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Fiber Integration on the Demailly Tower

FIBER INTEGRATION ON THE DEMAILLY TOWER

by Lionel DARONDEAU

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ABSTRACT. — The goal of this work is to provide a fiber integration formula on the Demailly tower, that avoids step-by-step elimination of horizontal cohomology classes, and that yields computational effectivity. A natural twist of the Demailly tower is introduced and a recursive formula for the total Segre class at k-th level is obtained. Then, by interpreting single Segre classes as coefficients, an iterated residue formula is derived.

RÉSUMÉ. — Le but de ce travail est de fournir une formule d'intégration le long des fibres de la tour de Demailly, évitant l'élimination pas-à-pas des classes de cohomologie horizontales, et permettant des calculs effectifs. Une modification naturelle de la tour de Demailly est introduite et une formule récursive pour la classe de Segre totale au niveau k est obtenue. Ensuite, l'interprétation des classes de Segre individuelles comme des coefficients mène à une formule de résidus itérés.

1. Introduction

Let $\overline{X}$ be a complex manifold and let D be a divisor on $\overline{X}$ with *normal crossings*, that is $D = \sum D_i$, where the components D_i are smooth irreducible divisors that meet transversally. For such a pair $(\overline{X}, D)$, one denotes by $T_{\overline{X}}(-\log D)$ the *logarithmic tangent bundle of $\overline{X}$ along D* ([19]).

Given a subbundle:

$$V \subset T_{\overline{X}}(-\log D) \subset T_{\overline{X}}$$

of the logarithmic tangent bundle, one constructs ([5, 7]), for any fixed order $\kappa \in \mathbb{N}$, the *logarithmic Demailly tower* of projectivized bundles:

$$\left(\overline{X}_\kappa, D_\kappa, V_\kappa\right) \to \left(\overline{X}_{\kappa-1}, D_{\kappa-1}, V_{\kappa-1}\right) \to \ldots$$
$$\ldots \to \left(\overline{X}_1, D_1, V_1\right) \to \left(\overline{X}_0, D_0, V_0\right) := \left(\overline{X}, D, V\right),$$

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having the main property that every holomorphic map $g \colon \mathbb{C} \to \overline{X} \setminus D$ lifts as maps:

$$g_{[i]} \colon \mathbb{C} \to \overline{X}_i \setminus D_i \qquad (i = 0, 1, \ldots, \kappa),$$

which depends only on the corresponding i-jet of g. Later on in Section §2, we will describe precisely this construction, central here.

For any two integers $j, k \in 0, 1, \ldots, \kappa$, the composition of the projections $\pi_i \colon \overline{X}_i \to \overline{X}_{i-1}$ yields a natural projection from the j-th level of the Demailly tower to the lower k-th level:

$$\pi_{j,k} := \pi_{k+1} \circ \cdots \circ \pi_j \colon \overline{X}_j \to \overline{X}_k.$$

The Demailly tower is of great importance in the study of the algebraic degeneracy of entire curves on $\overline{X} \setminus D$ (*cf.* the enlightening surveys [11, 21]). A first step towards the proof of algebraic degeneracy of entire curves is to prove the existence of a non zero polynomial P on $\overline{X}$ such that every non constant entire curve $g \colon \mathbb{C} \to \overline{X} \setminus D$ satisfies the algebraic differential equation:

$$P_{g(t)}\big(g'(t), g''(t), \ldots, g^{(\kappa)}(t)\big) = 0, \quad \text{for all } t \in \mathbb{C}.$$

Being by definition a projective vector bundle, the manifold $\overline{X}_i$ comes naturally equipped with a tautological line bundle, $\mathcal{O}_{\overline{X}_i}(-1)$, the multiples of which are usually denoted by $\mathcal{O}_{\overline{X}_i}(m) := (O_{\overline{X}_i}(-1)^\vee)^{\otimes m}$. The direct image:

$$\mathcal{O}\left(E_{\kappa,m}(V_0)^\star(\log D_0)\right) := (\pi_{\kappa,0})_\star \mathcal{O}_{\overline{X}_\kappa}(m)$$

is the sheaf of sections of a holomorphic bundle $E_{\kappa,m}(V_0)^\star(\log D_0)$, called the *Demailly-Semple bundle of jet differentials*, and a fundamental vanishing theorem ([5, 7]) states that for every global section:

$$P \in H^0\big(\overline{X}_\kappa, \mathcal{O}_{\overline{X}_\kappa}(m) \otimes \pi_{\kappa,0}^\star A^\vee\big) \simeq H^0\big(\overline{X}, E_{\kappa,m}(V_0)^\star(\log D_0) \otimes A^\vee\big),$$

with values in the dual $A^\vee$ of an ample line bundle $A \to \overline{X}$, one has as desired, for any κ-jet $(g', g'', \ldots, g^{(\kappa)})$ of non constant entire map $g \colon \mathbb{C} \to \overline{X} \setminus D$:

$$P_{g(t)}\big(g'(t), g''(t), \ldots, g^{(\kappa)}(t)\big) = 0, \quad \text{for all } t \in \mathbb{C}.$$

One has thus to ensure the existence of global sections of the line bundle $\mathcal{O}_{\overline{X}_\kappa}(m) \otimes \pi_{\kappa,0}^\star A^\vee$, possibly with $m \gg 1$.

One approach, with Schur bundles ([5]), consists in bounding positive even cohomology groups H^{2i} in order to use the Riemann-Roch theorem. In [22], in dimension 3, the author is able to bound the dimension of H^2 by use of the famous algebraic Morse inequalities ([5, 23]). Later in [17], the case of arbitrary dimension is completed, for high order jet differentials.

With a different approach, in [6] the case of arbitrary dimension is completed by use of a stronger version of algebraic Morse inequalities.

Another approach, developed in various contexts ([1, 3, 4, 8, 9, 10]), consists in applying the holomorphic Morse inequalities in order to prove the existence of sections of a certain more tractable subbundle of the bundle of jet differentials. One is led to establish the positivity of a certain intersection number on the κ-th level of the Demailly tower:

$$I = \int_{\overline{X}_\kappa} f\Big(c_1\big(\pi_{\kappa,1}^\star \mathcal{O}_{\overline{X}_1}(1)\big), \ldots, c_1\big(\pi_{\kappa,\kappa-1}^\star \mathcal{O}_{\overline{X}_{\kappa-1}}(1)\big), c_1\big(\mathcal{O}_{\overline{X}_\kappa}(1)\big)\Big),$$

where f is a polynomial of large degree:

$$\deg(f) = n + \kappa\,(\mathrm{rk}\,P(V)) = \dim(\overline{X}_\kappa),$$

in the first Chern classes $c_1\big(\pi_{\kappa,i}^\star \mathcal{O}_{\overline{X}_i}(1)\big)$.

When computing this intersection number, the standard strategy is to integrate along the fibers of the projections $\pi_{i,i-1}\colon \overline{X}_i \to \overline{X}_{i-1}$, until one obtains an intersection product on the basis $\overline{X}_0$, where the intersection of cohomology classes becomes simpler.

In [10], the authors use step-by-step elimination of Chern classes, and are able to disentangle the complex intrication between horizontal and vertical cohomology classes by a technical tour de force. These precise computations yield *effectivity*.

In [3], the author makes a clever use of *Segre classes* in order to avoid a large part of the computations, but on the other hand, effectivity cannot be reached.

In [1], the author uses equivariant geometry in order to prove a *residue formula* in several variables, that avoids step-by-step elimination and yields effectivity.

In the present paper, we combine ideas coming from these authors, in order to prove a similar residue formula in several variables, that is valid in a versatile geometric context, since it holds in any situation where the Demailly tower appears, *cf. e.g.* [18, 3]. Our proof borrows from [3] the technical simplification of the use of Segre classes, it yields computational effectivity as in [10], and it is in the very spirit of the formula of [1].

To enter into the details, by the Leray-Hirsch theorem ([13]), the cohomology ring $H^\bullet(\overline{X}_\kappa)$ of $\overline{X}_\kappa$ is the free module generated by the first Chern classes $c_1\big(\pi_{\kappa,i}^\star \mathcal{O}_{\overline{X}_i}(-1)\big)$ over the cohomology ring $H^\bullet(\overline{X}_0)$ of the basis $\overline{X}_0$, but the implementation of the computation in [4] suggests to naturally consider a different basis for the vertical cohomology by introducing the

line bundles:

$$L_i := \mathcal{O}_{\overline{X}_i}(-1) \otimes \pi_{i,i-1}^{\star}\mathcal{O}_{\overline{X}_{i-1}}(-1) \otimes \cdots \otimes \pi_{i,1}^{\star}\mathcal{O}_{\overline{X}_0}(-1) \qquad (i=1,\ldots,\kappa).$$

We will use the notation v_i for the first Chern class of the dual of this line bundle L_i (dropping the pullbacks):

$$v_i := c_1\big(L_i^{\vee}\big) = c_1\big(\mathcal{O}_{\overline{X}_i}(1)\big) + \cdots + c_1\big(\mathcal{O}_{\overline{X}_1}(1)\big) \qquad (i=1,\ldots,\kappa).$$

Note that this formula looks like a plain change of variables having inverse:

$$c_1\big(\mathcal{O}_{\overline{X}_i}(1)\big) = v_i - v_{i-1} \qquad (i=2,\ldots,\kappa),$$

thus, clearly, the polynomial f appearing in the intersection product I above has also a polynomial expression in terms of $v_1,\ldots,v_{\kappa}$. We will shortly provide a formula in order to integrate a polynomial under this new form, still denoted f.

Let K be a field. A *multivariate formal series* in κ variables with coefficients in K is a collection of coefficients in K, indexed by $\mathbb{Z}^{\kappa}$:

$$\Psi\colon \mathbb{Z}^{\kappa} \to K.$$

The space of formal series is naturally a K-vector space.

In analogy with polynomials, it is usual to denote, without convergence consideration:

$$\Psi(t_1,\ldots,t_{\kappa}) := \sum_{i_1,\ldots,i_{\kappa}\in\mathbb{Z}} \Psi(i_1,\ldots,i_{\kappa})\, t_1^{\,i_1} \cdots t_{\kappa}^{\,i_{\kappa}},$$

hence, in order to avoid confusion, we will write:

$$\big[t_1^{i_1} \cdots t_{\kappa}^{i_{\kappa}}\big]\big(\Psi(t_1,\ldots,t_{\kappa})\big) := \Psi(i_1,\ldots,i_{\kappa}),$$

to extract the coefficient indexed by $i_1,\ldots,i_{\kappa}$, that is the coefficient of the monomial $t_1^{i_1} \cdots t_{\kappa}^{i_{\kappa}}$ in the expansion of $\Psi(t_1,\ldots,t_{\kappa})$. The *support* of the formal series Ψ is the subset of indices at which Ψ is non zero:

$$\operatorname{supp}(\Psi) := \big\{\underline{i} \in \mathbb{Z}^{\kappa} : [t_1^{i_1} \cdots t_{\kappa}^{i_{\kappa}}](\Psi) \neq 0\big\}.$$

One defines the *Cauchy product* $\Psi_1\Psi_2$ of two formal power series:

$$(1.1) \qquad \Psi_1\Psi_2\colon (i_1,\ldots,i_{\kappa}) \mapsto \sum_{\underline{j}+\underline{k}=\underline{i}} [t_1^{j_1} \cdots t_{\kappa}^{j_{\kappa}}]\big(\Psi_1\big)\, [t_1^{k_1} \cdots t_{\kappa}^{k_{\kappa}}]\big(\Psi_2\big),$$

whenever the displayed sum is a finite sum for each κ-tuple:

$$\underline{i} := i_1,\ldots,i_{\kappa}.$$

For a fixed partial ordering on $\mathbb{Z}^{\kappa}$, when considering only the series having well ordered support, the Cauchy product of two such series is always meaningful, since the computation of the coefficient of each monomial involves

only finitely many terms. Moreover, for each choice of partial ordering, the set of formal series having well ordered support, equipped with the Cauchy product, forms a field ([20, Theorem 13.2.11]).

We give two examples of such fields. A *multivariate Laurent series* is a multivariate formal series, the support of which is well ordered for the standard product order on $\mathbb{Z}^\kappa$. An *iterated Laurent series* is a multivariate formal series, the support of which is well ordered for the lexicographic order on $\mathbb{Z}^\kappa$. The field of iterated Laurent series is an extension of the field of multivariate Laurent series.

After several Laurent expansions at the origin, any rational function becomes an iterated Laurent series (but not necessarily a multivariate Laurent series), as we will explain in more details later in §3.

We come back to the subbundle $V = V_0 \subset T_{\overline{X}_0}(-\log D_0)$. The *total Segre class* of this bundle $V_0 \to \overline{X}_0$:

$$s_\bullet(V_0) = 1 + s_1(V_0) + s_2(V_0) + \cdots + s_{\dim(\overline{X}_0)}(V_0)$$

is the inverse of the total Chern class of V_0 in $H^\bullet(\overline{X}_0)$. This notion is strongly related to integration along the fibers of a projective vector bundle ([12]). We will be more explicit about this relation below in §3.

We are now in position to state the main result of this work. Recall for $i = 1, \ldots, \kappa$ the notation $v_i := c_1(L_i^\vee)$, set:

$$r := \mathrm{rk}\, P(V) = \mathrm{rk}\, V - 1,$$

and introduce the (finite) generating series:

$$s_u(V_0) = 1 + u\, s_1(V_0) + u^2\, s_2(V_0) + \cdots + u^{\dim(\overline{X}_0)}\, s_{\dim(\overline{X}_0)}(V_0).$$

MAIN RESULT. — *For any polynomial:*

$$f \in H^\bullet(\overline{X}_0, V_0)[t_1, \ldots, t_\kappa],$$

in κ variables $t_1, \ldots, t_\kappa$, with coefficients in the cohomology ring $H^\bullet(\overline{X}_0, V_0)$, the intersection number:

$$I := \int_{\overline{X}_\kappa} f(v_1, \ldots, v_\kappa)$$

is equal to the Cauchy product coefficient:

$$I = [t_1^r \cdots t_\kappa^r]\left(\Phi_\kappa(t_1, \ldots, t_\kappa)\, I(t_1, \ldots, t_\kappa)\right),$$

where $\Phi_\kappa(t_1, \ldots, t_\kappa)$ is the universal rational function:

$$\Phi_\kappa(t_1, \ldots, t_\kappa) = \prod_{1 \leqslant i < j \leqslant \kappa} \frac{t_j - t_i}{t_j - 2\, t_i} \prod_{2 \leqslant i < j \leqslant \kappa} \frac{t_j - 2\, t_i}{t_j - 2\, t_i + t_{i-1}},$$

and where $I(t_1, \ldots, t_\kappa)$ is the multivariate Laurent polynomial involving only explicit data of the base manifold:

$$I(t_1, \ldots, t_\kappa) = \int_{\overline{X}_0} f(t_1, \ldots, t_\kappa) \, s_{1/t_1}(V_0) \cdots s_{1/t_\kappa}(V_0).$$

Concretely, the computation of this intersection number I, on the κ-th level $\overline{X}_\kappa$, can be brought down to the basis $\overline{X}_0$ as follows:

Step 1a. —   Compute on the basis $\overline{X}_0$ the intersection number with parameters $t_1, \ldots, t_\kappa$:

$$I(t_1, \ldots, t_\kappa) = \int_{\overline{X}_0} f(t_1, \ldots, t_\kappa) \, s_{1/t_1}(V_0) \cdots s_{1/t_\kappa}(V_0),$$

and obtain a multivariate Laurent polynomial in $t_1, \ldots, t_\kappa$ over $\mathbb{Q}$.

Step 1b. —   Expand the universal rational function $\Phi_\kappa(t_1, \ldots, t_\kappa)$ successively with respect to $t_1, t_2, \ldots$ up to t_κ. Obtain, not a multivariate Laurent series, but what has been called an iterated Laurent series, similarly denoted by $\Phi_\kappa(t_1, \ldots, t_\kappa)$ – notice the slanted Φ.

Step 2. —   Compute the Cauchy product $I(t_1, \ldots, t_\kappa) \, \Phi_\kappa(t_1, \ldots, t_\kappa)$ of the multivariate Laurent polynomial $I(t_1, \ldots, t_\kappa)$ and of the iterated Laurent series $\Phi_\kappa(t_1, \ldots, t_\kappa)$ in the field of iterated Laurent series over $\mathbb{Q}$. Lastly, extract the coefficient of the monomial $t_1^r \cdots t_\kappa^r$ in the obtained multivariate formal series, and receive the sought element $I \in \mathbb{Q}$.

Obstacles and forthcoming results

Really computing I proves to be quite delicate in practice. The first effective result in any dimension towards the Green-Griffiths conjecture was obtained in 2010 by Diverio, Merker and Rousseau ([10]), using step-by-step algebraic elimination, for entire curves $\mathbb{C} \to X_d \subset \mathbb{P}^{n+1}$ with values in generic hypersurfaces X_d of degree d in $\mathbb{P}^{n+1}$, with an estimated sufficient lower bound:

$$d \geqslant 2^{n^5}.$$

Some time after, in [1], Gergely Bérczi made a substantial progress by replacing the elimination step of [10] by an iterated residue formula, and he reached the lower bound:

$$d \geqslant n^{8n}.$$

Using our main result, the difficulty is that in general, Step 1b does not yield a single iterated Laurent series, but produces an involved product of

several iterated Laurent series. Then, it is very difficult to determine even the sign of any individual coefficient of $\Phi_\kappa(t_1, \ldots, t_\kappa)$, because this amounts to disentangle the large product:

$$\prod_{2 \leqslant i \leqslant j \leqslant \kappa} \frac{t_j - 2\,t_i}{t_j - 2\,t_i + t_{i-1}} = \prod_{2 \leqslant i \leqslant j \leqslant \kappa} \left(1 - \sum_{p=0}^{\infty} \frac{t_{i-1}(2\,t_i - t_{i-1})^p}{t_j^{p+1}} \right).$$

On the other hand, it is relatively easy to control the absolute value of these coefficients, using a convergent majorant series with positive coefficients, whence suppressing the problem of signs:

$$|\text{coeff}| \left(\prod_{2 \leqslant i \leqslant j \leqslant \kappa} \frac{t_j - 2\,t_i}{t_j - 2\,t_i + t_{i-1}} \right)$$

$$\leqslant \text{coeff} \left(\prod_{2 \leqslant i \leqslant j \leqslant \kappa} \frac{t_j - 2\,t_i}{t_j - 2\,t_i - t_{i-1}} \right)$$

$$= \text{coeff} \left(\prod_{2 \leqslant i \leqslant j \leqslant \kappa} \left(1 + \sum_{p=0}^{\infty} \frac{t_{i-1}(2\,t_i + t_{i-1})^p}{t_j^{p+1}} \right) \right)$$

– notice that $+t_{i-1}$ in the denominator becomes $-t_{i-1}$. And this allows us to use (plainly) the triangle inequality in [4], to attain an effective lower bound on the degree d of generic smooth hypersurfaces $X_d \subset \mathbb{P}^n$ such that all entire curves $\mathbb{C} \to \mathbb{P}^n \setminus X_d$ are algebraically degenerate:

$$d \geqslant (5n)^2\, n^n,$$

a lower bound which also holds for curves with values in a generic hypersurface $X_d \subset \mathbb{P}^{n+1}$.

A better understanding of the combinatorics of the series Φ_κ would allow to improve in a more subtle way this effective lower bound. It seems reasonable to reach an exponential bound:

$$d \overset{?}{\geqslant} (\mathsf{constant})^n.$$

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2. Demailly tower of (logarithmic) directed manifolds

A *directed manifold* is defined to be a couple (X, V) where X is a complex manifold, equipped with a (not necessary integrable) holomorphic subbundle $V \subset T_X$ of its holomorphic tangent bundle. There is a natural generalization of this definition in the logarithmic setting. A *log-directed manifold* is by definition a triple $(\overline{X}, D, V)$ where $(\overline{X}, D)$ is a log-manifold and the distribution $V \subset T_{\overline{X}}(-\log D)$ is a (not necessary integrable) subbundle of the logarithmic tangent bundle.

Given a log-directed manifold $(\overline{X}, D, V)$, following Dethloff and Lu [7], we construct the *Demailly tower of projectivized bundles* $(\overline{X}_i, D_i, V_i)$ on $\overline{X}$ by induction on $i \geqslant 0$. This construction is formally the same as the construction [5] of the Demailly tower in the so-called *compact case*, *i.e.* where there is no divisor D. The only slight modification to keep in mind in the genuine logarithmic setting is that V is a holomorphic subbundle of the logarithmic tangent bundle $T_{\overline{X}}(-\log D)$.

Projectivization of a log directed manifold ([5, 7])

Recall that for a vector bundle $E \to \overline{X}$ on a smooth manifold $\overline{X}$ with projective bundle of lines $\pi \colon P(E) \to \overline{X}$, the *tautological line bundle*:

$$\mathcal{O}_{P(E)}(-1) \to P(E)$$

is defined as the subbundle of the pullback bundle $\pi^\star E \to P(E)$ with trivial fiber.

The dual line bundle $\mathcal{O}_{P(E)}(1) := \mathcal{O}_{P(E)}(-1)^\vee$ fits into the following *Euler exact sequence* ([12, B.5.8]):

$$0 \to \mathcal{O}_{P(E)} \to \pi^\star E \otimes \mathcal{O}_{P(E)}(1) \to T_\pi \to 0,$$

where $T_\pi := \ker(\pi_\star)$ stands for the *relative tangent bundle* of $P(E)$ over $\overline{X}$, that itself fits into the following short exact sequence:

$$0 \to T_\pi \hookrightarrow T_{P(E)} \xrightarrow{\pi_\star} \pi^\star T_{\overline{X}} \to 0.$$

We now recall the inductive step $(\overline{X}', V', D') \xrightarrow{\pi} (\overline{X}, V, D)$ of the construction of the Demailly tower. Keep in mind that V is a subbundle of $T_{\overline{X}}(-\log D)$ and that V' has to be a subbundle of the logarithmic tangent bundle $T_{\overline{X}'}(-\log D')$.

Firstly, for $\overline{X}'$ take the total space $P(V)$ of the projective bundle of lines of V:

$$\overline{X}' := P(V) \xrightarrow{\pi} \overline{X},$$

and in order to make π a log-morphism set:

$$D' := \pi^{-1}(D) \subset \overline{X}'.$$

Next, by definition of the relative tangent bundle $T_\pi := \ker(\pi_\star)$ of the log-morphism π one has the following short exact sequence:

$$0 \to T_\pi \hookrightarrow T_{\overline{X}'}(-\log D') \xrightarrow{\pi_\star} \pi^\star T_{\overline{X}}(-\log D) \to 0,$$

and since by assumption $V \subset T_{\overline{X}}(-\log D)$, the tautological line bundle of $\overline{X}' = P(V)$ is a subbundle of the bundle in the right-hand slot:

$$\mathcal{O}_{\overline{X}'}(-1) \subset \pi^\star V \subset \pi^\star T_{\overline{X}}(-\log D),$$

whence one can define a subbundle $V' \subset T_{\overline{X}'}(-\log D')$ by taking:

$$V' := (\pi_\star)^{-1}\mathcal{O}_{\overline{X}'}(-1).$$

The only thing to verify in order to get a log directed manifold is that V' is a holomorphic subbundle of $T_{\overline{X}'}(-\log D')$. Since $(\pi_\star)^{-1}$ has maximal rank everywhere, as it is a bundle projection, this is the case ([7]).

By its very definition V' then fits into the following short exact sequence:

$$0 \to T_\pi \hookrightarrow V' \xrightarrow{\pi_\star} \mathcal{O}_{\overline{X}'}(-1) \to 0.$$

and thus the rank of V' is the same as the rank of V, because:

$$\mathrm{rk}(V') = \mathrm{rk}(P(V)) + 1 = \mathrm{rk}(V) - 1 + 1.$$

Starting from a bundle V_0 having rank $r+1$, by iterating the construction κ times, one thus gets a tower of projectivized bundles

$$(\overline{X}_\kappa, V_\kappa, D_\kappa) \xrightarrow{\pi_\kappa} \cdots \xrightarrow{\pi_2} (\overline{X}_1, V_1, D_1) \xrightarrow{\pi_1} (\overline{X}_0, V_0, D_0)$$

with $\mathrm{rk}\, V_i = r+1$ and $n_i := \dim \overline{X}_i = \dim(\overline{X}_0) + i\,(\mathrm{rk}\, P(V_0)) = n + i\,r$.

Existence of global jet differentials

The fibers of the Demailly-Semple bundle of jet differentials $E_{\kappa,m}(V_0)^\star$ $(\log D_0)$ carries much complexity ([15, 16]). In order to prove the existence of global jet differentials of order $\kappa = \dim(\overline{X})$, one is led to consider a much more tractable line bundle, constructed in [5, 6.13] as a linear combination with non negative integer coefficients $(a_1, \ldots, a_\kappa)$:

$$\mathcal{O}_{\overline{X}_\kappa}(a_1, a_2, \ldots, a_\kappa) := (\pi_{\kappa,1})^\star \mathcal{O}_{\overline{X}_1}(a_1) \otimes (\pi_{\kappa,2})^\star \mathcal{O}_{\overline{X}_2}(a_2) \otimes \cdots \otimes \mathcal{O}_{\overline{X}_k}(a_\kappa).$$

If $a_1 + \ldots + a_\kappa = m$, the direct image $(\pi_{\kappa,0})_\star \mathcal{O}_{\overline{X}_\kappa}(a_1, \ldots, a_\kappa)$ may be seen as a subbundle of the Demailly-Semple bundle of jet differentials ([5, 7]).

For a suitable choice of the parameters $a_1, \ldots, a_\kappa$, the line bundle $\mathcal{O}_{\overline{X}_\kappa}(a_1, \ldots, a_\kappa)$ has some positivity properties, that can be used together with the following Demailly-Trapani *algebraic Morse inequalities* ([23, 5]) in order to establish the existence of global jet differentials.

THEOREM (Weak algebraic Morse inequalities). — *For any holomorphic line bundle L on a N-dimensional compact manifold $\overline{X}$, that can be written as the difference $L = F \otimes G^\vee$ of two nef line bundles F and G, one has:*

$$h^0\big(\overline{X}, L^{\otimes k}\big) \geqslant k^N \, \frac{(F^N) - k\,(F^{N-1} \cdot G)}{N!} - o(k^N).$$

For a choice of $a_1, \ldots, a_\kappa \in \mathbb{N}^\kappa$ such that:

$$(2.1) \qquad a_{\kappa-1} > 2\,a_\kappa > 0 \quad ; \quad a_i \geqslant 3\,a_{i+1} \quad (i = 1, \ldots, \kappa - 2),$$

the line bundle $\mathcal{O}_{\overline{X}_\kappa}(a_1, \ldots, a_\kappa)$ is relatively ample along the fibers of $\overline{X}_\kappa$ over $\overline{X}$ (*cf.* [5]). It hence suffices to multiply it by a sufficiently positive power $\pi_{\kappa,0}^\star A^{\otimes l}$ of a given ample line bundle $A \to \overline{X}$, in order to get an ample (hence nef) line bundle (*cf.* [14]). On the other hand $\pi_{\kappa,0}^\star A^{\otimes l+1}$ is nef for it is the pullback of a nef line bundle.

It gives an expression of the line bundle $\mathcal{O}_{\overline{X}_\kappa}(a_1, \ldots, a_\kappa) \otimes (\pi_{\kappa,0})^\star A^\vee$ as the difference $F \otimes G^\vee$ of two nef line bundles:

$$F := \mathcal{O}_{\overline{X}_\kappa}(a_1, \ldots, a_\kappa) \otimes (\pi_{\kappa,0})^\star A^{\otimes l} \quad \text{and} \quad G := (\pi_{\kappa,0})^\star A^{\otimes l+1}$$

In order to prove the existence of global sections:

$$P \in H^0\Big(\overline{X}_\kappa, \mathcal{O}_{\overline{X}_\kappa}(a_1, \ldots, a_\kappa) \otimes (\pi_{\kappa,0})^\star A^\vee\Big),$$

it hence remains to show the positivity of the following intersection number:

$$I := \int_{\overline{X}_\kappa} c_1(F)^{n_\kappa} - n_\kappa \, c_1(F)^{n_\kappa - 1} c_1(G) \quad (n_\kappa = \dim \overline{X}_\kappa).$$

We will give a formula for computing such an intersection product.

3. Fiber Integration on the Demailly tower

It is convenient to bring down the computation to the basis and we will now provide a formula for this purpose. Noteworthy, the proof of this formula involves iterated Laurent series, in the same spirit as the residue formula of Berczi [1, 2]. However, we will not use equivariant geometry like this author, but only basic lemmas of intersection theory, more precisely some of the properties of Segre classes exposed in the book of Fulton [12, Chap. 3]. We now first briefly recall these properties.

Segre classes on the Demailly tower

To go down one level, from $\overline{X}_{i+1} = P(V_i)$ to $\overline{X}_i$, we will use the very definition of the j-th *Segre class* of a vector bundle $E \to X$ (having rank $r + 1$), namely the fiber integration formula:

$$(3.1) \qquad \int_X s_j(E)\, \alpha = \int_{P(E)} u^{j+r}\, p^\star \alpha \qquad (j \geqslant 0,\ \alpha \in H^\bullet X),$$

where:

$$p\colon P(E) \to X \quad \text{and} \quad u := c_1\big(\mathcal{O}_{P(E)}(-1)^\vee\big).$$

We want to apply this formula in order to eliminate the powers of the first Chern classes $v_1, \ldots, v_\kappa$ of the vector bundles $L_i \to \overline{X}_i$. We will proceed by induction.

It is well known that the total Segre class of a vector bundle is the same as the inverse of its total Chern class. Thus, the total Segre class enjoys the Whitney formula.

Because we will obtain a result that is independent of the geometric context, we will deliberately be ambiguous about it. The only property of the Demailly construction that we use in what follows is the existence of the two short exact sequences:

$$0 \to T_\pi \to V' \to \mathcal{O}_{\overline{X}'}(-1) \to 0$$

and:

$$0 \to \mathcal{O}_{\overline{X}'} \to \pi^\star V \otimes \mathcal{O}_{\overline{X}'}(1) \to T_\pi \to 0,$$

where $(\overline{X}', V') \overset{\pi}{\to} (\overline{X}, V)$ is the inductive step of the Demailly construction.

Now, consider the following observation: the twist by a line bundle does not change the projective bundle of lines of V', but only the transition functions. Moreover, one can twist short exact sequences by line bundles.

We can hence chose a line bundle L' on $\overline{X}'$ that makes the induction more easy. We will twist both short exact sequences by the same line bundle, because we do not want T_π to appear in the final formula below. Also, we do not want anymore the central term of the second short exact sequence to be a product of line bundles with different base spaces $\overline{X}$ and $\overline{X}'$ but rather want it to be the pullback of a single bundle on the lower level $\overline{X}$, that is:

$$\pi^\star V \otimes \mathcal{O}_{\overline{X}'}(1) \otimes L' = \pi^\star(V \otimes L),$$

for a certain $L \to \overline{X}$ (in practice given by the preceding induction steps). Consequently, we have to take:

$$(3.2) \qquad L' := \mathcal{O}_{\overline{X}'}(-1) \otimes \pi^\star L = \mathcal{O}_{P(V \otimes L)}(-1).$$

Notice that accordingly the term $\mathcal{O}_{\overline{X}'}(-1)$ can now be replaced by $\big(L' \otimes \pi^\star L^\vee\big)$ in the first exact sequence.

Once twisted by L', the above two short exact sequences become:

$$\begin{cases} 0 \to T_\pi \otimes L' \to V' \otimes L' \to \big(L' \otimes \pi^\star L^\vee\big) \otimes L' \to 0 \\ 0 \to L' \to \pi^\star\big(V \otimes L\big) \to T_\pi \otimes L' \to 0 \end{cases}.$$

By the Whitney formula, the first line yields:

$$s(V' \otimes L') = s(T_\pi \otimes L') \, s\big((L')^{\otimes 2} \otimes \pi^\star L^\vee\big),$$

while the second line yields:

$$\pi^\star s\big(V \otimes L\big) = s\big(T_\pi \otimes L'\big) \, s\big(L'\big).$$

Thus, we can eliminate T_π, as it was our intention, in order to get the induction formula:

$$s\big(V' \otimes L'\big) = \frac{s\big((L')^{\otimes 2} \otimes \pi^\star L^\vee\big)}{s\big(L'\big)} \, \pi^\star s\big(V \otimes L\big).$$

Now, for a line bundle $L \to X$, the total Segre class is the finite sum:

$$s_\bullet(L) = \big(1 - c_1(L^\vee)\big)^{-1} = 1 + c_1(L^\vee) + c_1(L^\vee)^2 + \cdots + c_1(L^\vee)^{\dim(X)}$$

– we use the first Chern class of the dual in order to have positive signs. Let $v := c_1\big(L^\vee\big)$ and $v' := c_1\big((L')^\vee\big)$. We get the induction formula:

$$(3.3) \qquad s\big(V' \otimes L'\big) = \varphi\big(v', v\big) \, \pi^\star s\big(V \otimes L\big),$$

where $\varphi(x, y)$ is the truncated double Taylor expansion of the rational function $(1 - x)\big(1 - 2x + y\big)^{-1}$:

$$\varphi\big(x, y\big) := (1 - x) \sum_{k=0}^{n_{\kappa - 1}} (2\,x - y)^k.$$

Considering (3.2), we construct the *ad hoc* sequence of line bundles $L_i \to \overline{X}_i$ by taking first the tautological line bundle $L_1 := \mathcal{O}_{\overline{X}_1}(-1)$ of V_0 and then the tautological line bundle of the twisted vector bundle $V_{i-1} \otimes L_{i-1}$:

$$L_i := \mathcal{O}_{\overline{X}_i}(-1) \otimes (\pi_i)^\star L_{i-1} = \mathcal{O}_{P(V_{i-1} \otimes L_{i-1})}(-1) \qquad (i=2,\dots,\kappa).$$

We will denote by v_i the first Chern class of the dual of this line bundle:

$$v_i := c_1\big(L_i^\vee\big) = c_1\big(\mathcal{O}_{P(V_{i-1} \otimes L_{i-1})}(1)\big).$$

Then, by (3.3), on has the following inductive formulas, where, for simplicity, we omit the pullbacks:

$$s\big(V_1 \otimes L_1\big) = \varphi(v_i,0)\, s\big(V_0\big)$$
$$\text{and} \quad s\big(V_i \otimes L_i\big) = \varphi(v_i,v_{i-1})\, s\big(V_{i-1} \otimes L_{i-1}\big), \qquad (i=2,\dots,\kappa).$$

Notice that we can reformulate the positivity property (2.1) by using the more explicit expression of the line bundles L_i:

$$L_i = \mathcal{O}_{\overline{X}_i}(-1) \otimes \cdots \otimes (\pi_{i,2})^\star \mathcal{O}_{\overline{X}_2}(-1) \otimes (\pi_{i,1})^\star \mathcal{O}_{\overline{X}_1}(-1) \qquad (i=1,\dots,\kappa).$$

and the inversion of these formulas

$$\mathcal{O}_{\overline{X}_i}(1) = L_i^\vee \otimes L_{i-1};$$

in analogy with $\mathcal{O}_{\overline{X}_\kappa}(a_1,\dots,a_\kappa) = a_1\, \mathcal{O}_{\overline{X}_1}(-1)^\vee + \cdots + a_\kappa\, \mathcal{O}_{\overline{X}_\kappa}(-1)^\vee$ consider, a linear combination:

$$L(a_1,\dots,a_\kappa) := a_1\, L_1^\vee + \cdots + a_\kappa\, L_\kappa^\vee$$

of the line bundles $L_i^\vee$, with non negative coefficients a_i, such that:

$$a_1 + 2\, a_2 + \cdots + \kappa\, a_\kappa = m \in \mathbb{N},$$

then, the line bundle $\pi_{\kappa,0}^\star L(a_1,\dots,a_\kappa)$ may be seen as a certain subbundle of $E_{\kappa,m}V_0^\star(\log D_0)$ and if:

$$a_{\kappa-1} > a_\kappa \geqslant 1 \quad ; \quad a_i \geqslant 2(a_{i+1} + \cdots + a_\kappa) \quad (i \leqslant \kappa-2).$$

the line bundle $L(a_1,\dots,a_\kappa)$ is relatively ample along the fibers of $\overline{X}_\kappa \to \overline{X}_0$.

PROPOSITION 3.4. — *For any polynomial in the first Chern classes* $v_1,\dots,v_{i+1}$ *having coefficients in (the pullback of) the cohomology of* $\overline{X}_0$:

$$f \in H^\bullet\big(\overline{X}_0\big)[v_1,\dots,v_i,v_{i+1}],$$

the following formula of integration along the fibers of $\overline{X}_{i+1} \to \overline{X}_i$ *holds:*

$$\int_{\overline{X}_{i+1}} f(v_1,\dots,v_i,v_{i+1}) = \big[t_{i+1}^r\big] \left(\int_{\overline{X}_i} f(v_1,\dots,v_i,t_{i+1})\, s_{1/t_{i+1}}\big(V_i \otimes L_i\big) \right).$$

Proof. — Firstly, by the Leray-Hirsch theorem:

$$H^\bullet\big(\overline{X}_0\big)[v_1, \ldots, v_i, v_{i+1}] = H^\bullet\big(\overline{X}_{i+1}\big)$$

Thus, f has values in the cohomology ring of $\overline{X}_{i+1}$.

The polynomial f is of the form:

$$f(v_1, \ldots, v_i, v_{i+1}) = \sum_{j=0}^{n_{i+1}} (v_{i+1})^j \, (\pi_{i+1})^\star f_j(v_1, \ldots, v_i).$$

By linearity, the formula will hold for any such sum, if it holds for every monomial:

$$v_{i+1}^j \, (\pi_{i+1})^\star f_j(v_1, \ldots, v_i).$$

Recall that the line bundles L_i are constructed by the inductive formula:

$$L_{i+1} = \mathcal{O}_{P(V_i)}(-1) \otimes \pi_{i+1}^\star L_i = \mathcal{O}_{P(V_i \otimes L_i)}(-1).$$

Thus, it can be thought of as the tautological line bundle of the projective bundle:

$$P(V_i \otimes L_i) \simeq P(V_i) =: \overline{X}_{i+1}.$$

Then the above fiber integration formula (3.1) yields at once:

$$(*) \quad \int_{\overline{X}_{i+1}} v_{i+1}^j \, (\pi_{i+1})^\star f_j(v_1, \ldots, v_i) = \int_{\overline{X}_i} s_{j-r}\big(V_i \otimes L_i\big) \, f_j(v_1, \ldots, v_i).$$

In particular, this integral is zero for indices j smaller than r.

The problem is now to obtain the individual Segre classes from the total Segre class. In that aim, we will use the formalism of generating functions. Recall that, in analogy with Chern polynomial, for a vector bundle $E \to X$ over a N dimensional manifold X, we have introduced the generating function $s_u(E)$ of the Segre classes of E, that is:

$$s_u(E) := s_0(E) + u\, s_1(E) + u^2 \, s_2(E) + \cdots + u^N \, s_N(E).$$

Then, by taking $t = 1/u$, we obtain a Laurent polynomial:

$$s_{1/t}(E) := \frac{s_0(E)}{t^0} + \frac{s_1(E)}{t^1} + \frac{s_2(E)}{t^2} + \cdots + \frac{s_N(E)}{t^N},$$

in which the $(j-r)$-th Segre class involved in the fiber integration appears as the coefficient:

$$s_{j-r}(E) = \big[1/t^{j-r}\big] s_{1/t}(E) = \big[t^r\big]\Big(t^j \, s_{1/t}(E)\Big).$$

Therefore, by replacing in the integration formula $(*)$:

$$\int_{\overline{X}_{i+1}} v_{i+1}^j \, (\pi_{i+1})^\star f_j(v_1, \ldots, v_i)$$

$$= \int_{\overline{X}_i} \left[t_{i+1}{}^r \right] \left(t_{i+1}^j \, s_{1/t_{i+1}} \left(V_i \otimes L_i \right) \right) f_j(v_1, \ldots, v_i).$$

$$= \left[t_{i+1}{}^r \right] \left(\int_{\overline{X}_i} t_{i+1}^j \, f_j(v_i, \ldots, v_1) \, s_{1/t_{i+1}} \left(V_i \otimes L_i \right) \right).$$

Notice that inside of the parenthesis there is the product of a monomial by a Laurent polynomial. Thus, only a finite number of terms are involved and there is no objection to switching the integral and the coefficient extraction.

The obtained formula is exactly the sought formula for the considered monomial:

$$v_{i+1}^j \, (\pi_{i+1})^\star f_j(v_1, \ldots, v_i),$$

and this ends the proof. $\qquad\square$

Iteration of the integration formula

In order to iterate the fiber integration, we introduce the following formalism: for $i = 0, 1, \ldots, \kappa$, we denote by $\underline{vt}_i$ the κ-tuple obtained from:

$$\underline{v} := (v_1, \ldots, v_\kappa)$$

by replacing the last i components $v_{\kappa-i+1}$, ..., v_κ by the corresponding parameters $t_{\kappa-i+1}$, ..., t_κ, i.e.:

$$\underline{vt}_i := (v_1, \ldots, v_{\kappa-i}, t_{\kappa-i+1}, \ldots, t_\kappa) \qquad (i=0,1,\ldots,\kappa).$$

With this notation the fiber integration formula (3.4) just above yields directly, that for any polynomial in the first Chern classes $v_1, \ldots, v_{i+1}$ having coefficients in (the pullback of) the cohomology of $\overline{X}_0$, being a Laurent polynomial in the formal parameters $t_{i+2}, \ldots, t_\kappa$:

$$f \in H^\bullet(\overline{X}_0)[v_1, \ldots, v_i, v_{i+1}][t_{i+2}, t_{i+2}^{-1}, \ldots, t_\kappa, t_\kappa^{-1}],$$

the following formula of integration along the fibers of $\overline{X}_{i+1} \to \overline{X}_i$ holds:

$$(3.5) \qquad \int_{\overline{X}_i} f\left(\underline{vt}_{\kappa-i}\right) = \left[t_i^r \right] \int_{\overline{X}_{i-1}} f\left(\underline{vt}_{\kappa-i+1}\right) s_{1/t_i}\left(V_{i-1} \otimes L_{i-1}\right).$$

Notice that in the above formula the form of the polynomial appearing in the integrand:

$$f\left(\underline{vt}_{\kappa-i+1}\right) s_{1/t_i}\left(V_{i-1} \otimes L_{i-1}\right) \in H^\bullet(\overline{X}_0)[v_1, \ldots, v_i][t_{i+1}, t_{i+1}^{-1}, \ldots, t_\kappa, t_\kappa^{-1}],$$

allows to iterate this formula in order to integrate along the fibers of $\overline{X}_{i-1} \to \overline{X}_{i-2}$. For short, we denote the appearing polynomial rings by:

$$\Lambda[\underline{vt}_{\kappa-i}] := H^\bullet(\overline{X}_0)[v_1, \ldots, v_i][t_{i+1}, t_{i+1}^{-1}, \ldots, t_\kappa, t_\kappa^{-1}] \quad (i=0,1,\ldots,\kappa).$$

One has thus:

$$\Lambda[\underline{vt}_0] = H^\bullet(\overline{X}_0)[\underline{v}] = H^\bullet(\overline{X}_\kappa) \quad \text{and} \quad \Lambda[\underline{vt}_\kappa] = H^\bullet(\overline{X}_0)[\underline{t}, \underline{t}^{-1}].$$

We have first to investigate the dependence with respect to v_i of the appearing power series $s_{1/t_{i+1}}(V_i \otimes L_i)$. The induction formula (3.1) precisely provides us with this information. Thanks to it, we can split the power series $s_{1/t}(V_i \otimes L_i)$ in two parts:

$$s_{1/t_j}(V_i \otimes L_i) = \underbrace{\varphi\left(\frac{v_i}{t_j}, \frac{v_{i-1}}{t_j}\right)}_{\in \Lambda[\underline{vt}_{\kappa-i}]} \underbrace{s_{1/t_j}(V_{i-1} \otimes L_{i-1})}_{\in \Lambda[\underline{vt}_{\kappa-i+1}]},$$

or for $i = 1$:

$$s_{1/t_j}(V_1 \otimes L_1) = \underbrace{\varphi\left(\frac{v_1}{t_j}, 0\right)}_{\in \Lambda[\underline{vt}_{\kappa-1}]} \underbrace{s_{1/t_j}(V_0 \otimes L_0)}_{\in H^\bullet(\overline{X}_0)[\underline{t},\underline{t}^{-1}]},$$

the first of which depends on v_i whereas the second does not.

Write for short:

$$\Phi_{k,l}(t_1, \ldots, t_\kappa) := \varphi\left(\frac{t_k}{t_l}, \frac{t_{k-1}}{t_l}\right) \qquad (k=2,\ldots,\kappa-1,\ k+1\leqslant l\leqslant\kappa),$$

and:

$$\Phi_{1,l}(t_1, \ldots, t_\kappa) := \varphi\left(\frac{t_1}{t_l}, 0\right) \qquad (2\leqslant l\leqslant\kappa),$$

in such way that, for any two positive integers $k < l$:

$$(3.6) \qquad s_{1/t_l}(V_k \otimes L_k) = \Phi_{k,l}(\underline{vt}_k) \, s_{1/t_l}(V_{k-1} \otimes L_{k-1}) \quad (1\leqslant k<l\leqslant\kappa).$$

Let Φ_i be the product of the terms in the i last lines of the array:

$$\begin{pmatrix} 1 & \Phi_{1,2} & \cdots\cdots\cdots\cdots\cdots\cdots & \Phi_{1,\kappa} \\ & & \Phi_{2,3} & \cdots\cdots\cdots & \Phi_{2,\kappa} \\ & & & & \\ & & & \Phi_{\kappa-1,\kappa} & \\ 1 & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & 1 \end{pmatrix},$$

that is the product of $(i(i-1)/2)$ terms:

$$\Phi_i(t_1, \ldots, t_\kappa) := \prod_{\kappa-i+1\leqslant k<l\leqslant\kappa} \Phi_{k,l}(t_1, \ldots, t_\kappa).$$

As an example, $\Phi_1(t_1, \ldots, t_\kappa) = 1$.

The following lemma will be used in order to isolate the variable $v_{\kappa-i}$, that is to eliminate after the i-th step of the fiber integration.

LEMMA 3.7 (Isolation of $v_{\kappa-i}$). — *For any $i = 0, 1, \ldots, \kappa - 1$ one has the following relation between Φ_i and Φ_{i+1}:*

$$\Phi_i\big(\underline{vt}_i\big) \prod_{j=\kappa-i+1}^{\kappa} s_{1/t_j}\big(V_{\kappa-i} \otimes L_{\kappa-i}\big)$$

$$= \Phi_{i+1}\big(\underline{vt}_i\big) \prod_{j=\kappa-i+1}^{\kappa} s_{1/t_j}\big(V_{\kappa-(i+1)} \otimes L_{\kappa-(i+1)}\big).$$

Proof. — Recall the induction formula displayed above:

$$\frac{s_{1/t_j}\big(V_i \otimes L_i\big)}{s_{1/t_j}\big(V_{i-1} \otimes L_{i-1}\big)} = \Phi_{i,j}\big(\underline{vt}_{\kappa-i}\big).$$

Thus, one has:

$$\frac{\prod_{j=\kappa-i+1}^{\kappa} s_{1/t_j}\big(V_{\kappa-i} \otimes L_{\kappa-i}\big)}{\prod_{j=\kappa-i+1}^{\kappa} s_{1/t_j}\big(V_{\kappa-(i+1)} \otimes L_{\kappa-(i+1)}\big)} = \prod_{j=\kappa-i+1}^{\kappa} \Phi_{\kappa-i,j}\big(\underline{vt}_{\kappa-i}\big).$$

Now, by definition of Φ_i and Φ_{i+1}:

$$\frac{\Phi_{i+1}}{\Phi_i}\big(\underline{vt}_{\kappa-i}\big) = \frac{\prod_{\kappa-i \leqslant k < l \leqslant \kappa} \Phi_{k,l}}{\prod_{\kappa-i+1 \leqslant k < l \leqslant \kappa} \Phi_{k,l}}\big(\underline{vt}_{\kappa-i}\big) = \prod_{l=\kappa-i+1}^{\kappa} \Phi_{\kappa-i,l}\big(\underline{vt}_{\kappa-i}\big).$$

Hence, we get the announced result. $\qquad\square$

Notice that in the right hand side of the obtained formula, only the first factor depends on $v_{\kappa-i}$.

This result is given by anticipation of the proof of main theorem (3.8). However we can already notice that, *e.g.*:

$$\Phi_1\big(\underline{vt}_1\big) \prod_{j=\kappa-1+1}^{\kappa} s_{1/t_j}\big(V_{\kappa-i} \otimes L_{\kappa-i}\big) = s_{1/t_\kappa}\big(V_{\kappa-1} \otimes L_{\kappa-1}\big),$$

is the term appearing in the first step of the fiber integration.

THEOREM 3.8 (Fiber Integration on the Demailly tower). — *Any polynomial:*

$$f \in H^\bullet(\overline{X}_0, V_0)[t_1, \ldots, t_\kappa],$$

in κ variables $t_1, \ldots, t_\kappa$, with coefficients in the cohomology ring $H^\bullet(\overline{X}_0, V_0)$, yields a cohomology class:

$$f\big(\underline{v}\big) = f\big(v_1, \ldots, v_\kappa\big) \in H^\bullet\big(\overline{X}_\kappa\big),$$

that can be integrated along the fibers of the projective bundle $\overline{X}_\kappa \to \overline{X}_0$ according to the formula:

$$\int_{\overline{X}_\kappa} f(\underline{v}) = [t_1^r \cdots t_\kappa^r] \left(\Phi_\kappa(\underline{t}) \int_{\overline{X}_0} f(\underline{t}) \, s_{1/t_1}(V_0) \cdots s_{1/t_\kappa}(V_0) \right).$$

Proof. — We will prove by induction that for $i = 0, 1, \ldots, \kappa$, one has:

$$\int_{\overline{X}_\kappa} f(\underline{v}) = [t_{\kappa-i+1}^r \cdots t_\kappa^r] \int_{\overline{X}_{\kappa-i}} f_i(\underline{vt}_i),$$

with:

$$f_i(\underline{vt}_i) := f(\underline{vt}_i) \, \Phi_i(\underline{vt}_i) \prod_{k=\kappa-i+1}^{\kappa} s_{1/t_k}(V_{\kappa-i} \otimes L_{\kappa-i}).$$

Then, for $i = \kappa$:

$$\int_{\overline{X}_\kappa} f(\underline{v}) = [t_1^r \cdots t_\kappa^r] \int_{\overline{X}_0} f_\kappa(\underline{t}),$$

with:

$$f_\kappa(\underline{t}) := f(\underline{t}) \, \Phi_\kappa(\underline{t}) \prod_{k=1}^{\kappa} s_{1/t_k}(V_0 \otimes L_0).$$

That is the desired formula because $L_0 = \mathcal{O}_{\overline{X}_0}$.

For $i = 0$, this is tautological. Now, assume that the formula holds for the index i, that is to say:

$$(*) \qquad \int_{\overline{X}_\kappa} f(\underline{v}) = [t_{\kappa-i+1}^r \cdots t_\kappa^r] \int_{\overline{X}_{\kappa-i}} f_i(\underline{vt}_i),$$

According to lemma (3.7), f_i can also be written:

$$f_i(\underline{vt}_i) = f(\underline{vt}_i) \, \Phi_{i+1}(\underline{vt}_i) \underbrace{\prod_{k=\kappa-i+1}^{\kappa} s_{1/t_k}(V_{\kappa-(i+1)} \otimes L_{\kappa-(i+1)})}_{\in \Lambda[\underline{vt}_{i+1}]}.$$

Now applying lemma (3.5):

$$\int_{\overline{X}_{\kappa-i}} f_i(\underline{vt}_i) = [t_{\kappa-i}^r] \int_{\overline{X}_{\kappa-i-1}} f_i(\underline{vt}_{i+1}) s_{1/t_{\kappa-i}}(V_{\kappa-(i+1)} \otimes L_{\kappa-(i+1)}).$$

It remains to state that:

$$f_i(\underline{vt}_{i+1}) s_{1/t_{\kappa-i}}(V_{\kappa-(i+1)} \otimes L_{\kappa-(i+1)}) =$$

$$f(\underline{vt}_{i+1}) \, \Phi_{i+1}(\underline{vt}_{i+1}) \prod_{k=\kappa-i+1}^{\kappa} s_{1/t_k}(V_{\kappa-(i+1)} \otimes L_{\kappa-(i+1)})$$

$$s_{1/t_{\kappa-i}}(V_{\kappa-(i+1)} \otimes L_{\kappa-(i+1)}).$$

Here, we recognize the expression:

$$f_{i+1}\big(\underline{v}t_{i+1}\big) = f\big(\underline{v}t_{i+1}\big)\,\Phi_{i+1}\big(\underline{v}t_{i+1}\big)\prod_{k=\kappa-i}^{\kappa} s_{1/t_k}(V_{\kappa-(i+1)}\otimes L_{\kappa-(i+1)}).$$

Thus, we can replace the integrand in order to get:

$$\int_{\overline{X}_{\kappa-i}} f_i\big(\underline{v}t_i\big) = \big[t_{\kappa-i}^r\big]\int_{\overline{X}_{\kappa-i-1}} f_{i+1}\big(\underline{v}t_{i+1}\big).$$

Using the induction hypothesis $(*)$, one finally gets the desired formula, for the index $i+1$:

$$\int_{\overline{X}_\kappa} f\big(\underline{v}\big) = \big[t_{\kappa-i+1}^r\cdots t_\kappa^r\big]\int_{\overline{X}_{\kappa-i}} f_i\big(\underline{v}t_i\big)$$

$$= \big[t_{\kappa-i}^r\cdots t_\kappa^r\big]\int_{\overline{X}_{\kappa-(i+1)}} f_{i+1}\big(\underline{v}t_{i+1}\big).$$

This complete the proof. □

Laurent series expansion of rational functions

Before going further, we give now more details about the Laurent series expansion mentioned above in the introduction (page 33).

In the univariate case, we will denote by $K((t))$ the space of Laurent series. Equipped with the Cauchy product, it becomes a field. Indeed, the following *geometric series formula* is valid in the ring of formal power series:

$$(1-X)^{-1} = \sum_{i\geqslant 0} X^i,$$

and it allows to define the formal inverse (for the Cauchy product) of any Laurent series of the form:

$$\Psi = \sum_{i\geqslant N}\Psi_i\,t^i$$

with initial coefficient $\Psi_N\neq 0$, as follows:

$$(3.9)\qquad \Psi^{-1} = \frac{1}{\Psi_N\,t^N}\sum_{k\geqslant 0}\bigg(-\sum_{j\geqslant 1}\frac{\Psi_{N+j}}{\Psi_N}\,t^j\bigg)^k,$$

because the computation of the coefficient of any power of t in the later expression involves only a finite number of appearing k-th powers. The result is indeed a Laurent series, since its support is visibly bounded from below.

A direct consequence is that any rational function enjoys a natural Laurent expansion. Indeed, the support of a polynomial $Q \in K[t]$, considered as a formal series, is finite. Thus, it is naturally a Laurent series. Then, by formula (3.9) above, we can construct a formal inverse of Q in the field of Laurent series. Now, any rational function of the form:

$$\frac{P(t)}{Q(t)} = P(t)\,Q^{-1}(t),$$

with also $P \in R[t]$, can be expanded as a Laurent series: it suffices to use the multiplication rule (1.1) in order to compute the product (in the field of Laurent series) of the numerator P by the formal inverse "Q^{-1}" obtained after using the expansion rule (3.9). This yields an injective morphism of fields:

$$\Psi^0 \colon K(t) \hookrightarrow K((t)),$$

that we call *Laurent expansion of rational functions at the origin*.

In the multivariate case, in order to *unequivocally* expand a rational function of several variables $t_1, \ldots, t_\kappa$ under the form of a generalized Laurent series, it is necessary to assign at first a total ordering to the variables t_i (consider the example of $(t_1 - t_2)^{-1}$). Then, working step by step in the univariate setting (taking account of the ordering of the variables), one easily convinces oneself that the successive series expansions at zero yield an injective morphism of fields:

$$\Psi^0 \colon K(t_\kappa)(t_{\kappa-1})\ldots(t_1) \hookrightarrow K((t_\kappa))((t_{\kappa-1}))\ldots((t_1)),$$

that we call *Laurent expansion of rational functions at the origin, under the assumption $t_1 \ll \cdots \ll t_{\kappa-1} \ll t_\kappa \ll 1$*. The map Ψ^0 is indeed injective because its image contains only summable series, therefore its left inverse is the successive summation for $t_\kappa, t_{\kappa-1} \ldots, t_1$.

Here, the notation $t_1 \ll t_2 \ll \cdots \ll t_\kappa \ll 1$ express the idea that for two integers $k < k'$ the variable t_k is infinitely smaller than any (positive or negative) power of the variable $t_{k'}$. To compute the iterated Laurent series expansion of a rational function $Q \in K(t_1, \ldots, t_\kappa)$, we first expand Q at the origin as a rational function of t_1, formally considering any rational expression made of constants of K, and variables $t_2, \ldots, t_\kappa$ as elements of the field of coefficients. Then, when expanding the coefficients of the resulting series, we forget t_1 and we have: $t_2 \ll t_3 \ll \ldots t_\kappa \ll 1$. We iterate the procedure until we get $t_\kappa \ll 1$, that is the one-dimensional case.

Thus, an element of the ring:

$$K\langle\!\langle t_1, \ldots, t_\kappa \rangle\!\rangle := K((t_\kappa))((t_{\kappa-1}))\ldots((t_1)),$$

should be seen as a Laurent series in t_1 whose coefficients are Laurent series in t_2 whose coefficients are Laurent series in t_3 and so on... Accordingly such an element is called an *iterated Laurent series.*

It is a bigger space than the space of multivariate Laurent series. A formal series Ψ is an element of $K\langle\!\langle t_1,\ldots,t_\kappa\rangle\!\rangle$ if and only if its support is well ordered for the lexicographic order. This condition is clearly weaker than to be bounded from below for the standard product order on $\mathbb{Z}^\kappa$ (consider again the example of $(t_1 - t_2)^{-1}$).

We can extend the coefficient extraction operator to the field of rational functions $K(t_1,\ldots,t_\kappa)$ by using the injection Ψ^0. For a rational function $Q \in K(t_1,\ldots,t_\kappa)$, we always imply the assumption $t_1 \ll \cdots \ll t_\kappa \ll 1$ and we define the coefficient extraction operator:

$$\left[t_1^{i_1}\cdots t_\kappa^{i_\kappa}\right](Q) := \left[t_1^{i_1}\cdots t_\kappa^{i_\kappa}\right]\big(\Psi^0(Q)\big).$$

This convention in turn allows us to define the (Cauchy) product of a rational function by an iterated Laurent series, by using the same formalism as in (1.1).

Integration formula

We are now in position to state a more tractable version of formula (3.8):

THEOREM 3.10 (Fiber Integration on the Demailly tower). — *For any polynomial:*

$$f \in H^\bullet(\overline{X}_0, V_0)[t_1,\ldots,t_\kappa],$$

in κ variables $t_1,\ldots,t_\kappa$, with coefficients in the cohomology ring $H^\bullet(\overline{X}_0, V_0)$, having total degree at most n_κ, the cohomology class:

$$f\big(\underline{v}\big) = f\big(v_1,\ldots,v_\kappa\big) \in H^\bullet\big(\overline{X}_\kappa\big),$$

can be integrated along the fibers of the projective bundle $\overline{X}_\kappa \to \overline{X}_0$ according to the formula:

$$\int_{\overline{X}_\kappa} f\big(\underline{v}\big) = \left[t_1^r\cdots t_\kappa^r\right]\left(\Phi_\kappa\big(\underline{t}\big)\int_{\overline{X}_0} f\big(\underline{t}\big)\,s_{1/t_1}(V_0)\cdots s_{1/t_\kappa}(V_0)\right),$$

where Φ_κ is the universal rational function:

$$\Phi_\kappa(t_1,\ldots,t_\kappa) = \prod_{1\leqslant i<j\leqslant\kappa}\frac{t_j - t_i}{t_j - 2\,t_i}\prod_{2\leqslant i<j\leqslant\kappa}\frac{t_j - 2\,t_i}{t_j - 2\,t_i + t_{i-1}}.$$

Proof. — The product Φ_κ can be reshaped as follows:

$$\Phi_\kappa(t_1,\ldots,t_\kappa) = \prod_{j=2}^{\kappa-1} \underbrace{\frac{t_j - t_1}{t_j - 2\,t_1}}_{\Phi_{1,j}(\underline{t})} \; \prod_{i=2}^{j-1} \underbrace{\frac{t_j - t_i}{t_j - 2\,t_i + t_{i-1}}}_{\Phi_{i,j}(\underline{t})}.$$

For $1 \leqslant i < j \leqslant \kappa - 1$, let:

$$R_{i,j}(\underline{t}) := \Psi^0(\Phi_{i,j}(\underline{t})) - \Phi_{i,j}(\underline{t}).$$

By coming back to the definitions, it is immediate to see that these remainders are:

$$R_{1,j}(\underline{t}) = \left(1 - \frac{t_i}{t_j}\right) \sum_{k>n_{\kappa-1}} \frac{(2t_i)^k}{t_j^k}$$

$$\text{and} \quad R_{i,j}(\underline{t}) = \left(1 - \frac{t_i}{t_j}\right) \sum_{k>n_{\kappa-1}} \frac{(2t_i - t_{i-1})^k}{t_j^k} \quad (2\leqslant i<j\leqslant\kappa-1).$$

Noteworthy, the supports of these remainders satisfy:

$$\operatorname{supp} R_{i,j} \subset \big\{(i_1,\ldots,i_\kappa)\colon i_j < -n_{\kappa-1}\big\} \cap \bigcap_{k>j} \big\{(i_1,\ldots,i_\kappa)\colon i_k = 0\big\}.$$

One can write:

$$\Psi^0(\Phi(\underline{t})) = \prod_{\text{entries}} \begin{pmatrix} 1 & \Phi_{1,2} + R_{1,2} & \cdots & \Phi_{1,\kappa} + R_{1,\kappa} \\ & & & \\ & & & \\ & & \Phi_{\kappa-1,\kappa} + R_{\kappa-1,\kappa} \\ 1 & \cdots\cdots\cdots\cdots & & 1 \end{pmatrix}.$$

We clean inductively this array of the remainder in the k-th column. Let array_j be the array deduced from the above array by dropping the remainders in the j last columns. We will show that:

$$\big[t_1^r\cdots t_\kappa^r\big]\Big(\prod \mathsf{array}_j\, I(t_1,\ldots,t_\kappa)\Big) = \big[t_1^r\cdots t_\kappa^r\big]\Big(\prod \mathsf{array}_{j+1}\, I(t_1,\ldots,t_\kappa)\Big)$$

where:

$$I(t_1,\ldots,t_\kappa) := \int_{\overline{X}_0} f(t_1,\ldots,t_\kappa)\, s_{1/t_1}(V_0)\cdots s_{1/t_\kappa}(V_0).$$

We generalize the notation $I(t_1,\ldots,t_\kappa)$ by setting:

$$I(t_1,\ldots,t_i) := \big[t_{i+1}^r\cdots t_\kappa^r\big]\Big(I(\underline{t})\prod_{k=i+1}^{\kappa-1}\prod_{j=1}^{k-1}\Phi_{j,k}(\underline{t})\Big) \quad (i=0,1,\ldots,\kappa).$$

Then we claim (find the proof below):

$$\operatorname{supp} I(t_1, \ldots, t_{\kappa-j}) \subset \{(i_1, \ldots, i_{\kappa-j}) \colon i_{\kappa-j} \leqslant n_\kappa\}.$$

On the other hand, one is easily convinced that:

$$\prod \operatorname{array}_j = \prod \operatorname{array}_{j+1} + R_j \sum_{l=\kappa-j+1}^{\kappa-1} \prod_{k=1}^{l-1} \Phi_{k,l}(\underline{t}),$$

where R_j is a given series such that:

$$\operatorname{supp} R_j(t_1, \ldots, t_{\kappa-j}) \subset \{(i_1, \ldots, i_{\kappa-j}) \colon i_{\kappa-j} < -n_{\kappa-1}\}.$$

It is now clear that the second part cannot contribute to the coefficient of $t_{\kappa-j}^r$ in:

$$[t_{\kappa-j+1}^r \cdots t_\kappa^r]\left(\prod \operatorname{array}_j I(\underline{t})\right),$$

because the degree of $t_{\kappa-j}$ in the corresponding term:

$$[t_{\kappa-j+1}^r \cdots t_\kappa^r]\left(R_j(t_1, \ldots, t_{\kappa-j}) \sum_{l=\kappa-j+1}^{\kappa-1} \prod_{k=1}^{l-1} \Phi_{k,l}(\underline{t})\, I(\underline{t})\right)$$

$$= R_j(t_1, \ldots, t_{\kappa-j})\, I(t_1, \ldots, t_{\kappa-j})$$

is *strictly* less than $-n_{\kappa-1} + n_\kappa = r$.

Consequently:

$$[t_{\kappa-j}^r \cdots t_\kappa^r]\left(\prod \operatorname{array}_j I(t_1, \ldots, t_\kappa)\right)$$

$$= [t_{\kappa-j}^r \cdots t_\kappa^r]\left(\prod \operatorname{array}_{j+1} I(t_1, \ldots, t_\kappa)\right),$$

and by extraction of the coefficient of the monomial $t_1^r \cdots t_{\kappa-j-1}^r$, as announced:

$$[t_1^r \cdots t_\kappa^r]\left(\prod \operatorname{array}_j I(t_1, \ldots, t_\kappa)\right) = [t_1^r \cdots t_\kappa^r]\left(\prod \operatorname{array}_{j+1} I(t_1, \ldots, t_\kappa)\right).$$

An induction finishes the proof because:

$$\Psi^0(\Phi_\kappa) = \prod \operatorname{array}_0 \quad \text{and} \quad \Phi_\kappa = \prod \operatorname{array}_\kappa. \qquad \square$$

We have added a lot of non contributive terms, however in practice ([4]), the above reformulation of (3.8) is more efficient, because it takes account of the convergence of the series at stake.

Finally, we prove our claim above in the proof, that:

$$\operatorname{supp} I(t_1, \ldots, t_{\kappa-j}) \subset \{(i_1, \ldots, i_{\kappa-j}) \colon i_{\kappa-j} \leqslant n_\kappa\}.$$

Actually we will be more precise and show that for $j = 1, \ldots, \kappa$:

$$\operatorname{supp} I(t_1, \ldots, t_j) \subset \{(i_1, \ldots, i_j) \colon i_j \leqslant n_j\}.$$

Proof. — In order to prove this statement, it is easier to work with genuine polynomials, and not Laurent polynomials. An important remark is that for any two integers $k < l$, the Laurent polynomial $t_l^{n_{\kappa-1}}\Phi_{k,l}(\underline{t})$ is in fact a genuine polynomial, having degree $n_{\kappa-1}$. Thus, we rather consider:

$$t_j^n I(t_1,\ldots,t_j)$$
$$= \big[t_{j+1}^{m_{j+1}}\cdots t_{\kappa-1}^{m_{\kappa-1}} t_\kappa^{n+r}\big]\bigg(\big(t_j^n\cdots t_\kappa^n I(\underline{t})\big)\prod_{l=j+1}^{\kappa-1}\prod_{k=1}^{l-1}\big(t_l^{n_{\kappa-1}}\Phi_{k,l}(\underline{t})\big)\bigg),$$

where $m_l = n + r + (l-1)(n_{\kappa-1})$.

The appearing polynomial in $t_j, t_{j+1},\ldots,t_\kappa$ has degree:

$$\deg_{t_j,\ldots,t_\kappa}\bigg(\big(t_j^n\cdots t_\kappa^n I(\underline{t})\big)\prod_{l=j+1}^{\kappa-1}\prod_{k=1}^{l-1}\big(t_l^{n_{\kappa-1}}\Phi_{k,l}(\underline{t})\big)\bigg)$$
$$\leqslant n_\kappa + n + \sum_{l=j+1}^{\kappa-1}(m_l-r)+n.$$

Extracting the coefficient of $\big[t_{j+1}^{m_{j+1}}\cdots t_{\kappa-1}^{m_{\kappa-1}} t_\kappa^{n+r}\big]$ decrease the degree by at least:

$$\sum_{l=j+1}^{\kappa-1} m_l + (n+r).$$

Finally we get a polynomial in t_j having degree:

$$\deg_{t_j}\big(t_j^n I(t_1,\ldots,t_j)\big) \leqslant n + n_\kappa - r - \sum_{l=j+1}^{\kappa-1} r = n + n_j.$$

Thus, as announced:

$$\deg_{t_j}\big(I(t_1,\ldots,t_j)\big) \leqslant n_j. \qquad \square$$

Notice that our theorem holds as well without the (natural) technical assumption on the degree of f.

BIBLIOGRAPHY

[1] G. BÉRCZI, "Thom polynomials and the Green-Griffiths conjecture", http://arxiv.org/abs/1011.4710.

[2] ———, "Moduli of map germs, Thom polynomials and the Green-Griffiths conjecture", in *Contributions to algebraic geometry*, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2012, p. 141-167.

[3] D. BROTBEK, "Hyperbolicity related problems for complete intersection varieties", *Compos. Math.* **150** (2014), no. 3, p. 369-395.

[4] L. DARONDEAU, "On the Logarithmic Green-Griffiths Conjecture", International Mathematics Research Notices (2015), to appear.

[5] J.-P. DEMAILLY, "Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials", in *Algebraic geometry—Santa Cruz 1995*, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, p. 285-360.

[6] ———, "Holomorphic Morse inequalities and the Green-Griffiths-Lang conjecture", *Pure Appl. Math. Q.* **7** (2011), no. 4, Special Issue: In memory of Eckart Viehweg, p. 1165-1207.

[7] G.-E. DETHLOFF & S. S.-Y. LU, "Logarithmic jet bundles and applications", *Osaka J. Math.* **38** (2001), no. 1, p. 185-237.

[8] S. DIVERIO, "Differential equations on complex projective hypersurfaces of low dimension", *Compos. Math.* **144** (2008), no. 4, p. 920-932.

[9] ———, "Existence of global invariant jet differentials on projective hypersurfaces of high degree", *Math. Ann.* **344** (2009), no. 2, p. 293-315.

[10] S. DIVERIO, J. MERKER & E. ROUSSEAU, "Effective algebraic degeneracy", *Invent. Math.* **180** (2010), no. 1, p. 161-223.

[11] S. DIVERIO & E. ROUSSEAU, *A survey on hyperbolicity of projective hypersurfaces*, Publicações Matemáticas do IMPA. [IMPA Mathematical Publications], Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2011, [On the title page: A survey on hiperbolicity of projective hypersurfaces], x+109 pages.

[12] W. FULTON, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998, xiv+470 pages.

[13] A. HATCHER, *Algebraic topology*, Cambridge University Press, Cambridge, 2002, xii+544 pages.

[14] R. LAZARSFELD, *Positivity in algebraic geometry. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 48, Springer-Verlag, Berlin, 2004, Classical setting: line bundles and linear series, xviii+387 pages.

[15] J. MERKER, "Jets de Demailly-Semple d'ordres 4 et 5 en dimension 2", *Int. J. Contemp. Math. Sci.* **3** (2008), no. 17-20, p. 861-933.

[16] ———, "Application of computational invariant theory to Kobayashi hyperbolicity and to Green-Griffiths algebraic degeneracy", *J. Symbolic Comput.* **45** (2010), no. 10, p. 986-1074.

[17] ———, "Algebraic Differential Equations for Entire Holomorphic Curves in Projective Hypersurfaces of General Type: Optimal Lower Degree Bound", in *Geometry and Analysis on Manifolds* (T. Ochiai, T. Mabuchi, Y. Maeda, J. Noguchi & A. Weinstein, eds.), Progress in Mathematics, vol. 308, Springer International Publishing, 2015, p. 41-142.

[18] C. MOUROUGANE, "Families of hypersurfaces of large degree", *J. Eur. Math. Soc. (JEMS)* **14** (2012), no. 3, p. 911-936.

[19] J. NOGUCHI, "Logarithmic jet spaces and extensions of de Franchis' theorem", in *Contributions to several complex variables*, Aspects Math., E9, Vieweg, Braunschweig, 1986, p. 227-249.

[20] D. S. PASSMAN, *The algebraic structure of group rings*, Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1977, xiv+720 pages.

[21] M. PĂUN, "Techniques de construction de différentielles holomorphes et hyperbolicité"", in *Séminaire Bourbaki (Paris)*, 2012.

[22] E. Rousseau, "Équations différentielles sur les hypersurfaces de $\mathbb{P}^4$", *J. Math. Pures Appl. (9)* **86** (2006), no. 4, p. 322-341.

[23] S. Trapani, "Numerical criteria for the positivity of the difference of ample divisors", *Math. Z.* **219** (1995), no. 3, p. 387-401.

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