \beta) \end{aligned}$$

so the Theorem follows since

$$e^{-4||S(f)||_\infty} \leqslant E(\xi,\eta,\beta)E(\eta,\xi,\beta) \leqslant e^{4||S(f)||_\infty}$$

□

5. Marked length spectrum, geodesic conjugacies, and Moebius structure at infinity

The following Theorem follows from results of Bourdon ([1]), Hamenstadt ([5]) and Otal ([8]), we give a proof for the benefit of the reader.

Theorem 5.1. — *(Bourdon, Hamenstadt, Otal) Let X, Y be closed n-dimensional Riemannian manifolds with sectional curvatures bounded above by -1, and let $\tilde{X}, \tilde{Y}$ denote their universal covers. Then the following are equivalent:*

(1) The marked length spectra of X and Y coincide, i.e. there is an isomorphism $\Phi : \pi_1(X) \to \pi_1(Y)$ such that $l_Y \circ \Phi = l_X$.
(2) There is an equivariant Moebius map $f : \partial\tilde{X} \to \partial\tilde{Y}$
(3) There is a homeomorphism $\phi : T^1X \to T^1Y$ conjugating the geodesic flows.

Proof. — We prove:

1. (1) $\Rightarrow$ (2): It is well known that the isomorphism Φ induces an equivariant homeomorphism $f : \partial\tilde{X} \to \partial\tilde{Y}$ such that $f \circ \gamma = \Phi(\gamma) \circ f$ for $\gamma \in \pi_1(X)$ (with $\pi_1(X), \pi_1(Y)$ identified with groups of homeomorphisms of $\partial\tilde{X}, \partial\tilde{Y}$).

   Let h_X, h_Y denote the topological entropies of the geodesic flows of X and Y. For $t \geqslant 0$, let $\nu_X(t), \nu_Y(t)$ denote the number of conjugacy classes $[\gamma], [\gamma']$ in $\pi_1(X), \pi_1(Y)$ with $l_X(\gamma) \leqslant t, l_Y(\gamma') \leqslant t$. Then by hypothesis, $\nu_X(t) \equiv \nu_Y(t)$. Hence from Bowen's formula for the topological entropy ([3]) we have

$$h_X = \lim_{t\to+\infty} \frac{\log(\nu_X(t))}{t} = \lim_{t\to+\infty} \frac{\log(\nu_Y(t))}{t} = h_Y$$

   Let μ_X, μ_Y denote the Bowen-Margulis currents on $\partial^2\tilde{X}, \partial^2\tilde{Y}$; these are the geodesic currents corresponding to the Bowen-Margulis measures on T^1X, T^1Y, the unique invariant measures of maximal entropy. Then it follows from Bowen's formula for the Bowen-Margulis measure ([3]) that for any fixed small $\epsilon > 0$,

$$\mu_X = \lim_{t\to+\infty} \frac{1}{N_{\epsilon,X}(t)} \sum_{[\gamma]\in CO_{\epsilon,X}(t)} \delta_{[\gamma]}$$

where $CO_{\epsilon,X}(t)$ is the set of conjugacy classes $[\gamma]$ in $\pi_1(X)$ with $l_X(\gamma) \in [t-\epsilon, t+\epsilon]$, $N_{\epsilon,X}(t)$ is the cardinality of $CO_{\epsilon,X}(t)$, and $\delta_{[\gamma]}$ denotes the atomic geodesic current associated to a conjugacy class $[\gamma]$.

Since Φ preserves lengths, it follows that $(f \times f)_*(\mu_X) = \mu_Y$. We now recall Kaimanovich's formula for the Bowen-Margulis current ([6]),

$$d\mu_X(\xi, \eta) = \frac{d\nu_{x,X}(\xi) d\nu_{x,X}(\eta)}{(\rho_x(\xi,\eta))^{2h_X}}$$

where $x \in \tilde{X}$ (the right-hand side above is independent of the choice of x) and $\nu_{x,X}$ is the Patterson-Sullivan measure on $\partial \tilde{X}$ based at the point x.

Claim. — *For any $x \in \tilde{X}, y \in \tilde{Y}$, the map f is absolutely continuous with respect to the Patterson-Sullivan measures $\nu_{x,X}, \nu_{y,Y}$.*

Proof of Claim. — Let $A \subset \partial X$ such that $\nu_{x,X}(A) = 0$. Let $U, V \subset \partial X$ be closed disjoint balls in $(\partial X, \rho_x)$, let δ denote the minimum distance between points of U and V. Let $A' = A \cap U$. Then we have

$$\begin{aligned} \nu_{y,Y}(f(A'))\nu_{y,Y}(f(V)) &\leqslant \mu_Y(f(A') \times f(V)) \\ &= \mu_X(A' \times V) \\ &\leqslant \frac{\nu_{x,X}(A')\nu_{x,X}(V)}{\delta^{2h_X}} = 0 \end{aligned}$$

hence $\nu_{y,Y}(f(A')) = 0$. It follows that $\nu_{y,y}(f(A)) = 0$. This proves the claim. ☐

Let g be the Radon-Nikodym derivative of $f_*^{-1}\nu_{y,Y}$ with respect to $\nu_{x,X}$. Then the equality $(f \times f)_*(\mu_X) = \mu_Y$ implies that for μ_X-a.e. $(\xi, \eta) \in \partial^2 \tilde{X}$ we have

$$\frac{\rho_y(f(\xi), f(\eta))^{2h_Y}}{\rho_x(\xi,\eta)^{2h_X}} = g(\xi)g(\eta) \ ,$$

in particular the above equality holds for (ξ, η) in a dense subset $A \subset \partial^2 \tilde{X}$. Since $h_X = h_Y$, it follows that f preserves cross-ratios of quadruples in the dense subset $\partial^2 A \subset \partial^4 \tilde{X}$, and hence preserves all cross-ratios, since cross-ratios are continuous.

2. (2) $\Rightarrow$ (3): Let $\phi : T^1\tilde{X} \to T^1\tilde{Y}$ be the geodesic conjugacy induced by f, as given by Proposition 4.4. Then it is easy to see that ϕ is equivariant, hence induces a geodesic conjugacy $\overline{\phi} : T^1X \to T^1Y$.
3. (3) $\Rightarrow$ (1): The conjugacy ϕ induces an equivariant conjugacy $\tilde{\phi} : T^1\tilde{X} \to T^1\tilde{Y}$, which is uniformly continuous since ϕ is uniformly continuous, hence by Theorem 1.1 there is a Moebius homeomorphism

$f : \partial\tilde{X} \to \partial\tilde{Y}$ such that $\mathcal{E}_Y(\tilde{\phi}(\gamma)) = (f \times f) \circ \mathcal{E}_X(\gamma)$. Moreover f is equivariant because $\tilde{\phi}$ is. Identifying $\pi_1(X), \pi_1(Y)$ with groups of homeomorphisms of $\partial\tilde{X}, \partial\tilde{Y}$, we obtain a map

$$\begin{aligned} \Phi : \pi_1(X) &\to \pi_2(Y) \\ g &\mapsto f \circ g \circ f^{-1} \end{aligned}$$

which is clearly an isomorphism.

Each $g \in \pi_1(X)$ has a unique attracting and a unique repelling fixed point on $\partial\tilde{X}$, denoted ξ_g^+, ξ_g^- respectively. For any $\gamma \in \mathcal{G}\tilde{X}$ with $\mathcal{E}_{\tilde{X}}(\gamma) = (\xi_g^-, \xi_g^+)$, we have $g(\gamma'(0)) = \phi_{t_1}^{\tilde{X}}(\gamma'(0))$, where $t_1 = l_X(g)$. Now $f(\xi_g^+), f(\xi_g^-)$ are the attracting and repelling fixed points of $\Phi(g)$, and $\tilde{\phi}(\gamma) \in \mathcal{G}\tilde{Y}$ satisfies $\mathcal{E}_{\tilde{X}}(\tilde{\phi}(\gamma)) = (f(\xi_g^-), f(\xi_g^+))$ (we are abusing notation writing $\tilde{\phi}$ also for the induced map $\mathcal{G}\tilde{X} \to \mathcal{G}\tilde{Y}$). Hence $\Phi(g)(\tilde{\phi} \circ \gamma'(0)) = \phi_{t_2}^{\tilde{Y}}(\tilde{\phi} \circ \gamma'(0))$ where $t_2 = l_Y(\Phi(g))$.

Since $\tilde{\phi}$ is equivariant and is a geodesic conjugacy, we also have

$$\Phi(g)(\tilde{\phi} \circ \gamma'(0)) = \tilde{\phi}(g(\gamma'(0))) = \tilde{\phi}(\phi_{t_1}^{\tilde{X}}(\gamma'(0))) = \phi_{t_1}^{\tilde{Y}}(\tilde{\phi} \circ \gamma'(0)).$$

Since the time-t-map of the geodesic flow of $\tilde{Y}$ has no fixed points for $t \neq 0$, we must have $t_1 = t_2$, i.e. $l_Y(\Phi(g)) = l_X(g)$. □

We obtain as a corollary the following:

Theorem 5.2. — *Let X, Y be closed n-dimensional Riemannian manifolds with sectional curvatures bounded above by -1, and let $\tilde{X}, \tilde{Y}$ be their universal covers. If $f : \partial\tilde{X} \to \partial\tilde{Y}$ is an equivariant C^1 conformal map, then f is Moebius.*

Proof. — Let $\phi : T^1\tilde{X} \to T^1\tilde{Y}$ be the geodesic conjugacy given by Proposition 4.4. Then the equivariance of f implies that of ϕ, hence ϕ is the lift of a conjugacy $\overline{\phi} : T^1X \to T^1Y$ which is uniformly continuous, hence ϕ is uniformly continuous. It then follows from Theorem 1.1 that f is Moebius. □

6. Nearest points and almost isometric extension of Moebius maps

Let X be a proper geodesically complete CAT(-1) space such that ∂X has at least four points, and let $\mathcal{M} = \mathcal{M}(\partial X, \rho_x)$. Since the image of the isometric embedding $X \to \mathcal{M}$ is closed in $\mathcal{M}$ and the space $\mathcal{M}$ is proper, it follows that for all $\rho \in \mathcal{M}$ there exists $x \in X$ minimizing $d_{\mathcal{M}}(\rho, \rho_y)$ over $y \in X$.

THEOREM 6.1. — *The image of the map $i_X : X \to \mathcal{M}$ is $\frac{1}{2}\log 2$-dense in $\mathcal{M}$.*

Proof. — Given $\rho \in \mathcal{M}$ let $x \in X$ minimize $d_{\mathcal{M}}(\rho, \rho_y)$ over $y \in X$. Let $\lambda = \sup \log \frac{d\rho}{d\rho_x} = d_{\mathcal{M}}(\rho, \rho_x)$, let $Z \subset \partial X$ be the set where $\log \frac{d\rho}{d\rho_x} = \lambda$ and let $\xi_0 \in Z$.

Suppose that $\lambda > \frac{1}{2}\log 2$. Then for any $\xi \in Z$, by the Geometric Mean Value Theorem we have

$$1 \geqslant \rho(\xi_0, \xi)^2 = \rho_x(\xi_0, \xi)^2 \frac{d\rho}{d\rho_x}(\xi_0) \frac{d\rho}{d\rho_x}(\xi) = \rho_x(\xi_0, \xi)^2 e^{2\lambda} > \rho_x(\xi_0, \xi)^2 \cdot 2$$

hence $\max_{\xi \in Z} \rho_x(\xi_0, \xi) < 1/\sqrt{2}$. It follows that there is an open neighbourhood $N \supset Z$ and $\epsilon > 0$ such that $\angle^{(-1)}\xi x \xi_0 \leqslant \pi/2 - \epsilon$ for all $\xi \in N$. By monotonicity of comparison angles, for any $y \in [x, \xi_0)$, we also have $\angle^{(-1)}\xi x y \leqslant \pi/2 - \epsilon$ for all $\xi \in N$, so $\cos(\angle^{(-1)}\xi x y) \geqslant \delta_0$ for some $\delta_0 > 0$. Now let $\lambda' = \sup_{\xi \in \partial X - N} \log \frac{d\rho}{d\rho_x}(\xi), \delta_1 = \lambda - \lambda' > 0$, then, using Lemma 3.7, let $t_0 < \delta_1/3$ be such that, for $y \in [x, \xi_0)$ at distance t from x, we have

$$g_{x,y}(\xi) = t\cos(\angle^{(-1)}\xi x y) + o(t)$$

where $||o(t)||_\infty < t\delta_0/2$ for $t \leqslant t_0$. Then, using the Chain Rule, we have for $\xi \in N$ and $0 < t \leqslant t_0$, letting $y \in [x, \xi_0)$ be the point at distance t from x,

$$\log \frac{d\rho}{d\rho_y}(\xi) = \log \frac{d\rho}{d\rho_x}(\xi) - g_{x,y}(\xi) \leqslant \lambda - t\delta_0 + t\delta_0/2 < \lambda,$$

while for $\xi \in \partial X - N$ and $0 < t \leqslant t_0$ we have

$$\log \frac{d\rho}{d\rho_y}(\xi) = \log \frac{d\rho}{d\rho_x}(\xi) - g_{x,y}(\xi) \leqslant \lambda' + t + t\delta_0/2 \leqslant \lambda - \delta_1 + 2\delta_1/3 < \lambda$$

hence for $0 < t \leqslant t_0$ we have $d_{\mathcal{M}}(\rho, \rho_y) < d_{\mathcal{M}}(\rho, \rho_x)$, a contradiction. $\square$

THEOREM 6.2. — *If X is a metric tree then the map $i_X : X \to \mathcal{M}$ is a surjective isometry.*

Proof. — Suppose not, let $\rho \in \mathcal{M}$ be a point not in the image, let $x \in X$ minimize $d_{\mathcal{M}}(\rho, \rho_y)$ over $y \in X$. Let $\lambda = \sup \log \frac{d\rho}{d\rho_x} > 0$, let $Z \subset \partial X$ be the set where $\log \frac{d\rho}{d\rho_x} = \lambda$ and let $\xi_0 \in Z$. Then for all $\xi \in Z$, we have $1 \geqslant \rho_x(\xi_0, \xi)e^{\lambda}$ hence $\rho_x(\xi_0, \xi) \leqslant e^{-\lambda}$. Let $0 < \lambda' < \lambda$, and choose a neighbourhood $N \supset Z$ such that $\rho_x(\xi_0, \xi) \leqslant e^{-\lambda'}$ for all $\xi \in N$. Letting y_0 be the point on the ray $[x, \xi)$ at distance λ' from a, since X is a tree it follows that the segment $[x, y_0]$ is contained in all the rays $[x, \xi), \xi \in N$. Hence for $0 < t \leqslant \lambda'$, it follows that $\angle^{(-1)}\xi x y = 0$ for all $\xi \in N$, where y is the point on $[x, \xi_0)$ at distance t from x. Thus $\cos(\angle^{(-1)}\xi x y) = 1$ for all $\xi \in N$, and now the same argument as in the proof of Theorem 6.1 above shows that we may choose $0 < t_0 < \lambda'$ such that for $0 < t \leqslant t_0$ we have $d_{\mathcal{M}}(\rho, \rho_y) < d_{\mathcal{M}}(\rho, \rho_x)$, a contradiction. $\square$

Now let X, Y be proper geodesically complete CAT(-1) spaces such that ∂X has at least four points, let $f : \partial X \to \partial Y$ be a Moebius homeomorphism, and let $\mathcal{M}_X = \mathcal{M}(\partial X, \rho_x), \mathcal{M}_Y = \mathcal{M}(\partial Y, \rho_y)$ where $x \in X, y \in Y$. Let $g = f^{-1}$, then for $\rho \in \mathcal{M}_X$ we can define the pull-back metric $g_* \rho$ on ∂Y by $g_*\rho(\xi, \xi') := \rho(g(\xi), g(\xi')), \xi, \xi' \in \partial Y$. Since g is Moebius it follows easily that $g_*\rho \in \mathcal{M}_Y$. We can therefore define a map

$$\hat{f} : \mathcal{M}_X \to \mathcal{M}_Y$$
$$\rho \mapsto g_* \rho$$

which it is easy to see is a surjective isometry.

We define a nearest-point projection map for X,

$$\pi_X : \mathcal{M}_X \to X$$
$$\rho \mapsto a$$

by choosing for each $\rho \in \mathcal{M}_X$ a point $a \in X$ minimizing $d_{\mathcal{M}_X}(\rho, \rho_x), x \in X$ (not necessarily unique), and similarly we define a map $\pi_Y : \mathcal{M}_Y \to Y$. We can now prove the Theorems 1.5 and 1.6:

Proof of Theorem 1.5. — Define $F : X \to Y$ by $F = \pi_Y \circ \hat{f} \circ i_X$. Then by Theorem 6.1 for $x, x' \in X$, letting $y = F(x), y' = F(x')$ we have

$$\begin{aligned}
|d_Y(y, y') - d_X(x, x')| &= |d_{\mathcal{M}_Y}(\rho_y, \rho'_y) - d_{\mathcal{M}_Y}(\hat{f}(\rho_x), \hat{f}(\rho_{x'}))| \\
&\leqslant |d_{\mathcal{M}_Y}(\rho_y, \rho_{y'}) - d_{\mathcal{M}_Y}(\hat{f}(\rho_x), \rho_{y'})| \\
&\quad + |d_{\mathcal{M}_Y}(\hat{f}(\rho_x), \rho_{y'}) - d_{\mathcal{M}_Y}(\hat{f}(\rho_x), \hat{f}(\rho_{x'}))| \\
&\leqslant d_{\mathcal{M}_Y}(\rho_y, \hat{f}(\rho_x)) + d_{\mathcal{M}_Y}(\rho_{y'}, \hat{f}(\rho_{x'})) \leqslant \log 2
\end{aligned}$$

so F is a $(1, \log 2)$-quasi-isometry. Given $y \in Y$, by Theorem 6.1 we may choose $x \in X$ such that $d_{\mathcal{M}_X}(f_*\rho_y, \rho_x) \leqslant \frac{1}{2} \log 2$, then by definition of F,

$$\begin{aligned}
d_Y(F(x), y) = d_{\mathcal{M}_Y}(\rho_{F(x)}, \rho_y) &\leqslant d_{\mathcal{M}_Y}(\rho_{F(x)}, \hat{f}(\rho_x)) + d_{\mathcal{M}_Y}(\hat{f}(\rho_x), \rho_y) \\
&\leqslant d_{\mathcal{M}_Y}(\rho_y, \hat{f}(\rho_x)) + d_{\mathcal{M}_Y}(\hat{f}(\rho_x), \rho_y) \\
&= 2 d_{\mathcal{M}_X}(f_*\rho_y, \rho_x) \\
&\leqslant \log 2
\end{aligned}$$

thus the image of F is $\log 2$-dense in Y.

It follows from the above that F has a continuous extension $\partial F : \partial X \to \partial Y$, it remains to prove that $\partial F = f$. Let $\xi \in \partial X, x \in X$ and let $a \in X$ converge to ξ along the ray $[x, \xi)$. Let $y = F(x), b = F(a), \lambda = d_Y(y, b)$, then $b \to \eta = \partial F(\xi), \lambda \geqslant d_X(x, a) - \log 2 \to \infty$ as $a \to \xi$. Extend $[y, b]$ to a geodesic ray $[y, \eta')$ where $\eta' \in \partial Y$, then $\frac{d\rho_b}{d\rho_y}(\eta') = e^{\lambda}$ and $b \to \eta$ implies $\eta' \to \eta$. By the Chain Rule,

$$|| \log \frac{d\rho_b}{d\rho_y} - \log \frac{dg_*\rho_a}{dg_*\rho_x} ||_\infty \leqslant d_{\mathcal{M}_Y}(\rho_b, g_*\rho_a) + d_{\mathcal{M}_Y}(\rho_y, g_*\rho_x) \leqslant \log 2$$

and $\log \frac{dg_*\rho_a}{dg_*\rho_x}(f(\xi)) = d_X(x,a) \geqslant d_Y(y,b) - \log 2$, hence

$$\log \frac{d\rho_b}{d\rho_y}(f(\xi)) \geqslant \log \frac{dg_*\rho_a}{dg_*\rho_x}(f(\xi)) - \log 2 \geqslant \lambda - 2\log 2$$

so $\frac{d\rho_b}{d\rho_y}(f(\xi)) \geqslant e^{\lambda}/4$, thus

$$1 \geqslant \rho_b(f(\xi), \eta')^2 = \rho_y(f(\xi), \eta')^2 \frac{d\rho_b}{d\rho_y}(f(\xi)) \frac{d\rho_b}{d\rho_y}(\eta') \geqslant \rho_y(f(\xi), \eta')^2 e^{2\lambda}/4$$

hence $\rho_y(f(\xi), \eta') \to 0$, and $\eta' \to \eta$, so $f(\xi) = \eta = \partial F(\xi)$. □

Proof of Theorem 1.6. — For X, Y proper geodesically complete metric trees such that ∂X has at least four points, by Theorem 6.2 we have surjective isometries $i_X : X \to \mathcal{M}_X, \hat{f} : \mathcal{M}_X \to \mathcal{M}_Y, i_Y^{-1} : \mathcal{M}_Y \to Y$, and it is clear that the map F defined above equals the composition of these isometries, hence is a surjective isometry $X \to Y$ extending f. □

Proof of Theorem 1.8. — The assertion (1) follows immediately from Theorem 1.1 and 1.5. For the assertion (2), Theorem 1.1 and Theorem 1.4 give us an isometry $F : X \to Y$ with $f = \partial F$ a Moebius homeomorphism. Given $y \in Y$, choose a bi-infinite geodesic $\gamma' \in \mathcal{G}Y$ with $y \in \gamma'(\mathbb{R})$, let $\gamma \in \mathcal{G}X$ be a geodesic whose endpoints map to those of γ' under f, then F maps the image of γ onto the image of γ', in particular y belongs to the image of F, hence F is surjective. □

Finally we prove Theorem 1.7 on almost isometric extension of C^1 conformal maps with bounded integrated Schwarzian. The proof proceeds along similar lines to the proof of Theorem 1.5.

Let (Z, ρ_0) be a compact metric space. We assume Z has no isolated points, and that ρ_0 is diameter one and antipodal. We define the set of metrics

$$\begin{aligned} Conf(Z, \rho_0) := \{\rho | \rho \text{ is a diameter one antipodal metric on } Z \text{ s.t.} \\ id : (Z, \rho_0) \to (Z, \rho) \text{ is } C^1 \text{ conformal}\} \end{aligned}$$

Note that $\mathcal{M}(Z, \rho_0) \subset Conf(Z, \rho_0)$. For $\rho_1, \rho_2 \in Conf(Z, \rho_0)$, the derivative $\frac{d\rho_2}{d\rho_1}$ is a continuous function on Z so we can define

$$d_{Conf}(\rho_1, \rho_2) := \max_{\xi \in Z} \left| \log \frac{d\rho_2}{d\rho_1}(\xi) \right|$$

Then it is easy to see that d_{Conf} is a pseudo-metric on $Conf(Z,\rho_0)$ (though not necessarily a metric) extending the metric $d_{\mathcal{M}}$ on $\mathcal{M}(Z,\rho_0)$. Any C^1 conformal map between compact metric spaces $f : Z_1 \to Z_2$ induces a natural bijective isometry of pseudo-metric spaces $\hat{f} : Conf(Z_1) \to Conf(Z_2)$ by push-forward of metrics.

Now let X be a simply connected complete Riemannian manifold with sectional curvatures satisfying $-b^2 \leqslant K \leqslant -1$, let Y be a proper geodesically complete CAT(-1) space and let $f : \partial X \to \partial Y$ be a C^1 conformal map with bounded integrated Schwarzian. We let $Conf(\partial X) = Conf(\partial X, \rho_x), Conf(\partial Y) = Conf(\partial Y, \rho_y)$ for some $x \in X, y \in Y$ (note the definition does not depend on the choice of x and y), and let $\hat{f} : Conf(\partial X) \to Conf(\partial Y)$ be the induced isometry. We note that

$$\frac{d\hat{f}(\rho_x)}{d\rho_y} \circ f = 1/df_{\rho_x,\rho_y}$$

for all $x \in X, y \in Y$.

Lemma 6.3. — *For all $x \in X, y \in Y$,*

$$\left| \min_{\xi\in\partial X} \log df_{\rho_x,\rho_y}(\xi) + \max_{\xi\in\partial X} \log df_{\rho_x,\rho_y}(\xi) \right| \leqslant 4||S(f)||_\infty$$

Moreover

$$\max_{\xi\in\partial X} |\log df_{\rho_x,\rho_y}(\xi)| \leqslant -\min_{\xi\in\partial X} \log df_{\rho_x,\rho_y}(\xi) + 4||S(f)||_\infty$$

Proof. — Let $\lambda = \max_{\xi\in\partial X} \log df_{\rho_x,\rho_y}(\xi), \mu = \min_{\xi\in\partial X} \log df_{\rho_x,\rho_y}(\xi)$. Let $\eta \in \partial X$ minimize $\log df_{\rho_x,\rho_y}$. Choose $\xi \in \partial X$ such that $\rho_y(f(\xi), f(\eta)) = 1$, then we have, using Theorem 4.10,

$$\begin{aligned} e^\lambda e^\mu &\geqslant df_{\rho_x,\rho_y}(\xi) df_{\rho_x,\rho_y}(\eta) \\ &\geqslant \left(\frac{\rho_y(f(\xi), f(\eta))}{\rho_x(\xi,\eta)} \right)^2 e^{-4||S(f)||_\infty} \\ &\geqslant e^{-4||S(f)||_\infty} \end{aligned}$$

so $\lambda + \mu \geqslant -4||S(f)||_\infty$. For the other inequality, let $\eta \in \partial X$ maximize $\log df_{\rho_x,\rho_y}$, choose $\xi \in \partial X$ such that $\rho_x(\xi,\eta) = 1$, then again by Theorem 4.10, we have

$$\begin{aligned} e^\lambda e^\mu &\leqslant df_{\rho_x,\rho_y}(\xi) df_{\rho_x,\rho_y}(\eta) \\ &\leqslant \left(\frac{\rho_y(f(\xi), f(\eta))}{\rho_x(\xi,\eta)} \right)^2 e^{4||S(f)||_\infty} \\ &\leqslant e^{4||S(f)||_\infty} \end{aligned}$$

This proves the first assertion above. For the second, let $L = \max_{\xi\in\partial X} |\log df_{\rho_x,\rho_y}(\xi)|$. Then either $L = -\mu$ or $L = \lambda \leqslant -\mu + 4||S(f)||_\infty$ by the first assertion. $\square$

Lemma 6.4. — *For all $x \in X$, there exists $y \in Y$ such that $d_{Conf}(\hat{f}(\rho_x), \rho_y) \leqslant \frac{1}{2}\log 2 + 6||S(f)||_\infty$.*

Proof. — Given $x \in X$, define the function $\phi : Y \to \mathbb{R}$ by $\phi(y) = \max_{\xi\in\partial Y} \log \frac{d\hat{f}(\rho_x)}{d\rho_y}(\xi)$. Note ϕ is 1-Lipschitz (since $i_Y : Y \to Conf(\partial Y)$ is an isometry). Let $y_n \in Y$ be a sequence such that $\phi(y_n) \to \inf_{y\in Y} \phi(y)$. Then by Lemma 6.3,

$$\begin{aligned} d_{Conf}(\rho_{y_n}, \hat{f}(\rho_x)) &= \max_{\xi\in\partial X} |\log df_{\rho_x,\rho_{y_n}}(\xi)| \\ &\leqslant -\min_{\xi\in\partial X} \log df_{\rho_x,\rho_{y_n}}(\xi) + 4||S(f)||_\infty \\ &= \max_{\xi\in\partial X}(-\log df_{\rho_x,\rho_{y_n}}(\xi)) + 4||S(f)||_\infty \\ &= \max_{\xi\in\partial Y} \log \frac{d\hat{f}(\rho_x)}{d\rho_{y_n}}(\xi) + 4||S(f)||_\infty \\ &= \phi(y_n) + 4||S(f)||_\infty \end{aligned}$$

Since the sequence $\{\phi(y_n)\}$ is bounded above, by the triangle inequality $d_{Conf}(\rho_{y_n}, \rho_{y_m})$ is bounded independent of m, n, hence so is $d_Y(y_n, y_m)$. Thus we have a convergent subsequence $y_{n_k} \to z \in Y$, and $\phi(z) = \lim \phi(y_{n_k}) = \inf_{y\in Y} \phi(y)$.

Claim. — *Let $\lambda = \phi(z)$, then $\lambda \leqslant \frac{1}{2}\log 2 + 2||S(f)||_\infty$.*

Proof of Claim. — Suppose $\lambda > \frac{1}{2}\log 2 + 2||S(f)||_\infty$. Let $Z \subset \partial Y$ be the set where $\log \frac{d\hat{f}(\rho_x)}{d\rho_z} = \lambda$, and let $\xi_0 \in Z$. Then for any $\xi \in Z$, by Theorem 4.10 we have:

$$\begin{aligned} 1 &\geqslant \hat{f}(\rho_x)(\xi_0,\xi)^2 \\ &\geqslant \rho_z(\xi_0,\xi)^2 \frac{d\hat{f}(\rho_x)}{d\rho_z}(\xi_0)\frac{d\hat{f}(\rho_x)}{d\rho_z}(\xi) e^{-4||S(f)||_\infty} \\ &= \rho_z(\xi_0,\xi)^2 e^{2\lambda} e^{-4||S(f)||_\infty} \\ &> 2\rho_z(\xi_0,\xi)^2 \end{aligned}$$

thus $\rho_z(\xi_0,\xi) < 1/\sqrt{2}$. It follows that there is an open neighbourhood $N \supset Z$ and $\epsilon > 0$ such that $\angle^{(-1)}\xi z \xi_0 \leqslant \pi/2 - \epsilon$ for all $\xi \in N$. By monotonicity of comparison angles, for any $y \in [z, \xi_0)$, we also have $\angle^{(-1)}\xi z y \leqslant \pi/2 - \epsilon$ for all $\xi \in N$, so $\cos(\angle^{(-1)}\xi z y) \geqslant \delta_0$ for some $\delta_0 > 0$. Now let $\lambda' =$

$\sup_{\xi \in \partial Y - N} \log \frac{d\hat{f}(\rho_x)}{d\rho_z}(\xi), \delta_1 = \lambda - \lambda' > 0$, then, using Lemma 3.7, let $t_0 < \delta_1/3$ be such that, for $y \in [z, \xi_0)$ at distance t from z, we have

$$g_{z,y}(\xi) = t\cos(\angle^{(-1)}\xi z y) + o(t)$$

where $||o(t)||_\infty < t\delta_0/2$ for $t \leqslant t_0$. Then, using the Chain Rule, we have for $\xi \in N$ and $0 < t \leqslant t_0$, letting $y \in [z, \xi_0)$ be the point at distance t from z,

$$\log \frac{d\hat{f}(\rho_x)}{d\rho_y}(\xi) = \log \frac{d\hat{f}(\rho_x)}{d\rho_z}(\xi) - g_{z,y}(\xi) \leqslant \lambda - t\delta_0 + t\delta_0/2 < \lambda,$$

while for $\xi \in \partial X - N$ and $0 < t \leqslant t_0$ we have

$$\log \frac{d\hat{f}(\rho_x)}{d\rho_y}(\xi) = \log \frac{d\hat{f}(\rho_x)}{d\rho_z}(\xi) - g_{z,y}(\xi) \leqslant \lambda' + t + t\delta_0/2 \leqslant \lambda - \delta_1 + 2\delta_1/3 < \lambda$$

hence for $0 < t \leqslant t_0$ we have $\phi(y) < \phi(z)$, a contradiction. This proves the Claim. □

Now it follows from Lemma 6.3 that

$$\begin{aligned} d_{Conf}(\hat{f}(\rho_x), \rho_z) &= \max_{\xi \in \partial Y} |\log df_{\rho_x, \rho_z}(\xi)| \\ &\leqslant - \min_{\xi \in \partial Y} \log df_{\rho_x, \rho_z}(\xi) + 4||S(f)||_\infty \\ &= \lambda + 4||S(f)||_\infty \\ &\leqslant \frac{1}{2}\log 2 + 6||S(f)||_\infty \end{aligned}$$

□

We can now prove Theorem 1.7:

Proof of Theorem 1.7. — By the same argument as in the previous Lemma, for each $x \in X$ we may choose a point $F(x) \in Y$ which minimizes $d_{Conf}(\hat{f}(\rho_x), \rho_y)$ over $y \in Y$, and moreover we have $d_{Conf}(\hat{f}(\rho_x), \rho_{F(x)}) \leqslant \frac{1}{2}\log 2 + 6||S(f)||_\infty$. This defines a map $F : X \to Y$.

For $p, q \in X$, let $u = F(p), v = F(q)$, then we have

$$\begin{aligned} |d_Y(u,v) - d_X(p,q)| &= |d_{Conf}(\rho_u, \rho_v) - d_{Conf}(\hat{f}(\rho_p), \hat{f}(\rho_q))| \\ &\leqslant d_{Conf}(\rho_u, \hat{f}(\rho_p)) + d_{Conf}(\rho_v, \hat{f}(\rho_q)) \\ &\leqslant \log 2 + 12||S(f)||_\infty \end{aligned}$$

thus F is a $(1, \log 2 + 12||S(f)||_\infty)$-quasi-isometry. And thus F has a continuous extension to the boundary $\partial F : \partial X \to \partial Y$.

We prove $\partial F = f$. Let $\xi \in \partial X, x \in X$ and let $a \in X$ converge to ξ along the ray $[x, \xi)$. Let $y = F(x), b = F(a), \lambda = d_Y(y, b)$, then $b \to \eta = \partial F(\xi), \lambda \geqslant d_X(x, a) - \log 2 - 12||S(f)||_\infty \to \infty$ as $a \to \xi$. Extend $[y, b]$ to a geodesic ray $[y, \eta')$ where $\eta' \in \partial Y$, then $\frac{d\rho_b}{d\rho_y}(\eta') = e^{\lambda}$ and $b \to \eta$ implies $\eta' \to \eta$. By the Chain Rule,

$$\begin{aligned}\left|\log \frac{d\rho_b}{d\rho_y} - \log \frac{d\hat{f}(\rho_a)}{d\hat{f}(\rho_x)}\right| &\leqslant d_{Conf}(\rho_b, \hat{f}(\rho_a)) + d_{Conf}(\rho_y, \hat{f}(\rho_x)) \\ &\leqslant \log 2 + 12||S(f)||_{\infty}\end{aligned}$$

and $\log \frac{d\hat{f}(\rho_a)}{d\hat{f}(\rho_x)}(f(\xi)) = d_X(x, a) \geqslant d_Y(y, b) - \log 2 - 12||S(f)||_{\infty}$, hence

$$\begin{aligned}\log \frac{d\rho_b}{d\rho_y}(f(\xi)) &\geqslant \log \frac{d\hat{f}(\rho_a)}{d\hat{f}(\rho_x)}(f(\xi)) - \log 2 - 12||S(f)||_{\infty} \\ &\geqslant \lambda - 2\log 2 - 24||S(f)||_{\infty}\end{aligned}$$

so $\frac{d\rho_b}{d\rho_y}(f(\xi)) \geqslant Ce^{\lambda}$ for some constant $C > 0$, thus

$$1 \geqslant \rho_b(f(\xi), \eta')^2 = \rho_y(f(\xi), \eta')^2 \frac{d\rho_b}{d\rho_y}(f(\xi)) \frac{d\rho_b}{d\rho_y}(\eta') \geqslant \rho_y(f(\xi), \eta')^2 Ce^{2\lambda}$$

hence $\rho_y(f(\xi), \eta') \to 0$, so $f(\xi) = \eta = \partial F(\xi)$.

Finally, if Y is also a simply connected complete Riemannian manifold with sectional curvatures satisfying $-b^2 \leqslant K \leqslant -1$, then, given $y \in Y$, we may apply Lemma 6.4 to the map f^{-1} to obtain $x \in X$ such that $d_{Conf}(\hat{f}^{-1}(\rho_y), \rho_x) \leqslant \frac{1}{2}\log + 6||S(f)||_{\infty}$ (note $||S(f^{-1})||_{\infty} = ||S(f)||_{\infty}$). Then by definition of F,

$$\begin{aligned}d_Y(F(x), y) = d_{Conf}(\rho_{F(x)}, \rho_y) &\leqslant d_{Conf}(\rho_{F(x)}, \hat{f}(\rho_x)) + d_{Conf}(\hat{f}(\rho_x), \rho_y) \\ &\leqslant d_{Conf}(\rho_y, \hat{f}(\rho_x)) + d_{Conf}(\hat{f}(\rho_x), \rho_y) \\ &= 2d_{Conf}(\hat{f}^{-1}(\rho_y), \rho_x) \\ &\leqslant \log 2 + 12||S(f)||_{\infty}\end{aligned}$$

thus the image of F is $\log 2 + 12||S(f)||_{\infty}$-dense in Y. □

7. Dynamical classification of Moebius self-maps

Let X be a proper geodesically complete CAT(-1) space whose boundary has at least four points. We use the results of the previous section to prove the dynamical classification of Moebius self-maps of ∂X stated in Theorem 1.9:

Proof of Theorem 1.9. — Let $f : \partial X \to \partial X$ be a Moebius homeomorphism. As in the previous section choose and fix a nearest point projection $\pi_X : \mathcal{M}(\partial X) \to X$, so for all $\rho \in \mathcal{M}(\partial X)$, the visual metric ρ_{x_0}, where

$x_0 = \pi(\rho)$, minimizes $d_{\mathcal{M}}(\rho, \rho_x), x \in X$. Note in particular that π_X is a $(1, \log 2)$-quasi-isometry, $\pi_X \circ i_X = id_X$ and $d_{\mathcal{M}}(\rho, i_X \circ \pi_X(\rho)) \leqslant \frac{1}{2}\log 2, \rho \in \mathcal{M}(\partial X)$, i.e. $i_X \circ \pi_X$ is at a uniformly bounded distance from $id_{\mathcal{M}(\partial X)}$.

Define as in the proof of Theorem 1.5 a sequence of $(1, \log 2)$-quasi-isometric extensions $(F_n : X \to X)_{n\in\mathbb{Z}}$ of the maps $(f^n : \partial X \to \partial X)_{n\in\mathbb{Z}}$ by putting $F_n = \pi_X \circ \widehat{f^n} \circ i_X$ where $\widehat{f^n} : \mathcal{M}(\partial X) \to \mathcal{M}(\partial X)$ denotes the isometry induced by f^n. Note that $\widehat{f^n} = \hat{f}^n$ and $F_0 = id_X$. It is easy to see that since $i_X \circ \pi_X$ is at a bounded distance from $id_{\mathcal{M}(\partial X)}$, for any $m, n \in \mathbb{Z}$ the maps $F_m \circ F_n = \pi_X \circ \widehat{f^m} \circ (i_X \circ \pi_X) \circ \widehat{f^n} \circ i_X, F_n \circ F_m = \pi_X \circ \widehat{f^n} \circ (i_X \circ \pi_X) \circ \widehat{f^m} \circ i_X$ and $F_{m+n} = \pi_X \circ \widehat{f^{m+n}} \circ i_X$ are all within bounded distance of each other.

We note that by the definition of F_n, for any $x \in X$, the maps $f^n : (\partial X, \rho_x) \to (\partial X, \rho_{F_n(x)})$ are uniformly $\sqrt{2}$-bi-Lipschitz.

Since the maps F_n are uniform $(1, \log 2)$-quasi-isometries, it is clear that the set of accumulation points in ∂X of a sequence $(F_n(x))_{n\in\mathbb{Z}}$ is independent of the choice of $x \in X$. We denote this set by Λ. We observe that if $\xi \in \Lambda$, then there is a sequence (n_k) such that for any $x \in X$, $F_{n_k}(x) \to \xi$, in particular $F_{n_k}(F_1(x)) \to \xi$, hence $F_1(F_{n_k}(x)) \to \xi$ (as the two sequences are within bounded distance of each other), and since F_1 has boundary value f, it follows that $F_1(F_{n_k}(x)) \to f(\xi)$, hence $\xi = f(\xi)$. Thus all points of Λ are fixed points of f. We now consider three cases:

Case 1. — $\Lambda = \emptyset$: Then for any $x \in X$, the sequence $(F_n(x))_{n\in\mathbb{Z}}$ is bounded, so the metrics ρ_x and $\rho_{F_n(x)}$ are uniformly bi-Lipschitz to each other independent of n, and it follows from the observation made above that the maps $f^n : (\partial X, \rho_x) \to (\partial X, \rho_x)$ are uniformly bi-Lipschitz, so we are in Case 1 of Theorem 1.9, the elliptic case.

Case 2. — $\Lambda = \{\xi_0\}$: Then for any $x \in X$, $F_n(x) \to \xi_0$ as $|n| \to +\infty$. We claim that $f^n(\xi) \to \xi_0$ as $|n| \to +\infty$ for all $\xi \in \partial X$, i.e. we are in Case 2 of Theorem 1.9, the parabolic case.

Suppose not, then there is a $\xi \neq \xi_0$ such that some subsequence $f^{n_k}(\xi)$ converges to a $\xi_1 \neq \xi_0$. Fix $x \in X$ belonging to the geodesic $\gamma = (\xi_0, \xi)$. The images $F_{n_k}(\gamma)$ are uniform $(1, \log 2)$-quasi-geodesics with endpoints $\xi_0, f^{n_k}(\xi)$, with the endpoints $f^{n_k}(\xi)$ uniformly bounded away from ξ_0, hence there is a ball B of fixed radius around x such that $F_{n_k}(\gamma)$ intersects B for all k. Choose for each k a point $y_k \in F_{n_k}(\gamma) \cap B$. Then $d(y_k, F_{n_k}(x)) \to +\infty$ as $k \to +\infty$. The distances $d(y_k, F_{n_k}(x)), d(F_{-n_k}(y_k), F_{-n_k}(F_{n_k}(x)))$ differ by a uniformly bounded amount (since F_{-n_k}'s are uniform quasi-isometries), as do the distances $d(F_{-n_k}(y_k), F_{-n_k}(F_{n_k}(x)))$, $d(F_{-n_k}(y_k), x)$

(since the maps $F_{-n} \circ F_n$ are within uniformly bounded distance of the identity), hence $d(F_{-n_k}(y_k), x) \to +\infty$.

The horospherical distances $B(\xi_0, F_{-n_k}(y_k), x)$, $B(\xi_0, F_{n_k}(F_{-n_k}(y_k)), F_{n_k}(x))$ differ by a uniformly bounded amount (since the maps F_{n_k} are uniform quasi-isometries with boundary maps f^{n_k} fixing ξ_0), as do $B(\xi_0, F_{n_k}(F_{-n_k}(y_k)), F_{n_k}(x)), B(\xi_0, y_k, F_{n_k}(x))$ (since the maps $F_{-n} \circ F_n$ are within uniformly bounded distance of the identity), and clearly $B(\xi_0, y_k, F_{n_k}(x)) \to +\infty$, hence $B(\xi_0, F_{-n_k}(y_k), x) \to +\infty$. Since the points $F_{-n_k}(y_k)$ lie on uniform quasi-geodesics $F_{-n_k} \circ F_{n_k}(\gamma)$ with fixed endpoints ξ_0, ξ and $d(F_{-n_k}(y), x) \to +\infty$, it follows that $F_{-n_k}(y_k) \to \xi$. Since the points y_k are within uniformly bounded distance of x and the maps F_{-n_k} are uniform quasi-isometries, it follows that $F_{-n_k}(x) \to \xi$, a contradiction.

Case 3. — The set Λ has at least two points: Then pick two distinct points $\xi_+, \xi_- \in \Lambda$, and fix a point x on the geodesic $\gamma = (\xi_+, \xi_-)$. We may assume (replacing f by f^{-1} if necessary) that there is a subsequence $F_{n_k}(x) \to \xi_+$ with $n_k \to +\infty$.

We claim first that $f^{n_k}(\xi) \to \xi_+$ for all $\xi \in \partial X - \{\xi_-\}$. If not, then there is a $\xi \neq \xi_-$ such that, after passing to a further subsequence if necessary, the distances $\rho_x(\xi_+, f^{n_k}(\xi))$ are bounded below by a constant $\epsilon > 0$. Since the points $F_{n_k}(x)$ converge to ξ_+ and lie on uniform quasi-geodesics $F_{n_k}(\gamma)$ with fixed endpoints ξ_+, ξ_-, it follows that $\rho_{F_{n_k}(x)}(\xi_-, f^{n_k}(\xi)) \to 0$. However the maps $f^{n_k} : (\partial X, \rho_x) \to (\partial X, \rho_{F_{n_k}(x)})$ are uniformly bi-Lipschitz, hence the sequence $\rho_{F_{n_k}(x)}(\xi_-, f^{n_k}(\xi))$ is bounded below by a positive constant times $\rho_x(\xi_-, \xi)$, and does not tend to zero, a contradiction.

We now claim that $f^n(\xi) \to \xi_+$ for all $\xi \in \partial X - \{\xi_-\}$ as $n \to +\infty$. Denoting by $df_{p,q}(\xi)$ the derivative of the conformal map $f : (\partial X, \rho_p) \to (\partial X, \rho_q)$ at a point $\xi \in \partial X$, we have

$$\begin{aligned}
(df_{x,x}(\xi^+))^{n_k} &= df^{n_k}_{x,x}(\xi_+) \\
&= df^{n_k}_{x,F_{n_k}(x)}(\xi_+) \cdot \frac{d\rho_x}{d\rho_{F_{n_k}(x)}}(\xi_+) \\
&\to 0
\end{aligned}$$

since $df^{n_k}_{x,F_{n_k}(x)}(\xi_+)$ is bounded above (by $\sqrt{2}$) and $\frac{d\rho_x}{d\rho_{F_{n_k}(x)}}(\xi_+) \to 0$ (as the points $F_{n_k}(x)$ converge to ξ_+ and lie along uniform quasi-geodesics $F_{n_k}(\gamma)$ with fixed endpoints ξ_+, ξ_-). It follows that $df_{x,x}(\xi_+) < 1$, hence there is a neighbourhood U of ξ_+ such that $f^n(\xi) \to \xi_+$ as $n \to +\infty$ for all $\xi \in U$. Now given $\xi \in \partial X - \{\xi_-\}$, there is a k such that $f^{n_k}(\xi) \in U$, hence it follows that $f^n(\xi) \to \xi_+$ as $n \to +\infty$.

Now there is a sequence of integers m_k with $|m_k| \to +\infty$ such that $F_{m_k}(x) \to \xi_-$. By the argument given above, we must have $m_k \to -\infty$ (otherwise there would be a sequence of positive integers tending to infinity with f^n converging pointwise on $\partial X - \{\xi_+\}$ to ξ_-, contradicting the conclusion of the previous paragraph). It follows from the same argument as above that $f^n(\xi) \to \xi_-$ as $n \to -\infty$ for all $\xi \in \partial X - \{\xi_+\}$. Hence we are in Case 3 of Theorem 1.9, the hyperbolic case. □

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Manuscrit reçu le 14 janvier 2014,
accepté le 3 avril 2014.

Kingshook BISWAS
RKM Vivekananda University
P.O. Belur Math
WB-711 202, (India)
kingshook@rkmvu.ac.in