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CODIMENSION TWO INDEX OBSTRUCTIONS TO POSITIVE SCALAR CURVATURE

by Bernhard HANKE, Daniel PAPE & Thomas SCHICK (*)

Abstract. — We derive a general obstruction to the existence of Riemannian metrics of positive scalar curvature on closed spin manifolds in terms of hypersurfaces of codimension two. The proof is based on coarse index theory for Dirac operators that are twisted with Hilbert C^*-module bundles.

Along the way we give a complete and self-contained proof that the minimal closure of a Dirac type operator twisted with a Hilbert C^*-module bundle on a complete Riemannian manifold is a regular and self-adjoint operator on the Hilbert C^*-module of L^2-sections of this bundle.

Moreover, we give a new proof of Roe's vanishing theorem for the coarse index of the Dirac operator on a complete non-compact Riemannian manifold whose scalar curvature is uniformly positive outside of a compact subset. This proof immediately generalizes to Dirac operators twisted with flat Hilbert C^*-module bundles.

Résumé. — Nous dérivons une obstruction générale à l'existence d'une métrique à courbure scalaire positive sur une variété compacte spin, qui est basée sur des sous-variétés de codimension deux. La preuve utilise la théorie d'indice grossier (synonymement "indice a grande échelle") pour l'opérateur de Dirac tordu par un fibré de C^*-modules Hilbertiens.

En cours de route nous donnons une preuve complète et indépendant du fait que la clôture minimale d'un opérateur de type Dirac sur une variété complète, tordu par un fibré de C^*-modules Hilbertiens, est régulière et auto-adjointe comme operateur non-borné sur le C^*-module Hilbertien des sections L^2-intégrables de ce fibré.

En outre, nous donnons une preuve nouvelle du théorème de Roe affirmant que l'indice grossier de l'opérateur de Dirac est nul pour une variété Riemannienne complète non-compacte avec courbure scalaire uniformement positive en dehors d'un sous-ensemble compact. Notre preuve se généralise immédiatement aux operateurs de Dirac tordu par un fibré plat de C^*-modules Hilbertiens.

Keywords: index theory, positive scalar curvature, codimension 2, hypersurface, Mishchenko-Fomenko index, large scale geometry, coarse geometry, large scale index theory, coarse index theory.
Math. classification: 46L80, 19K56, 19L64, 53C20, 58J22, 53C27.
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1. Introduction

A central theme of geometric topology in recent decades asks whether a given smooth manifold admits a Riemannian metric with positive scalar curvature. On spin manifolds the most powerful obstructions to existence of such metrics are based on index theory for the Dirac operator. Indeed the Schrödinger-Lichnerowicz formula [30] implies that on a spin manifold with uniformly positive scalar curvature the Dirac operator is invertible and hence its index, suitably defined if the manifold is not compact, has to vanish.

Rosenberg [23, 22, 24] used Dirac operators twisted with flat Hilbert C^*-module bundles whose indices lie in the K-theory of C^*-algebras in order to obtain particularly strong obstructions to the existence of positive scalar curvature metrics. In particular, using the Mishchenko bundle, the canonical flat $C^*\pi_1(M)$-bundle on M, one obtains the Rosenberg index obstruction $\alpha(M) \in K_*(C^*\pi_1(M)))$.

Roe [20] developed coarse index theory to define meaningful indices of Dirac operators on non-compact complete manifolds. This can also be used to gain interesting information for compact manifolds by passing to non-compact covering spaces.

Gromov and Lawson [4, Theorem 7.5] found an intriguing obstruction to positive scalar curvature based on submanifolds of codimension two: if M is a closed aspherical spin manifold with a hypersurface N of codimension two with trivial normal bundle such that N is enlargeable and $\pi_1(N)$ injects into $\pi_1(M)$, then M does not admit a Riemannian metric of positive scalar curvature.

The main purpose of this paper is to illuminate this result from an index theoretic perspective. Our proof is based on coarse index theory for Dirac operators twisted with Hilbert C^*-module bundles. This allows us to prove the following statement, where in particular the asphericity of M in [4] is weakened to the vanishing of the second homotopy group.

Theorem 1.1. — *Let M be a closed connected spin manifold with $\pi_2(M) = 0$. Assume that $N \subset M$ is a codimension two submanifold with trivial normal bundle and that the induced map $\pi_1(N) \to \pi_1(M)$ is injective. Assume that the Rosenberg index of N does not vanish: $0 \neq \alpha(N) \in K_*(C^*\pi_1(N))$.*

Then M does not admit a Riemannian metric of positive scalar curvature.

Remark 1.2. — Here and above one can use either the reduced or the maximal group C^*-algebra. The first one is closely connected to the Baum-Connes and the strong Novikov conjecture, but a priori the latter one might lead to stronger obstructions. The material in the paper at hand is independent of which group C^*-algebra is used so that we will not distinguish them in our notation.

Remark 1.3. — In [36, Theorem 3.4] a result close to our Theorem 1.1 is stated without any assumption on $\pi_2(M)$. Unfortunately, the statement of [36, Theorem 3.4] is wrong, the manifolds $N = T^n$ and $M = (T^n \times S^2)\#(T^n \times S^2)$, $n > 2$, providing counterexamples. Our correct formulation of Theorem 1.1 had been established long before [36] appeared, and the authors of the present paper reported on it at several occasions in seminars and at conferences.

Remark 1.4. — The concept of enlargeability is not used in our paper; it is entirely based on properties of the Rosenberg index. Because enlargeable spin manifolds have non-vanishing Rosenberg index [7, Theorem 1.2], [5], [6, Theorem 1.5], this is no loss of generality.

Coarse index theory as developed by Roe is based on functional calculus for the (unbounded) Dirac operator. In our context we are dealing with Dirac operators twisted by Hilbert C^*-module bundles so that, in order to apply the functional calculus in [12], it is required to establish regularity and self-adjointness of their closures. In our opinion this fact is not well-documented in the literature and hence we decided to give a self-contained and complete proof of the following theorem, which might be of independent interest. Throughout our paper, A denotes a complex unital C^*-algebra.

Theorem 1.5. — *Let M be a complete Riemannian manifold, and let $E \to M$ be a smooth Hilbert A-module bundle with finitely generated projective fibers, which is equipped with a connection ∇, compatible with the inner product.*

Let D be any Dirac type operator on a complex Dirac bundle $S \to M$ and D_E its twist with (E, ∇). Then the closure of D_E is a densely defined, regular and self-adjoint operator on the Hilbert A-module $L^2(M, S \otimes E)$ of L^2-sections of $S \otimes E$.

We follow the program of Vassout [33] who proves a corresponding statement for foliations, based on the existence of a suitable pseudodifferential calculus.

Remark 1.6. — We could locate a couple of accounts of the result for compact manifolds, which however, for our taste, were quite sketchy and did not cover the case of non-compact manifolds. Zadeh [35, Lemma 2.1] offers an alternative proof that also includes the case of non-compact manifolds. However, it is based on properties of the wave operators e^{itD_E}, whose existence is assumed in [35] without further reference. We believe that a construction of these operators is possible, independent of a general functional calculus for D_E (which would depend on normality and self-adjointness of this operator and therefore would render the argument circular). But we could not find a detailed construction in the literature. Therefore we decided to give a full and independent proof of Theorem 1.5 in Section 2 below.

A final crucial ingredient of our proof of the codimension two obstruction in Theorem 1.1 is a generalized vanishing theorem for the coarse index on non-compact manifolds.

Theorem 1.7 (Partial vanishing theorem). — *Let (M, g) be a complete connected non-compact Riemannian spin manifold such that, outside of a compact subset, the scalar curvature is uniformly positive. Let $E \to M$ be a Hilbert A-module bundle as in Theorem 1.5 above and assume that this bundle is flat. Then the coarse index* $\mathrm{ind}(D_E) \in K_*(C^*(M; A))$ *vanishes.*

Remark 1.8. — The special case of this result with $A = \mathbb{C}$ and trivial E has been stated in [20, Proposition 3.11 and following remark] without proof. Only recently, Roe [18] published a full proof of this special case, using the theory of Friedrichs extensions of unbounded operators. Zadeh [36, Theorem 3.1] offers a proof of Theorem 1.7, again based on Friedrichs extensions. We feel that this is not completely satisfactory. Although the concept of Friedrichs extensions for unbounded operators on Hilbert A-modules should exist, it has not been developed yet, to the best of our knowledge. In particular the regularity of the resulting operator must be taken care of.

We present a proof in the spirit of Roe's coarse index theory, based on functional calculus and unit propagation of the wave operator. This proof first appeared in the second author's doctoral thesis [16, Theorem 0.2.1]. We expect that it can be generalized to other interesting situations, notably to perturbations of the signature operator, as they show up in proofs of the homotopy invariance of higher signatures, compare [11].

Remark 1.9. — The codimension two obstruction in 1.1 has a slight strengthening: even *stably* M does not admit a metric of positive scalar curvature. Here, "stably" means that for every simply connected closed 8-dimensional spin manifold B with $\hat{A}(B) = 1$, i.e. for any so-called *Bott* *manifold*, and for every $l \geqslant 0$ the manifold $M \times B^l$ does not admit a metric of positive scalar curvature. This simply follows by applying the codimension two obstruction theorem to $N \times B^l \subset M \times B^l$.

The *stable Gromov-Lawson-Rosenberg conjecture* [25, Conjecture 4.17] states that a closed spin manifold M stably admits a metric with positive scalar curvature if and only if its Rosenberg index $\alpha(M) \in KO_*(C^*_{\mathbb{R}}\pi_1(M))$ vanishes.

Stolz [31, 32] proved that the stable Gromov-Lawson-Rosenberg conjecture holds for all manifolds whose fundamental groups satisfy the strong Novikov conjecture. Recall that the *unstable* version of [25, Conjecture 4.8] is not true [27]. The construction of a corresponding example uses the *codimension one* obstruction of Schoen and Yau [29], which is based on minimal hypersurfaces and is independent from index theory.

Arguing in a rather indirect manner using Stolz's theorem it follows that under the assumptions of 1.1 not only $\alpha(N)$, but also the Rosenberg index $\alpha(M) \in K_*(C^*\pi_1(M))$ is non-zero, if $\pi_1(M)$ satisfies the strong Novikov conjecture,

However, we have not been able to prove non-vanishing of the Rosenberg index $\alpha(M)$ in complete generality in the situation of Theorem 1.1. We leave this as an open question. In view of the possibility that $\alpha(M)$ could be zero, one might speculate whether Theorem 1.1 can be used in the end to establish counterexamples to the strong Novikov conjecture.

Remark 1.10. — We formulate and prove our theorem in the context of complex C^*-algebras and the complex Dirac operator. Firstly, this is most suited to the approach to coarse index theory as developed by Roe, and secondly the literature on self adjoint regular operators and their functional calculus is much more complete in this case. Nonetheless, we expect that all of our results can be generalized to real C^*-algebras and the real Dirac operator, which indeed furnish the most efficient context for geometric applications of the index theory of Dirac operators.

2. Regularity and self-adjointness of Dirac operators twisted with Hilbert-module bundles

In this section we prove in detail that twisted Dirac type operators on complete Riemannian manifolds have regular and self-adjoint closures. As a preparation, we recall some basics about Hilbert C^*-modules and regular, self-adjoint operators on them.

Let A be a (complex) C^*-algebra, which in our paper is assumed to be unital throughout. Recall that a Hilbert A-module is a right A-module with an A-valued inner product satisfying a number of axioms, see page 4 in [13], that serves as our main reference for the theory of Hilbert C^*-modules and unbounded operators. We emphasize that, in contrast to usual Hilbert spaces, a closed submodule $\mathcal{H}_0 \subset \mathcal{H}$ and the orthogonal submodule $\mathcal{H}_0^\perp \subset \mathcal{H}$ do not complement each other in general, i.e. usually $\mathcal{H}_0 \oplus \mathcal{H}_0^\perp \subsetneq \mathcal{H}$. If the opposite inclusion holds one says that $\mathcal{H}_0$ *has an orthogonal complement*.

Let $\mathcal{H}_1$ as well as $\mathcal{H}_2$ be Hilbert A-modules. An *operator* from $\mathcal{H}_1$ to $\mathcal{H}_2$ is an A-linear map $T\colon \mathrm{dom}(T) \to \mathcal{H}_2$ on a submodule $\mathrm{dom}(T)$ of $\mathcal{H}_1$. The latter is called the *domain* of T. One calls T *densely defined* if $\overline{\mathrm{dom}(T)} = \mathcal{H}_1$. An operator S from $\mathcal{H}_1$ to $\mathcal{H}_2$ is called an *extension* of T, written $T \subset S$, if $\mathrm{dom}(T) \subset \mathrm{dom}(S)$ and $Tx = Sx$ holds for each $x \in \mathrm{dom}(T)$. The *graph* of T is denoted by $G(T)$:

$$G(T) := \{(x, y) \in \mathcal{H}_1 \oplus \mathcal{H}_2\,;\, x \in \mathrm{dom}(T) \text{ and } y = Tx\}\,.$$

One calls T *closed* if $G(T)$ is closed in $\mathcal{H}_1 \oplus \mathcal{H}_2$. The operator T is called *closable* if it admits a closed extension. This is equivalent to the existence of an operator S with $G(S) = \overline{G(T)}$. In this case S is the smallest closed extension of T, called the *closure* and usually denoted $\overline{T}$ or $T_{\min}$. Its domain is

$$\mathrm{dom}(T_{\min}) = \\ \{x \in \mathcal{H}_1 \,|\, \exists\, (x_n)_{n\in\mathbb{N}} \subset \mathrm{dom}(T) \text{ with } x_n \to x \text{ and } Tx_n \to T_{\min}x\}\,.$$

For a densely defined operator $T\colon \mathrm{dom}(T) \to \mathcal{H}_2$ we set

$$\mathrm{dom}(T^*) := \{y \in \mathcal{H}_2 \,|\, \exists z \in \mathcal{H}_1 \text{ with } \langle Tx, y\rangle = \langle x, z\rangle\ \forall x \in \mathrm{dom}(T)\}\,.$$

The element z appearing on the right is unique and we can define the *adjoint operator* of T as the operator $T^*\colon \mathrm{dom}(T^*) \to \mathcal{H}_1$ given by $T^*y = z$. Note that T^* is a closed operator and that for a closable operator T, we have $T^* = \overline{T}^*$.

The operator T is called *adjointable*, if $\mathrm{dom}(T) = \mathcal{H}_1$ and $\mathrm{dom}(T^*) = \mathcal{H}_2$. Adjointable operators are automatically bounded, but bounded operators are not necessarily adjointable, see [13, p. 8]. Because every densely defined operator has an adjoint by definition, we will prefer the term *bounded adjointable* instead of *adjointable* in order to avoid any confusion. The space of bounded adjointable operators is denoted $\mathcal{L}_A(\mathcal{H}_1, \mathcal{H}_2)$, or briefly $\mathcal{L}_A(\mathcal{H}_1)$ if $\mathcal{H}_1 = \mathcal{H}_2$.

The subspace of *A-compact operators* is the closure of the A-linear span of operators of the form $x \mapsto \langle x, a\rangle\, b$ where $a \in \mathcal{H}_1$, $b \in \mathcal{H}_2$.

DEFINITION 2.1. — *Let* $T\colon \mathrm{dom}(T) \to \mathcal{H}_2$ *be an operator with* $\mathrm{dom}(T) \subset \mathcal{H}_1$*. One calls* T regular *if*

1. T *is densely defined and closed,*
2. T^* *is densely defined,*
3. *the graph of* T *(a closed subset of* $\mathcal{H}_1 \oplus \mathcal{H}_2$*) has an orthogonal complement.*

We now come to a useful criterion for regularity and self-adjointness. Recall that a densely defined operator T is called *symmetric*, if $T \subset T^*$ and *self-adjoint*, if $T = T^*$. Because T^* is a closed operator by [13, p. 95], symmetric operators are closable and self-adjoint operators are closed.

THEOREM 2.2 (Characterization of self-adjoint, regular operators). — *Let* T *be a closed, densely defined and symmetric operator on the Hilbert* A*-module* $\mathcal{H}$*. Then the following are equivalent:*

1. T *is self-adjoint and regular,*
2. $T + i$, $T - i$ *both have dense range.*

Proof. — By [13, Lemma 9.8], if T is self-adjoint and regular then $T \pm i$ both have dense range.

Conversely, assume $T \pm i$ both have dense range. By [13, Lemma 9.7], the assumptions imply that $T + i$ and $T - i$ are injective and have closed range. Therefore $T \pm i$ both are bijective (and in particular both operators have range $\mathcal{H}$).

As T is symmetric, T^* is an extension of T, and therefore $T^* \pm i$ are extensions of $T \pm i$. As already $T \pm i$ is surjective, T^* is a proper extension of T if and only if both operators $T^* \pm i$ have non-trivial kernel. But for $x \in \ker(T^* + i)$ and $y \in \mathrm{dom}(T)$ we have

$$0 = \langle (T^* + i)x, y\rangle = \langle x, (T - i)y\rangle,$$

and since $T - i$ is surjective, $x = 0$. Therefore $T^* = T$, i.e. the assumption implies that T is self-adjoint. Finally [13, Lemma 9.8] implies that T is also regular. □

2.1. Regularity and self-adjointness of twisted Dirac operators

Let (M, g) be a complete Riemannian manifold, let $S \to M$ be a complex Dirac bundle on M and let $D\colon \Gamma^\infty(M, S) \to \Gamma^\infty(M, S)$ be the corresponding Dirac type operator acting on the sections of S, see [21, Definition 3.4] or [14, Definition II.5.2]. The main examples we have in mind are the

Dirac operator of a Riemannian spin manifold, the de Rham operator of a general Riemannian manifold, the signature operator of an oriented Riemannian manifold, or the Dolbeault operator of a Kähler manifold.

In addition, let A be a unital complex C^*-algebra and E a smooth Hilbert A-module bundle whose fibers are finitely generated projective Hilbert A-modules, equipped with a metric connection ∇^E. We obtain the twisted Dirac operator D_E acting on smooth sections $\Gamma^\infty(M, S \otimes E)$. Note that the bundle $S \otimes E \to M$ inherits the structure of a Hilbert A-module bundle so that the Riemannian metric on M allows us to define an A-valued inner product on the space $\Gamma^\infty_{\mathrm{cpt}}(M, S \otimes E)$ of compactly supported smooth sections of this bundle. This inner product is given by the formula

$$\langle s_1, s_2 \rangle := \int_M \langle s_1(x), s_2(x) \rangle_{S_x \otimes E_x} \, d\lambda_g(x),$$

where λ_g is the measure associated with g. The corresponding completion is the Hilbert A-module $L^2(M, S \otimes E)$, by definition.

Theorem 2.3. — *Let (M, g) be a complete Riemannian manifold and let (E, ∇^E) be a smooth finitely generated projective Hilbert A-module bundle with metric connection, then*

$$D_E \colon \Gamma^\infty_{\mathrm{cpt}}(M, S \otimes E) \to \Gamma^\infty_{\mathrm{cpt}}(M, S \otimes E)$$

is closable in $L^2(M, S \otimes E)$ and the minimal closure is regular and self-adjoint as unbounded Hilbert A-module operator. It is the unique self-adjoint extension of D_E.

Proof. — Recall from the proof of [21, Proposition 3.11] that D_E with domain equal to $\Gamma^\infty_{\mathrm{cpt}}(M, S \otimes E)$ is symmetric and hence closable.

We first deal with the case when M is compact. In this context, Mishchenko and Fomenko [15] developed a pseudodifferential calculus for operators on smooth sections of $S \otimes E$ which the following properties, among others:

1. The identity is an operator of order 0 in the pseudodifferential calculus.
2. The operator D_E is an operator of order 1 in the pseudodifferential calculus.
3. The operator D_E is has a parametrix, i.e. there are operators Q of order -1 and R, T of order $-\infty$ in the calculus such that $D_E Q = 1 - R$ and $Q D_E = 1 - T$.
4. Each operator P of order $\leqslant 0$ in the pseudodifferential calculus extends (uniquely) to a bounded adjointable operator $\overline{P}$ on $L^2(M, S \otimes E)$.

(5) If P is an operator of order < 0 in the calculus then its bounded adjointable extension is an A-compact operator.
(6) Each operator in the calculus has a formal adjoint of the same order, for D_E the formal adjoint is D_E.
(7) If P is an operator of order $\leqslant 0$ in the calculus, then its adjoint, which is necessarily equal to the adjoint of its closure, is the closure of its formal adjoint.

Now it turns out that these are exactly the properties needed for a proof of regularity and self-adjointness of the closure of D_E, first given, to our knowledge, in [33, Proposition 3.4.9]. Alternatively, in [26, Section 1] an argument due to Skandalis for the statement is sketched which even works in the case of Lipschitz manifolds. As the thesis [33] is only available on a university homepage without permanent link, we repeat this proof here for the reader's convenience.

It is based on the following three properties:

(1) $\overline{D_E} \circ \overline{Q} = \overline{D_E \circ Q}$ and $\overline{D_E} \circ \overline{T} = \overline{D_E \circ T}$, and these operators are bounded (hence everywhere defined),
(2) $\operatorname{dom}(\overline{D_E}) = \operatorname{im}(\overline{Q}) + \operatorname{im}(\overline{T})$,
(3) $\overline{D_E} = (D_E)^*$, in particular $\overline{D_E}$ is self-adjoint.

To establish $\overline{D_E Q} \subset \overline{D_E} \circ \overline{Q}$ let x be in the domain of $\overline{D_E Q}$. By definition, this means there are smooth sections x_n converging to x such that $D_E Q(x_n)$ converges to $y := \overline{D_E Q}(x)$. Now, Q has negative order and therefore a bounded closure $\overline{Q}$ and hence $\lim_{n\to\infty} Q x_n = z = \overline{Q}(x)$. By definition of $\overline{D_E}$, z is in the domain of D_E and $\overline{D_E}(z) = y$. So, indeed $\overline{D_E Q} \subset \overline{D_E} \circ \overline{Q}$. Now $D_E Q$ has order zero and therefore its closure is bounded, in particular everywhere defined, so that $\overline{D_E Q}$ has no proper extension and therefore we have the required equality.

This argument also shows that $\overline{D_E T} = \overline{D_E} \circ \overline{T}$. This finishes the proof of (1).

For (2), if x is in the domain of $\overline{D_E}$ then by definition there are smooth sections x_n such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} D_E x_n = y = \overline{D_E}(x)$. Then $Q D_E x_n = x_n - T x_n$ where Q and T have continuous closures, and hence, passing to the limits

$$\overline{Q}(\overline{D_E}(x)) = x - \overline{T}(x)$$

or in other words $x = \overline{Q}(y) + \overline{T}(x) \in \operatorname{im}(\overline{Q}) + \operatorname{im}(\overline{T})$.

Conversely, (1) implies that $\operatorname{im}(\overline{Q}) \subset \operatorname{dom}(\overline{D_E})$ and $\operatorname{im}(\overline{T}) \subset \operatorname{dom}(\overline{D_E})$.

To prove (3), the self-adjointness of the operator $\overline{D_E}$, recall that by symmetry $\overline{D_E} \subset (D_E)^*$. For the converse inclusion, recall that $D_E Q = 1-R$

and therefore $(D_E Q)^* = 1 - R^*$. For adjoints of compositions one always has $Q^* D_E^* \subset (D_E Q)^*$. Because Q^* is the closure of the formal adjoint of Q, which is bounded as it is of negative order, $\mathrm{dom}(D_E^*) = \mathrm{dom}(Q^* D_E^*)$ and for $x \in \mathrm{dom}(D_E^*)$ we have (using $Q^* D_E^* \subset (D_E Q)^* = 1 - R^*$)

$$Q^*(D_E^*(x)) + R^* x = x,$$

so $\mathrm{dom}(D_E^*) \subset \mathrm{im}(Q^*) + \mathrm{im}(R^*)$.

Taking the formal adjoint of the parametrix equations $D_E Q = 1 - R$, $QD_E = 1 - T$, we see that also the formal adjoint of Q is a parametrix of D_E, with error terms the formal adjoints of R, T, but with the roles of R and T exchanged in the parametrix equations. Because the closures of the formal adjoints of Q and R are equal to Q^* and R^* by property (vii) of the functional calculus, an argument analogous to the one employed for (1) shows that $\mathrm{im}(Q^*) + \mathrm{im}(R^*) \subset \mathrm{dom}(\overline{D_E})$. Hence we altogether have $\mathrm{dom}(D_E^*) \subset \mathrm{dom}(\overline{D_E})$.

It remains to prove the regularity of $\overline{D_E}$, i.e. we have to show that its graph is complemented. Write $\mathcal{H}$ for the Hilbert module of L^2-sections of $S \otimes E$. By (1) and (2),

$$G(\overline{D_E}) = \{(\overline{Q}x + \overline{T}y, \overline{D_E Q}x + \overline{D_E T}y) \mid (x,y) \in \mathcal{H} \times \mathcal{H}\} = U(\mathcal{H} \oplus \mathcal{H})$$

where $U\colon \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ is the bounded adjointable operator with $U(x,y) = (\overline{Q}x + \overline{T}y, \overline{D_E Q}x + \overline{D_E T}y)$, using that $Q, T, D_E Q, D_E T$ are all operators of non-positive order. As a graph of a closure, $U(\mathcal{H} \oplus \mathcal{H})$ is closed. By [13, Theorem 3.2], the image of U and hence the graph of $\overline{D_E}$ has an orthogonal complement.

Now we treat the general case where M is complete, but not compact. We will reduce this case to the compact one. For this we use Theorem 2.2 to show that $\overline{D_E}$ is self-adjoint and regular. The argument is inspired by the proof of essential self-adjointness of the untwisted Dirac operator on complete manifolds in [3, Proposition 1.3.5] and by the treatment of [35, Lemma 2.1]. We will use three basic and well-known features:

(1) Given a compact submanifold $K \subset M$ with boundary, there exists a closed Riemannian manifold (M', g') equipped with a finitely generated projective Hilbert A-module bundle $(E', \nabla^{E'})$ with metric connection, a Dirac bundle $(S', \nabla^{S'})$, a submanifold $K' \subset M'$ and a diffeomorphism $\psi\colon K \to K'$ such that ψ preserves all the structure (restricted to K and K', respectively). Specifically, $\psi^* g'|_{K'} = g_{|K}$ and $(\psi^* E', \psi^* \nabla^{E'}) \cong (E, \nabla^E)_{|K}$ as bundles with connections and $(\psi^* S', \psi^* \nabla^{S'}) \cong (S, \nabla^S)_{|K}$ as Dirac bundles.

(2) Because M is complete, for each compact subset $K \subset M$ and each $\varepsilon > 0$ there is a smooth function $\phi\colon M \to [0,1]$ with compact support such that $\phi_{|K} = 1$ and such that $\|\mathrm{grad}(\phi)\|_\infty \leqslant \varepsilon$.

(3) For each smooth function $\phi\colon M \to \mathbb{R}$ with compact support, the commutator of multiplication by ϕ and D_E extends to a bounded operator on $L^2(M, S \otimes E)$ with norm bounded by $\|\mathrm{grad}(\phi)\|_\infty$. More precisely, the commutator is given by Clifford multiplication with $\mathrm{grad}(\phi)$.

Let $s \in \Gamma^\infty_{\mathrm{cpt}}(M, S \otimes E)$. For given $\varepsilon > 0$ choose a function $\phi\colon M \to [0,1]$ with compact support which is identically equal to 1 on the support of s and with $\|\mathrm{grad}(\phi)\|_\infty \leqslant \varepsilon$. Then choose a compact Riemannian manifold (M', g') and bundles $(E', \nabla^{E'})$ as well as $(S', \nabla^{S'})$ with an isometry $\psi\colon K \to K'$ where K is a compact manifold with boundary containing the 1-neighborhood of the support of ϕ.

In the sequel functions and sections with support in $K \subset M$ or the corresponding set K' in M' will be transported back and forth using this isometry without further comment. For example, we interchangeably think of ϕ as a function on M' and s as a section of $S' \otimes E'|_{M'}$.

Because M' is compact the closure of the twisted Dirac operator $D_{E'}$ on M' acting on sections of $S' \otimes E'$ is already shown to be regular and self-adjoint. Hence by Theorem 2.2 we can find an element $x \in \mathrm{dom}(\overline{D_{E'}})$ such that $(\overline{D_{E'}} + i)x = s$. We obtain

$$\langle s, s\rangle = \langle (\overline{D_E} + i)x, (\overline{D_E} + i)x\rangle = \langle \overline{D_E}x, \overline{D_E}x\rangle + \langle x, x\rangle \geqslant \langle x, x\rangle \in A_+.$$

Now ϕx is defined on M and belongs to $\mathrm{dom}(\overline{D_E})$ by (3). Moreover,

$$(\overline{D_E} + i)(\phi\, x) = (\overline{D_{E'}} + i)(\phi x) = [\overline{D_{E'}}, \phi]x + \phi\,(\overline{D_{E'}} + i)x\ .$$

Here, now $\phi\,(\overline{D_{E'}} + i)x = \phi\, s = s$. On the other hand, $\|[\overline{D_{E'}}, \phi]\| \leqslant \|\mathrm{grad}(\phi)\|_\infty \leqslant \varepsilon$ so that $\|[\overline{D_{E'}}, \phi]x\| \leqslant \varepsilon\|x\| \leqslant \varepsilon\|s\|$.

It follows that s lies in the closure of the image of $\overline{D_E} + i$ and therefore that $\overline{D_E} + i$ has dense range as $\Gamma^\infty_{\mathrm{cpt}}(M, S \otimes E)$ is dense in $L^2(M, S \otimes E)$. In the same way it is shown that $\overline{D_E} - i$ has dense range. This implies the theorem by Theorem 2.2. □

3. Positive scalar curvature, partial vanishing, and coarse index

We now introduce the coarse index of the Dirac operator D_E on a complete spin manifold (M, g), twisted by a smooth Hilbert A-module bundle

E, and prove the vanishing result Theorem 1.7. For simplicity we will use the notation D_E for the densely defined, self-adjoint and regular closure $\overline{D_E}$ of D_E, see Theorem 2.3.

3.1. The coarse index

The construction of the index is based on the functional calculus for regular and self-adjoint operators on Hilbert A-modules from [13, Chapter 9 and 10] and [12, Section 3]. We will first recall this functional calculus in a form needed for our purpose.

Theorem 3.1 (Continuous functional calculus). — *Let $C(\mathbb{R})$ be the $*$-algebra of continuous complex valued functions on $\mathbb{R}$. Let T be a (possibly unbounded) regular, self adjoint operator on the Hilbert A-module $\mathcal{H}$. Then there is a $*$-preserving linear map*

$$\pi_T \colon C(\mathbb{R}) \to \mathcal{R}_A(\mathcal{H}) \,,\; f \mapsto f(T)$$

with values in the set of regular operators on $\mathcal{H}$, which has the following properties.

- *π_T restricts to a C^*-algebra homomorphism $\pi_T \colon C_b(\mathbb{R}) \to \mathcal{L}_A(\mathcal{H})$ on the set $C_b(\mathbb{R})$ of bounded complex valued functions on $\mathbb{R}$.*
- *If $|f| \leqslant |g|$, then $\mathrm{dom}(g(T)) \subset \mathrm{dom}(f(T))$.*
- *(Strong continuity) If $(f_n)_{n\in\mathbb{N}}$ is a sequence in $C(\mathbb{R})$ which is dominated by $F \in C(\mathbb{R})$, i.e. $|f_n| \leqslant |F|$ for all n, and if $f_n \to f$ uniformly on compact subsets of $\mathbb{R}$, then $\pi_T(f_n)x \to \pi_T(f)x$ for each $x \in \mathrm{dom}(F(T))$.*
- *$\pi_T(\mathrm{Id}) = T$.*
- *If $f \in C_b(\mathbb{R})$ and $F \in C(\mathbb{R})$ is defined by $F(t) = t \cdot f(t)$, then $\mathrm{dom}(T) \subset \mathrm{dom}(F(T))$ and for all $x \in \mathrm{dom}(T)$ we have $F(T)x = Tf(T)x = f(T)Tx$. If F is bounded, then $\mathrm{im}(f(T)) \subset \mathrm{dom}(T)$, and we have $F(T) = Tf(T)$ as bounded operators on $\mathcal{H}$.*

Let (M, g) be a complete Riemannian spin manifold and (E, ∇) a smooth Hilbert A-module bundle on M with metric connection and with finitely generated projective fibers. We will now define the coarse index $\mathrm{ind}(D_E)$, following [20]. It is an element of $K_*(C^*(M; A))$, the K-theory of the coarse C^*-algebra $C^*(M; A)$ of M with coefficients in A. This C^*-algebra was introduced in [8] and its definition will be recalled shortly. The definition of $\mathrm{ind}(D_E)$ uses the functional calculus for self-adjoint (unbounded) Hilbert A-module operators in Theorem 3.1.

We will work with the Hilbert A-module $\mathcal{H} := L^2(M, S \otimes E)$, on which $C_0(M)$, the C^*-algebra of all complex valued continuous functions on M vanishing at infinity, acts by pointwise multiplication. The corresponding representation is denoted $\rho\colon C_0(M) \to \mathcal{L}_A(\mathcal{H})$. The following definition generalizes the corresponding notions from [20, Chapter 3] to the Hilbert A-module $\mathcal{H}$.

Definition 3.2. — *Let $T \in \mathcal{L}_A(\mathcal{H})$.*

- *T is* locally compact *if $T \circ \rho(f)$ and $\rho(f) \circ T$ are A-compact operators for all $f \in C_0(M)$.*
- *T is called* pseudolocal *if the commutator $[T, \rho(f)]$ is A-compact for any $f \in C_0(M)$.*
- *T has* finite propagation *if there exists $R > 0$ such that $\rho(f) \circ T \circ \rho(g)$ vanishes for all $f, g \in C_0(M)$ with $d(\mathrm{supp}(f), \mathrm{supp}(g)) \geqslant R$. In this case we say that T has* propagation bounded by *R.*
- *The* Roe C^*-algebra *associated with ρ is the sub-C^*-algebra of $\mathcal{L}_A^*(\mathcal{H})$ generated by all locally compact operators with finite propagation. It will be denoted by $C^*(M; A)$.*
- *If $X \subset M$ is closed, we define $C^*(X \subset M; A)$ as the closed ideal of $C^*(M; A)$ generated by locally compact operators T of finite propagation which are* supported near *X, i.e. such that there is $R > 0$ with $T\rho(f) = 0$ and $\rho(f)T = 0$ for all $f \in C_0(M)$ with $d(\mathrm{supp}(f), X) \geqslant R$.*

Remark 3.3. — We suppress the dependence on the bundle $S \otimes E$ in the notation $C^*(M; A)$. This is justified by the following functoriality results [8, Lemma 5.4, Proposition 5.5].

For any Lipschitz map $f\colon M \to N$ and Hilbert A-module bundles $E \to M$, $F \to N$ so that the fibers of $F \to N$ are large enough - adding a trivial bundle will always suffice if M has positive dimension - there are canonical C^*-algebra homomorphisms $f_*\colon C^*(M; A) \to C^*(N; A)$, obtained by conjugation with an isometry between the Hilbert A-modules of sections of these bundles. The induced map on K-theory is functorial in f. For $f = \mathrm{id}_M\colon M \to M$ we can arrange that f_* is an isomorphism, if the fibers of $E \to M$ and $F \to N$ are large enough.

Proposition 3.4. — *Using the functional calculus of Subsection 3.1 we define the wave operator group $\{\exp(isD_E)\}_{s\in\mathbb{R}}$ which consists of unitary operators. It satisfies the wave equation: for $u \in \mathrm{dom}(D_E)$,*

$$\frac{d}{ds} \exp(isD_E)u = iD_E \exp(isD_E)u.$$

Moreover, each $\exp(isD_E)$ *is a finite propagation operator with propagation* $|s|$.

Proof. — Because the function $t \to \exp(ist)$ is bounded on $\mathbb{R}$, the operators $\exp(isD_E)$ are bounded adjointable and unitary by the properties of the functional calculus Theorem 3.1.

For fixed $s \in \mathbb{R}$, $\big(\exp(i(s+h)t) - \exp(ist)\big)/h$ converges to $it\exp(ist)$ uniformly for $t \in [-R, R]$ for each R and with a uniform bound of the difference quotients by $|1+t|$. The claim about the wave equation then follows from the strong continuity property in Theorem 3.1.

The unit propagation property is a standard fact which follows from a priori energy estimates. The proof given in [9, Proposition 10.3.1] only uses properties of the wave equation, some elementary properties of the functional calculus and the fact that, for a smooth function $g\colon M \to \mathbb{R}$, the commutator $[D_E, \rho(g)]$ is equal to Clifford multiplication with the gradient of g. It therefore generalizes immediately from the case of operators on Hilbert spaces treated in [9, Proposition 10.3.1] to the unbounded operator D_E on the Hilbert A-module $\mathcal{H}$. □

Definition 3.5. — *An odd function* $\chi \in C(\mathbb{R})$ *is called* normalizing function *if* $\chi(t) \to \pm 1$ *as* $t \to \pm\infty$.

Lemma 3.6. — *For the functional calculus of the regular self-adjoint operator* D_E *the following assertions hold.*

a) *For any* $\varphi \in C^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ *with* $\hat\varphi \in C^\infty_{\mathrm{cpt}}(\mathbb{R})$, *one has*

$$\varphi(D_E)u = \frac{1}{2\pi}\int_{\mathbb{R}} \hat\varphi(s)\, \exp(isD_E)u\, ds$$

for all compactly supported smooth sections $u \in \Gamma^\infty_{\mathrm{cpt}}(M, S\otimes E)$. *Further, the operator* $\varphi(D_E)$ *is locally compact and of finite propagation.*

b) *For arbitrary* $\varphi \in C_0(M)$ *we have* $\varphi(D_E) \in C^*(M; A)$.

c) *If* $\chi \in C_b(\mathbb{R})$ *is a normalizing function then* $\chi(D_E)$ *is a norm limit of bounded, self-adjoint finite propagation operators.*

Proof. — For a) we first assume that $\mathrm{supp}(\hat\varphi) \subset [-R, R]$ for $R > 0$. By Theorem 3.1 the compactly supported integrand is continuous and the integral is defined as a limit of Riemann sums. Moreover we have $\varphi(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat\varphi(s)\exp(ist)\, ds$, again as a limit of Riemann sums (and uniformly for t in compact subsets of $\mathbb{R}$), by the Fourier inversion theorem. The equation in a) now follows from the continuity statement in Theorem 3.1.

That $\varphi(D)$ has finite propagation is an immediate consequence of the integral representation of this operator, since the wave operators $\exp(isD_E)$ have finite propagation $|s|$.

To prove local compactness of $\varphi(D_E)$, let f be a compactly supported smooth function on M. Note that the propagation of $\varphi(D_E)$ is bounded by R. Let g be a compactly supported smooth function which is identically equal to 1 on the R-neighborhood of the support of f. Then $\varphi(D_E)\rho(f) = \rho(g)\varphi(D_E)\rho(f)$ and $\rho(f)\varphi(D_E) = \rho(f)\varphi(D_E)\rho(g)$.

Next, as in Subsection 2.1, when reducing from complete to compact manifolds, we can find an isometry of a suitable neighborhood of $\mathrm{supp}(g)$ with target a suitable subset of a compact manifold M_1, covered by an isometry of E to a Hilbert A-module bundle E_1 on M_1, when both bundles are restricted to the respective subsets of M and M_1.

This induces an isometry which conjugates $\rho(g)\varphi(D_E)\varphi(f)$ to the corresponding operator $\rho(g_1)\varphi(D_{E_1})\rho(f_1)$ on M_1. This assertion uses the integral representation and the fact that the family $\exp(isD_E)u$ on M is conjugated to the corresponding family $\exp(isD_{E_1})u_1$ on M_1 as long as $\mathrm{supp}(u) \subset \mathrm{supp}(f)$ and $|s| \leqslant R$. Here we observe that the latter is the unique solution of the wave equation for a given initial function u, which follows immediately from the a priori energy estimates for the wave operator mentioned in the proof of Proposition 3.4.

Now we use the parametrix Q_1 for D_{E_1} of order -1 with remainder R_1 of order $-\infty$ such that $D_{E_1}Q_1 = 1 - R_1$. Composing with the bounded operator $\varphi(D_E)$ from the left leads to the equation

$$\varphi(D_{E_1}) = (\varphi(D_{E_1})D_{E_1}) \circ Q_1 + \varphi(D_{E_1}) \circ R_1.$$

Here $\varphi(D_{E_1})D_{E_1}$ which is defined a priori only on $\mathrm{dom}(D_{E_1})$, can be extended to a bounded operator on $\mathcal{H}$, as $t \mapsto t\varphi(t)$ is bounded, and Q_1, R_1 are A-compact because they are of negative order in the pseudo-differential calculus on M_1. Consequently, since the A-compact operators are an ideal in the bounded operators also $\rho(g_1)\varphi(D_{E_1})\rho(f_1)$ and its conjugate $\rho(g)\varphi(D_E)\rho(f)$ are A-compact. The same argument implies that $\rho(f)\varphi(D_E)\rho(g)$ is A-compact.

The claim about arbitrary $\varphi \in C_0(M)$ follows from the usual density argument.

We now prove c). The function $f(x) = \frac{x}{\sqrt{1+x^2}}$ is a normalizing function, any other such function χ satisfies $\chi - f \in C_0(\mathbb{R})$. Because of b) it suffices to prove the statement for f. Now, $f(x) = xg(x)$ with $g(x) = (1+x^2)^{-1/2}$. We construct a sequence of bounded continuous functions g_n such that the functions $x \mapsto xg_n(x)$ are also bounded and

(1) $\lim_{n\to\infty} \|xg_n(x) - xg(x)\|_\infty = 0$.
(2) g_n has a smooth Fourier transform with compact support.

Then the sequence of bounded operators $D_E g_n(D_E)$ converges by the functional calculus in norm to $D_E g(D_E) = f(D_E)$. Moreover, $g_n(D_E)$ has finite propagation exactly by the same Fourier inversion argument which showed that $\varphi(D_E)$ has finite propagation. As D_E itself has propagation 0, the composition $D_E g_n(D_E)$ also has finite propagation. Hence assertion c) holds with χ replaced by f.

To construct g_n, consider first the Fourier transform $\hat{g}(\xi)$. This is, up to a constant, the modified Bessel function $K_0(|\xi|)$ of the second kind [1, p. 376]. The following Lemma 3.7 shows that this function is square integrable, smooth outside 0 and of Schwartz type as $\xi \to \pm\infty$, meaning that $\lim_{|\xi|\to\infty} |\xi^k \frac{d^l \hat{g}}{d\xi^l}(\xi)| = 0$ for all k, l. Choose smooth cutoff functions $\phi_n \colon \mathbb{R} \to [0,1]$ with $\|\phi_n^{(k)}\|_\infty \leqslant 1$ for $k = 0, 1$ and such that $\phi_n(\xi) = 1$ for $|\xi| \leqslant n$. Set $\hat{g}_n = \phi_n \hat{g}$ and let g_n be the Fourier transform of $\hat{g}_n$. Being equal to the convolution of the L^2-function g and the Schwartz function $\hat{\phi}_n$ the function g_n is bounded and continuous. We obtain

$$|x(g(x) - g_n(x))| \leqslant \int_{\mathbb{R}} |(\hat{g} - \hat{g}_n)'(\xi)|\, d\xi \leqslant \int_{|\xi|\geqslant n} |(\hat{g}(1-\phi_n))'(\xi)|\, d\xi \xrightarrow{n\to\infty} 0$$

as $\hat{g}(\xi)$ is rapidly decreasing for $|\xi| \to \infty$. Furthermore, the last inequality also shows that $x \mapsto xg_n$ is bounded for all n. □

Lemma 3.7. — *Set $g(x) = \frac{1}{\sqrt{1+x^2}}$. Then $g \in L^2(\mathbb{R})$ and its Fourier transform $\hat{g}$ has the following properties:*

(1) *For each $k > 0$ and each $0 \leqslant l < k$ the function $\frac{d^l}{d\xi^l}(\xi^k \hat{g}(\xi))$ belongs to $L^2(\mathbb{R})$.*
(2) *The restriction of $\hat{g}$ to $\mathbb{R} \setminus \{0\}$ is smooth*
(3) *The restriction of $\xi^k \frac{d^l}{d\xi^l}\hat{g}$ to $\mathbb{R} \setminus (-1,1)$ is bounded for each $k, l \in \mathbb{N}$.*

Proof. — An explicit calculation shows that $x^l \frac{d^k}{dx^k} g(x)$ belongs to $L^2(\mathbb{R})$ for $l < k$ and to $L^1(\mathbb{R})$ for $l < k+1$.

For the Fourier transforms, we therefore get that $\frac{d^l}{d\xi^l}\left(\xi^k \hat{g}\right)$ belongs to $L^2(\mathbb{R})$ for $l < k$ and to $L^\infty(\mathbb{R})$ for $l < k+1$. The Sobolev embedding theorem then implies that $\xi^k \hat{g}(\xi)$ belongs to $C^{k-1}(\mathbb{R})$. As ξ^k is smooth and invertible outside the origin, this implies that $\hat{g}$ is smooth on $\mathbb{R} \setminus \{0\}$. Calculating $\frac{d^l}{d\xi^l}(\xi^k \hat{g})$ with the product rule, by induction on l we establish that the restriction of $\xi^k \frac{d^l}{d\xi^l}\hat{g}$ to $\mathbb{R} \setminus (-1,1)$ is bounded for each k, l. □

Let $\mathcal{M}$ be the sub-C^*-algebra of $\mathcal{L}_A(\mathcal{H})$ generated by all operators with finite propagation. Then $C^*(M;A) \subset \mathcal{M}$ is a C^*-ideal (for the ideal property use an argument similar to the third paragraph in the proof of Lemma 3.6). Consider the associated six-term exact sequence

$$\begin{array}{ccccc} K_0(C^*(M;A)) & \longrightarrow & K_0(\mathcal{M}) & \longrightarrow & K_0(\mathcal{M}/C^*(M;A)) \\ \partial_1 \big\uparrow & & & & \big\downarrow \partial_0 \\ K_1(\mathcal{M}/C^*(M;A)) & \longleftarrow & K_1(\mathcal{M}) & \longleftarrow & K_1(C^*(M;A)) \end{array} \tag{3.1}$$

Definition 3.8. — *If* $\dim(M)$ *is odd,* $\frac{1+\chi(D_E)}{2}$ *belongs to* $\mathcal{M}$ *and is a projection modulo* $C^*(M;A)$*, as* $\chi(D_E)^2 - I \in C^*(M;A)$*. The* coarse index $\operatorname{ind}(D_E)$ *is then defined as*

$$\operatorname{ind}(D_E) := \partial_0[\frac{1}{2}(1+\chi(D_E))] \in K_1(C^*(M;A)).$$

If $\dim(M)$ *is even, the decomposition of the spinor bundle* $S \to M$ *in even and odd parts induces a decomposition* $L^2(M, S \otimes E) = \mathcal{H}_0 \oplus \mathcal{H}_1$ *for which*

$$D_E = \begin{bmatrix} 0 & D_1 \\ D_0 & 0 \end{bmatrix}.$$

Then $U^*\chi(D_E)_0$ *belongs to* $\mathcal{M}$*, where* $U\colon \mathcal{H}_0 \to \mathcal{H}_1$ *is a unitary embedding*[(1)] $\cdot \operatorname{id}_M$*, and* $U^*\chi(D_E)_0$ *is unitary modulo* $C^*(M;A)$ *as* $\chi^2(D_E) - I \in C^*(M;A)$*, i.e.* $U^*\chi(D_E)_0$ *represents an element in* $K_1(\mathcal{M}/C^*(M;A))$*.*

Then the coarse index $\operatorname{ind}(D_E)$ *is defined as*

$$\operatorname{ind}(D_E) := \partial_1[U^*\chi(D_E)_0] \in K_0(C^*(M;A)).$$

3.2. The vanishing theorem

The following vanishing theorem generalizes an analogous result from [20, Proposition 3.11 and the following Remark], [19] and [34, Proposition 4] for the spin Dirac operator to the case of the spin Dirac operator twisted with a flat Hilbert A-module bundle.

Definition 3.9. — *Let* M *be a complete Riemannian manifold and* A *a unital* C^**-algebra. A closed subset* $X \subset M$ *is called* coarsely A-negligible *if the inclusion induces the zero homomorphism* $0 = i_*\colon K_*(C^*(X \subset M;A)) \to K_*(C^*(M;A))$*.*

---

(1) See [10, p. 91 f] for the definition and a proof for the existence of such an isometry. This notion is used for the functoriality of Remark 3.3

Note that, by functoriality, every closed subset of a coarsely A-negligible set is itself coarsely A-negligible. Secondly, note that, by definition, $C^*(U_R(X) \subset M; A) = C^*(X \subset M; A)$ for any closed R-neighborhood $U_R(X)$ of X.

Proposition 3.10. — *If M is a complete connected non-compact Riemannian manifold then every compact subset $K \subset M$ is coarsely A-negligible for any A.*

Proof. — Choose an isometric embedding $\gamma\colon \mathbb{R}_+ \to M$. This is possible because M is complete, connected and non-compact, see [2, p. 92]. Then, because a compact set has finite diameter, $K \subset U_R(\gamma(\mathbb{R}_+))$ for R sufficiently large.

It remains to show that $K_*(C^*(\gamma(\mathbb{R}_+) \subset M; A)) = 0$, so that $\gamma(\mathbb{R}_+)$ is A-negligible. This follows from $K_*(C^*(\gamma(\mathbb{R}_+) \subset M; A)) \cong K_*(\mathbb{R}_+; A)$ by [28, Proposition 2.9] and $K_*(\mathbb{R}_+; A) = 0$ by an Eilenberg swindle argument as carried out in [20, Proposition 9.4]. Compare [28, Proposition 2.6] for the generalization to Hilbert A-module coefficients. ☐

Theorem 3.11. — *Let (M, g) be a complete Riemannian spin manifold with uniformly positive scalar curvature outside of an A-negligible set X. Let $E \to M$ be a smooth finitely generated projective Hilbert A-module bundle equipped with a flat metric connection. Then the coarse index $\operatorname{ind}(D_E) \in K_*(C^*(M; A))$ of the twisted Dirac operator of (M, g) vanishes.*

In particular, if M is non-compact connected and has uniformly positive scalar curvature outside a compact set, then $\operatorname{ind}(D_E) = 0$ for any flat Hilbert A-module bundle E as above.

Proof. — We use the notation from diagram (3.1) and consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^*(M;A) & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{M}/C^*(M;A) & \longrightarrow & 0 \\ & & {\scriptstyle i_*}\uparrow & & \| & & \uparrow & & \\ 0 & \longrightarrow & C^*(X \subset M;A) & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{M}/C^*(X \subset M;A) & \longrightarrow & 0 \end{array} \tag{3.2}$$

In Proposition 3.15 below we will construct a normalizing function χ for which $\chi(D_E)^2$ equals I modulo $C^*(X \subset M; A)$. This implies that $[\frac{1}{2}(1+\chi(D_E))]$ lifts to $K_0(C^*(X \subset M; A))$ (if $\dim M$ is odd) and $[U^*\chi(D_E)_0]$ lifts to $K_1(C^*(X \subset M; A))$ (if $\dim M$ is even).

Using the commutativity of

$$\begin{array}{ccc} K_{*+1}(\mathcal{M}/C^*(X \subset M; A)) & \xrightarrow{\partial_{*+1}} & K_*(C^*(X \subset M; A)) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_* = \mathbf{0}} \\ K_{*+1}(\mathcal{M}/C^*(M; A)) & \xrightarrow{\partial_{*+1}} & K_*(C^*(M; A)) \end{array}$$

in the six-term exact sequence, we obtain the desired result by the A-negligibility of X. ☐

To prove that $I - \chi(D_E)^2 \in C^*(X \subset M; A)$ for a suitable χ, we use the following criterion.

Lemma 3.12. — *An operator $T \in C^*(M; A)$ belongs to $C^*(X \subset M; A)$ for a closed subset X if and only if for each $\varepsilon > 0$ there is $R > 0$ such that for each $u \in L^2(M, S \otimes E)$ with support outside the R-neighborhood $U_R(X)$ we have*

$$\|Tu\| \leqslant \varepsilon \, \|u\|. \tag{3.3}$$

Proof. — If T is a norm limit of operators which are supported in R-neighborhoods of X, the inequality (3.3) obviously holds.

Conversely, write $T = \lim T_n$ with operators T_n which are locally compact and of finite propagation. Using the estimate, we have to modify the the operators T_n such that they are in addition supported in a bounded neighborhood of X. For this we choose cutoff functions $\phi_n \colon M \to [0, 1]$ which are supported in $U_{2n}(X)$ and which are identically equal to 1 in $U_n(X)$. Then $T_n\rho(\phi_n)$ are still locally compact, of some finite propagation P_n, and, in addition, supported in $U_{2n+P_n}(X)$, i.e. $(T_n\rho(\phi_n))\rho(f) = 0$, if $\operatorname{supp}(f) \cap U_{2n+P_n}(X) = \emptyset$.

We only have to show that T_n and $T_n\rho(\phi_n)$ are close in operator norm if n is sufficiently large. For $\varepsilon > 0$ choose R as in the assumption and $n > R$. Then

$$\|(T_n - T_n\rho(\phi_n))u\| = \|T_n(1 - \rho(\phi_n))u\| \leqslant \varepsilon\|(1 - \rho(\phi_n))u\| \leqslant \varepsilon\|u\|,$$

for each $u \in L^2(M, S \otimes E)$, as $(1 - \rho(\phi_n))u$ has support outside the R-neighborhood of X. Therefore $\|T_n - T_n\rho(\phi_n)\| \leqslant \varepsilon$. ☐

Before we can prove the required Proposition 3.15, we need two further preparatory lemmas. The first is a standard property of the Fourier transform, its proof is left to the reader.

Lemma 3.13. — *Let $f \in C^\infty_{\mathrm{cpt}}(\mathbb{R})$. Then for each $\delta > 0$ there exists a smooth L^1-function f_δ with compactly supported Fourier transform and such that for all $x \in \mathbb{R}$ and for $j = 0, 1, 2$ we have $|x^j\,(f(x) - f_\delta(x))| \leqslant \delta$.*

Lemma 3.14. — *Let $f \in C^\infty_{\mathrm{cpt}}(\mathbb{R})$ with $f \geqslant 0$. Then for each $\varepsilon > 0$ there exists a decomposition $f = f_\varepsilon + g_\varepsilon$ and $S(\varepsilon) > 0$ with the following properties:*

1. *$f_\varepsilon = F_\varepsilon^2$ with F_ε a smooth L^1-function and $\operatorname{supp}(\hat{F}_\varepsilon) \subset [-S(\varepsilon), S(\varepsilon)]$.*
2. *$\sup\{|x^j\, g_\varepsilon(x)|\,;\, x \in \mathbb{R}\} \leqslant \varepsilon$ for each $j = 0, 1, 2$.*

Proof. — Set $F := f^{1/2}$, choose $A > 0$ with $\operatorname{supp}(F) \subset [-A, A]$. Let $\varepsilon > 0$. Approximate F by $H \in C^\infty(\mathbb{R})$ with $\|F - H\| \leqslant \frac{\varepsilon}{(A+1)^2 \cdot 4(\|F\|+1)}$, $\operatorname{supp}(H) \subset [-A-1, A+1]$ and $\|H\| \leqslant \|F\| + 1$. Here and in the remainder of the proof we use the maximum norm on $C^\infty_{\mathrm{cpt}}(\mathbb{R})$.

Choose $\delta \leqslant \frac{\varepsilon}{4(\|F\|+1)}$, $\delta \leqslant 1$. By Lemma 3.13, H admits a decomposition $H = H_\delta + R_\delta$ where H_δ is a smooth L^1-function with compactly supported Fourier transform, such that $\sup|x^j R_\delta(x)| \leqslant \delta$ for $j = 0, 1, 2$, and $\|H_\delta\| \leqslant \|H\|$ and with $\operatorname{supp}(\hat{H}_\delta) \subset [-R(\delta), R(\delta)]$ for suitable $R(\delta) > 0$. The estimate on R_δ implies $\|H_\delta\| \leqslant \|H\| + \delta$.

Set $f_\varepsilon := H_\delta^2$ and $F_\varepsilon := H_\delta$. Then (i) holds with $S(\varepsilon) := R(\delta)$. Finally, we obtain (ii) from the following estimate for $j = 0, 1, 2$

$$
\begin{aligned}
|x^j (f(x) - f_\varepsilon(x))| &= |x^j (F(x) - F_\varepsilon(x))\,(F(x) + F_\varepsilon(x))| \\
&\leqslant |x^j (F(x) - H_\delta(x))|\,\|F + H_\delta\| \\
&\leqslant \big(|x^j (F(x) - H(x))| + |x^j (H(x) - H_\delta(x))|\big)\,\|F + H_\delta\| \\
&\leqslant \big(|x^j (F(x) - H(x))| + |x^j R_\delta(x)|\big)\,(\|F\| + \|H_\delta\|) \\
&\leqslant \big((A+1)^j \|F - H\| + \delta\big)\,(\|F\| + \|H\| + \delta) \\
&\leqslant \left(\frac{\varepsilon}{4(\|F\|+1)} + \frac{\varepsilon}{4(\|F\|+1)}\right)(\|F\| + \|F\| + 1 + 1) \;=\; \varepsilon\,.
\end{aligned}
$$

□

Proposition 3.15. — *In the situation of Theorem 3.11, there is a normalizing function χ such that $\chi(D_E)^2 - I \in C^*(X \subset M; A)$.*

Proof. — Let $s_0 > 0$ be such that $s := \mathrm{scal} \geqslant s_0$ outside of X, choose $0 < R < (s_0/4)^{1/2}$ and let χ be such that $\varphi := 1 - \chi^2$ is compactly supported in $[-R, R]$.

Let $\varepsilon > 0$. We will derive the inequality

$$\|\varphi(D_E)u\|^2 \leqslant \left(\frac{s_0}{4} - R^2\right)^{-1} \left(\frac{s_0}{4} + 1\right) \varepsilon\,\|u\|^2 \tag{3.4}$$

for each $u \in \Gamma^\infty_{\mathrm{cpt}}(M, S)$ with $\operatorname{supp}(u)$ outside of $B(K; 3\,S(\varepsilon))$ for an $S(\varepsilon) > 0$. Using Lemma 3.12 this implies that $\varphi(D_E) \in C^*(X \subset M; A)$, as required.

In order to obtain (3.4) we use Lemma 3.14 and write $f = \varphi^2 = f_\varepsilon + g_\varepsilon$ with

(1) $f_\varepsilon = F_\varepsilon^2$ with $F_\varepsilon \in C^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\operatorname{supp}(\hat{F}_\varepsilon) \subset [-S(\varepsilon), S(\varepsilon)]$,
(2) $\sup\{|x^j\, g_\varepsilon(x)|\,;\, x \in \mathbb{R}\} \leqslant \varepsilon$ for each $j = 0, 1, 2$.

From the functional calculus of Theorem 3.1 and the self-adjointness of D_E we obtain the following estimates for each $u \in \Gamma^\infty_{\mathrm{cpt}}(M, S \otimes E)$:

$$\langle D_E^2 f(D_E)u, u\rangle = \langle D_E^2 \varphi(D_E)^2 u, u\rangle \leqslant R^2\, \langle \varphi(D_E)u, \varphi(D_E)u\rangle \tag{3.5}$$

$$\langle D_E^2 g_\varepsilon(D_E)u, u\rangle \leqslant \varepsilon\, \|u\|^2 \cdot 1_A \tag{3.6}$$

$$\langle g_\varepsilon(D_E)u, u\rangle \leqslant \varepsilon\, \|u\|^2 \cdot 1_A \tag{3.7}$$

The first inequality uses $D_E\varphi(D_E) = m(D_E)\varphi(D_E)$ for a suitable function $m\colon \mathbb{R} \to \mathbb{R}$ with $\|m\|_\infty \leqslant R$ as $\operatorname{supp}(\varphi) \subset [-R, R]$ and the second inequality is based on the estimate

$$\langle D_E^2 g_\varepsilon(D_E)u, u\rangle \leqslant \|\langle D_E^2 g_\varepsilon(D_E)u, u\rangle\| \cdot 1_A \leqslant \|D_E^2 g_\varepsilon(D_E)\| \cdot \|u\|^2 \cdot 1_A$$

which follows from the inequality $0 \leqslant a \leqslant \|a\|_A \cdot 1_A$ for $a \in A_+$ and the Cauchy-Schwarz inequality for Hilbert A-modules. The third inequality above is derived in the same manner.

Using that D_E has unit propagation speed one sees that the inclusion $\operatorname{supp}(\hat{F}_\varepsilon) \subset [-S(\varepsilon), S(\varepsilon)]$ from (i) implies

$$\operatorname{supp}(F_\varepsilon(D_E)u) \subset B(\operatorname{supp}(u), S(\varepsilon)).$$

In particular, $\operatorname{supp}(F_\varepsilon(D_E)u)$ is outside of $B(X, S(\varepsilon))$ if $\operatorname{supp}(u)$ is outside of $B(X; 3\, S(\varepsilon))$.

From the Schrödinger-Lichnerowicz formula

$$D_E^2 = \Delta + s/4\ ,$$

where Δ is a positive operator and s denotes multiplication with the scalar curvature function, we obtain for such u:

$$\begin{aligned} &\langle D_E^2 f_\varepsilon(D_E)u, u\rangle \\ &\quad = \langle F_\varepsilon^2(D_E) D_E^2 u, u\rangle \\ &\quad = \langle \left\{\Delta + \frac{s}{4}\right\} F_\varepsilon(D_E)u, F_\varepsilon(D_E)u\rangle \\ &\quad = \langle \left\{\Delta + \left(\frac{s - s_0}{4}\right)\right\} F_\varepsilon(D_E)u, F_\varepsilon(D_E)u\rangle + \frac{s_0}{4}\,\langle f_\varepsilon(D_E)u, u\rangle \\ &\quad \geqslant \frac{s_0}{4}\,\langle f_\varepsilon(D_E)u, u\rangle\ . \end{aligned} \tag{3.8}$$

Here we have used that $\operatorname{supp}(F_\varepsilon(D_E)u)$ is outside of X and hence that $s \geqslant s_0$ holds there, hence $\Delta + (s - s_0)/4$ acts as a positive operator on

$F_\varepsilon(D_E)u$. Using (3.5)–(3.8) one finally obtains (3.4) from the following inequality in A

$$\begin{aligned}
\left(\frac{s_0}{4}-R^2\right)&\langle\varphi(D_E)u,\varphi(D_E)u\rangle \\
&\leqslant \frac{s_0}{4}\langle f(D_E)u,u\rangle - \langle D_E^2 f(D_E)u,u\rangle \\
&\leqslant \frac{s_0}{4}\langle f(D_E)u,u\rangle - \langle D_E^2 f_\varepsilon(D_E)u,u\rangle - \langle D_E^2 g_\varepsilon(D_E)u,u\rangle \\
&\leqslant \frac{s_0}{4}\langle f(D_E)u,u\rangle - \langle D_E^2 f_\varepsilon(D_E)u,u\rangle + \varepsilon\,\|u\|^2\cdot 1_A \\
&\leqslant \frac{s_0}{4}\langle f(D_E)u,u\rangle - \frac{s_0}{4}\langle f_\varepsilon(D_E)u,u\rangle + \varepsilon\|u\|^2\cdot 1_A \\
&= \frac{s_0}{4}\langle g_\varepsilon(D_E)u,u\rangle + \varepsilon\,\|u\|^2\cdot 1_A \\
&\leqslant \varepsilon\left(\frac{s_0}{4}+1\right)\|u\|^2\cdot 1_A\ ,
\end{aligned}$$

which implies the required inequality in $\mathbb{R}_+$ after applying the norm of A. □

4. Codimension two index obstruction to positive scalar curvature

In [35, Theorem 2.6] Roe's partitioned manifold index theorem [20, Theorem 4.4] was generalized to Dirac operators twisted with Hilbert A-module bundles. Since this version will be used in the proof of our main result, we will briefly restate it here.

Theorem 4.1. — *Let M be an odd-dimensional complete spin manifold with $\dim(M)\geqslant 3$ and let $N\subset M$ be a closed submanifold of codimension one with trivial normal bundle, which divides M into two parts M_0 and M_1 with common boundary N. Denote with D_E the spin Dirac operator twisted by the Hilbert A-module bundle $E\to M$.*

Let $\varphi_N\colon K_1(C^*(M;A))\to K_0(A)$ be the generalization to Hilbert A-module bundles of the homomorphism defined by the partitioning hypersurface as in [20, Section 4]. *Then*

$$\varphi_N(\mathrm{ind}(D_{M,E}))=\mathrm{ind}(D_{N,E_{|N}})$$

where $\mathrm{ind}(D_{N,E_{|N}})\in K_0(A)$ *is the classical Mishchenko-Fomenko index of the Dirac operator on the compact manifold N twisted by $E|_N$.*

Recall that on an arbitrary connected manifold M with fundamental group π, we have the canonical flat Mishchenko line bundle $\mathcal{V}(M):=$

$\widetilde{M} \times_\pi C^*\pi \to M$, a Hilbert $C^*\pi$-bundle, where $C^*\pi$ is the reduced or maximal group C^*-algebra for π, respectively. If M is a closed connected spin manifold, the Mishchenko-Fomenko index of the Dirac operator twisted by $\mathcal{V}(M)$, denoted $\mathrm{ind}(D_{\mathcal{V}(M)}) \in K_n(C^*\pi)$, is called the *Rosenberg index* and often written $\alpha(M) := \mathrm{ind}(D_{\mathcal{V}(M)})$. Here $n = \dim(M)$. If M is compact, standard arguments (see e.g. [17, Section 2.1]) show that this can also be viewed as the coarse index $\mathrm{ind}(D_{\mathcal{V}(M)})$ applied to the *compact* manifold M in which case $C^*(M; C^*\pi)$ is canonically Morita equivalent to $C^*\pi$ and therefore has the same K-theory.

By one of the many possible definitions of the Baum-Connes assembly map this is the image of the K-homology class of $B\pi$ represented by $[M]$ in the Baum-Douglas picture of K-homology under the Baum-Connes assembly map.

The following suspension result is well known and essentially contained in [23], compare in particular Proposition 2.9 and its proof (and the references therein) and the proof of Theorems 2.11 and 3.1 in [23].

Proposition 4.2. — *We have $C^*(\pi\times\mathbb{Z}) = C^*\pi \otimes C^*\mathbb{Z}$ and $K_1(C^*\mathbb{Z}) \cong \mathbb{Z}$. The last isomorphism is induced by the generator $e = \mathrm{ind}(D_{\mathcal{V}(S^1)}) \in K_1(C^*\mathbb{Z})$, the Rosenberg index of the Dirac operator on S^1, where S^1 carries the canonical orientation and any one of the two possible spin structures.*

For an arbitrary closed spin manifold M we have the product formula

$$\begin{aligned} \mathrm{ind}(D_{\mathcal{V}(M\times S^1)}) &= \mathrm{ind}(D_{\mathcal{V}(M)}) \otimes e \\ &\in K_n(C^*\pi_1(M)) \otimes K_1(C^*\mathbb{Z}) \subset K_{n+1}(C^*\pi_1(M\times S^1)) \end{aligned}$$

relating the Rosenberg indices of $M\times S^1$ and M, where we use the inclusion $K_n(A)\otimes K_1(C^*\mathbb{Z}) \hookrightarrow K_{n+1}(A\otimes C^*\mathbb{Z})$ coming from the Künneth theorem.

We can now state and prove the following result, which implies Theorem 1.1.

Theorem 4.3. — *Let M be a connected closed manifold with $\dim(M) \geqslant 3$ and $W \subset M$ a connected submanifold of codimension zero with boundary ∂W. Additionally, assume that the following holds:*

1. *The boundary ∂W is connected.*
2. *The second homotopy group of M vanishes: $\pi_2(M) = 0$.*
3. *The Hurewicz map $\mathrm{hur}_1 \colon \pi_1(\partial W) \to H_1(\partial W)$ is injective when restricted to the kernel $\ker(i_*) \subset \pi_1(\partial W)$ of the map induced by the inclusion map $i \colon \partial W \to W$.*

(4) *The inclusion map* $j\colon W \to M$ *induces a monomorphism*

$$j_*\colon \pi_1(W) \to \pi_1(M).$$

Then the following holds:

(1) *Let* $p\colon \overline{M} \to M$ *be the covering corresponding to the subgroup* $j_*(\pi_1(W))$ *of* $\pi_1(M)$*, and* $\overline{W} \subset \overline{M}$ *be a lift as isometric copy of* W *to* $\overline{M}$*, which exists by the choice of this covering. Denote by* $\mathbf{D}(\overline{M}, \overline{W})$ *the double of the manifold* $\overline{M} \setminus \mathrm{int}(\overline{W})$*. This double is partitioned by* $\partial\overline{W}$*. There exists an extension of the Mishchenko line bundle* $\mathcal{V}(\partial\overline{W})$ *over* $\partial\overline{W}$ *to a flat bundle* $\mathcal{E}$ *over* $\mathbf{D}(\overline{M}, \overline{W})$*.*

(2) *If* M *is a spin manifold and* $W = N \times \mathbf{D}^2$ *is a tubular neighborhood of a connected and closed submanifold* $N \subset M$ *with* $\mathrm{codim}(N) = 2$ *and trivial normal bundle, then (3) is automatically satisfied and if* $\alpha(N) \neq 0$ *then the manifold* M *does not admit a metric of positive scalar curvature.*

The condition (2) in Theorem 4.3 is necessary. For example consider $M := N \times S^2$ with some N which has non-trivial $\hat{A}$-genus. Then M does admit a metric with positive scalar curvature, but all the other assumptions are satisfied by a tubular neighborhood W of one copy of N in $N \times S^2$.

Remark 4.4. — If M, N are as in part b) of Theorem 4.3, and $\alpha(N) \neq 0$ then the index of $\pi_1(N)$ in $\pi_1(M)$ is necessarily infinite. Otherwise, passing to the finite covering $\overline{M}$ the complement $\overline{M} \setminus \overline{W}$ is a compact spin bordism between $N \times S^1$ and the empty set, over which the Mishchenko bundle of $N \times S^1$ extends.

By bordism invariance of the index of the twisted Dirac operator we have $\alpha(N \times S^1) = 0$ and therefore also $\alpha(N) = 0$, as explained in Proposition 4.2.

Corollary 4.5. — *Let* N *be a closed connected spin manifold with* $\pi_2(N) = 0$ *and* $\alpha(N) \neq 0$ *in* $K_*(C^*\pi_1(N))$*. Let* X *be the total space of a fiber bundle* $N \hookrightarrow X \to \Sigma$ *with fiber* N *over a compact surface* Σ *different from* S^2 *or* $\mathbb{R}P^2$*. If the spin structure on* N *extends to a compatible spin structure on* X*, then* X *does not admit a Riemannian metric with positive scalar curvature.*

Proof. — We view N as fiber over some point in Σ. Local triviality of the bundle implies the existence of a trivialized tubular neighborhood W of N in X. By assumption Σ is neither $\mathbb{R}P^2$ nor S^2. Hence $\pi_2(\Sigma) = 0$ and the long exact homotopy sequence of the bundle implies that the inclusion $j\colon W \to X$ is π_1-injective. By the same reasoning $\pi_2(X) = 0$. So (2) and (4) of Theorem 4.3 are satisfied. □

Proof of Theorem 4.3. — We consider the connected covering $p\colon \overline{M} \to M$ corresponding to the subgroup $j_*(\pi_1(W))$ of $\pi_1(M)$. The inclusion map $j\colon W \to M$ lifts to an injection $\overline{j}\colon \overline{W} \to \overline{M}$ which is an π_1-isomorphism, where $\overline{W}$ is homeomorphic to W via p.

(a) We will show subsequently that the inclusion map $k\colon \partial\overline{W} \to \overline{M} \setminus \overline{W}$ induces an injection on π_1 and that there exists a homomorphism $r\colon \pi_1(\overline{M} \setminus \overline{W}) \to \pi_1(\partial\overline{W})$ satisfying $r \circ k_* = \mathrm{id}$, i.e., k_* is a split injection. From this it follows that $\mathcal{E} := (Br \circ c)^*\mathcal{V}(B\pi_1(\partial\overline{W}))$ satisfies $k^*\mathcal{E} \cong \mathcal{V}(\partial\overline{W})$ if c is the classifying map of the universal covering of $\overline{M} \setminus \overline{W}$.

Injectivity of k_*: Let $i\colon \partial\overline{W} \hookrightarrow \overline{W}$ be the inclusion. Then the diagram

$$\begin{tikzcd}
\pi_1(\partial\overline{W}) \arrow[r, "k_*"] \arrow[d, "i_*"'] & \pi_1(\overline{M} \setminus \overline{W}) \arrow[d, "m_*"] \\
\pi_1(\overline{W}) \arrow[r, "\overline{j}_*", "\cong"'] & \pi_1(\overline{M})
\end{tikzcd} \tag{4.1}$$

is by the van Kampen theorem a pushout diagram (the right vertical arrow is induced by the inclusion $m\colon \overline{M} \setminus \overline{W} \to \overline{M}$). Since $\overline{j}_*$ is an isomorphism one has $\ker(k_*) \subset \ker(i_*)$. Therefore, if $[\alpha] \in \ker(k_*)$ then the loop α is both null-homotopic as a map to $\overline{M} \setminus \overline{W}$ and as a map to $\overline{W}$. This allows us to construct a singular sphere $\sigma\colon S^2 \to \overline{M}$ which maps the lower and upper hemisphere S^2_- and S^2_+, into $\overline{W}$ and $\overline{M} \setminus \overline{W}$, respectively, and whose restriction of σ to the equator $S^1 \subset S^2$ is α. By assumption (2) we have $\pi_2(\overline{M}) = \pi_2(M) = 0$ and hence $\sigma_*[S^2] = 0$ in singular homology. Therefore, by the construction of the boundary operator ∂ of the Mayer-Vietoris sequence of the triad $(\overline{M}, \overline{M} \setminus \overline{W}, \overline{W})$ also $\partial(\sigma_*[S^2]) = \alpha_*[S^1] = \mathrm{hur}_1[\alpha] = 0$. But this in conjunction with (3) implies $[\alpha] = 0$, proving that k_* is injective.

Let now N be the normal closure of $k_*(\ker(i_*))$ in $\pi_1(\overline{M} \setminus \overline{W})$. We get the bigger commutative diagram

$$\begin{tikzcd}
\pi_1(\partial\overline{W}) \arrow[r, "k_*"] \arrow[d] & \pi_1(\overline{M} \setminus \overline{W}) \arrow[d] \\
\pi_1(\partial\overline{W})/\ker(i_*) \arrow[r, "\overline{k_*}"] \arrow[d, "\overline{i_*}"'] & \pi_1(\overline{M} \setminus \overline{W})/N \arrow[d, "\overline{m_*}"] \\
\pi_1(\overline{W}) \arrow[r, "\overline{j}_*", "\cong"'] & \pi_1(\overline{M})
\end{tikzcd}$$

Because the outer square is a pushout diagram and the map $\pi_1(\overline{M}\setminus\overline{W}) \to \pi_1(\overline{M}\setminus\overline{W})/N$ is surjective, also the lower square is a pushout square. Now, as $\bar{i}_*$ is injective and $\bar{j}_*$ is an isomorphism, commutativity implies that $\bar{k}_*$ is injective. By the universal property of an amalgamated product, the lower pushout diagram is an amalgamed product. Using standard properties of amalgamated products (both factors $\pi_1(\overline{W})$ and $\pi_1(\overline{M}\setminus\overline{W})/N$ inject, and their intersection is precisely $\pi_1(\partial\overline{W})/\ker(i_*)$), and because $\bar{j}_*$ is an isomorphism, also $\bar{k}_*$ is an isomorphism. In particular, we have the product decomposition

$$\pi_1(\overline{M}\setminus\overline{W}) = \pi_1(\partial\overline{W})\cdot N. \tag{4.2}$$

Existence of r: Since $\overline{M}$ and $\overline{W}$ are path connected, $\pi_1(\overline{W}) \to \pi_1(\overline{M})$ is an isomorphism and since by assumption (ii) $\pi_2(\overline{M}) = 0$, the pair $(\overline{M},\overline{W})$ is 2-connected. By the relative Hurewicz theorem, $H_*(\overline{M},\overline{W}) = 0$ for $j = 0,1,2$. By excision the groups $H_1(\overline{M}\setminus\overline{W},\partial\overline{W})$ and $H_2(\overline{M}\setminus\overline{W},\partial\overline{W})$ are also trivial. In particular, $H_1(k)$ is an isomorphism. From (4.1) we obtain the following diagram

$$\begin{tikzcd} \pi_1(\partial\overline{W}) \arrow[r, "k_*"] \arrow[d, "i_*\times \mathrm{hur}_1"'] & \pi_1(\overline{M}\setminus\overline{W}) \arrow[d, "m_*\times \mathrm{hur}_1"] \\ \pi_1(\overline{W})\times H_1(\partial\overline{W}) \arrow[r, "\cong", "\bar{j}_*\times H_1(k)"'] & \pi_1(\overline{M})\times H_1(\overline{M}\setminus\overline{W}) \end{tikzcd} \tag{4.3}$$

which commutes by the naturality of the Hurewicz homomorphism. The lower horizontal arrow in (4.3) is an isomorphism as $\bar{j}_*$ and $H_1(k)$ are isomorphisms. Furthermore, our assumption (3) implies the injectivity of $i_*\times\mathrm{hur}_1$. This allows us to regard $\pi_1(\partial\overline{W})$ as subgroup of $\pi_1(\overline{M})\times H_1(\overline{M}\setminus\overline{W})$ via the injection given by the composition of this injection with the lower horizontal arrow in (4.3). Using the product decomposition (4.2) and the facts that the image of N in $\pi_1(\overline{M})$ is trivial and the image of the normal closure N of $\ker(i_*)$ in the abelian group $H_1(\overline{M}\setminus\overline{W})$ is equal to the image of $\ker(i_*)$, the right vertical arrow then surjects onto this subgroup $\pi_1(\partial\overline{W})$ of $\pi_1(\overline{M})\times H_1(\overline{M}\setminus\overline{W})$, giving rise to the required split r of k_*.

(b) Assume first that M is odd dimensional. Denote by W a trivial tubular neighbourhood of N. Then W is a zero-codimensional submanifold of M. The manifold $\mathbf{D}(\overline{M},\overline{W})$ admits a spin structure and is partitioned by the boundary $\partial\overline{W}\cong N\times S^1$ of $\overline{W}$. By part (a) there is a flat bundle $\mathcal{E}$ over $\mathbf{D}(\overline{M},\overline{W})$ which extends the Mishchenko line bundle $\mathcal{V}(\partial\overline{W})$ over

$\partial\overline{W}$. By Theorem 4.1 we have:

$$\varphi_{\partial\overline{W}}(\mathrm{ind}(D_{\mathbf{D}(\overline{M},\overline{W}),\mathcal{E}})) = \mathrm{ind}(D_{\partial\overline{W},\mathcal{V}(\partial\overline{W})}) \in K_0(C^*\pi_1(\partial\overline{W}))\ . \tag{4.4}$$

On the other hand, using Proposition 4.2,

$$\mathrm{ind}(D_{\partial\overline{W},\mathcal{V}(\partial\overline{W})}) = \alpha(\partial\overline{W}) = \alpha(\partial W) = \alpha(N\times S^1) = \alpha(N)\otimes e\ . \tag{4.5}$$

Since we assume $\alpha(N)\neq 0$, by Proposition 4.2 we can conclude

$$\mathrm{ind}(D_{\mathbf{D}(\overline{M},\overline{W}),\mathcal{E}}) \neq 0. \tag{4.6}$$

We conclude the proof by contradiction as follows. If M admits a metric of positive scalar curvature, then $\overline{M}$ admits a metric of uniformly positive scalar curvature. We can use this metric (deformed in a neighborhood of $\partial\overline{W}$ to get a smooth metric) to obtain a Riemannian metric with uniformly positive scalar curvature outside of a compact neighbourhood of $\partial\overline{W}$ on $\mathbf{D}(\overline{M},\overline{W})$. But since the bundle $\mathcal{E}$ is flat, Equation (4.6) and Theorem 3.11 imply that $\mathbf{D}(\overline{M},\overline{W})$ has no metric with uniformly positive scalar curvature outside of a compact subset. Hence M cannot admit a metric of positive scalar curvature.

Now assume that M is even-dimensional. In this case we replace the pair (M,N) by $(M\times S^1, N\times S^1)$. Since N has trivial normal bundle in M the normal bundle of $N\times S^1$ in $M\times S^1$ is trivial. Also the fundamental group of the submanifold still injects into the fundamental group of the ambient manifold. Since

$$\alpha(N\times S^1)\neq 0 \Longleftrightarrow \alpha(N)\otimes e\neq 0 \iff \alpha(N)\neq 0 \tag{4.7}$$

it follows from the previous paragraph that $M\times S^1$ admits no metric of positive scalar curvature. So in particular M has no such metric. ☐

Remark 4.6. — It should be possible to generalize the results of this paper in the following directions:

- Using real C^*-operators and Cl_n-linear versions, more refined invariants in the K-theory of real group C^*-algebras should be defined for which the same kind of vanishing result holds, and which should give rise to stronger obstructions to positive scalar curvature. Note that the (stable) Gromov-Lawson-Rosenberg conjecture concerns the real Dirac operator and the corresponding Rosenberg index.
- Using suitable further twists, as developed systematically by Stolz, compare e.g. [25, Section 5] one should be able to extend the theory to non-spin manifolds and even non-orientable manifolds, provided the universal covering carries a spin structure.

– The partitioned manifold index theorem underlying our approach has generalizations to multi-partitioned manifolds [28]. It should be possible, at least in special, iterated situations, to generalize the codimension two obstruction of Theorem 4.3 to even higher codimensions. For example, think of the following situation: one is given a codimension two hypersurface N_1 of a manifold M which itself contains a codimension two hypersurface H, for example let $H = N_1 \cap N_2$ be the intersection of two codimension two hypersurfaces. Is the Rosenberg index of H an obstruction to positive scalar curvature of M (under an appropriate assumption on fundamental groups and the vanishing of higher homotopy groups of M)?

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