ANNALES

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Todor MILANOV & Yefeng SHEN

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Tome 66, nº 1 (2016), p. 271-330.

<http://aif.cedram.org/item?id=AIF_2016__66_1_271_0>

Ann. Inst. Fourier, Grenoble
66, 1 (2016) 271-330

GLOBAL MIRROR SYMMETRY FOR INVERTIBLE SIMPLE ELLIPTIC SINGULARITIES

by Todor MILANOV & Yefeng SHEN

Abstract. — A simple elliptic singularity can be described in terms of a marginal deformation of an invertible polynomial. The choice of the polynomials and its marginal deformation are not unique. In this paper, following the earlier work of Krawitz-Shen and Milanov-Ruan, we investigate the global mirror symmetry phenomenon for simple elliptic singularities. We prove that the mirror symmetry for each family is governed by a certain system of hypergeometric equations. We conjecture that the Saito-Givental theory of the family at any special limit is mirror to either the Gromov-Witten theory of an elliptic orbifold projective line or the Fan-Jarvis-Ruan-Witten theory of an invertible polynomial, and the limits are classified by the Milnor number of the singularity and the j-invariant at the special limit. We prove the conjecture holds at all special limits of the Fermat polynomials and at the Gepner points in all other cases.

Résumé. — Une singularité simple elliptique peut être décrite en termes d'une déformation marginale d'un polynôme inversible. Le choix du polynôme et de sa déformation n'est pas unique. Dans ce papier, suivant les travaux de Krawitz-Shen et Milanov-Ruan, nous regardons la symétrie miroir globale pour les singularités simples elliptiques. Nous prouvons que la symétrie miroir pour chaque famille est règlée par un certain système d'équations hypergéométriques. Nous conjecturons que la théorie de Saito-Givental de la famille à une limite spéciale est liée soit à la théorie de Gromov-Witten d'une droite projective orbifold elliptique, soit à la théorie Fan-Jarvis-Ruan-Witten d'un polynôme inversible. Les limites sont classifiées par le nombre de Milnor de la singularité, et par le j-invariant à la limite spéciale. Nous vérifions la conjecture pour toutes les limites spéciales des polynômes de Fermat, et pour tous les points de Gepner dans les autres cas.

1. Introduction

In the famous mirror symmetry paper [5], the authors described a duality of Calabi-Yau 3-folds that exchanges the A-model with the B-model. The A-model contains information such as Kähler structure and Gromov-Witten

Keywords: mirror symmetry, simple elliptic singularities.
Math. classification: 14N35, 14B05.

invariants while the B-model contains information such as complex structures and periods integrals. However, this picture is not complete since the complex moduli usually has a nontrivial topology while the Kähler moduli does not. Consider the quintic 3-fold as an example. The mirror is a family of quintic 3-folds

$$\sum_{i=1}^{5} X_i^5 - 5\psi \prod_{i=1}^{5} X_i = 0$$

quotient by $(\mathbb{Z}/5\mathbb{Z})^3$, with $\psi \in \mathbb{P}^1$. This new family contain special limits $\psi = 0, \infty$ and fifth roots of unity, which are referred to as the Gepner point, the large complex structure limit point and the conifold limits. The mirror theorem asserts that the contractible Kähler moduli of quintic 3-fold is mirror to a neighborhood of the large complex structure limit [19, 27].

On the other hand, it is implicit in physics that we should study the entire complex moduli and all the special limits. This global point of view leads to BCOV-holomorphic anomaly equation [4] and recent spectacular physics predictions of the Gromov-Witten invariants of quintic 3-fold up to genus 52 [23]. In this paper we have yet another important motivation to establish global mirror symmetry. Namely, the Gromov–Witten invariants of the elliptic orbifold lines are known to be weak quasi-modular forms (see [30]). Using global mirror symmetry for simple elliptic singularities we prove that the invariants are holomorphic at the cusps, i.e., they are quasi-modular forms (see Section 1.3).

Landau-Ginzburg phases are introduced as part of the global picture, to describe the neighborhood of the Gepner point, or its mirror. Recently, a candidate of Landau-Ginzburg A-model has been constructed by Fan, Jarvis and Ruan based on a proposal of Witten [16, 15]. It is now called the Fan-Jarvis-Ruan-Witten theory (FJRW theory). It is a Gromov-Witten type theory which counts solutions of Witten equations. Based on this construction, Ruan proposed a mathematical formulation of Landau-Ginzburg/Calabi-Yau (LG/CY) correspondence [33]. This connects the FJRW theory and Gromov-Witten theory for a pair of same initial data. In [10], Chiodo and Ruan addressed the idea of global mirror symmetry to build a bridge for LG/CY correspondence. In short, in this picture, the FJRW theory is formulated as the mirror theory for the Gepner point.

1.1. The LG/CY correspondence via global mirror symmetry

Let us first briefly recall the general setup for the LG/CY correspondence. Recall that a polynomial W is called *quasi-homogeneous* if there are rational weights q_i for each X_i, such that

$$W(\lambda^{q_1}X_1,\ldots,\lambda^{q_N}X_N)=\lambda W(X_1,\ldots,X_N),\quad \forall\lambda\in\mathbb{C}^*.$$

The polynomial W is called *non-degenerate* if: (1) W has an isolated critical point at the origin; (2) W contains no monomial of the form X_iX_j for $i\neq j$. According to Saito [35], the choices of all $q_i\in(0,\frac{1}{2}]$ are unique. Let $W(\mathbf{x})$ be a quasi-homogeneous non-degenerate polynomial,

$$W(\mathbf{x})=\sum_{i=1}^{s}\prod_{j=1}^{N}X_j^{a_{ij}},\quad \mathbf{x}=(X_1,\ldots,X_N).$$

We say that W is *invertible*, if its *exponent matrix* $E_W=(a_{ij})_{s\times N}$ is an invertible matrix. A diagonal matrix $\operatorname{diag}(\lambda_1,\ldots,\lambda_N)$ is called a *diagonal symmetry* of W if

$$W(\lambda_1X_1,\ldots,\lambda_NX_N)=W(X_1,\ldots,X_N),\quad \lambda_i\in\mathbb{C}^*.$$

Let G_W be *the group of all diagonal symmetries* of W. It contains an element

$$J_W=\operatorname{diag}\left(\exp(2\pi\sqrt{-1}q_1),\ldots,\exp(2\pi\sqrt{-1}q_N)\right).$$

If the *Calabi-Yau condition* $(\sum_i q_i=1)$ holds, $X_W=\{W=0\}$ is a Calabi-Yau hypersurface in the weighted projective space $\mathbb{P}^{N-1}(c_1,\ldots,c_N)$, where $\gcd(c_1,\ldots,c_n)=1$ and $q_i=c_i/d$ for a common denominator d. The element J_W acts trivially on X_W, while for any group G such that $\langle J_W\rangle\subseteq G\subseteq G_W$, the group $\widetilde{G}=G/\langle J_W\rangle$ acts faithfully on X_W. The LG/CY correspondence [33] predicts that the ancestor potential $(\mathscr{A}_{W,G}^{\mathrm{FJRW}})$ of the FJRW theory for (W,G) is the same as the total ancestor potential $(\mathscr{A}_{\mathcal{X}}^{\mathrm{GW}})$ of the GW theory for $\mathcal{X}=X_W/\widetilde{G}$, up to analytic continuation and the quantization of a symplectic transformation. Both $\mathscr{A}_{W,G}^{\mathrm{FJRW}}$ and $\mathscr{A}_{\mathcal{X}}^{\mathrm{GW}}$ will be defined in Section 3.

For an invertible polynomial W, its *transpose* W^T is the unique invertible polynomial such that $E_{W^T}=(E_W)^T$, where $(E_W)^T$ is the transpose matrix of E_W. The role of the transpose W^T in mirror symmetry was first studied in [3] by Berglund and Hübsch. Later, Krawitz introduced a mirror group G^T[24]. Now a pair (W^T,G^T) is referred to as the Berglund-Hübsch-Krawitz mirror (BHK mirror) of a pair (W,G).

In order to describe the analytic continuation in the LG/CY correspondence for the pair (W,G), Chiodo and Ruan [10] addressed the idea of global mirror symmetry. They proposed to consider a global LG B-model for the BHK mirror (W^T,G^T). Such a global moduli space contains a Gepner point and a large complex structure limit point. Then the FJRW theory is formulated as the mirror theory for the Gepner point, and the GW theory is formulated as the mirror theory for the large complex structure limit point. The LG/CY correspondence is obtained by connecting the Gepner point and the large complex structure limit point on the global moduli. This works extremely well for $G = G_W$. In this case, the mirror group G^T is the trivial group and the Saito-Givental theory of W^T is expected to be the right object of the global LG B-model. If $G \neq G_W$, a global Calabi-Yau B-model [9, 8] is used to replace the LG B-model for the genus zero theory. However, a mathematical theory for the higher genus of such a global B-model is not available yet. On the other hand, Costello and Li has a different approach to construct a higher genus on the special limits for both CY B-model and LG B-model [12, 26].

1.2. Special limits in Saito-Givental theory

In this paper, we will study the special limits (see Definition 1.1 below) in the Saito-Givental theory of a one-parameter family deformation of invertible simple elliptic singularities (ISES for brevity) and their geometric mirrors. All ISESs are classified in Table 1.1, of type $E_{\mu-2}^{(1,1)}, \mu = 8, 9, 10$.

Let W be an ISES in Table 1.1. Saito constructed a flat structure on the miniversal deformation space $\mathcal{S}$ of W using primitive forms [37, 34]. The primitive form depends on the choice of W and *a marginal monomial* ϕ_{-1}, with ϕ_{-1} has degree 1 in $\mathscr{Q}_W$, the *Jacobian algebra* of W. Givental's higher-genus formalism [21, 20] defines *a total ancestor potential* $\mathscr{A}_W^{\mathrm{SG}}(\mathbf{s})$ for every semisimple point $\mathbf{s} \in \mathcal{S}$. More details of the Saito-Givental theory will be introduced in Section 2.

In this paper, we define *the special limits* as follows.

Definition 1.1. — *Let $W_\sigma = W + \sigma\phi_{-1}$ be a family of simple elliptic singularities along marginal monomial ϕ_{-1}. Let $p_1, \ldots, p_l \in \mathbb{C}$ such that for $\sigma = p_i$, the point $\mathbf{x} = 0$ is not an isolated critical point of the polynomial W_σ. We call σ (or $\mathbf{s} = (\sigma, 0 \ldots, 0)$) a special limit for the Saito-Givental theory of W, if*

$$\sigma = 0,\ p_1, \ldots, p_l,\ \infty\ \in \mathbb{C} \cup \{\infty\}.$$

We denote the punctured plane $\mathbb{C} - \{p_1, \ldots, p_l\}$ by Σ. We point out that the point $\mathbf{s} = (\sigma, \mathbf{0})$ is not semisimple. Thus Givental's formula $\mathscr{A}_W^{\mathrm{SG}}(\mathbf{s})$ does not apply directly for such points. However, according to [25, 30, 29, 11], at some special limit points σ, it is still possible to find a limit of $\mathscr{A}_W^{\mathrm{SG}}(\mathbf{s})$,

Table 1.1. Invertible simple elliptic singularities

	$E_6^{(1,1)}$	$E_7^{(1,1)}$	$E_8^{(1,1)}$
Fermat	$X_1^3 + X_2^3 + X_3^3$	$X_1^4 + X_2^4 + X_3^2$	$X_1^6 + X_2^3 + X_3^2$
Fermat+Chain	$X_1^2X_2 + X_2^3 + X_3^3$	$X_1^3X_2 + X_2^4 + X_3^2$ $X_1^2X_2 + X_2^2 + X_3^4$	$X_1^4X_2 + X_2^3 + X_3^2$ $X_1^3X_2 + X_2^2 + X_3^3$
Fermat+Loop	$X_1^2X_2 + X_1X_2^2 + X_3^3$	$X_1^3X_2 + X_1X_2^3 + X_3^2$	
Chain	$X_1^2X_2 + X_2^2X_3 + X_3^3$	$X_1^3X_2 + X_2^2X_3 + X_3^2$	
Loop	$X_1^2X_2 + X_2^2X_3 + X_1X_3^2$		

which we will call a *Saito-Givental limit*

$$\mathscr{A}_W^{\mathrm{SG}}(\sigma) = \lim_{\mathbf{s}\to(\sigma,\mathbf{0})} \mathscr{A}_W^{\mathrm{SG}}(\mathbf{s}).$$

The proof of the existence of such a limit is quite sutble. We will explain it in Section 4.1.

Our goal is to study the Saito-Givental limit $\mathscr{A}_W^{\mathrm{SG}}(\sigma)$ at the special limit points, when they exist.

Definition 1.2. — *Let σ be a special limit of W. Then $\sigma = 0$ is called a Gepner point. Two special limits are isomorphic if the Saito-Givental limit $\mathscr{A}^{\mathrm{SG}}(\sigma)$ at each point exists and the two limits match.*

- *We say σ is an FJRW-point (or a (W', G')-FJRW point) if there exist a Saito-Givental limit $\mathscr{A}_W^{\mathrm{SG}}(\sigma)$ at σ, and a pair (W', G'), such that $\mathscr{A}_W^{\mathrm{SG}}(\sigma) = \mathscr{A}_{W',G'}^{\mathrm{FJRW}}$, where $\mathscr{A}_{W',G'}^{\mathrm{FJRW}}$ is the total ancestor potential of the FJRW theory for (W', G').*
- *We say σ is a GW-point (or an $\mathcal{X}$-GW point) if there exist a Saito-Givental limit $\mathscr{A}_W^{\mathrm{SG}}(\sigma)$ at σ, and an orbifold $\mathcal{X}$, such that $\mathscr{A}_W^{\mathrm{SG}}(\sigma) = \mathscr{A}_{\mathcal{X}}^{\mathrm{GW}}$, where $\mathscr{A}_{\mathcal{X}}^{\mathrm{GW}}$ is the total ancestor potential of $\mathcal{X}$.*

Usually, when the Gepner point $\sigma = 0$ for W is a (W^T, G_{W^T})-FJRW point, this is always referred to as *the LG-LG mirror symmetry* [10].

Let E_σ be the elliptic curve in $\mathbb{P}^2(c_1, c_2, c_3)$, defined by $W_\sigma = 0$. Let $j(\sigma)$ be the j-invariant of E_σ and μ be the Milnor number of W. Based on the calculations in [30, 25], we propose the following conjecture to understand the mirror symmetry and classification of the special limits for invertible simple elliptic singularities.

Conjecture 1.3. — *Let W be an invertible polynomial of type $E_{\mu-2}^{(1,1)}$, then the Saito-Givental limit $\mathscr{A}_W^{\mathrm{SG}}(\sigma)$ exists at any special limit point in Definition 1.1. Moreover,*

a) *The Saito-Givental theory at a special limit σ is isomorphic to either a FJRW theory of a simple elliptic singularity or a GW theory of an elliptic orbifold $\mathbb{P}^1$.*
b) *The isomorphism classes of the special limits σ are one-to-one correspondent to the set of pairs $(\mu, j(\sigma)) \in \{8, 9, 10\} \times \{0, 1728, \infty\}$.*

In higher dimensions, such as quintic case, the points $p_1, \ldots, p_l$ are usually referred to as conifold points. It is still not yet known how to construct a geometric mirror for a conifold point. If Conjecture 1.3 holds, then there is no conifold point for invertible simple singularities at all.

Our first result is that Conjecture 1.3 is true for Gepner points.

Theorem 1.4. — *Let W be an invertible polynomial of type $E^{(1,1)}_{\mu-2}$; then*

a) *If W^T belongs to Tables 3.1, 3.2, or 3.3, then the Gepner point $\sigma = 0$ of W is a (W^T, G_{W^T})-FJRW point. In other words, the LG-LG mirror symmetry holds for such (W^T, G_{W^T}) and its BHK mirror $(W, \{1\})$.*
b) *The Gepner point $\sigma = 0$ is always an FJRW point. Its isomorphism class is determined by $(\mu, j(\sigma))$, with $\mu = 8, 9, 10$, $j(\sigma) = 0$ or 1728.*

Note that in Theorem 1.4 we have excluded some of the polynomials appearing in Table 1.1. This is because we do not know how to compute all the FJRW invariants for them (see Section 3.3).

In this paper, we prove

Theorem 1.5. — *For Fermat simple elliptic singularities, Conjecture 1.3 is true.*

Let us point out that $\sigma = \infty$ for the Fermat $E^{(1,1)}_8$ polynomial is an FJRW-point and all other special limits $\sigma \neq 0$ are GW-points. This is a little bit surprising because usually in a one-parameter B-model family [13], the point $\sigma = \infty$ is a GW-point. As a corollary, we get various correspondences of LG/LG-type or of LG/CY-type.

Corollary 1.6. — *For a given ISES with a fixed marginal deformation, the total ancestor potentials of the GW theory and the FJRW theories that are mirror partners at the special limit points are related by analytic continuation and quantizations of symplectic transformations.*

1.3. Modularity

This is the 4th in a series of papers in which we investigate mirror symmetry for simple elliptic singularities. One of the main motivations at the beginning was to prove that the Gromov–Witten (GW) invariants of certain elliptic orbifold lines are quasi-modular forms. Using the mirror symmetry results of Krawitz–Shen [25], Milanov–Ruan [30] proved the GW invariants of any genus for orbifolds $\mathbb{P}^1_{3,3,3}$,$\mathbb{P}^1_{4,4,2}$ and $\mathbb{P}^1_{6,3,2}$ are quasi-modular forms on an appropriate finite index subgroup $\Gamma(W)$ ($\Gamma(W)$ depends on the type of the singularity W) of $\mathrm{SL}_2(\mathbb{Z})$. Besides $W = X_1^3 + X_2^3 + X_3^3$, the subgroup $\Gamma(W)$ however was left undetermined and also the definition of a quasi-modular form was relaxed by allowing finite order poles at the cusps of $\Gamma(W)$, i.e., Milanov-Ruan proved that the GW invariants are *weak* quasi-modular forms. The group $\Gamma(W)$ for the Fermat polynomials W of type $E_{\mu-2}^{(1,1)}$, $\mu=9,10$ is computed in [32]. In this paper we prove that the weak quasi-modular forms are holomorphic at the cusps, i.e., Theorem 1.5 implies the following corollary.

Corollary 1.7. — *Let W be a Fermat simple elliptic singularity. For any g, genus-g Saito-Givental correlation functions are holomorphic near the special limits.*

This completes the proof that the GW invariants of the elliptic orbifold lines are quasi-modular forms. It would be interesting to see whether this helps to express the higher genus GW invariants explicitly in closed forms, as polynomials of ring generators of quasi-modular forms, as in [23].

1.4. Plan of the paper

The paper is organized as follows. In Section 2, we recall Givental's construction of the B-model Gromov-Witten type potential in the setting of Saito's theory of primitive forms. We also derive the Picard-Fuchs equations satisfied by the various period integrals. In Section 3, we discuss the two types of geometric theories: the Gromov-Witten theory of elliptic orbifold lines and the FJRW theory of simple elliptic singularities. We also recall the reconstruction theorem in both theories. In Sections 4, we establish the LG-LG mirror symmetry for Gepner points (Theorem 1.4) by comparing the B-model constructed in Section 2 and the FJRW A-models constructed in Section 3. In Section 5, we establish the global mirror symmetry for Fermat polynomials by proving Theorem 1.5.

1.5. Acknowledgement

We thank Yongbin Ruan for his insight and support for this project. Both authors would like to thank Kentaro Hori, Hiroshi Iritani, and Kyoji Saito for many stimulating conversations. The first author benefited from conversations with Satoshi Kondo and Charles Siegel, while the second author would like to thank Alessandro Chiodo, Igor Dolgachev, Huijun Fan, Tyler Jarvis, Jeffrey Lagarias, Sijun Liu, Noriko Yui and Jie Zhou for helpful discussions. We thank Arthur Greenspoon for editorial assistance. The work of the first author is supported by JSPS Grant-In-Aid 26800003 and by the World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan. The work of the second author is partially supported by NSF grant DMS-1159156.

2. Global B-model for simple elliptic singularities

Let W be an invertible polynomial from Table 1.1. We would like to recall Saito's theory of primitive forms which yields a Frobenius structure on $\mathcal{S}$. Following Givental's higher genus reconstruction formalism we will introduce the total ancestor potential of W. We also derive a system of hypergeometric equations that determines the restriction of the flat coordinates of the Frobenius manifold $\mathcal{S}$ to Σ.

2.1. Miniversal deformations

In this paper, we are interested in invertible polynomials with 3 variables and the weights $q_i = 1/a_i$ $(1 \leqslant i \leqslant 3)$ for some positive integers a_i satisfying the Calabi-Yau condition

$$(a_1, a_2, a_3) = (3,3,3),\ (4,4,2),\ \text{and } (6,3,2).$$

We denote the corresponding classes of invertible polynomials respectively by $E_6^{(1,1)}$, $E_7^{(1,1)}$, and $E_8^{(1,1)}$. Modulo permutation of the variables $X_i (1 \leqslant i \leqslant N)$ there are 13 types of invertible polynomials (see Table 1.1). We refer to these polynomials as invertible simple elliptic singularities. Let $\mathscr{Q}_W$ be the *Jacobian algebra* of W,

$$\mathscr{Q}_W = \mathbb{C}[X_1, X_2, X_3]/(\partial_{X_1} W, \partial_{X_2} W, \partial_{X_3} W).$$

The dimension of $\mathscr{Q}_W$ is called the *multiplicity* of the critical point or *Milnor number* and it will be denoted by μ. Let us fix a degree-1 monomial $\phi_{\mathbf{m}}(\mathbf{x}) = X_1^{m_1} X_2^{m_2} X_3^{m_3}$, $\mathbf{m} = (m_1, m_2, m_3)$ whose projection in $\mathscr{Q}_W$ is non-zero. Let us construct a deformation of W of the following form:

$$W_\sigma(\mathbf{x}) = W(\mathbf{x}) + \sigma\, \phi_{\mathbf{m}}(\mathbf{x}), \quad \sigma \in \Sigma, \tag{2.1}$$

where $\Sigma \subset \mathbb{C}$ is the set of all $\sigma \in \mathbb{C}$ such that $W_\sigma(\mathbf{x})$ has only isolated critical points. Such deformations do not change the multiplicity of the critical point at $\mathbf{x} = 0$. The polynomials (2.1) are families of simple elliptic singularities of type $E_{\mu-2}^{(1,1)}$ (see [36]).

There exists a set $\mathfrak{R}$ of weighted homogeneous monomials

$$\phi_{\mathbf{r}}(\mathbf{x}) = X_1^{r_1} X_2^{r_2} X_3^{r_3}, \quad \mathbf{r} = (r_1, r_2, r_3) \in \mathfrak{R}, \tag{2.2}$$

such that their projections in $\mathscr{Q}_{W_\sigma}$ form a basis for all $\sigma \in \Sigma$. There is precisely one monomial of top degree, which may be chosen to be $\phi_{\mathbf{m}}$, i.e., we may assume that $\mathbf{m} = (m_1, m_2, m_3) \in \mathfrak{R}$. Let us point out that the proof that such a basis exists is done on a case-by-case basis. More generally, we consider a miniversal deformation (see e.g. [2]) of W

$$F(\mathbf{s},\mathbf{x}) = W(\mathbf{x}) + \sum_{\mathbf{r}\in\mathfrak{R}} s_{\mathbf{r}}\,\phi_{\mathbf{r}}(\mathbf{x}). \tag{2.3}$$

It is convenient to adopt two notations for the deformation parameters. Namely, put

$$\mathbf{s} = \{s_{\mathbf{r}}\}_{\mathbf{r}\in\mathfrak{R}} = (s_{-1}, s_0, s_1, \ldots, s_{\mu-2}),$$

where the second equality is obtained by putting an order on the elements $\mathbf{r}\in\mathfrak{R}$ and enumerating them with the integers from -1 to $\mu-2$ in such a way that

$$s_{-1} = s_{\mathbf{m}} = \sigma, \quad s_0 = s_{\mathbf{0}}, \quad \mathbf{0} = (0,0,0)\in\mathfrak{R}.$$

The moduli space of miniversal deformations, i.e., the range of the parameters $s_{\mathbf{r}}$ is then defined to be the affine space $\mathcal{S} = \Sigma\times\mathbb{C}^{\mu-1}$. Furthermore, each $s_{\mathbf{r}}$ is assigned a degree so that $F(\mathbf{s},\mathbf{x})$ is weighted-homogeneous of degree 1. Note that the parameter $s_{\mathbf{m}} = \sigma$ has degree 0. Following the terminology in physics we call $s_{\mathbf{m}}$ and $\phi_{\mathbf{m}}$ *marginal*. Note that $W_\sigma(\mathbf{x})$ is the restriction of $F(\mathbf{s},\mathbf{x})$ to the subspace Σ of marginal deformations. Except for Fermat case, there is more than one choice of a marginal monomial. For example, $X_1X_2X_3$, $X_1^4X_3$ are both marginal for $W = X_1^3X_2 + X_2^2 + X_3^3$.

2.2. Saito's theory

Let C be the critical variety of the miniversal deformation $F(\mathbf{s},\mathbf{x})$ (see (2.3)), i.e., the support of the sheaf

$$\mathcal{O}_C := \mathcal{O}_X/\langle\partial_{X_1}F, \partial_{X_2}F, \partial_{X_3}F\rangle,$$

where $X = \mathcal{S}\times\mathbb{C}^3$. Let $q : X\to\mathcal{S}$ be the projection on the first factor. The Kodaira–Spencer map ($\mathcal{T}_{\mathcal{S}}$ is the sheaf of holomorphic vector fields on $\mathcal{S}$)

$$\mathcal{T}_{\mathcal{S}} \longrightarrow q_*\mathcal{O}_C, \quad \partial/\partial s_i \mapsto \partial F/\partial s_i \bmod (F_{X_1}, F_{X_2}, F_{X_3})$$

is an isomorphism, which implies that for any $\mathbf{s}\in\mathcal{S}$, the tangent space $T_{\mathbf{s}}\mathcal{S}$ is equipped with an associative commutative multiplication $\bullet_{\mathbf{s}}$ depending holomorphically on $\mathbf{s}\in\mathcal{S}$. If in addition we have a volume form $\omega = g(\mathbf{s},\mathbf{x})d^3\mathbf{x}$, where $d^3\mathbf{x} = dX_1\wedge dX_2\wedge dX_3$ is the standard volume form, then $q_*\mathcal{O}_C$ (hence $\mathcal{T}_{\mathcal{S}}$ as well) is equipped with the *residue pairing*:

$$\Big\langle\psi_1, \psi_2\Big\rangle = \frac{1}{(2\pi i)^3}\int_{\Gamma_\epsilon}\frac{\psi_1(\mathbf{s},\mathbf{y})\psi_2(\mathbf{s},\mathbf{y})}{F_{y_1}F_{y_2}F_{y_3}}\,\omega, \tag{2.4}$$

where $\mathbf{y} = (y_1, y_2, y_3)$ is a unimodular coordinate system for the volume form, i.e., $\omega = d^3\mathbf{y}$, and Γ_ϵ is a real 3-dimensional cycle supported on $|F_{X_i}| = \epsilon$ for $1 \leqslant i \leqslant 3$.

Given a semi-infinite cycle

$$\mathcal{A} \in \varprojlim H_3(\mathbb{C}^3, (\mathbb{C}^3)_{-m}; \mathbb{C}) \cong \mathbb{C}^\mu, \tag{2.5}$$

where

$$(\mathbb{C}^3)_m = \{\mathbf{x} \in \mathbb{C}^3 \mid \mathrm{Re}(F(\mathbf{s},\mathbf{x})/z) \leqslant m\}. \tag{2.6}$$

Let $d_{\mathcal{S}}$ be the de Rham differential on $\mathcal{S}$. Put

$$J_{\mathcal{A}}(\mathbf{s}, z) = (-2\pi z)^{-3/2}\, z d_{\mathcal{S}} \int_{\mathcal{A}} e^{F(\mathbf{s},\mathbf{x})/z}\omega, \tag{2.7}$$

The oscillatory integrals $J_{\mathcal{A}}$ are by definition sections of the cotangent sheaf $\mathcal{T}^*_{\mathcal{S}}$.

According to Saito's theory of primitive forms [37, 34], there exists a volume form ω such that the residue pairing is flat and the oscillatory integrals satisfy a system of differential equations. Let us point out that the form ω is multi-valued analytic with respect to the parameter $\mathbf{s} \in \mathcal{S}$, so it is analytic on the universal cover $\widetilde{\mathcal{S}}$ of $\mathcal{S}$. If we switch the coordinate system from $\mathbf{s}$ to a system of flat-homogeneous coordinates $\mathbf{t} = (t_{-1}, t_0, \ldots, t_{\mu-2})$, then the differential equations have the form

$$z\partial_i J_{\mathcal{A}}(\mathbf{t}, z) = \partial_i \bullet_{\mathbf{t}} J_{\mathcal{A}}(\mathbf{t}, z), \quad \partial_i := \partial/\partial t_i \ (-1 \leqslant i \leqslant \mu - 2), \tag{2.8}$$

and the multiplication is defined by identifying vectors and covectors via the residue pairing. Using the residue pairing, the flat structure, and the Kodaira–Spencer isomorphism we have

$$T^*\widetilde{\mathcal{S}} \cong T\widetilde{\mathcal{S}} \cong \widetilde{\mathcal{S}} \times T_0\mathcal{S} \cong \widetilde{\mathcal{S}} \times \mathcal{Q}_W.$$

Due to homogeneity, the integrals satisfy a differential equation:

$$(z\partial_z + E) J_{\mathcal{A}}(\mathbf{t}, z) = \Theta\, J_{\mathcal{A}}(\mathbf{t}, z), \quad z \in \mathbb{C}^* \tag{2.9}$$

where E is the *Euler vector field*

$$E = \sum_{i=-1}^{\mu-2} d_i t_i \partial_i, \quad (d_i := \deg t_i = \deg s_i),$$

and Θ is the so-called *Hodge grading operator*

$$\Theta : \mathcal{T}^*_S \to \mathcal{T}^*_S, \quad \Theta(dt_i) = \left(\frac{1}{2} - d_i\right) dt_i.$$

The compatibility of the system (2.8)–(2.9) implies that the residue pairing, the multiplication, and the Euler vector field give rise to a *conformal*

Frobenius structure on the universal cover $\widetilde{S}$ of conformal dimension 1. We refer to B. Dubrovin [14] for the definition and more details on Frobenius structures and to C. Hertling [22] or to Saito–Takahashi [38] for more details on constructing a Frobenius structure from a primitive form.

2.3. The primitive forms

The classification of primitive forms in general is a very difficult problem. In the case of simple elliptic singularities however, all primitive forms are known (see [37, 34]). They are given by $\omega = d^3\mathbf{x}/\pi_A(\sigma)$, where $\pi_A(\sigma)$ is the period (2.11). As we will prove below, these periods are solutions to the hypergeometric equation (2.12), so a primitive form may be equivalently fixed by fixing a solution to the differential equation that does not vanish on Σ. Note that since $\pi_A(\sigma)$ is multi-valued function, the corresponding Frobenius structure on $\mathcal{S}$ is multi-valued as well. In other words, the primitive form gives rise to a Frobenius structure on the universal cover $\widetilde{\mathcal{S}} \cong \mathbb{H} \times \mathbb{C}^{\mu-1}$.

The key to the primitive form is the Picard-Fuchs differential equation for the periods of the so-called *elliptic curve at infinity*

$$E_\sigma := \left\{ [X_1 : X_2 : X_3] \in \mathbb{CP}^2(c_1, c_2, c_3) \,\middle|\, W_\sigma = 0 \right\}, \tag{2.10}$$

where $c_i = d/a_i$, $1 \leqslant i \leqslant 3$ and d is the least common multiple of a_1, a_2, and a_3. Note that E_σ are the fibers of an elliptic fibration over $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ whose non-singular fibers are parametrized by $\Sigma \subset \mathbb{C} \subset \mathbb{CP}^1$. Note that $\mathrm{Res}_{E_\sigma}\Omega$, where

$$\Omega := \frac{dX_1 \wedge dX_2 \wedge dX_3}{dW_\sigma},$$

is a Calabi-Yau form of the elliptic curve E_σ. For every $A \in H_1(E_\sigma)$, we define

$$\pi_A(\sigma) = \int_A \mathrm{Res}_{E_\sigma}\Omega. \tag{2.11}$$

It is well known that the period integrals are solutions to a Fuchsian differential equation. In particular, we obtain the following lemma,

Lemma 2.1. — *Let $\delta = \sigma\partial/\partial\sigma$, the elliptic period $\pi_A(\sigma)$ described above satisfies the Picard-Fuchs equation*

$$\begin{gathered} \delta(\delta-1)\ \pi_A(\sigma) = C\,\sigma^l(\delta + l\alpha)(\delta + l\beta)\ \pi_A(\sigma), \\ \alpha + \beta = 1 - \frac{1}{l}, \quad C = \prod_{i=1}^{3}\left(-\frac{l_i}{l}\right)^{l_i}. \end{gathered} \tag{2.12}$$

If we put $x = C\,\sigma^l$, $\gamma = \alpha+\beta$, *the equation* (2.11) *becomes a hypergeometric equation*

$$x(1-x)\frac{d^2\pi_A}{dx^2} + \Big(\gamma - (1+\alpha+\beta)x\Big)\frac{d\pi_A}{dx} - (\alpha\beta)\,\pi_A = 0 \tag{2.13}$$

We call (α,β,γ) *the weights system of the hypergeometric equation. The explicit values are listed in Tables 2.1, 2.2 and 2.3 below.*

In particular, $\Sigma = \mathbb{C}\backslash\{p_1,\dots,p_l\}$, where p_i are the singularities of the Picard–Fuchs equation (2.12). All the singular points are

$$p_i = C^{-1/l}\eta^i, \quad 1 \leqslant i \leqslant l, \quad \eta = \exp(2\pi\sqrt{-1}/l), \tag{2.14}$$

For our purposes, we will give a proof of this Lemma in Section 2.4 following the approach of S. Gährs (see [18]). To find out α,β and γ, we will need *the mirror weight* q_i^T, which is the weight of X_i in the BHK mirror W^T and *a charge vector* $(l_1,l_2,l_3,-l) \in \mathbb{Z}^4$ by choosing the minimal $l \in \mathbb{Z}_{>0}$ such that

$$(l_1,l_2,l_3) = l\,\mathbf{m}\,E_W^{-1}, \quad \mathbf{m} = (m_1,m_2,m_3). \tag{2.15}$$

Table 2.1. $E_6^{(1,1)}$

W	m_1,m_2,m_3	l_1,l_2,l_3,l	q_1^T,q_2^T,q_3^T	α,β,γ
$X_1^3+X_2^3+X_3^3$	$1,1,1$	$1,1,1,3$	$\frac13,\frac13,\frac13$	$\frac13,\frac13,\frac23$
$X_1^2X_2+X_2^3+X_3^3$	$2,0,1$	$3,-1,1,3$	$\frac12,\frac16,\frac13$	$\frac16,\frac12,\frac23$
$X_1^2X_2+X_2^3+X_3^3$	$0,2,1$	$0,2,1,3$	$\frac12,\frac16,\frac13$	$\frac1{12},\frac7{12},\frac23$
$X_1^2X_2+X_1X_2^2+X_3^3$	$1,1,1$	$1,1,1,3$	$\frac13,\frac13,\frac13$	$\frac13,\frac13,\frac23$
$X_1^2X_2+X_1X_2^2+X_3^3$	$2,0,1$	$4,-2,1,,3$	$\frac13,\frac13,\frac13$	$\frac1{12},\frac7{12},\frac23$
$X_1^2X_2+X_2^2X_3+X_3^3$	$2,0,1$	$2,-1,1,2$	$\frac12,\frac14,\frac14$	$\frac14,\frac14,\frac12$
$X_1^2X_2+X_2^2X_3+X_3^3$	$0,3,0$	$0,3,-1,2$	$\frac12,\frac14,\frac14$	$\frac1{12},\frac5{12},\frac12$
$X_1^2X_2+X_2^2X_3+X_3^3$	$0,1,2$	$0,1,1,2$	$\frac12,\frac14,\frac14$	$\frac14,\frac14,\frac12$
$X_1^2X_2+X_2^2X_3+X_1X_3^2$	$1,1,1$	$1,1,1,3$	$\frac13,\frac13,\frac13$	$\frac13,\frac13,\frac23$
$X_1^2X_2+X_2^2X_3+X_1X_3^2$	$3,0,0$	$4,-2,1,3$	$\frac13,\frac13,\frac13$	$\frac1{12},\frac7{12},\frac23$

Table 2.2. $E_7^{(1,1)}$

W	m_1, m_2, m_3	l_1, l_2, l_3, l	q_1^T, q_2^T, q_3^T	α, β, γ
$X_1^4 + X_2^4 + X_3^2$	$2, 2, 0$	$1, 1, 0, 2$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$
$X_1^3X_2 + X_2^4 + X_3^2$	$4, 0, 0$	$4, -1, 0, 3$	$\frac{1}{3}, \frac{1}{2}, \frac{1}{6}$	$\frac{1}{12}, \frac{7}{12}, \frac{2}{3}$
$X_1^3X_2 + X_2^4 + X_3^2$	$1, 3, 0$	$1, 2, 0, 3$	$\frac{1}{3}, \frac{1}{2}, \frac{1}{6}$	$\frac{1}{12}, \frac{7}{12}, \frac{2}{3}$
$X_1^2X_2 + X_2^2 + X_3^4$	$2, 0, 2$	$2, -1, 1, 2$	$\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$
$X_1^2X_2 + X_2^2 + X_3^4$	$0, 1, 2$	$0, 1, 1, 2$	$\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$
$X_1^3X_2 + X_1X_2^3 + X_3^2$	$4, 0, 0$	$3, -1, 0, 2$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$	$\frac{1}{12}, \frac{5}{12}, \frac{1}{2}$
$X_1^3X_2 + X_1X_2^3 + X_3^2$	$2, 2, 0$	$1, 1, 0, 2$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$
$X_1^3X_2 + X_2^2X_3 + X_3^2$	$1, 1, 1$	$1, 1, 1, 3$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}, \frac{1}{3}, \frac{2}{3}$
$X_1^3X_2 + X_2^2X_3 + X_3^2$	$1, 3, 0$	$1, 4, -2, 3$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{12}, \frac{7}{12}, \frac{2}{3}$
$X_1^3X_2 + X_2^2X_3 + X_3^2$	$4, 0, 0$	$4, -2, 1, 3$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{12}, \frac{7}{12}, \frac{2}{3}$

Table 2.3. $E_8^{(1,1)}$

W	m_1, m_2, m_3	l_1, l_2, l_3, l	q_1^T, q_2^T, q_3^T	α, β, γ
$X_1^6 + X_2^3 + X_3^2$	$4, 1, 0$	$1, 2, 0, 3$	$\frac{1}{6}, \frac{1}{3}, \frac{1}{2}$	$\frac{1}{12}, \frac{7}{12}, \frac{2}{3}$
$X_1^3X_2 + X_2^2 + X_3^3$	$1, 1, 1$	$1, 1, 1, 3$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}, \frac{1}{3}, \frac{2}{3}$
$X_1^3X_2 + X_2^2 + X_3^3$	$4, 0, 1$	$4, -2, 1, 3$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{12}, \frac{7}{12}, \frac{2}{3}$
$X_1^4X_2 + X_2^3 + X_3^2$	$2, 2, 0$	$1, 1, 0, 2$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$
$X_1^4X_2 + X_2^3 + X_3^2$	$6, 0, 0$	$3, -1, 0, 2$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$	$\frac{1}{12}, \frac{5}{12}, \frac{1}{2}$

2.4. Picard-Fuchs equations

Let us denote by

$$X_{\mathbf{s}} = \{\mathbf{x} \in \mathbb{C}^3 \mid F(\mathbf{s}, \mathbf{x}) = 1\}, \quad \mathbf{s} \in \mathcal{S}.$$

The points $\mathbf{s}$ for which $X_{\mathbf{s}}$ is singular form an analytic hypersurface in $\mathcal{S}$ called the *discriminant*. Its complement in $\mathcal{S}$ will be denoted by $\mathcal{S}'$. We will be interested in the period integrals

$$\Phi_{\mathbf{r}}(\mathbf{s}) = \int \phi_{\mathbf{r}}(x)\, \frac{d^3\mathbf{x}}{dF}, \quad \phi_{\mathbf{r}}(\mathbf{x}) = X_1^{r_1} X_2^{r_2} X_3^{r_3}, \quad \mathbf{r} = (r_1, r_2, r_3) \in \mathbb{Z}^3_{\geqslant 0}.$$

They are sections of the middle (or vanishing) cohomology bundle on $\mathcal{S}'$ formed by $H^2(X_{\mathbf{s}}, \mathbb{C})$. Slightly abusing the notation, we denote the restriction to $s_{-1} = \sigma$, $s_i = 0 (0 \leqslant i \leqslant \mu - 2)$ by $\Phi_{\mathbf{r}}(\sigma)$. Following the idea of [18], we first obtain a GKZ (Gelfand–Kapranov–Zelevinsky) system of differential equations for the periods. Using that the period integrals are not polynomial in σ (they have singularities at the punctures of Σ) we can reduce the GKZ system to a Picard-Fuchs equation.

2.4.1. The GKZ system

In order to derive the GKZ system, we slightly modify the polynomial W. By definition $W(\mathbf{x}) = \sum_{i=1}^3 \phi_{\mathbf{a}_i}(\mathbf{x})$, where $\mathbf{a}_i$ are the rows of the matrix E_W. Put

$$W_{\mathbf{v},\sigma}(\mathbf{x}) = \sum_{i=1}^{3} v_i\, \phi_{\mathbf{a}_i}(\mathbf{x}) + \sigma\, \phi_{-1}(\mathbf{x}),$$

where $\mathbf{v} = (v_1, v_2, v_3)$ are some complex parameters. For simplicity. we omit $\mathbf{v}$ in the notation if $\mathbf{v} = (1, 1, 1)$. Let us write $X_\sigma^{\mathbf{v},\lambda} = \{\mathbf{x} \in \mathbb{C}^3 \mid W_{\mathbf{v},\sigma}(\mathbf{x}) = \lambda\}$. Then we define the period integrals

$$\Phi_{\mathbf{r}}^{\mathbf{v},\lambda}(\sigma) = \int \phi_{\mathbf{r}}(\mathbf{x})\, \frac{d^3\mathbf{x}}{dW_{\mathbf{v},\sigma}}, \quad \mathbf{r} = (r_1, r_2, r_3) \in \mathbb{Z}^3_{\geqslant 0}; \tag{2.16}$$

again one should think that the above integral is a section of the vanishing cohomology for $W_{\mathbf{v},\sigma}(\mathbf{x})$. The vanishing cohomology bundle is equipped with a Gauss–Manin connection ∇. The following formulas are well known (see e.g. [2])

$$\begin{aligned} \nabla_{\partial/\partial\lambda} \int \theta &= \int \frac{d\theta}{dW_{\mathbf{v},\sigma}} \\ \nabla_{\partial/\partial v_i} \int \theta &= -\int \frac{\partial W_{\mathbf{v},\sigma}}{\partial v_i}\, \frac{d\theta}{dW_{\mathbf{v},\sigma}} + \int \mathrm{Lie}_{\partial/\partial v_i} \theta, \end{aligned} \tag{2.17}$$

where θ is a 2-form on $\mathbb{C}^3$ possibly depending on the parameters $\mathbf{v}$. Finally, note that rescaling $X_i \mapsto \lambda^{q_i} X_i (1 \leqslant i \leqslant 3)$ yields

$$\Phi_{\mathbf{r}}^{\mathbf{v},\lambda}(\sigma) = \lambda^{\deg \phi_{\mathbf{r}}}\, \Phi_{\mathbf{r}}^{\mathbf{v},1}(\sigma).$$

Let $\delta_i = v_i \partial/\partial v_i (1 \leqslant i \leqslant 3)$ and $\delta = \sigma\partial/\partial\sigma$.

Lemma 2.2. — *The period integral $\Phi^{\mathbf{v},\lambda}_{\mathbf{r}}$ satisfies the system of differential equations:*

$$\begin{aligned} \partial_\sigma^l \prod_{i:l_i<0} \partial_{v_i}^{-l_i}\, \Phi &= \prod_{i:l_i>0} \partial_{v_i}^{l_i}\, \Phi, \quad 1\leqslant i\leqslant 3;\\ (\delta_1,\delta_2,\delta_3)\, E_W\, \Phi + (m_1,m_2,m_3)\delta\, \Phi &= -(1+r_1,1+r_2,1+r_3)\Phi. \end{aligned}$$

Proof. — Using (2.17) we get the following differential equations:

$$\partial_{v_i}\, \Phi^{\mathbf{v},\lambda}_{\mathbf{r}} = -\partial_\lambda\, \Phi^{\mathbf{v},\lambda}_{\mathbf{r}+\mathbf{a}_i}, \quad 1\leqslant i\leqslant 3,$$

and

$$\partial_\sigma\, \Phi^{\mathbf{v},\lambda}_{\mathbf{r}} = -\partial_\lambda\, \Phi^{\mathbf{v},\lambda}_{\mathbf{r}+\mathbf{m}}, \quad \mathbf{m}=(m_1,m_2,m_3),$$

where $\phi_{\mathbf{m}}(\mathbf{x})$ is the marginal monomial. The first differential equation is equivalent to the identity

$$l\, m_k - \sum_{i,l_i<0} a_{ik}\, l_i = \sum_{j,l_j>0} a_{jk}\, l_j\,.$$

which is true by definition (see (2.15)). For the second equation, using the above formulas we get that the i-th entry on the LHS is

$$-\partial_\lambda \int X_i\, \phi_{\mathbf{r}}(X)\, \frac{d^3\mathbf{x}}{dW_{\mathbf{v},\sigma}} = -(r_i+1)\int \phi_{\mathbf{r}}(X)\, \frac{d^3\mathbf{x}}{dW_{\mathbf{v},\sigma}},$$

where we used formulas (2.17) again. □

Let us define the row-vector

$$\zeta = (\zeta_1,\zeta_2,\zeta_3) = \mathbf{r}\, E_W^{-1}, \quad \mathbf{r}=(r_1,r_2,r_3)\in\mathbb{Z}^3_{\geqslant 0}. \tag{2.18}$$

Note also that the weights (q_1^T,q_2^T,q_3^T) of the mirror polynomial W^T are precisely

$$(q_1^T,q_2^T,q_3^T) = (1,1,1)\, E_W^{-1} \tag{2.19}$$

Lemma 2.3. — *Let $C=\prod_{i=1}^3(-l_i/l)^{l_i}$, the period integral $\Phi_{\mathbf{r}}(\sigma)$ is in the kernel of the differential operator:*

$$\begin{aligned} &\sigma^{-l}\prod_{k=0}^{l-1}(\delta-k)\prod_{i,l_i<0}\prod_{k=0}^{-l_i-1}\left(\delta+\frac{l\left(q_i^T+\zeta_i+k\right)}{l_i}\right)\\ &\qquad\qquad - C\prod_{i,l_i>0}\prod_{k=0}^{l_i-1}\left(\delta+\frac{l\left(q_i^T+\zeta_i+k\right)}{l_i}\right), \end{aligned} \tag{2.20}$$

Proof. — Using the second equation in Lemma 2.2 we can express the derivatives $\partial_{v_i}=v_i^{-1}\delta_i$ in terms of δ. Substituting in the first equation we get a higher order differential equation in σ only. It remains only to notice that the resulting equation is independent of v and λ. □

2.4.2. Picard-Fuchs equation

Let $q_0^T = 0, l_0 = -l, \zeta_0 = 0$, and set

$$\beta_{i,k} = \frac{1}{l_i}(q_i^T + \zeta_i + k), \quad 0 \leqslant k \leqslant |l_i| - 1. \tag{2.21}$$

The differential operator in (2.20) is the product of a Bessel differential operator

$$\prod_{i,k}(\delta + l\,\beta_{i,k}) \tag{2.22}$$

and an operator of the form

$$\prod_{i',k'}(\delta + l\,\beta_{i',k'}) - C\sigma^l \prod_{i'',k''}(\delta + l\,\beta_{i'',k''}). \tag{2.23}$$

This is done by factoring out the common left divisors in the two summands. There is no pair (i', k') and (i'', k'') in the operator (2.23), such that, $\beta_{i',k'} + 1 = \beta_{i'',k''}$.

Lemma 2.4. — *The numbers* (2.21) *satisfy the following identity:*

$$\sum_{i:l_i>0}\sum_{k=0}^{l_i-1}\beta_{i,k} - \sum_{0\leqslant j\leqslant 3:l_j<0}\sum_{k'=0}^{-l_j-1}(1+\beta_{j,k'}) = \deg\phi_{\mathbf{r}}.$$

Proof. — By definition

$$LHS = \sum_{i:l_i>0}\left(\frac{l_i-1}{2} + q_i^T + \zeta_i\right) - \sum_{j;l_j<0}\left(-l_j - \frac{-l_j-1}{2} - q_j^T - \zeta_j\right) - \frac{l-1}{2}$$
$$= \sum_{i=0}^{3}\left(q_i^T + \zeta_i + \frac{l_i-1}{2}\right) - \frac{l-1}{2} = \sum_{i=1}^{3}\zeta_i = \deg\phi_{\mathbf{r}}. \qquad \square$$

As a consequence, we get a proof of Lemma 2.1.

Proof of Lemma 2.1. — To begin with, Lemma 2.3 (with $\zeta_i = 0$) implies

$$\prod_{i:\,l_i<0}\prod_{k=0}^{-l_i-1}\left(\delta + \frac{l}{l_i}(q_i^T + k)\right)\Phi = C\sigma^l \prod_{i:\,l_i>0}\prod_{k=0}^{l_i-1}\left(\delta + \frac{l(q_i^T+k)}{l_i}\right)\Phi. \tag{2.24}$$

The various values of q_i^T and l_i are listed in Tables 2.1, 2.2 and 2.3. We make the following observations:

- If the RHS of (2.24) contains a term $\delta + j$ with $j \in \mathbb{Z}, 1 \leqslant j \leqslant l-1$, then the the reduced equation (2.23) has the form that we claimed.
- For $l = 3$, $\delta + 1$ is always a factor of the RHS of (2.24).

- If $l_i < 0$, then for all $0 \leqslant k \leqslant -l_i - 1$, $l - \frac{l(q_i^T + k)}{l_i}$ is always a factor of the RHS of (2.24).

This completes the proof of Lemma 2.1. □

The action of the operator (2.23) on a period integral is again a period integral. The latter is holomorphic at $\sigma = 0$; therefore, if it is in the kernel of the Bessel operator (2.22), it must be a polynomial in σ. But a non-zero period integral cannot be a polynomial. In other words the period $\Phi_{\mathbf{r}}(\sigma)$ is a solution to the Picard-Fuchs equation corresponding to the differential operator (2.23). In particular, we can check

LEMMA 2.5. — *If W is a Fermat simple elliptic singularity. Let $x = C\sigma^l$; then either*

$$(1 - x)\frac{\partial}{\partial x}\Phi_{\mathbf{r}} = (\deg \phi_{\mathbf{r}})\,\Phi_{\mathbf{r}}, \tag{2.25}$$

or $\Phi_{\mathbf{r}}$ satisfies a hypergeometric equation

$$x(1 - x)\frac{\partial^2 \Phi_{\mathbf{r}}}{\partial x^2} + (\gamma_{\mathbf{r}} - (1 + \alpha_{\mathbf{r}} + \beta_{\mathbf{r}})\,x)\,\frac{\partial \Phi_{\mathbf{r}}}{\partial x} - (\alpha_{\mathbf{r}}\beta_{\mathbf{r}})\,\Phi_{\mathbf{r}} = 0, \tag{2.26}$$

where the weights $(\alpha_{\mathbf{r}}, \beta_{\mathbf{r}}, \gamma_{\mathbf{r}})$ follow from (2.23) *and satisfy*

$$\alpha_{\mathbf{r}} + \beta_{\mathbf{r}} - \gamma_{\mathbf{r}} = \deg \phi_{\mathbf{r}}. \tag{2.27}$$

Moreover, for $\mathbf{r} = \mathbf{0}$, $\Phi_{\mathbf{r}}$ satisfies (2.26) *for all invertible simple elliptic singularities W.*

The first part of the Lemma and the identity (2.27) are corollaries of Lemma 2.4. Unfortunately, we do not have a general combinatorial rule to determine which indexes (i', k') and (i'', k'') should appear in (2.23). In other words, the second part of the Lemma is proved by straightforward computation, case by case.

Example 2.6. — For $W = X_1^6 + X_2^3 + X_3^2$, $\phi_{\mathbf{m}} = X_1^4 X_2$, since $r_3 = 0$, we write $\mathbf{r} = (r_1, r_2)$ instead of (r_1, r_2, r_3). The weights of the hypergeometric equations for $\Phi_{\mathbf{r}}$ are

$$(\alpha_{\mathbf{r}}, \beta_{\mathbf{r}}, \gamma_{\mathbf{r}}) = \left(\frac{1 + r_1}{12}, \frac{7 + r_1}{12}, \frac{2 - r_2}{3}\right). \tag{2.28}$$

2.5. Givental's theory

The goal in this subsection is to define the total ancestor potential $\mathscr{A}_W^{\mathrm{SG}}(\mathbf{s})$ at semisimple point $\mathbf{s}$ for W. Following Givental, we introduce the

Table 2.4. Weights of periods for Fermat $E_8^{(1,1)}$

$\phi_{\mathbf{r}}$	X_1	X_2	X_1^2	X_1X_2	X_1^3	$X_1^2X_2$	X_1^4	$X_1^3X_2$
$\alpha_{\mathbf{r}}, \beta_{\mathbf{r}}, \gamma_{\mathbf{r}}$	$\frac{1}{6}, \frac{2}{3}, \frac{2}{3}$	$\frac{1}{12}, \frac{7}{12}, \frac{1}{3}$	$\frac{1}{4}, \frac{3}{4}, \frac{2}{3}$	$\frac{1}{6}, \frac{2}{3}, \frac{1}{3}$	$\frac{1}{3}, \frac{5}{6}, \frac{2}{3}$	$\frac{1}{4}, \frac{3}{4}, \frac{1}{3}$	$\frac{5}{12}, \frac{11}{12}, \frac{2}{3}$	$\frac{1}{3}, \frac{5}{6}, \frac{1}{3}$

vector space $\mathcal{H} = \mathcal{Q}_W((z))$ of formal Laurent series in z^{-1} with coefficients in $\mathcal{Q}_W$, equipped with the symplectic structure

$$\Omega(f(z), g(z)) = \mathrm{res}_{z=0}(f(-z), g(z))dz.$$

Using the polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_+ = \mathcal{Q}_W[z]$ and $\mathcal{H}_- = \mathcal{Q}_W[[z^{-1}]]z^{-1}$ we identify $\mathcal{H}$ with the cotangent bundle $T^*\mathcal{H}_+$.

Let $s \in \mathcal{S}$ be a semi-simple point, i.e., the critical values u_i of F ($1 \leqslant i \leqslant \mu$) form locally near s a coordinate system. Let us also fix a path from $0 \in \mathcal{S}$ to s, so that we have a fixed branch of the flat coordinates. Then we have an isomorphism

$$\Psi_{\mathbf{s}} : \mathbb{C}^\mu \to H, \quad e_i \mapsto \sqrt{\Delta_i}\, \partial_{u_i}$$

where Δ_i is determined by $(\partial/\partial u_i, \partial/\partial u_j) = \delta_{ij}/\Delta_i$. It is well known that $\Psi_{\mathbf{s}}$ diagonalizes the Frobenius multiplication and the residue pairing, i.e.,

$$e_i \bullet e_j = \sqrt{\Delta_i} e_i \delta_{i,j}, \quad (e_i, e_j) = \delta_{ij}.$$

Let $\mathcal{S}_{\mathrm{ss}}$ be the set of all semi-simple points. The complement $\mathcal{K} = \mathcal{S} \setminus \mathcal{S}_{\mathrm{ss}}$ is an analytic hypersurface also known as the *caustic*. It corresponds to deformations, s.t., F has at least one non-Morse critical point. By definition

$$\mathcal{S}_{\mathrm{ss}} \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^\mu, H), \quad \mathbf{s} \mapsto \Psi_{\mathbf{s}}$$

is a multi-valued analytic map.

The system of differential equations (2.8) and (2.9) admits a unique formal asymptotical solution of the type

$$\Psi_{\mathbf{s}} R_{\mathbf{s}}(z) e^{U_{\mathbf{s}}/z}, \quad R_{\mathbf{s}}(z) = 1 + R_{\mathbf{s},1} z + R_{\mathbf{s},2} z^2 + \cdots$$

where $U_{\mathbf{s}}$ is a diagonal matrix with entries $u_1(\mathbf{s}), \ldots, u_\mu(\mathbf{s})$ on the diagonal and $R_{\mathbf{s},k} \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^\mu, \mathbb{C}^\mu)$. We refer to [14, 21] for more details and proofs.

2.5.1. The quantization formalism

Let us fix a Darboux coordinate system on $\mathcal{H}$ given by the linear functions q_k^i, $p_{k,i}$ defined as follows:

$$q_k^i(\mathbf{f}) = \Omega(dt_i(-z)^{-k-1}, \mathbf{f}), \quad p_{k,i}(\mathbf{f}) = -\Omega(\partial_i z^k, \mathbf{f}), \quad \mathbf{f} \in \mathcal{H},$$

where $\{dt_i\}_{i\in\mathfrak{R}}$ is a basis of H dual to $\{\partial_i\}$ with respect to the residue pairing.

If $R = e^{A(z)}$, where $A(z)$ is an infinitesimal symplectic transformation, then we define $\widehat{R}$ as follows. Since $A(z)$ is infinitesimal symplectic, the map $\mathbf{f} \in \mathcal{H} \mapsto A\mathbf{f} \in \mathcal{H}$ defines a Hamiltonian vector field with Hamiltonian given by the quadratic function $h_A(\mathbf{f}) = \frac{1}{2}\Omega(A\mathbf{f}, \mathbf{f})$. By definition, the quantization of e^A is given by the differential operator $e^{\widehat{h_A}}$, where the quadratic Hamiltonians are quantized according to the following rules:

$$(p_{k',e'}p_{k'',e''})\widehat{\ } = \hbar\frac{\partial^2}{\partial q_{k'}^{e'}\partial q_{k''}^{e''}}, \quad (p_{k',e'}q_{k''}^{e''})\widehat{\ } = (q_{k''}^{e''}p_{k',e'})\widehat{\ } = q_{k''}^{e''}\frac{\partial}{\partial q_{k'}^{e'}},$$
$$(q_{k'}^{e'}q_{k''}^{e''})\widehat{\ } = q_{k'}^{e'}q_{k''}^{e''}/\hbar.$$

Note that the quantization defines a projective representation of the Poisson Lie algebra of quadratic Hamiltonians:

$$[\widehat{F}, \widehat{G}] = \{F, G\}\widehat{\ } + C(F, G),$$

where F and G are quadratic Hamiltonians and the values of the cocycle C on a pair of Darboux monomials is non-zero only in the following cases:

$$C(p_{k',e'}p_{k'',e''}, q_{k'}^{e'}q_{k''}^{e''}) = \begin{cases} 1 & \text{if } (k', e') \neq (k'', e''), \\ 2 & \text{if } (k', e') = (k'', e''). \end{cases} \tag{2.29}$$

2.5.2. The total ancestor potential

By definition, the Kontsevich-Witten tau-function is the following generating series:

$$\mathcal{D}_{\mathrm{pt}}(\hbar; q(z)) = \exp\Big(\sum_{g,n}\frac{1}{n!}\hbar^{g-1}\int_{\overline{\mathcal{M}}_{g,n}}\prod_{i=1}^{n}(q(\psi_i)+\psi_i)\Big), \tag{2.30}$$

where $q(z) = \sum_k q_k z^k$, $(q_0, q_1, \ldots)$ are formal variables, ψ_i $(1 \leqslant i \leqslant n)$ are the first Chern classes of the cotangent line bundles on $\overline{\mathcal{M}}_{g,n}$. The function is interpreted as a formal series in $q_0, q_1 + \mathbf{1}, q_2, \ldots$ whose coefficients are Laurent series in $\hbar$..

Let $\mathbf{s} \in \mathcal{S}_{\mathrm{ss}}$ be *a semi-simple* point. Motivated by Gromov–Witten theory Givental introduced the notion of the *total ancestor potential* of a semi-simple Frobenius structure (see [21, 20]). In our settings, the definition takes the form

$$\mathscr{A}_W^{\mathrm{SG}}(\mathbf{s}) = \mathscr{A}_{\mathbf{s}}(\hbar; \mathbf{q}) := \widehat{\Psi_{\mathbf{s}}}\, \widehat{R_{\mathbf{s}}}\, e^{\widehat{U_{\mathbf{s}}/z}} \prod_{i=1}^{\mu} \mathcal{D}_{\mathrm{pt}}(\hbar\, \Delta_i(\mathbf{s}); {}^{i}\mathbf{q}(z) \sqrt{\Delta_i(\mathbf{s})}) \tag{2.31}$$

where

$$\mathbf{q}(z) = \sum_{k=0}^{\infty} \sum_{j \in \mathfrak{R}} q_k^j\, z^k \partial t_j, \quad {}^{i}\mathbf{q}(z) = \sum_{k=0}^{\infty} {}^{i}q_k\, z^k.$$

The quantization $\widehat{\Psi_s}$ is interpreted as the change of variables

$$\sum_{i=1}^{\mu} {}^{i}\mathbf{q}(z) e_i = \Psi_{\mathbf{s}}^{-1} \mathbf{q}(z) \quad \text{i.e.} \quad {}^{i}q_k \sqrt{\Delta_i} = \sum_{j \in \mathfrak{R}} (\partial u^i / \partial t_j)\, q_k^j. \tag{2.32}$$

3. Geometric limits: GW theory and FJRW theory

The proof of Theorem 1.4 relies on a certain reconstruction property of the mirror GW invariants and FJRW invariants. All the invariants are defined by intersections of cohomologies on $\overline{\mathcal{M}}_{g,n}$ associated with some Cohomological Field Theory (CohFT). According to Krawitz–Shen [25], starting with a certain initial set of 3- and 4-point genus-0 correlators, we can determine the remaining invariants using only the axioms of a CohFT. The goal in this section is to introduce the CohFTs relevant for Theorem 1.4 and to compute explicitly the initial data of correlators needed for the reconstruction.

3.1. Cohomological Field Theories

Let H be a vector space of dimension N with a unit $\mathbf{1}$ and a non-degenerate pairing η. Without loss of generality, we always fix a basis of H, say $\mathscr{S} := \{\partial_i, i = 0, \ldots, N-1\}$, and we set $\partial_0 = \mathbf{1}$. Let $\{\partial^j\}$ be the dual basis such that $\eta(\partial_i, \partial^j) = \delta_i^j$. A CohFT Λ is a set of multi linear maps $\{\Lambda_{g,n}\}$, for each stable genus g curve with n marked points, i.e., $2g - 2 + n > 0$,

$$\Lambda_{g,n} : H^{\otimes n} \longrightarrow H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{C}).$$

Furthermore, Λ satisfies a set of axioms (CohFT axioms) described below:

(1) (S_n-invariance) For any $\sigma \in S_n$, and $\gamma_1, \ldots, \gamma_n \in H$, then

$$\Lambda_{g,n}(\gamma_{\sigma(1)}, \ldots, \gamma_{\sigma(n)}) = \Lambda_{g,n}(\gamma_1, \ldots, \gamma_n).$$

(2) (Gluing tree) Let

$$\rho_{tree} : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g,n}$$

where $g = g_1 + g_2, n = n_1 + n_2$, be the morphism induced from gluing the last marked point of the first curve and the first marked point of the second curve; then

$$\rho^*_{tree}\big(\Lambda_{g,n}(\gamma_1, \ldots, \gamma_n)\big) = \sum_{\alpha,\beta \in \mathscr{S}} \Lambda_{g_1,n_1+1}(\gamma_1, \ldots, \gamma_{n_1}, \alpha)\eta^{\alpha,\beta}\Lambda_{g_2,n_2+1}(\beta, \gamma_{n_1+1}, \ldots, \gamma_n).$$

Here $\big(\eta^{\alpha,\beta}\big)_{N\times N}$ is the inverse matrix of $\big(\eta(\alpha,\beta)\big)_{N\times N}$.

(3) (Gluing loop) Let

$$\rho_{loop} : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n},$$

be the morphism induced from gluing the last two marked points; then

$$\rho^*_{loop}\big(\Lambda_{g,n}(\gamma_1, \ldots, \gamma_n)\big) = \sum_{\alpha,\beta \in \mathscr{S}} \Lambda_{g-1,n+2}(\gamma_1, \ldots, \gamma_n, \alpha, \beta)\eta^{\alpha,\beta}.$$

(4) (Pairing)

$$\int_{\overline{\mathcal{M}}_{0,3}} \Lambda_{0,3}(\mathbf{1}, \gamma_1, \gamma_2) = \eta(\gamma_1, \gamma_2).$$

If in addition the following axiom holds:

(5) (Flat identity) Let $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the forgetful morphism; then

$$\Lambda_{g,n+1}(\gamma_1, \ldots, \gamma_n, \mathbf{1}) = \pi^*\Lambda_{g,n}(\gamma_1, \ldots, \gamma_n).$$

then we say that Λ is a CohFT with a *flat identity*.

If Λ is a CohFT; then there is a natural formal family of CohFTs $\Lambda(\mathbf{t})$. Namely,

$$\Lambda(\mathbf{t})_{g,n}(\gamma_1, \ldots, \gamma_n) = \sum_{k=0}^{\infty} \frac{1}{k!}\,\pi_*\left(\Lambda_{g,n+k}(\gamma_1, \ldots, \gamma_n, \mathbf{t}, \ldots, \mathbf{t})\right),$$

where $\pi : \overline{\mathcal{M}}_{g,n+k} \to \overline{\mathcal{M}}_{g,n}$ is the morphism forgetting the last k marked points. Note that $\Lambda(\mathbf{t})_{0,3}$ induces a family of Frobenius multiplications $\bullet_{\mathbf{t}}$ on (H, η), defined by

$$\eta(\alpha \bullet_{\mathbf{t}} \beta, \gamma) = \int_{\overline{\mathcal{M}}_{0,3}} \Lambda(\mathbf{t})_{0,3}(\alpha, \beta, \gamma). \tag{3.1}$$

It is well known that [28] the genus-0 part of the CohFT $\{\Lambda(\mathbf{t})_{0,n}\}$ is equivalent to a Frobenius manifold $(H, \eta, \bullet_{\mathbf{t}})$, in the sense of Dubrovin [14]. We call the vector space H the *state space* of the CohFT.

3.1.1. The total ancestor potential of a CohFT

For a given CohFT Λ, the *correlator functions* are by definition the following formal series in $\mathbf{t} \in H$:

$$\text{(3.2)} \quad \left\langle \alpha_1 \cdot \psi_1^{k_1}, \dots, \alpha_n \cdot \psi_n^{k_n} \right\rangle_{g,n}^{\Lambda} (\mathbf{t}) = \int_{\overline{\mathcal{M}}_{g,n}} \Lambda(\mathbf{t})_{g,n}(\alpha_1, \dots, \alpha_n)\, \psi_1^{k_1} \dots \psi_n^{k_n} \,,$$

where ψ_i is the i-th psi class on $\overline{\mathcal{M}}_{g,n}$, $\alpha_i \in H$, and $k_i \in \mathbb{Z}_{\geqslant 0}$. The value of a correlator function at $\mathbf{t} = 0$ is called simply a *correlator*. We call g the genus of the correlator function and each $\alpha_i \cdot \psi_i^{k_i}$ is called a *descendant* (resp. *non-descendant*) *insertion* if $k_i > 0$ (resp. $k_i = 0$).

For each basis $\{\partial_i\}$ in H, we fix a sequence of formal variables $\{q_k^i\}_{k=0}^{\infty}$ and define

$$\mathbf{q}(z) = \sum_{k=0}^{\infty} \sum_{i=0}^{N-1} q_k^i \, \partial_i \, z^k \;;$$

then the *genus-g ancestor potential* is the following generating function:

$$\mathscr{F}_g^{\Lambda(\mathbf{t})}(\mathbf{q}) := \sum_n \frac{1}{n!} \Big\langle \mathbf{q}(\psi_1) + \psi_1, \dots, \mathbf{q}(\psi_n) + \psi_n \Big\rangle_{g,n}^{\Lambda} (\mathbf{t}),$$

where each correlator should be expanded multi linearly in $\mathbf{q}$ and the resulting correlators are evaluated according to (3.2). Let us point out that we have assumed that the CohFT has a flat identity $\mathbf{1} \in H$ and we have incorporated the *dilaton shift* in our function, so that $\mathscr{F}_g^{\Lambda(\mathbf{t})}$ is a formal series in q_k, $k \neq 1$ and $q_1 + \mathbf{1}$. Finally, the *total ancestor potential of a CohFT* $\Lambda(\mathbf{t})$ is defined by

$$\text{(3.3)} \qquad \mathscr{A}^{\Lambda(\mathbf{t})}(\hbar; \mathbf{q}) := \exp\left(\sum_{g=0}^{\infty} \hbar^{2g-2} \mathscr{F}_g^{\Lambda(\mathbf{t})}(\mathbf{q}) \right).$$

3.2. GW theory of elliptic orbifold $\mathbb{P}^1$

Let $\mathcal{X} := \mathbb{P}^1_{a_1,a_2,a_3}$ be the orbifold-$\mathbb{P}^1$ with 3 orbifold points, such that, the i-th one has isotropy group $\mathbb{Z}/a_i\mathbb{Z}$. We are interested in 3 cases: $(a_1, a_2, a_3) =$

$(3,3,3),(4,4,2),(6,3,2)$. Together with $\mathbb{P}^1_{2,2,2,2}$, they correspond to orbifold-$\mathbb{P}^1$s that are quotients of an elliptic curve by a finite group. The Chen-Ruan cohomology $H^*_{\mathrm{CR}}(\mathbb{P}^1_{a_1,a_2,a_3})$ has the following form:

$$H^*_{\mathrm{CR}}(\mathbb{P}^1_{a_1,a_2,a_3}) = \bigoplus_{i=1}^{3}\bigoplus_{j=1}^{a_i-1}\mathbb{C}[\Delta_{ij}]\bigoplus\mathbb{C}[\Delta_{01}]\bigoplus\mathbb{C}[\Delta_{02}].$$

where $\Delta_{01}=1$ and Δ_{02} is the Poincaré dual to a point. The classes $\Delta_{ij}(1\leqslant i\leqslant 3, 1\leqslant j\leqslant a_i-1)$ are in one-to-one correspondence with the twisted sectors. The latter are just orbifold points, and we define Δ_{ij} to be the unit in the cohomology of the corresponding twisted sector. Our index makes the complex degrees

$$\deg\Delta_{ij}=j/a_i,\quad 1\leqslant i\leqslant 3,\quad 1\leqslant j\leqslant a_i-1.$$

$\mathcal{X}$ is a compact Kähler orbifold. Let $\overline{\mathcal{M}}^{\mathcal{X}}_{g,n,d}$ be the moduli space of degree-d stable maps from a genus-g orbi-curve, equipped with n marked points, to $\mathcal{X}$. Let us denote by π the forgetful map, and by ev_i the evaluation at the i-th marked point

$$\overline{\mathcal{M}}_{g,n}\quad\stackrel{\pi}{\longleftarrow}\quad\overline{\mathcal{M}}^{\mathcal{X}}_{g,n+k,d}\quad\stackrel{\mathrm{ev_i}}{\longrightarrow}\quad I\mathcal{X}\ .$$

The moduli space is equipped with a virtual fundamental cycle $[\overline{\mathcal{M}}^{\mathcal{X}}_{g,n,d}]$. Let the maps $(\Lambda^{\mathcal{X}})_{g,n,d}: H^*_{\mathrm{CR}}(\mathbb{P}^1_{a_1,a_2,a_3})^{\otimes n}\longrightarrow H^*(\overline{\mathcal{M}}_{g,n};\mathbb{C})$ be defined by

$$(\Lambda^{\mathcal{X}})_{g,n,d}(\alpha_1,\dots,\alpha_n):=\pi_*\left([\overline{\mathcal{M}}^{\mathcal{X}}_{g,n,d}]\cap\prod_{i=1}^{n}\mathrm{ev}_i^*(\alpha_i)\ \right).$$

Then the set of maps

$$\left\{(\Lambda^{\mathcal{X}})_{g,n}:=\sum_{d}(\Lambda^{\mathcal{X}})_{g,n,d}\,q^d\right\}$$

form a CohFT with state space $H^*_{CR}(\mathbb{P}^1_{a_1,a_2,a_3})$, where $q=e^{t_{02}}$ and t_{02} parametrizes the class Δ_{02}. The total ancestor potential $\mathscr{A}^{\mathrm{GW}}_{\mathcal{X}}$ of $\mathcal{X}$ is by definition (3.3) the total ancestor potential of the CohFT $\Lambda^{\mathcal{X}}(\mathbf{t}=\mathbf{0})$. For more details on orbifold Gromov–Witten theory, we refer to [6]. We will use the *ancestor Gromov-Witten invariants of* $\mathcal{X}$ that is defined by the maps $(\Lambda^{\mathcal{X}})_{g,n,d}$,

$$\left\langle\alpha_1\cdot\psi_1^{k_1},\dots,\alpha_n\cdot\psi_n^{k_n}\right\rangle_{g,n,d}=\int_{\overline{\mathcal{M}}_{g,n}}(\Lambda^{\mathcal{X}})_{g,n,d}(\alpha_1,\dots,\alpha_n)\,\psi_1^{k_1}\dots\psi_n^{k_n}\ .$$

For $\mathbb{P}^1_{a_1,a_2,a_3}$, the orbifold Poincaré pairing takes the form

$$\left\langle \Delta_{i_1j_1}, \Delta_{i_2j_2} \right\rangle = \begin{cases} \left(\delta_{i_1,i_2}\delta_{j_1+j_2,a_k}\right)/a_k, & k = i_1, i_1+i_2 \neq 0; \\ \delta_{j_1+j_2,3}, & i_1 = i_2 = 0. \end{cases} \tag{3.4}$$

It is easy to compute that the above 3-point correlators are

$$\begin{aligned} &\left\langle \Delta_{i_1j_1}, \Delta_{i_2j_2}, \Delta_{i_3j_3} \right\rangle_{0,3,0} \\ &= \begin{cases} 1/a_k, & i_1 = i_2 = i_3 = k \in \{1,2,3\}, \quad j_1+j_2+j_3 = a_k; \\ \left\langle \Delta_{i_2j_2}, \Delta_{i_3j_3} \right\rangle, & (i_1, j_1) = (0,1); \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.5}$$

Recall the Chen-Ruan orbifold cup product $\bullet$ on $H^*_{CR}(\mathbb{P}^1_{a_1,a_2,a_3};\mathbb{C})$ is defined by pairing and 3-point correlators (3.1). According to Krawitz and Shen [25], we have the following reconstruction result.

Lemma 3.1. — *The Gromov-Witten total ancestor potential $\mathscr{A}^{\mathrm{GW}}_{\mathcal{X}}$ of elliptic orbifold $\mathcal{X} = \mathbb{P}^1_{3,3,3}$, $\mathbb{P}^1_{4,4,2}$, $\mathbb{P}^1_{6,3,2}$ is determined by the following initial data: the Poincaré pairing, the Chen-Ruan product, and the correlator $\langle \Delta_{1,1}, \Delta_{2,1}, \Delta_{3,1} \rangle^{\mathcal{X}}_{0,3,1} = 1$.*

In particular, using Lemma 3.1, one can construct a mirror map to identify the Gromov–Witten theory of the above orbifolds and the Saito–Givental theory associated to certain ISES (see [25, 30]). The genus 0 reconstruction of the GW theory for those orbifolds are obtained by Satake and Takahashi independently [39]. Moreover, they also proved the isomorphism between the Frobenius manifold from Gromov-Witten theory of $\mathbb{P}^1_{3,3,3}$ and Saito's Frobenius manifold of $X_1^3 + X_2^3 + X_3^3 + \sigma X_1X_2X_3$ at $\sigma = \infty$.

On the other hand, since the mirror symmetry identifies the correlation functions with certain period integrals, by analyzing the monodromy of the period integrals, one can prove that the Gromov-Witten invariants are quasi-modular forms on some finite index subgroups of the modular group, [30, 32]. In particular, the non-zero, genus-0, 3-point correlators are modular forms of weight 1. Let us list the first few terms of their Fourier series. For $\mathcal{X} = \mathbb{P}^1_{3,3,3}$, the following correlators are weight-1 modular forms on $\Gamma(3)$ (where $q = e^{2\pi i\tau}$)

$$\begin{cases} \left\langle \Delta_{1,1}, \Delta_{2,1}, \Delta_{3,1} \right\rangle_{0,3} = q + q^4 + 2q^7 + 2q^{13} + \cdots = \frac{\eta(9\tau)^3}{\eta(3\tau)} \\ \left\langle \Delta_{1,1}, \Delta_{1,1}, \Delta_{1,1} \right\rangle_{0,3} = 1/3 + 2q^3 + 2q^9 + 2q^{12} + \cdots = \frac{3\eta(9\tau)^3+\eta(\tau)^3}{3\eta(3\tau)}, \end{cases}$$

For $\mathcal{X} = \mathbb{P}^1_{4,4,2}$, the following correlators are weight-1 modular forms on $\Gamma(4)$,

$$\begin{cases} \left\langle \Delta_{1,1}, \Delta_{2,1}, \Delta_{3,1} \right\rangle_{0,3} = q + 2q^5 + q^9 + 2q^{13} + \cdots \\ \left\langle \Delta_{1,1}, \Delta_{1,1}, \Delta_{1,2} \right\rangle_{0,3} = 1/4 + q^4 + q^8 + q^{16} + \cdots \end{cases}$$

For $\mathcal{X} = \mathbb{P}^1_{6,3,2}$, the following correlators are weight-1 modular forms on $\Gamma(6)$

$$\begin{cases} \left\langle \Delta_{1,1}, \Delta_{2,1}, \Delta_{3,1} \right\rangle_{0,3} = q + 2q^7 + 2q^{13} + 2q^{19} + \cdots \\ \left\langle \Delta_{1,1}, \Delta_{1,1}, \Delta_{1,4} \right\rangle_{0,3} = 1/6 + q^6 + q^{18} + q^{24} + \cdots \end{cases}$$

3.3. FJRW theory of invertible simple elliptic singularities

For any non-degenerate, quasi-homogeneous polynomial W, Fan–Jarvis–Ruan, following a suggestion of Witten, introduced a family of moduli spaces and constructed a virtual fundamental cycle. The latter gives rise to a cohomological field theory, which is now called the FJRW theory. We remark that FJRW theory is defined for a pair (W, G), where W is a quasi-homogeneous non-degenerate polynomial and $G \subset G_W$ is a so-called *admissible group* when it contains the exponential grading element $(\mathbf{e}[q_1], \cdots, \mathbf{e}[q_n]) \in G_W$ (see Proposition 2.3.5 in [16]). In our paper, we make use of the FJRW theories associated with the invertible simple elliptic singularities, i.e., W is one of the polynomials in Table 1, and $G = G_W$. Let us briefly review the FJRW theory only for such W and refer to [16, 15] for the general case and more details.

3.3.1. FJRW vector space and axioms

Recall the *group of diagonal symmetries* G_W of the polynomial W is

$$G_W := \left\{ (\lambda_1, \lambda_2, \lambda_3) \in (\mathbb{C}^*)^3 \,\middle|\, W(\lambda_1 X_1, \lambda_2 X_2, \lambda_3 X_3) = W(X_1, X_2, X_3) \right\}.$$

The *FJRW state space* $\mathscr{H}_{W,G_W}$ (or $\mathscr{H}_W$ for short) is the direct sum of all G_W-invariant relative cohomology:

$$\mathscr{H}_W := \bigoplus_{h \in G_W} H_h, \quad H_h := H^*(\mathbb{C}_h; W_h^\infty; \mathbb{C})^{G_W}. \tag{3.6}$$

Here $\mathbb{C}_h (h \in G_W)$ is the h-invariant subspace of $\mathbb{C}^3$, W_h is the restriction of W to $\mathbb{C}_h$ and $\mathrm{Re} W_h$ is the real part of W_h, and $W_h^\infty = (\mathrm{Re} W_h)^{-1}(M, \infty)$, for some $M \gg 0$.

The vector space $H_h (h \in G_W)$ has a natural grading given by the degree of the relative cohomology classes. However, for the purposes of the FJRW theory we need a modification of the standard grading. Namely, for any $\alpha \in H_h$, we define

$$\deg_W \alpha := \frac{\deg \alpha}{2} + \sum_{i=1}^{3} (\Theta_i^h - q_i),$$

This is half of $\deg_W \alpha$ in [16]. deg α is the degree of α as a relative cohomology class in $H^*(\mathbb{C}_h; W_h^\infty; \mathbb{C})$ and

$$h = \left(\mathbf{e}[\Theta_1^h], \mathbf{e}[\Theta_2^h], \mathbf{e}[\Theta_3^h]\right) \in (\mathbb{C}^*)^3, \quad \Theta_i^h \in [0,1) \cap \mathbb{Q}$$

where for $y \in \mathbb{R}$, we put $\mathbf{e}[y] := \exp(2\pi\sqrt{-1}y)$. Clearly the numbers Θ_i^h are uniquely determined from h. For any $\alpha \in H_h$, we define

$$\Theta(\alpha) := h. \tag{3.7}$$

The elements in H_h are called *narrow* (resp. *broad*) and H_h is called a *narrow sector* (resp. *broad sector*) if $\mathbb{C}_h = \{0\}$ (resp. $\mathbb{C}_h \neq \{0\}$). For invertible simple elliptic singularities, the space $H^*(\mathbb{C}_h; W_h^\infty; \mathbb{Q})$ is one-dimensional for all narrow sectors H_h. We always choose a generator $\alpha \in H_h$ such that

$$\alpha := 1 \in H^*(\mathbb{C}_h; W_h^\infty; \mathbb{Q})^{G_W}. \tag{3.8}$$

In general, in order to describe the broad sectors, we have to represent the relative cohomology classes by differential forms; then there is an identification (see [16] and the references there)

$$\Big(\mathscr{H}_{W,G}, \langle\ ,\ \rangle\Big) \equiv \Big(\bigoplus_{h\in G} H_h^B, \sum_{h\in G} \mathrm{Res}_h\Big), \tag{3.9}$$

where if $\mathbb{C}_h = \{0\}$, then $H_h^B = \mathbb{C}\omega_h$, and if $\mathbb{C}_h \neq \{0\}$, then $H_h^B := (\mathscr{Q}_{W_h}\omega_h)^G$. Here ω_h is the restriction of the standard volume form to the fixed locus $\mathbb{C}_h$. The bi-linear map $\mathrm{Res}_h : H_h^B \times H_{h^{-1}}^B \to \mathbb{C}$ is the residue pairing of W_h if $\mathbb{C}_h \neq \{0\}$, and it sends $(\omega_h, \omega_{h^{-1}})$ to 1 if $\mathbb{C}_h = \{0\}$. $\langle\ ,\ \rangle$ is a non-degenerate pairing induced from the intersection of relative homology cycles. There exists a basis of the narrow sectors such that the pairing $\langle v_1, v_2\rangle$, $v_i \in H_{h_i}$, is 1 if $h_1 h_2 = 1$ and 0 otherwise. The vectors in the broad sectors are orthogonal to the vectors in the narrow sectors. In order to compute the pairing on the broad sectors one needs to use the identification (3.9) and compute an appropriate residue pairing. In our case however, we can express all invariants using narrow sectors only. So a more detailed description of the broad sectors is not needed. We refer to [16] for more details.

Let (W,G) be an admissible pair. Let $W=\sum_{a=1}^s M_a$, where M_a is the a-th monomial of W. A *W-spin structure* on a genus-g Riemann surface C with n marked orbifold points $(z_1,\dots,z_n)$ is a collection of N (N is the number of variables in W) orbifold line bundles $\mathcal{L}_1,\dots,\mathcal{L}_N$ on C and isomorphisms

$$\psi_a : M_a(\mathcal{L}_1,\dots,\mathcal{L}_N)\to\omega_C(-z_1-\cdots-z_n),$$

where ω_C is the dualizing sheaf on C and the multiplication is just the tensor product among the orbifold line bundles. The orbifold line bundles have a monodromy near each marked point z_i which determines an element $h_i\in G$. In particular, if H_{h_i} is a narrow (resp. broad) sector we say that the marked point is narrow (resp. broad). For fixed g,n, and $h_1,\dots,h_n\in G$, Fan-Jarvis-Ruan (see [16]) constructed the compact moduli space $\overline{\mathscr{W}}_{g,n}(h_1,\cdots,h_n)$ of nodal Riemann surfaces equipped with a W-spin structure. In this compactification the line bundles $(\mathcal{L}_1,\dots,\mathcal{L}_N)$ are allowed to be orbifold at the nodes in such a way that the monodromy around each node is an element of G as well. The moduli space has a decomposition into a disjoint union of moduli subspaces $\overline{\mathscr{W}}_{g,n}(\Gamma_{h_1,\dots,h_n})$ consisting of W-spin structures on curves C whose dual graph is $\Gamma_{h_1,\dots,h_n}$. Recall that the dual graph of a nodal curve C is a graph whose vertices are the irreducible components of C, edges are the nodes, and tails are the marked points. The latter are decorated by elements $h_i\in G$, so the tails of our graphs are also colored respectively. We omit the subscript $(h_1,\dots,h_n)$ whenever the decoration is understood from the context. The connected component $\overline{\mathscr{W}}_{g,n}(\Gamma_{h_1,\dots,h_n})$ is naturally stratified by fixing the monodromy transformations around the nodes, i.e., the strata are in one-to-one correspondence with the colorings of the edges of the dual graph $\Gamma_{h_1,\cdots,h_n}$.

Fan–Jarvis–Ruan constructed a virtual fundamental cycle $[\overline{\mathscr{W}}_{g,n}(\Gamma)]^{\mathrm{vir}}$ of $\overline{\mathscr{W}}_{g,n}(\Gamma)$ (see [15]), which gives rise to a CohFT

$$\Lambda_{g,n}^{W,G} : (\mathscr{H}_{W,G})^{\otimes n}\longrightarrow H^*(\overline{\mathcal{M}}_{g,n}).$$

For brevity put $\Lambda_{g,n}^W$ for $\Lambda_{g,n}^{W,G_W}$. We define *the total ancestor FJRW potential* $\mathscr{A}_W^{\mathrm{FJRW}}$ of (W,G_W) from the CohFT Λ^W using (3.3).

Finally, let us list some general properties of the FJRW correlators of a simple elliptic singularity W, see [16] for the proofs.

- (Selection rule) If the correlator $\left\langle\alpha_1\cdot\psi_1^{k_1},\dots,\alpha_n\cdot\psi_n^{k_n}\right\rangle_{g,n}^W$ is nonzero; then

$$\sum_{i=1}^n \deg_W(\alpha_i)+\sum_{i=1}^n k_i=2g-2+n. \tag{3.10}$$

- (Line bundle criterion). If the moduli space $\overline{\mathscr{W}}_{g,n}(h_1,\ldots,h_n)$ is non-empty, then the degree of the desingularized line bundle $|\mathcal{L}_j|$ is an integer, i.e.

$$\deg(|\mathcal{L}_j|)=q_j(2g-2+n)-\sum_{k=1}^{n}\Theta_j^{h_k}\in\mathbb{Z}. \tag{3.11}$$

- (Concavity) Suppose that all marked points are narrow, π is the morphism from the universal curve to $\overline{\mathscr{W}}_{g,n}(h_1,\ldots,h_n)$ and $\pi_*\bigoplus_{i=1}^{3}\mathcal{L}_i=0$ holds; then

$$[\overline{\mathscr{W}}_{g,n}(h_1,\ldots,h_n)]^{\mathrm{vir}}=c_{\mathrm{top}}\left(-R^1\pi_*\bigoplus_{i=1}^{3}\mathcal{L}_i\right)\cap[\overline{\mathscr{W}}_{g,n}(h_1,\ldots,h_n)]. \tag{3.12}$$

Let $\alpha_i=1\in H_{h_i}$, $1\leqslant i\leqslant 4$ be the generators (cf. (3.8)). The concavity formula (3.12) implies that $\Lambda_{0,4}^{W}(\alpha_1,\ldots,\alpha_4)\in H^*(\overline{\mathcal{M}}_{0,4},\mathbb{C})$. According to the orbifold Grothendieck-Riemann-Roch formula (see [7], Theorem 1.1.1),

$$\begin{aligned}&\Lambda_{0,4}^{W}(\alpha_1,\ldots,\alpha_4)\\&=\sum_{i=1}^{3}\left(\frac{B_2(q_i)}{2}\kappa_1-\sum_{j=1}^{4}\frac{B_2(\Theta_i^{h_j})}{2}\psi_j+\sum_{\Gamma\in\mathbf{\Gamma}_{0,4,W}(h_1,\ldots,h_4)}\frac{B_2(\Theta_i^{h_\Gamma})}{2}[\Gamma]\right),\end{aligned} \tag{3.13}$$

where κ_1 is the first kappa class defined by Mumford, B_2 is the second Bernoulli polynomial

$$B_2(y)=y^2-y+\frac{1}{6},$$

$[\Gamma]$ is the boundary class on $\overline{\mathcal{M}}_{g,n}$ corresponding to the graph Γ, and $\mathbf{\Gamma}_{0,4,W}(h_1,\ldots,h_4)$ is the set of graphs with one edge decorated by G_{W^T}. The graph Γ has 4 tails decorated by h_1,h_2,h_3,h_4 and its edge is decorated by h_Γ and h_Γ^{-1}. If the moduli space $\overline{\mathscr{W}}_{0,4}(h_1,\ldots,h_4)$ is non-empty, each component satisfies (3.11). It is easy to see that the formula does not depend on the choice of h_Γ or h_Γ^{-1}.

3.3.2. Generators of the FJRW ring.

From now on, we will consider W as an ISES in Table (1.1). Let (W^T,G_{W^T}) be the Berglund-Hübsch-Krawitz mirror of $(W,\{1\})$ (see its definition in Section 1.1). We will compare the FJRW theory for (W^T,G_{W^T}) with the Saito-Givental theory for W.

Since $\mathscr{H}_{W^T}:=\mathscr{H}_{W^T,G_{W^T}}$ is the state space of a CohFT, it has a Frobenius algebra structure, where the multiplication $\bullet$ is defined by pairing and 3-point correlators (3.1). For all invertible W, M. Krawitz (see [24]) constructed a ring isomorphism

$$\mathscr{H}_{W^T} \cong \mathscr{Q}_W. \tag{3.14}$$

Next, we give an explicit description of the generators of $\mathscr{H}_{W^T}$ and the ring isomorphism for W, which is an ISES in Table (1.1). For a more general description, we refer the interested readers to [24], [17] and [1].

For every ISES W^T, there exists a 3-tuple $(a,b,c) \in \mathbb{Z}^3_{>0}$ such that

$$G_{W^T} \cong \mu_a \times \mu_b \times \mu_c, \quad \mu_k = \mathbb{Z}/k\mathbb{Z}.$$

We assume $a \geqslant b \geqslant c$ and omit the factor μ_1. For example, for $W^T = X_1^3 + X_1X_2^4 + X_3^2$

$$G_{W^T} = \left\{(\lambda_1,\lambda_2,\lambda_3)\Big|\lambda_1^3 = \lambda_1\lambda_2^4 = \lambda_3^2 = 1\right\} \cong \mu_{12} \times \mu_2.$$

Let $h = (i,j,k) \in \mu_a \times \mu_b \times \mu_c \cong G_{W^T}$, H_h is a narrow sector and $H_h \cong \mathbb{C}$, if

$$1 \leqslant i < a, \quad 1 \leqslant j < b, \quad 1 \leqslant k < c,$$

In this case, we denote a generator of H_h by

$$\mathbf{e}_{i,j,k} := 1 \in H_h = H^0(\mathbb{C}_h; W_h^\infty; \mathbb{Q}).$$

Example 3.2. — We compute the FJRW ring for loop singularity W^T, with $W \in E_6^{(1,1)}$.

$$\begin{aligned} W^T &= X_1^2X_3 + X_1X_2^2 + X_2X_3^2,\\ G_{W^T} &= \left\{\mathbf{e}_i = \left(\mathbf{e}[\frac{i}{9}], \mathbf{e}[\frac{4i}{9}], \mathbf{e}[-\frac{2i}{9}]\right), i = 1,\dots,8\right\} \cong \mu_8. \end{aligned}$$

All nonzero 3-point genus-0 correlators are

$$\begin{cases} \left\langle \mathbf{e}_1,\mathbf{e}_1,\mathbf{e}_1\right\rangle_{0,3} = \left\langle \mathbf{e}_4,\mathbf{e}_4,\mathbf{e}_4\right\rangle_{0,3} = \left\langle \mathbf{e}_7,\mathbf{e}_7,\mathbf{e}_7\right\rangle_{0,3} = -2;\\ \left\langle \mathbf{e}_3,\mathbf{e}_i,\mathbf{e}_{9-i}\right\rangle_{0,3} = \left\langle \mathbf{e}_1,\mathbf{e}_4,\mathbf{e}_7\right\rangle_{0,3} = 1. \end{cases}$$

The first row uses Index Zero Axiom (see [16]) and the second row uses Concavity Axiom (3.12). It is easy to see $\mathbf{e}_3$ is the identity element and the ring relations are

$$2\,\mathbf{e}_1 \bullet \mathbf{e}_4 + \mathbf{e}_7^2 = 2\,\mathbf{e}_4 \bullet \mathbf{e}_7 + \mathbf{e}_1^2 = 2\,\mathbf{e}_7 \bullet \mathbf{e}_1 + \mathbf{e}_4^2 = 0.$$

Thus we obtain a ring isomorphism between $\mathscr{H}_{W^T}$ and $\mathscr{Q}_W$:

$$\rho_1 = \mathbf{e}_4 \mapsto X_1, \quad \rho_2 = \mathbf{e}_1 \mapsto X_2, \quad \rho_3 = \mathbf{e}_7 \mapsto X_3.$$

For all 13 types of ISESs with a maximal admissible group, there is a unique narrow sector ρ_{-1}, with $\deg_{W^T}(\rho_{-1}) = 1$ and

$$\Theta(\rho_{-1}) := \left(1 - q_1^T, 1 - q_2^T, 1 - q_3^T\right).$$

There are 13 types of ISESs, but only 9 of them do not have broad generators. The narrow sectors have the advantage that we can use the powerful concavity axiom (3.12). Combined with the remaining properties of the correlators and the WDVV equations this allows us to reconstruct all genus-0 FJRW invariants. According to the reconstruction theorem in [25], we can also reconstruct the higher genus FJRW invariants, i.e., the total ancestor potential function $\mathscr{A}_{W^T}^{\mathrm{FJRW}}$.

In the remaining 4 cases, we know how to offset the complication of having broad generators for $W^T = X_1^2 + X_1X_2^2 + X_2X_3^3$. The maximal abelian group is of order 12. Its FJRW vector space has eight generators:

$$\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_5, \mathbf{e}_7, \mathbf{e}_9, \mathbf{e}_{11}, R_4, R_8.$$

Here R_4 and R_8 are the cohomology classes represented by the following forms:

$$R_h = dX_1 \wedge dX_2 \in H^2(\mathbb{C}_h; W_h^\infty; \mathbb{Q}), \quad h = 4, 8 \in G_{W^T}.$$

Note that R_4 and R_8 are G_{W^T}-invariant elements in $\mathscr{Q}_{W_h}\omega_h$ where $h \in G_{W^T}$ acts on each factor X_i and dX_i as multiplication by $\mathbf{e}[q_i^T]$. Although one of the ring generators (R_4) is broad, we have enough WDVV equations to reconstruct the correlators containing broad sectors from correlators with only narrow elements and apply the concavity axioms.

For the other three types of ISESs, we can still compute some genus-0 4-point correlators with broad sectors, but we do not know how to reconstruct the complete theory only from correlators with narrow elements. In other words, for 10 out of the 13 ISESs, we can compute all the FJRW invariants. These cases and the corresponding ring generators ρ_i of the FJRW ring $\mathscr{H}_{W^T}$ are listed in tables below.

3.3.3. Classification of ring structure

According to isomorphism (3.14), in order to classify the FJRW rings for ISES, it is enough to classify the Jacobian algebras. For any special limit in the Saito-Givental theory for ISES, we need the ring structure for the classification. Let us first focus on special limits at $\sigma = 0$. According to Saito [36], simple elliptic singularities are classified by their Milnor number and the elliptic curve at infinity. It follows that the Jacobian algebras

Table 3.1. *Generators of the FJRW ring* $\mathscr{H}_{W^T}$, $W \in E_6^{(1,1)}$

W^T	G_{W^T}	$\Theta(\mathbf{e}_{i,j,k})$	ρ_1	ρ_2	ρ_3	ρ_{-1}
$X_1^3+X_2^3+X_3^3$	μ_3^3	$\mathbf{e}[\frac{i}{3}],\mathbf{e}[\frac{j}{3}],\mathbf{e}[\frac{k}{3}]$	$\mathbf{e}_{2,1,1}$	$\mathbf{e}_{1,2,1}$	$\mathbf{e}_{1,1,2}$	$\mathbf{e}_{2,2,2}=\rho_1\rho_2\rho_3$
$X_1^2X_3+X_1X_2^2+X_2X_3^2$	μ_8	$\mathbf{e}[\frac{i}{9}],\mathbf{e}[\frac{4i}{9}],\mathbf{e}[-\frac{2i}{9}]$	$\mathbf{e}_4$	$\mathbf{e}_1$	$\mathbf{e}_7$	$\mathbf{e}_6=\rho_1\rho_2\rho_3$
$X_1^2+X_1X_2^2+X_2X_3^3$	μ_{12}	$\mathbf{e}[\frac{i}{2}],\mathbf{e}[-\frac{i}{4}],\mathbf{e}[\frac{i}{12}]$	R_4	$\mathbf{e}_1$	$\mathbf{e}_7$	$\mathbf{e}_9=\rho_2\rho_3^2$

Table 3.2. *Generators of the FJRW ring* $\mathscr{H}_{W^T}$, $W \in E_7^{(1,1)}$

W^T	G_{W^T}	$\Theta(\mathbf{e}_{i,j,k})$	ρ_1	ρ_2	ρ_{-1}
$X_1^4+X_2^4+X_3^2$	$\mu_4^2\times\mu_2$	$\mathbf{e}[\frac{i}{4}],\mathbf{e}[\frac{j}{4}],\mathbf{e}[\frac{1}{2}]$	$\mathbf{e}_{2,1}$	$\mathbf{e}_{1,2}$	$\mathbf{e}_{2,2}=\rho_1^2\rho_2^2$
$X_1^3X_2+X_1X_2^3+X_3^2$	$\mu_8\times\mu_2$	$\mathbf{e}[-\frac{3i}{8}],\mathbf{e}[\frac{i}{8}],\mathbf{e}[\frac{1}{2}]$	$\mathbf{e}_1$	$\mathbf{e}_5$	$\mathbf{e}_6=\rho_1^2\rho_2^2$
$X_1^3+X_1X_2^4+X_3^2$	$\mu_{12}\times\mu_2$	$\mathbf{e}[\frac{-i}{3}],\mathbf{e}[\frac{i}{12}],\mathbf{e}[\frac{1}{2}]$	$\mathbf{e}_1$	$\mathbf{e}_5$	$\mathbf{e}_{10}=\rho_1\rho_2^3$
$X_1^3+X_1X_2^2+X_2X_3^2$	μ_{12}	$\mathbf{e}[\frac{i}{3}],\mathbf{e}[-\frac{i}{6}],\mathbf{e}[\frac{i}{12}]$	$\mathbf{e}_5$	$\mathbf{e}_1$	$\mathbf{e}_8=\rho_1\rho_2\rho_3$

Table 3.3. Generators of the FJRW ring $\mathscr{H}_{W^T}$, $\quad W \in E_8^{(1,1)}$

W^T	G_{W^T}	$\Theta(\mathbf{e}_{i,j,k})$	ρ_1	ρ_2	ρ_{-1}
$X_1^6 + X_2^3 + X_3^2$	$\mu_6 \times \mu_3 \times \mu_2$	$\mathbf{e}[\frac{i}{6}], \mathbf{e}[\frac{j}{3}], \mathbf{e}[\frac{1}{2}]$	$\mathbf{e}_{2,1}$	$\mathbf{e}_{1,2}$	$\mathbf{e}_{5,2} = \rho_1^4\rho_2$
$X_1^3 + X_1X_2^2 + X_3^3$	$\mu_6 \times \mu_3$	$\mathbf{e}[\frac{-i}{3}], \mathbf{e}[\frac{i}{6}], \mathbf{e}[\frac{j}{3}]$	$\mathbf{e}_{1,1}$	$\mathbf{e}_{2,2}$	$\mathbf{e}_{4,2} = \rho_1\rho_2\rho_3$
$X_1^4 + X_1X_2^3 + X_3^2$	$\mu_{12} \times \mu_2$	$\mathbf{e}[\frac{-i}{4}], \mathbf{e}[\frac{i}{12}], \mathbf{e}[\frac{1}{2}]$	$\mathbf{e}_2$	$\mathbf{e}_7$	$\mathbf{e}_9 = \rho_1^2\rho_2^2$

Table 3.4. Classification of ring structure

$\big(\mu, j(0)\big)$	W
(8,0)	$X_1^3 + X_2^3 + X_3^3, X_1^2X_2 + X_2^3 + X_3^3, X_1^2X_2 + X_1X_2^2 + X_3^3, X_1^2X_2 + X_2^2X_3 + X_1X_3^2$
(8,1728)	$X_1^2X_2 + X_2^2X_3 + X_3^3$
(9,0)	$X_1^3X_2 + X_2^4 + X_3^2, X_1^3X_2 + X_2^2X_3 + X_3^2$
(9,1728)	$X_1^4 + X_2^4 + X_3^2, X_1^2X_2 + X_2^2 + X_3^4, X_1^3X_2 + X_1X_2^3 + X_3^2$
(10,0)	$X_1^6 + X_2^3 + X_3^2, X_1^3X_2 + X_2^2 + X_3^3$
(10,1728)	$X_1^4X_2 + X_2^3 + X_3^2$

of the ISES with 3 variables can be classified into 6 isomorphic classes, parametrized by the pair consisting of the Milnor number $\mu = \dim \mathscr{Q}_W$ and $j(0)$, the j-invariant of $E_{\sigma=0}$:

For any two polynomials in the same list, it is easy to find a linear map between the generators X_1, X_2, X_3 of the corresponding Jacobian algebras, such that it induces a ring isomorphism. Let us point out that the choice of such linear maps is not unique in general. In Section 4.4, we can always adjust some constants such that the ring isomorphism will be extended to an isomorphism of Frobenius manifold, as well as an isomorphism of the corresponding ancestor total potential.

3.3.4. Reconstruction of all genera FJRW invariants

For an ISES W^T, its total ancestor potential $\mathscr{A}_{W^T}^{\mathrm{FJRW}}$ can be reconstructed from genus-0 primary correlators. This technique is already used in [25] for three special examples of ISESs. As the reconstruction procedures used there only require tautological relations on cohomology of moduli spaces of curves and the FJRW ring structure, we can easily generalize to all other examples. We sketch the general procedures here and refer to [25] for readers who are interested in more details. There are three steps.

First, we express the correlators of genus at least 2 and the correlators with descendant insertions in terms of correlators of genus-0 or genus-1 with non-descendant insertions (called *primary correlators*). This step is based on a tautological relation which splits a polynomial of ψ-classes and κ-classes with higher degree to a linear combination of products of boundary classes and polynomials of ψ-classes and κ-classes of lower degrees. This is called g-reduction. The reason why g-reduction works in our case is that the Selection rule imposes a constraint on the degree of the polynomials involving ψ- and κ- classes (see Theorem 6.2.1 in [16]). In general, for an arbitrary CohFT this argument fails and one has to use other methods (e.g. Teleman's reconstruction theorem).

Next, we reconstruct the non-vanishing genus-1 primary correlators from genus 0 primary correlators using Getzler's relation. The latter is a relation in $H^4(\overline{\mathcal{M}}_{1,4})$, which gives identities involving the FJRW corrletors with genus 0 and 1. In order to obtain the desired reconstruction identity, i.e., to express genus-1 in terms of genus-0 correlators, one has to make an appropriate choice of the insertions corresponding to the 4 marked points in $\overline{\mathcal{M}}_{1,4}$ (see Theorem 3.9 in [25]).

Finally, to reconstruct the genus-0 correlators we use the WDVV equations. We say that a homogeneous element $\alpha \in \mathscr{H}_{W^T}$ is *primitive* if it cannot be decomposed as a product $a' \bullet a''$ of two elements a' and a'' of non-zero degrees. We also say that a genus-0 correlator is a *basic correlator* if there are at most two non-primitive insertions, neither of which is the identity. We use the WDVV equation to rewrite a primary genus-0 correlator which contains several non-primitive insertions to correlators with fewer non-primitive insertions and correlators with a fewer number of marked points. Again the Selection rule should be taken into account in order to obtain a bound for the number of marked points. It turns out that all correlators are determined by the basic correlators with at most four marked points (see Lemma 3.7 in [25]).

LEMMA 3.3. — *For an invertible simple elliptic singularity W^T the total ancestor FJRW potential $\mathscr{A}^{\mathrm{FJRW}}_{W^T}$ of (W^T, G_{W^T}) is reconstructed from the pairing, the FJRW ring structure constants and the 4-point basic correlators with one of the insertions being a top degree element.*

3.3.5. The 4-point genus-0 FJRW invariants

We introduce the following notation.

DEFINITION 3.4. — *Let $\Xi(\rho_1, \rho_2, \rho_3)$ be a degree 1 monomial with leading coefficient 1. For simplicity, we denote by $\langle \Xi, \rho_{-1}\rangle^{W^T}_{0,4}$ a basic correlator such that the first three insertions give a factorization of Ξ.*

This is well defined because the WDVV equations guarantee that $\langle \Xi, \rho_{-1}\rangle^{W^T}_{0,4}$ does not depend on the choices of the factorization. For example, let $\Xi(\rho_1, \rho_2, \rho_3) = \rho_1^2\rho_2^2$; then the notation $\langle \Xi, \rho_{-1}\rangle^{W^T}_{0,4}$ represents any of the following choices of correlators:

$$\left\langle \rho_1, \rho_1, \rho_2^2, \rho_{-1}\right\rangle^{W^T}_{0,4}, \left\langle \rho_1, \rho_2, \rho_1\rho_2, \rho_{-1}\right\rangle^{W^T}_{0,4}, \left\langle \rho_2, \rho_2, \rho_1^2, \rho_{-1}\right\rangle^{W^T}_{0,4}.$$

But it does not represent $\left\langle 1, \rho_1, \rho_1\rho_2^2, \rho_{-1}\right\rangle^{W^T}_{0,4}$, which is not a basic correlator.

LEMMA 3.5. — *Let W^T be an ISES; then the total FJRW potential $\mathscr{A}^{\mathrm{FJRW}}_{W^T}$ for (W^T, G_{W^T}) can be reconstructed from the FJRW algebra, and the basic 4-point FJRW correlators $\langle \Xi, \rho_{-1}\rangle^{W^T}_{0,4}$. Furthermore, if W^T is an ISES as in Tables 3.1, 3.2, or 3.3, then*

$$\left\langle \Xi(\rho_1, \rho_2, \rho_3), \rho_{-1}\right\rangle^{W^T}_{0,4} = \begin{cases} q_i^T & \text{if } \Xi = M_i, \\ 0 & \text{otherwise.} \end{cases} \tag{3.15}$$

where M_i are the homogeneous monomials such that $W = M_1 + M_2 + M_3$.

Proof. — In [25, Theorem 3.4], it was proved for three special simple elliptic singularities $W^T = X_1^3+X_2^3+X_3^3, X_1^3+X_1X_2^2+X_2X_3^2$ and $X_1^3+X_1X_2^2+X_3^3$, their FJRW correlators with symmetry group G_{W^T} can be reconstructed from their FJRW algebra and some basic 4-point correlators. We apply the same method to all cases of simple elliptic singularities here. Finally, using WDVV equations in each case, it is again not hard to verify all 4-point basic correlators without insertion ρ_{-1} can be reconstructed too.

For the second part of the lemma, we use WDVV and concavity to compute FJRW correlators. We show the argument works for singularities of Fermat type and of loop type. Other cases are similar. For a Fermat type singularity, put $M_i = X_i^{1/q_i^T}$, since all insertions are narrow, we apply the Concavity Axiom (3.13) to compute

$$\left\langle \rho_i, \rho_i, \rho_i^{1/q_i^T-2}, \rho_{-1}\right\rangle_{0,4}^{W^T}. \tag{3.16}$$

Note that $\deg \mathscr{L}_i = -2$ and the degree shifting numbers are $(2q_i^T, 2q_i^T, 1-q_i^T, 1-q_i^T)$, thus the dual graphs will have $\Theta_\Gamma = 0, 0, 1-3q_i^T$. The correlator (3.16) becomes

$$\frac{1}{2}\left(B_2(q_i^T)+B_2(1-3q_i^T)+2B_2(0)-2B_2(q_i^T)-2B_2(1-q_i^T)\right) = q_i^T.$$

For loop type, $W^T = X_1^2X_3+X_1X_2^2+X_2X_3^2$. Let us compute $\left\langle \rho_1, \rho_1, \rho_2, \rho_{-1}\right\rangle_{0,4}^{W^T}$, which is not concave. However, the Concavity Axiom (3.13) implies

$$\left\langle \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_7, \mathbf{e}_2\right\rangle_{0,4}^{W^T} = -\frac{2}{9}.$$

On the other hand, WDVV equations show

$$\begin{cases} \left\langle \mathbf{e}_1\bullet\mathbf{e}_4, \mathbf{e}_4, \mathbf{e}_7, \mathbf{e}_2\right\rangle_{0,4}^{W^T} + \left\langle \mathbf{e}_1, \mathbf{e}_4, \mathbf{e}_4, \mathbf{e}_7\bullet\mathbf{e}_2\right\rangle_{0,4}^{W^T} = \left\langle \mathbf{e}_7\bullet\mathbf{e}_4, \mathbf{e}_4, \mathbf{e}_1, \mathbf{e}_2\right\rangle_{0,4}^{W^T}; \\ \left\langle \mathbf{e}_4\bullet\mathbf{e}_4, \mathbf{e}_1, \mathbf{e}_7, \mathbf{e}_2\right\rangle_{0,4}^{W^T} + \left\langle \mathbf{e}_4, \mathbf{e}_4, \mathbf{e}_1, \mathbf{e}_7\bullet\mathbf{e}_2\right\rangle_{0,4}^{W^T} = \left\langle \mathbf{e}_4\bullet\mathbf{e}_7, \mathbf{e}_1, \mathbf{e}_4, \mathbf{e}_2\right\rangle_{0,4}^{W^T}; \end{cases}$$

We observe up to symmetry, $\left\langle \mathbf{e}_5, \mathbf{e}_1, \mathbf{e}_7, \mathbf{e}_2\right\rangle_{0,4}^{W^T} = \left\langle \mathbf{e}_8, \mathbf{e}_1, \mathbf{e}_4, \mathbf{e}_2\right\rangle_{0,4}^{W^T}$. Recall the ring relations in Example 3.2, we obtain

$$\left\langle \rho_1, \rho_1, \rho_2, \rho_{-1}\right\rangle_{0,4}^{W^T} = \left\langle \mathbf{e}_4, \mathbf{e}_4, \mathbf{e}_1, \mathbf{e}_6\right\rangle_{0,4}^{W^T} = \frac{1}{3}.$$ □

4. Mirror symmetry at Gepner points

4.1. The Saito–Givental limit

Now we will discuss the existence of Saito-Givental limit at the special limit point of simple elliptic singularities. In general, Givental's formula (2.31) only defines a total ancestor potential at a semisimple points $\mathbf{s} \in S$, because the asymptotic operator $R_{\mathbf{s}}(z)$ (see Section 2.5) has singularities along the *caustic* $\mathcal{K} \subset \mathcal{S}$ consisting of non-semisimple $\mathbf{s}$. It is a famous question under which conditions the potentials at semisimple points extend to non-semisimple points. Teleman generalizes the formula to CohFT level and one would also ask whether such CohFTs extend to non-semisimple points as well. Both extension problems are highly nontrivial, see [40, 11, 29] for discussions. In particular, the extension problem of the Saito-Givental ancestor potentials at caustics is already solved for generic isolated singularity by the first author recently [29], using Eynard-Orantin recursion.

For simple elliptic singularities, the extension problem at the special limit points that appear in Theorem 1.4 and Theorem 1.5 can be solved by the reconstruction technique [25]. More explicitly, the extension follows from the convergence of certain generating functions, and the convergence relies on the reconstruction, both in genus zero and higher genus. Since the proof of the convergence is quite cumbersome, we do not repeat the process here. We briefly explain the argument and refer the readers to the proof of Theorem 1.2 in [25] for details.

The logic of our argument are seperated in two steps. In the first step, we identify the genus-0 generating functions with no descendent insertions between Saito-Givental theory at those special limit points and Gromov-Witten theory of elliptic orbifold $\mathbb{P}^1$s or FJRW theory of some simple elliptic singularities. We call the identification the *genus zero mirror symmetry*. The genus-0 generating functions of the Saito–Givental theory are well defined for all $\mathbf{s} \in \mathcal{S}$. At other points, we can simply treat them as meromorphic functions. According to the genus zero reconstruction, we only need to match a few genus zero invariants in Saito-Givental theory with those in GW theory or FJRW theory. In the final sections of this paper, we will do the calculations to match those invariants.

In the second step, we can use the reconstruction (both at genus zero and higher genus) of the correlation functions (see Section 3.2 and Section 3.3.4 for the genus zero part and [25] for the higher genus part) for both GW-point and FJRW-point to prove that the GW (or FJRW) total ancestor potential is convergent at the points, which are mirror to $\mathbf{s} \in \mathcal{S}$, such that $\sigma = s_{\mu-1}$ is sufficiently close to the special point and it coincides with the Saito–Givental ancestor potential $\mathscr{A}_W^{\mathrm{SG}}(\mathbf{s})$ (see (2.31)). An upshot is that $\mathscr{A}_W^{\mathrm{SG}}(\mathbf{s})$ extends holomorphically through the caustic $\mathcal{K}$ and the special limit points of all Fermat simple elliptic singularities.

All in all, as long as we can check the genus zero mirror symmetry at the special limit point σ, one can define the Saito-Givental limit $\mathscr{A}_W^{\mathrm{SG}}(\sigma)$ at those special limits σ by

$$\mathscr{A}_W^{\mathrm{SG}}(\sigma) := \lim_{\mathbf{s}\to(\sigma,\mathbf{0})} \mathscr{A}_W^{\mathrm{SG}}(\mathbf{s}). \tag{4.1}$$

There is an alternative way to proceed provided that we know that the genus-0 CohFTs are the same. It is based on Teleman's classification of semisimple CohFTs [40]. More explicitly, Coates and Iritani proved that for a smooth projective variety which satisfies the so-called *Genus-Zero Convergence* condition and *Analytic Semisimplicity* condition, then its GW total ancestor potential is convergent, in the sense that it is a rational element in some Fock space, see Theorem 6.5 in [11]. We refer the readers to [11] for the details of the explanation.

The same technique can be applied to the orbifold GW theory and FJRW theory. Then we only need to check the *Genus-Zero Convergence* condition and the *Analytic Semisimplicity* condition. In GW theory of elliptic orbifold $\mathbb{P}^1$s and FJRW theory of simple elliptic singularities, the first condition can be checked by explicit estimation of genus zero invariants based on the reconstruction; the second condition follows from genus zero mirror symmetry and the properties of isolated singularities. Again, the existence of the limit is guaranteed by the genus zero mirror symmetry and appropriate estimation. The advantage of this approach is that one can prove a stronger result. Namely, the higher-genus Saito–Givental CohFT (not only its ancestor potential) extends through the caustic $\mathcal{K}$ and the special limits of the Fermat simple ellitptic singularities. The details in the proofs of the above statements can be found in Lemma 3.2 in [30] for the case of ancestor potentials and in Proposition 5.5 in [31] for the CohFTs.

In the rest of this section, our goal is to prove Theorem 1.4. According to the reconstruction results in FJRW theory (see Lemma 3.5) we need to compute certain 3- and 4- point genus-0 correlators in Saito's theory and compare them to the ones in the mirror FJRW theory.

4.2. B-model 3-point genus-0 correlators

We continue to use the same notation as in Section 2.4. Namely, let $W = M_1 + M_2 + M_3$ be an ISES with a miniversal deformation given by a monomial $\phi_{\mathbf{m}}(\mathbf{x})$, $\mathbf{m} = (m_1, m_2, m_3)$. We choose a primitive form $\omega = d^3\mathbf{x}/\pi(\sigma)$ in a neighborhood of $\sigma = 0$, such that $\pi(\sigma)$ is the solution to the Picard-Fuchs equation (2.12) satisfying the initial conditions $\pi(0) = c, \pi'(0) = 0$, where the constant c is such that the residue pairing (see (2.4)) satisfies

$$\langle 1, \phi_{\mathbf{m}}\rangle|_{\mathbf{s}=0} = 1\,.$$

Let $\{t_{\mathbf{r}}\}$ be the flat coordinate system, such that $t_{\mathbf{r}}(0) = 0$ and the flat vector fields $\partial_{\mathbf{r}} := \partial/\partial t_{\mathbf{r}}$ agree with $\partial/\partial s_{\mathbf{r}}$ at $\mathbf{s} = 0$.

The primitive form induces an isomorphism between the tangent and the vanishing cohomology bundle via the following period mapping:

$$\begin{aligned}\partial/\partial t_{\mathbf{r}} \mapsto -\nabla^{-1}_{\frac{\partial}{\partial\lambda}}\nabla_{\frac{\partial}{\partial t_{\mathbf{r}}}}\int\frac{\omega}{dF} &= \int\delta_{\mathbf{r}}(\mathbf{s},\mathbf{x})\,\frac{\omega}{dF}\\ &= \int\delta_{\mathbf{r}}(\mathbf{s},\mathbf{x})\,\frac{1}{\pi(\sigma)}\,\frac{d^3\mathbf{x}}{dF},\end{aligned}\tag{4.2}$$

where $\delta_{\mathbf{r}}$ is some homogeneous polynomial (in $\mathbf{x}$) of degree $\deg(\phi_{\mathbf{r}})$. Note that the Kodaira–Spencer isomorphism takes the form

$$\partial/\partial t_{\mathbf{r}} \mapsto \delta_{\mathbf{r}}(\mathbf{s},\mathbf{x}) \bmod (F_{X_1}, F_{X_2}, F_{X_3}).\tag{4.3}$$

We know $\partial_{\mathbf{r}_1}\bullet\partial_{\mathbf{r}_2}$ is induced from multiplication in $q_*\mathcal{O}_C$, and the pairing is

$$\Big\langle\delta_{\mathbf{r}_1},\delta_{\mathbf{r}_2}\Big\rangle := \eta(\partial_{\mathbf{r}_1},\partial_{\mathbf{r}_2}) = \operatorname{Res}\frac{\delta_{\mathbf{r}_1}(\mathbf{s},\mathbf{x})\delta_{\mathbf{r}_2}(\mathbf{s},\mathbf{x})}{(\partial_{X_1}W_\sigma)\,(\partial_{X_2}W_\sigma)(\partial_{X_3}W_\sigma)}\frac{d^3\mathbf{x}}{\pi(\sigma)^2}.$$

By definition, the restriction of the 3-point correlators to the marginal direction is

$$\begin{aligned}\Big\langle\delta_{\mathbf{r}_1},\delta_{\mathbf{r}_2},\delta_{\mathbf{r}_3}\Big\rangle_{0,3} &= \Big\langle\delta_{\mathbf{r}_1},\delta_{\mathbf{r}_2}\cdot\delta_{\mathbf{r}_3}\Big\rangle\\ &= \operatorname{Res}\frac{\delta_{r_1}(\sigma,\mathbf{x})\delta_{r_2}(\sigma,\mathbf{x})\delta_{r_3}(\sigma,\mathbf{x})}{(\partial_{X_1}W_\sigma)\,(\partial_{X_2}W_\sigma)(\partial_{X_3}W_\sigma)}\,\frac{d^3\mathbf{x}}{\pi(\sigma)^2}.\end{aligned}\tag{4.4}$$

Note that the 3-point correlator depends only on the product $\Xi := \delta_{\mathbf{r}_1}\delta_{\mathbf{r}_2}\delta_{\mathbf{r}_3}$. Therefore we can simply use the notation $\langle\Xi\rangle_{0,3}$ instead. Finally, Definition (4.4) makes sense even if we replace $\delta_{\mathbf{r}}$, $\mathbf{r} = \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ by arbitrary polynomials, not only the ones that correspond to flat vector fields via (4.3).

4.3. B-model 4-point genus-0 correlators

Let $\mathscr{F}_0^{\mathrm{SG}}$ be the genus-0 generating functions for the Frobenius manifolds of miniversal deformations near the origin. By definition,

$$\Big\langle \delta_{\mathbf{r}_1}, \ldots, \delta_{\mathbf{r}_n} \Big\rangle_{0,n}^{\mathrm{SG}} = \frac{\partial^n \mathscr{F}_0^{\mathrm{SG}}}{\partial t_{\mathbf{r}_1} \ldots \partial t_{\mathbf{r}_n}}\Big|_{\mathbf{t}=0}.$$

Thus, using that $\partial/\partial\sigma = \delta_{\mathbf{m}}$ at $\sigma = 0$, we get

$$\Big\langle \delta_{\mathbf{r}_1}, \delta_{\mathbf{r}_2}, \delta_{\mathbf{r}_3}, \delta_{\mathbf{m}} \Big\rangle_{0,4}^{\mathrm{SG}} = \partial_\sigma \Big\langle \delta_{\mathbf{r}_1}, \delta_{\mathbf{r}_2}, \delta_{\mathbf{r}_3} \Big\rangle\Big|_{\sigma=0}. \tag{4.5}$$

In order to compute 4-point correlators of the form (4.5) it is enough to determine $\delta_{\mathbf{r}}(\sigma, \mathbf{x})$ up to linear terms in σ. To begin with, we notice that $\phi_{\mathbf{r}+\mathbf{m}}$ lies in the Jacobian ideal of W_σ. More precisely, the following Lemma holds.

Lemma 4.1. — *There are polynomials $g_{\mathbf{r},i} \in \mathbb{C}[\sigma, X_1, X_2, X_3]$ such that*

$$(1 - C\sigma^l)\, \phi_{\mathbf{r}+\mathbf{m}} = \sum_{i=1}^{3} g_{\mathbf{r},i}\, \partial_i W_\sigma.$$

This Lemma can be proved in all cases by using Saito's higher residue pairing. However, in what follows, we need an explicit formula for

$$g_{\mathbf{r}} := (g_{\mathbf{r},1}, g_{\mathbf{r},2}, g_{\mathbf{r},3}).$$

Therefore we verified the Lemma on a case-by-case basis. Some of our computations will be given below. The remaining cases are completely analogous.

There are several corollaries of Lemma 4.1. First of all, note that under the period map (4.2) the Gauss–Manin connection takes the form (2.8) (with $z \equiv -\partial_\lambda^{-1}$). It follows that if $\deg(\phi_{\mathbf{r}})$ is not integral, then the restriction of the section (4.2) of the vanishing cohomology bundle to the marginal deformation subspace must be flat, i.e., the sections

$$[\delta_{\mathbf{r}}\, \omega](\sigma) := \int \delta_{\mathbf{r}}(\sigma, \mathbf{x})\, \frac{\omega}{dW_\sigma}, \quad \deg(\phi_{\mathbf{r}}) \notin \mathbb{Z} \tag{4.6}$$

are independent of σ. Furthermore, using formulas (2.17) for the Gauss-Manin connection we get

$$(1 - C\sigma^l) \frac{\partial}{\partial\sigma} \Phi_{\mathbf{r}} = -\int \sum_{i=1}^{3} \partial_i\, g_{\mathbf{r},i}\, \frac{d^3\mathbf{x}}{dW_\sigma}$$

Both sides must have the same degree, i.e.,

$$(1 - C\sigma^l) \frac{\partial}{\partial\sigma} \Phi_{\mathbf{r}} = \sum_{\mathbf{r}'} c_{\mathbf{r},\mathbf{r}'}(\sigma) \Phi_{\mathbf{r}'}, \tag{4.7}$$

where the sum is over all $\mathbf{r}'$, such that $\deg \phi_{\mathbf{r}} = \deg \phi_{\mathbf{r}'}$ and $c_{\mathbf{r},\mathbf{r}'}(\sigma) \in \mathbb{C}[\sigma]$ are some polynomials.

LEMMA 4.2. — *Suppose* $\deg(\phi_{\mathbf{r}}) \notin \mathbb{Z}$*; then we have*

$$\delta_{\mathbf{r}} = \phi_{\mathbf{r}} - \sigma \sum_{\mathbf{r}',\mathbf{r}'\neq\mathbf{r}} c_{\mathbf{r},\mathbf{r}'}(0)\,\phi_{\mathbf{r}'} + O(\sigma^2), \tag{4.8}$$

where $O(\sigma^2)$ *denotes terms that have order of vanishing at* $\sigma = 0$ *at least 2.*

Proof. — Follows easily from (4.7). We omit the details. □

Let $M(X_1, X_2, X_3) \in \mathbb{C}[\mathbf{x}]$ be a weight-1 monomial with leading coefficient 1. Our next goal is to evaluate the following auxiliary expression (recall Definition 3.4):

$$\langle M, \phi_{\mathbf{m}}\rangle_{0,4} := \partial_\sigma \langle M\rangle_{0,3}\big|_{\sigma=0}\,.$$

LEMMA 4.3. — *The number* $\langle M, \phi_{\mathbf{m}}\rangle_{0,4}$ *is non-zero iff* $M = M_i$ *for some* $i = 1, 2, 3$. *In the latter cases the numbers are given as follows*

$$\Big(\langle M_1, \phi_{\mathbf{m}}\rangle_{0,4}, \langle M_2, \phi_{\mathbf{m}}\rangle_{0,4}, \langle M_3, \phi_{\mathbf{m}}\rangle_{0,4}\Big) = -(m_1, m_2, m_3)E_W^{-1}. \tag{4.9}$$

Proof. — For the second part, we apply the operators $X_i\,\partial_{X_i}$, $i = 1, 2, 3$, to the identity

$$M_1 + M_2 + M_3 = W_\sigma - \sigma\phi_{\mathbf{m}}(\mathbf{x})$$

and take the residue. We get

$$\langle M_1\rangle_{0,3}\, a_{1i} + \langle M_2\rangle_{0,3}\, a_{2i} + \langle M_3\rangle_{0,3}\, a_{3i} = -\sigma\, m_i\, \langle \phi_{\mathbf{m}}\rangle_{0,3}.$$

It remains only to differentiate with respect to σ and set $\sigma = 0$.

For the first part, because M is a weight-1 monomial with coefficient 1, we can use the relations in the Jacobian algebra of W_σ to rewrite M as a product of $\phi_{\mathbf{m}}$ and a function of σ. Let us write $M = h(\sigma)\,\phi_m$. For example, in the Fermat $E_6^{(1,1)}$ case,

$$X_1^3 = -3\sigma\phi_{111}; \quad (1 + \frac{\sigma^3}{27})\, X_1^2 X_2 = 0.$$

If $M \neq M_i$, $i = 1, 2, 3$, then $h(\sigma)$ either does not vanish at $\sigma = 0$ or vanishes at $\sigma = 0$ with order at least 2. In both cases, $\langle M, \phi_{\mathbf{m}}\rangle_{0,4}$ vanish. □

Now we are ready to compute the 4-point correlators that are needed for the reconstruction of the CohFT. Let $\delta_{\mathbf{r}}(\mathbf{s}, x)$, $\mathbf{r} = \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ be polynomials corresponding to the flat vector fields $\partial/\partial t_{\mathbf{r}}$ via the Kodaira–Spencer isomorphism (4.3). Put

$$\Xi(\mathbf{s}, \mathbf{x}) = \delta_{\mathbf{r}_1}(\mathbf{s}, x)\delta_{\mathbf{r}_2}(\mathbf{s}, x)\delta_{\mathbf{r}_3}(\mathbf{s}, x).$$

Note that $\Xi(0,\mathbf{x})$ is a homogeneous monomial (see (4.8)) with leading coefficient 1.

Lemma 4.4. — *The 4-point genus-0 correlators with a top degree insertion $\delta_{\mathbf{m}}$ are*

$$\Big\langle \delta_{\mathbf{r}_1}, \delta_{\mathbf{r}_2}, \delta_{\mathbf{r}_3}, \delta_{\mathbf{m}} \Big\rangle_{0,4}^{\mathrm{SG}} = \begin{cases} -q_i^T & \text{if } \Xi(0,\mathbf{x}) = M_i, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. — The same argument as in the proof of the first part of Lemma 4.3 also works for $\Xi(\sigma,\mathbf{x})$. Thus if $\Xi \neq M_i$, i=1,2,3, we have

$$\Big\langle \Xi, \delta_{\mathbf{m}} \Big\rangle_{0,4}^{\mathrm{SG}} = 0.$$

In order to finish the proof we need only to compute the correlators when $\Xi(0,\mathbf{x}) = M_i$ for some $i = 1,2,3$. Note that the diagonal entries of the matrix E_W are always at least 2 (see Table 1.1). Therefore, it is enough to compute the following correlators:

$$\begin{cases} \Big\langle \delta_{100}, \delta_{100}, \delta_{\mathbf{r}}, \delta_{\mathbf{m}} \Big\rangle_{0,4}^{\mathrm{SG}}, & \mathbf{r} = (a_{11}-2, a_{12}, a_{13}), \\ \Big\langle \delta_{010}, \delta_{010}, \delta_{\mathbf{r}}, \delta_{\mathbf{m}} \Big\rangle_{0,4}^{\mathrm{SG}}, & \mathbf{r} = (a_{21}, a_{22}-2, a_{23}), \\ \Big\langle \delta_{001}, \delta_{001}, \delta_{\mathbf{r}}, \delta_{\mathbf{m}} \Big\rangle_{0,4}^{\mathrm{SG}}, & \mathbf{r} = (a_{31}, a_{32}, a_{33}-2). \end{cases}$$

We do not have a uniform computation since we need to use Lemma 4.2, for which the coefficients $c_{\mathbf{r},\mathbf{r}'}(0)$ can be computed only on a case-by-case basis. Let us sketch the main steps of the computation in several examples, leaving the details and the remaining cases to the reader. We will make use of the notation

$$\delta(\sigma,\mathbf{x}) \approx \phi(\sigma,\mathbf{x}), \quad \delta, \phi \in \mathbb{C}[\mathbf{x}],$$

which means first order approximation at $\sigma = 0$, i.e., $\delta(\sigma,\mathbf{x}) - \phi(\sigma,\mathbf{x}) = O(\sigma^2)$.

Case 1. — $W = X_1^3 + X_2^3 + X_3^3 \in E_6^{(1,1)}$ and $\phi_{\mathbf{m}} = X_1X_2X_3$. Since W is symmetric in X_1, X_2, X_3 it is enough to compute only one of the correlators, say $\Xi = M_1$. After a straightforward computation (the notation is the same as in Lemma 4.1) we get

$$g_{100} = \Big(\frac{1}{3}\,\phi_{011}, -\frac{\sigma}{9}\,\phi_{002}, \frac{\sigma^2}{27}\,\phi_{101} \Big).$$

It follows that $\delta_{100} \approx \phi_{100}$ and then using formula (4.9) we get

$$\Big\langle \delta_{100}, \delta_{100}, \delta_{100}, \delta_{\mathbf{m}} \Big\rangle_{0,4}^{\mathrm{SG}} = -\frac{1}{3}.$$

Case 2. — $W = X_1^4 + X_2^4 + X_3^2 \in E_7^{(1,1)}$ and $\phi_{\mathbf{m}} = X_1^2 X_2^2$. In this case $M_3 = 0$ in the Jacobian algebra of W and W is symmetric in X_1 and X_2. It is enough to compute only one of the correlators, say the one with $\Xi(0,\mathbf{x}) = M_1$. We have

$$g_{100} = \Big(\frac{1}{4}\,\phi_{020}, -\frac{\sigma}{8}\,\phi_{110}, 0\Big).$$

It follows that

$$\delta_{100} \approx \phi_{100}, \quad \delta_{200} \approx \phi_{200} + \frac{\sigma}{4}\phi_{020}.$$

Using formula (4.9) we find

$$\Big\langle \delta_{100}, \delta_{100}, \delta_{400}, \delta_{\mathbf{m}} \Big\rangle_{0,4}^{\mathrm{SG}} = -\frac{1}{4}.$$

Case 3. — $W = X_1^3 X_2 + X_2^2 + X_3^3 \in E_8^{(1,1)}$ and $\phi_{\mathbf{m}} = X_1 X_2 X_3$. In this case, since $M_2 = 0$ in the Jacobian algebra, we need to compute two correlators. We have

$$\begin{pmatrix} g_{100} \\ g_{010} \\ g_{001} \end{pmatrix} = \begin{pmatrix} \frac{1}{3}\phi_{001} - \frac{\sigma^2}{54}\phi_{200} & \frac{\sigma^2}{18}\phi_{110} & -\frac{\sigma}{9}\phi_{010} \\ -\frac{1}{6}\phi_{201} + \frac{\sigma^2}{27}\phi_{110} & \frac{1}{2}\phi_{111} & -\frac{\sigma^2}{9}\phi_{210} \\ -\frac{\sigma}{9}\phi_{010} - \frac{\sigma^2}{54}\phi_{101} & \frac{\sigma^2}{9}\phi_{011} & \frac{1}{3}\phi_{110} \end{pmatrix}.$$

It follows that we have the following linear approximations:

$$\delta_{100} \approx \Phi_{100}, \quad \delta_{001} \approx \Phi_{001}, \quad \delta_{110} \approx \Phi_{110}.$$

The correlators then become

$$\Big\langle \delta_{100}, \delta_{100}, \delta_{110}, \delta_{\mathbf{m}} \Big\rangle_{0,4}^{\mathrm{SG}} = -\frac{1}{3}, \quad \Big\langle \delta_{001}, \delta_{001}, \delta_{001}, \delta_{\mathbf{m}} \Big\rangle_{0,4}^{\mathrm{SG}} = -\frac{1}{3}$$

Case 4. — The Fermat type $E_8^{(1,1)}$, i.e. $W = X_1^6 + X_2^3 + X_3^2$ and $\phi_{\mathbf{m}} = X_1^4 X_2$. In this case $M_3 = 0$, so again we have to compute two correlators. We have

$$g_{100} = \Big(\frac{1}{6}\phi_{020} + \frac{\sigma^2}{27}\phi_{200}, -\frac{2\sigma}{9}\phi_{300}, 0\Big),$$
$$g_{010} = \Big(- \frac{\sigma}{18}\phi_{300} + \frac{\sigma^2}{27}\phi_{110}, \frac{1}{3}\phi_{400}, 0\Big).$$

It follows that the first order approximations that we need are

$$\delta_{100} \approx \Phi_{100}, \quad \delta_{010} \approx \Phi_{010}, \quad \delta_{400} \approx \Phi_{400} + \frac{\sigma}{2}\Phi_{210}.$$

Formulas (4.9) and (4.5) imply

$$\Big\langle \delta_{100}, \delta_{100}, \delta_{400}, \delta_{\mathbf{m}} \Big\rangle_{0,4}^{\mathrm{SG}} = -\frac{1}{6}; \quad \Big\langle \delta_{010}, \delta_{010}, \delta_{010}, \delta_{\mathbf{m}} \Big\rangle_{0,4}^{\mathrm{SG}} = -\frac{1}{3}. \qquad \square$$

4.4. Proof of Theorem 1.4

The 4-point correlators in Lemma 3.5 and Lemma 4.4 have opposite signs. If $\rho_{-1}\mapsto\delta_{\mathbf{m}}$, we rescale the primitive form by (-1) and define

$$\mathscr{H}_{W^T}\to T^*\mathcal{S}_W,\quad \rho_{\mathbf{r}}\mapsto(-1)^{1-\deg\phi_{\mathbf{r}}}\delta_{\mathbf{r}},\quad \mathbf{r}=(r_1,r_2,r_3), \tag{4.10}$$

where $\rho_{\mathbf{r}}=\rho_1^{r_1}\rho_2^{r_2}\rho_3^{r_3}$. For the new basis, the 3-point correlators in the Saito–Givental theory do not change, while the 4-point correlators are rescaled by (-1). Lemma 3.5 and Lemma 4.4 imply that the map (4.10) identifies $\mathscr{A}^{\mathrm{FJRW}}_{W^T}$ and $\mathscr{A}^{\mathrm{SG}}_{W}(\sigma=0)$. If $\rho_{-1}\mapsto c\delta_{\mathbf{m}}$ for some nonzero constant $c>1$, we need to rescale the ring generators furthermore to obtain an suitable mirror map. Thus Theorem 1.4, a) is proved.

On the other hand, since we already computed all basic 4-point genus-0 correlators for Saito-Givental limit at Gepner point $\sigma=0$, we can use it here to identify two Gepner points of Saito-Givental theory, if they have isomorphic rings. We notice that all the ring structures at Gepner points are already listed in Section 3.3.3. The following Lemma gives a complete classification for the Gepner points, in the sense of Definition 1.2. In particular it gives a proof for part b) of Theorem 1.4.

Lemma 4.5. — *If W_1 and W_2 are invertible simple elliptic singularities, and $\mathscr{Q}_{W_1}\cong\mathscr{Q}_{W_2}$, then there exists a ring isomorphicm $\Psi:\mathscr{Q}_{W_1}\cong\mathscr{Q}_{W_2}$, such that Ψ preserves the Saito-Givental limits at Gepner point, i.e. $\Psi:\mathscr{A}^{\mathrm{SG}}_{W_1}(\sigma)=\mathscr{A}^{\mathrm{SG}}_{W_2}(\sigma)$ for $\sigma=0$.*

Proof. — We will construct explicitly linear isomorphisms Ψ inducing the ring isomorphisms; then one has to check that they also preserve the 4-point correlators in Lemma 4.4. For example, let $W_1=X_1^3+X_2^3+X_3^3$ and $W_2=X_1^2X_2+X_2^2X_3+X_1X_3^2$, both marginal $X_1X_2X_3$. We construct a ring homomorphism

$$\Psi:\mathscr{Q}_{W_1}\to\mathscr{Q}_{W_2}$$

which sends $\{Y_i=X_i/\lambda_i\}_{i=1}^3$, a basis of generators of $\mathscr{Q}_{W_1}$ to $\{\Psi(Y_i)\}_{i=1}^3\in\mathscr{Q}_{W_2}$,

$$(\Psi(Y_1),\Psi(Y_2),\Psi(Y_3))=(X_1,X_2,X_3)\begin{pmatrix}1&1&1\\1&\mathbf{e}[\frac{1}{3}]&\mathbf{e}[\frac{2}{3}]\\1&\mathbf{e}[\frac{2}{3}]&\mathbf{e}[\frac{1}{3}]\end{pmatrix}.$$

Here the parameters $\lambda_1,\lambda_2,\lambda_3$ are as follows

$$\lambda_1^4\lambda_2\lambda_3=\mathbf{e}[\frac{1}{3}]\,\lambda_1\lambda_2^4\lambda_3=\mathbf{e}[\frac{2}{3}]\,\lambda_1\lambda_2\lambda_3^4=-\frac{1}{27}. \tag{4.11}$$

We extend Ψ to be a ring isomorphism. Since, $\delta_{\mathbf{m}_1} = \delta_{\mathbf{m}_2} = X_1X_2X_3$, one can check

$$\begin{aligned}\Big\langle \Psi(X_1), \Psi(X_1), \Psi(X_1), \Psi(X_1X_2X_3)\Big\rangle_{0,4,W_2} \\ &= -27\lambda_1^4\lambda_2\lambda_3\Big\langle X_1, X_1, X_1, X_1X_2X_3\Big\rangle_{0,4,W_2} \\ &= -\frac{1}{3}.\end{aligned}$$

Similarly, due to the other two identities in (4.11), we can check all the 4-point genus-0 correlators listed in Lemma 4.4 matches. Thus $\Psi : \mathscr{A}^{\mathrm{SG}}_{W_1}(\sigma) = \mathscr{A}^{\mathrm{SG}}_{W_2}(\sigma)$ for $\sigma = 0$.

For each W, we can fix the choice of $\phi_{\mathbf{m}}$. We list the isomorphism for those examples in Table 4.1. We always choose $\{Y_i = X_i/\lambda_i\}_{i=1}^3$. Cases of other choices of $\phi_{\mathbf{m}}$ are obtained by rescaling the ring generators appropriately. □

Table 4.1. Classification of Gepner points

W	$\phi_{\mathbf{m}}$	Ring Generators	Constraints
$X_1^3+X_2^3+X_3^3$	$X_1X_2X_3$	(Y_1,Y_2,Y_3)	
$X_1^2X_2+X_2^3+X_3^3$	$X_2^2X_3$	$(\frac{X_1}{\sqrt{3}}+X_2,\frac{-X_1}{\sqrt{3}}+X_2,X_3)$	$-4\lambda_1^4\lambda_2=4\lambda_1\lambda_2^4=\lambda_3=1$
$X_1^2X_2+X_1X_2^2+X_3^3$	$X_1X_2X_3$	$(\frac{X_1}{\mathbf{e}[\frac{1}{6}]}+X_2,\frac{X_1}{\mathbf{e}[\frac{5}{6}]}+X_2,X_3)$	$-3\sqrt{-3}\lambda_1^4\lambda_2=-3\sqrt{-3}\lambda_1\lambda_2^4=\lambda_3=1$
$X_1^3X_2+X_2^4+X_3^2$	$X_1X_2^3$	(Y_1,Y_2)	
$X_1^3X_2+X_2^2X_3+X_3^2$	$X_1X_2X_3$	(X_1,X_2)	$-2\lambda_1^4\lambda_2^4=8\lambda_1\lambda_2^7=1$
$X_1^4+X_2^4+X_3^2$	$X_1^2X_2^2$	(Y_1,Y_2)	
$X_1^2X_2+X_2^2+X_3^4$	$X_1^2X_3^2$	(X_1,X_3)	$-4\lambda_1^6\lambda_2^2=\lambda_1^2\lambda_2^6=1$
$X_1^3X_2+X_1X_2^3+X_3^2$	$X_1^2X_2^2$	$(X_1+X_2,-X_1+X_2)$	$-64\lambda_1^6\lambda_2^2=64\lambda_1^2\lambda_2^6=1$
$X_1^6+X_2^3+X_3^2$	$X_1^4X_2$	(Y_1,Y_2)	
$X_1^3X_2+X_2^2+X_3^3$	$X_1X_2X_3$	(X_1,X_3)	$8\lambda_1^{10}\lambda_2=-2\lambda_1^4\lambda_2^4=1$

5. Global mirror symmetry for Fermat simple elliptic singularities

The goal in this section is to prove Theorem 1.5. For special limit $\sigma \neq 0, \infty$, it is enough to prove it only for one of the points $p_k = C^{-1/l}\eta^k$, $1 \leqslant k \leqslant l$. The remaining cases follow easily due to the $\mathbb{Z}/l\,\mathbb{Z}$-symmetry of Σ. For all those limits except $\sigma = \infty$ for $W = X_1^6 + X_2^3 + X_3^2$, according to the reconstruction Lemma 3.1, we need to construct an appropriate *mirror map* from the Chen–Ruan orbifold cohomology ring to the limit of Jacobian algebras, such that after choosing an appropriate primitive form, the Poincaré pairing is identified with the residue pairing and the 3-point correlator (see Lemma 3.1) is the same for both the Gromov–Witten and the Saito–Givental CohFTs. Finally, we also prove the Saito-Givental limit $\sigma = \infty$ for $W = X_1^6 + X_2^3 + X_3^2$ is isomorphic to an FJRW theory. This agrees with the physicists' prediction that the monodromy of the Gauss–Manin connection around the large volume limit point should be maximally unipotent, while as we will see below, the monodromy around $\sigma = \infty$ is diagonalizable.

The limit of the Saito-Givental theory of ISESs at $\sigma = \infty$ is already discussed in [30] for $(W, \phi_{\mathbf{m}} = X_1X_2X_3)$, where

$$W = X_1^3 + X_2^3 + X_3^3 \in E_6^{(1,1)}, \qquad X_1^3X_2 + X_2^2X_3 + X_3^2 \in E_7^{(1,1)},$$
$$X_1^3X_2 + X_2^2 + X_3^3 \in E_8^{(1,1)}.$$

Namely, it was proved that the Saito-Givental theory at $\sigma = \infty$ is mirror to the Gromov-Witten theory respectively of $\mathbb{P}^1_{3,3,3}, \mathbb{P}^1_{4,4,2}$ and $\mathbb{P}^1_{6,3,2}$.

5.1. Construction of a mirror map

We construct a mirror map based on solving the systems of hypergeometric equations. We will introduce explicit how to construct it near $\sigma = p_k$. For the special limit $\sigma = \infty$, the construction is similar.

5.1.1. Non-twisted sectors

The primitive form is chosen to be $\omega = d^3\mathbf{x}/\pi_A(\sigma)$, where the cycle $A \in H_1(E_\sigma)$ is invariant with respect to the local monodromy around $\sigma = p_k$. Recall that π_A is a solution to the hypergeometric equation (2.13) with weights (α,β,γ), where $\gamma=\alpha+\beta$. The invariance of A implies near $x=1$, we have

$$\pi_A(\sigma)=\lambda_W\, F_1^{(1)}(x):=\lambda_W\, {}_2F_1\left(\alpha,\beta,1,1-x\right), \quad \lambda_W\in\mathbb{C}^*. \tag{5.1}$$

Since the j-invariant of E_σ always has the form

$$j(\sigma)=\frac{P(\sigma)}{(1-C\sigma^l)^N}$$

for some polynomial $P(\sigma)\in\mathbb{C}[\sigma]$ and some integer N. We can always choose a second cycle $B\in H_1(E_\sigma)$, such that[(1)]

$$\pi_B(\sigma)=\frac{N\lambda_W}{2\pi\sqrt{-1}}\left(F_2^{(1)}(x)-\frac{\ln P(p_k)}{N}\,F_1^{(1)}(x)\right), \tag{5.2}$$

where $F_2^{(1)}(x)$ is also a solution to the hypergeometric equation (2.13) such that

$$F_2^{(1)}(x)=\ln(1-x)\,F_1^{(1)}(x)+\sum_{n=1}^{\infty}b_n(1-x)^n,$$

with

$$b_n=\frac{(\alpha)_n(\beta)_n}{(n!)^2}\left(\frac{1}{\alpha}+\cdots+\frac{1}{\alpha+n-1}+\frac{1}{\beta}+\cdots+\frac{1}{\beta+n-1}-2\Big(\frac{1}{1}+\cdots+\frac{1}{n}\Big)\right).$$

We can check the following τ is the modulus of the elliptic curve E_σ.

$$\tau:=\frac{\pi_B(\sigma)}{\pi_A(\sigma)}. \tag{5.3}$$

If we put $Q=\exp(2\pi\sqrt{-1}\tau)$, then the j-invariant always has a Q-expansion

$$j(\sigma)=\frac{1}{Q}+744+196884Q+\cdots$$

Note that the residue pairing implies

$$\Big\langle 1,\phi_{-1}\Big\rangle=\frac{1}{K(1-C\,\sigma^l)}, \tag{5.4}$$

where K is some fixed constant. Since the residue pairing must be identified with the Poincaré pairing, the mirror map should satisfy

$$\Delta_{01}\mapsto 1,\quad \Delta_{02}\mapsto K(1-C\,\sigma^l)\phi_{-1}(\mathbf{x})\pi_A^2. \tag{5.5}$$

The next step is to identify the divisor coordinate t_{02} in the orbifold GW theory and the modulus τ. In order to get the correct q-expansion, we define

$$q:=\exp(t_{-1}),\quad t_{-1}:=\frac{2\pi\sqrt{-1}}{L}\tau, \tag{5.6}$$

(1) The coefficients are chosen to match the expansion of the j-invariant, formula (5.3), and (5.6).

Here[(2)] $L = 3, 4, 6$ respectively for the elliptic orbifolds $\mathbb{P}^1_{3,3,3}, \mathbb{P}^1_{4,4,2}, \mathbb{P}^1_{6,3,2}$. This implies

(5.7)
$$\left\langle \Delta_{01}, \Delta_{02} \right\rangle \mapsto \left\langle 1, \frac{\partial}{\partial t_{-1}} \right\rangle = \frac{\partial \sigma}{\partial \tau} \frac{\partial \tau}{\partial t_{-1}} \left\langle 1, \frac{\partial}{\partial \sigma} \right\rangle = \frac{L}{2\pi\sqrt{-1}} \frac{\partial \sigma}{\partial \tau} \frac{1}{\pi_A^2} \left\langle 1, \phi_{-1} \right\rangle$$

is a constant. The last equality follows from equation (4.3) and (4.4). By formulas (5.1), (5.2), (5.3), (5.4), (5.6), we can fix the constant λ_W by choosing

$$\left\langle \Delta_{01}, \Delta_{02} \right\rangle \mapsto \left\langle 1, \frac{\partial}{\partial t_{-1}} \right\rangle = 1.$$

In computations below, for our convenience, we may choose $\pi_A(\sigma)$ different than formula (5.1) by a scalar. This will not change the Frobenius manifold structure since we can always rescale the ring generators to offset its influence.

5.1.2. Twisted sectors

In order to complete the construction, we need to identify the twisted cohomology classes Δ_{ij} with monomials $\delta_{\mathbf{r}}(\sigma, \mathbf{x})$. The key observation is that the sections

$$\int \delta_{\mathbf{r}}(\sigma, \mathbf{x}) \frac{\omega}{dW_\sigma} \tag{5.8}$$

of the vanishing cohomology bundle of W_σ are flat on the restriction of the Gauss–Manin connection to $\Sigma \times \{0\} \subset \Sigma \times \mathbb{C}^{\mu-1}$. In general, we need flatness on $\Sigma \times \mathbb{C}^{\mu-1}$. However, since we only need to match the invariants in Lemma 3.1 and Lemma 3.5, the above formula is enough. This way our choice of $\delta_{\mathbf{r}}$ depends on an invertible matrix of size $(\mu - 2) \times (\mu - 2)$. The matrix is decomposed into 1×1 and 2×2 blocks according to Lemma 2.5. In particular, the entries of a 2×2 block are obtained from a basis of solutions of hypergeometric equations (2.26) with weights $(\alpha_{\mathbf{r}}, \beta_{\mathbf{r}}, \gamma_{\mathbf{r}})$ near $x = 1$:

$$\begin{cases} F^{(1)}_{1,\mathbf{r}}(x) = {}_2F_1\left(\alpha_{\mathbf{r}}, \beta_{\mathbf{r}}; \alpha_{\mathbf{r}} + \beta_{\mathbf{r}} - \gamma_{\mathbf{r}} + 1; 1 - x\right), \\ F^{(1)}_{2,\mathbf{r}}(x) = {}_2F_1\left(\gamma_{\mathbf{r}} - \alpha_{\mathbf{r}}, \gamma_{\mathbf{r}} - \beta_{\mathbf{r}}; \gamma_{\mathbf{r}} - \alpha_{\mathbf{r}} - \beta_{\mathbf{r}} + 1; 1 - x\right) \\ \qquad\qquad \times (1 - x)^{\gamma_{\mathbf{r}} - \alpha_{\mathbf{r}} - \beta_{\mathbf{r}}}. \end{cases} \tag{5.9}$$

The correlation functions in the Saito–Givental CohFT are invariant with respect to the translation $t_{-1} \mapsto t_{-1} + 2\pi\sqrt{-1}$, see (5.6). The coefficient in front of q^d, $d \in \mathbb{Z}$, is called the *degree-d* part of the correlator function.

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(2) We remark that L is the order of the cyclic group μ_L, where each corresponding orbifold here can be obtained as a μ_L-quotient of some elliptic curve.

By taking the degree-0 part of the 3-point functions, we obtain a Frobenius algebra structure on the Jacobian algebra $\mathscr{Q}_W$ that under the mirror map should be identified with the Frobenius algebra corresponding to the Chen–Ruan orbifold (classical) cup product. Using also that the mirror map preserves homogeneity we obtain a system of equations for the matrix. It remains only to see that these equations have a solution. Let us list the explicit formulas for the mirror map. We omit the details of the computations, which by the way are best done with the help of some computer software–Mathematica.

5.2. Global mirror symmetry for $W_\sigma = X_1^3 + X_2^3 + X_3^3 + \sigma X_1 X_2 X_3$

The j-invariant

$$j(\sigma) = -\frac{\sigma^3(-216+\sigma^3)^3}{(27+\sigma^3)^3} = \frac{-27x(8+x)^3}{(1-x)^3}, \quad x = -\frac{\sigma^3}{27}.$$

$\Phi_{\mathbf{r}}$ satisfies a first order differential equation (2.25) and can be solved

$$\Phi_{\mathbf{r}} = (1-x)^{-\deg \phi_{\mathbf{r}}}\delta_{\mathbf{r}}.$$

Here $\delta_{\mathbf{r}}$ are some flat sections. The residue pairing implies

$$\left\langle X_1X_2X_3 \right\rangle := \operatorname{Res} \frac{X_1X_2X_3}{(\partial_{X_1}W_\sigma)(\partial_{X_2}W_\sigma)(\partial_{X_3}W_\sigma)} d^3\mathbf{x} = \frac{1}{27(1-x)}. \tag{5.10}$$

5.2.1. GW-point at root of unity at $x = 1$

We have $p_l = -3$ and

$$\pi_A(\sigma) = {}_2F_1\left(\frac{1}{3}, \frac{1}{3}; 1; 1-x\right),$$

$$\pi_B(\sigma) = \frac{3}{2\pi\sqrt{-1}}\left(F_2^{(1)}(x) - \frac{\ln P(p_k)}{3} \cdot F_1^{(1)}(x)\right).$$

The Fourier series of $1-x$ in $q = e^{2\pi i\tau/3}$ is

$$1 - x = -27q - 324q^2 - 2430q^3 - 13716q^4 + \cdots \tag{5.11}$$

A natural basis for the flat sections with non-integral degrees are

$$\delta_{\mathbf{r}} = (1-x)^{1/3}\phi_{\mathbf{r}}(\mathbf{x})\,\pi_A, \quad \delta_{\mathbf{r}'} = (1-x)^{2/3}\phi_{\mathbf{r}'}(\mathbf{x})\,\pi_A.$$

Here $\mathbf{r} = (1,0,0), (0,1,0), (0,0,1)$ and $\mathbf{r}' = (1,1,1) - \mathbf{r}$. Applying (4.4), we know the non-vanishing correlators $\left\langle \cdots \right\rangle_{0,3,0}$ are

$$\left\langle 1, \delta_{\mathbf{r}}, \delta_{\mathbf{r}'} \right\rangle_{0,3,0} = \left\langle \delta_{\mathbf{r}}, \delta_{\mathbf{r}}, \delta_{\mathbf{r}} \right\rangle_{0,3,0} = \left\langle \delta_{1,0,0}, \delta_{0,1,0}, \delta_{0,0,1} \right\rangle_{0,3,0} = \frac{1}{27}.$$

The mirror map is given by (5.5), (5.6), and it identifies the ring generators by

$$\begin{pmatrix}\Delta_{11}\\ \Delta_{21}\\ \Delta_{31}\end{pmatrix}\mapsto\begin{pmatrix}1 & 1 & 1\\ 1 & \mathbf{e}[\frac{2}{3}] & \mathbf{e}[\frac{1}{3}]\\ 1 & \mathbf{e}[\frac{1}{3}] & \mathbf{e}[\frac{2}{3}]\end{pmatrix}\begin{pmatrix}\delta_{1,0,0}\\ \delta_{0,1,0}\\ \delta_{0,0,1}\end{pmatrix}. \tag{5.12}$$

It is easy to check that this identification agrees with the Chen-Ruan orbifold cohomology ring of $\mathbb{P}^1_{3,3,3}$ (see (3.4), (3.5)). For example, from (5.12) we have

$$\Big\langle\Delta_{11},\Delta_{11},\Delta_{11}\Big\rangle_{0,3,0} \mapsto \sum_{\mathbf{r};\deg\phi_{\mathbf{r}}=1/3}\Big\langle\delta_{\mathbf{r}},\delta_{\mathbf{r}},\delta_{\mathbf{r}}\Big\rangle_{0,3,0}+6\Big\langle\delta_{1,0,0},\delta_{0,1,0},\delta_{0,0,1}\Big\rangle_{0,3,0}=\frac{1}{3}.$$

Finally, $\Big\langle\Delta_{11},\Delta_{21},\Delta_{31}\Big\rangle_{0,3,1}\mapsto 1$ for Lemma (3.1) follows from

$$\Big\langle\Delta_{11},\Delta_{21},\Delta_{31}\Big\rangle_{0,3} \mapsto \frac{(1-(1-x))^{1/3}-1}{9}\pi_A = q+q^4+2q^7+\cdots=\frac{\eta(9\tau)^3}{\eta(3\tau)}.$$

5.2.2. GW-point at infinity

For this limit, the mirror symmetry is already verified in [30]. It maps the twisted sector $\Delta_{i,j}$ to $\delta_{\mathbf{r}}$, where i-th index of $\mathbf{r}$ is j.

5.3. Global mirror symmetry for $W_\sigma = X_1^4+X_2^4+X_3^2+\sigma X_1^2X_2^2$

The j-invariant

$$j(\sigma)=\frac{16(12+\sigma^2)^3}{(4-\sigma^2)^2}=\frac{64(3+x)^3}{(1-x)^2},\quad x=\frac{\sigma^2}{4}.$$

For $\mathbf{r}=(10,01,11,21,12)$, $\Phi_{\mathbf{r}}$ satisfies a first order differential equations (2.25). On the other hand, both Φ_{20} and Φ_{02} satisfy a second order hypergeometric equation with weights $\alpha_{20}=\beta_{02}=3/4,\beta_{20}=\alpha_{02}=1/4,\gamma_{20}=\gamma_{02}=1/2$, which comes from the following system of first order differential equations:

$$\begin{cases}(4-\sigma^2)\,\partial_\sigma\Phi_{20}(\sigma)=\frac{\sigma}{2}\,\Phi_{20}(\sigma)-\Phi_{02}(\sigma);\\ (4-\sigma^2)\,\partial_\sigma\Phi_{02}(\sigma)=\frac{\sigma}{2}\,\Phi_{02}(\sigma)-\Phi_{20}(\sigma).\end{cases} \tag{5.13}$$

It follows that $\Phi_{02} = L\,\Phi_{20}$ (and $\Phi_{20} = L\,\Phi_{02}$) where L is the differential operator

$$L = -(4-\sigma^2)\partial_\sigma + \frac{\sigma}{2}.$$

Moreover, here the residue pairing implies

$$\left\langle X_1^2 X_2^2 \right\rangle = \frac{1}{32(1-x)}, \quad \left\langle X_1^4 \right\rangle = \left\langle X_2^4 \right\rangle = \frac{-x^{1/2}}{32(1-x)}, \quad \left\langle X_1^3 X_2 \right\rangle = \left\langle X_1 X_2^3 \right\rangle = 0. \tag{5.14}$$

5.3.1. GW-point at root of unity at $x=1$

We are looking for $\sigma = 2x^{1/2} = \pm 1$. We pick $p_2 = 2$. The Fourier series for $1-x$ in terms of $q = e^{2\pi i\tau/4}$ is

$$1 - x = 64q^2 - 1536q^4 + 19200q^6 + \cdots .$$

We choose

$$\pi_A(\sigma) = {}_2F_1\left(\frac{1}{4}, \frac{1}{4}; 1; 1-x\right),$$
$$\pi_B(\sigma) = \frac{2}{2\pi\sqrt{-1}}\left(F_2^{(1)}(x) - \frac{\ln P(p_k)}{2}\cdot F_1^{(1)}(x)\right).$$

Let us construct a basis of flat sections. For first order equations,

$$\delta_{\mathbf{r}}(\sigma, \mathbf{x}) = (x-1)^{\deg\phi_{\mathbf{r}}}\,\phi_{\mathbf{r}}(\mathbf{x})\,\pi_A(\sigma), \quad \mathbf{r} = (10, 01, 11, 21, 12) \tag{5.15}$$

Then for second order equations, we can obtain a pair of polynomials δ_{20} and δ_{02} that determine flat sections (5.8) by solving the following system:

$$\begin{pmatrix} \phi_{20}\,\pi_A \\ \phi_{02}\,\pi_A \end{pmatrix} = \begin{pmatrix} {}_2F_1\left(\frac14, \frac34; \frac32; 1-x\right) & (x-1)^{-1/2}{}_2F_1\left(\frac14, -\frac14; \frac12; 1-x\right) \\ {}_2F_1\left(\frac14, \frac34; \frac32; 1-x\right) & -(x-1)^{-1/2}{}_2F_1\left(\frac14, -\frac14; \frac12; 1-x\right) \end{pmatrix}\begin{pmatrix} \delta_{20} \\ \delta_{02} \end{pmatrix}. \tag{5.16}$$

Now the genus-0 3-point Saito-Givental correlators for flat sections $\delta_{\mathbf{r}}$ can be calculated using residue formula (5.14). Based on this, we can check that after a linear transformation, we actually match those flat sections to flat elements in Chen-Ruan cohomology of $\mathbb{P}^1_{4,4,2}$ via the following mirror map

$$\begin{cases} \Delta_{1,1} \mapsto \delta_{10} + \sqrt{-1}\delta_{01}, \quad \Delta_{1,2} \mapsto 2\sqrt{-1}\delta_{11} + 2\delta_{02}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \Delta_{1,3} \mapsto 4\sqrt{-1}\delta_{2,1} - 4\delta_{1,2}; \\ \Delta_{2,1} \mapsto \sqrt{-1}\delta_{10} + \delta_{01}, \quad \Delta_{2,2} \mapsto 2\sqrt{-1}\delta_{11} - 2\delta_{02}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \Delta_{2,3} \mapsto -4\delta_{2,1} + 4\sqrt{-1}\delta_{1,2}; \\ \Delta_{01} \mapsto 1, \qquad\qquad \Delta_{3,1} \mapsto 8\delta_{20}, \qquad \Delta_{02} \mapsto 32(1-x)\,\phi_{-1}\,\pi_A^2. \end{cases}$$

It is not hard to check this map matches all the pairing (3.4) using (5.14). For example,

$$\Big\langle 8\delta_{20}, 8\delta_{20} \Big\rangle = 16 \Big\langle \phi_{20} + \phi_{02}, \phi_{20} + \phi_{02} \Big\rangle \,{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; \frac{3}{2}; 1-x\right)^{-2} = \frac{1}{2}.$$

The last equality is a consequence of the hypergeometric identity

$${}_2F_1\left(\frac{1}{4}, \frac{3}{4}; \frac{3}{2}; 1-x\right)^2 (1 + x^{1/2}) = 2.$$

We can also check the mirror map matches Chen-Ruan product (3.5). For example,

$$\begin{aligned} \Big\langle \Delta_{1,1}, \Delta_{1,1}, \Delta_{1,2} \Big\rangle_{0,3} &\mapsto \frac{1}{8}\left({}_2F_1\left(\frac{1}{4}, -\frac{1}{4}; \frac{1}{2}; 1-x\right) + 1\right) {}_2F_1\left(\frac{1}{4}, \frac{1}{4}; 1; 1-x\right) \\ &= \frac{1}{4} + q^4 + q^8 + q^{16} + \cdots . \end{aligned}$$

The last equality follows from mirror map (5.3) and a hypergeometric identity

$${}_2F_1\left(\frac{1}{4}, -\frac{1}{4}; \frac{1}{2}; 1-x\right)^2 = \frac{1 + x^{1/2}}{2}.$$

The degree-1 genus-0 3-point correlator in Lemma 3.1 is verified by

$$\begin{aligned} \Big\langle \Delta_{11}, \Delta_{21}, \Delta_{31} \Big\rangle_{0,3} &\mapsto \frac{1}{8}(1-x)^{1/2}\,{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; \frac{3}{2}; 1-x\right) {}_2F_1\left(\frac{1}{4}, \frac{1}{4}; 1; 1-x\right) \\ &= q + 2q^5 + q^9 + 2q^{13} + \cdots . \end{aligned}$$

5.3.2. GW-point at infinity

Near the point $\sigma = \infty$, we choose

$$\begin{aligned} \pi_A(\sigma) &= \left(\frac{1}{4x}\right)^{1/4} {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{1}{x}\right), \\ \pi_B(\sigma) &= \frac{2}{2\pi\sqrt{-1}}\left(F_2^{(1)}(x) - \frac{\ln P(p_k)}{2} \cdot F_1^{(1)}(x)\right), \end{aligned}$$

where $F_2^{(1)}(x)$ is defined by (same b_n as in (5.2)),

$$F_2^{(1)}(x) = -(\ln x)\, F_1^{(1)}(x) + \sum_{n=1}^{\infty} b_n\, x^{-n}.$$

The Saito-Givental limit at this point is mirror to the Gromov-Witten theory of $\mathbb{P}^1_{4,4,2}$. We construct a mirror map (5.3) for the Kähler class. It implies an Fourier expansion

$$x^{-1} = 64q^4 - 2560q^8 + 84736q^{12} + ...$$

We choose the flat sections for the first order differential equations

$$\delta_{\mathbf{r}} = (x-1)^{\deg \phi_{\mathbf{r}}} \phi_{\mathbf{r}}\, \pi_A.$$

For the second order differential equations, we solve (5.13) to obtain

$$\begin{pmatrix} \phi_{20}\, \pi_A \\ \phi_{02}\, \pi_A \end{pmatrix} = \begin{pmatrix} x^{-3/4}{}_2F_1\left(\frac{3}{4}, \frac{5}{4}; \frac{1}{2}; \frac{1}{x}\right) & x^{-1/4}{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; \frac{1}{x}\right) \\ -2x^{-1/4}{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; \frac{1}{x}\right) & -\frac{1}{2}x^{-3/4}{}_2F_1\left(\frac{3}{4}, \frac{5}{4}; \frac{1}{2}; \frac{1}{x}\right) \end{pmatrix} \begin{pmatrix} \delta_{20} \\ \delta_{02} \end{pmatrix}. \tag{5.17}$$

The entries comes from the solutions of a hypergeometric equation (2.26) near $x = \infty$:

$$\begin{cases} F_1^{(\infty)}(x) = {}_2F_1\left(\alpha, \alpha-\gamma+1; \alpha-\beta+1; x^{-1}\right)\, x^{-\alpha}\,, \\ F_2^{(\infty)}(x) = {}_2F_1\left(\beta, \beta-\gamma+1; \beta-\alpha+1; x^{-1}\right)\, x^{-\beta}\,. \end{cases} \tag{5.18}$$

We construct a mirror map as follows

$$\begin{cases} \Delta_{1,1} \mapsto 2\delta_{10}, & \Delta_{1,2} \mapsto -4\sqrt{2}\delta_{02}, \quad \Delta_{1,3} \mapsto -8\delta_{1,2}; \\ \Delta_{2,1} \mapsto 2\sqrt{-1}\delta_{01}, & \Delta_{2,2} \mapsto -2\sqrt{2}\delta_{20}, \quad \Delta_{2,3} \mapsto 8\sqrt{-1}\delta_{2,1}; \\ \Delta_{01} \mapsto 1, & \Delta_{3,1} \mapsto 4\sqrt{-1}\delta_{11}, \quad \Delta_{02} \mapsto 32(1-x)\, X_1^2 X_2^2\, \pi_A^2. \end{cases} \tag{5.19}$$

Now, everything can be checked as before using Mathematica. In particular, we get

$$\Big\langle \Delta_{1,1}, \Delta_{2,1}, \Delta_{3,1} \Big\rangle_{0,3} \mapsto \pi_A(\sigma)/2 = q + 2q^5 + q^9 + \cdots\,.$$

5.4. Global mirror symmetry for $W_\sigma = X_1^6 + X_2^3 + X_3^2 + \sigma X_1^4 X_2$

The j-invariant

$$j(\sigma) = 1728 \frac{4\sigma^3}{27 + 4\sigma^3} = \frac{-1728x}{1-x}, \quad x = -\frac{4}{27}\sigma^3.$$

In order to construct the mirror map for twisted sectors, we have to find a basis of homogeneous flat sections (with non-integer degrees) of the Gauss–Manin connection. The periods corresponding to the polynomials $\phi_{(k,1)}$ and $\phi_{(k+2,0)}$ satisfy

$$(5.20) \quad \begin{cases} (27+4\sigma^3)\partial_\sigma \, \Phi_{(k+2,0)} = -(k+3)\sigma^2 \, \Phi_{(k+2,0)} - \dfrac{9(k+1)}{2} \Phi_{(k,1)} \\ (27+4\sigma^3)\partial_\sigma \, \Phi_{(k,1)} = \dfrac{3(k+3)\sigma}{2} \Phi_{(k+2,0)} - (k+1)\sigma^2 \Phi_{(k,1)} \end{cases}$$

Moreover, we know $\Phi_{(k,1)}$ satisfies a hypergeometric equation (see Section 2.4). Let

$$(5.21) \qquad L_k := \frac{2}{3(k+3)\sigma} \left((27+4\sigma^4)\partial_\sigma + (k+1)\sigma^2 \right), \quad k = 0, 1, 2$$

5.4.1. GW-point at root of unity at $x = 1$

Near $p_2 = \frac{3}{2}(-2)^{1/3}$, we choose $N = 1$,

$$\pi_A(\sigma) = {}_2F_1\left(\frac{1}{12}, \frac{7}{12}; 1; 1-x\right),$$

$$\pi_B(\sigma) = \frac{1}{2\pi\sqrt{-1}} \left(F_2^{(1)}(x) - \ln\, P(p_k) \cdot F_1^{(1)}(x) \right).$$

The Saito-Givental limit at this point is mirror to the Gromov-Witten theory of $\mathbb{P}^1_{6,3,2}$. The mirror map for Kähler class implies Fourier expansion

$$1 - x = -1728 q^6 - 1700352 q^{12} + \cdots$$

Up to some constant for each element, we choose the following map from the non-twisted sectors in GW theory to the flat sections,

$$\Delta_{0,1} \mapsto 1, \quad \Delta_{02} \mapsto 36(1-x) X_1^4 X_2 \pi_A^2.$$

For the twisted sectors, we choose flat sections from solutions of first order differential equations

$$\Delta_{11} \sim (1-x)^{1/6} \, \phi_{10} \, \pi_A, \quad \Delta_{15} \sim (1-x)^{5/6} \, \phi_{31} \, \pi_A.$$

Others are obtained by solving equations (5.20) near $x=1$,

$$\begin{aligned}
\Delta_{21} &\sim (1-x)^{1/3}\Big(F^{(1)}_{2,20}\,\phi_{01} + (-2)^{-1/3}\ F^{(1)}_{2,01}\,\phi_{20}\Big)\,\pi_A,\\
\Delta_{12} &\sim (1-x)^{1/3}\Big(F^{(1)}_{1,20}\,\phi_{01} - 3(-2)^{-1/3}\ F^{(1)}_{1,01}\,\phi_{20}\Big)\,\pi_A,\\
\Delta_{31} &\sim (1-x)^{1/2}\Big(F^{(1)}_{2,30}\,\phi_{11} + (-2)^{-1/3}\ F^{(1)}_{2,11}\,\phi_{30}\Big)\,\pi_A,\\
\Delta_{13} &\sim (1-x)^{1/2}\Big(F^{(1)}_{1,20}\,\phi_{11} - 2(-2)^{-1/3}\ F^{(1)}_{1,11}\,\phi_{30}\Big)\,\pi_A,\\
\Delta_{22} &\sim (1-x)^{2/3}\Big(F^{(1)}_{2,40}\,\phi_{21} + (-2)^{-1/3}\ F^{(1)}_{2,21}\,\phi_{40}\Big)\,\pi_A,\\
\Delta_{14} &\sim (1-x)^{2/3}\Big(F^{(1)}_{1,40}\,\phi_{21} - \frac{5}{3}(-2)^{-1/3}\ F^{(1)}_{1,21}\,\phi_{40}\Big)\,\pi_A,
\end{aligned}$$

The proportions can be fixed by identify the pairing and the ring structure constants. For the 3-point correlator we get

$$\Big\langle \Delta_{1,1}, \Delta_{2,1}, \Delta_{3,1}\Big\rangle_{0,3} \mapsto q + 2q^7 + \cdots$$

5.4.2. FJRW-point at infinity

The Picard-Fuchs equation for the periods π_A has weights $(\alpha,\beta,\gamma) = (1/12, 7/12, 2/3)$. Since $\alpha-\beta$ is not an integer, the monodromy is diagonalizable and we have the following basis of solutions (eigenvectors for the monodromy around $\sigma=\infty$) near $x=\infty$:

$$\begin{aligned}
\pi_{A_\infty} &:= x^{-1/12}{}_2F_1\Big(\frac{1}{12},\frac{5}{12};\frac{1}{2};x^{-1}\Big),\\
\pi_{B_\infty} &:= \lambda\ x^{-7/12}{}_2F_1\Big(\frac{7}{12},\frac{11}{12};\frac{3}{2};x^{-1}\Big),
\end{aligned}$$

where the constant λ will be fixed later on. Put

$$t_{-1} = \pi_{B_\infty}/\pi_{A_\infty} \approx \lambda\ x^{-\frac{1}{2}},$$

where $\approx$ means that we truncated terms of order $O(\sigma^2)$. It is easy to check (by using the differential equation for the periods) that when restricted to the subspace of marginal deformation, t_{-1} is a degree 0 flat coordinate, i.e., the residue pairing $\langle 1, \partial/\partial t_{-1}\rangle$ is a constant.

For first order differential equations, we get flat sections

$$A_{(1,0)} = (x-1)^{1/6}\phi_{10}\pi_A(\sigma), \quad A_{(3,1)} = (x-1)^{5/6}\phi_{31}\pi_A(\sigma).$$

The solutions to the differential equations (5.20) near $x=\infty$, are

$$\begin{pmatrix} \phi_{(k,1)}\,\pi_A(\sigma) \\ \phi_{(k+2,0)}\,\pi_A(\sigma) \end{pmatrix} = \begin{pmatrix} F^{(\infty)}_{1,(k,1)}(x) & F^{(\infty)}_{2,(k,1)}(x) \\ L_k F^{(\infty)}_{1,(k,1)}(x) & L_k F^{(\infty)}_{2,(k,1)}(x) \end{pmatrix} \begin{pmatrix} A_{(k,1)} \\ A_{(k+2,0)} \end{pmatrix}, \quad k=0,1,2$$

Here $F^{(\infty)}_{i,\mathbf{r}}(x)$ is from (5.18) and L_k is from (5.21). Let $c_{\mathbf{r}}$ be given below

$\mathbf{r} =$	10	01	20	11	30	21	40	31
$c_{\mathbf{r}}$	λ_1	λ_2	$-\frac{\lambda_1^2 C_0}{3}$	$\lambda_1\lambda_2$	$-\lambda_1^3 C_0$	$\lambda_1\lambda_2^2$	$\frac{4\lambda_1^4 C_0}{5}$	$\frac{2\lambda_1^5 C_0}{9}$

The constants appearing in the table are given as follows:

$$\lambda_1^6 = 24C_0^2, \quad \lambda_2^2 = \frac{\lambda_1^4}{C_0}, \quad C_0^3 = -\frac{27}{4}. \tag{5.22}$$

Let $\delta_{\mathbf{r}}(\mathbf{s},\mathbf{x})$ be polynomials, such that the geometric sections (see (4.6))

$$[\delta_{\mathbf{r}}\omega] = c_{\mathbf{r}} A_{\mathbf{r}}.$$

Now we compute the pairing and the necessary genus-0 correlators. The pairing is

$$\left\langle \delta_{10},\delta_{31}\right\rangle = \left\langle \delta_{01},\delta_{21}\right\rangle = \left\langle \delta_{20},\delta_{40}\right\rangle = \left\langle \delta_{11},\delta_{11}\right\rangle = \left\langle \delta_{30},\delta_{30}\right\rangle = 1.$$

All 3-point correlator functions that do not have insertion 1 (otherwise the correlator reduces to a 2-point one) have a limit at $\sigma=\infty$. The non-zero limits are as follows:

$$\left\langle \delta_{10},\delta_{10},\delta_{40}\right\rangle_{0,3} = \left\langle \delta_{10},\delta_{01},\delta_{11}\right\rangle_{0,3} = \left\langle \delta_{01},\delta_{01},\delta_{20}\right\rangle_{0,3} = 1.$$
$$\left\langle \delta_{10},\delta_{20},\delta_{30}\right\rangle_{0,3} = \left\langle \delta_{20},\delta_{20},\delta_{20}\right\rangle_{0,3} = -3.$$

In other words, the Jacobian algebra extends over $\sigma=\infty$. If we denote the extension by $\mathscr{Q}_{W^\infty}$, then it is not hard to see that δ_{10} and δ_{01} are generators and we have

$$\mathscr{Q}_{W^\infty} := \mathbb{C}[\delta_{10},\delta_{01}]/\left(4\delta_{10}^3\delta_{01}, \delta_{10}^4 + 3\delta_{01}^2\right).$$

Finally, we set

$$\lambda = \frac{\lambda_1^4\lambda_2}{54}.$$

The nonzero 4-point genus-0 basic correlators are

$$\Big\langle \delta_{01},\delta_{01},\delta_{01},\delta_{-1}\Big\rangle_{0,4}=\frac{\partial}{\partial t_{-1}}\Big\langle \lambda_2^3\left(x^{1/12}\Phi_{01}+\frac{3}{4C_0}x^{-1/4}\Phi_{20}\right)^3\Big\rangle\Big|_{x=\infty}$$
$$=-\frac{\lambda_2^2C_0}{4\lambda_1^4}\frac{\partial}{\partial t_{-1}}\left(\lambda\, x^{-1/2}\right)=-\frac{1}{4}.$$

and

$$\Big\langle \delta_{10},\delta_{10},\delta_{10}^2\delta_{01},\delta_{-1}\Big\rangle_{0,4}=-\frac{1}{4}.$$

Recall that ρ_1,ρ_2 are generators of the FJRW ring for $\Big(W'=X_1^4X_2+X_2^3+X_3^2,G_{W'}\Big)$. We construct the following mirror map from $\mathscr{H}_{W'}$ to $\mathscr{Q}_{W^\infty}$,

$$\Big(\rho_1,\rho_2\Big)\mapsto\Big((-1)^{5/6}\delta_{01},(-1)^{2/3}\delta_{10}\Big)$$

Using Lemma 3.5, it is easy to check that this map identifies the FJRW theory of $(W',G_{W'})$ to the Saito-Givental limit of $W_\sigma=X_1^6+X_2^3+X_3^2+\sigma X_1^4X_2$ at $\sigma=\infty$,

$$\mathscr{A}_{W'}^{\mathrm{FJRW}}=\lim_{\sigma\to\infty}\mathscr{A}_W^{\mathrm{SG}}(\sigma).$$

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Manuscrit reçu le 19 mai 2014,
révisé le 7 février 2015,
accepté le 7 mai 2015.

Todor MILANOV
Kavli Institute for the Physics and Mathematics of the Universe (WPI)
Todai Institutes for Advanced Study
The University of Tokyo
Kashiwa, Chiba 277-8583 (Japan)
todor.milanov@ipmu.jp

Yefeng SHEN
Department of Mathematics
Stanford University
Stanford, CA 94305 (USA)
yfshen@stanford.edu