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## ORDERING THE SPACE OF FINITELY GENERATED GROUPS

by Laurent BARTHOLDI & Anna ERSCHLER (\*)

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ABSTRACT. — We consider the oriented graph whose vertices are isomorphism classes of finitely generated groups, with an edge from  $G$  to  $H$  if, for some generating set  $T$  in  $H$  and some sequence of generating sets  $S_i$  in  $G$ , the marked balls of radius  $i$  in  $(G, S_i)$  and  $(H, T)$  coincide. We show that if a connected component of this graph contains at least one torsion-free nilpotent group  $G$ , then it consists of those groups which generate the same variety of groups as  $G$ . We show on the other hand that the first Grigorchuk group has infinite girth, and hence belongs to the same connected component as free groups.

The arrows in the graph define a preorder on the set of isomorphism classes of finitely generated groups. We show that a partial order can be imbedded in this preorder if and only if it is realizable by subsets of a countable set under inclusion.

We show that every countable group imbeds in a group of non-uniform exponential growth. In particular, there exist groups of non-uniform exponential growth that are not residually of subexponential growth and do not admit a uniform imbedding into Hilbert space.

RÉSUMÉ. — Nous considérons le graphe orienté dont les sommets sont les classes d'isomorphisme de groupes de type fini, avec une arête de  $G$  à  $H$  si, pour une partie génératrice de  $H$  et une suite de parties génératrices de  $G$ , les boules marquées de rayon de plus en plus grand coïncident dans  $G$  et  $H$ . Nous montrons que les composantes connexes de groupes nilpotents sans torsion sont leurs variétés, et qu'il y a une arête du premier groupe de Grigorchuk vers un groupe libre.

Les flèches dans ce graphe définissent un préordre sur l'ensemble des classes d'isomorphisme de groupes de type fini. Nous montrons qu'un ordre partiel se plonge dans ce préordre si et seulement s'il est réalisable par des ensembles d'un ensemble dénombrable pour l'inclusion.

Nous montrons que tout groupe dénombrable se plonge dans un groupe de croissance exponentielle non-uniforme. En particulier, il existe des groupes de croissance exponentielle non-uniforme qui ne sont pas résiduellement de croissance subexponentielle.

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*Keywords:* Topological space of marked groups, limit groups, varieties of groups, non-uniform exponential growth, universal statements and identities.

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## 1. Introduction

Our aim, in this paper, is to relate the following preorder on the set of isomorphism classes of finitely generated groups with asymptotic and algebraic properties of groups.

**DEFINITION 1.1.** — *Let  $G, H$  be finitely generated groups. We write  $G \rightsquigarrow H$ , and say that  $G$  preforms  $H$ , if the following holds. There exist a finite generating set  $T$  of  $H$  and a sequence of finite generating sets  $S_1, S_2, \dots$  of  $G$  with bijections  $S_n \rightarrow T$  such that, for all  $R \in \mathbb{N}$ , if  $n$  is large enough then the balls of radius  $R$  in the marked Cayley graphs of  $(G, S_n)$  and  $(H, T)$  are isomorphic.*

We denote by  $\mathcal{C}(G, S)$  the Cayley graph of the group  $G$  with respect to the generating set  $S$ . Its edges are marked with the generator they correspond to.

If  $G$  preforms  $H$ , then we also say that  $H$  is preformed by  $G$ .

Definition 1.1 can be interpreted in terms of the *Chabauty-Grigorchuk topology*, also called the *Cayley topology*, defined as follows. The space of marked groups is the set  $\mathcal{G}$  of pairs  $(G, S)$  with  $G$  a finitely generated group and  $S$  a finite ordered generating set, considered up to group isomorphism preserving the generating set. It is equipped with a natural topology, two marked groups  $(G, S)$  and  $(G', S')$  being close to each other if marked balls of large radius in the Cayley graphs  $\mathcal{C}(G, S)$  and  $\mathcal{C}(G', S')$  are isomorphic.

Chabauty considers this topological space in [20, §3]; he uses it to describe the space of lattices in locally compact groups. Gromov [30, pages 71–72] uses it to derive an effective version of his theorem on groups of polynomial growth. Grigorchuk [28] was the first to study this topology systematically; in particular, he uses it to construct groups of wildly-oscillating intermediate growth, by approximating them in  $\mathcal{G}$  by solvable groups. For generalities on the the space of marked groups, see [21].

Definition 1.1 may then be formulated as follows:  $G \rightsquigarrow H$  if and only if the closure of the isomorphism class of  $G$  in the Chabauty-Grigorchuk topology contains  $H$ .

In Definition 1.1, we require that the sets all  $S_n$  generate  $G$ ; otherwise we would obtain a coarser relation than our preorder, in which too many groups become equal. This coarser relation between  $G$  and  $H$  is an equivalent form of a definition due to Sela [56] that  $H$  is a  *$G$ -limit group*; see also [49]. A group  $A$  is a  *$G$ -limit group* if and only there exists a group containing  $A$  and preformed by  $G$ ; see the remark after Lemma 2.18. We are grateful to Guirardel for turning our attention to this point.

We stress that, in our definition, we consider limits in the space of marked groups of a fixed group, letting only its generating set vary. Various authors have already considered limits in the space of marked groups, not necessarily restricting to limits within one isomorphism class. Limits of one fixed group have been studied when this group is free: they coincide with limit groups (see [21, Thm 1.1]; see also §6.1 for more references). Zarzycki [65] considers groups that are preformed by Thompson’s group  $F$ , and gives some necessary conditions for HNN extensions to appear in this manner; Guyot [33, 34] considers groups that are preformed by  $G$  for some metabelian groups  $G$ , and identifies their closure in  $\mathcal{G}$ . On the other hand, groups that preform free groups are groups that have infinite girth for generating sets of fixed cardinality. These groups can be characterized as groups without almost-identities (see [48, Theorem 9] by Olshansky and Sapir; see also §6.2).

We recall that a *preorder* is a binary relation  $\lesssim$  such that  $A \lesssim C$  whenever  $A \lesssim B$  and  $B \lesssim C$  and such that  $A \lesssim A$  for all  $A$ . If furthermore ‘ $A \lesssim B$  and  $B \lesssim A$ ’ imply  $A = B$ , then it is an *order*. A preorder is *directed* if every finite subset has an upper bound. It is easy to see that the relation ‘ $\rightsquigarrow$ ’ is a preorder, and that  $G \rightsquigarrow H$  does not depend on the choice of a finite generating set in  $H$  (see Lemmas 2.2 and 2.1 in the next section). It is not difficult to see that the restriction of this relation to some classes of groups is an order; this happens, for example, for residually finite finitely presented groups, such as polycyclic groups (see Corollary 2.6). For some other classes of groups this is not true: for example, there exist solvable groups  $G$  admitting a continuum of non-isomorphic solvable groups which are equivalent to  $G$  under our preorder, that is, which both preform and are preformed by  $G$ . Nekrashevych gives in [46] examples of groups acting on rooted trees which are equivalent under our preorder.

In many cases, if  $A$  preforms  $B$ , then  $A$  “looks smaller” than  $B$ . Simple examples of this kind include:  $\mathbb{Z}^m \rightsquigarrow \mathbb{Z}^n$  if and only if  $m \leq n$ ; free groups satisfy  $\mathbb{F}_m \rightsquigarrow \mathbb{F}_n$  if and only if  $m \leq n$ ; and the  $n$ -generated free groups  $\mathbb{V}_n$  in the variety generated by a torsion-free nilpotent group of nilpotency class  $c$  satisfy, for  $m, n \geq c$ , the same relation  $\mathbb{V}_m \rightsquigarrow \mathbb{V}_n$  if and only if  $m \leq n$ , see Proposition A. On the other hand, it may happen that  $A$  preforms  $B$  and that the growth of  $A$  is asymptotically larger than the growth of  $B$ ; we consider this in more detail in §1.3.

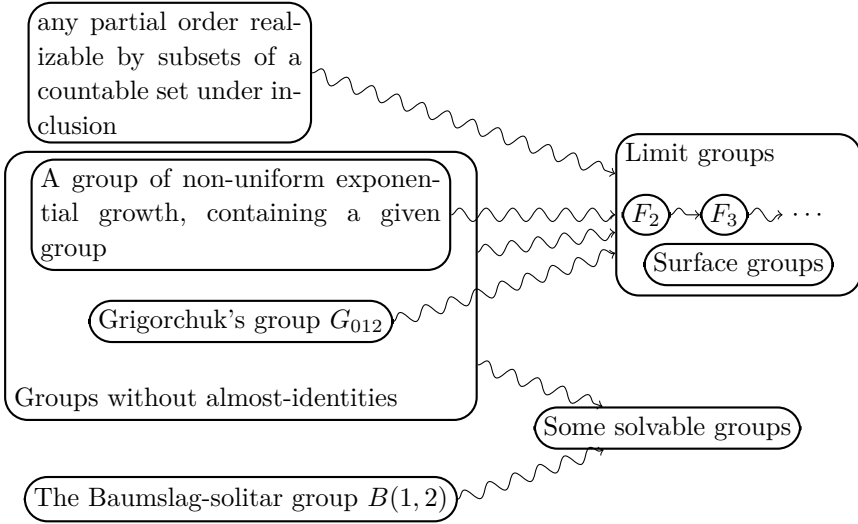


Figure 1.1. Some classes of groups and their relationship under  $\rightsquigarrow$

### 1.1. The structure of components

We view  $\rightsquigarrow$  as specifying the edge set of an oriented graph with vertex set the isomorphism classes of finitely generated groups. In studying this graph, we may consider independently the *connected components* of its underlying unoriented graph. What do they look like? Which components admit an initial vertex? a terminal vertex? Given a connected component, does it have an upper bound? What is the group of preorder preserving bijections of a given component? For which groups are their *strongly connected components* (consisting of groups both preforming and preformed by the given group) reduced to points, or have the cardinality of the continuum?

Unlike some other natural preorders, such as “being a subgroup”, “being a quotient group”, or “being larger” in the sense of Pride ( $G \succeq_P H$  if  $H_1$  is a quotient of  $G_1$ , for respective quotients  $G_1, H_1$  of finite-index subgroups of  $G, H$  by finite normal subgroups, see [53, 58]), the preorder that we consider in this paper has infinitely many connected components. An easily described component is the connected component of  $\mathbb{Z}$ : it contains all infinite abelian groups, and we describe the group of the order preserving bijections of this component in Proposition 3.7.

For a nilpotent group, its connected component is related to, but does not necessarily coincide with, the set of isomorphism classes of groups that generate the same variety as  $G$ .

PROPOSITION A (= Proposition 4.6). — *Let  $G$  be a finitely generated nilpotent group such that  $G$  and  $G/\text{Torsion}(G)$  generate the same variety (i.e. satisfy the same identities). Then, for all  $k \in \mathbb{N}$  large enough,  $G$  preforms the relatively free group of rank  $k$  in that variety.*

This implies in particular that the connected component of  $G$  coincides with the set of isomorphism classes of groups generating the same variety as  $G$ . We show, conversely, that if  $G$  and  $G/\text{Torsion}(G)$  generate different varieties then the connected component of  $G$  is always smaller than the set of isomorphism classes of groups generating the same variety as  $G$ , see Corollary 4.12.

In particular, every finite set of nilpotent groups as in Proposition A has a supremum with respect to our preorder. We believe, in fact, that this last statement holds for all virtually nilpotent groups.

We show, on the other hand, that the preorder types that can occur are quite general, even within solvable groups of class 3, or within groups that preform free groups:

THEOREM B (= Corollary 5.2 and Remark 6.9). — *Let  $(X, \lesssim)$  be a preorder. Then  $(\mathcal{G}/\cong, \rightsquigarrow)$  contains  $(X, \lesssim)$  as a subpreorder if and only if  $X$  has cardinality at most the continuum and all the partial orders it contains are imbeddable in the partial order of subsets of  $\mathcal{B}$  under inclusion, for a countable set  $\mathcal{B}$ .*

In particular, an order  $(X, \leq)$  is a subpreorder of  $(\mathcal{G}/\cong, \rightsquigarrow)$  if and only if  $(X, \leq)$  is imbeddable in the partial order of subsets of  $\mathcal{B}$  under inclusion.

Furthermore, the imbedding of  $(X, \lesssim)$  can then be chosen to be within the set of isomorphism classes of solvable groups of solubility class 3 or, alternatively, within the set of isomorphism classes of groups that preform  $\mathbb{F}_3$ .

Thomas studies in [58] the complexity, with respect to the Borelian structure on  $\mathcal{G}$ , of Pride’s “largeness” preorder and of the “being a quotient” preorder. He shows that these preorders are high in the Borel hierarchy (they are what is called  $\mathbf{K}_\sigma$ -universal). The preorder  $\rightsquigarrow$  differs from the above mentioned preorders even if we forget the underlying Borelian structure: the quotients and largeness preorders have chains with cardinality the continuum, while (by Theorem B) chains for  $\rightsquigarrow$  are countable.

## 1.2. Groups larger or smaller than a given group

Given a group  $G$ , how many groups preform  $G$ ? How many groups are preformed by  $G$ ? How big is the connected component of  $G$ ? What is its diameter?

We note that, if a group  $G$  is virtually nilpotent, then its component is countable. The number of groups that are preformed by  $G$  is countably infinite.

If  $G$  is a free group, a surface group, or more generally a non-abelian limit group (see §6.1), then there are countably many groups that are preformed by  $G$ , see [56, 40]. However, the connected component of  $G$  has the cardinality of the continuum, see Example 6.3. These results are special cases of the following. A group  $G$  is said to be *equationally noetherian* if for every system of equations over a finite set of variables there exists a finite system of equations having the same set of solutions (see [13] for more details of this definition). Baumslag, Myasnikov and Remeslennikov show in [13, Thm B1] that all linear groups are equationally noetherian. Ould Houcine proves in [49] that, if  $G$  is equationally noetherian, then there are at most countably many groups that are limits of  $G$ , in particular, there are at most countably many groups preformed by  $G$ .

We study the groups that preform free groups. Schleimer considers groups of unbounded girth (there are generating sets such that the smallest cycle in the Cayley graph is arbitrarily long) in an unpublished note [55], and they are intimately connected to groups that preform free groups, see Question 8.5. The latter are groups that do not satisfy an *almost-identity* [48, Theorem 9]: a word whose evaluation vanishes on every generating set. Olshansky and Sapir show in [48] that there are groups with non-trivial quasi-identities but no non-trivial identity.

In §6.3, we modify a criterion by Abért [1] about groups without identities to determine when a group has no almost-identity. This lets us answer negatively a question by Schleimer [55, Conjecture 6.2] that groups of unbounded girth have exponential word growth (see §1.3 for the definition of growth):

**THEOREM C** (= Corollary 6.12). — *The first Grigorchuk group  $G_{012}$  preforms  $\mathbb{F}_3$ .*

Extending an argument by Akhmedov (see [4]), we give a criterion for a wreath product with infinite acting group to preform a free group:

**PROPOSITION D** (= Proposition 6.15). — *Let  $G$  and  $H$  be finitely generated groups, and suppose that  $H$  is infinite. Then the restricted wreath*

product  $G \wr H := G^{(H)} \rtimes H$  preforms a free group if and only if at least one of the following conditions holds:

- (1)  $G$  does not satisfy any identity;
- (2)  $H$  does not satisfy any almost-identity.

From this, we deduce (see Remark 6.21) that the connected component of the free group has diameter at least 3; this is in contrast with the nilpotent case, see Proposition A. There are solvable groups, and infinite free Burnside groups, at distance 2 from a free group.

See also subsection 2.4 where we discuss groups that preform a group containing a given subgroup.

### 1.3. Growth of groups

We finally give in §7 new examples of groups of non-uniform exponential growth. Recall that, for a group  $G$  generated by a set  $S$ , its *growth function* counts the number  $\nu_{G,S}(R)$  of group elements expressible as a product of at most  $R$  generators. The group has *exponential growth* if  $\lambda_{G,S} := \lim \sqrt[R]{\nu_{G,S}(R)} > 1$  and *subexponential growth* otherwise; it then has *polynomial growth* if  $\nu_{G,S}$  is dominated by a polynomial, and *intermediate growth* otherwise. The existence of groups of intermediate growth is asked by Milnor in [19], and is answered by Grigorchuk in [28], by means of his group  $G_{012}$ .

A group  $G$  of exponential growth is said to have *uniform exponential growth* if furthermore  $\inf_S \lambda_{G,S} > 1$ . The existence of groups of non-uniform exponential growth is asked by Gromov in [31, Remarque 5.12]; see also [38]. The first examples were constructed by Wilson [64]; see also [8, 46, 63].

**THEOREM E** (= Corollary 7.3). — *Every countable group may be imbedded in a group  $G$  of non-uniform exponential growth.*

*Furthermore, let  $\alpha \approx 0.7674$  be the positive root of  $2^{3-3/\alpha} + 2^{2-2/\alpha} + 2^{1-1/\alpha} = 2$ . Then  $G$  may be required to have the following property: there is a constant  $K$  such that, for any  $R > 0$ , there exists a generating set  $S$  of  $G$  with*

$$\nu_{G,S}(r) \leq \exp(Kr^\alpha) \text{ for all } r \leq R.$$

Theorem E implies the existence of groups of non-uniform exponential growth that do not imbed uniformly into Hilbert space; this answers a question by Brioussell [16, after Proposition 2.5], who asks whether there exist



groups of non-uniform exponential growth without the Haagerup property. We also construct groups of non-uniform exponential growth that admit infinitely many distinct intermediate growth functions at different scales. Moreover, these examples can be constructed among groups that preform free groups and groups of intermediate growth.

The idea of the proof of Theorem E is as follows. We denote by  $G_{012}$  the first Grigorchuk group. It acts on the infinite binary tree  $\{0, 1\}^*$  and its boundary  $\{0, 1\}^\infty$ . We denote by  $X$  the orbit  $G_{012} \cdot 1^\infty$ . We prove in Corollary 7.2 that the group  $G \wr_X G_{012}$  has non-uniform exponential growth whenever  $G$  is a group of exponential growth. To prove Corollary 7.2 we show that  $G \wr_X G_{012}$  preforms a group of intermediate growth. (In fact, all known examples of groups of non-uniform exponential growth preform groups of intermediate growth, though the corresponding group of intermediate growth is not always given explicitly by their construction ; for more on this see Question 8.7).

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## 2. First properties and examples

LEMMA 2.1 (A special case of [21, Proposition 2.20]). — *The “for some generating set  $T$ ” in Definition 1.1 may be changed to “for every generating set  $T$ ”.*

*Proof.* — Assume  $G \overset{\sim}{\rightsquigarrow} H$ , that  $T$  generates  $H$  and that  $\mathcal{C}(G, S_n)$  coincides with  $\mathcal{C}(H, T)$  on ever larger balls. Write  $\tau_n : T \rightarrow S_n$  the bijections.

Let  $T'$  be another generating set of  $H$ ; write every  $t \in T'$  as a word  $w_t$  over  $T$ . Let  $k$  be the maximum of the lengths of the  $w_t$ . Consider the generating sets  $S'_n = \{\tau_n(w_t) : t \in T'\}$  of  $G$  obtained by replacing each  $T$ -letter in  $w_t$  by its corresponding element  $\tau_n(t) \in G$ .

Then, if  $\mathcal{C}(G, S_n) \cap B(1, R)$  is isomorphic to  $\mathcal{C}(H, T) \cap B(1, R)$ , then  $\mathcal{C}(G, S'_n) \cap B(1, \lfloor R/k \rfloor)$  is isomorphic to  $\mathcal{C}(H, T') \cap B(1, \lfloor R/k \rfloor)$ , since they are respective subsets in the isomorphic graphs  $\mathcal{C}(G, S_n) \cap B(1, R)$  and  $\mathcal{C}(H, T) \cap B(1, R)$ . □

LEMMA 2.2. — *The relation  $\overset{\sim}{\rightsquigarrow}$  is a preorder.*

*Proof.* — It is clear that  $G \overset{\sim}{\rightsquigarrow} G$  holds for all groups  $G$ .

Consider now  $G \overset{\sim}{\rightsquigarrow} H \overset{\sim}{\rightsquigarrow} K$ , and let  $U$  be a generating set for  $K$ . There are then generating sets  $T_n$  for  $H$ , in bijection with  $U$ , such that  $\mathcal{C}(H, T_n)$  and  $\mathcal{C}(K, U)$  agree in ever larger balls. For each  $n$ , there are generating sets  $S_{mn}$  for  $G$ , in bijection with  $T_n$ , such that  $\mathcal{C}(G, S_{mn})$  and  $\mathcal{C}(H, T_n)$  agree in ever larger balls.

Therefore, the generating sets  $S_{mn}$ , which are in bijection with  $U$ , are such that  $\mathcal{C}(G, S_{mn})$  and  $\mathcal{C}(K, U)$  agree in ever larger balls, which shows  $G \overset{\sim}{\rightsquigarrow} K$ . □

Let  $\mathbb{F}$  be the free group on infinitely many generators  $x_1, x_2, \dots$ , and consider the space  $\mathcal{G}$  of finitely generated groups  $(G, T)$  with marked generating set. This marking may be given by a homomorphism  $\mathbb{F} \rightarrow G$  such that almost all  $x_n$  map to 1; and this identifies  $\mathcal{G}$  with the set of normal subgroups of  $\mathbb{F}$  containing almost all the  $x_n$ . This turns  $\mathcal{G}$  into a locally compact Polish space. In this alternative terminology, we have the obvious

LEMMA 2.3. — *Let  $G, H$  be finitely generated groups. Then  $G \overset{\sim}{\rightsquigarrow} H$  if and only if for some (hence all) generating set  $T$ , the marked group  $(H, T)$  belongs to the closure of  $\{(G, S) : S \text{ generates } G\}$  in  $\mathcal{G}$ .*

We observe that if  $G \overset{\sim}{\rightsquigarrow} H$  and either  $G$  or  $H$  are finite, then  $G = H$ . We thus restrict ourselves to infinite, finitely generated groups.

LEMMA 2.4. — *Let  $G$  be a finitely generated group, and let  $H$  be a finitely presented group. If  $G \overset{\sim}{\rightsquigarrow} H$ , then  $G$  is a quotient of  $H$ .*

*Proof.* — Let  $T$  be a generating set of  $H$ , and let  $R$  be the maximal length of  $H$ 's relators in that generating set. If  $G \overset{\sim}{\rightsquigarrow} H$ , then there exists a generating set  $S$  for  $G$  such that  $\mathcal{C}(G, S)$  and  $\mathcal{C}(H, T)$  coincide in a ball of radius  $R$ ; so all relations of  $H$  hold in  $T$ . □

We note ([21, Example 2.4(e)]) that every residually finite group is a limit of finite groups; and conversely a finitely presented limit of finite groups is residually finite.

Shalom shows in [57] that every group  $G$  with Kazhdan's property (T) is a quotient of a finitely presented group with Kazhdan's property (T). In this manner, Champetier and Guirardel deduce in [21, Proposition 2.15]

that if  $G \overset{\sim}{\rightsquigarrow} H$  and  $G$  does *not* have Kazhdan's property (T), then neither does  $H$ .

There are isolated points in the space of groups; they are studied in [25]. Clearly, isolated groups are minimal elements for  $\overset{\sim}{\rightsquigarrow}$ ; but the converse is not true. For example,  $\mathbb{Z}$  and  $\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$  are minimal, but none of them is isolated.

## 2.1. Partial orders

On some classes of groups, the relation  $\overset{\sim}{\rightsquigarrow}$  is also antisymmetric, and therefore defines a partial order. Recall that a group  $G$  is Hopfian if every epimorphism  $G \rightarrow G$  is an automorphism.

LEMMA 2.5. — *Among Hopfian, finitely presented groups,  $\overset{\sim}{\rightsquigarrow}$  is an order relation. More generally, if  $G$  and  $H$  are finitely presented groups with  $G \overset{\sim}{\rightsquigarrow} H \overset{\sim}{\rightsquigarrow} G$  and  $G$  is Hopfian, then  $G$  and  $H$  are isomorphic.*

*Proof.* — From  $G \overset{\sim}{\rightsquigarrow} H$  and Lemma 2.4 we deduce that  $G$  is a quotient of  $H$ ; and similarly  $H$  is a quotient of  $G$ . Therefore we have epimorphisms  $G \rightarrow H \rightarrow G$ , and since  $G$  is Hopfian these epimorphisms are isomorphisms.  $\square$

COROLLARY 2.6. — *The relation  $\overset{\sim}{\rightsquigarrow}$  is an order relation on polycyclic groups, and on limit groups.*

*Proof.* — Polycyclic groups are known to be finitely presented and residually finite [39]. We will recall some known facts about limit groups in §6.1; for the proof of the corollary it suffices to know that limit groups are residually free and therefore residually finite, and that they are finitely presented; see §6.1 for references.

Since residually finite groups are Hopfian (see [44]), the corollary follows from Lemma 2.5.  $\square$

## 2.2. Identities, universal statements and varieties

Let  $G$  be a group. An *identity for  $G$*  is a non-trivial word  $w(x_1, x_2, \dots)$  in the free group on countably many generators, such that  $w(g_1, g_2, \dots) = 1$  for every choice of  $g_i \in G$ . Note that  $w$  is really a word in finitely many of the  $x_i$ 's, namely  $w = w(x_1, \dots, x_n)$  for some  $n \in \mathbb{N}$ .

An identity for  $G$  is a special case of a universal sentence: ‘ $\forall g_1, g_2 \dots (w = 1)$ ’. More generally, any well-formed expression made of conjunctions, disjunctions, equalities, and universal quantifiers, is a *positive universal sentence*. If furthermore negations are allowed, it is a *universal sentence*. The *variety* generated by a group  $G$  is the set of identities that it satisfies; and its (*positive*) *universal theory* is the set of (positive) universal sentences that it satisfies.

For example, consider the group  $G = \langle x, y, z \mid [x, y]z^{-1}, z^2, [x, z], [y, z] \rangle$ . It satisfies the identity  $[x_1, x_2]^2$ . It also satisfies the positive universal statement

$$\forall x_1, \dots, x_4 ([x_1, x_2] = 1 \vee [x_1, x_3] = 1 \vee \dots \vee [x_3, x_4] = 1).$$

As a last example, limits groups are known to be “commutative-transitive”; this is the universal statement

$$(2.1) \quad \forall x, y, z ([x, y] = 1 \wedge [y, z] = 1 \Rightarrow [x, z] = 1).$$

Note that this statement is not positive; rewriting it in terms of the primitives  $\vee, \wedge, \neg$  gives  $\forall x, y, z (\neg([x, y] = 1 \wedge [y, z] = 1) \vee [x, z] = 1)$ . An example of a positive statement appears in Example 4.13. For more details relating logic to the space of marked groups, see §6.1 and [21, §5]. In particular, the first assertion of the following lemma is [21, Proposition 5.2].

LEMMA 2.7.

- (1) If  $G \overset{\sim}{\rightsquigarrow} H$  and  $G$  satisfies a universal statement (e.g., an identity), then  $H$  satisfies it too.
- (2) If  $G \overset{\sim}{\rightsquigarrow} H$  and  $H$  is a finitely presented group satisfying a positive universal statement, then  $G$  satisfies it too.
- (3) If  $G \overset{\sim}{\rightsquigarrow} H$  and  $G$  is torsion-free, then  $H$  is torsion-free. More generally, if  $F$  is a finite subgroup of  $H$ , then  $F$  imbeds in  $G$ .

*Proof.* — Ad (1): consider a universal statement satisfied in  $G$ ; it is of the form  $\forall x_1, \dots, x_n (E)$  for a boolean expression  $E$  made of identities  $w_1, \dots, w_\ell$ . Let  $R$  be the maximal length  $w_1, \dots, w_\ell$ .

Consider arbitrary  $h_1, \dots, h_n \in H$ . Extend  $\{h_1, \dots, h_n\}$  to a generating set  $T$  of  $H$ , and find a generating set  $S$  of  $G$  such that the balls of radius  $R$  in  $\mathcal{C}(G, S)$  and  $\mathcal{C}(H, T)$  coincide. Let  $g_1, \dots, g_n$  be the generators of  $G$  that correspond to  $h_1, \dots, h_n$  respectively. Then  $w_i$  traces a path in  $\mathcal{C}(G, S)$  that remains in an  $R$ -neighbourhood of the origin, so  $w_i$  traces a closed loop in  $\mathcal{C}(G, S)$  if and only if it traces a closed loop in  $\mathcal{C}(H, T)$ ; therefore,  $w_i(h_1, \dots, h_n) = 1 \Leftrightarrow w_i(g_1, \dots, g_n) = 1$ , so  $E(h_1, \dots, h_n)$  follows from  $E(g_1, \dots, g_n)$ .

Ad (2): Lemma 2.4 shows that  $G$  is a quotient of  $H$ ; and positive universal statements are preserved by taking quotients.

Ad (3): consider a finite group  $F$ . Then the fact that  $F$  is *not* a subgroup of  $G$  is a universal statement: writing  $f_1, \dots, f_k$  the elements of  $F$ , with multiplication table  $f_i f_j = f_{m(i,j)}$ , the statement is  $\forall g_1, \dots, g_k (g_i = g_j \text{ for some } i \neq j \vee g_i g_j \neq g_{m(i,j)} \text{ for some } i, j)$ . Therefore (3) follows from (1).  $\square$

*Remark 2.8.* — It is essential not to allow negations in (2): a group with torsion, and moreover a torsion group, can preform a finitely presented torsion-free group — e.g., Grigorchuk’s group  $G$  preforms  $\mathbb{F}_3$ , see Corollary 6.12. In fact, if  $G \overset{\sim}{\rightsquigarrow} \mathbb{F}_n$  for some  $n$ , then  $G$  has the same positive universal theory as  $\mathbb{F}$ . However,  $G$  is universally equivalent to  $H$  if and only if  $G$  is a non-abelian limit group of Sela (see §6.1), that is, if  $\mathbb{F} \overset{\sim}{\rightsquigarrow} G$ .

The lemma implies in particular that if  $G$  is virtually nilpotent, then every group in the same connected component has the same language of positive universal statements. However, in any such connected component there are groups that are not universally equivalent to  $G$ .

*Remark 2.9.* — Timoshenko proves the following in [60, Thm 1]: let  $\mathbb{V}$  be a free group in a variety  $\mathcal{V}$ , and let  $H$  be a subgroup of  $\mathbb{V}$  that generates the same variety  $\mathcal{V}$ . Assume that  $\mathbb{V}$  is discriminating (for example, if  $\mathbb{V}$  is approximable by finite  $p$ -groups for an infinite sequence of primes  $p$ ; see §4.1). Then the universal theories of  $\mathbb{V}$  and of  $H$  coincide.

Recall that a group  $G$  is  $\omega$ -residually  $H$  if for every finite subset  $S$  of  $G$  there exists a homomorphism  $\phi_S: G \rightarrow H$  whose restriction to  $S$  is injective; see §6.1. Observe that if  $G$  is  $\omega$ -residually  $H$ , then  $H$  preforms a group  $K$  that contains  $G$  as a subgroup; if moreover the homomorphisms  $\phi_S$  can be chosen to be surjective, then  $H$  preforms  $G$ . Indeed, take  $G = \langle S \rangle$ ,  $H = \langle T \rangle$ , and  $K$  an accumulation point of  $\{\langle \phi_r(S) \cup T \rangle\}$  where  $\phi_r$  is injective on a ball of radius  $r$  in  $G$ .

Timoshenko proves, in [60, Lemma 2], that if  $G$  is a finitely presented group in a variety  $\mathcal{V}$  and  $H$  is a group universally equivalent to  $G$ , then  $G$  is  $\omega$ -residually  $H$ .

*Remark 2.10.* — For equationally noetherian groups (see §1.2), the condition “the universal theory of  $G$  is contained in the universal theory of  $H$ ” actually characterizes the fact that  $H$  is a  $G$ -limit group in the sense of Sela; see [43, Thm A] for this and other equivalences to being a  $G$ -limit group.

**2.3. Basic properties and examples related to varieties**

We defined varieties in §2.2 as collections of identities. Alternatively (see [47]), they are families of groups closed under taking subgroups, quotients and cartesian products, namely the class  $\mathcal{V}$  of all the groups that satisfy these identities. The variety  $\mathcal{V}$  is *finitely based* if it may be defined by finitely many identities. It is *finite* if all finitely generated groups in the variety are finite. Let  $\mathcal{V}$  be a variety, seen as a collection of identities. For a group  $G$ , one defines  $\mathcal{V}(G) = \langle w(g_1, g_2, \dots) : w \in \mathcal{V}, g_1, g_2, \dots \in G \rangle$ , the *verbal subgroup* of  $G$  corresponding to  $\mathcal{V}$ ; thus  $\mathcal{V}(G) = 1$  if and only if  $G$  belongs to the variety. The  $k$ -generated *relatively free group* is  $\mathbb{V}_k := \mathbb{F}_k/\mathcal{V}(\mathbb{F}_k)$ ; it belongs to  $\mathcal{V}$ , and every  $k$ -generated group in  $\mathcal{V}$  is a quotient of  $\mathbb{V}_k$ . A direct consequence of Lemma 2.7(1) is the

LEMMA 2.11. — *If  $G \overset{\sim}{\rightsquigarrow} H$  and  $G$  belongs to  $\mathcal{V}$ , then  $H$  belongs to  $\mathcal{V}$ .  $\square$*

We will consider, in later sections, the restriction of the relation  $\overset{\sim}{\rightsquigarrow}$  to varieties. Just as  $\mathcal{G}$  is a topology on the normal subgroups of  $\mathbb{F}_k$ , there is a topology  $\mathcal{G}(\mathcal{V})$  on the normal subgroups of  $\mathbb{V}_k$ , or equivalently on the normal subgroups of  $\mathbb{F}_k$  that contain  $\mathcal{V}(\mathbb{F}_k)$ . Directly from the definitions,

LEMMA 2.12 ([21, Lemma 2.2]). — *The natural map  $\mathcal{G}(\mathcal{V}) \hookrightarrow \mathcal{G}$  is a homeomorphism on its image, and the image is closed if and only if  $\mathbb{V}_k$  is finitely presented for all  $k \in \mathbb{N}$ .  $\square$*

LEMMA 2.13. — *Let  $\mathcal{V}$  be a finite variety. If  $G \overset{\sim}{\rightsquigarrow} H$ , then  $\mathcal{V}(G) \overset{\sim}{\rightsquigarrow} \mathcal{V}(H)$ .*

*Proof.* — Let  $H$  be generated by a set  $T = \{h_1, \dots, h_k\}$  of cardinality  $k$ , and let  $\mathbb{F}_k$  denote the free group on  $k$  generators  $x_1, \dots, x_k$ . Then  $\mathcal{V}(\mathbb{F}_k)$  admits a generating set of the form  $w(v_1, \dots)$  for some identities  $w$  in  $\mathcal{V}$  and some  $v_1, \dots \in \mathbb{F}_k$ . Then  $\mathcal{V}(H)$  is generated by the set  $T'$  of all corresponding  $w(v_1(h_1, \dots, h_k), \dots)$ .

Consider a generating set  $S = \{g_1, \dots, g_k\}$  of  $G$ , such that  $\mathcal{C}(G, S)$  coincides with  $\mathcal{C}(H, T)$  in a large ball; then  $S' = \{w(v_1(g_1, \dots, g_k), \dots), \dots\}$  generates  $\mathcal{V}(G)$ , and the Cayley graphs  $\mathcal{C}(\mathcal{V}(G), S')$  coincides with  $\mathcal{C}(\mathcal{V}(H), T')$  in a large ball.  $\square$

Given a variety  $\mathcal{V}$ , the *verbal product* of groups  $G_1, G_2, \dots, G_n$  is defined as follows: first set  $G = G_1 * G_2 \cdots * G_n$  the free product; then

$$\prod_{\mathcal{V}} G_i = \frac{G}{\mathcal{V}(G) \cap \langle [g_i, g_j] : g_i \in G_i^G, g_j \in G_j^G, i \neq j \rangle}.$$

For example, if  $\mathcal{V}$  is the variety of all groups, then  $\prod_{\mathcal{V}}$  is the free product; while if  $\mathcal{V}$  is the variety of abelian groups, then  $\prod_{\mathcal{V}}$  is the direct product.

Recall that the *wreath product* of two groups  $G_1, G_2$  is

$$G_1 \wr G_2 = \{f: G_2 \rightarrow G_1 \text{ of finite support}\} \rtimes G_2,$$

where  $G_2$  acts by shift on functions  $G_2 \rightarrow G_1$ .

LEMMA 2.14. — *Let  $G_1, G_2, H_1, H_2$  be groups, and assume  $G_1 \overset{\sim}{\rightsquigarrow} H_1$  and  $G_2 \overset{\sim}{\rightsquigarrow} H_2$ . Then*

- (1)  $G_1 \times G_2 \overset{\sim}{\rightsquigarrow} H_1 \times H_2$ ;
- (2)  $G_1 * G_2 \overset{\sim}{\rightsquigarrow} H_1 * H_2$ ;
- (3) *Let  $\mathcal{V}$  be a variety of groups. Then  $\prod_{\mathcal{V}} G_i \overset{\sim}{\rightsquigarrow} \prod_{\mathcal{V}} H_i$ ;*
- (4)  $G_1 \wr G_2 \overset{\sim}{\rightsquigarrow} H_1 \wr H_2$ .

*Proof.* — We start by (2), and argue that, for arbitrarily large  $R$ , we can make balls of radius  $R$  agree in respective Cayley graphs. For all  $i \in \{1, 2\}$ , let  $T_i$  generate  $H_i$ , and let  $S_i$  generate  $G_i$  in such a manner that balls of radius  $R$  coincide in  $\mathcal{C}(G_i, S_i)$  and  $\mathcal{C}(H_i, T_i)$ . Then  $T := \bigsqcup T_i$  generates  $H := *_i H_i$ , and the corresponding set  $S := \bigsqcup S_i$  generates  $*_i G_i$ . Balls of radius  $R$  coincide in  $\mathcal{C}(G, S)$  and  $\mathcal{C}(H, T)$ .

Ad (3), the relations imposed on  $*_i G_i$  and  $*_i H_i$  are formally defined by  $\mathcal{V}$ , so again balls of radius  $R$  in  $\mathcal{C}(\prod_{\mathcal{V}} G_i, S)$  and  $\mathcal{C}(\prod_{\mathcal{V}} H_i, T)$  coincide.

(1) is a special case of (3).

Ad (4), note that the relations giving  $G_1 \wr G_2$  from  $G := G_1 * G_2$  are  $[x_1^{x_2}, y_1^{y_2}]$  for all  $x_1, y_1 \in G_1$  and  $x_2, y_2 \in G_2 \setminus \{1\}$ . These relations do not exactly define a varietal product; but nevertheless there is a bijection between non-trivial elements of norm  $\leq R$  in  $G_2$  and  $H_2$ , and between elements of norm  $\leq R$  in  $G_1$  and  $H_1$ . The result again follows.  $\square$

Note that in (1) we can have  $G_1 \times C \overset{\sim}{\rightsquigarrow} H_1 \times C$  without having  $G_1 \overset{\sim}{\rightsquigarrow} H_1$ . We examine more carefully this for abelian groups in §3:

*Example 2.15.* — We have  $1 \times \mathbb{Z} \overset{\sim}{\rightsquigarrow} \mathbb{Z} \times \mathbb{Z}$ , yet 1 doesn't preform  $\mathbb{Z}$ .

For  $A = \mathbb{Z}/6 \times \mathbb{Z}$ ,  $B = \mathbb{Z}/35 \times \mathbb{Z}$ ,  $C = \mathbb{Z}/10 \times \mathbb{Z}$ ,  $D = \mathbb{Z}/21 \times \mathbb{Z}$ , we also have  $A \times B \overset{\sim}{\rightsquigarrow} C \times D$  while  $A, B, C, D$  are mutually incomparable.

*Proof.* — Consider  $\{(1, 0), (0, 1)\}$  a generating set of  $\mathbb{Z} \times \mathbb{Z}$ , and, for arbitrary  $R \in \mathbb{N}$ , the generating set  $\{(0, 1), (0, 2R + 1)\}$  of  $1 \times \mathbb{Z}$ . Their Cayley graphs agree on a ball of radius  $R$ .

For the second claim, note that  $A \times B$  is isomorphic to  $C \times D$ , but for any two groups among  $A, B, C, D$ , none is a quotient of the other.  $\square$

Similarly, in (2) we can have  $G_1 * C \overset{\sim}{\rightsquigarrow} H_1 * C$  without having  $G_1 \overset{\sim}{\rightsquigarrow} H_1$ . We will examine more closely the situation of free groups in §6.1; here and

in the sequel we use the notation  $\mathbb{F}_k$  for free groups on  $k$  generators. For now, we just mention the

*Example 2.16.* — Let  $G$  be a  $k$ -generated group. Then, for every  $m \geq 2$ , the free product  $G * \mathbb{F}_m$  preforms  $\mathbb{F}_{k+m} = \mathbb{F}_k * \mathbb{F}_m$ ; yet  $G$  need not preform  $\mathbb{F}_k$ , for example if  $G$  satisfies an identity.

*Proof.* — Let  $S$  generate  $\mathbb{F}_k$ , let  $T$  generate  $\mathbb{F}_m$ , and let  $\{g_1, \dots, g_k\}$  generate  $G$ . Then  $S \sqcup T$  generates  $\mathbb{F}_k * \mathbb{F}_m$ . In  $\mathbb{F}_m$ , there exist elements  $w_1, \dots, w_k$  such that no relation among them and  $T$ , of length  $\leq R$ , holds; consider the generating set  $\{g_1 w_1, \dots, g_k w_k\} \sqcup T$  of  $G * \mathbb{F}_m$ . Then no relation of length  $\leq R$  holds among them.  $\square$

Note finally that in (4) we may have  $G_1 \wr C \overset{\sim}{\rightsquigarrow} H_1 \wr C$  without having  $G_1 \overset{\sim}{\rightsquigarrow} H_1$ ; see §6.5 for more examples:

*Example 2.17.* — Consider  $A, B$  arbitrary groups, and an infinite group  $C$ . Then  $(A * B) \wr C \overset{\sim}{\rightsquigarrow} (A \times B) \wr C$ .

On the other hand, if  $A$  and  $B$  are non-trivial, finitely presented, and each satisfies an identity, then  $A * B$  does not satisfy the identities of  $A \times B$ , so  $A * B$  doesn't preform  $A \times B$  by Lemma 2.7(2).

*Proof.* — Let  $S, T, U$  be generating set of  $A, B, C$  respectively. Then, as generating set of  $(A \times B) \wr C$ , we consider  $S' \sqcup T' \sqcup U$ , in which  $S'$  corresponds to the generators of  $A$  supported at  $1 \in C$ , and similarly for  $T'$ .

For arbitrary  $R \in \mathbb{N}$ , choose  $x \in C$  of norm  $> R$ , and consider the following generating set  $S'' \sqcup T'' \sqcup U$  of  $(A * B) \wr C$ . The copy  $S''$  of  $S$  corresponds to the generators of  $A$  supported at  $1 \in C$ , while the copy  $T''$  corresponds to the generators of  $T$  supported at  $x$ .

Both  $(A \times B) \wr C$  and  $(A * B) \wr C$  are quotients of  $A * B * C$ ; in both cases, all relations of the form  $[s_1^h, s_2]$  and  $[t_1^h, t_2]$  are imposed for all  $h \neq 1$  and  $s_i \in S', t_i \in T'$ , respectively  $s_i \in S'', t_i \in T''$ . However, in the former case, all relations of the form  $[s^h, t]$  are also imposed for all  $h \in H$  and  $s \in S', t \in T'$ . In the latter case, these relations are only imposed for  $h \neq x$  and  $s \in S'', t \in T''$ . However, this distinction is invisible in the ball of radius  $R$ .  $\square$

### 2.4. Limits and prelimits of groups with a given subgroup or quotient

We explore, in this subsection, the ways in which the “preform” relation may be exchanged with the operation of taking subgroups and quotients.



We express these relationships as commutative diagrams, with quantifiers attached to the objects. We start by the following straightforward lemma.

LEMMA 2.18. — *If  $A \overset{\sim}{\rightsquigarrow} B$  and  $A$  is a subgroup of  $G$ , then there exists a group  $H$  containing  $B$  as a subgroup and satisfying  $G \overset{\sim}{\rightsquigarrow} H$ :*

$$\begin{array}{ccc} G & \overset{\sim}{\rightsquigarrow} & \exists H \\ \cup & & \cup \\ A & \overset{\sim}{\rightsquigarrow} & B. \end{array}$$

*Proof.* — Consider finite generating sets  $S_n$  of  $A$  and  $T$  of  $B$  such that  $(A, S_n)$  converges to  $(B, T)$  in the space  $\mathcal{G}$  of marked groups, as  $n \rightarrow \infty$ . Let  $S$  be a finite generating set of  $G$ . Set  $S'_n = S \sqcup S_n$ ; these define finite generating sets of  $G$ . Consider a subsequence  $(n_k)$  such that  $(G, S'_{n_k})$  converges in  $\mathcal{G}$ ; denote its limit by  $(H, U \sqcup V)$ .

In particular,  $(A, S_{n_k})$  converges to the subgroup  $\langle V \rangle$  of  $H$ . Since  $(A, S_n)$  converges to  $B$ , we conclude that  $\langle V \rangle$  is isomorphic to  $B$ .  $\square$

We remark that, in the notation of the lemma,  $B$  is a  $G$ -limit group in the sense of Sela. More generally; given a family of subgroups  $(A_n)$  of  $G$  such that  $(A_n, S_n)$  converges to  $B$ , the same conclusion holds; that is, there exists a group  $H$  containing  $B$  as a subgroup such that  $G$  preforms  $H$ . This shows therefore that  $B$  is a  $G$ -limit group in the sense of Sela if and only if there exists a group containing  $B$  as a subgroup and which is preformed by  $G$ .

LEMMA 2.19. — *If  $A \overset{\sim}{\rightsquigarrow} B$  and  $A$  is a quotient of  $G$ , then there exists a group  $H$  with  $G \overset{\sim}{\rightsquigarrow} H$  and  $B$  is a quotient of  $H$ :*

$$\begin{array}{ccc} G & \overset{\sim}{\rightsquigarrow} & \exists H \\ \downarrow & & \downarrow \\ A & \overset{\sim}{\rightsquigarrow} & B. \end{array}$$

*Proof.* — Let  $A, B$  be  $k$ -generated, with  $T$  a generating set for  $B$ . Since  $A$  preforms  $B$ , there exists a sequence of generating sets  $S_n$  of cardinality  $k$  such that  $(A, S_n) \rightarrow (B, T)$ . Without loss of generality, we may assume  $1 \in S_n$  for all  $n \in \mathbb{N}$ .

Let  $\pi: G \rightarrow A$  be the given epimorphism. Let  $G$  be  $\ell$ -generated. Then for each  $n \in \mathbb{N}$  there exists a generating set  $S'_n = S''_n \sqcup S'''_n$  of  $G$  such that  $S''_n$  maps bijectively to  $S_n$  under  $\pi$  and  $S'''_n$  maps to  $1 \in A$  and has cardinality  $\ell$ . Indeed first choose a generating set  $S'$  for  $G$  of cardinality  $\ell$ ; then, for each  $n \in \mathbb{N}$ , choose an arbitrary lift  $S'_n$  of  $S_n$ ; and multiply each  $g \in S'$  by an appropriate word in  $S'_n$  to obtain  $S'''_n$  mapping to 1.

Passing if need be to a subsequence, we can assume that  $(G, S'_n)$  converges in the space  $\mathcal{G}$  of marked groups. Denote the limit of the subsequence

by  $(H, T')$ , again with decomposition  $T' = T'' \sqcup T'''$ . Let us construct an epimorphism  $\rho: H \rightarrow B$ , showing that  $B$  is a quotient of  $H$ . Recall that  $T''$  is naturally in bijection with  $T$ , via  $S''_n$  and  $S_n$ . We define  $\rho$  on  $T''$  by this bijection, and put  $\rho(t) = 1$  for all  $t \in T'''$ .

To prove that  $\rho$  is a homomorphism, consider a word  $w(x_1, \dots, x_{k+\ell})$  with  $w(T') = 1$  in  $H$ . Since  $(G, S'_n)$  converges to  $(H, T')$ , for sufficiently large  $n \in \mathbb{N}$  we have  $w(S'_n) = 1$  in  $G$ . Let  $v(x_1, \dots, x_k)$  denote the word obtained from  $w$  by deleting its letters  $x_{k+1}, \dots, x_{k+\ell}$ . Since  $\pi$  is a homomorphism, we then have  $v(S_n) = 1$ , and therefore in the limit  $v(T) = 1$ . This is precisely the result of computing  $\rho(w(T'))$  letter by letter.

Finally,  $T$  is in the image of  $\rho$  so  $\rho$  is surjective. □

We may improve on Lemma 2.19 in case the quotient is by a verbal subgroup:

LEMMA 2.20. — *Let the group  $G$  be generated by a set of cardinality  $k$ , and let  $\mathcal{V}$  be a variety. If  $G/\mathcal{V}(G) \overset{\sim}{\rightsquigarrow} \mathbb{V}_k$ , then there exists a group  $H$  with  $G \overset{\sim}{\rightsquigarrow} H$  and  $\mathbb{V}_k = H/\mathcal{V}(H)$ :*

$$\begin{array}{ccc} G & \overset{\sim}{\rightsquigarrow} & \exists H \\ \downarrow & & \downarrow \\ G/\mathcal{V}(G) & \overset{\sim}{\rightsquigarrow} & \mathbb{V}_k. \end{array}$$

*Proof.* — We proceed first as in the proof of Lemma 2.19, to construct a group  $H$  and an epimorphism  $\rho: H \rightarrow \mathbb{V}_k$ .

On the one hand,  $\mathcal{V}(H) \subseteq \ker \rho$ , because  $\mathbb{V}_k$  belongs to  $\mathcal{V}$ . On the other hand, consider  $c \in \ker \rho$ , and write  $c = w(T)$  as a word in the generators  $T$  of  $H$ . Then  $\rho(w(T)) = 1$ , so  $w$  belongs to the variety  $\mathcal{V}(\mathbb{F}_k)$  because  $\mathbb{V}_k$  is relatively free. It follows that  $c$  belongs to  $\mathcal{V}(H)$ . □

LEMMA 2.21 ([21, Proposition 2.25]). — *If  $G \overset{\sim}{\rightsquigarrow} H$  and  $A$  is a quotient of  $G$ , then there exists a group  $B$  with  $A \overset{\sim}{\rightsquigarrow} B$  and  $B$  is a quotient of  $H$ :*

$$\begin{array}{ccc} G & \overset{\sim}{\rightsquigarrow} & H \\ \downarrow & & \downarrow \\ A & \overset{\sim}{\rightsquigarrow} & \exists B. \end{array}$$

Let us turn to the converse property: if  $A \overset{\sim}{\rightsquigarrow} B$  and  $B$  is a subgroup of  $H$ , does there exist a group  $G$  containing  $A$  with  $G \overset{\sim}{\rightsquigarrow} H$ ? Given a subgroup  $B$  of a group  $H$ , we say that the pair  $(H, B)$  satisfies the “prelimit of an overgroup” property if, whenever  $A$  is a group which preforms  $B$ , there

exists a group  $G$  which preforms  $H$  and contains  $A$ :

$$\begin{array}{ccc} \exists G & \overset{\sim}{\rightsquigarrow} & H \\ \cup & & \cup \\ \forall A & \overset{\sim}{\rightsquigarrow} & B. \end{array}$$

We then say that  $H$  has the “prelimit of an overgroup” property if  $(H, B)$  has that property for all finitely generated subgroups  $B$  of  $H$ .

QUESTION 2.22. — Which finitely generated groups have the “prelimit of an overgroup” property?

It is clear that if  $H$  has very few subgroups, for example if every proper subgroup of  $H$  is finite, then  $H$  has the “prelimit of an overgroup” property.

LEMMA 2.23. — All finitely generated abelian groups have the “prelimit of an overgroup” property.

*Proof.* — Inclusions of finitely generated abelian groups into one another can be decomposed into the following “elementary inclusions”:  $B \subseteq B \oplus \mathbb{Z}$ ,  $B \subseteq B \oplus \mathbb{Z}/a\mathbb{Z}$  and  $B \oplus \mathbb{Z}/a\mathbb{Z} \subseteq B \oplus \mathbb{Z}/ab\mathbb{Z}$ . Similarly, the cases to consider for  $A$  that preforms  $B$  are of the form  $\mathbb{Z} \oplus \mathbb{Z}/ac\mathbb{Z} \overset{\sim}{\rightsquigarrow} \mathbb{Z} \oplus \mathbb{Z}/a\mathbb{Z}$  and  $\mathbb{Z} \overset{\sim}{\rightsquigarrow} \mathbb{Z}^2$ . To prove the lemma, it suffices therefore to consider the following case:  $B = \mathbb{Z}^2 \oplus \mathbb{Z}/a\mathbb{Z}$  is a subgroup of  $H = \mathbb{Z}^2 \oplus \mathbb{Z}/ab\mathbb{Z}$ , and  $A = \mathbb{Z} \oplus \mathbb{Z}/ac\mathbb{Z}$  preforms  $B$ . We observe that in this case  $G := \mathbb{Z} \oplus \mathbb{Z}/abc\mathbb{Z}$  contains  $A$ , and preforms  $H$ . □

Example 2.24 (Groups without the “prelimit of an overgroup” property). There are finitely generated groups  $A \overset{\sim}{\rightsquigarrow} B \subseteq H$  such that there exists no group  $G$  with  $A \subseteq G \overset{\sim}{\rightsquigarrow} H$ .

Take indeed  $A = \mathbb{F}_2 \wr \mathbb{Z}$ ; it preforms  $B = \mathbb{Z}^2 \wr \mathbb{Z}$ , which is metabelian. By [12], every metabelian group imbeds in a finitely presented metabelian group  $H$ . If  $G \overset{\sim}{\rightsquigarrow} H$ , then  $G$  is a quotient of  $H$ . This shows that every group which preforms  $H$  is metabelian. Therefore, there are no groups that preform  $H$  that contain  $A$  as a subgroup.

Example 2.25 (Finitely presented groups without the “prelimit of an overgroup” property). — Here is another example of this kind. Consider a finitely presented infinite torsion-free simple group  $H$  containing a non-abelian free group  $B = \mathbb{F}_3$  as a subgroup; such groups were constructed by Burger and Mozes, see [18]. Set  $A = \mathbb{F}_2 \times \mathbb{Z}/2\mathbb{Z}$ ; then  $A \overset{\sim}{\rightsquigarrow} B$  and  $B \subseteq H$ . However, if  $G \overset{\sim}{\rightsquigarrow} H$ , then  $G = H$  because  $H$  is finitely presented and simple. However,  $H$  does not contain  $A$  because  $H$  is torsion-free.

It is usually not true that, if  $G$  preforms  $H$ , then the torsion of  $G$  and  $H$  coincide. Here is a partial result in this direction:

LEMMA 2.26. — *Let  $G$  and  $H$  be groups with  $H$  finitely presented and  $G \rightsquigarrow H$ , and let  $\mathcal{V}$  be a variety. Then*

- (1)  $\#\mathcal{V}(G) = \#\mathcal{V}(H)$ ;
- (2) if  $\mathcal{V}(G)$  is finite, then  $\mathcal{V}(G)$  is isomorphic to  $\mathcal{V}(H)$ .

*Proof.* — By Lemma 2.7(1), the group  $G$  is a quotient of  $H$ , so  $\mathcal{V}(G)$  is a quotient of  $\mathcal{V}(H)$ . In particular,  $\#\mathcal{V}(G) \leq \#\mathcal{V}(H)$ . Furthermore, if  $\mathcal{V}(H)$  is finite then Lemma 2.7(3) implies that  $\mathcal{V}(G)$  and  $\mathcal{V}(H)$  are isomorphic. It therefore remains to prove  $\#\mathcal{V}(G) \geq \#\mathcal{V}(H)$ . We will prove in fact that, if  $\#\mathcal{V}(H) \geq N$ , then  $\#\mathcal{V}(G) \geq N$ .

Choose generating sets  $S_n$  of  $G$  and  $T$  of  $H$ , of cardinality  $k$ , such that  $(G, S_n)$  converges to  $(H, T)$  is the space  $\mathcal{G}$  of marked groups.

Consider then  $N$  distinct elements  $h_1, \dots, h_N$  in  $\mathcal{V}(H)$ , and write each  $h_j = w_j(T)$  for a word  $w_j \in \mathcal{V}(\mathbb{F}_k)$ . Take  $R \in \mathbb{N}$  bigger than the length of each  $w_j$ , and let  $i$  be such that the balls of radius  $R$  in  $\mathcal{C}(G, S_i)$  and  $\mathcal{C}(H, T)$  coincide. Then the ball of radius  $R$  in  $\mathcal{C}(H, T)$  contains at least the  $N$  distinct elements  $h_1, \dots, h_N$  from  $\mathcal{V}(H)$ , so the ball of radius  $R$  in  $\mathcal{C}(G, S_n)$  also contains at least  $N$  distinct elements  $w_1(S_n), \dots, w_N(S_n)$  from  $\mathcal{V}(G)$ . □

### 2.5. Universal theories of solvable groups

For a group  $G$ , we denote by  $G^{(n)}$  its derived series, with  $G^{(0)} = G$  and  $G^{(n+1)} = [G^{(n)}, G^{(n)}]$ . In particular  $G^{(1)} = G'$  and  $G^{(2)} = G''$ .

Here is an example of metabelian group that preforms the free group in its variety. In the next sections, we will study when a nilpotent group preforms the free group in the variety it generates.

Example 2.27. — We have  $\mathbb{Z} \wr \mathbb{Z} \rightsquigarrow \mathbb{F}_2/\mathbb{F}_2''$ .

*Proof.* — Consider the presentation  $\langle a, t \mid [a, a^{t^m}] \forall m \rangle$  of  $\mathbb{Z} \wr \mathbb{Z}$ , and its generating sets  $S_n = \{t, t^n a\}$ . Write  $u = t^n a$ ; then  $[t, u] = [t, a]$ , and  $[t, u]^{t^x u^y}$  all have distinct supports, for  $|x|, |y| \leq n$ . □

Chapuis considers in [22] the universal theory of some solvable groups; he shows that  $\mathbb{F}_k/\mathbb{F}_k''$  and  $\mathbb{Z}^k \wr \mathbb{Z}^\ell$  have the same universal theory. An explicit description of that theory is given in [23]. On the other hand,  $\mathbb{Z} \wr \mathbb{Z} \wr \mathbb{Z}$  and  $\mathbb{F}_k/\mathbb{F}_k^{(3)}$  do not have the same theory.

Timoshenko proves in [59] that, if  $G_1, G_2$  have the same universal theory, and  $H_1, H_2$  have the same universal theory, then  $G_1 \wr H_1$  and  $G_2 \wr H_2$  have the same universal theory. He shows, however, that the varietal wreath product does not, in general, enjoy this property; in particular, it fails in the metabelian variety [61].

He also shows in [60] that, if  $G$  is the quotient of  $\mathbb{S}_{2,n} := \mathbb{F}_2/\mathbb{F}_2^{(n)}$  by a finitely generated normal subgroup, and has the same universal theory as  $\mathbb{S}_{2,n}$ , then either  $G \cong \mathbb{S}_{2,n}$  or  $G$  is the verbal wreath product  $\mathbb{Z} \wr \mathbb{Z}$  in the variety of soluble groups of class  $n - 1$ .

He also considers the universal theories of partially commutative metabelian groups in [62] and subsequent papers.

### 3. Abelian groups

By Corollary 2.6, the relation  $\overset{\sim}{\rightsquigarrow}$  is a partial order on the set of abelian groups. The following is straightforward.

LEMMA 3.1. — *For non-zero  $m, n \in \mathbb{N}$ , we have  $\mathbb{Z}^m \overset{\sim}{\rightsquigarrow} \mathbb{Z}^n$  if and only if  $m \leq n$ .*

*Proof.* — If  $\mathbb{Z}^m \overset{\sim}{\rightsquigarrow} \mathbb{Z}^n$ , then  $\mathbb{Z}^m$  is a quotient of  $\mathbb{Z}^n$  by Lemma 2.4, so  $m \leq n$ . Conversely, if  $m \leq n$ , then choose for  $\mathbb{Z}^n$  a basis  $T$  as generating set, and let  $\{e_1, \dots, e_m\}$  be a basis of  $\mathbb{Z}^m$ . For arbitrary  $R \in \mathbb{N}$ , choose  $S = \{e_1, \dots, e_m, Re_1, R^2e_1, \dots, R^{n-m}e_1\}$  as generating set for  $\mathbb{Z}^m$ , and note that  $\mathcal{C}(\mathbb{Z}^m, S)$  and  $\mathcal{C}(\mathbb{Z}^n, T)$  agree on a ball of radius  $R$ .  $\square$

We now show that all infinite abelian groups are in the same component of  $\overset{\sim}{\rightsquigarrow}$ , which has diameter 2; more precisely,

PROPOSITION 3.2. — *The restriction of  $\overset{\sim}{\rightsquigarrow}$  to infinite abelian subgroups is a net: a partial order in which every pair of elements has an upper bound.*

PROPOSITION 3.3. — *For infinite abelian finitely generated groups  $A, B$ , we have  $A \overset{\sim}{\rightsquigarrow} B$  if and only if  $A$  is a quotient of  $B$  via a map  $B \rightarrow A$  that is injective on the torsion of  $B$ .*

*Proof.* — If  $A \overset{\sim}{\rightsquigarrow} B$ , then  $A$  is a quotient of  $B$  by Lemma 2.4. Let  $R$  be larger than the order of the torsion of  $A$  and  $B$ , and let  $S, T$  be generating sets of  $A, B$  respectively such that  $\mathcal{C}(A, S)$  and  $\mathcal{C}(B, T)$  coincide in the ball of radius  $R$ . Then all torsion elements of  $B$  belong to that ball, and are mapped, by the identification of the ball, to torsion elements of  $A$ . This imbeds the torsion of  $B$  into that of  $A$ .

Conversely, consider an epimorphism  $B \twoheadrightarrow A$  that is injective on the torsion of  $B$ . Let  $B = G_0 \twoheadrightarrow G_1 \twoheadrightarrow \dots \twoheadrightarrow G_n = A$  be a maximal sequence of non-invertible epimorphisms. If we prove  $G_i \overset{\sim}{\twoheadrightarrow} G_{i-1}$  for all  $i = 1, \dots, n$ , then we have  $A \overset{\sim}{\twoheadrightarrow} B$  by Lemma 2.2, so we may restrict to a minimal epimorphism  $\pi: B \twoheadrightarrow A$ . Its kernel is thus infinite cyclic, and we have reduced to the case  $A = \mathbb{Z} \oplus \mathbb{Z}/(k\ell)\mathbb{Z}$  and  $B = \mathbb{Z}^2 \oplus \mathbb{Z}/k\mathbb{Z}$ .

In that case, we consider  $T = \{f_1, f_2, f_3\}$  the standard generating set for  $B$ , and denote by  $\{e_1, e_2\}$  the standard generators for  $A$ . For arbitrary  $R \in \mathbb{N}$ , we consider the generating set  $S = \{\ell e_1, e_2, e_1 + R e_2\}$  for  $A$ , and note that the balls of radius  $R$  in  $\mathcal{C}(B, T)$  and  $\mathcal{C}(A, S)$  coincide.  $\square$

*Proof of Proposition 3.2.* — Consider  $A, B$  abelian groups, written as

$$A = \bigoplus_{i=1}^a \mathbb{Z}/m_i\mathbb{Z}, \quad B = \bigoplus_{i=1}^b \mathbb{Z}/n_i\mathbb{Z}.$$

Then both groups preform  $\mathbb{Z}^{\max(a,b)}$ .  $\square$

**COROLLARY 3.4.** — *Let  $A$  be an infinite abelian group. Then  $A$  is torsion-free if and only if the set of groups that are preformed by  $A$  is linearly ordered.*

*Proof.* — If  $A = \mathbb{Z}^d$  and  $A \overset{\sim}{\twoheadrightarrow} B$ , then  $B = \mathbb{Z}^e$  for some  $e \geq d$ . The set of such  $B$  is order-isomorphic to  $\{d, d + 1, \dots\}$ .

Now suppose that  $A$  is not torsion-free. By Proposition 3.3, we have  $A \overset{\sim}{\twoheadrightarrow} \mathbb{Z}^d \oplus \mathbb{Z}/p\mathbb{Z}$  for some  $p > 1$  and  $d > 1$ . Then  $A \overset{\sim}{\twoheadrightarrow} \mathbb{Z}^{d+1}$  and  $A \overset{\sim}{\twoheadrightarrow} \mathbb{Z}^{d+1} \oplus \mathbb{Z}/p\mathbb{Z}$ , but these last groups are not comparable.  $\square$

Let us denote by  $\mathcal{A}$  the subset of  $\mathcal{G}$  consisting of abelian groups, and by  $\mathcal{A}/\cong$  the set of isomorphism classes of abelian groups; as we noted above,  $(\mathcal{A}/\cong, \overset{\sim}{\twoheadrightarrow})$  is a net.

**COROLLARY 3.5.** — *Every finite partial order is imbeddable in  $(\mathcal{A}/\cong, \overset{\sim}{\twoheadrightarrow})$ .*

*Proof.* — Let  $(X, \leq)$  be a partially ordered set. We identify  $x \in X$  with  $I_x := \{z \in X : z \geq x\}$ , and have  $I_y \subseteq I_x \Leftrightarrow x \leq y$ ; therefore, we assume without loss of generality that  $X$  is contained, for some  $N \in \mathbb{N}$ , in the partially ordered set of subsets of  $\{1, \dots, N\}$ , ordered under reverse inclusion.

Consider  $N$  distinct prime numbers  $p_1, \dots, p_N$ . For any subset  $U \subseteq \{1, \dots, N\}$ , consider the  $N + 1$ -generated group  $A_U$  defined by

$$A_U = \bigoplus_{i \in U} \mathbb{Z}/p_i\mathbb{Z} \oplus \mathbb{Z}^{1+N-\#U}.$$

Observe that the torsion subgroup of  $A_U$  is contained in the torsion group of  $A_{U'}$  if and only if  $U' \subseteq U$ . Observe also that if  $U' \subseteq U$ , then  $A_U$  is a quotient of  $A_{U'}$ . By Proposition 3.3, we get  $A_U \rightsquigarrow A_{U'}$  if and only if  $U' \subseteq U$ . □

*Remark 3.6.* — Some countable orders cannot be imbedded in  $(\mathcal{A}/\cong, \rightsquigarrow)$ ; for example,  $\mathbb{N} \cup \{\infty\}$ . Observe indeed that a countable increasing sequence of non-isomorphic abelian groups has no common upper bound in  $(\mathcal{A}/\cong, \rightsquigarrow)$ .

**PROPOSITION 3.7.** — *The group of order-preserving bijections of  $(\mathcal{A}/\cong, \rightsquigarrow)$  is the infinite symmetric group on a countable set. If we identify this countable set with the prime numbers, then the action on infinite abelian groups is as follows. A permutation  $p \mapsto \sigma(p)$  of the primes acts as*

$$(3.1) \quad \mathbb{Z}^d \oplus \mathbb{Z}/p_1^{\nu_1} \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_k^{\nu_k} \mathbb{Z} \mapsto \mathbb{Z}^d \oplus \mathbb{Z}/\sigma(p_1)^{\nu_1} \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/\sigma(p_k)^{\nu_k} \mathbb{Z}.$$

*Proof.* — As a countable set, we take the set  $\mathcal{P}$  of prime numbers. By Proposition 3.3, the group  $\mathfrak{S}$  of permutations of  $\mathcal{P}$  acts on  $(\mathcal{A}/\cong, \rightsquigarrow)$  by (3.1). We wish to prove that there are no other order-preserving bijections. We implement this in the following lemmas.

**LEMMA 3.8.** — *Every order-preserving bijection of infinite abelian groups fixes torsion-free abelian groups.*

*Proof.* — By Corollary 3.4, torsion-free abelian groups are characterized by the fact that the set of groups that they preform is linearly ordered. Let  $\phi$  be an order-preserving bijection. Observe that  $\phi$  must fix the minimal element  $\mathbb{Z}$ . Note that groups that are preformed by  $\mathbb{Z}$  are linearly ordered by  $\mathbb{N}$ , so admit no order isomorphism. Therefore,  $\phi(\mathbb{Z}^d) = \mathbb{Z}^d$  for any  $d \geq 1$ . □

**LEMMA 3.9.** — *Every order-preserving bijection of infinite abelian groups preserves the number of factors in a minimal decomposition as a product of cyclic groups.*

*Proof.* — Consider an infinite abelian group  $A$ , and let  $\ell$  be the minimal number of cyclic subgroups in the decomposition of  $A$  in a product of (finite or infinite) cyclic groups. Since  $A$  is infinite, at least one subgroup in the decomposition is infinite. We know that for any  $p \in \mathbb{N}$  the group  $\mathbb{Z} + p\mathbb{Z}$  preforms  $\mathbb{Z}^2$ , so  $A$  preforms  $\mathbb{Z}^\ell$ .

Observe also that for  $k < \ell$  the group  $A$  cannot be generated by  $k$  elements, so  $A$  is not a quotient of  $\mathbb{Z}^k$ . By Proposition 3.3,  $A$  doesn't preform  $\mathbb{Z}^k$  for  $k < \ell$ .

Let  $\phi$  be an order-preserving bijection. By Lemma 3.8, we have  $\phi(\mathbb{Z}^k) = \mathbb{Z}^k$  for all  $k \geq 1$ , so  $\phi(A)$  preforms  $\mathbb{Z}^\ell$  but not  $\mathbb{Z}^k$  for  $k < \ell$ . Therefore,  $\phi(A)$  requires precisely  $\ell$  factors in a minimal decomposition as a product of cyclic groups.  $\square$

LEMMA 3.10. — *Every order-preserving bijection  $\phi$  of infinite abelian groups preserves the number of finite and infinite factors in a minimal decomposition as a product of cyclic groups.*

*Proof.* — Let  $A$  be an infinite abelian group. Let  $t$  be the minimal number of finite cyclic groups in its decomposition into a product of cyclic ones, and let  $t + d$  be the minimal total number of finite cyclic groups in such decomposition. We have  $A = \mathbb{Z}^d \oplus \bigoplus_{i=1}^t \mathbb{Z}/n_i\mathbb{Z}$ , with  $n_i \geq 2$ . Observe that  $A$  is preformed by  $\mathbb{Z} \oplus \bigoplus_{i=1}^t \mathbb{Z}/n_i\mathbb{Z}$ , and thus is preformed by some group whose minimal total number of cyclic groups in a decomposition equals  $t + 1$ . Observe then that  $A$  is not preformed by any group for which this minimal number is  $\leq t$ . Indeed, if  $B$  preforms  $A$ , then  $B$  is an infinite group, so the number of infinite cyclic group in the decomposition is  $\geq 1$ . We know that the torsion subgroup of  $A$ , that is  $\bigoplus_{i=1}^t \mathbb{Z}/n_i\mathbb{Z}$ , is a subgroup of the torsion subgroup of  $B$ . Therefore, the minimal number of finite cyclic groups in the decomposition of  $B$  is at least  $t$ . The statement of the lemma now follows from the previous lemma.  $\square$

Consider now an order-preserving bijection  $\phi$  of abelian groups. Let us show that for every prime  $p$  there exists a prime  $q$  such that  $\phi(\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/q\mathbb{Z}$ . First observe that any group with non-trivial torsion and which preforms  $\mathbb{Z}^2$  has the form  $\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$  for some  $n \geq 2$ . If  $n$  is not a prime number, then  $n$  can be written as  $n = n_1 n_2$  with  $n_1, n_2 \geq 2$  and in this case  $\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$  preforms  $\mathbb{Z} \oplus \mathbb{Z}/n_1\mathbb{Z}$ . This implies that the groups of the form  $A = \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$  are characterized by the following properties:  $A$  is not torsion-free;  $A$  preforms  $\mathbb{Z}^2$ ; if  $B$  is such that  $A \overset{\sim}{\rightsquigarrow} B \overset{\sim}{\rightsquigarrow} \mathbb{Z}^2$  then either  $B = A$  or  $B = \mathbb{Z}^2$ . This implies that  $\phi(\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z})$  is isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}/q\mathbb{Z}$  for some prime  $q$ .

As we have already mentioned, every permutation of the primes induces an order-preserving permutation of infinite abelian group. It remains to prove that a permutation of infinite abelian groups is determined by its action on groups of the form  $\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ . Consider therefore such a permutation  $\phi$ , and assume that it fixes  $\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$  for all  $p \in \mathcal{P}$ . We wish to show that it fixes every abelian group.

LEMMA 3.11. — *Let  $\phi$  be an order-preserving bijection of the infinite abelian groups, such that  $\phi(\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$  for all primes  $p$ .*



Then for all  $k, m \geq 1$  we have  $\phi(\mathbb{Z}^k \oplus \mathbb{Z}/p^m\mathbb{Z}) = \mathbb{Z}^k \oplus \mathbb{Z}/p^m\mathbb{Z}$ .

*Proof.* — Set  $A = \mathbb{Z}^k \oplus \mathbb{Z}/p^m\mathbb{Z}$ . By Lemma 3.10, we have  $\phi(A) = \mathbb{Z}^k \oplus \mathbb{Z}/n\mathbb{Z}$  for some  $n \geq 2$ . We proceed by induction on  $m$  to show that  $A$  is fixed.

If  $m = 1$ , then  $A$  is preformed by  $\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$  which is fixed, so  $\phi(A)$  is also preformed by this group, and  $n|p$ . Since  $n \neq 1$ , we have  $n = p$  as required.

Consider then  $m \geq 2$ . We have  $A \overset{\sim}{\rightsquigarrow} \mathbb{Z}^{k+1} \oplus \mathbb{Z}/p^{m-1}\mathbb{Z}$ , which is fixed by induction, so  $p^{m-1}|n$ , and in fact  $p^{m-1} \neq n$  because  $\phi(A)$  does not belong to the set of groups of the form  $\mathbb{Z}^\ell \oplus \mathbb{Z}/p^{m-1}$  which are all fixed by  $\phi$ .

On the other hand,  $A$  doesn't preform any of the groups  $\mathbb{Z}^\ell \oplus \mathbb{Z}/q\mathbb{Z}$  for  $q \neq p$  prime, which are fixed, so  $\phi(A)$  doesn't preform any of these groups, and  $n = p^e$  for some  $e \geq m$ .

Now there are precisely  $m + 1$  groups between  $A$  and  $\mathbb{Z}^{k+2}$ , namely all  $\mathbb{Z}^{k+1} \oplus \mathbb{Z}/p^i\mathbb{Z}$  for  $i = 0, \dots, m$ . This feature distinguishes  $A$  from  $\mathbb{Z} \oplus \mathbb{Z}/p^e\mathbb{Z}$  for all  $e \neq m$ , and therefore  $A$  is fixed by  $\phi$ . □

LEMMA 3.12. — *Let  $\phi$  be an order-preserving bijection of the infinite abelian groups, such that  $\phi(\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$  for all primes  $p$ .*

*Then  $\phi$  fixes all groups of the form  $\mathbb{Z}^k \oplus C$  with  $C$  an abelian  $p$ -group.*

*Proof.* — By Lemma 3.10, we have  $\phi(\mathbb{Z}^k \oplus C) = \mathbb{Z}^k \oplus C'$  for a finite group  $C'$  with the same number of factors in a minimal decomposition as a product of cyclic groups.

Write  $C = \bigoplus_{i=1}^r \mathbb{Z}/p^{e_i}\mathbb{Z}$ , with  $1 \leq e_1 \leq e_2 \leq \dots \leq e_r$ . We proceed by induction on  $r$ , the case  $r = 1$  being covered by Lemma 3.11.

Write  $A = \mathbb{Z}^k \oplus C$ . Since, when  $\ell$  is large,  $A \overset{\sim}{\rightsquigarrow} \mathbb{Z}^\ell \oplus \mathbb{Z}/q\mathbb{Z}$  with  $q$  prime if and only if  $q = p$ , we find that  $C'$  is a  $p$ -group of the form  $\bigoplus_{i=1}^r \mathbb{Z}/p^{f_i}\mathbb{Z}$ , with  $1 \leq f_1 \leq \dots \leq f_r$ .

Consider  $B = \mathbb{Z}^{k+1} \oplus \bigoplus_{i=1}^{k-1} \mathbb{Z}/p^{e_i}\mathbb{Z}$ , which is fixed by induction. We have  $A \overset{\sim}{\rightsquigarrow} B$ , so  $\phi(A) \overset{\sim}{\rightsquigarrow} B$  and therefore  $f_1 = e_1, \dots, f_{r-1} = e_{r-1}, f_r \geq e_r$  by Proposition 3.3. It remains to prove  $f_r = e_r$ .

Again by induction, the group  $\mathbb{Z} \oplus B$  is fixed by  $\phi$ . There are  $e_r + 1$  groups between  $A$  and  $\mathbb{Z} \oplus B$ , namely  $B \oplus \mathbb{Z}/p^e\mathbb{Z}$  for  $e = 0, \dots, e_r$ . This distinguishes  $A$  among all  $\mathbb{Z}^k \oplus \bigoplus_{i=1}^{r-1} \mathbb{Z}/p^{e_i}\mathbb{Z} \oplus \mathbb{Z}/p^{f_r}\mathbb{Z}$  with  $f_r \geq e_r$ . □

We are ready to finish the proof of Proposition 3.7. Consider again  $\phi$  fixing all  $\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$  for  $p$  prime, and an abelian group  $A = \mathbb{Z}^k \oplus C$  with  $C$  finite; let us show that the torsion of  $\phi(A)$  is isomorphic to  $C$ .

First, by Lemma 3.11, we have  $\phi(A) = \mathbb{Z}^k \oplus C'$  for a finite group  $C'$ . Observe that, for  $\ell$  large and  $D$  a  $p$ -group,  $A$  preforms  $\mathbb{Z}^\ell \oplus D$  if and only if  $D$  is a subgroup of  $C$ . By Lemma 3.12, this group  $\mathbb{Z}^\ell \oplus D$  is fixed by  $\phi$ , so  $C$

and  $C'$  have the same  $p$ -subgroups. Since every abelian group is the product of its  $p$ -Sylow subgroups, it follows that  $C$  and  $C'$  are isomorphic.  $\square$

### 3.1. Virtually abelian groups

There are countably many components of virtually abelian groups, as we now show:

*Example 3.13.* — Let  $N_{2,2}$  be the group with presentation

$$N_{2,2} = \langle a, b \mid c = [a, b] \text{ central} \rangle,$$

and for every  $n \in \mathbb{N}$  let  $G_n$  be the virtually abelian group

$$N_{2,2,n} = N_{2,2} / \langle c^n \rangle = \langle a, b \mid [a, b]^n, [a, b] \text{ central} \rangle.$$

Then every  $N_{2,2,n}$  is virtually  $\mathbb{Z}^2$ , but if  $m \neq n$  then  $N_{2,2,n}$  and  $N_{2,2,m}$  belong to different components of  $\mathcal{G}/\cong$ .

*Proof.* — Without loss of generality, assume  $m < n$ , and let  $H$  belong to the component of  $N_{2,2,m}$ ; so there is a sequence  $N_{2,2,m} = H_0, H_1, \dots, H_\ell = H$  with  $H_i \overset{\sim}{\rightsquigarrow} H_{i-1}$  or  $H_{i-1} \overset{\sim}{\rightsquigarrow} H_i$  for all  $i = 1, \dots, \ell$ . By Lemma 2.7(1,2), every  $H_i$  is finitely presented and satisfies the identity  $[x, y]^m$ . However,  $N_{2,2,n}$  does not satisfy this identity.  $\square$

*Remark 3.14.* — If  $p$  is prime, then the set of groups preformed by  $N_{2,2,p}$  is precisely  $\{N_{2,2,p} \times \mathbb{Z}^\ell : \ell \in \mathbb{N}\}$ .

*Proof.* — Elements of  $N_{2,2,p}$  may uniquely be written in the form  $a^x b^y c^z$  for some  $x, y \in \mathbb{Z}$  and  $z \in \{0, \dots, p - 1\}$ . Consider a sequence of generating sets  $S_1, S_2, \dots$  of same cardinality  $k$ . Clearly, if each  $S_n$  is changed by a bounded number of Nielsen transformations, then without loss of generality one may assume (up to taking a subsequence) that the same transformations are applied to all  $S_n$ , and therefore the limit does not change.

Using at most  $pk$  transformations, the set  $S_n$ , whose elements we write as  $\{s_{n,1}, \dots, s_{n,k}\}$ , can be transformed in such a manner that two elements  $s_{n,1}, s_{n,2}$  generate  $N_{2,2,p}$  while the other  $s_{n,3}, \dots, s_{n,k}$  are of the form  $a^x b^y c^z$  with  $p|x$  and  $p|y$ , and therefore belong to the centre of  $N_{2,2,p}$ . Some of these elements will belong to  $\langle s_1, s_2 \rangle$  in the limit, and others will generate extra abelian factors.  $\square$

## 4. Nilpotent groups

Given a group  $G$ , we denote its lower central series by  $\gamma_1(G) = G$  and  $\gamma_{i+1}(G) = [G, \gamma_i(G)]$  for all  $i \geq 1$ . By  $N_{s,k} = \mathbb{F}_k/\gamma_{s+1}(\mathbb{F}_k)$  we denote the free nilpotent group of class  $s$  on  $k$  generators.

We study in this section the structure of connected components of nilpotent groups; our main result is that, if  $G/\text{Torsion}(G)$  generates the same variety as  $G$ , then the connected component of  $G$  is determined by the variety that it generates and conversely.

### 4.1. Free groups and subgroups in nilpotent varieties

Following [47, Def 17.12], a group  $G$  is said to be *discriminating* if, given any finite set  $\mathscr{W}$  of identities that do not hold in  $G$  (i.e., for every  $w \in \mathscr{W}$  there are  $g_1, g_2, \dots \in G$  with  $w(g_1, \dots) \neq 1$ ), all identities in  $\mathscr{W}$  can be falsified simultaneously (i.e. there are  $g_1, g_2, \dots \in G$  such that  $w(g_1, \dots) \neq 1$  for all  $w \in \mathscr{W}$ ). We will say  $G$  is *discriminating on  $k$  generators* if, given any finite set  $\mathscr{W}$  of identities in  $k$  letters that do not hold in  $G$  (i.e., for every  $w \in \mathscr{W}$  there are  $g_1, \dots, g_k \in G$  with  $w(g_1, \dots, g_k) \neq 1$ ), all identities in  $\mathscr{W}$  can be falsified simultaneously on a generating set (i.e. there exists a generating set  $\{g_1, \dots, g_k\}$  of  $G$  such that  $w(g_1, \dots, g_k) \neq 1$  for all  $w \in \mathscr{W}$ ).

Baumslag, Neumann, Neumann, and Neumann show in [14, Cor 2.17] that finitely generated torsion-free nilpotent groups are discriminating; see also [47, Thm 17.9]. If  $G$  is a nilpotent group with torsion, the matter is more delicate: Baumslag and the Neumanns prove in the same place that  $G$  is discriminating if and only if  $G$  and  $G/\text{Torsion}(G)$  generate the same variety.

**LEMMA 4.1.** — *Let  $G$  be a discriminating group, and let  $\mathcal{V}$  be the variety generated by  $G$ . Let  $\mathbb{V}_k := \mathbb{F}_k/\mathcal{V}(\mathbb{F}_k)$  be the free group on  $k$  generators in  $\mathcal{V}$ . Then for every  $k \in \mathbb{N}$  there exists a group  $H$  that is preformed by  $G$  and contains  $\mathbb{V}_k$  as a subgroup.*

*If furthermore  $G$  is discriminating on  $k$  generators, then  $G$  preforms  $\mathbb{V}_k$ .*

*Proof.* — Consider first a finite set of words  $\mathscr{W} \subset \mathbb{F}_k$  that are not identities of  $\mathbb{V}_k$ , that is  $w \in \mathbb{F}_k \setminus \mathcal{V}(\mathbb{F}_k)$  for all  $w \in \mathscr{W}$ . Observe that for each  $w \in \mathscr{W}$  there exist elements  $g_{w,1}, \dots, g_{w,k} \in G$  with  $w(g_{w,1}, \dots, g_{w,k}) \neq 1$ ; otherwise,  $w$  would be an identity in  $G$  and therefore would vanish on  $\mathbb{V}_k$ . Since  $G$  is discriminating, there exist  $g_{\mathscr{W},1}, \dots, g_{\mathscr{W},k} \in G$  such that  $w(g_{\mathscr{W},1}, \dots, g_{\mathscr{W},k}) \neq 1$  for all  $w \in \mathscr{W}$ .

We apply this with  $\mathscr{W}$  the set of words of length at most  $R$  in  $\mathbb{F}_k$  that are not identities in  $\mathbb{V}_k$ , and denote the resulting  $g_{\mathscr{W},1}, \dots, g_{\mathscr{W},k}$  by  $g_{R,1}, \dots, g_{R,k}$ .

Let  $S$  be a finite generating set for  $G$ , and put  $S_R = S \sqcup \{g_{R,1}, \dots, g_{R,k}\}$ . Choose an accumulation point  $(H, T)$  of the sequence  $(G, S_R)$  in the space  $\mathscr{G}$  of marked groups. Then  $H$  contains  $\mathbb{V}_k$  as the subgroup generated by the limit of  $\{g_{R,1}, \dots, g_{R,k}\}$ .

If  $G$  is discriminating on  $k$  generators, then we can take  $S = \emptyset$  in the previous paragraph, to see that  $H$  is isomorphic to the relatively free group  $\mathbb{V}_k$ . □

For a real constant  $C$ , let us say that the sequence of positive real numbers  $x_1, x_2, \dots, x_s$  grows at speed  $C$  if  $x_1 \geq C$  and  $x_{i+1} \geq x_i^C$  for  $i = 1, \dots, s - 1$ . Similarly, an unordered set  $\{x_1, \dots, x_s\}$  grows at speed  $C$  if it admits an ordering that grows at speed  $C$ .

LEMMA 4.2. — *Suppose that  $f_1, \dots, f_t$  are nonzero polynomials in  $s$  variables with real coefficients. Then there exists  $C$  such that  $f_i(x_1, \dots, x_s) \neq 0$  for all  $i = 1, \dots, t$  whenever  $(x_1, \dots, x_s)$  grows at speed  $C$ .*

*Proof.* — It suffices to prove the statement for a single polynomial  $f$ . Let  $x_1^{e_1} \dots x_s^{e_s}$  be the lexicographically largest monomial in  $f$ ; namely,  $e_s$  is maximal among all monomials in  $f$ ; then  $e_{s-1}$  is maximal among monomials of degree  $e_s$  in  $x_s$ ; etc. Then this monomial dominates  $f$  as  $(x_1, \dots, x_s)$  grows faster and faster. □

LEMMA 4.3. — *Consider  $d \geq 1$ . Then for all  $e \geq d + 1$  and all  $C > 0$  there exists a set of numbers  $\{x_{1,1}, x_{1,2}, \dots, x_{1,d}, x_{2,1}, \dots, x_{e,1}, \dots, x_{e,d}\}$  growing at speed  $C$  and such that  $\{(x_{1,1}, \dots, x_{1,d}), \dots, (x_{e,1}, \dots, x_{e,d})\}$  is a generating set for  $\mathbb{Z}^d$ .*

*Proof.* — It suffices to prove the statement for  $e = d + 1$ . We start by proving the following claim by induction on  $n = 1, \dots, d$ : there exists an  $n \times n$  integer matrix  $(x_{i,j})$  whose coefficients grow at speed  $C$ , and such that for every  $k = 1, \dots, n$  the determinant of the upper left corner  $(x_{i,j} : 1 \leq i, j \leq k)$  is a prime number  $p_k$ , with all primes  $p_1, \dots, p_n$  distinct.

The induction starts by setting  $x_{1,1} = p_1$  for some prime number  $p_1 > C$ .

Assume then that an  $(n - 1) \times (n - 1)$  matrix  $A_{n-1} = (x_{i,j})$  has been constructed, with entries growing at speed  $C$  and determinant a prime number  $p_{n-1}$ .

First, an  $n$ th row  $(x_{n,1}, \dots, x_{n,n-1})$  may be added to  $A_{n-1}$  in such a manner that the entries still grow at speed  $C$ , and the determinant  $d_n$  of  $A'_{n-1} = (x_{i,j} : i \neq n - 1)$  is coprime to  $p_{n-1}$ . Indeed the coefficients

$x_{n,1}, \dots, x_{n,n-2}$  may be chosen arbitrarily as long as they grow fast enough. Then increasing  $x_{n,n-1}$  increases the determinant of  $A'_{n-1}$  by  $p_{n-2}$  which is coprime to  $p_{n-1}$ ; and sufficiently increasing this coefficient makes the augmented matrix  $A''_{n-1} = (x_{i,j} : i \leq n)$  still grow at speed  $C$ .

Then an  $n$ th column may be added to  $A''_{n-1}$  as follows. Start by choosing  $x_{1,n}, \dots, x_{n-2,n}$  arbitrarily as long as they grow fast enough, without fixing  $x_{n-1,n}$  and  $x_{n,n}$  yet. Call  $A_n$  the resulting matrix. Then increasing  $x_{n-1,n}$  decreases the determinant of  $A_n$  by  $d_n$ , while increasing  $x_{n,n}$  increases the determinant of  $A_n$  by  $p_{n-1}$ . Since  $d_n$  and  $p_{n-1}$  are coprime, there exist choices of  $x_{n-1,n}$  and  $x_{n,n}$  such that  $A_n$  has determinant 1; and the entries of  $A_n$  grow at speed  $C$ , except perhaps for  $x_{n,n}$ .

Now, by Dirichlet's theorem, there exist arbitrarily large primes  $p_n$  that are  $\equiv 1 \pmod{p_{n-1}}$ . For such a prime  $p_n = 1 + ap_{n-1}$ , add  $a$  to the entry  $x_{n,n}$  yielding a matrix  $A_n$  of determinant  $p_n$ . Choosing  $a$  large enough makes the coefficients of  $A_n$  grow at speed  $C$ .

To prove the lemma, consider a  $d \times d$  matrix  $A$  with integer entries growing at speed  $C$  and determinant  $p$ . Its rows generate a subgroup of  $\mathbb{Z}^d$  of prime index, and a single extra generator, with fast growing entries that are coprime to  $p$ , gives the desired generating set.  $\square$

We are ready to sharpen [14, Cor 2.17], claiming that torsion-free nilpotent groups are discriminating:

LEMMA 4.4. — *Let  $G$  be a torsion-free  $k$ -generated nilpotent group. Then, for each  $N > k$ , the group  $G$  is discriminating on  $N$  generators.*

*Proof.* — We start by considering more generally poly- $\mathbb{Z}$  groups, namely groups  $G$  admitting a sequence of subgroups  $G = G_1 \triangleright G_2 \triangleright \dots \triangleright G_{\ell+1} = 1$  such that  $G_i/G_{i+1} \cong \mathbb{Z}$  for all  $i$ .

If  $G$  is torsion-free nilpotent and  $(Z_i)$  denotes its ascending central series (defined inductively by  $Z_0 = 1$  and  $Z_{i+1}/Z_i = Z(G/Z_i)$ ), then each  $Z_{i+1}/Z_i$  is free abelian, so the ascending central series can be refined to a series in which successive quotients are  $\mathbb{Z}$ .

Choose for all  $i = 1, \dots, \ell$  a generator of  $G_i/G_{i+1}$ , and lift it to an element  $u_i \in G_i$ . Then every  $g \in G$  may uniquely be written in the form  $g = u_1^{\xi_1} \dots u_\ell^{\xi_\ell}$ , and the integers  $\xi_1, \dots, \xi_\ell$  determine the element  $g$ , which we write  $u^\xi$ . Philip Hall proves in [36, Thm 6.5] that products and inverses are given by polynomials, in the sense that if  $u^\xi u^\eta = u^\zeta$  and  $(u^\xi)^{-1} = u^\chi$ , then  $\zeta_i$  and  $\chi_i$  are polynomials in  $\{\xi_1, \dots, \xi_\ell, \eta_1, \dots, \eta_\ell\}$  and  $\{\xi_1, \dots, \xi_\ell\}$  respectively. In particular, every identity  $w \in \mathscr{W}$ , in  $N$  variables, is a polynomial in the exponents  $\xi_{1,1}, \dots, \xi_{\ell,N}$  of its arguments  $x_1, \dots, x_N$  written as  $u^{\xi_1}, \dots, u^{\xi_N}$ .

By Lemma 4.3, there exist sequences with arbitrarily fast growth that generate the abelianization of  $G$ ; and by Lemma 4.2 the identities in  $\mathscr{W}$  will not vanish on these generators, if their growth is fast enough. Finally, since  $G$  is nilpotent, a sequence of elements generates  $G$  if and only if it generates its abelianization.  $\square$

LEMMA 4.5. — *Let  $G$  be a finitely generated nilpotent group such that  $G$  and  $G/\text{Torsion}(G)$  generate the same variety. Then  $G$  preforms a torsion-free nilpotent group.*

*Proof.* — Infinite, finitely generated nilpotent groups have infinite abelianization; we apply Lemma 2.20 to  $G$  and the variety  $\mathscr{V}$  of abelian groups. Since every infinite abelian group preforms a free abelian group, we assume without loss of generality that  $G$  has torsion-free abelianization.

Assume that  $G$  is  $k$ -generated, and consider  $N > k$  and  $R > 0$ . Consider the set  $\mathscr{W}(R)$  of all words  $w$  of length at most  $R$  in  $N$  variables such that, for some  $g_1, \dots, g_N \in G$ , the evaluation  $w(g_1, \dots, g_N)$  is a non-trivial torsion element in  $G$ . In particular, such  $w$  are not identities in  $G$ . Since  $G$  and  $G/\text{Torsion}(G)$  generate the same variety, none of these words is an identity in  $G/\text{Torsion}(G)$ . Since  $G/\text{Torsion}(G)$  is a torsion-free nilpotent group, Lemma 4.4 implies that  $\mathscr{W}(R)$  is discriminated by an  $N$ -element generating set of  $G/\text{Torsion}(G)$ , which we denote by  $S'_R$ . Let  $S_R$  denote a preimage in  $G$  of  $S'_R$ . Since the abelianization of  $G$  is torsion-free, it is isomorphic (under the natural quotient map) to the abelianization of  $G/\text{Torsion}(G)$ . Therefore,  $S_R$  generates the abelianization of  $G$ , so generates  $G$ .

Let  $(H, T)$  be an accumulation point of the sequence  $(G, S_R)$  in the space  $\mathscr{G}$  of marked groups. Observe that  $H$  is torsion-free. Indeed, by Lemma 2.7(3) the torsion of  $H$  imbeds in that of  $G$ ; and if  $a$  is a torsion element of  $G$ , then for all  $R$  large enough there are words  $w \in \mathscr{W}(R)$  that assume the value  $a$ . By construction of  $S_R$ , the value  $a$  is not taken by a word of length  $\leq R$  in  $S_R$ , so  $a$  does not have a limit in  $H$ .  $\square$

PROPOSITION 4.6. — *Let  $G$  be a  $k$ -generated nilpotent group, and assume that  $G$  and  $G/\text{Torsion}(G)$  generate the same variety,  $\mathscr{V}$ .*

*Then, for every  $N > k$ , the group  $G$  preforms  $\mathbb{V}_N$ .*

*Consequently, the connected component of  $G$  for the relation  $\rightsquigarrow$  has diameter 2.*

*Proof.* — By Lemma 4.5, we may assume that  $G$  is torsion-free nilpotent. By Lemma 4.4, the group  $G$  is discriminating on  $N$  generators. By Lemma 4.1, the group  $G$  preforms  $\mathbb{V}_N$ .  $\square$

*Remark 4.7.* — The assumption that  $G$  is torsion-free is essential for the first claim of the proposition above. Consider indeed the variety of nilpotent groups of nilpotent class 2 in which every commutator is of order  $p$ . This variety is generated, e.g., by the group  $N_{2,2,p}$  from Example 3.13. However, there does not even exist any group preformed by  $G$  and containing  $\mathbb{V}_3$  as a subgroup, because the torsion in  $\mathbb{V}_3$  is larger than the torsion in  $N_{2,2,p}$ .

*Remark 4.8.* — Let  $\mathcal{V}$  be a nilpotent variety. Then, if  $\mathbb{V}_m \overset{\sim}{\rightsquigarrow} \mathbb{V}_n$ , then  $m \leq n$ .

*Proof.* — Since  $\mathbb{V}_n$  is finitely presented,  $\mathbb{V}_m$  is a quotient of  $\mathbb{V}_n$ . The abelianization of  $\mathbb{V}_n$  is  $n$ -generated, so the abelianization of any quotient of  $\mathbb{V}_n$  is also  $n$ -generated, so  $m \leq n$ .  $\square$

Proposition 4.6 has the following

**COROLLARY 4.9.** — *Consider a nilpotent variety  $\mathcal{V}$  generated by a group  $G$  such that  $G/\text{Torsion}(G)$  also generates  $\mathcal{V}$ . Let  $c$  be the nilpotency class of  $G$ .*

*For  $m, n > c$ , we have  $\mathbb{V}_m \overset{\sim}{\rightsquigarrow} \mathbb{V}_n$  if and only if  $m \leq n$ .*

*Proof.* — It is known from [47, Thm 35.11] that  $\mathbb{V}_m$  generates  $\mathcal{V}$  as soon as  $m \geq c$ .  $\square$

*Remark 4.10.* — Consider a nilpotent variety  $\mathcal{V}$  generated by a torsion-free nilpotent group. For small  $m, n$  the free groups  $\mathbb{V}_m$  and  $\mathbb{V}_n$  need not belong to the same component. For example, if  $\mathcal{V}$  the variety of nilpotent groups of class 5 then  $\mathbb{V}_2$  does not generate  $\mathcal{V}$ , since it is metabelian but  $\mathbb{V}_3$  is not. See [47, 35.33] for details.

## 4.2. When generators of a variety lie in different components

We will see that if  $G$  and  $G/\text{Torsion}(G)$  lie in different varieties then the variety of  $G$  contains infinitely many connected components under  $\overset{\sim}{\rightsquigarrow}$ .

**LEMMA 4.11.** — *Let  $G$  be a nilpotent group such that the varieties generated by  $G$  and  $G/\text{Torsion}(G)$  are different. There exists a variety  $\mathcal{V}$  such that the verbal subgroup  $\mathcal{V}(G)$  is non-trivial and finite.*

*Proof.* — First recall that torsion elements of a nilpotent group  $G$  form a finite subgroup of  $G$ . Since  $G$  and  $G/\text{Torsion}(G)$  generate different varieties, there exists an identity  $w$  of  $G/\text{Torsion}(G)$  that is not an identity in  $G$ . Set  $\mathcal{V} = \{w\}$ ; then  $\mathcal{V}(G)$  is non-trivial and is contained in the torsion of  $G$ , hence finite.  $\square$

**COROLLARY 4.12.** — *Let  $G$  be a nilpotent group and let  $\mathcal{V}$  be the variety that it generates. The connected component of  $G$  coincides with the set of groups generating  $\mathcal{V}$  if and only if  $G/\text{Torsion}(G)$  generates  $\mathcal{V}$ . If this is not the case, the set of groups generating  $\mathcal{V}$  consists of infinitely many connected components for the relation  $\rightsquigarrow$ .*

*Proof.* — If  $G/\text{Torsion}(G)$  generates  $\mathcal{V}$ , the corollary follows from Proposition 4.6. Assume now that  $G/\text{Torsion}(G)$  does not generate  $\mathcal{V}$ . Then by Lemma 4.11 there exists a variety  $\mathcal{W}$  such that the verbal subgroup  $\mathcal{W}(G)$  is non-trivial and finite. Observe that a verbal subgroup of a direct product is the product of its verbal subgroups. Therefore, for all  $n \in \mathbb{N}$ , the verbal subgroups  $\mathcal{W}(\times_n G)$  are non-isomorphic. By Lemma 2.26, all the groups  $\times_n G$  lie in distinct connected components. However, they all generate  $\mathcal{V}$ . □

### 4.3. Examples and illustrations

In the variety of abelian groups, the following is true: if  $G$  is a quotient of  $H$  and the torsion of  $H$  imbeds in the torsion of  $G$  under the quotient map, then  $G \rightsquigarrow H$ . This is not true anymore among nilpotent groups.

*Example 4.13.* — Consider the groups  $G = N_{2,2}$  and  $H = N_{2,2} \times N_{2,2}$ , see Example 3.13. Then both  $G$  and  $H$  are torsion-free, and  $G$  is a quotient of  $H$ . However,  $G$  doesn't preform  $H$ .

*Proof.* — Consider the following universal statement:

$$\forall a, b, c, z (([a, b] = 1 \wedge [a, c] = 1 \wedge [b, c] \neq 1) \Rightarrow [a, z] = 1).$$

It states that if  $a$  commutes with two non-commuting elements  $b$  and  $c$ , then  $a$  is central.

This property does not hold in  $H$ : take  $a, z$  the generators of the first  $N_{2,2}$  and  $b, c$  the generators of the second one.

On the other hand, in  $N_{2,2}$ , this property holds. Indeed if  $[a, b] = 1$  then the image of  $\{a, b\}$  in  $N_{2,2}/Z(N_{2,2}) \cong \mathbb{Z}^2$  lies in a cyclic subgroup; Similarly the image of  $\{a, c\}$  lies in a cyclic subgroup; so either  $a$  is central or the image of  $\{b, c\}$  lies in a cyclic subgroup. □

*Example 4.14.* — As soon as the nilpotency class is allowed to grow beyond 4, there exist nilpotent varieties whose free groups are not virtually free nilpotent. For example, consider the group  $G = \mathbb{F}_3 / \langle \mathbb{F}_3'', \gamma_5(\mathbb{F}_3) \rangle$ . This group is nilpotent of class 4, and is an iterated central extension of 29



copies of  $\mathbb{Z}$ . The 3-generated free nilpotent groups of class 3 and 4 have respectively 14 and 32 cyclic factors, so  $G$  is not commensurable to either. This is easily seen in the (Malcev) Lie algebra associated with these groups.

LEMMA 4.15. — *Let  $G$  be a non-virtually abelian nilpotent group. Then the connected component of  $G$  is not isomorphic, as partially ordered set, to the component of abelian groups.*

*Proof.* — In the component of abelian groups, the following holds: for any  $A$  there exists  $B$  with  $A \rightsquigarrow B$  and such that the set of groups that are preformed by  $B$  is linearly ordered. We claim that the connected component of  $G$  does not have this property.

More precisely, for any non-virtually abelian nilpotent  $G$ , we construct incomparable groups  $H_1, H_2$  that are both preformed by  $G$ .

Since  $G$  is not virtually abelian,  $[G, G]$  is infinite. Then both  $G$  and  $[G, G]$  have infinite abelianization, so that  $G$  maps onto  $N_{2,2}$ , the free nilpotent group of class 2 on 2 generators. Since  $N_{2,2} \rightsquigarrow N_{2,k}$  for all  $k \geq 2$ , there exists by Lemma 2.19 a group  $H_1$  such that  $\gamma_2(H_1)/\gamma_3(H_1)$  has arbitrarily large rank, in particular rank larger than that of  $\gamma_2(G)/\gamma_3(G)$ . Set then  $H_2 = G \times \mathbb{Z}^d$  for  $d$  larger than the rank of  $H_1/\gamma_2(H_1)$ . Then  $H_1$  is not a quotient of  $H_2$ , because  $\gamma_2(H_1)/\gamma_3(H_1)$  is not a quotient of  $\gamma_2(H_2)/\gamma_3(H_2)$ ; and  $H_2$  is not a quotient of  $H_1$ , because  $H_2/\gamma_2(H_2)$  is not a quotient of  $H_1/\gamma_2(H_1)$ .  $\square$

## 5. Imbeddability of orders. Solvable groups

We characterize the preorders (transitive, reflexive relations) that can be imbedded in the preorder of groups up to isomorphism, under the relation  $\rightsquigarrow$ . We show in this manner that  $\rightsquigarrow$  has a rich structure, even when restricted to solvable groups of class 3.

In this section, we view  $\rightsquigarrow$  as a preorder on  $\mathcal{G}$ , defined by  $(G, S) \rightsquigarrow (H, T)$  if and only if  $G \rightsquigarrow H$ . For  $X$  a set, we denote by  $\mathcal{P}(X)$  the family of subsets of  $X$ .

PROPOSITION 5.1. — *Let  $\mathcal{B}$  be a countably infinite set, and let  $\mathcal{X}$  have the cardinality of the continuum. Put on  $\mathcal{P}(\mathcal{B}) \times \mathcal{X}$  the preorder*

$$(X, c) \lesssim (Y, c') \text{ if and only if } X \supseteq Y.$$

*Then the preorders  $(\mathcal{G}, \rightsquigarrow)$  and  $(\mathcal{P}(\mathcal{B}) \times \mathcal{X}, \lesssim)$  imbed into each other.*

We note that  $(\mathcal{P}(\mathcal{B}) \times \mathcal{X}, \lesssim)$  is the relation obtained by the partial order on subsets of  $\mathcal{B}$  by inclusion; its equivalence classes (strongly connected components) have the cardinality of the continuum. We also remark that  $(\mathcal{P}(\mathcal{B}), \subseteq)$  is isomorphic to  $(\mathcal{P}(\mathcal{B}), \supseteq)$ , via the map  $X \mapsto \mathcal{B} \setminus X$ .

**COROLLARY 5.2.** — *A preorder imbeds in  $(\mathcal{G}/\cong, \rightsquigarrow)$  if and only if it imbeds in  $(\mathcal{P}(\mathcal{B}) \times \mathcal{X}, \lesssim)$ . In particular, a partial order imbeds in  $(\mathcal{G}/\cong, \rightsquigarrow)$  if and only if it is realizable by subsets of a countable set under inclusion.*

*Proof.* — Proposition 5.1 yields imbeddings between  $\mathcal{G}$  and  $\mathcal{P}(\mathcal{B}) \times \mathcal{X}$ . We therefore have an imbedding of  $\mathcal{G}/\cong$  into  $\mathcal{P}(\mathcal{B}) \times \mathcal{X}$ .

Conversely, isomorphism classes of groups in  $\mathcal{G}$  are countable, because there are countably many homomorphisms between finitely generated groups. On the other hand, equivalence classes in  $\mathcal{P}(\mathcal{B}) \times \mathcal{X}$  are uncountable; so there exists an imbedding  $\mathcal{P}(\mathcal{B}) \times \mathcal{X} \rightarrow \mathcal{P}(\mathcal{B}) \times \mathcal{X}$ , which is the identity on its first argument, and such that its image imbeds in  $\mathcal{G}/\cong$ .  $\square$

*Proof of Proposition 5.1,  $\Leftarrow$ .* — Consider first the space  $\mathcal{G}$  of marked groups. For every  $k, R \in \mathbb{N}$ , there are finitely many possibilities for the marked graphs  $B(1, R)$  of degree  $\leq k$  that may appear in the Cayley graphs of these groups; letting  $k, R$  range over  $\mathbb{N}$ , we obtain a countable collection  $\mathcal{B}$  of finite graphs. Now to each  $(G, S') \in \mathcal{G}$  we associate the subset  $\mathcal{O}_G$  of  $\mathcal{B}$  consisting of all marked balls that may appear in Cayley graphs  $\mathcal{C}(G, S)$ , as we let  $S$  range over generating sets of  $G$ . Clearly,  $G \rightsquigarrow H$  if and only if  $\mathcal{O}_H \subseteq \mathcal{O}_G$ .

We deduce that  $(\mathcal{G}, \rightsquigarrow)$  imbeds in  $(\mathcal{P}(\mathcal{B}), \subseteq)$ . We can make this map injective by taking  $\mathcal{X} = \mathcal{P}(\mathbb{F})$ , and mapping  $(G, S)$  to  $(\mathcal{O}_G, \ker(\mathbb{F} \rightarrow G))$ , for the natural map  $\mathbb{F} \rightarrow G$  presenting  $G$ .  $\square$

To construct the imbedding in the other direction, we begin by a general construction. Let  $P$  be a group. Consider first the free nilpotent group  $N_{2,P}$  of class 2 on a generating set indexed by  $P$ . Denote its generators by  $a_p$  for  $p \in P$ , and for  $p, q \in P$  write  $c_{p,q} := [a_p, a_q]$ . Written additively, we have  $c_{p,p} = 0$ , and  $c_{p,q} = -c_{q,p}$  for all  $p, q \in P$ . Define then  $\overline{N}_{2,P}$  as the quotient of  $N_{2,P}$  by the relations  $c_{p,q} = c_{pr,qr}$  for all  $p, q, r \in P$ . Finally let  $H(P)$  be the semidirect product  $P \ltimes \overline{N}_{2,P}$ , for the action  $a_p \cdot q := a_{pq}$ . The centre of  $H(P)$  is generated by the images of the  $c_{p,q}$ . Let  $P_+ \subseteq P \setminus \{1\}$  contain precisely one element out of each pair  $\{p, p^{-1}\}$ ; then  $\{c_{1,p}\}$  freely generates the centre of  $H(P)$ . If  $S$  be a generating set for  $P$ , then  $S \cup \{a_1\}$  generates  $H(P)$ .

The case  $P = \mathbb{Z}$  is considered by Hall in [35, §3]; he introduced this group in order to construct  $2^{\aleph_0}$  non-isomorphic solvable finitely generated groups (of solvability length 3).

In this proof, we take  $P = \mathbb{Z}^2$ , and for convenience  $(\mathbb{Z}^2)_+ = \{(m, n) \in \mathbb{Z}^2 : m > 0 \text{ or } m = 0 < n\}$ . We abbreviate  $H(\mathbb{Z}^2)$  as  $H$ , generated by  $\{x, y, a\}$  with  $\{x, y\}$  the standard generators of  $\mathbb{Z}^2$  and  $a = a_{(0,0)}$ .

A *prime colouring* is a function  $\phi: (\mathbb{Z}^2)_+ \rightarrow \{1\} \cup \{\text{primes}\}$ ; it extends to a function still written  $\phi: \mathbb{Z}^2 \rightarrow \mathbb{Z}$  by  $\phi(-z) = -\phi(z)$  and  $\phi(0) = 0$ . Given a prime colouring  $\phi$ , we define the *standard quotient*  $H_\phi$  of  $H$  as the quotient of  $H$  by all the relations  $c_{1,z}^{\phi(z)} = 1$ , as  $z$  ranges over  $(\mathbb{Z}^2)_+$ . Clearly,

LEMMA 5.3. — *A standard central quotient  $H_\phi$  contains an element of order  $p$  if and only if there exists  $z \in (\mathbb{Z}^2)_+$  such that  $\phi(z) = p$ .*

*If  $H_\phi \overset{\sim}{\rightsquigarrow} H_\psi$ , then the set of primes in  $\psi(\mathbb{Z})$  is contained in the set of primes in  $\phi(\mathbb{Z})$ . □*

Let  $I$  be a set of primes. A prime colouring  $\phi$  is  *$I$ -universal* if its values lie in  $I$  and it contains every finite  $I$ -colouring, in the following sense: for every  $R \in \mathbb{N}$  and every function  $\theta: \{-R, \dots, R\}^2 \cap (\mathbb{Z}^2)_+ \rightarrow I \cup \{1\}$ , there exists  $M \in \mathbf{SL}_2(\mathbb{Z})$  such that  $\theta(z) = \phi(M(z))$  for all  $z \in \{-R, \dots, R\}^2 \cap (\mathbb{Z}^2)_+$ .

LEMMA 5.4. — *For every set  $I$  of primes of cardinality  $\geq 2$ , there exist a continuum of  $I$ -universal colourings.*

*Proof.* — One enumerates all finite  $I$ -colourings, and constructs  $\phi$  step by step. At each step, only finitely many values of  $\phi$  have been specified, say within the box  $\{-S, \dots, S\}^2$ , and we want to extend  $\phi$  using the partial colouring  $\theta: \{-R, \dots, R\}^2 \cap (\mathbb{Z}^2)_+ \rightarrow I \cup \{1\}$ . A large enough  $M \in \mathbf{SL}_2(\mathbb{Z})$  can be found such that  $M(\{-R, \dots, R\}^2 \cap \{-S, \dots, S\}^2) = \{(0, 0)\}$ , for example  $M = \begin{pmatrix} (S+1)(S+R+1)+1 & S+1 \\ S+R+1 & 1 \end{pmatrix}$ . Extend  $\phi$  by setting  $\phi(M(z)) = \theta(z)$  for all  $z \in \{-R, \dots, R\}^2 \cap (\mathbb{Z}^2)_+$ . Once this is done for all  $R \in \mathbb{N}$ , set finally  $\phi(z) = 1$  at unspecified values in  $(\mathbb{Z}^2)_+$ .

To obtain a continuum of different  $I$ -universal colourings, note that countably many matrices  $M_0, M_1, \dots$  were used in the construction, and the only condition was that they had to be sufficiently far away from the identity. Fix a finite-index subgroup  $\Gamma \subset \mathbf{SL}_2(\mathbb{Z})$ . Then, given a subset  $C \subseteq \mathbb{N}$ , one may choose the matrices  $M_i$  as above, and additionally such that  $M_i \in \Gamma \Leftrightarrow i \in C$ . This encodes  $C$  into the constructed colouring. □

*Proof of Proposition 5.1,  $\Leftarrow$ .* — We are ready to imbed  $\mathcal{P}(\mathcal{B}) \times \mathcal{X}$  into  $\mathcal{G}$ . Without loss of generality, we may assume that  $\mathcal{B}$  is the set of primes  $\geq 10$ .

Given  $X \subseteq \mathcal{B}$ , consider  $I = \{2, 3\} \cup X$ . By Lemma 5.4, there exist continuously many  $I$ -universal prime colourings  $\phi_{I,C}$ , parameterized by  $C \subseteq \mathbb{N}$ . Let  $H_{X,C}$  be the central quotient  $H_{\phi_{X \cup \{2,3\},C}}$ , and note that the  $(H_{X,C}, \{x, y, a\})$  are distinct points of  $\mathcal{G}$  for distinct  $(X, C)$ . We have therefore defined an imbedding  $\mathcal{P}(\mathcal{B}) \times \mathcal{P}(\mathbb{N}) \rightarrow \mathcal{G}$ .

On the one hand, if  $H_{X,C} \rightsquigarrow H_{Y,C'}$ , then  $X \supseteq Y$  by Lemma 5.3. On the other hand, if  $X \supseteq Y \subseteq \mathcal{B}$  and  $C, C' \subseteq \mathbb{N}$ , then consider the prime colourings  $\phi, \psi$  with  $H_{X,C} = H_\phi$  and  $H_{Y,C'} = H_\psi$ , and choose  $T = \{x, y, a\}$  as generating set of  $H_\psi$ . Consider an arbitrary  $R \in \mathbb{N}$ . Then the restriction of  $\psi$  to  $\{-R, \dots, R\}^2$  is a finite  $(\{2, 3\} \cup Y)$ -colouring, and therefore a finite  $(\{2, 3\} \cup X)$ -colouring; so there exists  $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Z})$  such that  $\psi$  and  $\phi \circ M$  agree on  $\{-R, \dots, R\}^2$ . Consider the generating set  $S = \{x^\alpha y^\beta, x^\gamma y^\delta, a\}$  of  $H_\psi$ ; then the Cayley graphs  $\mathcal{C}(H_\psi, T)$  and  $\mathcal{C}(H_\phi, S)$  agree on a ball of radius  $R$ . □

*Remark 5.5.* — By Lemma 2.7(3), if  $A \rightsquigarrow B$  and  $F$  is a finite subgroup of  $B$ , then  $F$  imbeds in  $A$ . In general, if  $F$  is a torsion subgroup of  $B$ , this need not be true. There exist finitely generated solvable groups  $A \rightsquigarrow B$ , such that  $B$  contains the divisible group  $\mathbb{Q}/\mathbb{Z}$ , while  $A$  does not contain any divisible elements.

*Proof of remark.* — We modify the proof of Proposition 5.1. Before, we enumerated finite  $I$ -colourings  $\theta: \{-R, \dots, R\}^2 \cap (\mathbb{Z}^2)_+ \rightarrow I \cup \{1\}$  and imposed the relations  $c_{1,M(z)}^{\theta(z)} = 1$ , for appropriate  $M \in \mathbf{SL}_2(\mathbb{Z})$ . Now, we enumerate  $(\mathbb{Z}^2)_+$  as  $\{p_1, p_2, \dots\}$ , and we impose relations on  $H$  step-by-step. At each step, only finitely many of the  $c_{1,z}$  will have been affected by the relations; we call the corresponding  $z \in \mathbb{Z}^2$  *bound*.

For each  $N = 1, 2, \dots$ , we find  $M \in \mathbf{SL}_2(\mathbb{Z})$  such that  $M(\{p_1, \dots, p_N\})$  is disjoint from all bound  $z \in \mathbb{Z}^2$ . We impose the relations  $c_{1,M(p_1)} = 1$  and  $c_{1,M(p_i)}^i = c_{1,M(p_{i-1})}$  for all  $i = 2, \dots, N$ . Finally, we set  $c_{1,z} = 1$  for all unbound  $z \in \mathbb{Z}^2$ .

We call the resulting central quotient  $G$ , and note that it is solvable, and that its torsion is the subgroup generated by the  $c_{1,z}$ ; this group is a direct sum of cyclic groups, and in particular is not divisible.

On the other hand, let  $(H, T)$  be the limit of  $(G, S_M)$  in the space  $\mathcal{G}$  of marked groups, along the generating sets  $S_M = \{x^\alpha y^\beta, x^\gamma y^\delta, a\}$  corresponding to the matrices  $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Z})$  used in the construction of  $G$ . Then  $H$  contains a copy of  $\mathbb{Q}/\mathbb{Z}$ , with the limit of  $c_{1,M(p_i)}$  playing the role of  $1/i!$ . □

## 6. The connected component of free groups

We concentrate, in this section, on those groups that either preform or are preformed by free groups. Both of these classes have already been thoroughly investigated; the first are known as “limit groups”, and the second as “groups without almost-identities”.

### 6.1. Limit groups

Groups that are preformed by free groups are known as “limit groups”. This section reviews some known facts about them; we refer to the recent expositions [15, 50, 42].

Benjamin Baumslag considers  $\omega$ -residually free groups in [11]. They are groups  $G$  such that for all  $n$  and all distinct  $g_1, \dots, g_n \in G$  there exists a homomorphism  $\pi: G \rightarrow \mathbb{F}$  to a free group such that all  $\pi(g_1), \dots, \pi(g_n)$  are distinct. Baumslag proves in particular that  $G$  is  $\omega$ -residually free if and only if it is both residually free and commutative-transitive (see Equation 2.1).

Remeslennikov proves in [54] that the following are equivalent for a residually free group: it is  $\omega$ -residually free; it is universally free (namely has the same universal theory as a free group); it is commutative transitive (see Equation 2.1). All three statements are characterizations of non-abelian *limit groups*. The terminology was introduced by Sela, referring to limits of epimorphisms onto free groups.

Champetier and Guirardel show in [21] that  $G$  is a limit group if and only if it is a limit of subgroups of free groups. In other words,  $G$  is a non-abelian limit group if and only if  $\mathbb{F}_2 \overset{\sim}{\twoheadrightarrow} G$ .

Kharlampovich-Myasnikov [40, 41] and Sela [56] prove that limit groups are finitely presented.

### 6.2. Groups with no almost-identities

Groups that preform free groups will be shown to be “groups with no almost-identities”. We write  $G \overset{\sim}{\twoheadrightarrow} \mathbb{F}$  if there exists  $k \in \mathbb{N}$  such that  $G \overset{\sim}{\twoheadrightarrow} \mathbb{F}_k$ ; equivalently,  $G \overset{\sim}{\twoheadrightarrow} \mathbb{F}_k$  for all  $k$  large enough.

We begin by some elementary observations and examples. We include the proofs for convenience of the reader.

LEMMA 6.1 (See [55] and [21, Example 2.4(d)]). — We have  $\mathbb{F}_m \overset{\sim}{\rightsquigarrow} \mathbb{F}_n$  if and only if  $m \leq n$ .

More precisely, let  $\{x_1, \dots, x_m\}$  be a basis of  $\mathbb{F}_m$  and let  $S_R$  be, for all  $R \in \mathbb{N}$ , a set of  $n - m$  words of length at least  $2R$  satisfying the  $C'(1/6)$  small cancellation condition. Then  $(\mathbb{F}_m, \{x_1, \dots, x_m\} \cup S_R)$  converges to  $(\mathbb{F}_n, \text{basis})$  in  $\mathcal{G}$ .

*Proof.* — Consider  $m \leq n$ . Let  $S = \{x_1, \dots, x_m\}$  be a basis of  $\mathbb{F}_m$ . Given  $R > 0$ , consider a set  $S_R := \{w_1, \dots, w_{n-m}\}$  such that each word  $w_i$  has length larger than  $2R$ , and  $\{w_1, \dots, w_{n-m}\}$  satisfies the  $C'(1/6)$  small cancellation condition. The presentation  $\langle x_1, \dots, x_m, y_1, \dots, y_{n-m} \mid y_1 w_1, \dots, y_{n-m} w_{n-m} \rangle$  then defines the free group  $\mathbb{F}_m$ , and also satisfies the  $C'(1/6)$  small cancellation condition. By Greendlinger’s Lemma [27], the shortest relation in it has length larger than  $2R$ , so the ball of radius  $R$  in  $\mathcal{C}(\mathbb{F}_m, \{x_1, \dots, x_m\} \cup S_R)$  coincides with that in  $\mathbb{F}_n$ .

Conversely, if  $\mathbb{F}_m \overset{\sim}{\rightsquigarrow} \mathbb{F}_n$  then  $\mathbb{F}_m$  is a quotient of  $\mathbb{F}_n$ , by Lemma 2.4, so  $m \leq n$ . □

LEMMA 6.2 (See [55, Lemma 5.1]). — If  $G$  be an  $s$ -generated group which admits  $\mathbb{F}_m$  as a quotient, for some  $m \geq 2$ , then  $G$  preforms a free group on  $m + s$  elements.

*Proof.* — Let  $\{g_1, \dots, g_s\}$  generate  $G$ , and let  $g'_1, \dots, g'_s$  be the projections of the  $g_i$  to  $\mathbb{F}_m$ . Let also  $h_1, \dots, h_m \in G$  project to a basis  $x_1, \dots, x_m$  of  $\mathbb{F}_m$ . Let  $N$  be the maximal length of a  $g'_i$  in the basis  $\{x_1, \dots, x_m\}$ .

For each  $R > 0$ , consider words  $w_1, \dots, w_s$  in  $\{x_1, \dots, x_m\}$  of length at least  $R$  and satisfying the small cancellation condition  $C'(1/6)$ . Consider the generating set  $S_R = \{h_1, \dots, h_m, g_1 w_1(h_1, \dots, h_m), \dots, g_s w_s(h_1, \dots, h_m)\}$  of  $G$  of cardinality  $m + s$ . Its projection to  $\mathbb{F}_m$  is  $\{x_1, \dots, x_m, g'_1 w_1, \dots, g'_s w_s\}$ . These elements may be rewritten as words of length at most  $N + 1$  in  $\{x_1, \dots, x_m, w_1, \dots, w_s\}$ . Therefore, by Lemma 6.1, no relation of length  $\leq R/(N + 1)$  holds among these elements. □

Example 6.3. — For every group  $A$  and every  $m \geq 2$  we have  $A \times \mathbb{F}_m \overset{\sim}{\rightsquigarrow} \mathbb{F}$ ,  $A * \mathbb{F}_m \overset{\sim}{\rightsquigarrow} \mathbb{F}$  and  $A \wr \mathbb{F}_m \overset{\sim}{\rightsquigarrow} \mathbb{F}$ .

In particular, there exists a continuum of non-isomorphic groups that preform free groups.

Remark 6.4. — If  $A$  preforms a non-abelian free group and  $A$  is a quotient of  $B$  then  $B$  preforms a non-abelian free group.

*Proof.* — By Lemma 2.19 we know that  $B$  preforms some group  $C$  that admits a non-abelian free group as a quotient. By Lemma 6.2 we know that

$C$  preforms a non-abelian free group. Therefore,  $B$  preforms a non-abelian free group.  $\square$

By Lemma 2.7(1), if  $G$  satisfies an identity then  $G$  doesn't preform a free group. However, this does not characterize groups that preform free groups.

LEMMA 6.5. — *Given words  $w_1, \dots, w_\ell \in \mathbb{F}_n$ , there exists a word  $w \in \mathbb{F}_n$  such that, for every group  $G$ , the identity  $w$  is satisfied in  $G$  as soon as at least one identity  $w_i$  is satisfied.*

*Proof.* — Construct words  $v_1, \dots, v_\ell \in \mathbb{F}_n$  inductively as follows:  $v_1 := w_1$ ; and for  $i \geq 2$ , if  $v_{i-1}$  and  $w_i$  have a common power  $v_{i-1}^a = w_i^b = z$  then  $v_i := z$ , while otherwise  $v_i := [v_{i-1}, w_i]$ .

Observe that  $v_\ell$  is non-trivial, and  $v_\ell(g, h) = 1$  if  $w_i(g, h) = 1$  for some  $i \in \{1, \dots, \ell\}$ . Therefore  $w = v_\ell$  is the required identity.  $\square$

COROLLARY 6.6. — *A group satisfies no identity if and only if it preforms a group containing a non-abelian free subgroup.*

*Proof.* — If a group  $G$  satisfies an identity, then so does any group that is preformed by it; so no group which is preformed by  $G$  may have a non-abelian free subgroup.

Conversely, consider a group  $G$  which satisfies no identity. Let the set  $S$  generate  $G$ . For every  $R > 0$ , apply the previous lemma to the set  $\{w_1, \dots, w_\ell\}$  of non-trivial words of length at most  $R$  in  $\mathbb{F}_2$ . Let  $w$  be the resulting identity. Since it does not hold in  $G$ , there are  $g_R, h_R$  be such that  $w(g_R, h_R) \neq 1$ , so  $v(g_R, h_R) \neq 1$  for every word  $v$  of length at most  $R$ . Consider the generating set  $S_R = S \cup \{g_R, h_R\}$  of  $G$ . Take a converging subsequence, in  $\mathcal{G}$ , of the marked groups  $\mathcal{C}(G, S_R)$ , and let  $\mathcal{C}(H, T)$  be its limit. Then the last two elements of  $T$  generate a free subgroup  $\mathbb{F}_2$  of  $H$ .  $\square$

Akhmedov [3] and Olshansky and Sapir [48] give the following definition. Let  $G$  be a  $k$ -generated group. A non-trivial word  $w(x_1, \dots, x_k)$  is a  $k$ -almost-identity for  $G$  if  $w(g_1, \dots, g_k) = 1$  for all  $g_1, \dots, g_k \in G$  such that  $\{g_1, \dots, g_k\}$  generates  $G$ . The group  $G$  satisfies an almost-identity if for all  $k \in \mathbb{N}$  there exists a  $k$ -almost-identity satisfied by  $G$ .

COROLLARY 6.7 (Olshansky & Sapir, [48, Thm 9]). — *A group preforms a free group if and only if it satisfies no almost-identity. More precisely,  $G \rightsquigarrow \mathbb{F}_k$  if and only if  $G$  is  $k$ -generated and satisfies no  $k$ -almost-identity.*

*Proof.* — If  $G$  satisfies a  $k$ -almost-identity and  $G \rightsquigarrow H$ , then  $H$  satisfies the same almost-identity; therefore  $H$  cannot be free.

Conversely, consider a  $k$ -generated group  $G$  which satisfies no  $k$ -almost-identity. For every  $R > 0$ , apply the previous lemma to the set  $\{w_1, \dots, w_\ell\}$  of non-trivial words of length at most  $R$  in  $\mathbb{F}_k$ . Let  $w$  be the resulting word. Since it is not a almost-identity satisfied by  $G$ , there exists a generating set  $S_R := \{g_{R,1}, \dots, g_{R,k}\}$  of  $G$  such that  $w(g_{R,1}, \dots, g_{R,k}) \neq 1$ , so  $v(g_{R,1}, \dots, g_{R,k}) \neq 1$  for every word  $v$  of length at most  $R$ . Take a converging subsequence, in  $\mathcal{G}$ , of the marked groups  $\mathcal{C}(G, S_R)$ , and let  $\mathcal{C}(H, T)$  be its limit. Then  $H$  is a free group of rank  $k$ .  $\square$

Following an idea sketched by Schleimer in [55, §4], Olshansky and Sapir show in [48] that there are groups with almost-identities but without identities; see also [3, §4].

*Example 6.8 (Schleimer, Olshansky & Sapir).* — There exist groups without identities, but with almost-identities. For all  $n$  large enough, such an example is the group  $\mathbb{F}_2 / \langle w^n : w \notin \mathbb{F}_2^n[\mathbb{F}_2, \mathbb{F}_2] \rangle$ .

It is known that the following groups preform  $\mathbb{F}$ :

- (1) Non-elementary hyperbolic groups (see Akhmedov [4], with a refinement in by Olshansky and Sapir [48] on the number of generators of the free group); furthermore, [48, Remark 5] states, using results of Osin, that “strongly relatively hyperbolic groups” have infinite girth;
- (2) linear groups [4];
- (3) one-relator groups [4];
- (4) Thompson’s group  $F$  (Brin shows in [17] that it preforms  $\mathbb{F}_2$ , and Akhmedov, Stein and Taback give a slightly worse estimate in [5]).

Akhmedov also shows that there exist amenable groups that preform  $\mathbb{F}$ . We show later in this section that there are groups of intermediate growth (e.g. the first Grigorchuk group) that preform free groups.

*Remark 6.9.* — Any order satisfying the assumption of Corollary 5.2 is imbeddable in the set of groups that preform  $\mathbb{F}$ .

*Proof.* — If  $G$  preforms  $H$ , then Lemma 2.14(1) implies that  $G \times \mathbb{F}_m$  preforms  $H \times \mathbb{F}_m$ . Observe, by considering the torsion subgroups, that the converse is true for the groups used in the proof of Proposition 5.1.  $\square$

### 6.3. A criterion à la Abért for having no almost-identity

We start by recalling a general result by Abért [1, Thm 1.1]. Consider a group  $G$  acting by permutations on a set  $X$ . Say that  $G$  separates  $X$  if,



for every finite  $Y \subseteq X$ , the fixed point set of the fixator  $G_Y$  of  $Y$  is equal to  $Y$ . It follows that  $G_Y$  has infinite orbits on  $X \setminus Y$ : if  $xG_Y \subset X \setminus Y$  were a finite orbit, then the fixator of  $Y \cup xG_Y \setminus \{x\}$  would also fix  $x$ . Abért proves that if  $G$  separates  $X$  then  $G$  satisfies no identity.

In the theorem below we strengthen the assumption of Abért's theorem in order to get a criterion for absence of almost-identities, not only identities. Recall that the *Frattini subgroup*  $\Phi(G)$  of a group  $G$  is the intersection of its maximal subgroups. It is the maximal subgroup of  $G$  such that  $S$  generates  $G$  if and only if  $S\Phi(G)$  generates  $G/\Phi(G)$ . Equivalently, if  $\{s_1, \dots, s_k\}$  generates  $G$ , then  $\{s_1g_1, \dots, s_kg_k\}$  also generates  $G$ , for arbitrary  $g_1, \dots, g_k \in \Phi(G)$ .

**THEOREM 6.10.** — *Let  $G$  separate the set  $X$  on which it acts on the right, and assume that  $\Phi(G)$  has finite index in  $G$ . Then  $G$  satisfies no almost-identity.*

*Proof.* — We follow [1, Thm 1.1]. Let  $k$  be large enough that  $G$  can be  $k$ -generated, and let  $w = w(x_1, \dots, x_k) = v_1 \dots v_\ell$  be a non-trivial reduced word in  $\mathbb{F}_k$ . Write  $w_n = v_1 \dots v_n$  for all  $n \in \{0, \dots, \ell\}$ . Fix a point  $p_0 \in X$ . A tuple  $(g_1, \dots, g_k) \in G^k$  is called *distinctive* for  $w$  if all the points  $p_n = p_0w_n(g_1, \dots, g_k)$ , for  $n = 0, \dots, \ell$ , are distinct. This implies in particular  $p_\ell \neq p_0$ , so  $w(g_1, \dots, g_k) \neq 1$ .

We prove by induction on  $n = 0, \dots, \ell$  that there exists a distinctive tuple  $(g_1, \dots, g_k)$  for  $w_n$  such that  $\{g_1, \dots, g_k\}$  generates  $G$ . The case  $n = 0$  follows from the fact that  $G$  can be  $k$ -generated; we choose any generating sequence  $(g_1, \dots, g_k)$ .

By induction, we may assume that  $p_0, \dots, p_{n-1}$  are all distinct. Put

$$Y = \{p_i : v_{i+1} = v_n \text{ for } i \leq n-1, \text{ or } v_i = v_n^{-1}\}.$$

If  $v_n = x_j$ , then we modify  $g_j$  into  $h_j := cg_j$  for some  $c \in \Phi(G) \cap G_Y$  to be chosen later, while if  $v_n = x_j^{-1}$  then we modify  $g_j$  into  $h_j := g_jc$ . In all cases, we leave the other  $g_i$  unchanged, and write  $h_i := g_i$  for all  $i \neq j$ . Clearly  $(h_1, \dots, h_k)$  still generates  $G$ .

For  $i = 1, \dots, n-1$ , we have  $p_i = p_{i-1}v_i(g_1, \dots, g_k) = p_{i-1}v_i(h_1, \dots, h_k)$  since  $c \in G_Y$ . From  $v_{n-1} \neq v_n^{-1}$  we get  $p_{n-1} \notin Y$ , so the  $G_Y$ -orbit of  $p_{n-1}$  is infinite because the action is separating, and its  $(G_Y \cap \Phi(G))$ -orbit is infinite too because  $\Phi(G)$  has finite index. Therefore, we may choose  $c \in G_Y \cap \Phi(G)$  such that

$$p_{n-1}c \notin \{p_iv_n(g_1, \dots, g_k) : i = 1, \dots, n-1\},$$

from which  $p_n = p_{n-1}v_n(h_1, \dots, h_k) \notin \{x_0, \dots, x_{n-1}\}$  and  $(h_1, \dots, h_k)$  is distinctive for  $w_n$ . □

### 6.4. The first Grigorchuk group

We now show that the first Grigorchuk group  $G_{012}$  satisfies no almost-identity, and therefore preforms a non-abelian free group. We begin by recalling  $G_{012}$ 's construction.

Consider the following recursively defined transformations  $a, b, c, d$  of  $\{0, 1\}^\infty$ : for  $\omega \in \{0, 1\}^\infty$ ,

$$\begin{aligned} (0\omega)a &= 1\omega & (1\omega)a &= 0\omega, \\ (0\omega)b &= 0(\omega a) & (1\omega)b &= 1(\omega c), \\ (0\omega)c &= 0(\omega a) & (1\omega)c &= 1(\omega d), \\ (0\omega)d &= 0\omega & (1\omega)d &= 1(\omega b). \end{aligned}$$

This action is continuous and measure-preserving; it permutes the clopens  $\{v\{0, 1\}^\infty : v \in \{0, 1\}^*\}$ , preserving the length of  $v$ . We call such actions *arborical*. The first Grigorchuk group  $G_{012}$  is  $\langle a, b, c, d \rangle$ ; see [29, 6] for its origins, and [37, Chapter VIII] for a more recent introduction.

Recall that a group  $G$  acting arborically with dense orbits on a Cantor set  $\Sigma^\infty$  is *weakly branched* if, for every finite word  $v \in \Sigma^*$ , there exists  $g \in G$  which acts non-trivially in the clopen  $v\Sigma^\infty \subseteq \Sigma^\infty$  but fixes its complement. It is known that  $G_{012}$  is weakly branched.

LEMMA 6.11. — *If  $G$  is weakly branched, then it separates  $\Sigma^\infty$ .*

*Proof.* — Consider a finite subset  $Y \subset \Sigma^\infty$ , and  $\omega \in \Sigma^\infty \setminus Y$ . Let  $v \in \Sigma^*$  be a prefix of  $\omega$  that is not a prefix of any element of  $Y$ . Let  $H$  denote the stabilizer  $v\Sigma^\infty$ , and let  $K \triangleleft H$  be the set of  $g \in G$  that fix  $\Sigma^\infty \setminus v\Sigma^\infty$ .

Since  $G$  has dense orbits on  $\Sigma^\infty$ , its subgroup  $H$  has dense orbits on  $v\Sigma^\infty$ . Assume for contradiction that  $K$  fixes  $\omega$ ; then, since  $K$  is normal in  $H$ , it fixes  $\omega H$  which is dense in  $v\Sigma^\infty$ , so  $K = 1$ , contradicting the hypothesis that  $G$  is weakly branched. □

COROLLARY 6.12. — *The first Grigorchuk group  $G_{012}$  preforms  $\mathbb{F}_3$ . In particular,  $G_{012}$  has infinite girth.*

Note that this gives a negative answer to a question of Schleimer, who conjectures in [55, Conjecture 6.2] that all groups with infinite girth have exponential growth.

*Proof.* — Lemma 6.11 shows that  $G_{012}$  separates  $\{0, 1\}^\infty$ . Pervova proves in [51] that all maximal subgroups of  $G_{012}$  have index 2; so the Frattini subgroup of  $G_{012}$  satisfies  $\Phi(G_{012}) = [G_{012}, G_{012}]$ . Theorem 6.10 then shows that  $G_{012}$  satisfies no almost-identity, so  $G_{012} \overset{\sim}{\rightsquigarrow} \mathbb{F}_3$  since  $G_{012}$  is 3-generated.  $\square$

Note that Pervova proves, in [52], that a large class of 2-generated groups, called “GGs groups”, satisfy the same condition that all of their maximal subgroups are normal, and hence contain the derived subgroup. Every weakly branched GGS group preforms  $\mathbb{F}_2$ , following the same argument as in 6.12; and most GGS groups are weakly branched, those that are not constituting a few, well-understood exceptions.

### 6.5. Permutational wreath products

We return to wreath products, and consider a more general situation. Let  $A$  be a group, and let  $G$  be a group acting on a set  $X$ . Recall that the *permutational wreath product* is the group

$$A \wr_X G = \{f: X \rightarrow A \text{ finitely supported}\} \rtimes G,$$

with the standard action at the source of  $G$  on functions  $X \rightarrow A$ . The *standard wreath product*  $A \wr G$  is then the wreath product in which  $X = G$  carries the regular  $G$ -action.

We extend the notion of Cayley graph to sets with a group action (they are sometimes called *Schreier graphs*). If  $G = \langle T \rangle$ , we denote by  $\mathcal{C}(X, U)$  the graph with vertex set  $X$  and an edge from  $x$  to  $xt$  for all  $x \in X, t \in T$ .

LEMMA 6.13. — *Let  $A = \langle a_1, \dots, a_k \rangle$  be an arbitrary group, and let  $G = \langle T \rangle$  be a group acting transitively on an infinite set  $X$ . Fix a point  $x_1 \in X$ , and assume that, for all  $R \in \mathbb{N}$ , there exist  $x_2, \dots, x_k \in X$ , at distance  $> R$  from each other and from  $x_1$  in  $\mathcal{C}(X, T)$ , such that the balls of radius  $R$  around  $x_1$  and  $x_i$  are isomorphic for all  $i = 2, \dots, k$ . Let  $e_1, \dots, e_k$  denote the orders of  $a_1, \dots, a_k$  respectively. Then*

$$A \wr_X G \overset{\sim}{\rightsquigarrow} (C_{e_1} \times \dots \times C_{e_k}) \wr_X G.$$

*Proof.* — We adapt the argument in Example 2.17. As generating set of  $(C_{e_1} \times \dots \times C_{e_k}) \wr_X G$ , we consider  $\{b_1, \dots, b_k\} \sqcup T$ , in which  $b_i$  corresponds to the generator of  $C_{e_i}$  supported at  $x_0 \in X$ .

For arbitrary  $R \in \mathbb{N}$ , choose  $x_1, \dots, x_k \in X$  as in the Lemma’s hypotheses, and consider the following generating set  $\{s_1, \dots, s_k\} \sqcup T$  of  $A \wr_X G$ :

the generator  $s_i$  corresponds to the generator  $a_i$  of the copy of  $A$  supported at  $x_i$ .

Both  $\prod C_{e_i} \wr_X G$  and  $A \wr_X G$  are quotients of  $(\ast_i C_{e_i}) \ast G$ ; for the former, the additional relations are  $[b_i, g]$  for all  $i \in \{1, \dots, k\}$  and  $g \in G_{x_0}$ , and  $[b_i^g, b_j]$  for all  $i, j \in \{1, \dots, k\}$  and  $g \in G$ .

For the latter, the additional relations are  $[s_i, g]$  for all  $i \in \{1, \dots, k\}$  and  $g \in G_{x_i}$ , and  $[s_i^g, s_j]$  for all  $i, j \in \{1, \dots, k\}$  and  $g \in G$  with  $x_i g \neq x_j$ , and  $w(s_1^{g_1}, \dots, s_k^{g_k})$  for every relation  $w(a_1, \dots, a_k) = 1$  in  $A$  and every  $g_1, \dots, g_k \in G$  such that  $x_i g_i = x_j g_j$  for all  $i, j$ .

Our conditions imply that these two sets of relations agree on a ball of radius  $R$ . □

Our main example is as follows. Let  $X$  be the orbit of  $0^\infty$  under  $G_{012}$ .

**COROLLARY 6.14.** — *For every finitely generated group  $G$ , there exists an abelian group  $B$  such that  $G \wr_X G_{012} \overset{\sim}{\rightsquigarrow} B \wr_X G_{012}$ .*

*Proof.* — Let  $\{a_1, \dots, a_k\}$ , of respective orders  $e_1, \dots, e_k$ , generate  $G$ . Define  $B = C_{e_1} \times \dots \times C_{e_k}$ . Choose  $x_1 = 0^\infty$ , and for  $R \in \mathbb{N}$  choose distinct words  $v_2, \dots, v_k \in \{0, 1\}^*$  of length  $2\lceil \log_2 R \rceil$ . Set  $x_i = v_i 0^\infty$  for  $i = 2, \dots, k$ . Since the action of  $G_{012}$  is contracting, the  $R$ -balls around the  $x_i$  are isomorphic. The conclusion follows from Lemma 6.13. □

### 6.6. A necessary and sufficient condition for standard wreath products

**PROPOSITION 6.15.** — *Consider a wreath product  $W = G \wr H$  with  $H$  infinite. Then  $G \wr H \overset{\sim}{\rightsquigarrow} \mathbb{F}$  if and only if one of the following holds:*

- (1)  $G$  does not satisfy any identity;
- (2)  $H$  does not satisfy any almost-identity.

We split the proof in a sequence of lemmas. The following generalizes the construction in [4, Lemma 2.3] and the main result of that paper:

**LEMMA 6.16.** — *Let  $G$  be a  $k$ -generated group that satisfies no identity, and let  $H$  be an infinite group. Then  $G \wr H$  preforms  $\mathbb{F}_{k+1} \ast H$ , and hence preforms  $\mathbb{F}$  in view of Lemma 6.2.*

*Proof.* — Fix generating sets  $S = \{g_1, \dots, g_k\}$  of  $G$  and  $T$  of  $H$ ; we then identify  $g_i$  with the function  $H \rightarrow G$  supported at  $\{1\} \subset H$  at taking value  $g_i$  at 1.

By Lemma 6.6 and Lemma 2.14(4) it is sufficient to consider the case in which  $G$  contains a non-abelian free subgroup. Given  $R > 0$ , we construct the following generating set of  $G \wr H$ . Let  $B$  denote the ball of radius  $(k+1)R$  in  $H$ . Since  $G$  contains a free subgroup, it also contains a free subgroup  $\mathbb{F}_B$  of rank  $\#B$ . Let  $w$  be a function  $G \rightarrow H$ , supported at  $B$ , whose image is a basis of  $\mathbb{F}_B$ . Choose also  $h \in H \setminus B$ , and  $h_1, \dots, h_k \in H$  such that  $\|h_i\| = Ri$  for all  $i = 1, \dots, k$ . Consider then the set

$$U = \{w, w^{h_1} g_1^h, \dots, w^{h_k} g_k^h\} \cup T.$$

It is clear that  $U$  generates  $G \wr H$ . Consider a word  $u$  of length  $\leq R$  in  $U^{\pm 1}$ . Assume that it contains no relation in  $H$  (that would come from the  $T$  letters). If  $u$  is non-trivial, then it contains at least one term  $w^{h_i} g_i^h$ . Concentrating on what happens in  $B$ , we see generators of  $\mathbb{F}_B$  that cannot cancel, because to do so they would have to come from a term  $(w^{h_i} g_i^h)^{-1}$ , which would imply that  $u$  was not reduced, or from a term  $(w^{h_j} g_j^h)^{-1}$  via conjugation by a word of length at least  $R$  in  $T$ .

Therefore, the relations of length  $\leq R$  that appear in  $\mathcal{C}(G \wr H, U)$  are precisely those of  $\mathcal{C}(H, T)$ . □

LEMMA 6.17. — *If  $H$  satisfies no almost-identity, then  $G \wr H$  preforms a non-abelian free group.*

*Proof.* — Let  $H$  be  $k$ -generated. Since  $H$  does not satisfy any  $k$ -almost-identity, it preforms  $\mathbb{F}_k$  by Corollary 6.7. By Lemma 2.14(4), we get  $G \wr H \rightsquigarrow G \wr \mathbb{F}_k$ . Then  $G \wr \mathbb{F}_k$  admits  $\mathbb{F}_k$  as a quotient, hence by Lemma 6.2 preforms a non-abelian free group. □

If two groups satisfy an identity, then so does their wreath product. An analogous statement is valid for almost-identities:

LEMMA 6.18. — *Suppose that the group  $G$  satisfies an identity, and that for all  $k \in \mathbb{N}$  there is a  $k$ -almost-identity in  $H$ . Then for all  $k \in \mathbb{N}$  the wreath product  $G \wr H$  satisfies a  $k$ -almost-identity.*

*Proof.* — Let  $k \in \mathbb{N}$  be given, let  $v(x_1, \dots, x_m)$  be an identity for  $G$ , and let  $w(x_1, \dots, x_k)$  be an almost-identity for  $H$  on generating sets of cardinality  $k$ .

Let  $\{s_1, \dots, s_k\}$  be a generating set for  $G \wr H$ . Its projection to  $H$  then is a generating set for  $H$ , so  $w(s_1, \dots, s_k)$  belongs to the base  $G^H$  of  $G \wr H$ . For  $a_1, \dots, a_m \in \mathbb{F}_k$  to be determined later, let us consider the word

$$u(x_1, \dots, x_k) = v(w(x_1, \dots, x_k)^{a_1}, \dots, w(x_1, \dots, x_k)^{a_m}).$$

We clearly have  $u(s_1, \dots, s_k) = 1$ , so  $u$  is an almost-identity in  $G \wr H$ . We only have to choose the  $a_i \in \mathbb{Z}$  in such a way that  $u$  is not the trivial word.

Since  $w$  is a non-trivial word, there exists  $a \in \mathbb{F}_k$  such that  $\langle w, a \rangle$  is a free group of rank 2. Observe that  $\{w^{a^n} : n \in \mathbb{N}\}$  freely generates a free subgroup  $E$  of  $\mathbb{F}_k$ . Select then  $a_i = a^i$ . Then, since  $v$  is a non-trivial word,  $v(w^{a_1}, \dots, w^{a_m})$  is a non-trivial element of  $E$  and therefore of  $\mathbb{F}_k$ .  $\square$

*Example 6.19 (A solvable group in the component of free groups).* — Consider  $A = \mathbb{F}_2 \wr \mathbb{Z}$  and  $B = \mathbb{Z}^2 \wr \mathbb{Z}$ . Then  $B$  is solvable of class 2. By Lemma 6.13, the group  $A$  preforms  $B$ . Since  $\mathbb{F}_2$  satisfies no identity and since  $\mathbb{Z}$  is infinite, Lemma 6.16 implies that  $A$  preforms a free group.

In summary,  $A$  preforms a solvable group, and also preforms a non-abelian free group.

*Example 6.20 (A group of bounded torsion in the component of free groups).* — Let  $p \geq 3$  be such that there exist infinite finitely generated groups of  $p$ -exponent (any sufficiently large prime  $p$  has such property, see [2]). Let  $H$  be an infinite  $s$ -generated group of exponent  $p$ . Set  $A = (*^s \mathbb{Z}/p\mathbb{Z}) \wr H$  and  $B = (\mathbb{Z}/p\mathbb{Z})^s \wr H$ . By Lemma 6.13, the group  $A$  preforms  $B$ .

Observe that  $*^s \mathbb{Z}/p\mathbb{Z}$  contains a non-abelian free subgroup and therefore satisfies no identity. Since  $H$  is infinite, Lemma 6.16 implies that  $A$  preforms a free group. Clearly  $B$  is a torsion group of exponent  $p^2$ .

### 6.7. Distance between finitely generated groups

Given two finitely generated groups  $A$  and  $B$ , let us denote by  $\text{dist}_{\rightsquigarrow} (A, B)$  the distance between  $A$  and  $B$  in the (oriented) graph corresponding to the ‘ $\rightsquigarrow$ ’ preorder. It is the minimal length  $\ell$  of a chain of groups  $A = A_0, A_1, \dots, A_\ell = B$  such that either  $A_{i-1} \rightsquigarrow A_i$  or  $A_i \rightsquigarrow A_{i-1}$  for all  $i = 1, \dots, \ell$ . We also write  $\text{dist}_{\rightsquigarrow} (A, B) = \infty$  if  $A$  and  $B$  are in distinct connected components.

If  $A$  is a torsion-free nilpotent group, then we have seen in Proposition 4.6 that the diameter of the connected component that contains  $A$  is equal to two.

Examples 6.19 and 6.20 exhibit solvable groups and groups of bounded exponent at distance 2 from some non-abelian free group.

In contrast to the nilpotent case, the diameter of the connected component that contains non-abelian free groups is at least 3:

*Remark 6.21.* — If  $A$  is a finitely presented group satisfying an identity (for example, a finitely presented solvable group), then  $\text{dist}_{\rightsquigarrow} (A, \mathbb{F}_m) \geq 3$

for all  $m \geq 2$ . Indeed, any group that is preformed by  $A$  satisfies the same identity. Any group that preforms  $A$  is a quotient of  $A$  (since  $A$  is finitely presented) and hence also satisfies the same identity. This implies that all groups that are preformed by or preform  $A$  are at distance at least 2 from non-abelian free groups. Therefore, the distance from  $A$  to free groups is at least 3.

Before we discuss in more detail some groups from Remark 6.21, we need the following

*Example 6.22.* — Consider  $p \geq 2$ , and let

$$\mathbf{BS}(1, p) = \langle a, t \mid t^{-1}at = a^p \rangle$$

be a solvable Baumslag-Solitar group. Then  $\mathbf{BS}(1, p)$  preforms  $\mathbb{Z} \wr \mathbb{Z}^2$ .

*Proof.* — We write  $A = \mathbf{BS}(1, p)$ . Fix sequences  $(m_R), (n_R)$  in  $\mathbb{N}$  such that  $m_R, n_R$  are relatively prime,  $m_R \rightarrow \infty, n_R \rightarrow \infty$  and  $n_R/m_R \rightarrow \infty$ . For example,  $m_R = i$  and  $n_R = i^2 + 1$  will do.

Consider the generating set  $\{a, x_R = t^{n_R}, y_R = t^{m_R}\}$  of  $A$ . Let us prove that  $(A, S_R)$  subconverges to  $\mathbb{Z} \wr \mathbb{Z}^2 = \langle a, x, y \mid [b, c], [a, a^{x^i y^j}] \forall i, j \in \mathbb{Z} \rangle$  in  $\mathcal{G}$ .

Observe that  $a, x_R, y_R$  satisfy all the relations satisfied by  $a, x, y$  in  $\mathbb{Z} \wr \mathbb{Z}^2$ . Therefore,  $(A, S_R)$  subconverges to a quotient  $(\mathbb{Z} \wr \mathbb{Z}^2)/N$  of  $\mathbb{Z} \wr \mathbb{Z}^2$ . Furthermore,  $(\langle t \rangle, \{x_R, y_R\})$  converges to  $(\mathbb{Z}^2, \{x, y\})$ , so  $N$  maps to the trivial subgroup of  $\mathbb{Z}^2$  under the natural projection  $\mathbb{Z} \wr \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ .

Now every element of  $\mathbb{Z} \wr \mathbb{Z}^2$  may uniquely be written in the form  $w(a, x, y) = \prod_{i, j \in \mathbb{Z}} a^{\ell_{i, j} x^i y^j} x^p y^q$ , and if this element maps trivially to  $\mathbb{Z}^2$  then  $p = q = 0$ .

Let us therefore assume by contradiction that there exists a non-trivial word  $w(a, x, y) = \prod_{i, j \in \mathbb{Z}} a^{\ell_{i, j} x^i y^j}$  with  $w(a, x_R, y_R) = 1$  for all sufficiently large  $R$ .

The group  $A$  is isomorphic to  $\mathbb{Z}[1/p] \rtimes \mathbb{Z}$ , with the generator of  $\mathbb{Z}$  acting on  $\mathbb{Z}[1/p]$  by multiplication by  $p$ . Since  $w(a, x, y)$  maps trivially to  $\mathbb{Z}^2$ , we have  $w(a, x_R, y_R) \in \mathbb{Z}[1/p]$ , and in fact under this identification

$$w(a, x_R, y_R) = \sum_{i, j \in \mathbb{Z}} \ell_{i, j} p^{in_R + jm_R}.$$

Let  $(i, j) \in \mathbb{Z}^2$  be lexicographically maximal such that  $\ell_{i, j} \neq 0$ ; that is,  $\ell_{i', j'} = 0$  if  $i' > i$  or if  $i' = i$  and  $j' > j$ . Set  $N = \sum_{i, j \in \mathbb{Z}} |\ell_{i, j}|$ . For  $R$  sufficiently large, we have  $p^{in_R + jm_R} > N p^{i'n_R + j'm_R}$  whenever  $(i', j') \in \mathbb{Z}^2$  is such that  $\ell_{i', j'} \neq 0$ . For such  $R$ , we have  $|w(a, x_R, y_R)| \geq p^{in_R + jm_R} -$

$\sum_{(i',j') \neq (i,j)} \ell_{i',j'} p^{i'n_R + j'm_R} > 0$ , contradicting the hypothesis that  $w$  is a relation in the limit of  $(A, S_R)$ . □

*Example 6.23 (Groups at distance 3 from free groups).* — The distance between solvable Baumslag Solitar groups and free groups is equal to 3.

*Proof.* — Consider  $p \geq 2$  and  $A = \mathbf{BS}(1, p)$  a solvable Baumslag-Solitar group. Since  $A$  is finitely presented and solvable, Remark 6.21 implies that the distance from  $A$  to free groups is at least 3.

By Example 6.22 we know that  $A$  preforms  $\mathbb{Z} \wr \mathbb{Z}^2$ . Since  $\mathbb{Z} \overset{\sim}{\rightsquigarrow} \mathbb{Z}^2$ , we know by Lemma 2.14 that  $\mathbb{Z} \wr \mathbb{Z}^2 \overset{\sim}{\rightsquigarrow} \mathbb{Z}^2 \wr \mathbb{Z}^2$ , so  $A \overset{\sim}{\rightsquigarrow} \mathbb{Z}^2 \wr \mathbb{Z}^2$ . By Lemma 6.16,  $\mathbb{F}_2 \wr \mathbb{Z}^2$  preforms a free group. We therefore have a chain  $A \overset{\sim}{\rightsquigarrow} \mathbb{Z}^2 \wr \mathbb{Z}^2 \overset{\sim}{\rightsquigarrow} \mathbb{F}_2 \wr \mathbb{Z}^2 \overset{\sim}{\rightsquigarrow} \mathbb{F}_4$ , and  $\text{dist}_{\overset{\sim}{\rightsquigarrow}}(A, \mathbb{F}_4) \leq 3$ .

On the other hand, if we had  $\text{dist}_{\overset{\sim}{\rightsquigarrow}}(A, \mathbb{F}_4) = 2$  then either there would exist  $B$  with  $A \overset{\sim}{\rightsquigarrow} B \overset{\sim}{\rightsquigarrow} \mathbb{F}_4$ ; this is impossible because  $B$  would then be both solvable and preformed by a free group; or there would exist  $B$  with  $A \overset{\sim}{\rightsquigarrow} B \overset{\sim}{\rightsquigarrow} \mathbb{F}_4$ ; and again  $B$  would be both solvable and without almost-identities. □

### 7. Groups of non-uniform exponential growth

Let  $G$  be a group generated by a set  $S$ . The *growth function* of  $G$  with respect to  $S$ ,

$$\nu_{G,S}(R) = \#B(1, R) \subseteq \mathcal{C}(G, S),$$

counts the number of group elements that may be expressed using at most  $R$  generators. This function depends on  $S$ , but only mildly; if for two functions  $\gamma, \delta: \mathbb{N} \rightarrow \mathbb{N}$  one defines  $\gamma \lesssim \delta$  whenever there exists a constant  $k \in \mathbb{N}_+$  such that  $\gamma(R) \leq \delta(kR)$ , and  $\gamma \sim \delta$  whenever  $\gamma \lesssim \delta \lesssim \gamma$ , then the  $\sim$ -equivalence class of  $\nu_{G,S}$  is independent of  $S$ .

The group  $G$  has *polynomial growth* if  $\nu_{G,S}(R) \lesssim R^d$  for some  $d$ ; then necessarily  $G$  is virtually nilpotent and  $\nu_{G,S}(R) \sim R^d$  for some  $d \in \mathbb{N}$ , by [30, 10]. On the other hand, if  $\nu_{G,S}(R) \gtrsim b^R$  for some  $b > 1$ , then  $\nu_{G,S}(R) \sim 2^R$  and  $G$  has *exponential growth*; this happens for free groups, and more generally for groups containing a free subsemigroup. If  $G$  has neither polynomial nor exponential growth, then it has *intermediate growth*. The existence of groups of intermediate growth, asked by Milnor [19], is proven by Grigorchuk in [28].

Set  $\lambda_{G,S} = \lim \sqrt[R]{\nu_{G,S}(R)}$ ; the limit exists because  $\nu_{G,S}$  is submultiplicative ( $\nu_{G,S}(R_1 + R_2) \leq \nu_{G,S}(R_1)\nu_{G,S}(R_2)$ ). Reformulating the above



definitions, we say  $G$  that has *subexponential growth* if  $\lambda_{G,S} = 1$  for some and hence all  $S$ ; that  $G$  has *exponential growth* if  $\lambda_{G,S} > 1$ ; and that  $G$  has *uniform exponential growth* if  $\inf_S \lambda_{G,S} > 1$ . The existence of groups of non-uniform exponential growth, asked by Gromov [31, Remarque 5.12], is proven by Wilson in [64].

LEMMA 7.1. — *If  $G \overset{\sim}{\rightsquigarrow} H$ , then  $\inf_S \lambda_{G,S} \leq \inf_T \lambda_{H,T}$ . In particular, if  $G$  has exponential growth and  $H$  has subexponential growth, then  $G$  has non-uniform exponential growth.*

*Proof.* — For every  $\epsilon > 0$ , there exists a generating set  $T$  for  $H$  such that  $\lambda_{H,T} < \inf_{T'} \lambda_{H,T'} + \epsilon$ . There exists then  $R \in \mathbb{N}$  such that  $\nu_{H,T}(R)^{1/R} \leq \lambda_{H,T} + \epsilon$ . Choose then a generating set  $S$  for  $G$  such that the balls of radius  $R$  in  $\mathcal{C}(G, S)$  and  $\mathcal{C}(H, T)$  agree. Then  $\nu_{G,S}(R) = \nu_{H,T}(R)$ , so  $\lambda_{G,S} \leq \nu_{H,T}(R)^{1/R}$  because growth functions are submultiplicative. Therefore, for all  $\epsilon > 0$  there exists  $S$  generating  $G$  such that  $\lambda_{G,S} \leq \inf_{T'} \lambda_{H,T'} + 2\epsilon$ .  $\square$

Note that the inequality in Lemma 7.1 can be strict; for example, the Grigorchuk group  $G_{012}$ , has intermediate growth, yet  $G_{012} \overset{\sim}{\rightsquigarrow} \mathbb{F}_3$ .

COROLLARY 7.2. — *For every group  $G$  of exponential growth, the wreath product  $G \wr_X G_{012}$  of  $G$  with the Grigorchuk group  $G_{012}$  acting on  $X = 0^\infty G_{012}$  has non-uniform exponential growth.*

*Proof.* — From Corollary 6.14 we get  $G \wr_X G_{012} \overset{\sim}{\rightsquigarrow} B \wr_X G_{012}$  for an abelian group  $B$ . It is proven in [9, Thm A] that  $B \wr_X G_{012}$  has subexponential growth, in fact of the form  $\exp(R^\alpha)$  if  $B$  is finite, non-trivial, and of the form  $\exp(R^\alpha \log R)$  if  $B$  is infinite, for some constant  $\alpha < 1$ , see Corollary 7.3. The claim then follows from Lemma 7.1.  $\square$

COROLLARY 7.3. — *Every countable group may be imbedded in a group of non-uniform exponential growth.*

Furthermore, let  $\alpha \approx 0.7674$  be the positive root of  $2^{3-3/\alpha} + 2^{2-2/\alpha} + 2^{1-1/\alpha} = 2$ . Then the group of non-uniform exponential growth  $G$  has the following property: there is a constant  $K$  such that, for any  $R > 0$ , there exists a generating set  $S$  of  $G$  with

$$\nu_{G,S}(r) \leq \exp(Kr^\alpha) \text{ for all } r \leq R.$$

In particular, there exist groups of non-uniform exponential growth that do not imbed uniformly into Hilbert space.

*Proof.* — Let  $G$  be a countable group. Imbed first  $G$  into a finitely generated group  $H$ . Without loss of generality, assume that  $H$  has exponential

growth (if needed, replace  $H$  by  $H \times \mathbb{F}_2$ ), and that the generators of  $H$  are torsion elements.

By Corollary 6.14, the group  $H \wr_X G_{012}$  preforms  $A \wr_X G_{012}$  for a finite abelian group  $A$ . Since  $A \wr_X G_{012}$  has growth  $\sim \exp(R^\alpha)$ , the first claim follows.

The second claim follows from the first, since there exist groups  $G$  that do not imbed into Hilbert space [32]; and the property of not imbedding into Hilbert space is inherited from subgroups.  $\square$

Briussel asks in [16, after Proposition 2.5] whether there exist groups of non-uniform exponential growth and without the Haagerup property. Recall that a group has the Haagerup property if it admits a proper affine action on Hilbert space; this property is also known as “a-T-menability”, see [24]. It is clear that any group with the Haagerup property can be uniformly imbedded into Hilbert space. Therefore, Corollary 7.3 implies in particular that there exist groups of non-uniform exponential growth that do not have the Haagerup property.

### 7.1. Non-uniform non-amenability

Let  $G$  be a group generated by a finite set  $S$ . By Følner’s criterion,  $G$  is *non-amenable* if the isoperimetric constant

$$\alpha_S := \inf_{F \subset G \text{ finite}} \#(FS \setminus F) / \#F$$

satisfies  $\alpha_S > 0$ . Arzhantseva et al. [7] call  $G$  *non-uniformly non-amenable* if  $G$  is non-amenable, but  $\inf_S \alpha_S = 0$ .

If  $G$  has non-uniform exponential growth and is non-amenable, then it is non-uniformly non-amenable. However, there are groups of uniform exponential growth that are non-uniformly non-amenable. Clearly, if  $G$  preforms an amenable group, then  $G$  may not be uniformly non-amenable:

*Example 7.4.* —  $\mathbb{F}_2 \wr \mathbb{Z}$  has uniform exponential growth, but is non-uniformly non-amenable.

*Proof.* — The group  $\mathbb{F}_2 \wr \mathbb{Z}$  maps onto  $\mathbb{Z}^2 \wr \mathbb{Z}$ , which is solvable and of exponential growth; so its growth is uniformly exponential, and the same holds for  $\mathbb{F}_2 \wr \mathbb{Z}$ .

By Lemma 6.13, we also have  $\mathbb{F}_2 \wr \mathbb{Z} \rightsquigarrow \mathbb{Z}^2 \wr \mathbb{Z}$ , so  $\mathbb{F}_2 \wr \mathbb{Z}$  preforms an amenable group, so is not uniformly non-amenable.  $\square$

## 8. Open problems and questions

QUESTION 8.1. — *Is every non-virtually nilpotent group in the connected component of the free group?*

A positive answer to the following question would imply a negative answer to the question by Olshansky: “Is there a variety other than virtually nilpotent or free in which the relatively free group is finitely presented?”

QUESTION 8.2. — *Do two nilpotent groups belong to the same connected component if and only if they have the same positive universal theory?*

We answer positively the question above in the case of nilpotent groups  $G$  such that  $G$  and  $G/\text{Torsion}(G)$  generate the same variety.

We show in Remark 6.21 that the diameter of the free group’s component is at least three:

QUESTION 8.3. — *What is the diameter of the connected component of the free group?*

The following question complements the previous one; we show in Proposition 4.6 that its answer is positive, in particular, in the case of torsion-free nilpotent groups. Guyot considers limits of dihedral groups in [33], and shows that they are semidirect products of (a finitely generated abelian group with cyclic torsion subgroup) by  $\mathbb{Z}/2$ , the latter acting by  $-1$ . His result implies that the groups preformed by the infinite dihedral group form a directed set.

QUESTION 8.4. — *Is every connected component of virtually nilpotent groups directed, namely, is it a partially ordered set in which every finite subset has an upper bound?*

If  $G \overset{\sim}{\rightsquigarrow} \mathbb{F}_k$ , then there are generating sets  $S_n$  for  $G$ , of cardinality  $k$ , such that the girth of  $\mathcal{C}(G, S_n)$  tends to infinity.

QUESTION 8.5. — *If a finitely generated group  $G$  has infinite girth, does one have  $G \overset{\sim}{\rightsquigarrow} \mathbb{F}_k$  for some  $k \in \mathbb{N}$ ?*

In other words, the question asks whether in the definition of girth one can always chose a sequence of generating sets with a bounded number of generators.

Cornulier and Mann ask in [26, Question 18]: “Does there exist a group of intermediate growth that satisfies an identity?” The following question

is also open: “Does there exist a group of non-uniform exponential growth that satisfies an identity?” So as to better determine which groups preform free groups, we ask:

QUESTION 8.6. — *Does there exist a group of intermediate growth that satisfies an almost-identity? Does there exist a group of non-uniform exponential growth that satisfies an almost-identity?*

A well-known question by Adyan asks: “Are there finitely presented groups of intermediate growth?” Such a group would not be preformed by a group of exponential growth. The following question by Mann is also open [45, Problem 4]: “Are there finitely presented groups of non-uniform exponential growth?”

Given a group  $G$  of non-uniform exponential growth, it admits generating sets  $S_n$  with growth rate tending to 1. If furthermore the cardinalities of the  $S_n$  are bounded, then a subsequence of  $(G, S_n)$  converges to a group of intermediate growth.

QUESTION 8.7. — *Does there exist a group of non-uniform exponential growth that doesn't preform a group of subexponential (equivalently, intermediate) growth?*

QUESTION 8.8. — *Does there exist a group  $G$  such that, for every finitely generated group  $A$  of non-polynomial growth, there exists a group  $H$  with  $G \overset{\sim}{\rightsquigarrow} H$  and the growth of  $A$  and  $H$  are equivalent?*

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