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PROJECTIVE INVARIANTS OF AN ORTHOGONAL ENNUPLE IN A FINSLER SPACE

by H. D. PANDE (*)

1. Introduction.

We consider an n-dimensional Finsler space F_n with the fundamental metric function $F(x, \dot{x})$. This fundamental function is positive homogeneous of the first degree in $\dot{x}^i$, it is > 0 for $\Sigma(\dot{x}^i)^2 \neq 0$ and the quadratic form $(\partial^2 F^2/\partial\dot{x}^i\partial\dot{x}^j)\xi^i\xi^j$ is positive definite in the variables ξ^i. The metric tensor is given by

$$(1.1) \qquad g_{ij}(x, \dot{x}) \stackrel{\text{def}}{=\!=} \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2(x, \dot{x}) \ (^1), \ (^2)$$

This tensor is symmetricin the in dices i, j and positive homogeneous of degree zero in $\dot{x}^i$. The contravariant components $g^{ij}(x, \dot{x})$ of the metric tensor is determined by

$$(1.2) \qquad g^{ij}(x, \dot{x})g_{jk}(x, \dot{x}) = \delta^i_k \quad \begin{cases} 1 & \text{if} \quad k = i. \\ 0 & \text{if} \quad k \neq i \end{cases}$$

The covariant components of the unit vector along the direction of the element of support $(x^i, \dot{x}^i)$ are given by

$$(1.3) \qquad l_i(x, \dot{x}) = \dot{\partial}_i F(x, \dot{x}).$$

The covariant derivative of a vector $X^i(x, \dot{x})$, depending on the element of support, with respect to x^k in the sense of

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(1) $\partial_i = \partial/\partial x^i$ and $\dot{\partial}_i = \partial/\partial\dot{x}^i$.

(2) Numbers in brackets refer to the references at the end of the paper.

Cartan is given by

$$(1.4) \qquad X^i_{|k}(x, \dot{x}) = (\partial_k X^i) - (\dot{\partial}_j X^i) G^j_k + X^j \Gamma^{*i}_{jk},$$

where

$$(1.5a) \qquad G^i_k(x, \dot{x}) \stackrel{\text{def}}{=\!=} \dot{\partial}_k G^i(x, \dot{x}),$$

$$(1.5b) \qquad 2G^i(x, \dot{x}) \stackrel{\text{def}}{=\!=} \gamma^i_{jk}(x, \dot{x}) \dot{x}^j \dot{x}^k,$$

$\gamma^i_{jk}(x, \dot{x})$ being the Christoffel's symbols of second kind [1] and $\Gamma^{*i}_{jk}(x, \dot{x})$ are the Cartan's connection coefficients symmetric in their lower indices and homogeneous of degree zero in their directional arguments. We have [1]

$$(1.6) \qquad G^i_{jk}(x, \dot{x}) \dot{x}^j = \Gamma^{*i}_{jk}(x, \dot{x}) \dot{x}^j = G^i_k(x, \dot{x}),$$

where $G^i_{jk}(x, \dot{x}) \stackrel{\text{d\&f}}{=\!=} \dot{\partial}_k G^i_j((x, \dot{x})$.

Let $\lambda_{(a)}\{a = 1, 2, \ldots, n\}$ be the unit tangents of n-congruences of an orthogonal ennuple. The subscript « a » in the paranthesis simply distinguishes one congruence from the other. The covariant and contravariant components of $\lambda_{(a)}$ will respectively be denoted by $\lambda_{(a)i}$ and $\lambda^i_{(a)}$. Since n-congruences are mutually orthogonal, we have [2]

$$(1.7) \qquad g_{ij}(x, \dot{x}) \lambda^i_{(a)} \lambda^j_{(b)} = \delta_{ab},$$

where the Kronecker delta $\delta_{ab} = \begin{cases} 1, & \text{if} \quad a = b \\ 0, & \text{if} \quad a \neq b \end{cases}$. We have the Ricci coefficients of rotation, given by [2, 3]

$$(1.8) \qquad Y_{abc}(x, \dot{x}) \stackrel{\text{def}}{=\!=} \lambda^i_{(a)|j} \lambda_{(b)i} \lambda^j_{(c)},$$

where the symbol $|$ denotes the covariant derivative with respect to x^k in the sense of Cartan and

$$(1.9) \qquad \mu^i_{(m)}(x, \dot{x}) \stackrel{\text{def}}{=\!=} \sum_h Y_{mhm} \lambda^i_{(h)}.$$

The geometric entities $\mu^i_{(m)}(x, \dot{x})$ are called the first curvature vector of a curve of congruence in Finsler space [3].

2. Projective transformation.

The equation of a geodesic

$$(2.1) \qquad \frac{d^2 x^i}{ds^2} + \Gamma^{*i}_{jk}(x, dx/ds) \frac{dx^j}{ds} \frac{dx^k}{ds} = 0$$

assumes the following form by the transformation of its parameter s to t [4]:

$$\dot{x}^l \left\{ \frac{d^2x^i}{dt^2} + \Gamma^*_{jk}(x, \dot{x})\dot{x}^j\dot{x}^k \right\} - \dot{x}^i \left\{ \frac{d^2x^l}{dt^2} + \Gamma^{*l}_{jk}(x, \dot{x})\dot{x}^j\dot{x}^k \right\} = 0, \tag{2.2}$$

where

$$\Gamma^{*i}_{jk}(x, \dot{x}) = \Gamma^{*i}_{kj}(x, \dot{x}). \tag{2.3}$$

The equation (2.2) remains unchanged if we replace the Cartan's connection coefficient $\Gamma^{*i}_{jk}(x, \dot{x})$ by a new symmetric coefficient $\overline{\Gamma}^{*i}_{jk}(x, \dot{x})$, given by [6]

$$\overline{\Gamma}^{*i}_{jk}(x, \dot{x}) \stackrel{\text{def}}{=} \Gamma^{*i}_{jk}(x, \dot{x}) + 2\delta^i_{(j}p_{k)} + p_{jk}\dot{x}^i, \tag{2.4}$$

where $p_k(x, \dot{x})$ is a covariant vector, positively homogeneous of degree zero in its directional arguments and

$$p_{jk}(x, \dot{x}) \stackrel{\text{def}}{=} \dot{\partial}_j p_k(x, \dot{x}). \tag{2.5}$$

Définition 2.1. — *Let* F_n *and* $\overline{F}_n$ *be two spaces with fundamental tensor* $g_{ij}(x, \dot{x})$ *and* $\overline{g}_{ij}(x, \dot{x})$ *at the corresponding points. Then the spaces are said to be in geodesic correspondence if their geodesics are the same and we shall call* (2.4)*a « projective change » of the Cartan's function* $\Gamma^{*i}_{jk}(x, \dot{x})$.

Contracting (2.4) with respect to the indices i and j, we get

$$\overline{\Gamma}^{*\gamma}_{\gamma k}(x, \dot{x}) = \Gamma^{*\gamma}_{\gamma k}(x, \dot{x}) + (n+1)p_k(x, \dot{x}). \tag{2.6}$$

Differentiating (2.6) with respect to $\dot{x}^l$, we obtain

$$\dot{\partial}_l \overline{\Gamma}^{*\gamma}_{\gamma k}(x, \dot{x}) = \dot{\partial}_l \Gamma^{*\gamma}_{\gamma k}(x, \dot{x}) + (n+1)p_{lk}(x, \dot{x}). \tag{2.7}$$

3. Projective invariants.

Theorem 3.1. — *If* $\lambda^i_{(a)}(x)$ *and* $\lambda_{(a)i}(x)$ *are the contravariant and covariant components of an orthogonal ennuple, then the following geometric entities are invariant under the projective change:*

$$A^i_k(x, \dot{x}) \stackrel{\text{def}}{=} \lambda^i_{(a)|k} - \frac{1}{n+1}\lambda^j_{(a)} \left\{ 2 \sum_b \lambda_{(b)m}\delta^i_{(j}\lambda^m_{(b)|k)} \right\}, \tag{3.1}$$

and

$$\text{(3.2)} \qquad A^{*i}_{k}(x, \dot{x}) \overset{\text{def}}{=\!=} \sum_{a} \lambda_{(a)i} \lambda^{i}_{(a)|k} - \Gamma^{*\gamma}_{\gamma k}.$$

Proof. — If we denote by $\lambda^{i}_{(a)|\bar{k}}$ the covariant derivative of $\lambda^{i}_{(a)}$ in the sense of Cartan for the connection coefficients $\overline{\Gamma}^{*i}_{jk}(x, \dot{x})$, then we have

$$\text{(3.3)} \qquad \lambda^{i}_{(a)|\bar{k}} = \eth_{k} \lambda^{i}_{(a)} + \lambda^{j}_{(a)} \overline{\Gamma}^{*i}_{jk}.$$

Hence we get in consequence of (2.4)

$$\text{(3.4)} \qquad \lambda^{i}_{(a)|\bar{k}} - \lambda^{i}_{(a)|k} = \lambda^{j}_{(a)} \{ 2\delta^{i}_{(j} p_{k)} + p_{jk} \dot{x}^{i} \}.$$

Multiplying (3.4) by $\lambda_{(a)i}$ throughout and summing with respect to a and using the orthogonality condition (1.7), we obtain

$$\text{(3.5)} \qquad \sum_{a} \lambda_{(a)i} (\lambda^{i}_{(a)|\bar{k}} - \lambda^{i}_{(a)|k}) = (n+1) p_{k}.$$

Eliminating the vector $p_{k}(x, \dot{x})$ from equations (3.4) and (3.5), we get

$$\text{(3.6)} \qquad \lambda^{i}_{(a)|\bar{k}} - \lambda^{i}_{(a)|k} = \frac{1}{n+1} \lambda^{j}_{(a)} \Big[\delta^{i}_{j} \sum_{b} \lambda_{(b)m} (\lambda^{m}_{(b)|\bar{k}} - \lambda^{m}_{(b)|k}) + \delta^{i}_{k} \sum_{b} \lambda_{(b)m} (\lambda^{m}_{(b)|\bar{j}} - \lambda^{m}_{(b)|j}) \Big].$$

Again, with the help of (2.6), equation (3.5) yields

$$\text{(3.7)} \qquad \sum_{a} \lambda_{(a)i} (\lambda^{i}_{(a)|\bar{k}} - \lambda^{i}_{(a)|k}) = \overline{\Gamma}^{*\gamma}_{\gamma k} - \Gamma^{*\gamma}_{\gamma k},$$

which gives us (3.2).

THEOREM 3.2. — *When* F_n *and* $\overline{F}_n$ *are in geodesic correspondence, we have the following geometric entities which are invariant under the projective change:*

$$\text{(3.8)} \qquad C^{i}_{k}(x, \dot{x}) \overset{\text{def}}{=\!=} \lambda^{i}_{(a)|k} - \frac{1}{n+1} \lambda^{j}_{(a)} \{ 2\delta^{i}_{(j} \Gamma^{*\gamma}_{k)\gamma} + \dot{x}^{i} \eth_{j} \Gamma^{*\gamma}_{\gamma k} \},$$

and

$$\text{(3.9)} \qquad C^{*i}_{k} x \; \dot{x}) = \lambda^{i}_{(a)|k} - \frac{1}{n+1} \lambda^{j}_{(a)} \Big\{ \sum_{b} 2\lambda_{(b)m} \delta^{i}_{(j} \lambda^{m}_{(b)|k)} + \dot{x}^{i} \eth_{j} \Gamma^{*\gamma}_{\gamma k} \Big\}.$$

Proof. — Using equations (2.6), (2.7) and (3.4), we get

$$(3.10)\quad \lambda^i_{(a)|\bar{k}} - \lambda^i_{(a)|k} = \frac{1}{n+1}\lambda^j_{(a)}[\delta^i_j(\bar{\Gamma}^{*\gamma}_{\gamma k} - \Gamma^{*\gamma}_{\gamma k}) + \delta^i_k(\bar{\Gamma}^{*\gamma}_{\gamma j} - \Gamma^{*\gamma}_{\gamma j}) + \dot{x}^i\dot{\partial}_j(\bar{\Gamma}^{*\gamma}_{\gamma k} - \Gamma^{*\gamma}_{\gamma k})],$$

which yields the result (3.8).

Again, eliminating $p_k(x, \dot{x})$ and $p_{lk}(x, \dot{x})$ from equations (2.7), (3.4) and (3.5), we obtain

$$(3.11)\quad \lambda^i_{(a)|\bar{k}} - \lambda^i_{(a)|k} = \frac{1}{n+1}\lambda^j_{(a)}\sum_b \lambda_{(b)m}[\delta^i_j(\lambda^m_{(b)|\bar{k}} - \lambda^m_{(b)|k}) + \delta^i_k(\lambda^m_{(b)|\bar{j}} - \lambda^m_{(b)|j})] + \frac{\dot{x}^i}{n+1}\lambda^j_{(a)}\,\dot{\partial}_j(\bar{\Gamma}^{*\gamma}_{\gamma k} - \Gamma^{*\gamma}_{\gamma k}),$$

which gives us (3.9).

THEOREM 3.3. — *When* F_n *and* $\bar{F}_n$ *are in geodesic correspondence, we have the following projective invariant geometric entities:*

$$(3.12)\quad S_{abc}(x, \dot{x}) \overset{\text{def}}{=\!=} Y_{abc} - \frac{1}{n+1}(\delta_{ab}\lambda^k_{(c)}\Gamma^{*\gamma}_{\gamma k} + \delta_{bc}\lambda^j_{(a)}\Gamma^{*\gamma}_{\gamma j}) - \frac{1}{n+1}\lambda^j_{(a)}\lambda_{(b)i}\lambda^k_{(c)}\dot{x}^i\dot{\partial}_j\Gamma^{*\gamma}_{\gamma k},$$

and

$$(3.13)\quad S^*_{acd}(x, \dot{x}) \overset{\text{def}}{=\!=} Y_{acd} - \frac{1}{n+1}\Big[\sum_b (\delta_{ca}Y_{bbd} + \delta_{cd}Y_{bba})\Big] - \frac{1}{n+1}\lambda^j_{(a)}\lambda_{(c)i}\lambda^k_{(d)}\dot{x}^i\dot{\partial}_j\Gamma^{*\gamma}_{\gamma k},$$

and

$$(3.14)\quad S_b(x, \dot{x}) \overset{\text{def}}{=\!=} \sum_a Y_{aab} - \lambda^k_{(b)}\Gamma^{*\gamma}_{\gamma k},$$

where Y_{abc} are Ricci coefficients of rotation.

Proof. — Multiplying (3.10) by the product $\lambda_{(b)i}\lambda^k_{(c)}$ and using the orthogonality relation (1.7), we get

$$(3.15)\quad \bar{Y}_{abc} - \frac{1}{n+1}(\delta_{ab}\lambda^k_{(c)}\bar{\Gamma}^{*\gamma}_{\gamma k} + \delta_{bc}\lambda^j_{(a)}\bar{\Gamma}^{*\gamma}_{\gamma j} + \lambda^j_{(a)}\lambda_{(b)i}\lambda^k_{(c)}\dot{x}^i\dot{\partial}_j\bar{\Gamma}^{*\gamma}_{\gamma k}) = Y_{abc} - \frac{1}{n+1}(\delta_{ab}\lambda^k_{(c)}\Gamma^{*\gamma}_{\gamma k} + \delta_{bc}\lambda^j_{(a)}\Gamma^{*\gamma}_{\gamma j} + \lambda^j_{(a)}\lambda_{(b)i}\lambda^k_{(c)}\dot{x}^i\dot{\partial}_j\Gamma^{*\gamma}_{\gamma k}),$$

where the projectively transformed Ricci coefficients of rotation are given by

$$(3.16) \qquad \overline{\Upsilon}_{abc} \overset{\text{def}}{=\!=} \lambda^{i}_{(a)|\overline{k}} \lambda_{(b)i} \lambda^{k}_{(c)}.$$

Similarly, multiplying (3.11) by the product $\lambda_{(c)i}\lambda^{k}_{(d)}$ and using the orthogonal relation (1.7), we obtain (3.13).

Again, multiplying (3.7) by $\lambda^{k}_{(b)}$ and making use of (1.7), we get

$$(3.17) \qquad \sum_{a} \overline{\Upsilon}_{aab} - \lambda^{k}_{(b)} \overline{\Gamma}^{*\gamma}_{\gamma k} = \sum_{a} \Upsilon_{aab} - \lambda^{k}_{(b)} \Gamma^{*\gamma}_{\gamma k},$$

which shows that $S_b(x, \dot{x})$ are invariant under the projective change.

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BIBLIOGRAPHY

[1] H. Rund, The differential Geometry of Finsler spaces, *Springer-Verlag*, Berlin (1959).

[2] B. B. Sinha, Projective Invariants, *Maths. Student*, XXXIII, No. 2 and 3 (1965), 121-127.

[3] R. S. Mishra, On the congruence of curves through points of a subspace imbedded in a Riemann space, *Ann. Soc. Sci.*, Bruxelles (1951), 109-115.

[4] T. Y. Thomas, Differential Geometry of generalised spaces, Cambridge (1934).

[5] L. Berwald, On the projective geometry of paths, *Ann. Maths.*, 37 (1936), 879-898.

[6] H. D. Pande, The projective transformation in a Finsler space, *Atti della Accad. Naz.*, Lincei. Rendiconti, XLIII, No 6 (1967), 480-484.

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