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LOCAL COMPACTNESS AND CARTESIAN PRODUCTS OF QUOTIENT MAPS AND k-SPACES

by Ernest MICHAEL (1)

1. Introduction.

In 1948, J.H.C. Whitehead [8; Lemma 4] proved that, if X is locally compact Hausdorff, then the Cartesian product (2) $i_X \times g$ is a quotient map (3) for every quotient map g. Using this result, D.E. Cohen proved in [1; 3.2] that, if X is locally compact Hausdorff, then $X \times Y$ is a k-space (4) for every k-space Y. The principal purpose of this note is to show that these results are the best possible, in the sense that, if a regular space X is not locally compact, then the conclusions of both results are false. (That the conclusions are false without *some* restrictions on X is well known; see, for instance, Bourbaki [2, p. 151, Exercise 6] and C.H. Dowker [4; p. 563]).

Our main results are formally stated and proved in sections 2 and 3, while section 4 contains analogous results for sequential spaces, and section 5 considers the special case where X is metrizable.

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(2) If $f_i : X_i \to Y_i$ $(i = 1, 2)$, the product $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is defined by $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$. We use i_X to denote the identity map on X.

(3) A map $f : X \to Y$ is a *quotient* map if a set $V \subset Y$ is open in Y if and only if $f^{-1}(V)$ is open in X.

(4) A topological space X is a *k-space* if a subset A of X is closed whenever $A \cap K$ is closed in K for every compact $K \subset X$. All locally compact spaces and all first-countable spaces are k-spaces.

I am grateful to S.P. Franklin and A.H. Stone for a valuable conversation over a Mexican dinner during an Arizona sandstorm.

2. Products of quotient maps.

THEOREM 2.1. — *The following properties of a regular* [5] *space* X *are equivalent.*

(a) X *is locally compact.*

(b) $i_X \times g$ *is a quotient map for every quotient map* g.

(c) $i_X \times g$ *is a quotient map for every closed compact-covering* [6] *map* g *with domain and range paracompact* k*-spaces.*

Proof. — The implication (a) $\rightarrow$ (b) is the theorem of J.H.C. Whitehead quoted in the introduction, and (b) $\rightarrow$ (c) is obvious because continuous closed maps are quotient maps. It remains to prove (c) $\rightarrow$ (a).

Suppose X is not locally compact at some $x_0 \in X$. Let $\{U_\alpha\}_{\alpha \in A}$ be a local base at x_0. Then, for all $\alpha \in A$, the closure $\overline{U}_\alpha$ is not compact, and thus has a well ordered family $\{F_\lambda\}_{\lambda < \lambda(\alpha)}$ of non-empty closed subsets whose intersection is empty [7]. We assume that the collection of all the well-ordered index sets $\Lambda_\alpha = \{\lambda : \lambda \leqslant \lambda(\alpha)\}$, with $\alpha \in A$, is disjoint. Topologize each Λ_α with the order topology, which makes it compact Hausdorff. Let Λ denote the topological sum $\Sigma_{\alpha \in A} \Lambda_\alpha$, and let Y be the space obtained from Λ by identifying all the final points $\lambda(\alpha) \in \Lambda_\alpha$ to a single point $y_0 \in Y$. Clearly Λ is a paracompact k-space, and it is easy to check directly that so is Y. Let $g : \Lambda \rightarrow Y$ be the quotient map. Clearly g is closed, and g is compact-covering because every compact subset of Y is contained in the union of

[5] I am grateful K. A. Baker for informing me that, while our proof of (c) $\rightarrow$ (a) makes essential use of regularity, (b) $\rightarrow$ (a) can nevertheless be proved for all Hausdorff spaces X by constructing a separate proof in case X is not regular. I dont't know whether (c) $\rightarrow$ (a) remains true for all Hausdorff X.

[6] A continuous map $f : X \rightarrow Y$ is *compact-covering* if every compact subset of Y is the image of some compact subset of X.

[7] This follows from [6; p. 163 H] and the fact that every simply ordered set has a cofinal well-ordered subset.

finitely many $g(\Lambda_\alpha)$. It remains to show that $h = i_X \times g$ is not a quotient map.

For each $\alpha \in A$ and $\lambda \in \Lambda_\alpha$, let $E_\lambda = \bigcap_{\nu<\lambda} F_\nu$. Then $E_{\lambda(\alpha)} = \emptyset$, and $E_\lambda \supset F_\lambda \neq \emptyset$ if $\lambda < \lambda(\alpha)$. For each $\alpha \in A$, define $S_\alpha \subset X \times \Lambda_\alpha$ by

$$S_\alpha = \bigcup\{E_\lambda \times \{\lambda\} : \lambda \in \Lambda_\alpha\}.$$

Then S_α is clearly closed in $X \times \Lambda_\alpha$. Define $S \subset X \times Y$ by

$$S = \bigcup_{\alpha \in A} h(S_\alpha).$$

Let us show that $h^{-1}(S)$ is closed in $X \times \Lambda$, but that S is not closed in $X \times Y$.

To see that $h^{-1}(S)$ is closed in $X \times \Lambda$, it suffices to check that $h^{-1}(S) \cap (X \times \Lambda_\alpha)$ is closed in $X \times \Lambda_\alpha$ for all α. But, since $E_{\lambda(\alpha)} = \emptyset$ for all α,

$$h^{-1}(S) \cap (X \times \Lambda_\alpha) = S_\alpha,$$

and S_α is indeed closed in $X \times \Lambda_\alpha$.

To see that S is not closed in $X \times Y$, note first that $(x_0, y_0) \notin S$. However, if $U \times V$ is a neighborhood of (x_0, y_0) is $X \times Y$, then $\overline{U}_\beta \subset U$ for some $\beta \in A$; if we pick $\lambda \in g^{-1}(V) \cap \Lambda_\beta$ with $\lambda \neq \lambda_\beta$, then

$$\emptyset \neq h(E_\lambda \times \{\lambda\}) \subset (U \times V) \cap S.$$

Hence $(x_0, y_0) \in \overline{S}$, and that completes the proof.

3. Products of k-spaces.

THEOREM 3.1. — *The following properties of a regular* (5) *space* X *are equivalent.*

(*a*) X *is locally compact.*

(*b*) $X \times Y$ *is a k-space for every k-space* Y.

(*c*) $X \times Y$ *is a k-space for every paracompact k-space* Y.

Proof. — The implication $(a) \to (b)$ is the result of D.E. Cohen quoted in the introduction, and $(b) \to (c)$ is obvious. It remains to prove $(c) \to (a)$.

Suppose X is not locally compact. Then Theorem 2.1 implies that there exists a compact-covering map $g : \Lambda \to Y$, with Y a paracompact k-space, such that $i_X \times g$ is not a quotient map. Since g is compact- covering, so is $i_X \times g$. Now it is easy to show [7; Lemma 11.2] that any compact-covering map whose range is a Hausdorff k-space must be a quotient map. Since $i_X \times g$ is not a quotient map, it follows that $X \times Y$ is not a k-space. That completes the proof.

4. Two analogous results.

S. P. Franklin has pointed out that Theorems 2.1 and 3.1 have simple analogues in case the domain of g in Theorem 2.1, or the space Y in Theorem 3.1, are assumed to be sequential. Recall that a space Y is called *sequential* [5] if a subset A of Y is closed whenever $A \cap S$ is closed in S in for every convengent sequence (including the limit) S in Y. Since such S are compact, every sequential space is clearly a k-space. Moreover, quotients of sequential spaces are always sequential, and sequential spaces are precisely the quotients of (locally compact) metrizable spaces (see [5]).

For each infinite cardinal $\mathfrak{m}$, let $D_\mathfrak{m}$ denote the discrete space of cardinality $\mathfrak{m}$, let $Y_\mathfrak{m}$ be the quotient space obtained from $D_\mathfrak{m} \times [0, 1]$ by identifying all points in $D_\mathfrak{m} \times \{0\}$ (i.e. $Y_\mathfrak{m}$ is the cone over $D_\mathfrak{m}$), and let $g_\mathfrak{m} : D_\mathfrak{m} \times [0, 1] \to Y_\mathfrak{m}$ be the quotient map.

By the *pointwise weight* of a space X we will mean the smallest cardinal $\mathfrak{m}$ such that each $x \in X$ has a neighborhood base of cardinality $\leqslant \mathfrak{m}$.

Theorem 4.1. — *The following properties of a regular space X are equivalent.*

a) X is locally countably compact.

b) $i_X \times g$ is a quotient map for every quotient map g with sequential domain.

c) $i_X \times g_\mathfrak{m}$ is a quotient map, where $\mathfrak{m}$ is the pointwise weight of X.

Proof. — $(a) \to (b)$. This proof goes just like J. H. C. Whitehead's proof [8; Lemma 4] that $(a) \to (b)$ in Theorem 2.1. In fact, Whitehead's proof is based on the fact that if U is an open subset of a product space $E \times F$, and if $C \subset F$ is compact, then $\{x \in E : \{x\} \times C \subset U\}$ is open in E. It is easy to check that, if E is sequential, this conclusion remains valid if C is only assumed to be countably compact.

$(b) \to (c)$ Obvious.

$(c) \to (a)$ Suppose X is not locally countably compact. Examing the proof of Theorem 2.1, one sees that then there are only m space Λ_α, and each Λ_α can be chosen to be a convengent sequence or, if one prefers, a closed interval. In the latter case, the map g constructed in the proof of Theorem 2.1 is precisely g_m. That completes the proof.

THEOREM 4.2. — *The following properties of a regular sequential space* X *are equivalent.*

a) X *is locally countably compact.*

b) $X \times Y$ *is sequential for every sequential space* Y.

c) $X \times Y_{\mathfrak{m}}$ *is a k-space, where* $\mathfrak{m}$ *is the pointwise weight of* X.

Proof. — $(a) \to (b)$. This follows immediately from T. K. Boehme [1; Theorem] and S. P. Franklin [5; Proposition 1.10].

$(b) \to (c)$. Obvious.

$(c) \to (a)$. This follows from 4.1 $(c) \to (a)$ in the same way that 3.1 $(c) \to (a)$ followed from 2.1 $(c) \to (a)$. That completes the proof.

BIBLIOGRAPHIE

[1] T. K. BOEHME, Linear s-spaces, Proc. Symp. Convergence Structures, U. of Oklahoma, 1965.

[2] N. BOURBAKI, Topologie Générale, Chapters 1 and 2, 3rd ed., Hermann, 1961.

[3] D. E. COHEN, Spaces with weak topology, *Quart. J. Math.*, Oxford Ser. (2) 5 (1954), 77-80.

[4] C. H. DOWKER. Topology of metric complexes, *Amer. J. Math.*, 74 (1952), 555-577.
[5] S. P. FRANKLIN, Spaces in which sequences suffice, *Fund. Math.*, 57 (1965), 107-115.
[6] J. L. KELLEY, General Topology, VanNostrand, (1955).
[7] E. MICHAEL, $\aleph_0$-spaces, *J. Math. Mech.*, 15 (1966), 983-1002.
[8] J. H. C. WHITEHEAD, A note on a theorem of Borsuk, *Bull. Amer. Math. Soc*, 54 (1958), 1125-1132.

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