ANNALES

DE

L'INSTITUT FOURIER

Semyon DYATLOV

Spectral gaps for normally hyperbolic trapping

SPECTRAL GAPS FOR NORMALLY HYPERBOLIC TRAPPING

by Semyon DYATLOV (*)

———

ABSTRACT. — We establish a resonance free strip for codimension 2 symplectic normally hyperbolic trapped sets with smooth incoming/outgoing tails. An important application is wave decay on Kerr and Kerr–de Sitter black holes. We recover the optimal size of the strip and give an $o(h^{-2})$ resolvent bound there. We next show existence of deeper resonance free strips under the r-normal hyperbolicity assumption and a pinching condition. We also give a lower bound on the one-sided cutoff resolvent on the real line.

RÉSUMÉ. — Cet article démontre l'existence d'une bande sans résonances pour des ensembles captés normalement hyperboliques de codimension 2, dont les variétés entrantes/sortantes sont lisses. Une application importante est la décroissance exponentielle des ondes pour les trous noirs de Kerr et Kerr–de Sitter. On retrouve la taille optimale de la bande et on y donne une borne $o(h^{-2})$ de la résolvante. On démontre alors l'existence de bandes plus profondes sans résonances si l'ensemble capté est r-normalement hyperbolique et satisfait une condition de pincement. On donne aussi une borne inférieure sur la norme de la résolvante tronquée sur l'axe réel.

1. Introduction

This paper is a collection of results regarding resonance free strips (also known as *spectral gaps*) and resolvent estimates in the presence of *normally hyperbolic trapping*. Such trapping has received a lot of attention recently because of its connection with exponential decay of waves on black hole

backgrounds, exponential decay of correlations for contact Anosov flows, and applications to molecular chemistry. See below for an introduction to the role of resolvent bounds in decay estimates.

In [28], Wunsch and Zworski showed existence of a small spectral gap for symplectic normally hyperbolic trapped sets under the assumption that the incoming/outgoing tails $\Gamma_\pm$ are smooth and have codimension 1 in the phase space; this includes subextremal Kerr and Kerr–de Sitter black holes. More recently, Nonnenmacher and Zworski [23] extended this result by (a) assuming much weaker regularity of $\Gamma_\pm$, namely that their tangent spaces at the trapped set have merely continuous dependence on the base point (b) making no assumption on the codimension of $\Gamma_\pm$ (c) establishing the optimal size of the gap. The assumptions of [23] apply to more general situations, including contact Anosov flows.

The first result of this paper (Theorem 1.1) is a spectral gap of optimal size under the assumptions of [28] (and the orientability of $\Gamma_\pm$). The novelty compared to [23] is the $o(h^{-2})$ resolvent bound in this gap. The short proof presented here is also more direct than those in [28, 23] because it relies on regular semiclassical analysis rather than exotic symbol calculus, at a cost of not recovering the extensions (a), (b) discussed above.

We next show (Theorem 1.3) the existence of deeper resonance free strips, under the additional assumptions of *r-normal hyperbolicity* and *pinching*. The second gap and a Weyl law for resonances in between the two gaps were previously proved in [11] under the same assumptions. The present paper essentially removes the projector Π from the method of [11], working directly with the pseudodifferential operators Θ_j^+ instead (see the discussion of the proof below). This shows the existence of additional resonance free strips, but does not recover the Weyl law or the structure of Π (which is important in understanding wave decay, see [10]). For a different yet related setting of Pollicott–Ruelle resonances for contact Anosov flows, existence of multiple gaps under a pinching condition and a Weyl law for the first band of resonances was proved in [15]. Our methods also bear some similarities to the recent work [12] on resonances for the geodesic flow on hyperbolic quotients; the horocyclic operators $\mathcal{U}_-$ of [12] play the same role as the operators Θ_j^+ in this paper.

We finally show a lower bound of $h^{-1}\sqrt{\log(1/h)}$ on the one-sided cutoff resolvent on the real line, complementing the upper bounds of [3, 8] – see (1.12) and §5.

Motivation

We first give a brief introduction to resolvent estimates on a model example. Assume that (M, g) is a Riemannian manifold isometric to the Euclidean space $\mathbb{R}^n$ outside of a compact set, and n is odd. Then [14, §4.2] the resolvent

$$R_g(\omega) := (-\Delta_g - \omega^2)^{-1} : L^2(M) \to H^2(M), \quad \operatorname{Im} \omega > 0$$

admits a meromorphic continuation to $\omega \in \mathbb{C}$ as a family of operators $L^2_{\operatorname{comp}}(M) \to H^2_{\operatorname{loc}}(M)$, and its poles are called *resonances*. We say that R_g has a *spectral gap of size $\nu > 0$ with loss of $m \geqslant 0$ derivatives*, if there exists $C_0 > 0$ such that

$$(1.1) \quad \|\chi R_g(\omega)\chi\|_{L^2 \to L^2} \leqslant C_\chi |\omega|^{-1+m}, \quad |\operatorname{Re}\omega| \geqslant C_0, \quad \operatorname{Im}\omega \in [-\nu, 1]$$

for all $\chi \in C_0^\infty(M)$. If (1.1) holds, then each solution to the wave equation

$$(\partial_t^2 - \Delta_g)u = 0, \quad u|_{t=0} = \chi u_0 \in H^1_{\operatorname{comp}}(M), \quad u_t|_{t=0} = \chi u_1 \in L^2_{\operatorname{comp}}(M)$$

satisfies the *resonance expansion* (assuming there are no resonances on $\{\operatorname{Im}\omega = -\nu\}$)

$$u(t, x) = \sum_j \sum_{k=0}^{J(\omega_j)-1} t^k e^{-it\omega_j} u_{jk}(x) + u_R(t, x),$$

with the sum above over the finitely many resonances in $\{\operatorname{Im}\omega \geqslant -\nu\}$ with multiplicities $J(\omega_j)$, and the remainder is exponentially decaying in $t \geqslant 0$:

$$\|e^{\nu t}\chi u_R\|_{H^1_{t,x}(\{t>0\})} \leqslant C_\chi(\|u_0\|_{H^{1+m}} + \|u_1\|_{H^m}).$$

See for instance [9, Proposition 2.1] for the proof. We see that the size ν of the spectral gap gives the rate of exponential decay of the remainder, while m is the number of derivatives lost in the estimate. Whether or not (1.1) holds depends on the structure of the *trapped set* for the geodesic flow $\varphi_t : T^*M \to T^*M$

$$K = \{(x, \xi) \in T^*M \mid |\xi|_g \in (1/2, 2), \ \exists V \Subset T^*M : \varphi_t(x, \xi) \in V$$
$$\text{for all } t \in \mathbb{R}\}$$

The bound (1.1) is equivalent to the estimate

$$(1.2) \quad \begin{aligned} \|\chi R_h(\lambda)\chi\|_{L^2 \to L^2} &\leqslant C_\chi h^{-1-m}, \\ \operatorname{Re}\lambda = \pm 1, \quad \operatorname{Im}\lambda &\in [-\nu h, h], \quad 0 < h \ll 1 \end{aligned}$$

where $R_h(\lambda) = h^{-2}R_g(h^{-1}\lambda)$ is the semiclassical resolvent:

$$(1.3) \quad R_h(\lambda) = (-h^2\Delta_g - \lambda^2)^{-1} : L^2(M) \to H^2(M), \quad \operatorname{Im}\lambda > 0.$$

The bounds (1.2) for $\operatorname{Re}\lambda = 1$ and $\operatorname{Re}\lambda = -1$ are equivalent (by taking adjoints), thus we assume that $\operatorname{Re}\lambda = 1$.

By the gluing method of Datchev–Vasy [6, §4.1], it suffices to prove the bound for the model resolvent

$$(1.4) \qquad \|(-h^2\Delta_{g'} - \lambda^2 - iQ')^{-1}\|_{L^2(X)\to L^2(X)} \leqslant Ch^{-1-m},$$

where (X, g') is a compact Riemannian manifold which contains a part X_M isometric to $(M \cap B_{R_0}, g)$ where B_{R_0} is the ball of radius $R_0 \gg 1$ in $\mathbb{R}^n$, and $Q' \in \Psi_h^2(X)$ is a semiclassical pseudodifferential operator (see §2 for notation) such that $\sigma(Q') \geqslant 0$ everywhere and Q' is elliptic on $X \setminus X_M$ and supported away from $X_M \cap B_{R_0/2}$. The operator Q', as well as the operators Q'', Q introduced below, are called *complex absorbing operators* and generalize complex absorbing potentials used for instance in quantum chemistry.

To make the setup of (1.6) apply, we need to make the dependence on λ linear. For that, take self-adjoint compactly microlocalized operators $P, Q'' \in \Psi_h^{\text{comp}}(X)$ such that

$$-h^2\Delta_{g'} - iQ' = (P - iQ'')^2 + \mathcal{O}(h^\infty) \quad \text{microlocally on } \{1/2 < |\xi|_{g'} < 2\}.$$

See for instance [18, Lemma 4.6]; moreover, we have in $\{1/2 < |\xi|_{g'} < 2\}$,

$$\sigma(P) - i\sigma(Q'') = \sqrt{|\xi|_{g'}^2 - i\sigma(Q')},$$

where $\sqrt{\cdot}$ maps positive numbers to positive numbers, so in particular $\sigma(Q'') \geqslant 0$ everywhere. Then (see [11, Lemma 4.3] for a more general argument)

$$-h^2\Delta_{g'} - \lambda^2 - iQ' = (P - iQ'' + \lambda)(P - iQ'' - \lambda)$$
$$\text{microlocally on } \{1/2 < |\xi|_{g'} < 2\}.$$

By the elliptic estimate [13, Proposition 2.4], it suffices to prove a bound on $A(-h^2\Delta_{g'} - \lambda^2 - iQ')^{-1}$ for some $A \in \Psi_h^{\text{comp}}(X)$ which is elliptic on

$$\Omega := \{|\xi|_{g'}^2 - i\sigma(Q') = 1\} \subset \{x \in X_M, \ |\xi|_g = 1\}.$$

Since $P - iQ'' + \lambda$ is elliptic on Ω, we reduce (1.4) to the bound

$$(1.5) \qquad \|(P - iQ - \lambda)^{-1}\|_{L^2(X)\to L^2(X)} \leqslant Ch^{-1-m},$$

where $Q \in \Psi_h^0(X)$, $\sigma(Q) \geqslant 0$ everywhere, and $Q = Q''$ microlocally near Ω. In this paper, we prove the bound (1.5) for the case when the trapped set K has a *normally hyperbolic* structure.

Setup

Let X be a compact manifold with a fixed volume form and

$$(1.6) \qquad R(\lambda; h) := (P(h) - iQ(h) - \lambda)^{-1} : L^2(X) \to L^2(X),$$

where $P(h), Q(h) \in \Psi_h^0(X)$ (henceforth denoted simply P, Q) are self-adjoint semiclassical pseudodifferential operators of order 0 with principal symbols p, q (see §2 for the semiclassical notation used) and $\lambda \in \mathbb{C}$ satisfies $\lambda = \mathcal{O}(h)$. We furthermore assume that $q \geqslant 0$ everywhere and $q(x, \xi) > 0$ for $|\xi|$ large enough. The family $R(\lambda; h)$ is meromorphic in λ, and we call its poles *resonances*. (The resonances for the model resolvent (1.6) need not coincide with resonances for the scattering resolvent (1.3), but a spectral gap for (1.6) implies a spectral gap for (1.3).)

As a particular application, [25] shows exponential decay of linear waves on Kerr–de Sitter black holes (modulo a finite dimensional space) using a resolvent estimate of type (1.7). We refer the reader to [25, 10] for further discussion of the relation between resolvent bounds and wave decay, and for an overview of previous results on wave decay for black holes. We remark that the analysis of trapping is critical for the understanding of semilinear and quasilinear wave equations on black holes – see the work of Hintz–Vasy [21, 19]. In particular, the loss of regularity in the decay estimate (quantified by $m > 0$ in (1.1)) is the reason why [19] has to invoke Nash–Moser theory.

We also point out that normally hyperbolic (in fact, r-normally hyperbolic) trapped sets appear naturally in the semiclassical theory of chemical reaction dynamics, see [17] for a physical description and [23, Remark 1.1] for a mathematical explanation.

Our results also hold (with the same proofs) in the general framework of [11, §4.1], which does not use complex absorbing operators and applies to a variety of scattering problems. In fact, [11, §§4.2, 8.1] reduces the general case microlocally to a neighborhood of the trapped set, providing an alternative to the gluing method discussed in the motivation section above.

We make the following normally hyperbolic trapping assumptions (see Figure 1.1; these assumptions provide a definition of $\Gamma_\pm$ and K in our framework):

(1) $\Gamma_\pm$ are codimension 1 orientable C^∞ submanifolds of T^*X such that $\Gamma_\pm \cap \{p = 0\} \cap \{q = 0\}$ are compact;

(2) if $(x, \xi) \in \{p = 0\} \setminus \Gamma_\pm$, then $e^{\mp tH_p}(x, \xi) \in \{q > 0\}$ for some $t \geqslant 0$;

(3) the Hamiltonian field H_p is tangent to $\Gamma_\pm$;
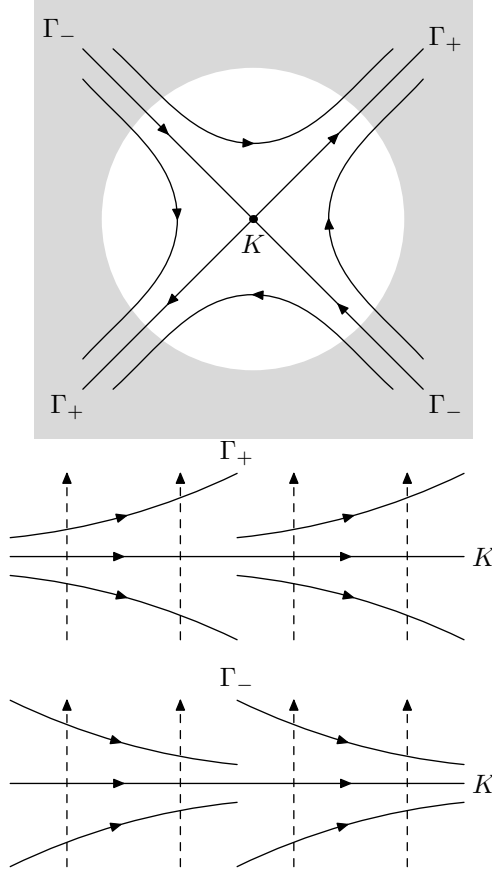
Figure 1.1. An illustration of normally hyperbolic trapping, with the Hamiltonian flow lines of p on the whole T^*X shown on the left and the restriction of the flow to $\Gamma_\pm$, on the right. The shaded region is the set $\{q > 0\}$ where the complex absorbing operator takes over. The dashed lines are the flow lines of $H_{\varphi_\pm}$, see §2.2.

(4) $\Gamma_\pm$ intersect transversely, $K = \Gamma_+ \cap \Gamma_-$ is called the *trapped set* and we assume that $\mathrm{WF}_h(Q) \cap K \cap \{|p| \leqslant \delta\} = \emptyset$ for $\delta > 0$ small enough;

(5) K is a symplectic codimension 2 submanifold of T^*X;

(6) if $v \in T_K\Gamma_\pm$, then $de^{\mp tH_p} \cdot v$ exponentially approaches $TK \subset T_K\Gamma_\pm$ as $t \to +\infty$.

Assumptions (1)–(6) hold for subextremal Kerr and Kerr–de Sitter black holes with small cosmological constant, see [28, §2], [25, §6.4], and [10, §3.2]. Note that, as in [11], $\Gamma_\pm$ are open subsets of the full incoming/outgoing tails of the flow cut off to a small neighborhood of $\{p = 0\} \cap \{q = 0\}$ (and thus are noncompact manifolds without boundary); as in [25] (see also [10, §3.5]), one needs to embed the Kerr(–de Sitter) trapped set into a compact manifold without boundary.

We furthermore define $0 < \nu_{\min} \leqslant \nu_{\max}$ as the maximal and the minimal numbers such that for each $\varepsilon > 0$ there exists a constant C such that for each $v \in T_K \Gamma_\pm$

$$C^{-1} e^{-(\nu_{\max}+\varepsilon)t} |\pi(v)| \leqslant |\pi(de^{\mp tH_p} \cdot v)| \leqslant C e^{-(\nu_{\min}-\varepsilon)t} |\pi(v)|, \quad t \geqslant 0,$$

where $\pi : T_K \Gamma_\pm \to T_K \Gamma_\pm$ is any fixed smooth linear projection map whose kernel is equal to TK. In other words, $\nu_{\min}$ and $\nu_{\max}$ are the minimal and maximal expansion rates in directions transversal to the trapped set.

Results

Our first result is a resonance free strip with a polynomial resolvent bound:

THEOREM 1.1. — *For each $\varepsilon > 0$ and h small enough depending on ε (see Figure 1.2)*

(1.7)
$$\|R(\lambda)\|_{L^2(X) \to L^2(X)} = o(h^{-2})$$
$$\text{if } |\lambda| = \mathcal{O}(h), \ \text{Im}\,\lambda > -(\nu_{\min} - \varepsilon)h/2.$$

Remarks 1.2.

(i) Theorem 1.1 also extends to the case when P, Q are operators acting on sections of some vector bundle $\mathcal{E}$ over X, as long as they are self-adjoint with respect to some smooth inner product on the fibers of $\mathcal{E}$, and their principal symbols (see for instance [13, §C.1] for a definition) are equal to $p \cdot \text{Id}_{\mathcal{E}}, q \cdot \text{Id}_{\mathcal{E}}$ for $p, q \in C^\infty(X)$. We can also relax the assumption that $Q = \mathcal{O}(h^\infty)$ microlocally near $K \cap \{p = 0\}$, requiring instead that $q = 0$ and $\sigma(h^{-1}Q) \geqslant 0$ near $K \cap \{p = 0\}$ (as an endomorphism of $\mathcal{E}$). Indeed, the proof of Lemma 3.1 still applies, and the relation (3.9) becomes a lower bound on $\mu(e^{-tH_p}(U_\delta))$. We leave the details to the reader.
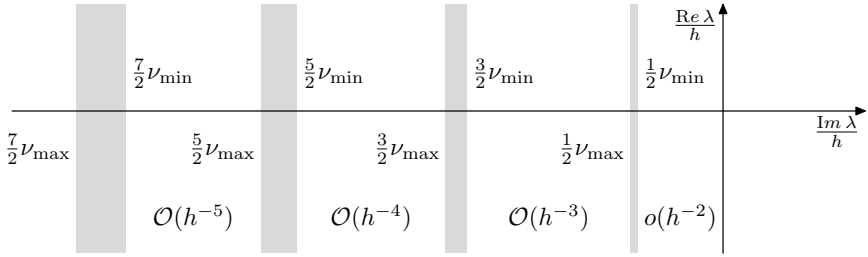
Figure 1.2. The resonance free strips of Theorems 1.1 and 1.3 (ignoring ε) are in white, with bounds on the resolvent $R(\lambda)$ written in the strips and (potential) bands of resonances in between the strips shaded.

(ii) We note that polynomial resolvent bounds are known in a variety of other situations – see the review of Wunsch [27] for an overview of the subject and [4] for other recent results. Also the much earlier work of Gérard–Sjöstrand [16] treated normally hyperbolic trapping in the analytic category with (implicit) exponential resolvent bounds.

We now make the stronger assumption that the trapping is *r-normally hyperbolic* for large r, namely

$$(1.8) \qquad\qquad \nu_{\min} > r\mu_{\max},$$

where $\mu_{\max}$ is the maximal expansion rate along K, namely $\|de^{tH_p}|_{TK}\| = \mathcal{O}(e^{(\mu_{\max}+\varepsilon)t})$ as $|t| \to \infty$, for all $\varepsilon > 0$. The condition (1.8) means that the rate of expansion in directions transversal to K is much larger than the rate of expansion along K, and it holds for Kerr(–de Sitter) black holes (see the references above).

Under the assumption (1.8) and an appropriate pinching condition, we exhibit deeper resonance free strips with polynomial resolvent bounds:

THEOREM 1.3. — *Assume that the trapping is r-normally hyperbolic for all r. Fix $m \geqslant 1$ and assume the pinching condition*

$$(m + 1/2)\nu_{\min} > (m - 1/2)\nu_{\max}.$$

Then for each small $\varepsilon > 0$ and h small enough depending on ε, we have (see Figure 1.2)

$$(1.9) \quad \begin{aligned} \|R(\lambda)\|_{L^2(X)\to L^2(X)} &\leqslant Ch^{-m-2} \quad \text{if } |\lambda| = \mathcal{O}(h), \\ &\text{and } h^{-1}\operatorname{Im}\lambda \in [-(m+1/2)\nu_{\min}+\varepsilon, -(m-1/2)\nu_{\max}-\varepsilon]. \end{aligned}$$

Remarks 1.4.

(i) Since semiclassical arguments only require finitely many derivatives, it is straightforward to see that Theorem 1.3 is also valid for trapping which is r-normally hyperbolic for some fixed r, where r is chosen large enough depending only on m and on the dimension of X. Such trapping is structurally stable under perturbations of the symbol p, see [22] and [11, §5.2].

(ii) Unlike Theorem 1.1, we only prove Theorem 1.3 in the scalar case, rather than for operators on general vector bundles, and for $Q = \mathcal{O}(h^\infty)$ near $K \cap \{p = 0\}$.

(iii) The pinching condition is true (for all m) for the Schwarzschild(–de Sitter) black hole. In the Kerr case, it was studied for $m = 1$ in [10, Figure 2(a) and §3.3].

(iv) A more careful argument, using in part the proof of Theorem 1.1, shows that the bound (1.9) could be improved to $\mathcal{O}(h^{-m-1})$ for $h^{-1} \operatorname{Im} \lambda \geqslant -m\nu_{\min} + \varepsilon$ and to $o(h^{-m-2})$ elsewhere in the strip, and more optimal bounds can be obtained using complex interpolation. The transition from one bound to the other at $h^{-1} \operatorname{Im} \lambda \geqslant -m\nu_{\min}$ is similar in spirit to the transition from the nontrapping bound $\mathcal{O}(h^{-1})$ in the upper half-plane to the $o(h^{-2})$ bound in the first gap. The method of Theorem 1.1 produces the slightly better bound of $o(h^{-m-2})$ because Lemma 3.2 applies under slightly more relaxed conditions on the norm of $\Theta^+ u$ than Lemma 4.4.

Ideas of the proofs

The proof of Theorem 1.3 is based on constructing pseudodifferential operators Θ_j^+, W_j^+ such that microlocally near $K \cap \{p = 0\}$, we have

$$\Theta_m^+ \dots \Theta_0^+ (P - \lambda) = (P - ihW_{m+1}^+ - \lambda)\Theta_m^+ \dots \Theta_0^+ + \mathcal{O}(h^\infty).$$

The principal symbol of W_{m+1}^+ is bounded from below by $(m+1)(\nu_{\min} - \varepsilon)$ near $K \cap \{p = 0\}$, which has the effect of shifting the spectral parameter λ into the upper half-plane. Since there are no resonances in the upper half-plane, we obtain for m large enough depending on $\operatorname{Im} \lambda$, if $(P - iQ - \lambda)u = \mathcal{O}(h^\infty)$, then

(1.10) $\qquad \Theta_m^+ \dots \Theta_0^+ u = \mathcal{O}(h^\infty) \quad$ microlocally near $K \cap \{p = 0\}$.

This is a pseudodifferential equation on u. The principal symbol of each Θ_j^+ is a defining function φ_+ of Γ_+, and the equation (1.10) gives information

about the phase space behavior of u along the Hamiltonian flow lines of φ_+, which are transversal to K (see Figure 1.1). Together with the information about the decay of u as one approaches K, which depends on $\operatorname{Im}\lambda$, one can show that λ has to avoid the strips in (1.9).

A good example to keep in mind, considered in §5, is

$$P_0 = \frac{h}{i}(x\partial_x + 1/2).$$

The relevant solutions to the equation $(P_0 - \lambda)u = 0$ are those whose wavefront set lies on $\Gamma_+ = \{\xi = 0\}$, which in particular means that u has to be smooth. This gives the resonances $\lambda_m = -(m + \frac{1}{2})ih$, $m = 0, 1, \ldots$ with resonant states $u_m(x) = x^m$ and the operator ∂_x maps u_m to mu_{m-1}. For the case of P_0, we can then take $\Theta_j^+ = \frac{h}{i}\partial_x$ and $W_j^+ = j$.

In the case of Theorem 1.1, we cannot construct the operators Θ_j^+, however a rougher version of the operator Θ_0^+ gives enough information on the concentration of u along the flow lines of H_{φ_+} to still obtain an estimate. The proof of Theorem 1.1 in given in §3 and the proof of Theorem 1.3 is given in §4. Both sections use preliminary dynamical and analytical constructions of §2.

Bounds on the real line

For λ on the real line, it is shown in [28] that

$$(1.11) \qquad \|R(\lambda)\|_{L^2 \to L^2} \leqslant Ch^{-1}\log(1/h), \quad \text{if } |\lambda| = \mathcal{O}(h), \ \operatorname{Im}\lambda = 0.$$

The results of Burq–Zworski [3] and Datchev–Vasy [8] (see also the papers of Burq [1, 2], Vodev [26], and Datchev [5] for bounds on the two-sided cutoff resolvent) deduce the following improved cutoff resolvent bound:

$$(1.12) \quad \|R(\lambda)A\|_{L^2 \to L^2} \leqslant Ch^{-1}\sqrt{\log(1/h)}, \quad \text{if } |\lambda| = \mathcal{O}(h), \ \operatorname{Im}\lambda = 0,$$

if A is an h-pseudodifferential operator such that $\mathrm{WF}_h(A) \cap K = \emptyset$. The bound (1.12) can be improved to $\mathcal{O}(h^{-1})$ (which is the estimate in the case there is no trapping) if one puts A on both sides of $R(\lambda)$, or if the principal symbol of A vanishes on Γ_- – see [7, 20].

In §5, we show that the bound (1.12) is sharp by giving an example of an operator $P - iQ$ and A for which the corresponding lower cutoff resolvent bound holds.

2. Preliminaries

The proofs in this paper rely on the methods of *semiclassical analysis*. We refer the reader to [29] for an introduction to the subject, and to [13, Appendix C.2] for the notation used here. In particular, we use the algebra $\Psi_h^0(X)$ of semiclassical pseudodifferential operators with symbols in the class $S^0(X)$. We moreover require that symbols of the elements of $\Psi_h^0(X)$ are *classical* in the sense that they have an asymptotic expansion in non-negative integer powers of h. Denote by

$$\sigma : \Psi_h^0(X) \to S^0(X)$$

the principal symbol map and recall the standard identities for $A, B \in \Psi_h^0(X)$,
(2.1)
$$\sigma(AB) = \sigma(A)\sigma(B), \quad \sigma(A^*) = \overline{\sigma(A)}, \quad \sigma(h^{-1}[A, B]) = -i\{\sigma(A), \sigma(B)\}.$$

Moreover, each $A \in \Psi_h^0(X)$ acts $L^2(X) \to L^2(X)$ with norm bounded uniformly in h. In fact, we have the bound [29, Theorem 5.1]

$$(2.2) \qquad \limsup_{h \to 0} \|A\|_{L^2(X) \to L^2(X)} \leqslant \|\sigma(A)\|_\infty := \sup_{T^*X} |\sigma(A)|.$$

We will mostly use the subalgebra of *compactly microlocalized* pseudodifferential operators, $\Psi_h^{\mathrm{comp}}(X) \subset \Psi_h^0(X)$. For $A \in \Psi_h^{\mathrm{comp}}(X)$, its *wavefront set* $\mathrm{WF}_h(A)$ is a compact subset of T^*X. We say that $A = B + \mathcal{O}(h^\infty)$ microlocally in some set $U \subset T^*X$ if $\mathrm{WF}_h(A - B) \cap U = \emptyset$.

We will often consider sequences $u(h_j) \in L^2(X)$, where $h_j \to 0$ is a sequence of positive numbers; we typically suppress the dependence on j and simply write $u = u(h_j)$ and $h = h_j$. For such a sequence, we say that $u = O(h^\delta)$ microlocally in some open set $U \subset T^*X$, if $\|Au\|_{L^2} = O(h^\delta)$ for each $A \in \Psi_h^{\mathrm{comp}}(X)$ such that $\mathrm{WF}_h(A) \subset U$. The notion of $u = o(h^\delta)$ in U is defined similarly.

We may also consider distributions in $L^2(X; \mathcal{E})$, where $\mathcal{E}$ is some smooth vector bundle over X with a prescribed inner product. One can take pseudodifferential operators acting on sections of $\mathcal{E}$, see [13, Appendix C.1], and the principal symbol of such an operator is a section of the endomorphism bundle $\mathrm{End}(\mathcal{E})$ over T^*X. However, we will only consider *principally scalar* operators, that is, operators whose principal symbols have the form $a(x, \xi)\,\mathrm{Id}_\mathcal{E}$. (Equivalently, these are the operators $A \in \Psi^k(X; \mathrm{End}(\mathcal{E}))$ such that for all $B \in \Psi^k(X; \mathrm{End}(\mathcal{E}))$, we have $[A, B] = \mathcal{O}(h)$.) The formulas (2.1) still hold for principally scalar operators. We henceforth suppress $\mathcal{E}$ in the notation.

2.1. Semiclassical defect measures

Our proofs rely on the following definition, designed to capture the concentration of a sequence of L^2 functions in phase space:

DEFINITION 2.1. — *Assume that $h_j \to 0$ and $u = u(h_j) \in L^2(X)$ is bounded uniformly in L^2 as $j \to \infty$. We say that u converges to a nonnegative Radon measure μ on T^*X if for each $A = A(h) \in \Psi_h^{\mathrm{comp}}(X)$, we have*

$$(2.3) \qquad \langle A(h_j)u(h_j), u(h_j) \rangle \to \int_{T^*X} \sigma(A) \, d\mu \quad \text{as } j \to \infty.$$

See [29, Chapter 5] for an introduction to defect measures; in particular,

- for each sequence $u(h_j)$ uniformly bounded in L^2, there exists a subsequence h_{j_k} such that $u(h_{j_k})$ converges to some μ [29, Theorem 5.2];
- we have $\mu(U) = 0$ for some open $U \subset T^*X$ if and only if $u = o(1)$ microlocally on U.

We now show the basic properties of semiclassical measures corresponding to approximate solutions of differential equations:

LEMMA 2.2 (Ellipticity). — *Take $P \in \Psi_h^0(X)$ and denote $p = \sigma(P)$. Assume that $u = u(h_j)$ converges to some measure μ and $Pu = o(1)$ microlocally in some open set $U \subset T^*X$. Then $\mu(U \cap \{p \neq 0\}) = 0$.*

Proof. — By (2.1), we have for each $A \in \Psi_h^{\mathrm{comp}}(X)$ with $\mathrm{WF}_h(A) \subset U$,

$$\int_{T^*X} \sigma(A) p \, d\mu = \lim_{h \to 0} \langle APu, u \rangle = 0.$$

Since $\sigma(A)p$ can be any function in $C_0^\infty(U \cap \{p \neq 0\})$, we have $\mu(U \cap \{p \neq 0\}) = 0$. $\qquad\square$

LEMMA 2.3 (Propagation). — *Take $P, W \in \Psi_h^0(X)$, denote $p = \sigma(P)$, $w = \sigma(W)$, and assume that $P^* = P$. Assume that $u = u(h_j)$ converges to some measure μ and denote $f := (P - ihW)u$. Then for each $a \in C_0^\infty(T^*X)$ and for each $Y \in \Psi_h^{\mathrm{comp}}(X)$ such that $Y = 1 + \mathcal{O}(h^\infty)$ microlocally in a neighborhood of $\mathrm{supp}\, a$,*

$$(2.4) \quad \left| \int_{T^*X} (H_p - 2\,\mathrm{Re}\, w) a \, d\mu \right| \leqslant 2\|a\|_\infty \cdot \limsup_{h \to 0} (h^{-1}\|Yf\|_{L^2} \cdot \|Yu\|_{L^2}).$$

In particular, if $f = o(h)$ microlocally in a neighborhood of $\mathrm{supp}\, a$, then

$$\int_{T^*X} (H_p - 2\,\mathrm{Re}\, w) a \, d\mu = 0.$$

Proof. — Take $A \in \Psi_h^{\mathrm{comp}}(X)$ with $\sigma(A) = a$ and $Y = 1 + \mathcal{O}(h^\infty)$ near $\mathrm{WF}_h(A)$. Without loss of generality, we may assume that $W^* = W$; indeed, one can put the imaginary part of W into P. Then we have

$$(2.5) \qquad \frac{\langle Af, u \rangle - \langle Au, f \rangle}{ih} = \frac{\langle A(P - ihW)u, u \rangle - \langle (P + ihW)Au, u \rangle}{ih}$$
$$= \langle ((ih)^{-1}[A, P] - (AW + WA))u, u \rangle.$$

By (2.1), $\sigma\big((ih)^{-1}[A, P] - (AW + WA)\big) = H_p a - 2wa$; therefore, the right-hand side of (2.5) converges in absolute value to the left-hand side of (2.4). The left-hand side of (2.5) is equal to

$$\frac{\langle AYf, Yu \rangle - \langle AYu, Yf \rangle}{ih} + \mathcal{O}(h^\infty),$$

and its limit as $h \to 0$ is bounded by the right-hand side of (2.4) by (2.2). $\qquad \square$

2.2. Dynamical preliminaries

In this section, we review several properties of normally hyperbolic trapping, assuming that $p, q, \Gamma_\pm, K$ satisfy properties (1)–(6) listed in the introduction. We start with the following restatement of [11, Lemma 5.1], see also [28, Lemma 4.1]. To be able to quantize the functions $\varphi_\pm, c_\pm$, we multiply them by a cutoff to obtain compactly supported functions, but only require their properties to hold in a neighborhood U of $K \cap \{p = 0\}$.

LEMMA 2.4. — *Fix small $\varepsilon > 0$. Then there exists a bounded neighborhood U of $K \cap \{p = 0\}$ and functions $\varphi_\pm \in C_0^\infty(T^*X)$ such that $\mathrm{WF}_h(Q) \cap U = \emptyset$ and*

(1) *for $\delta > 0$ small enough, the set*

$$(2.6) \qquad U_\delta := \{|\varphi_+| < \delta, \ |\varphi_-| < \delta, \ |p| < \delta\} \cap U$$

is compactly contained in U;

(2) $\Gamma_\pm \cap U = \{\varphi_\pm = 0\} \cap U$;

(3) $H_p \varphi_\pm = \mp c_\pm \varphi_\pm$ *on U, where $c_\pm \in C_0^\infty(T^*X)$ satisfy*

$$0 < \nu_{\min} - \varepsilon \leqslant c_\pm \leqslant \nu_{\max} + \varepsilon \quad \text{on } U;$$

(4) $\{\varphi_+, \varphi_-\} > 0$ *on U.*

Note that in particular, for $\delta > 0$ small enough and all $t \geqslant 0$,

$$(2.7) \qquad e^{-tH_p}(U_\delta \cap \Gamma_+) \subset \{|\varphi_-| < e^{-(\nu_{\min} - \varepsilon)t}\delta\} \cap U_\delta \cap \Gamma_+.$$

Moreover, the map

$$(2.8) \qquad\qquad (\rho, s) \mapsto e^{sH_{\varphi_+}}(\rho), \quad \rho \in K, \ s \in \mathbb{R}$$

is a diffeomorphism from some neighborhood of $(K \cap \{p = 0\}) \times \{s = 0\}$ in $K \times \mathbb{R}$ onto some neighborhood of $K \cap \{p = 0\}$ in Γ_+.

For Theorem 1.3, we also need existence of solutions to the transport equation on Γ_+:

LEMMA 2.5. — *Assume that the flow is r-normally hyperbolic in the sense of (1.8), for all r. Fix small $\delta > 0$ and take U_δ defined by (2.6). Then for each $f \in C^\infty(\Gamma_+ \cap U_\delta)$, there exists unique $u \in C^\infty(\Gamma_+ \cap U_\delta)$ such that*

$$(2.9) \qquad\qquad (H_p + c_+)u = f.$$

Proof. — We use [11, Lemma 5.2], which gives existence and uniqueness of $v \in C^\infty(\Gamma_+ \cap U_\delta)$ such that $H_p v = g$ and $v|_K = 0$, for each $g \in C^\infty(\Gamma_+ \cap U_\delta)$ such that $g|_K = 0$.

We claim that there exists a function $G \in C^\infty(\Gamma_+ \cap U_\delta)$ such that

$$H_p F = c_+ F, \quad F := e^G \{\varphi_+, \varphi_-\}^{-1} \varphi_-.$$

Indeed, G needs to solve the equation

$$H_p G = c_+ - c_- + \frac{H_p\{\varphi_+, \varphi_-\}}{\{\varphi_+, \varphi_-\}},$$

and we use [11, Lemma 5.2] since the right-hand side of this equation vanishes on K.

Now (2.9) becomes

$$H_p(Fu) = Ff,$$

and it remains to invoke [11, Lemma 5.2] once again.   □

2.3. Basic estimates

We now derive some estimates for the operator $P - iQ - \lambda$, where P, Q satisfy the assumptions of Theorem 1.1. We start by using the complex absorbing operator Q to reduce the analysis to a neighborhood of $K \cap \{p = 0\}$, where K is the trapped set:

LEMMA 2.6. — *Assume that $h_j \to 0$, $u = u(h_j) \in L^2(X)$ is bounded uniformly in h, and*

$$\|(P - iQ - \lambda)u\|_{L^2} = \mathcal{O}(h^{\alpha+1})$$

for some constant $\alpha > 0$ and $\lambda = \mathcal{O}(h)$. Then:

1. $u = \mathcal{O}(h^\alpha)$ microlocally on the complement of $\{p = 0\} \cap \Gamma_+$.
2. If u converges to some measure μ in the sense of Definition 2.1, then μ is supported on $\{p = 0\} \cap \Gamma_+$. If moreover $\|u\|_{L^2} \geqslant c > 0$ for some constant c, then for each neighborhood U of $\{p = 0\} \cap K$, we have $\mu(U) > 0$.

Same statements are true if we replace $\mathcal{O}(h^{\alpha+1}), \mathcal{O}(h^\alpha)$ by $o(h^{\alpha+1}), o(h^\alpha)$ and $\alpha \geqslant 0$.

Proof. — Take $A \in \Psi_h^0(X)$ such that $\mathrm{WF}_h(A) \cap \{p = 0\} \cap \Gamma_+ = \emptyset$. Then

$$(2.10) \qquad \|Au\|_{L^2} \leqslant Ch^{-1}\|(P - iQ - \lambda)u\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2}.$$

This follows from the elliptic estimate [13, Proposition 2.4] and propagation of singularities with a complex absorbing operator [13, Proposition 2.5], since for each $(x, \xi) \in \mathrm{WF}_h(A)$, there exists $t \geqslant 0$ such that $e^{-tH_p}(x, \xi) \in \{p - iq \neq 0\}$.

The estimate (2.10) immediately implies part 1 of the lemma, as well as the first statement of part 2. To see the second statement of part 2, it suffices to use the following estimate, valid for each $B \in \Psi_h^{\mathrm{comp}}(X)$ such that $\sigma(B) \neq 0$ on $\{p = 0\} \cap K$:

$$(2.11) \qquad \|u\|_{L^2} \leqslant Ch^{-1}\|(P - iQ - \lambda)u\|_{L^2} + \|Bu\|_{L^2}.$$

To prove (2.11), we again use the elliptic estimate and propagation of singularities, noting that for each $(x, \xi) \in T^*X$, there exists $t \geqslant 0$ that $e^{-tH_p}(x, \xi) \in \{p - iq \neq 0\} \cup \{\sigma(B) \neq 0\}$, see [11, Lemma 4.1]. □

The next lemma is a generalization of the statement that there are no resonances in the upper half-plane (which, keeping in mind Lemma 2.6, is the special case with $W = 0$ and $\mathrm{Im}\,\lambda \geqslant ch$). See [11, §8.2] for a proof using positive commutator estimates directly instead of going through semiclassical measures. One could also replace $\mathrm{Re}\,\sigma(W)$ in (2.12) by its finite time avarage average along the flow of H_p on $K \cap \{p = 0\}$, see for instance [24, Theorem 3.2].

LEMMA 2.7. — Fix small $\delta > 0$; we use the set U_δ defined in (2.6). Assume that $A, B, B_1 \in \Psi_h^{\mathrm{comp}}(X)$ satisfy (see Figure 2.1)

— $\mathrm{WF}_h(A) \subset U_{3\delta/2}$ and $A = 1 + \mathcal{O}(h^\infty)$ microlocally on $\overline{U_\delta}$;
— $\mathrm{WF}_h(B) \subset U_{3\delta} \cap \{|\varphi_+| > \delta/2\}$ and $B = 1 + \mathcal{O}(h^\infty)$ microlocally on $\overline{U_{2\delta}} \cap \{|\varphi_+| \geqslant \delta\}$;
— $\mathrm{WF}_h(B_1) \subset U_{3\delta}$ and $B_1 = 1 + \mathcal{O}(h^\infty)$ microlocally on $\overline{U_{2\delta}}$.

Take $W \in \Psi_h^{\mathrm{comp}}(X)$ and $\lambda = \mathcal{O}(h)$ such that

$$(2.12) \qquad \mathrm{Re}\,\sigma(W) + h^{-1}\mathrm{Im}\,\lambda \geqslant c > 0 \quad \text{on } U_{3\delta}$$
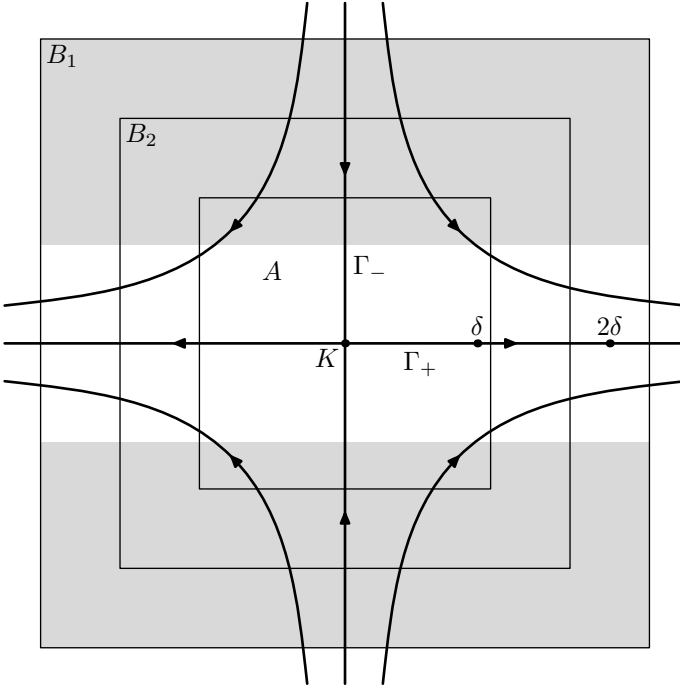
Figure 2.1. The flow H_p and the wavefront sets of A, B, B_1, B_2; here $\mathrm{WF}_h(B)$ is the shaded portion of $\mathrm{WF}_h(B_1)$. The horizontal axis is φ_- and the vertical axis is φ_+.

for some constant c. Then for each $u \in L^2(X)$,

$$\|Au\|_{L^2} \leqslant Ch^{-1}\|B_1(P - ihW - \lambda)u\|_{L^2} + C\|Bu\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2}.$$

Proof. — Take $B_2 \in \Psi_h^{\mathrm{comp}}(X)$ such that $\mathrm{WF}_h(B_2) \subset U_{2\delta}$ and $B_2 = 1 + \mathcal{O}(h^\infty)$ microlocally on $\overline{U_{3\delta/2}}$. We first claim that for each fixed $\varepsilon_0 > 0$, there exists a constant C such that for h small enough,

$$(2.13) \quad \|Au\|_{L^2} \leqslant Ch^{-1}\|B_2(P - ihW - \lambda)u\|_{L^2} + C\|Bu\|_{L^2} + \varepsilon_0\|u\|_{L^2}.$$

We argue by contradiction. If (2.13) does not hold, then there exist sequences $h_j \to 0$, $\lambda = \lambda(h_j)$, and $u = u(h_j) \in L^2$ such that $\|u\|_{L^2} \leqslant 1$, $(P - ihW - \lambda)u = o(h)$ microlocally in $U_{3\delta/2}$, and $\|Bu\|_{L^2} = o(1)$, yet $\|Au\|_{L^2}$ is separated away from zero.

By passing to a subsequence, we may assume that u converges to some measure μ in the sense of Definition 2.1. Since for each $(x, \xi) \in U_{3\delta/2} \setminus \Gamma_+$, there exists $t \geqslant 0$ such that $e^{-tH_p}(x, \xi) \in \{\sigma(B) = 1\}$ and $e^{-sH_p}(x, \xi) \in$

$U_{3\delta/2}$ for $s \in [0, t]$, by propagation of singularities [13, Proposition 2.5] we see that $u = o(1)$ microlocally in $U_{3\delta/2} \setminus \Gamma_+$ and thus

$$(2.14) \qquad \mu\big(U_{3\delta/2} \setminus \Gamma_+\big) = 0.$$

Passing to a subsequence, we may assume that $h^{-1}\lambda \to \omega \in \mathbb{C}$. By Lemma 2.3, for each $a \in C_0^\infty(U_{3\delta/2})$,

$$\int_{T^*X} H_p a \, d\mu = 2 \int_{T^*X} (\operatorname{Re} \sigma(W) + \operatorname{Im} \omega) a \, d\mu.$$

By (2.7), (2.14), and (2.12), it follows that for each nonnegative $a \in C_0^\infty(U_{3\delta/2})$,

$$\int_{\Gamma_+ \cap U_{3\delta/2}} a \circ e^{tH_p} \, d\mu \geqslant e^{2ct} \int_{\Gamma_+ \cap U_{3\delta/2}} a \, d\mu, \quad t \geqslant 0.$$

However, the left-hand side is bounded uniformly as $t \to +\infty$ (by $\|a\|_\infty \cdot \mu(U_{3\delta/2})$), therefore $\mu(U_{3\delta/2}) = 0$ and thus $\|Au\|_{L^2} = o(1)$, giving a contradiction and proving (2.13).

Take $B_2' \in \Psi_h^{\mathrm{comp}}(X)$ such that $B_2' = 1 + \mathcal{O}(h^\infty)$ microlocally on $\mathrm{WF}_h(B_2)$. We apply (2.13) to $B_2'u$ to get for each $\varepsilon_0 > 0$,

$$(2.15) \qquad \begin{aligned} \|Au\|_{L^2} &\leqslant Ch^{-1}\|B_2(P - ihW - \lambda)u\|_{L^2} + C\|Bu\|_{L^2} \\ &\qquad + \varepsilon_0\|B_2'u\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2}. \end{aligned}$$

Now, for a correct choice of B_2', for each $(x, \xi) \in \mathrm{WF}_h(B_2') \cap \{p = 0\}$, there exists $t \geqslant 0$ such that $e^{-tH_p}(x, \xi) \in \{\sigma(A) \neq 0\} \cup \{\sigma(B) \neq 0\}$ and $e^{-sH_p}(x, \xi) \in \{\sigma(B_1) \neq 0\}$ for $s \in [0, t]$. By propagation of singularities, we then have

$$(2.16) \qquad \begin{aligned} \|B_2'u\|_{L^2} &\leqslant Ch^{-1}\|B_1(P - ihW - \lambda)u\|_{L^2} + C\|Bu\|_{L^2} \\ &\qquad + C\|Au\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2}. \end{aligned}$$

It remains to combine (2.15) with (2.16) and take ε_0 small enough. $\qquad\square$

3. First gap

In this section, we prove Theorem 1.1. We argue by contradiction. Assume that the estimate (1.7) does not hold; then (replacing ε by 2ε) there exist sequences $h_j \to 0$, $\lambda = \lambda(h_j)$, and $u = u(h_j) \in L^2(X)$ such that

$$\begin{aligned} \|u\|_{L^2} &= 1, & \|(P - iQ - \lambda)u\|_{L^2} &= \mathcal{O}(h^2), \\ |\lambda| &= \mathcal{O}(h), & \operatorname{Im} \lambda &> -(\nu_{\min} - 2\varepsilon)h/2. \end{aligned}$$

By Lemma 2.6, we have

$$(3.1) \qquad u = \mathcal{O}(h) \quad \text{microlocally on } T^*X \setminus \Gamma_+.$$

Take small $\delta > 0$ and let $U_\delta \subset T^*X$ be defined in (2.6). Recall that $Q = \mathcal{O}(h^\infty)$ microlocally on $U_{3\delta}$; therefore,

$$(3.2) \qquad (P - \lambda)u = \mathcal{O}(h^2) \quad \text{microlocally on } U_{3\delta}.$$

Take $\Theta_+, W_+ \in \Psi_h^{\mathrm{comp}}(X)$ such that

$$\sigma(\Theta_+) = \varphi_+, \quad \sigma(W_+) = c_+, \quad \Theta_+^* = \Theta_+,$$

where φ_+, c_+ are constructed in Lemma 2.4.

LEMMA 3.1. — *For* $\operatorname{Im}\lambda > -(\nu_{\min} - 2\varepsilon)h$ *(which is weaker than the assumption of Theorem 1.1),*

$$(3.3) \qquad \Theta_+ u = \mathcal{O}(h) \quad \text{microlocally on } U_\delta.$$

Proof. — Since $H_p \varphi_+ = -c_+ \varphi_+$ on $U_{3\delta}$, we have

$$(3.4) \qquad [P, \Theta_+] = ihW_+\Theta_+ + \mathcal{O}(h^2)_{\Psi_h^{\mathrm{comp}}} \quad \text{microlocally on } U_{3\delta}.$$

Applying Θ_+ to (3.2) and using (3.4), we obtain

$$(3.5) \qquad (P - ihW_+ - \lambda)\Theta_+ u = \mathcal{O}(h^2) \quad \text{microlocally on } U_{3\delta}.$$

We have $\sigma(W_+) = c_+ \geqslant \nu_{\min} - \varepsilon$ on $U_{3\delta}$; therefore, $\sigma(W_+) + h^{-1}\operatorname{Im}\lambda \geqslant \varepsilon > 0$ on $U_{3\delta}$. It remains to apply Lemma 2.7 to $\Theta_+ u$ and use (3.1) and (3.5). $\square$

Now, passing to a subsequence of h_j, we may assume that (in the sense of Definition 2.1)

$$u \to \mu, \quad h^{-1}\operatorname{Im}\lambda \to \omega \in \mathbb{C}.$$

By Lemma 2.6, we have

$$(3.6) \qquad \mu(U_\delta) > 0, \quad \mu(U_\delta \setminus \Gamma_+) = 0.$$

We now show that μ is Lipschitz in the direction transversal to K. Recall that φ_-, constructed in Lemma 2.4, is a defining function of K on $\Gamma_+ \cap U_\delta$.

LEMMA 3.2. — *There exists a constant C such that for each $\delta_0 > 0$,*

$$(3.7) \qquad \mu\big(U_\delta \cap \{|\varphi_-| < \delta_0\}\big) \leqslant C\delta_0.$$

Proof. — We may assume that δ_0 is arbitrarily small. Applying Lemma 2.3 to (3.3), we see that there exists a constant C such that for each $a \in C_0^\infty(\Gamma_+ \cap U_\delta)$,

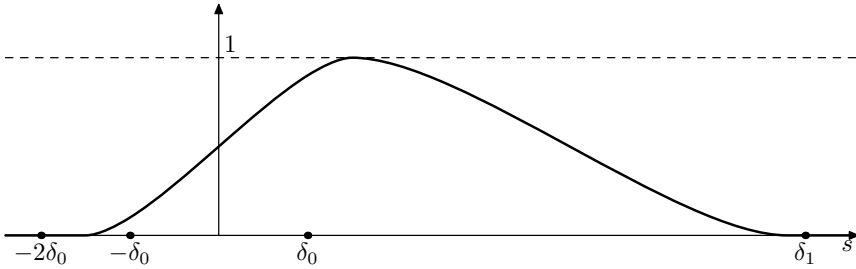$$(3.8) \qquad \left| \int_{\Gamma_+ \cap U_\delta} H_{\varphi_+} a \, d\mu \right| \leqslant C\|a\|_\infty.$$

Figure 3.1. The function $\tilde{a}$ used in the proof of Lemma 3.2. Here $\delta_1 > 0$ is fixed and $\delta_0 \to 0$.

To see (3.7), we need to apply this estimate to a function a depending on δ_0 which, written in coordinates (ρ, s) of (2.8), has the form $a = \tilde{a}(s)\chi(\rho)$ with $\chi \in C_0^\infty(K \cap \{|p| < \delta\})$ equal to 1 on $K \cap \{p = 0\}$ and

$$\operatorname{supp}\tilde{a} \subset (-2\delta_0, \delta_1), \quad \|\tilde{a}\|_\infty \leqslant 1, \quad \partial_s\tilde{a} \geqslant -\frac{2}{\delta_1},$$

$$\partial_s\tilde{a} \geqslant \frac{1}{3\delta_0} \quad \text{for } |s| \leqslant \delta_0$$

for some fixed small $\delta_1 \in (0, \delta)$ independent of δ_0, see Figure 3.1. Then

$$\int_{\Gamma_+ \cap U_\delta} H_{\varphi_+} a\, d\mu \;\geqslant\; \frac{1}{3\delta_0}\mu\big(\{|s| \leqslant \delta_0\}\big) - \frac{2}{\delta_1}\mu\big(\{\delta_0 \leqslant |s| \leqslant \delta_1\}\big)$$

The left-hand side is bounded by a δ_0-independent constant by (3.8); so is the second term on the right-hand side (since μ is a finite measure). Multiplying both sides by $3\delta_0$ and using that $\Gamma_+ \cap U_\delta \cap \{|\varphi_-| \leqslant \delta_0\} \subset \{|s| \leqslant C\delta_0\}$ for some constant C, we obtain (3.7). □

We now have by Lemma 2.3 applied to (3.2), for each $a \in C_0^\infty(U_\delta)$,

$$\int_{\Gamma_+ \cap U_\delta} H_p a\, d\mu = 2\operatorname{Im}\omega \int_{\Gamma_+ \cap U_\delta} a\, d\mu.$$

For $t \geqslant 0$, since $e^{-tH_p}(\Gamma_+ \cap U_\delta) \subset \Gamma_+ \cap U_\delta$, we have

$$(3.9) \qquad\qquad \mu(e^{-tH_p}(U_\delta)) = e^{2t\operatorname{Im}\omega}\mu(U_\delta).$$

However, by (2.7), (3.6), and (3.7), as $t \to +\infty$

$$\mu(e^{-tH_p}(U_\delta)) \;\leqslant\; \mu\big(U_\delta \cap \{|\varphi_-| < \delta e^{-(\nu_{\min}-\varepsilon)t}\}\big) \;\leqslant\; Ce^{-(\nu_{\min}-\varepsilon)t}.$$

However, since $\operatorname{Im}\omega \geqslant -(\nu_{\min} - 2\varepsilon)/2$, we see that $e^{2t\operatorname{Im}\omega}$ decays exponentially slower than $e^{-(\nu_{\min}-\varepsilon)t}$ as $t \to +\infty$. Since $\mu(U_\delta) > 0$ by (3.6), we arrive to a contradiction, finishing the proof of Theorem 1.1.

4. Further gaps for r-normally hyperbolic trapping

In this section, we prove Theorem 1.3. For small $\delta > 0$, let U_δ be defined by (2.6).

4.1. Auxiliary microlocal construction

The proof relies on the following statement, which in particular makes it possible to improve the remainder in (3.4) to $\mathcal{O}(h^\infty)$. For $Z = 0$, the operators Θ, W below were previously constructed in [11, Proposition 7.1].

LEMMA 4.1. — *Assume that the trapping is r-normally hyperbolic for all r, in the sense of (1.8). Then for each $Z \in \Psi_h^{\mathrm{comp}}(X)$, there exist $\Theta, W \in \Psi_h^{\mathrm{comp}}(X)$ such that*

$$(4.1) \qquad \Theta(P - ihZ) = (P - ihW)\Theta + \mathcal{O}(h^\infty) \quad \text{microlocally on } U_{3\delta};$$

$$(4.2) \qquad \sigma(\Theta) = \varphi_+, \quad \sigma(W) = \sigma(Z) + c_+ \quad \text{on } U_{3\delta}.$$

Proof. — We construct operators

$$\Theta_k, W_k \in \Psi_h^{\mathrm{comp}}(X), \quad k = 0, 1, \ldots,$$

such that microlocally in $U_{3\delta}$, we have for some $R_k \in \Psi_h^{\mathrm{comp}}(X)$,

$$(4.3) \qquad \Theta_k(P - ihZ) = (P - ihW_k)\Theta_k + h^{k+2}R_k + \mathcal{O}(h^\infty)$$

and take Θ, W to be the asymptotic limits as $k \to +\infty$ of these sequences of operators.

We use induction on k. For $k = 0$, take any Θ_0, W_0 satisfying in $U_{3\delta}$,

$$\sigma(\Theta_0) = \varphi_+, \quad \sigma(W_0) = \sigma(Z) + c_+,$$

then (4.3) follows immediately from (2.1) and the fact that $H_p\varphi_+ = -c_+\varphi_+$ on $U_{3\delta}$.

Assume now that $k > 0$ and (4.3) holds for $k - 1$; we construct the operators Θ_k and W_k in the form

$$\Theta_k := \Theta_{k-1} + h^k\Omega_k, \quad W_k := W_{k-1} + h^kY_k,$$

where $\Omega_k, Y_k \in \Psi_h^{\mathrm{comp}}(X)$. We rewrite (4.3) for k as the following statement, valid microlocally in $U_{3\delta}$:

$$[P, \Omega_k] + ih(\Omega_k Z - W_{k-1}\Omega_k - Y_k\Theta_{k-1}) = hR_{k-1} + \mathcal{O}(h^2)_{\Psi_h^{\mathrm{comp}}}.$$

and this translates to the following equation on the principal symbols[†] of Ω_k, Y_k in $U_{3\delta}$:

$$(H_p + c_+)\sigma(\Omega_k) + \varphi_+ \sigma(Y_k) \;=\; i\sigma(R_{k-1}).$$

Since we can choose $\sigma(Y_k)$ arbitrarily and φ_+ is a defining function of Γ_+, it suffices to find Ω_k whose symbol solves the transport equation

$$(H_p + c_+)\sigma(\Omega_k) \;=\; i\sigma(R_{k-1}) \quad \text{on } \Gamma_+ \cap U_{3\delta}.$$

The existence of such a symbol follows from Lemma 2.5 (which is where we use r-normal hyperbolicity), finishing the construction of Θ_k, W_k. $\quad\square$

Iterating Lemma 4.1, we obtain

LEMMA 4.2. — *Assume that the trapping is r-normally hyperbolic for all r. Then there exist operators*

$$\Theta_m^+, W_m^+ \in \Psi_h^{\text{comp}}(X), \quad m = 0, 1, \dots$$

such that $W_0^+ = 0$ and

(4.4) $\Theta_m^+(P - ihW_m^+) = (P - ihW_{m+1}^+)\Theta_m^+ + \mathcal{O}(h^\infty)$ *microlocally on $U_{3\delta}$.*

Moreover, we have

$$\sigma(\Theta_m^+) = \varphi_+, \quad \sigma(W_m^+) = mc_+ \quad \text{on } U_{3\delta}.$$

4.2. Measures associated to resonant states and proof of Theorem 1.3

Assume that $u = u(h_j) \in L^2(X)$ is bounded uniformly in h and define $u_0, u_1, \dots \in L^2(X)$ by the relations

(4.5) $$u_0 := u, \quad u_{m+1} := \Theta_m^+ u_m.$$

LEMMA 4.3. — *Assume that for some $\alpha \geqslant 0$ and $\lambda = \mathcal{O}(h)$,*

(4.6) $$\|(P - iQ - \lambda)u\|_{L^2} = o(h^{\alpha+1}).$$

Then for each $m \geqslant 0$,

(4.7) $\quad (P - ihW_m^+ - \lambda)u_m = o(h^{\alpha+1})$ *microlocally on $U_{3\delta}$;*

(4.8) $\quad u_m = o(h^\alpha)$ *microlocally on $U_{3\delta} \setminus (\Gamma_+ \cap \{p = 0\})$.*

[†] This is the place where we cannot take P to act on sections of a vector bundle, as R_{k-1} and thus Ω_k, Y_k need not be principally scalar and a more complicated transport equation would be required.

Moreover, if

(4.9) $\operatorname{Im} \lambda > -m(\nu_{\min} - \varepsilon)h + \varepsilon h,$

then $u_m = o(h^\alpha)$ microlocally on U_δ.

Proof. — The condition (4.7) for $m = 0$ follows directly from (4.6) (since $Q = \mathcal{O}(h^\infty)$ microlocally on $U_{3\delta}$); for $m > 0$, it follows by induction from (4.4). The condition (4.8) for $m = 0$ follows from Lemma 2.6, and is still true for $m \geqslant 0$ since u_m is obtained by applying an h-pseudodifferential operator to u.

The last statement of this lemma follows from (4.7), (4.8) by Lemma 2.7, as on $U_{3\delta}$, we have $\sigma(W_m^+) = mc_+ \geqslant m(\nu_{\min} - \varepsilon)$ and thus $\sigma(W_m^+) + h^{-1}\operatorname{Im}\lambda \geqslant \varepsilon > 0$. □

We now obtain information on semiclassical measures of solutions to $\Theta_m^+ v = o(h)$, which generalizes the Lipschitz property of Lemma 3.2. It is possible to give a more precise description of the measure along the Hamiltonian flow lines of φ_+; for Θ_0^+, this is done in [11, §8.5, Theorem 4].

LEMMA 4.4. — *Assume that $v = v(h_j) \in L^2(X)$ converges to some measure μ in the sense of Definition 2.1, and $\mu(U_\delta) > 0$, $\mu(U_\delta \setminus \Gamma_+) = 0$. Assume also that for some m,*

$$\Theta_m^+ v = o(h) \quad \text{microlocally in } U_\delta.$$

Then for some constant C and for each $\delta_0 \in (0,1)$, we have

$$C^{-1}\delta_0 \leqslant \mu\big(U_\delta \cap \{|\varphi_-| < \delta_0\}\big) \leqslant C\delta_0.$$

Proof. — Let $b := \sigma(h^{-1}\operatorname{Im}\Theta_m^+)$, where we denote $\operatorname{Im} A := \frac{1}{2i}(A - A^*)$. By Lemma 2.3, we see that for each $a \in C_0^\infty(U_\delta)$,

$$\int_{U_\delta \cap \Gamma_+} (H_{\varphi_+} + 2b)a\, d\mu = 0.$$

It remains to note that in the coordinates (2.8), we have for each $a \in C_0^\infty(U_\delta)$

$$\int a\, d\mu = \int e^{2\psi} a\, d\tilde{\mu}(\rho)ds$$

where $\psi \in C^\infty(U_\delta \cap \Gamma_+)$ is some fixed solution to the equation $H_{\varphi_+}\psi = b$ and $\tilde{\mu}$ is some measure on K. □

Lemma 4.4 makes it possible to show bounds on approximate solutions to $\Theta_m^+ v = 0$ and $(P - ihW_m^+ - \lambda)v = 0$ for $\operatorname{Im}\lambda$ outside of a certain interval:

LEMMA 4.5. — *Let $A, B_1 \in \Psi_h^{\text{comp}}(X)$ satisfy*

  — $\operatorname{WF}_h(A) \subset U_{3\delta/2}$ *and* $A = 1 + \mathcal{O}(h^\infty)$ *microlocally on* $\overline{U_\delta}$;

— $\mathrm{WF}_h(B_1) \subset U_{3\delta}$ and $B_1 = 1 + \mathcal{O}(h^\infty)$ microlocally on $\overline{U_{2\delta}}$.

Fix m and assume that $\lambda = \mathcal{O}(h)$ is such that

(4.10) $h^{-1} \operatorname{Im} \lambda \notin [-(m + 1/2)(\nu_{\max} + \varepsilon) - \varepsilon, -(m + 1/2)(\nu_{\min} - \varepsilon) + \varepsilon]$.

Then for each $v \in L^2(X)$,

$$\|Av\|_{L^2} \leqslant Ch^{-1}\big(\|B_1(P - ihW_m^+ - \lambda)v\|_{L^2} + \|B_1\Theta_m^+ v\|_{L^2}\big) + \mathcal{O}(h^\infty)\|v\|_{L^2}.$$

Proof. — Arguing similarly to the proof of Lemma 2.7 (removing B from (2.13) and (2.15) and replacing B by $B_1\Theta_m^+$ in (2.16)), we see that it suffices to show that if $h_j \to 0$ and $v = v(h_j)$ satisfies $\|v\|_{L^2} \leqslant 1$ and

(4.11) $(P - ihW_m^+ - \lambda)v = o(h)$ microlocally in $U_{3\delta/2}$,

(4.12) $\Theta_m^+ v = o(h)$ microlocally in $U_{3\delta/2}$,

then $v = o(1)$ microlocally in $U_{3\delta/2}$.

Passing to a subsequence, we may assume that v converges to some measure μ in the sense of Definition 2.1 and $h^{-1}\lambda \to \omega \in \mathbb{C}$, where

(4.13) $\operatorname{Im} \omega \notin [-(m + 1/2)(\nu_{\max} + \varepsilon) - \varepsilon, -(m + 1/2)(\nu_{\min} - \varepsilon) + \varepsilon]$.

We have $\mu(U_{3\delta/2} \setminus \Gamma_+) = 0$ by Lemma 2.2 applied to (4.12). By Lemma 2.3 applied to (4.11), we see that for each $a \in C_0^\infty(U_{3\delta/2})$,

$$\int_{\Gamma_+ \cap U_{3\delta/2}} H_p a \, d\mu = 2 \int_{\Gamma_+ \cap U_{3\delta/2}} (\operatorname{Im} \omega + \sigma(W_m^+))a \, d\mu;$$

for $t \geqslant 0$, since $e^{-tH_p}(\Gamma_+ \cap U_{3\delta/2}) \subset \Gamma_+ \cap U_{3\delta/2}$, we have by (4.13) either (when $\operatorname{Im} \omega$ is too big)

(4.14) $\mu(e^{-tH_p}(\Gamma_+ \cap U_{3\delta/2})) \geqslant e^{-(\nu_{\min} - 3\varepsilon)t}\mu(U_{3\delta/2})$

or (when $\operatorname{Im} \omega$ is too small)

(4.15) $\mu(e^{-tH_p}(\Gamma_+ \cap U_{3\delta/2})) \leqslant e^{-(\nu_{\max} + 3\varepsilon)t}\mu(U_{3\delta/2})$.

Here we used that $\sigma(W_m^+) = mc_+ \in [m(\nu_{\min} - \varepsilon), m(\nu_{\max} + \varepsilon)]$ on $U_{3\delta/2}$. Using Lemma 4.4 with (4.12) and observing that for $t \geqslant 0$, by Lemma 2.4

$$\{|\varphi_-| < \tfrac{3\delta}{2}e^{-(\nu_{\max} + \varepsilon)t}\} \cap \Gamma_+ \cap U_{3\delta/2} \quad \subset \quad e^{-tH_p}(\Gamma_+ \cap U_{3\delta/2}),$$
$$e^{-tH_p}(\Gamma_+ \cap U_{3\delta/2}) \quad \subset \quad \{|\varphi_-| < \tfrac{3\delta}{2}e^{-(\nu_{\min} - \varepsilon)t}\} \cap \Gamma_+ \cap U_{3\delta/2}$$

both (4.14) and (4.15) imply, by taking the limit $t \to +\infty$, that $\mu(U_{3\delta/2}) = 0$ and thus $v = o(1)$ microlocally on $U_{3\delta/2}$, finishing the proof. $\qquad\square$

We can now prove Theorem 1.3. We argue by contradiction. If (1.9) does not hold, then (changing ε) there exist sequences $h_j \to 0$ and $u = u(h_j) \in L^2(X)$ such that

$$\|u\|_{L^2} = 1, \quad \|(P - iQ - \lambda)u\|_{L^2} = o(h^{m+2}),$$

where $\lambda = \lambda(h_j) = \mathcal{O}(h_j)$ satisfies

(4.16) $h^{-1} \operatorname{Im} \lambda \in \big(-(m+1/2)(\nu_{\min} - \varepsilon) + \varepsilon, -(m-1/2)(\nu_{\max} + \varepsilon) - \varepsilon \big).$

Define $u_0, \ldots, u_{m+1}$ by (4.5); then by Lemma 4.3, $u_{m+1} = o(h^{m+1})$ microlocally in U_δ. Here we used the following corollary of (4.16):

$$h^{-1} \operatorname{Im} \lambda > -(m+1)(\nu_{\min} - \varepsilon) + \varepsilon.$$

Using Lemma 4.5 for $v = u_m, u_{m-1}, \ldots, u_0$ (and a decreasing sequence of δ's), we get $u_j = o(h^j)$ microlocally in $U_{3^{j-m-1}\delta}$ for $j = 0, 1, \ldots, m+1$. Here we used the following corollary of (4.16):

$$h^{-1} \operatorname{Im} \lambda \notin [-(j+1/2)(\nu_{\max}+\varepsilon)-\varepsilon, -(j+1/2)(\nu_{\min}-\varepsilon)+\varepsilon], \quad 0 \leqslant j \leqslant m.$$

In particular, we see that $u = u_0 = o(1)$ microlocally in the neighborhood $U_{3^{-m-1}\delta}$ of $K \cap \{p = 0\}$, giving a contradiction with Lemma 2.6 and finishing the proof of Theorem 1.3.

5. A one-dimensional example

In this section, we show a lower bound on the cutoff resolvent for $\lambda = 0$ and a specific operator with normally hyperbolic trapping. We start with the model operator (which we later put into the framework discussed in the introduction)

$$P_0 = xhD_x + \frac{h}{2i} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \quad D := \frac{1}{i}\partial.$$

Note that $P_0^* = P_0$.

We construct an approximate solution u_0 for the equation $P_0 u = 0$ by truncating the exact solutions $(x_\pm)^{-1/2}$ in the frequency space. More precisely, take a function $\chi \in \mathscr{S}(\mathbb{R})$ such that $\hat{\chi} \in C_0^\infty\big((1/2, 1)\big)$ and define

$$\psi(x) = \operatorname{sgn} x \int_0^x |x|^{-1/2}|y|^{-1/2}\chi(y)\,dy.$$

Here $\hat{\chi}$ denotes the Fourier transform of χ:

$$\hat{\chi}(\xi) = \int_{\mathbb{R}} e^{-ix\xi}\chi(x)\,dx.$$

Then (as can be seen by considering the Taylor expansion of χ at 0) we have

$$\psi \in C^\infty(\mathbb{R}), \quad P_0\psi = -ih\chi.$$

Moreover, we have

(5.1) $$\psi(x) = \psi_\pm |x|^{-1/2} + \mathcal{O}(|x|^{-\infty}) \quad \text{as } x \to \pm\infty,$$

where the constants $\psi_\pm$ are given by

$$\psi_\pm = \pm \int_0^{\pm\infty} |y|^{-1/2}\chi(y)\, dy.$$

We choose χ so that $\psi_\pm \neq 0$; this is possible since

$$\psi_\pm = \frac{e^{\pm \frac{i\pi}{4}}}{2\sqrt{\pi}} \int_0^\infty \xi^{-1/2}\hat{\chi}(\xi)\, d\xi.$$

Define

$$u_0(x;h) := h^{-1/2}\psi(x/h), \quad f_0(x;h) := -ih^{1/2}\chi(x/h),$$

then $P_0 u_0 = f_0$. We have

(5.2) $$\mathrm{WF}_h(u_0) \subset \{\xi = 0\} \cup \{x = 0,\ \xi \in [0,1)\}.$$

Indeed, if $a \in C_0^\infty\big((0,\infty)\big)$, then by (5.1),

$$a(x)u_0(x) = \psi_+ a(x)x^{-1/2} + \mathcal{O}(h^\infty);$$

since $\psi_+ a(x)x^{-1/2}$ is smooth and independent of h, we see by [29, §8.4.2] that $\mathrm{WF}_h(au_0) \subset \{\xi = 0\}$. Same is true when $a \in C_0^\infty\big((-\infty,0)\big)$. Moreover, since $(x\partial_x + \frac{1}{2})\psi = \chi$, we have $(\xi\partial_\xi + \frac{1}{2})\hat{\psi} = -\hat{\chi}$; since $\partial_x\psi \in L^2$, we have $\xi\hat{\psi}(\xi) \in L^2$ and the fact that $\mathrm{supp}\,\hat{\chi} \subset (1/2,1)$ implies $\mathrm{supp}\,\hat{\psi} \subset [0,1)$. This in turn implies that $\mathrm{WF}_h(u_0) \subset \{\xi \in [0,1)\}$, finishing the proof of (5.2). Since $\mathrm{supp}\,\hat{\chi} \subset (1/2,1)$ and χ is Schwartz, we also have

(5.3) $$\mathrm{WF}_h(f_0) \subset \{x = 0,\ \xi \in (1/2,1)\}.$$

We furthermore calculate using (5.1)

(5.4) $$\|u_0\|_{L^2(-1,1)} = \sqrt{|\psi_+|^2 + |\psi_-|^2}\sqrt{\log(1/h)} + \mathcal{O}(1),$$
$$\|f_0\|_{L^2(\mathbb{R})} = \mathcal{O}(h).$$

We now put u_0, f_0, P_0 into the framework discussed in the introduction. Put

$$X = \mathbb{S}^1 = \mathbb{R}/(6\mathbb{Z});$$

we view X as the interval $[-3,3]$ with the endpoints glued together. We consider an operator

$$P \in \Psi_h^{\mathrm{comp}}(X), \quad P = P^*; \quad P = P_0 \quad \text{microlocally on } \{|x| \leqslant 2,\ |\xi| \leqslant 2\}.$$

Note that by (5.2), we have $Pu_0 = f_0 + \mathcal{O}(h^\infty)_{C^\infty}$ on $(-2, 2)$.

We next consider $q_0 \in C^\infty(X)$ such that $q_0 = 0$ on $[-1, 1]$, $q_0 \geqslant 0$ everywhere, and $q_0 = 1$ outside of $[-3/2, 3/2]$. Take $\chi_1 \in C_0^\infty(-2, 2)$ such that $\chi_1 = 1$ on $[-3/2, 3/2]$ and define

$$u_1 = \chi_1 u_0, \quad f_1 = (P - iq_0)u_1 - \chi_1 f_0 = [P, \chi_1]u_0 - iq_0\chi_1 u_0 + \mathcal{O}(h^\infty)_{C^\infty}.$$

In particular, $f_1 = \mathcal{O}(h^\infty)$ on $[-1, 1]$ and $\mathrm{WF}_h(f_1) \subset \{\xi = 0\}$. Using ODE theory for the operator $P_0 - iq_0 = \frac{h}{i}(x\partial_x + \frac{1}{2} + h^{-1}q_0)$, we construct $u_2 \in C^\infty(X)$ such that $u_2 = 0$ on $[-1, 1]$, $u_2 = \mathcal{O}(e^{-\varepsilon/h})$ outside of $[-7/4, 7/4]$, $\mathrm{WF}_h(u_2) \subset \{\xi = 0\}$, and

$$(P - iq_0)u_2 = -f_1 + \mathcal{O}(h^\infty)_{C^\infty}.$$

Finally, take a self-adjoint operator $Q_1 \in \Psi_h^0$ such that $\mathrm{WF}_h(Q_1) \subset \{|\xi| \geqslant 3/2\}$, $\sigma(Q_1) \geqslant 0$ everywhere, and $\sigma(Q_1) > 0$ for $|\xi| \geqslant 2$, denote by Q_0 the multiplication operator by q_0, and put

$$Q := Q_0 + Q_1, \quad \sigma(Q) = q_0 + \sigma(Q_1).$$

Then the operator $P - iQ : L^2(X) \to L^2(X)$ satisfies the assumptions (1)–(6) of the introduction, with $\nu_{\min} = \nu_{\max} = 1$ and

$$\Gamma_+ = \{\xi = 0, \ |x| < 2\}, \quad \Gamma_- = \{x = 0, \ |\xi| < 2\}, \quad K = \{x = \xi = 0\}.$$

We have

$$(P - iQ)u = \chi_1 f_0 + \mathcal{O}(h^\infty), \quad u := u_1 + u_2.$$

On the other hand, (5.4) implies

$$\|u\|_{L^2} \geqslant C^{-1}\sqrt{\log(1/h)}, \quad \|\chi_1 f_0\|_{L^2} = \mathcal{O}(h).$$

This means that for each $A \in \Psi_h^{\mathrm{comp}}(X)$ such that

$$A = 1 + \mathcal{O}(h^\infty) \quad \text{microlocally on } V := \{x = 0, \ \xi \in (1/2, 1)\},$$

we have

$$\|R(0)A\|_{L^2 \to L^2} \geqslant C^{-1}h^{-1}\sqrt{\log(1/h)}.$$

Note that $V \subset \Gamma_-$ and $V \cap \Gamma_+ = \emptyset$, therefore we can choose A to be microlocalized away from the trapped set K.

BIBLIOGRAPHY

[1] N. BURQ, "Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel", *Acta Math.* **180** (1998), no. 1, p. 1-29.

[2] ———, "Lower bounds for shape resonances widths of long range Schrödinger operators", *Amer. J. Math.* **124** (2002), no. 4, p. 677-735.

[3] N. Burq & M. Zworski, "Geometric control in the presence of a black box", *J. Amer. Math. Soc.* **17** (2004), no. 2, p. 443-471 (electronic).

[4] H. Christianson, "Dispersive estimates for manifolds with one trapped orbit", *Comm. Partial Differential Equations* **33** (2008), no. 7-9, p. 1147-1174.

[5] K. Datchev, "Quantitative limiting absorption principle in the semiclassical limit", *Geom. Funct. Anal.* **24** (2014), no. 3, p. 740-747.

[6] K. Datchev & A. Vasy, "Gluing semiclassical resolvent estimates via propagation of singularities", *Int. Math. Res. Not. IMRN* (2012), no. 23, p. 5409-5443.

[7] ———, "Propagation through trapped sets and semiclassical resolvent estimates", *Ann. Inst. Fourier (Grenoble)* **62** (2012), no. 6, p. 2347-2377 (2013).

[8] ———, "Semiclassical resolvent estimates at trapped sets", *Ann. Inst. Fourier (Grenoble)* **62** (2012), no. 6, p. 2379-2384 (2013).

[9] S. Dyatlov, "Exponential energy decay for Kerr–de Sitter black holes beyond event horizons", *Math. Res. Lett.* **18** (2011), no. 5, p. 1023-1035.

[10] ———, "Asymptotics of linear waves and resonances with applications to black holes", *Comm. Math. Phys.* **335** (2015), no. 3, p. 1445-1485.

[11] ———, "Resonance projectors and asymptotics for *r*-normally hyperbolic trapped sets", *J. Amer. Math. Soc.* **28** (2015), no. 2, p. 311-381.

[12] S. Dyatlov, F. Faure & C. Guillarmou, "Power spectrum of the geodesic flow on hyperbolic manifolds", to appear in Analysis and PDE, http://arxiv.org/abs/1403.0256.

[13] S. Dyatlov & M. Zworski, "Dynamical zeta functions for Anosov flows via microlocal analysis", to appear in Annales de l'ENS, http://arxiv.org/abs/1306.4203.

[14] ———, "Mathematical theory of scattering resonances", http://math.mit.edu/~dyatlov/res/.

[15] F. Faure & M. Tsujii, "The semiclassical zeta function for geodesic flows on negatively curved manifolds", http://arxiv.org/abs/1311.4932.

[16] C. Gérard & J. Sjöstrand, "Resonances en limite semiclassique et exposants de Lyapunov", *Comm. Math. Phys.* **116** (1988), no. 2, p. 193-213.

[17] A. Goussev, R. Schubert, W. Holger & W. Stephen, "Quantum theory of reactive scattering in phase space", *Adv. Quant. Chem.* **60** (2010), p. 269-332.

[18] A. Grigis & J. Sjöstrand, *Microlocal analysis for differential operators*, London Mathematical Society Lecture Note Series, vol. 196, Cambridge University Press, Cambridge, 1994, An introduction, iv+151 pages.

[19] P. Hintz & A. Vasy, "Global well-posedness of quasilinear wave equations on asymptotically de Sitter spaces", http://arxiv.org/abs/1311.6859.

[20] ———, "Non-trapping estimates near normally hyperbolic trapping", http://arxiv.org/abs/1311.7197.

[21] ———, "Semilinear wave equations on asymptotically de Sitter, Kerr–de Sitter and Minkowski spacetimes", http://arxiv.org/abs/1306.4705.

[22] M. W. Hirsch, C. C. Pugh & M. Shub, *Invariant manifolds*, Lecture Notes in Mathematics, Vol. 583, Springer-Verlag, Berlin-New York, 1977, ii+149 pages.

[23] S. Nonnenmacher & M. Zworski, "Decay of correlations for normally hyperbolic trapping", http://arxiv.org/abs/1302.4483.

[24] J. Royer, "Limiting absorption principle for the dissipative Helmholtz equation", *Comm. Partial Differential Equations* **35** (2010), no. 8, p. 1458-1489.

[25] A. Vasy, "Microlocal analysis of asymptotically hyperbolic and Kerr-de Sitter spaces (with an appendix by Semyon Dyatlov)", *Invent. Math.* **194** (2013), no. 2, p. 381-513.

[26] G. Vodev, "Exponential bounds of the resolvent for a class of noncompactly supported perturbations of the Laplacian", *Math. Res. Lett.* **7** (2000), no. 2-3, p. 287-298.

[27] J. Wunsch, "Resolvent estimates with mild trapping", `http://arxiv.org/abs/1209.0843`.

[28] J. Wunsch & M. Zworski, "Resolvent estimates for normally hyperbolic trapped sets", *Ann. Henri Poincaré* **12** (2011), no. 7, p. 1349-1385.

[29] M. Zworski, *Semiclassical analysis*, Graduate Studies in Mathematics, vol. 138, American Mathematical Society, Providence, RI, 2012, xii+431 pages.

Semyon DYATLOV
Department of Mathematics
Massachusetts Institute of Technology
77 Massachusetts Ave
Cambridge, MA 02139 (USA)
dyatlov@math.mit.edu