



ANNALES

DE

L'INSTITUT FOURIER

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Tome 66, n° 5 (2016), p. 2101-2155.

http://aif.cedram.org/item?id=AIF_2016__66_5_2101_0



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CHARACTER GROUPS OF HOPF ALGEBRAS AS INFINITE-DIMENSIONAL LIE GROUPS

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ABSTRACT. — In this article character groups of Hopf algebras are studied from the perspective of infinite-dimensional Lie theory. For a graded and connected Hopf algebra we obtain an infinite-dimensional Lie group structure on the character group with values in a locally convex algebra. This structure turns the character group into a Baker–Campbell–Hausdorff–Lie group which is regular in the sense of Milnor. Furthermore, we show that certain subgroups associated to Hopf ideals become closed Lie subgroups of the character group.

If the Hopf algebra is not graded, its character group will in general not be a Lie group. However, we show that for any Hopf algebra the character group with values in a weakly complete algebra is a pro-Lie group in the sense of Hofmann and Morris.

RÉSUMÉ. — Dans cet article, nous étudions les groupes de caractères des algèbres de Hopf du point de vue de la théorie de Lie de dimension infinie. Pour une algèbre de Hopf connexe et graduée, nous munissons le groupe de caractères d'une structure de groupe de Lie de dimension infinie, à valeurs dans une algèbre localement convexe. Cette structure permet de voir le groupe de caractères comme un groupe de Lie de Baker–Campbell–Hausdorff, qui est régulier au sens de Milnor. De plus, nous montrons que certains sous-groupes associés aux idéaux de Hopf sont alors des sous-groupes de Lie du groupe de caractères.

Si l'algèbre de Hopf n'est pas graduée, son groupe de caractères ne sera pas un groupe de Lie, en général. Cependant, nous montrons que pour une algèbre de Hopf quelconque, le groupe de caractères à valeurs dans une algèbre faiblement complète est un groupe pro-Lie au sens de Hofmann et Morris.

Keywords: real analytic, infinite-dimensional Lie group, Hopf algebra, continuous inverse algebra, Butcher group, weakly complete space, pro-Lie group.

Math. classification: 22E65, 16T05, 43A40, 58B25, 46H30, 22A05.

(*) The research on this paper was partially supported by the project *Topology in Norway* (NRC project 213458) and *Structure Preserving Integrators, Discrete Integrable Systems and Algebraic Combinatorics* (NRC project 231632). We thank Antoine Julien for his services in translating the text to French.

Introduction and statement of results

Hopf algebras and their character groups appear in a variety of mathematical and physical contexts. To name just a few, they arise in non-commutative geometry, renormalisation of quantum field theory (see [7, 10]) and numerical analysis (cf. [4]). We recommend [5] as a convenient introduction to Hopf algebras and their historical development.

In their seminal work [8, 9] Connes and Kreimer associate to the group of characters of a Hopf algebra of Feynman graphs a Lie algebra. It turns out that this (infinite-dimensional) Lie algebra is an important tool to analyse the structure of the character group. In fact, the character group is then called “infinite-dimensional Lie group” meaning that it is a projective limit of finite-dimensional Lie groups with an associated infinite-dimensional Lie algebra. These structures enable the treatment of certain differential equations on the group which are crucial to understand the renormalisation procedure. Moreover, on a purely algebraic level it is always possible to construct a Lie algebra associated to the character group of a Hopf algebra. Thus it seems natural to ask whether the differential equations and the Lie algebras are connected to some kind of Lie group structure on the character group. Indeed, in [3] the character group of the Hopf algebra of rooted trees was turned into an infinite-dimensional Lie group. Its Lie algebra is closely related to the Lie algebra constructed in [7] for the character group.

These observations hint at a general theme which we explore in the present paper. Our aim is to study character groups (with values in a commutative locally convex algebra) of a Hopf algebra from the perspective of infinite-dimensional Lie theory. We base our investigation on a concept of C^r -maps between locally convex spaces known as Keller’s C_c^r -theory⁽¹⁾ [21] (see [12, 28, 30] for streamlined expositions and Appendix A for a quick reference). In the framework of this theory, we treat infinite-dimensional Lie group structures for character groups of Hopf algebras. If the Hopf algebra is graded and connected, it turns out that the character group can be made into an infinite-dimensional Lie group. We then investigate Lie theoretic properties of these groups and some of their subgroups. In particular, the Lie algebra associated to the Lie group structure on the group of characters turns out to be the Lie algebra of infinitesimal characters.

⁽¹⁾ Although Keller’s C_c^r -theory is in general inequivalent to the “convenient setting” of calculus [22], in the important case of Fréchet spaces both theories coincide (e.g. Example 4.7).

The character group of an arbitrary Hopf algebra can in general not be turned into an infinite-dimensional Lie group and we provide an explicit example for this behaviour. However, it turns out that the character group of an arbitrary Hopf algebra (with values in a finite-dimensional algebra) is always a topological group with strong structural properties, i.e. it is always the projective limit of finite-dimensional Lie groups. Groups with these properties — so called pro-Lie groups — are accessible to Lie theoretic methods (cf. [18]) albeit they may not admit a differential structure.

We now go into some more detail and explain the main results of the present paper. Let us recall first the definition of the character group of a Hopf algebra $(\mathcal{H}, m_{\mathcal{H}}, 1_{\mathcal{H}}, \Delta_{\mathcal{H}}, \varepsilon_{\mathcal{H}}, S_{\mathcal{H}})$ over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Fix a commutative locally convex algebra B . Then the character group $G(\mathcal{H}, B)$ of \mathcal{H} with values in B is defined as the set of all unital algebra characters

$$G(\mathcal{H}, B) := \left\{ \phi \in \text{Hom}_{\mathbb{K}}(\mathcal{H}, B) \mid \begin{array}{l} \phi(ab) = \phi(a)\phi(b), \forall a, b \in \mathcal{H} \\ \text{and } \phi(1_{\mathcal{H}}) = 1_B \end{array} \right\},$$

together with the convolution product $\phi \star \psi := m_B \circ (\phi \otimes \psi) \circ \Delta_{\mathcal{H}}$.

Closely related to this group is the Lie algebra of infinitesimal characters

$$\mathfrak{g}(\mathcal{H}, B) := \{ \phi \in \text{Hom}_{\mathbb{K}}(\mathcal{H}, B) \mid \phi(ab) = \varepsilon_{\mathcal{H}}(a)\phi(b) + \varepsilon_{\mathcal{H}}(b)\phi(a) \}$$

with the commutator Lie bracket $[\phi, \psi] := \phi \star \psi - \psi \star \phi$.

It is well known that for a certain type of Hopf algebra (e.g. graded and connected) the exponential series induces a bijective map $\mathfrak{g}(\mathcal{H}, B) \rightarrow G(\mathcal{H}, B)$. In this setting, the ambient algebra $(\text{Hom}_{\mathbb{K}}(\mathcal{H}, B), \star)$ becomes a locally convex algebra with respect to the topology of pointwise convergence and we obtain the following result.

THEOREM A. — *Let \mathcal{H} be a graded and connected Hopf algebra and B a commutative locally convex algebra, then the group $G(\mathcal{H}, B)$ of B -valued characters of \mathcal{H} is a (\mathbb{K} -analytic) Lie group.*

The Lie algebra of $G(\mathcal{H}, B)$ is the Lie algebra $\mathfrak{g}(\mathcal{H}, B)$ of infinitesimal characters.

Note that this Lie group structure recovers the Lie group structure on the character group of the Hopf algebra of rooted trees which has been constructed in [3]. For further information we refer to Example 4.7.

We then investigate the Lie theoretic properties of the character group of a graded connected Hopf algebra. To understand these results first recall the notion of regularity for Lie groups.

Let G be a Lie group modelled on a locally convex space, with identity element $\mathbf{1}$, and $r \in \mathbb{N}_0 \cup \{\infty\}$. We use the tangent map of the left translation

$\lambda_g: G \rightarrow G$, $x \mapsto gx$ by $g \in G$ to define $g.v := T_1\lambda_g(v) \in T_gG$ for $v \in T_1(G) =: \mathbf{L}(G)$. Following [15], G is called C^r -semiregular if for each C^r -curve $\gamma: [0, 1] \rightarrow \mathbf{L}(G)$ the initial value problem

$$\begin{cases} \eta'(t) = \eta(t).\gamma(t) \\ \eta(0) = \mathbf{1} \end{cases}$$

has a (necessarily unique) C^{r+1} -solution $\text{Evol}(\gamma) := \eta: [0, 1] \rightarrow G$. If further the map

$$\text{evol}: C^r([0, 1], \mathbf{L}(G)) \rightarrow G, \gamma \mapsto \text{Evol}(\gamma)(1)$$

is smooth, we call G C^k -regular. If G is C^r -regular and $r \leq s$, then G is also C^s -regular. A C^∞ -regular Lie group G is called *regular (in the sense of Milnor)* — a property first defined in [28]. Every finite-dimensional Lie group is C^0 -regular (cf. [30]). Several important results in infinite-dimensional Lie theory are only available for regular Lie groups (see [15, 28, 30], cf. also [22] and the references therein).

Concerning the Lie theoretic properties of the character groups our results subsume the following theorem.

THEOREM B. — *Let \mathcal{H} be a graded and connected Hopf algebra and B be a commutative locally convex algebra.*

- (a) *Then $G(\mathcal{H}, B)$ is a Baker–Campbell–Hausdorff–Lie group which is exponential, i.e. the Lie group exponential map is a global \mathbb{K} -analytic diffeomorphism.*
- (b) *If B is sequentially complete then $G(\mathcal{H}, B)$ is a C^0 -regular Lie group.*

To illustrate the importance of regularity, let us digress and consider the special case of the \mathbb{C} -valued character group of the Hopf algebra of Feynman graphs. In the Connes–Kreimer theory of renormalisation of quantum field theories, a crucial step is to integrate paths in the Lie algebra associated to this group. These integrals, called “time-ordered exponentials” (cf. [10, Definition 1.50]), encode information on the renormalisation procedure. By [10, Chapter 1, 7.2] the “main advantage” of the time-ordered differentials is that they are the solutions of certain differential equation on the character group (see [10, Proposition 1.51 (3)]). In view of the Lie group structure induced by Theorem A on the character group, these differential equations turn out to be the differential equations of regularity of the Lie group. Hence, in our language loc. cit. proves that the \mathbb{C} -valued characters of the Hopf algebra of Feynman graphs are regular in the sense of Milnor.

Returning to the results contained in the present paper, we turn to a class of closed subgroups of character groups which are closed Lie subgroups. For a Hopf ideal \mathcal{J} of a Hopf algebra \mathcal{H} , consider the annihilator

$$\text{Ann}(\mathcal{J}, B) := \{\phi \in \text{Hom}_{\mathbb{K}}(\mathcal{H}, B) \mid \phi(a) = 0_B, \forall a \in \mathcal{J}\}.$$

Then $\text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ becomes a subgroup and we obtain the following result.

THEOREM C. — *Let \mathcal{H} be a connected and graded Hopf algebra and B be a commutative locally convex algebra.*

- (a) *Then $\text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ is a closed Lie subgroup of $G(\mathcal{H}, B)$ whose Lie algebra is $\text{Ann}(\mathcal{J}, B) \cap \mathfrak{g}(\mathcal{H}, B)$.*
- (b) *There is a canonical isomorphism of (topological) groups*

$$\text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B) \cong G(\mathcal{H}/\mathcal{J}, B),$$

where \mathcal{H}/\mathcal{J} is the quotient Hopf algebra. If \mathcal{H}/\mathcal{J} is a connected and graded Hopf algebra (e.g. \mathcal{J} is a homogeneous ideal) then this map is an isomorphism of Lie groups.

Note that in general \mathcal{H}/\mathcal{J} will not be graded and connected. In these cases the isomorphism $\text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B) \cong G(\mathcal{H}/\mathcal{J}, B)$ extends the construction of Lie group structures for character groups to Hopf algebras which are quotients of connected and graded Hopf algebras. However, this does not entail that the character groups of all Hopf algebras are infinite-dimensional Lie groups. In general, the character group will only be a topological group with respect to the topology of pointwise convergence. We refer to Example 4.11 for an explicit counterexample of a character group which can not be turned into an infinite-dimensional Lie group.

Finally, we consider a class of character groups of (non-graded) Hopf algebras whose topological group structure is accessible to Lie theoretic methods. The class of characters we consider are character groups with values in an algebra which is “weakly complete”, i.e. the algebra is as a topological vector space isomorphic to \mathbb{K}^I for some index set I . All finite-dimensional algebras are weakly complete, we refer to the Diagram B.1 and Appendix C for more information. Then we obtain the following result:

THEOREM D. — *Let \mathcal{H} be an arbitrary Hopf algebra and B be a commutative weakly complete algebra. Then the following holds*

- (a) *the topological group $G(\mathcal{H}, B)$ is a projective limit of finite-dimensional Lie groups (a pro-Lie group in the sense of [18]).*

A pro-Lie group is associated to a Lie algebra which we identify for $G(\mathcal{H}, B)$:

- (b) *the pro-Lie algebra $\mathcal{L}(G(\mathcal{H}, B))$ of the pro-Lie group $G(\mathcal{H}, B)$ is the Lie algebra of infinitesimal characters $\mathfrak{g}(\mathcal{H}, B)$.*

The pro-Lie group structure of the character group is exploited in the theory of renormalisation (cf. [10]). We refer to Remark 5.8 for concrete examples of computations in loc. cit. which use the pro-Lie structure.

Note that pro-Lie groups are in general only topological groups without a differentiable structure attached to them. However, these groups admit a Lie theory which has been developed in the extensive monograph [18]. The results on the pro-Lie structure are somewhat complementary to the infinite-dimensional Lie group structure. If the Hopf algebra \mathcal{H} is graded and connected and B is a commutative weakly complete algebra, then the pro-Lie group structure of $G(\mathcal{H}, B)$ is compatible with the infinite-dimensional Lie group structure of $G(\mathcal{H}, B)$ obtained in Theorem A.

1. Linear maps on (connected) coalgebras

In this preliminary section we collect first some basic results and notations used throughout the paper (also cf. Appendices A–C). Most of the results are not new, however, we state them together with a proof for the reader's convenience.

Notation 1.1. — We write $\mathbb{N} := \{1, 2, 3, \dots\}$, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. In this article (with the exception of Appendix C), \mathbb{K} denotes either the field \mathbb{R} of real or the field \mathbb{C} of complex numbers.

Notation 1.2 (Terminology).

- (a) By the term *(co-)algebra*, we always mean an (co-)associative unital \mathbb{K} -(co-)algebra.
- (b) The *unit group* or *group of units* of an algebra \mathcal{A} is the group of its invertible elements and is denoted by \mathcal{A}^\times .
- (c) A *locally convex space* is a locally convex Hausdorff topological vector space and a *weakly complete space* is a locally convex space which is topologically isomorphic to \mathbb{K}^I for an index set I (see Definition C.1).
- (d) A *locally convex algebra (weakly complete) algebra* is a topological algebra whose underlying topological space is locally convex (weakly complete). Cf. also Lemma 5.3.
- (e) Finally, a *continuous inverse algebra (CIA)* is a locally convex algebra with an open unit group and a continuous inversion.

If we want to emphasize that an algebraic structure, such as a vector space or an algebra, carries no topology, we call it an *abstract vector space* or *abstract algebra*, etc.

Throughout this section, let $\mathcal{C} = (\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ denote an abstract coalgebra and let B denote a locally convex topological algebra, e.g. a Banach algebra.

DEFINITION 1.3 (Algebra of linear maps on a coalgebra). — We consider the locally convex space

$$A := \text{Hom}_{\mathbb{K}}(\mathcal{C}, B)$$

of all \mathbb{K} -linear maps from \mathcal{C} into B . We will give this space the topology of pointwise convergence, i.e. we embed A into the product space $B^{\mathcal{C}}$ with the product topology.

The space A becomes a unital algebra with respect to the convolution product (cf. [33, Section IV])

$$\star: A \times A \rightarrow A, (h, g) \mapsto m_B \circ (h \otimes g) \circ \Delta_{\mathcal{C}}.$$

Here $m_B: B \otimes B \rightarrow B$ is the algebra multiplication. The unit with respect to \star is the map $1_A = u_B \circ \varepsilon_{\mathcal{C}}$ where we defined $u_B: \mathbb{K} \rightarrow B, z \mapsto z1_B$.

One of the most interesting choices for the target algebra B of $A = \text{Hom}_{\mathbb{K}}(\mathcal{C}, B)$ is $B = \mathbb{K}$: In this case, A is the algebraic dual of the abstract vector space \mathcal{C} with the weak*-topology.

LEMMA 1.4. — Consider the algebra $A = \text{Hom}_{\mathbb{K}}(\mathcal{C}, B)$ with the convolution \star as above. The map $\star: A \times A \rightarrow A$ is continuous, whence (A, \star) is a locally convex algebra,

Proof. — Since the range space $A = \text{Hom}_{\mathbb{K}}(\mathcal{C}, B)$ carries the topology of pointwise convergence, we fix an element $c \in \mathcal{C}$. We write $\Delta_{\mathcal{C}}(c) \in \mathcal{C} \otimes \mathcal{C}$ in Sweedler’s sigma notation (see [20, Notation 1.6] or [33, Section 1.2]) as a finite sum

$$\Delta_{\mathcal{C}}(c) = \sum_{(c)} c_1 \otimes c_2.$$

Then the convolution product $\phi \star \psi$ evaluated at point c is of the form:

$$\begin{aligned} (\phi \star \psi)(c) &= m_B \circ (\phi \otimes \psi) \circ \Delta_{\mathcal{C}}(c) = m_B \circ (\phi \otimes \psi) \left(\sum_{(c)} c_1 \otimes c_2 \right) \\ &= \sum_{(c)} m_B(\phi(c_1) \otimes \psi(c_2)) = \sum_{(c)} \phi(c_1) \cdot \psi(c_2). \end{aligned}$$

This expression is continuous in (ϕ, ψ) since point evaluations are continuous as well as multiplication in the locally convex algebra B . □

Remark 1.5. — Note that the multiplication of the locally convex algebra B is assumed to be a continuous bilinear map $B \times B \rightarrow B$. However, we did not need to put a topology on the space $B \otimes B$ nor did we say anything about the continuity of the linear map $m_B: B \otimes B \rightarrow B$.

LEMMA 1.6 (Properties of the space A). — Let $A = \text{Hom}_{\mathbb{K}}(\mathcal{C}, B)$ as above.

- (a) As a locally convex space (without algebra structure), the space A is isomorphic to B^I , where the cardinality of the index set I is equal to the dimension of \mathcal{C} .
- (b) If the vector space \mathcal{C} is of countable dimension and B is a Fréchet space, A is a Fréchet space as well.
- (c) The locally convex algebra A is (Mackey/sequentially) complete if and only if the algebra B is (Mackey/sequentially) complete.

Proof.

- (a) A linear map is uniquely determined by its valued on a basis $(c_i)_{i \in I}$ of \mathcal{C} .
- (b) As a locally convex space $A \cong B^I$. Since I is countable and B is a Fréchet space, A is a countable product of Fréchet spaces, whence a Fréchet space.
- (c) By part (a), B is a closed vector subspace of A . So every completeness property of A is inherited by B . On the other hand, products of Mackey complete (sequentially complete, complete) spaces are again of this type. \square

The terms *abstract gradings* and *dense gradings* used in the next lemma are defined in Definition B.1 and B.2, respectively.

LEMMA 1.7. — Let \mathcal{C} be an abstract coalgebra, let B be a locally convex algebra, and set $A = \text{Hom}_{\mathbb{K}}(\mathcal{C}, B)$ as above.

- (a) If \mathcal{C} admits an (abstract) grading $\mathcal{C} = \bigoplus_{n=0}^{\infty} \mathcal{C}_n$, the bijection

$$(1.1) \quad A = \text{Hom}_{\mathbb{K}} \left(\bigoplus_{n=0}^{\infty} \mathcal{C}_n, B \right) \rightarrow \prod_{n=0}^{\infty} \text{Hom}_{\mathbb{K}}(\mathcal{C}_n, B), \quad \phi \mapsto (\phi|_{\mathcal{C}_n})_{n \in \mathbb{N}_0}$$

turns A into a densely graded algebra with respect to the grading $(\text{Hom}_{\mathbb{K}}(\mathcal{C}_n, B))_{n \in \mathbb{N}_0}$.

- (b) If in addition \mathcal{C} is connected and B is a CIA, then A is a CIA as well.

Proof.

- (a) It is clear that the map (1.1) is an isomorphism of topological vector spaces. Via this dualisation, the axioms of the graded coalgebra (see Definition B.1(c)) directly translate to the axioms of densely graded locally convex algebra (see Definition B.2(b)).
- (b) By Lemma B.7, we know that the densely graded locally convex algebra A is a CIA, if $A_0 = \text{Hom}_{\mathbb{K}}(\mathcal{C}_0, B)$ is a CIA. Since we assume that \mathcal{C} is connected, this means that $\mathcal{C} \cong \mathbb{K}$ and hence $A_0 \cong B$. The assertion follows. \square

From Lemma 1.7 and Lemma A.6 one easily deduce the following proposition.

PROPOSITION 1.8 (A^\times is a Lie group). — *Let \mathcal{C} be an abstract graded connected coalgebra and let B be a Mackey complete CIA. Then the unit group A^\times of the densely graded algebra $A = (\text{Hom}_{\mathbb{K}}(\mathcal{C}, B), \star)$ is a BCH-Lie group. The Lie algebra of the group A^\times is $(A, [\cdot, \cdot])$, where $[\cdot, \cdot]$ denotes the usual commutator bracket.*

Furthermore, the Lie group exponential function of A^\times is given by the exponential series, whence it restricts to the exponential function constructed in Lemma B.5.

THEOREM 1.9 (Regularity of A^\times). — *Let \mathcal{C} be an abstract graded connected coalgebra and let B be a Mackey complete CIA. As above, we set $A := (\text{Hom}_{\mathbb{K}}(\mathcal{C}, B), \star)$ and assume that B is commutative or locally m-convex (i.e. the topology is generated by a system of submultiplicative seminorms).*

- (a) *The Lie group A^\times is C^1 -regular.*
- (b) *If in addition, the space B is sequentially complete, then A^\times is C^0 -regular.*

In both cases the associated evolution map is even \mathbb{K} -analytic.

Proof. — Since B is a commutative CIA or locally m-convex, the algebra B has the (GN)-property by Lemma A.8. The algebra $A := \text{Hom}_{\mathbb{K}}(\mathcal{C}, B)$ is densely graded with $A_0 \cong B$ by Lemma 1.7. We claim that since $A_0 = B$ has the (GN)-property, the same holds for A (the details are checked in Lemma 1.10 below).

By Lemma 1.6, we know that A and B share the same completeness properties, i.e. the algebra A is Mackey complete if B is so and the same hold for sequential completeness. In conclusion, the assertion follows directly from Lemma A.9. \square

LEMMA 1.10. — *Let A be a densely graded algebra and denote the dense grading by $(A_N)_{N \in \mathbb{N}_0}$. Then A has the (GN)-property (see Definition A.7) if and only if the subalgebra A_0 has the (GN)-property.*

Proof. — Let us first see that the condition is necessary. Pick a continuous seminorm p_0 on A_0 . Then $P := p_0 \circ \pi_0$ is a continuous seminorm on A (where $\pi_0: A \rightarrow A_0$ is the canonical projection). Following Definition A.7 there is a continuous seminorm Q on A and a number $M > 0$ such that the following condition holds

$$(1.2) \quad \forall n \in \mathbb{N} \text{ and each } (\bar{a}_1, \dots, \bar{a}_n) \in A^n \text{ with } Q(\bar{a}_i) \leq 1 \text{ for } 1 \leq i \leq n, \\ \text{we have } P(\bar{a}_1 \cdots \bar{a}_n) \leq M^n.$$

Now we set $q_0 := Q|_{A_0}$ and observe that q_0 is a continuous seminorm as the inclusion $A_0 \rightarrow A$ is continuous and linear. A trivial computation now shows that p_0, q_0 and M satisfy (1.2). We conclude that A_0 has the (GN)-property.

For the converse assume that A_0 has the (GN)-property and fix a continuous seminorm P on A . The topology on A is the product topology, i.e. it is generated by the canonical projections $\pi_N: A \rightarrow A_N$. Hence we may assume that P is of the form

$$P = \max_{0 \leq N \leq L} (p_N \circ \pi_N)$$

where each $p_N: A_N \rightarrow [0, \infty[$ is a continuous seminorm on A_N . The number $L \in \mathbb{N}_0$ is finite and remains fixed for the rest of the proof.

The key idea is here that P depends only on a finite number of spaces in the grading. Now the multiplication increases the degree of elements except for factors of degree 0. However, these contributions can be controlled by the (GN)-property in A_0 .

We will now construct a continuous seminorm Q on A and a number $M > 0$ such that (1.2) holds.

Construction of the seminorm Q . Let $w = (w_1, w_2, \dots, w_r) \in \mathbb{N}^r$ (where $r \in \mathbb{N}$) be a multi-index and denote by $|w|$ the sum of the entries of the multi-index w . Define for $r \leq L$ and $w \in \mathbb{N}^r$ with $|w| \leq L$ a continuous r -linear map

$$f_w: A_{w_1} \times A_{w_2} \times \cdots \times A_{w_r} \rightarrow A_{|w|}, (b_1, \dots, b_r) \mapsto b_1 \cdots b_r,$$

Since $L < \infty$ is fixed and the w_i are strictly positive for $1 \leq i \leq r$, there are only finitely many maps f_w of this type. This allows us to define for each $1 \leq k \leq L$ a seminorm q_k on A_k with the following property: For all

$r \leq L$ and $w \in \mathbb{N}^r$ with $|w| \leq L$ we obtain an estimate

$$(1.3) \quad p_{|w|}(f_w(b_1, \dots, b_r)) \leq q_{w_1}(b_1) \cdot q_{w_2}(b_2) \cdots q_{w_r}(b_r).$$

Consider for each $N \leq L$ the continuous trilinear map

$$g_N: A_0 \times A_N \times A_0 \rightarrow A_N, (c, d, e) \mapsto c \cdot d \cdot e.$$

As there are only finitely many of these maps, we can define a seminorm q_0 on A_0 and a seminorm q_N^\sim on A_N for each $1 \leq N \leq L$ such that

$$(1.4) \quad q_N(g_N(c, d, e)) \leq q_0(c)q_N^\sim(d)q_0(e) \text{ holds for } 1 \leq N \leq L.$$

Enlarging the seminorm q_0 , we may assume that $q_0 \geq p_0$.

Now we use the fact that the subalgebra A_0 has the (GN)-property. Hence, there is a continuous seminorm q_0^\sim on A_0 and a number $M_0 \geq 1$ such that for $n \in \mathbb{N}$ and elements $c_i \in A_0$, $1 \leq i \leq n$ with $q_0^\sim(c_i) \leq 1$ we have

$$(1.5) \quad q_0(c_1 \cdots c_n) \leq M_0^n.$$

Finally, we define the seminorm Q via

$$Q := \max_{0 \leq k \leq L} (q_k^\sim \circ \pi_k).$$

Clearly Q is a continuous seminorm on A . Moreover, we set

$$M := M_0^{2(L+1)} \cdot (L + 1).$$

The seminorms P, Q and the constant M satisfy (1.2). Let $n \in \mathbb{N}$ and $(\bar{a}_1, \dots, \bar{a}_n) \in A^n$ with $Q(\bar{a}_j) \leq 1$ be given. It remains to show that $P(\bar{a}_1 \cdots \bar{a}_n) \leq M^n$. Each element \bar{a}_j can be written as a converging series

$$\bar{a}_j = \sum_{k=0}^{\infty} a_j^{(k)} \text{ with } a_j^{(k)} \in A_k.$$

Plugging this representation into P , we obtain the estimate

$$\begin{aligned} P(\bar{a}_1 \cdots \bar{a}_n) &= P \left(\sum_{N=0}^{\infty} \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha|=N}} a_1^{(\alpha_1)} \cdots a_n^{(\alpha_n)} \right) \\ &= \max_{0 \leq N \leq L} p_N \left(\sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha|=N}} a_1^{(\alpha_1)} \cdots a_n^{(\alpha_n)} \right) \leq \max_{0 \leq N \leq L} \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha|=N}} p_N \left(a_1^{(\alpha_1)} \cdots a_n^{(\alpha_n)} \right). \end{aligned}$$

For fixed $0 \leq N \leq L$ the number of summands in this sum is bounded from above by $(N + 1)^n \leq (L + 1)^n$ since for the entries of α there are at most $N + 1$ choices. We claim that each summand can be estimated as

$$(1.6) \quad p_N \left(a_1^{(\alpha_1)} \cdots a_n^{(\alpha_n)} \right) \leq \left(M_0^{2(L+1)} \right)^n .$$

If this is true then one easily deduces that $P(\bar{a}_1 \cdots \bar{a}_n) \leq (L + 1)^n \cdot (M_0^{2(L+1)})^n = M^n$ and the assertion follows.

Hence we have to prove that (1.6) holds. To this end, fix $0 \leq N \leq L$ and $\alpha \in \mathbb{N}_0^n$.

Case $N = 0$. — Then $\alpha = (0, \dots, 0)$ and we have

$$p_0 \left(a_1^{(0)} \cdots a_n^{(0)} \right) \leq q_0 \left(a_1^{(0)} \cdots a_n^{(0)} \right) \stackrel{(1.5)}{\leq} M_0^n \leq \left(M_0^{2(L+1)} \right)^n .$$

Case $N \geq 1$. — The product $a_1^{(\alpha_1)} \cdots a_n^{(\alpha_n)}$ may contain elements from the subalgebra A_0 and elements from the subspaces A_k with $k \geq 1$. Combining each element contained in A_0 with elements to the left or the right, we rewrite the product as

$$a_1^{(\alpha_1)} \cdots a_n^{(\alpha_n)} = b_1 \cdots b_r$$

for some $r \leq \min\{n, L\}$. Deleting all zeroes from α , we obtain a multi-index $w \in \mathbb{N}^r$. Now by construction $b_k \in A_{w_k}$ is a product $b_k = c_k \cdot d_k \cdot e_k$, where each $d_k \in A_{w_k}$ is one of the a_j and c_k and e_k are finite products of A_0 -factors in the original product.

Since each c_k is a product of at most n elements of A_0 , all of which have q_0^\sim -norm at most 1, we may apply (1.5) to obtain the estimate:

$$q_0(c_k) \leq M_0^n .$$

For the same reason, we have the corresponding estimate $q_0(e_k) \leq M_0^n$.

Combining these results, we derive

$$\begin{aligned} p_N \left(a_1^{(\alpha_1)} \cdots a_n^{(\alpha_n)} \right) &= p_N(b_1 \cdots b_r) \stackrel{(1.3)}{\leq} \prod_{k=1}^r q_{w_k}(b_k) = \prod_{k=1}^r q_{w_k}(c_k d_k e_k) \\ &\stackrel{(1.4)}{\leq} \prod_{k=1}^r \underbrace{q_0(c_k)}_{\leq M_0^n} \cdot \underbrace{q_{w_k}^\sim(d_k)}_{\leq 1} \cdot \underbrace{q_0(e_k)}_{\leq M_0^n} \leq \prod_{k=1}^r (M_0^2)^n = (M_0^{2r})^n \\ &\leq \left(M_0^{2(L+1)} \right)^n \end{aligned} \quad \square$$

2. Characters on graded connected Hopf algebras

In this section we construct Lie group structures on character groups of (graded and connected) Hopf algebras.

Notation 2.1. — From now on $\mathcal{H} = (\mathcal{H}, m_{\mathcal{H}}, u_{\mathcal{H}}, \Delta_{\mathcal{H}}, \varepsilon_{\mathcal{H}}, S_{\mathcal{H}})$ will denote a fixed Hopf algebra and we let B be a fixed commutative locally convex algebra. Using only the coalgebra structure of \mathcal{H} , we obtain the locally convex algebra

$$A := (\text{Hom}_{\mathbb{K}}(\mathcal{H}, B), \star) \quad (\text{see Lemma 1.4}).$$

Note that our framework generalises the special case $B = \mathbb{K}$ which is also an interesting case. For example, the Hopf algebra of rooted trees (see Example 4.6) is a connected, graded Hopf algebra and its group of \mathbb{K} -valued characters turns out to be the Butcher group from numerical analysis (cf. Example 4.7).

We will now consider groups of characters of Hopf algebras:

DEFINITION 2.2. — A linear map $\phi: \mathcal{H} \rightarrow B$ is called (B -valued) character if it is a homomorphism of unital algebras, i.e.

$$(2.1) \quad \phi(ab) = \phi(a)\phi(b) \text{ for all } a, b \in \mathcal{H} \text{ and } \phi(1_{\mathcal{H}}) = 1_B.$$

Another way of saying this is that ϕ is a character, if

$$(2.2) \quad \phi \circ m_{\mathcal{H}} = m_B \circ (\phi \otimes \phi) \text{ and } \phi(1_{\mathcal{H}}) = 1_B.$$

The set of characters is denoted by $G(\mathcal{H}, B)$.

LEMMA 2.3. — The set of characters $G(\mathcal{H}, B)$ is a closed subgroup of (A^{\times}, \star) . With the induced topology, $G(\mathcal{H}, B)$ is a topological group. Inversion in this group is given by the map $\phi \mapsto \phi \circ S_{\mathcal{H}}$ and the unit element is $1_A := u_B \circ \varepsilon_{\mathcal{H}}: \mathcal{H} \rightarrow B, x \mapsto \varepsilon_{\mathcal{H}}(x)1_B$.

Proof. — The fact that the characters of a Hopf algebra form a group with respect to the convolution product is well-known, see for example [26, Proposition II.4.1 3)]. Note that in loc. cit. this is only stated for a connected graded Hopf algebra although the proof does not use the grading at all.

The closedness of $G(\mathcal{H}, B)$ follows directly from Definition 2.2 and the fact that we use the topology of pointwise convergence on A . Continuity of the convolution product was shown in Lemma 1.4. Inversion is continuous as the precomposition with the antipode is obviously continuous with respect to pointwise convergence. □

Our goal in this section is to turn the group of characters into a Lie group. Hence, we need a modelling space for this group. This leads to the following definition:

DEFINITION 2.4. — *A linear map $\phi \in \text{Hom}_{\mathbb{K}}(\mathcal{H}, B)$ is called an infinitesimal character if*

$$(2.3) \quad \phi \circ m_{\mathcal{H}} = m_B \circ (\phi \otimes \varepsilon_{\mathcal{H}} + \varepsilon_{\mathcal{H}} \otimes \phi),$$

which means for $a, b \in \mathcal{H}$ that $\phi(ab) = \phi(a)\varepsilon_{\mathcal{H}}(b) + \varepsilon_{\mathcal{H}}(a)\phi(b)$.

We denote by $\mathfrak{g}(\mathcal{H}, B)$ the set of all infinitesimal characters.

LEMMA 2.5. — *The infinitesimal characters $\mathfrak{g}(\mathcal{H}, B)$ form a closed Lie subalgebra of $(A, [\cdot, \cdot])$, where $[\cdot, \cdot]$ is the commutator bracket of (A, \star) .*

Proof. — As for $G(\mathcal{H}, B)$, the closedness follows directly from the definition. The fact that the infinitesimal characters form a Lie subalgebra is well-known, see for example [26, Proposition II.4.2]. □

Remark 2.6. — From now on we assume for the rest of this section that the Hopf algebra \mathcal{H} is *graded and connected* (see Definition B.1). Thus by Lemma 1.7 the locally convex algebra $A = \text{Hom}_{\mathbb{K}}(\mathcal{H}, B)$ is densely graded.

Every infinitesimal character $\phi \in \mathfrak{g}(\mathcal{H}, B)$ maps $1_{\mathcal{H}}$ to $0_{\mathbb{K}} \cdot 1_B$ since $\phi(1_{\mathcal{H}} \cdot 1_{\mathcal{H}}) = \phi(1_{\mathcal{H}})\varepsilon_{\mathcal{H}}(1_{\mathcal{H}}) + \varepsilon_{\mathcal{H}}(1_{\mathcal{H}})\phi(1_{\mathcal{H}}) = 2\phi(1_{\mathcal{H}})$. Now $\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$ is assumed to be connected and we have $\mathcal{H}_0 = \mathbb{K}1_{\mathcal{H}}$, whence $\phi|_{\mathcal{H}_0} = 0$. Translating this to the densely graded algebra A , we observe $\mathfrak{g}(\mathcal{H}, B) \subseteq \mathcal{I}_A$ (the ideal of all elements which vanish on \mathcal{H}_0 , cf. Definition B.2). Similarly, a character maps $1_{\mathcal{H}}$ to 1_B by definition. Hence, $G(\mathcal{H}, B) \subseteq 1_A + \mathcal{I}_A$.

THEOREM 2.7. — *Let \mathcal{H} be an abstract graded connected Hopf algebra \mathcal{H} . For any commutative locally convex algebra B , the group $G(\mathcal{H}, B)$ of B -valued characters of \mathcal{H} is a (\mathbb{K} -analytic) Lie group.*

Furthermore, we observe the following properties

- (i) *The Lie algebra $\mathbf{L}(G(\mathcal{H}, B))$ of $G(\mathcal{H}, B)$ is the Lie algebra $\mathfrak{g}(\mathcal{H}, B)$ of infinitesimal characters with the commutator bracket $[\phi, \psi] = \phi \star \psi - \psi \star \phi$.*
- (ii) *$G(\mathcal{H}, B)$ is a BCH-Lie group which is exponential, i.e. the exponential map is a global \mathbb{K} -analytic diffeomorphism and is given by the exponential series.*
- (iii) *The model space of $G(\mathcal{H}, B)$ is a Fréchet space whenever \mathcal{H} is of countable dimension (e.g. a Hopf algebra of finite type) and B is a Fréchet space (e.g. B is finite-dimensional or a Banach algebra).*

In the special case that B is a weakly complete algebra, the modelling space $\mathfrak{g}(\mathcal{H}, B)$ is weakly complete as well.

Proof. — The locally convex algebra $A = \text{Hom}_{\mathbb{K}}(\mathcal{H}, B)$ is densely graded. Hence, by Lemma B.5, the exponential series converges on the closed vector subspace \mathcal{I}_A and defines a $C_{\mathbb{K}}^{\omega}$ -diffeomorphism:

$$\exp_A: \mathcal{I}_A \rightarrow 1_A + \mathcal{I}_A, \phi \mapsto \exp[\phi].$$

This implies that the closed vector subspace $\mathfrak{g}(\mathcal{H}, B)$ is mapped to a closed analytic submanifold of $1_A + \mathcal{I}_A \subseteq A$. By Lemma B.10, we obtain a commutative diagram

$$(2.4) \quad \begin{array}{ccc} \mathcal{I}_A & \xrightarrow{\exp_A} & 1_A + \mathcal{I}_A \\ \subseteq \uparrow & & \uparrow \subseteq \\ \mathfrak{g}(\mathcal{H}, B) & \xrightarrow{\text{Exp} := \exp_A|_{\mathfrak{g}(\mathcal{H}, B)}} & G(\mathcal{H}, B) \end{array}$$

which shows that the group $G(\mathcal{H}, B)$ is a closed analytic submanifold of $1_A + \mathcal{I}_A \subseteq A$.

The group multiplication is a restriction of the continuous bilinear map $\star: A \times A \rightarrow A$ to the analytic submanifold $G(\mathcal{H}, B) \times G(\mathcal{H}, B)$ and since the range space $G(\mathcal{H}, B) \subseteq A$ is closed this ensures that the restriction is analytic as well. Inversion in $G(\mathcal{H}, B)$ is composition with the antipode map $G(\mathcal{H}, B) \rightarrow G(\mathcal{H}, B), \phi \mapsto \phi \circ S$ (see Lemma 2.3). This is a restriction of the continuous linear (and hence analytic) map $A \rightarrow A, \phi \mapsto \phi \circ S$ to closed submanifolds in the domain and range and hence inversion is analytic as well. This shows that $G(\mathcal{H}, B)$ is an analytic Lie group.

- (i) The map $\text{Exp}: \mathfrak{g}(\mathcal{H}, B) \rightarrow G(\mathcal{H}, B), \phi \mapsto \exp_A(\phi)$ is an analytic diffeomorphism. Hence, we take the tangent map at point 0 in the diagram (2.4) to obtain:

$$\begin{array}{ccc} \mathcal{I}_A & \xrightarrow{T_0 \exp_A} & \mathcal{I}_A \\ \subseteq \uparrow & & \uparrow \subseteq \\ \mathfrak{g}(\mathcal{H}, B) & \xrightarrow{T_0 \text{Exp}} & T_{1_A}(G(\mathcal{H}, B)) \end{array}$$

By Lemma B.6(b) we know that $T_0 \exp_A = \text{id}_A$ which implies $T_0 \text{Exp} = \text{id}_{\mathfrak{g}(\mathcal{H}, B)}$. Hence as locally convex spaces $\mathbf{L}(G(\mathcal{H}, B)) = T_{1_A}(G(\mathcal{H}, B)) = \mathfrak{g}(\mathcal{H}, B)$. It remains to show that the Lie bracket on $T_{1_A}(G(\mathcal{H}, B))$ induced by the Lie group structure is the commutator bracket on $\mathfrak{g}(\mathcal{H}, B)$. This is what we will show in the following:

For a fixed $\phi \in G(\mathcal{H}, B)$, the inner automorphism

$$c_\phi: G(\mathcal{H}, B) \rightarrow G(\mathcal{H}, B), \psi \mapsto \phi \star \psi \star (\phi \circ S)$$

is a restriction of continuous linear map on the ambient space A . Taking the derivative of c_ϕ we obtain the usual formula for the adjoint action

$$\mathbf{L}(c_\phi) = \text{Ad}(\phi): \mathfrak{g}(\mathcal{H}, B) \rightarrow \mathfrak{g}(\mathcal{H}, B), \psi \mapsto \phi \star \psi \star (\phi \circ S).$$

For fixed $\psi \in \mathfrak{g}(\mathcal{H}, B)$, this formula can be considered as a restriction of a continuous polynomial in ϕ . We then define $\text{ad} := T_{1_A} \text{Ad}(\cdot).\psi$ (cf. [30, Example II.3.9]) which yields the Lie bracket of ϕ and ψ in $\mathfrak{g}(\mathcal{H}, B)$:

$$(2.5) \quad [\phi, \psi] = \text{ad}(\phi).\psi = \phi \star \psi + \psi \star (\phi \circ S).$$

Recall that the antipode is an anti-coalgebra morphism by [25, Proposition 1.3.1]. Thus the map $A \rightarrow A, f \mapsto f \circ S$ is an anti-algebra morphism which is continuous with respect to the topology of pointwise convergence. We conclude for an infinitesimal character ϕ that $\exp_A(\phi \circ S) = \exp_A(\phi) \circ S = (\exp_A(\phi))^{-1}$, i.e. $\exp_A(\phi \circ S)$ is the inverse of $\exp_A(\phi)$ with respect to \star . As \exp_A restricts to a bijection from $\mathfrak{g}(\mathcal{H}, B)$ to $G(\mathcal{H}, B)$ we derive from Lemma B.6(a) that $\phi \circ S = -\phi$. This implies together with (2.5) that $[\phi, \psi] = \phi \star \psi - \psi \star \phi$.

- (ii) We already know that the exponential series defines an analytic diffeomorphism $\text{Exp}: \mathfrak{g}(\mathcal{H}, B) \rightarrow G(\mathcal{H}, B)$ between Lie algebra and Lie group. It only remains to show that Exp is the exponential function of the Lie group. To this end let $\phi \in \mathfrak{g}(\mathcal{H}, B)$ be given. The analytic curve

$$\gamma_\phi: (\mathbb{R}, +) \rightarrow G(\mathcal{H}, B), t \mapsto \text{Exp}(t\phi)$$

is a group homomorphism by Lemma B.6(a) and we have $\gamma'_\phi(0) = \phi$ by Lemma B.6(b). By definition of a Lie group exponential function (see [30, Definition II.5.1]) we have $\exp(\phi) = \gamma_\phi(1) = \text{Exp}(\phi)$.

- (iii) Assume that the underlying vector space of the algebra \mathcal{H} is of countable dimension. The Lie algebra $\mathfrak{g}(\mathcal{H}, B)$ is a closed vector subspace of A , the latter is Fréchet by Lemma 1.6(b). Hence, $\mathfrak{g}(\mathcal{H}, B)$ is Fréchet as well.

If B is weakly complete then $\mathfrak{g}(\mathcal{H}, \mathbb{K})$ is a closed vector subspace of a weakly complete space by Lemma 1.6(a). Since closed subspaces of weakly complete spaces are again weakly complete (e.g. [18, Theorem A2.11]), the assertion follows. □

Remark 2.8.

- (a) Character groups of (graded and connected) Hopf algebras with values in locally convex algebras arise naturally in application in physics. Namely, in renormalisation of quantum field theories one considers characters of the Hopf algebra of Feynman graphs with values in algebras of polynomials or the algebra of germs of meromorphic functions (see e.g. [26, 10]).
- (b) In Theorem 2.7 we did not need to assume that the Hopf algebra is of finite type, i.e. that the steps of the grading are all finite-dimensional. Moreover, the Lie group structure does not depend on the grading on the Hopf algebra \mathcal{H} , i.e. if \mathcal{H} admits two connected gradings, then the Lie group structures induced by Theorem 2.7 coincide. Note that the Hopf algebra of Feynman graphs admits two connected gradings which are natural and of interest in physics (cf. [10, Proposition 1.30] and [31]).

Remark 2.9. — Consider the special case of a \mathbb{K} -Hopf algebra \mathcal{H} which is graded, connected and commutative. Recall that an *affine group scheme* is a covariant representable functor from the category $\mathbf{CommAlg}_{\mathbb{K}}$ of commutative algebras over \mathbb{K} to the category of groups. The functor

$$G(\mathcal{H}, \cdot): \mathbf{CommAlg}_{\mathbb{K}} \longrightarrow \mathbf{Groups},$$

which sends \mathbb{K} -algebras to their associated character groups and an algebra morphism $f: A \rightarrow B$ to $G(\mathcal{H}, f): G(\mathcal{H}, A) \rightarrow G(\mathcal{H}, B)$, $\phi \mapsto f \circ \phi$, is an affine group scheme (cf. e.g. [34, 1.4]). In this case, [29, Proposition 4.13] implies that the group $G(\mathcal{H}, B)$ is a projective limit (in \mathbf{Groups}) of linear affine groups (see also [34]). We will encounter a similar phenomenon (albeit in the case of an arbitrary Hopf algebra) in Section 5.

Assume further that the Hopf algebra \mathcal{H} is of finite type. Then the Milnor–Moore Theorem [29, Theorem 5.18] asserts that \mathcal{H} can be recovered as the graded dual of the universal enveloping algebra of a dense Lie subalgebra of $\mathfrak{g}(\mathcal{H}, \mathbb{K})$. Hence the affine group scheme is completely determined by the Lie algebra of infinitesimal characters (cf. [10, p. 76–77] for a full account).⁽²⁾ However, we remark that our results do not rely on the Milnor–Moore Theorem or techniques for affine group schemes.

⁽²⁾ In general, the Lie algebra encountered in the Milnor–Moore Theorem is not the full Lie algebra of infinitesimal characters (a fact which is somewhat opaque in [10]). This is due to the fact that one needs to take the restricted dual, i.e. the direct sum of the duals of the finite-dimensional steps of the grading. The Lie algebra obtained from the restricted dual is a dense subalgebra of the infinitesimal characters which are contained in the full dual, cf. Lemma 1.7.

We now turn to the question whether the Lie group constructed in Theorem 2.7 is a *regular* Lie group. To derive the regularity condition, we need to restrict our choice of target algebras to *Mackey complete* algebras. Let us note first the following result.

LEMMA 2.10. — *Let \mathcal{H} be a graded and connected Hopf algebra and B be a commutative locally convex algebra. Then $G(\mathcal{H}, B)$ and $\mathfrak{g}(\mathcal{H}, B)$ are contained in a closed subalgebra of $A = (\text{Hom}_{\mathbb{K}}(\mathcal{H}, B), \star)$ which is a densely graded continuous inverse algebra.*

Proof. — Consider the closed subalgebra $\Lambda := (\mathbb{K}1_{A_0}) \times \prod_{n \in \mathbb{N}} A_n$. By Lemma B.8 this algebra is densely graded (with respect to the grading induced by A) and a continuous inverse algebra. Note that $\mathcal{I}_A = \ker \pi_0$ and $\pi_0(1_A + \mathcal{I}_A) = 1_A$ imply that \mathcal{I}_A and $1_A + \mathcal{I}_A$ are contained in Λ . Thus Remark 2.6 shows that Λ contains $G(\mathcal{H}, B)$ and $\mathfrak{g}(\mathcal{H}, B)$. \square

THEOREM 2.11. — *Let \mathcal{H} be a graded and connected Hopf algebra and B be a Mackey complete locally convex algebra.*

- (a) *The Lie group $G(\mathcal{H}, B)$ is C^1 -regular.*
- (b) *If in addition, B is sequentially complete, then $G(\mathcal{H}, B)$ is even C^0 -regular.*

In both cases, the associated evolution map is even a \mathbb{K} -analytic map.

Proof. — We have seen in Lemma 2.10 that $G(\mathcal{H}, B)$ and $\mathfrak{g}(\mathcal{H}, B)$ are contained in a closed subalgebra Λ of $A = (\text{Hom}_{\mathbb{K}}(\mathcal{H}, B), \star)$. By construction Λ is a densely graded CIA. Note that Λ inherits all completeness properties from B since it is closed in A and this space inherits its completeness properties from B by Lemma 1.6. Combine Lemma 2.3 and Lemma B.7(a) to see that $G(\mathcal{H}, B)$ is a closed subgroup of the group of units of the CIA Λ . Theorem 2.7 and Lemma A.6 show that Λ^\times and $G(\mathcal{H}, B)$ are BCH–Lie groups and thus [30, Theorem IV.3.3] entails that $G(\mathcal{H}, B)$ is a closed Lie subgroup of Λ^\times .

Since $\Lambda_0 \cong \mathbb{K}$ (cf. Lemma B.8) is a commutative CIA, we deduce from Lemma A.8 that Λ_0 has the (GN)-property. Hence Lemma 1.10 shows that also Λ has the (GN)-property. We deduce from Lemma A.9 that the unit group Λ^\times is a C^1 -regular Lie group which is even C^0 -regular if B is sequentially complete. For the rest of this proof, fix $k \in \{0, 1\}$ and let B be sequentially complete if $k = 0$.

Our first goal is to show that $G(\mathcal{H}, B)$ is C^k -semiregular, i.e. that every C^k -curve into the Lie algebra $\mathfrak{g}(\mathcal{H}, B)$ admits a C^{k+1} -evolution in the character group (cf. [15]). To this end, let us recall the explicit regularity condition for the unit group.

Step 1: The initial value problem for C^k -regularity in the group Λ^\times . The Lie group Λ^\times is open in the CIA Λ . Take the canonical identification $T\Lambda^\times \cong \Lambda^\times \times \Lambda$. Recall that the group operation of Λ^\times is the restriction of the bilinear convolution $\star: \Lambda \times \Lambda \rightarrow \Lambda$. Consider for $\theta \in \Lambda^\times$ the left translation $\lambda_\theta(h) := \theta \star h, h \in \Lambda^\times$. Then the identification of the tangent spaces yields $T_{1_\Lambda} \lambda_\theta(X) = \theta \star X$ for all $X \in T_{1_\Lambda} \Lambda^\times = \Lambda$. Summing up, the initial value problem associated to C^k -regularity of Λ^\times becomes

$$(2.6) \quad \begin{cases} \eta'(t) = \eta(t) \cdot \gamma(t) = T_{1_\Lambda} \lambda_{\eta(t)}(\gamma(t)) = \eta(t) \star \gamma(t) & t \in [0, 1], \\ \eta(0) = 1_\Lambda \end{cases}$$

where $\gamma \in C^k([0, 1], \Lambda)$.

Fix $\gamma \in C^k([0, 1], \mathfrak{g}(\mathcal{H}, B))$. Now $\mathfrak{g}(\mathcal{H}, B) \subseteq \Lambda$ holds and Λ^\times is C^k -regular. Thus γ admits a C^{k+1} -evolution η in Λ^\times , i.e. $\eta: [0, 1] \rightarrow \Lambda^\times$ is of class C^{k+1} and solves (2.6) with respect to γ . We will now show that η takes its values in $G(\mathcal{H}, B)$.

Step 2: An auxiliary map to see that $G(\mathcal{H}, B)$ is C^k -semiregular. Consider

$$F: [0, 1] \times \mathcal{H} \times \mathcal{H} \rightarrow B, (t, x, y) \mapsto \eta(t)(xy) - \eta(t)(x)\eta(t)(y).$$

If F vanishes identically, the evolution η is multiplicative for each fixed t . Note that as η is a C^{k+1} -curve and Λ carries the topology of pointwise convergence, for each $(x, y) \in \mathcal{H} \times \mathcal{H}$ the map $F_{x,y} := F(\cdot, x, y): [0, 1] \rightarrow B$ is a C^{k+1} -map. Furthermore, $F(0, \cdot, \cdot) \equiv 0$ as $\eta(0) = 1_\Lambda \in G(\mathcal{H}, B)$. Thus for each pair $(x, y) \in \mathcal{H} \times \mathcal{H}$ the fundamental theorem of calculus yields

$$(2.7) \quad F(t, x, y) = F_{x,y}(t) = \int_0^t \frac{\partial}{\partial t} F_{x,y}(t) dt.$$

In the following formulae, abbreviate $\eta_t := \eta(t)$ and $\gamma_t := \gamma(t)$ to shorten the notation. To evaluate the expression (2.7) we compute the derivative of $F_{x,y}$ as

$$(2.8) \quad \begin{aligned} \frac{\partial}{\partial t} F_{x,y}(t) &= \frac{\partial}{\partial t} \eta_t(xy) - \left(\frac{\partial}{\partial t} \eta_t(x) \right) \eta_t(y) - \left(\frac{\partial}{\partial t} \eta_t(y) \right) \eta_t(x) \\ &\stackrel{(2.6)}{=} [\eta_t \star \gamma_t](xy) - [\eta_t \star \gamma_t](x)\eta_t(y) - [\eta_t \star \gamma_t](y)\eta_t(x). \end{aligned}$$

We use Sweedler’s sigma notation to write $\Delta_{\mathcal{H}}(x) = \sum_{(x)} x_1 \otimes x_2$ and $\Delta_{\mathcal{H}}(y) = \sum_{(y)} y_1 \otimes y_2$. As $\Delta_{\mathcal{H}}$ is an algebra homomorphism, the convolution

in (2.8) can then be rewritten as

$$(2.9) \quad \frac{\partial}{\partial t} F_{x,y}(t) = \sum_{(x)(y)} \eta_t(x_1 y_1) \gamma_t(x_2 y_2) - \sum_{(x)} \eta_t(x_1) \gamma_t(x_2) \eta_t(y) - \sum_{(y)} \eta_t(y_1) \gamma_t(y_2) \eta_t(x).$$

Recall that the curve γ takes its values in the infinitesimal characters, whence we have the identity $\gamma_t(ab) = \varepsilon(a)\gamma_t(b) + \varepsilon(b)\gamma_t(a)$. Plugging this into the first summand in (2.9) and using that η_t is linear for all t we obtain the identity

$$(2.10) \quad \begin{aligned} & \sum_{(x)} \sum_{(y)} \eta_t(x_1 y_1) \gamma_t(x_2 y_2) \\ &= \sum_{(x)} \sum_{(y)} (\eta_t(\varepsilon(x_2) x_1 y_1) \gamma_t(y_2) + \eta_t(x_1 (\varepsilon(y_2) y_1)) \gamma_t(x_2)) \\ &= \sum_{(y)} \eta_t(x y_1) \gamma_t(y_2) + \sum_{(x)} \eta_t(x_1 y) \gamma_t(x_2). \end{aligned}$$

As B is commutative inserting (2.10) into (2.9) yields

$$(2.11) \quad \begin{aligned} \frac{\partial}{\partial t} F_{x,y}(t) &= \sum_{(y)} (\eta_t(x y_1) - \eta_t(x) \eta_t(y_1)) \gamma_t(y_2) \\ &+ \sum_{(x)} (\eta_t(x_1 y) - \eta_t(x_1) \eta_t(y)) \gamma_t(x_2). \end{aligned}$$

Since η_t is linear for each fixed t it suffices to check that η_t is multiplicative for all pairs of elements in a set spanning the vector space \mathcal{H} . As \mathcal{H} is graded, the homogeneous elements span the vector space \mathcal{H} . We will now use the auxiliary mapping F and its partial derivative to prove that the evolution η_t is multiplicative on all homogeneous elements in \mathcal{H} (whence on all elements in \mathcal{H}).

Step 3: The evolution $\eta(t)$ is multiplicative on \mathcal{H}_0 and maps $1_{\mathcal{H}}$ to 1_B . The Hopf algebra \mathcal{H} is graded and connected, i.e. $\mathcal{H} = \bigoplus_{n \in \mathbb{N}_0} \mathcal{H}_n$ and $\mathcal{H}_0 = \mathbb{K}1_{\mathcal{H}}$. By construction this entails $\Delta_{\mathcal{H}}(\mathcal{H}_0) \subseteq \mathcal{H}_0 \otimes \mathcal{H}_0$ and the infinitesimal character γ vanishes on \mathcal{H}_0 . Thus for $x, y \in \mathcal{H}_0$ we have $\frac{\partial}{\partial t} F_{x,y}(t) = 0$ for all $t \in [0, 1]$ by (2.11). We conclude from (2.7) for $x, y \in \mathcal{H}_0$ the formula

$$\eta(t)(xy) - \eta(t)(x)\eta(t)(y) = F(t, x, y) = 0 \quad \forall t \in [0, 1].$$

Hence $\eta(t)(xy) = \eta(t)(x)\eta(t)(y)$ for all elements in degree 0. Specialising to $x = 1_{\mathcal{H}} = y$ we see that $\eta(t)(1_{\mathcal{H}})$ is an idempotent in the CIA B .

Furthermore, since $\eta(t) \in \Lambda^\times$, we have $1_B = 1_\Lambda(1_{\mathcal{H}}) = \eta(t) \star (\eta(t))^{-1}(1_{\mathcal{H}})$, whence $\eta(t)(1_{\mathcal{H}}) \in B^\times$ for all $t \in [0, 1]$. As B^\times is a group it contains only one idempotent, i.e. $\eta(t)(1_{\mathcal{H}}) = 1_B$.

Step 4: The evolution $\eta(t)$ is multiplicative for all homogeneous elements. As \mathcal{H} is connected, η_t is linear and $\eta_t(1_{\mathcal{H}}) = 1_B$ holds, we see that (2.11) vanishes if either x or y are contained in \mathcal{H}_0 . We conclude from (2.7) that

$$\eta_t(xy) = \eta_t(x)\eta_t(y) \quad \forall t \in [0, 1] \text{ if } x \text{ or } y \text{ are contained in degree } 0$$

Denote for a homogeneous element $x \in \mathcal{H}$ by $\deg x$ its degree with respect to the grading. To prove that η_t is multiplicative for elements of higher degree and $t \in [0, 1]$ we proceed by induction on the sum of the degrees of x and y . Having established multiplicativity of η_t if at least one element is in \mathcal{H}_0 , we have already dealt with the cases $\deg x + \deg y \in \{0, 1\}$.

Induction step for $\deg x + \deg y \geq 2$. — We assume that for homogeneous elements a, b with $\deg a + \deg b \leq \deg x + \deg y - 1$ the formula $\eta_t(ab) = \eta_t(a)\eta_t(b)$ holds.

Since \mathcal{H} is connected, for each $z \in \mathcal{H}_n$ with $n \geq 1$ the coproduct can be written as

$$\Delta_{\mathcal{H}}(z) = z \otimes 1_{\mathcal{H}} + 1_{\mathcal{H}} \otimes z + \tilde{\Delta}(z)$$

where $\tilde{\Delta}(z) = \sum_{(\tilde{z})} \tilde{z}_1 \otimes \tilde{z}_2 \in \varepsilon_{\mathcal{H}}^{-1}(0) \otimes \varepsilon_{\mathcal{H}}^{-1}(0)$ is the reduced coproduct. Note that by construction the elements \tilde{z}_1, \tilde{z}_2 are homogeneous of degree strictly larger than 0. Let us plug this formula for the coproduct into (2.11). We compute for the first sum in (2.11):

$$\begin{aligned} & \sum_{(y)} (\eta_t(xy_1) - \eta_t(x)\eta_t(y_1))\gamma_t(y_2) \\ &= (\eta_t(xy) - \eta_t(x)\eta_t(y)) \underbrace{\gamma_t(1_{\mathcal{H}})}_{=0} + \underbrace{(\eta_t(x1_{\mathcal{H}}) - \eta_t(x)\eta_t(1_{\mathcal{H}}))}_{=0} \gamma_t(y) \\ (2.12) \quad & \quad \quad \quad + \sum_{(\tilde{y})} (\eta_t(x\tilde{y}_1) - \eta_t(x)\eta_t(\tilde{y}_1))\gamma_t(\tilde{y}_2) \\ &= \sum_{(\tilde{y})} \underbrace{(\eta_t(x\tilde{y}_1) - \eta_t(x)\eta_t(\tilde{y}_1))}_{C_{x, \tilde{y}_1} :=} \gamma_t(\tilde{y}_2) \end{aligned}$$

By construction we have $\deg \tilde{y}_1, \deg \tilde{y}_2 \in [1, \deg y - 1]$. Then $\deg \tilde{y}_1 < \deg y$ implies $\deg x + \deg \tilde{y}_1 < \deg x + \deg y$ and thus C_{x, \tilde{y}_1} vanishes by the induction assumption.

As the two sums in (2.11) are symmetric, interchanging the roles of x and y together with an analogous argument as above shows that also the second

sum vanishes. Hence, arguing as in Step 3, we see that $\eta_t(xy) = \eta_t(x)\eta_t(y)$ holds for all homogeneous elements $x, y \in \mathcal{H}$.

In conclusion, the evolution $\eta: [0, 1] \rightarrow \Lambda^\times$ of γ takes its values in the closed subgroup $G(\mathcal{H}, B)$ and thus $G(\mathcal{H}, B)$ is C^k -semiregular.

Step 5: $G(\mathcal{H}, B)$ is C^k -regular. Let $\iota: \mathfrak{g}(\mathcal{H}, B) \rightarrow \Lambda$ be the canonical inclusion mapping. Consider $\iota_*: C^k([0, 1], \mathfrak{g}(\mathcal{H}, B)) \rightarrow C^k([0, 1], \Lambda)$, $c \mapsto \iota \circ c$. As ι is continuous and linear the map ι_* is continuous and linear by [16, Lemma 1.2], whence smooth and even \mathbb{K} -analytic.

Let $\text{evol}_\Lambda: C^k([0, 1], \Lambda) \rightarrow \Lambda^\times$ be the (smooth) evolution map of the C^k -regular Lie group Λ^\times . Then the map

$$\text{evol}: C^k([0, 1], \mathfrak{g}(\mathcal{H}, B)) \rightarrow \Lambda^\times, \text{evol}_{\Lambda^\times} \circ \iota_*$$

is \mathbb{K} -analytic by Theorem 1.9 and maps a C^k -curve in the Lie algebra of $G(\mathcal{H}, B)$ to its time 1 evolution. As the closed subgroup $G(\mathcal{H}, B)$ is C^k -semiregular by Step 4, evol factors through a \mathbb{K} -analytic map

$$\text{evol}_{G(\mathcal{H}, B)}: C^k([0, 1], \mathfrak{g}(\mathcal{H}, B)) \rightarrow G(\mathcal{H}, B).$$

Summing up, $G(\mathcal{H}, B)$ is C^k -regular and the evolution map is \mathbb{K} -analytic. □

3. Subgroups associated to Hopf ideals

So far, we were only able to turn the character group of a graded connected Hopf algebra \mathcal{H} into a Lie group. In this section, we will show that the character group of a quotient \mathcal{H}/\mathcal{J} can be regarded as a closed Lie subgroup of the character group of \mathcal{H} and thus carries a Lie group structure as well. See Remark 4.10 for examples of Hopf ideals and quotients arising in renormalisation of quantum field theories. It should be noted that this does not imply that the character group of every Hopf algebra can be endowed with a Lie group structure (see Example 4.11).

DEFINITION 3.1 (Hopf ideal). — *Let \mathcal{H} be a Hopf algebra. We say $\mathcal{J} \subseteq \mathcal{H}$ is a Hopf ideal if the subset \mathcal{J} is*

- (a) a two-sided (algebra) ideal,
- (b) a coideal, i.e. $\varepsilon(\mathcal{J}) = 0$ and $\Delta(\mathcal{J}) \subseteq \mathcal{J} \otimes \mathcal{H} + \mathcal{H} \otimes \mathcal{J}$ and
- (c) stable under the antipode, i.e. $S(\mathcal{J}) \subseteq \mathcal{J}$.

Let \mathcal{H} be a graded Hopf algebra. Then we call \mathcal{J} homogeneous if for all $c \in \mathcal{J}$ with $c = \sum_{i=1}^n c_i$ and each c_i homogeneous we have $c_i \in \mathcal{J}$ for $1 \leq i \leq n$.

DEFINITION 3.2 (Quotient Hopf algebra and the annihilator of an ideal).
 Let \mathcal{H} be a Hopf algebra and let $\mathcal{J} \subseteq \mathcal{H}$ be a Hopf ideal.

- (a) The quotient vector space \mathcal{H}/\mathcal{J} carries a natural Hopf algebra structure (see [33, Theorem 4.3.1.]). This structure turns the canonical quotient map $q: \mathcal{H} \rightarrow \mathcal{H}/\mathcal{J}$ into a morphism of Hopf algebras.
- (b) Let B be a locally convex algebra. Then the algebra $\text{Hom}_{\mathbb{K}}(\mathcal{H}/\mathcal{J}, B)$ is canonically isomorphic to the annihilator of \mathcal{J} :

$$\text{Ann}(\mathcal{J}, B) = \{ \phi \in \text{Hom}_{\mathbb{K}}(\mathcal{H}, B) \mid \phi(\mathcal{J}) = 0_B \}$$

which is a closed unital subalgebra of $\text{Hom}_{\mathbb{K}}(\mathcal{H}, B)$.

If \mathcal{H} is graded and the ideal \mathcal{J} is homogeneous then the grading of \mathcal{H} induces a natural grading on the quotient \mathcal{H}/\mathcal{J} . However, like the example of the universal enveloping algebra as a quotient of the tensor algebra (see Examples 4.1 and 4.2) shows, there are interesting ideals which occur naturally but are not homogeneous.

LEMMA 3.3. — Let \mathcal{J} be a Hopf ideal of the Hopf algebra \mathcal{H} with quotient mapping $q: \mathcal{H} \rightarrow \mathcal{H}/\mathcal{J}$. Let B be a commutative locally convex algebra. Then $\text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ is a closed subgroup of the topological group $G(\mathcal{H}, B)$. Furthermore it is isomorphic as a topological group to $G(\mathcal{H}/\mathcal{J}, B)$ via the following isomorphism:

$$q_*: G(\mathcal{H}/\mathcal{J}, B) \rightarrow \text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B), \phi \mapsto \phi \circ q.$$

Proof. — We first prove that $\text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ is a closed subgroup. It is stable under the group product and contains the unit because $\text{Ann}(\mathcal{J}, B)$ is a unital subalgebra. To see that it is stable under inversion recall from Lemma 2.3 that inversion in $G(\mathcal{H}, B)$ is given by precomposition with the antipode. Hence for $\phi \in \text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ we find $\phi^{-1}(\mathcal{J}) = \phi \circ S(\mathcal{J}) \subseteq \phi(\mathcal{J}) = 0$. Finally, $\text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ is closed as a subset of $G(\mathcal{H}, B)$ because $\text{Ann}(\mathcal{J}, B)$ is closed in A , and $G(\mathcal{H}, B)$ carries the subset topology.

The map $q_*: G(\mathcal{H}/\mathcal{J}, B) \rightarrow \text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ is clearly an isomorphism of groups. Continuity of q_* and $(q_*)^{-1}$ follows from the fact that we use pointwise convergence on all spaces. □

THEOREM 3.4. — Let \mathcal{H} be a graded connected Hopf algebra and $\mathcal{J} \subseteq \mathcal{H}$ be a (not necessarily homogeneous) Hopf ideal. Furthermore, we fix a commutative locally convex algebra B . Then

- (i) $G(\mathcal{H}/\mathcal{J}, B) \cong \text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B) \subseteq G(\mathcal{H}, B)$ is a closed Lie subgroup, and even an exponential BCH–Lie group.

- (ii) $\text{Ann}(\mathcal{J}, B) \cap \mathfrak{g}(\mathcal{H}, B) \subseteq \mathfrak{g}(\mathcal{H}, B)$ is a closed Lie subalgebra, and a BCH–Lie algebra.
- (iii) The map \exp restricts to a global \mathbb{K} -analytic diffeomorphism $\text{Ann}(\mathcal{J}, B) \cap \mathfrak{g}(\mathcal{H}, B) \rightarrow \text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$.
- (iv) If \mathcal{J} is homogeneous then the Lie group structure on $\text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ agrees with the one already obtained on $G(\mathcal{H}/\mathcal{J}, B)$.

Proof.

- (ii) $\text{Ann}(\mathcal{J}, B) \cap \mathfrak{g}(\mathcal{H}, B)$ is a Lie subalgebra because $[\phi, \psi] = \phi \star \psi - \psi \star \phi$ and $\text{Ann}(\mathcal{J}, B)$ is stable under convolution. It is closed because $\text{Ann}(\mathcal{J}, B)$ is closed in A . As a closed Lie subalgebra of a BCH–Lie algebra, $\text{Ann}(\mathcal{J}, B) \cap \mathfrak{g}(\mathcal{H}, B)$ is also a BCH–Lie algebra.
- (iii) From Theorem 2.7 we deduce that it suffices to prove that \exp restricts to a bijection $\text{Ann}(\mathcal{J}, B) \cap \mathfrak{g}(\mathcal{H}, B) \rightarrow \text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$.

Recall from Theorem 2.7 that the Lie group exponential map of $G(\mathcal{H}, \mathcal{J})$ is a global diffeomorphism which is given on $\mathfrak{g}(\mathcal{H}, B) \subseteq \mathcal{I}_A$ by a convergent power series. Hence \exp maps elements in a closed unital subalgebra into the subalgebra, i.e. $\exp(\phi) \in \text{Ann}(\mathcal{J}, B)$ for each $\phi \in \text{Ann}(\mathcal{J}, B) \cap \mathcal{I}_A$ and thus

$$\exp(\text{Ann}(\mathcal{J}, B) \cap \mathfrak{g}(\mathcal{H}, B)) \subseteq \text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B).$$

The logarithm \log on $G(\mathcal{H}, B) \subseteq (1_A + \mathcal{I}_A)$ is also given by a power series which converges on $\text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ and by the same argument, we obtain:

$$\log(\text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)) \subseteq \text{Ann}(\mathcal{J}, B) \cap \mathfrak{g}(\mathcal{H}, B).$$

In conclusion, \exp restricts to a bijection $\text{Ann}(\mathcal{J}, B) \cap \mathfrak{g}(\mathcal{H}, B) \rightarrow \text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ as desired.

- (i) It now follows from (ii), (iii) and [30, Theorem IV.3.3] that the group $\text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ is a closed Lie subgroup, and an exponential BCH–Lie group.
- (iv) We have already seen that $G(\mathcal{H}/\mathcal{J}, B)$ and $\text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ are isomorphic as topological groups and that $G(\mathcal{H}/\mathcal{J}, B)$ and also $\text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ are Baker–Campbell–Hausdorff–Lie groups. The Automatic Smoothness Theorem [30, Theorem IV.1.18] implies that we have in fact (\mathbb{K} -analytic) Lie group isomorphisms. \square

Note that when \mathcal{J} is homogeneous and B is Mackey complete, it follows from Theorem 3.4(iv) that the Lie subgroup $\text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ is again a regular Lie group. Namely, with Theorem 2.11 we obtain a regularity result:

COROLLARY 3.5. — *Let \mathcal{H} be a graded connected Hopf algebra and B be a commutative and Mackey complete locally convex algebra. Furthermore, let $\mathcal{J} \subseteq \mathcal{H}$ be a homogeneous Hopf ideal, then the Lie subgroup $\text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ is a C^1 -regular Lie group. If B is in addition sequentially complete, then $\text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ is C^0 -regular.*

Note that we have not established the regularity condition of $\text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ for non-homogeneous \mathcal{J} . Hence, we pose the following problem:

PROBLEM. — *Let \mathcal{H} be a graded and connected Hopf algebra and B be a commutative and Mackey complete locally convex algebra. Is the Lie group $\text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ C^k -regular (with $k \in \mathbb{N}_0 \cup \{\infty\}$) if \mathcal{J} is a non-homogeneous Hopf Ideal? It suffices to prove that $\text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ is a semiregular Lie subgroup of $G(\mathcal{H}, B)$. However, the idea used to prove Theorem 2.11 seems to carry over only to homogeneous ideals.*

In the special case that B is a weakly complete algebra (e.g. a finite-dimensional algebra) we deduce from Remark 5.8(c) the following corollary.

COROLLARY 3.6. — *Let \mathcal{H} be a connected graded Hopf algebra and B be a commutative and weakly complete locally convex algebra. Furthermore, let $\mathcal{J} \subseteq \mathcal{H}$ be a Hopf ideal. Then the Lie subgroup $\text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ is regular.*

4. (Counter-)examples for Lie groups arising as Hopf algebra characters

In this section we give several examples for Lie groups arising from the construction in the last section. In the literature many examples for graded and connected Hopf algebras are studied (we refer the reader to [5] and the references and examples therein). In particular, the so called combinatorial Hopf algebras provide a main class of examples for graded and connected Hopf algebras (see [24] for an overview). A prime example for a combinatorial Hopf algebra is the famous Connes–Kreimer Hopf algebra of rooted trees. Its character group corresponds to the Butcher group from numerical analysis and we discuss it as our main example below. Furthermore, we discuss several (counter-)examples to statements in Theorem 2.7 for characters of Hopf algebras which are *not* graded.

Tensor algebras and universal enveloping algebras

Example 4.1 (Tensor algebra). — Consider an abstract vector space \mathcal{V} . Then the tensor algebra

$$T(\mathcal{V}) := \bigoplus_{n=0}^{\infty} \mathcal{V}^{\otimes n} \text{ with } \mathcal{V}^{\otimes n} := \underbrace{\mathcal{V} \otimes \cdots \otimes \mathcal{V}}_n$$

has a natural structure of a graded connected Hopf algebra which we denote by $(T(\mathcal{V}), \otimes, u, \Delta, \varepsilon, S)$ where

$$\Delta(v) = 1 \otimes v + v \otimes 1 \text{ and } S(v) = -v \text{ for } v \in \mathcal{V}.$$

By Theorem 2.7 the character group of $T(\mathcal{V})$ is a BCH–Lie group. This group can be described explicitly.

Every linear functional on \mathcal{V} has a unique extension to a character of the Hopf algebra $T(\mathcal{V})$, yielding a bijection to the algebraic dual \mathcal{V}^* :

$$\Phi: G(T(\mathcal{V}), \mathbb{K}) \rightarrow \mathcal{V}^*, \phi \mapsto \phi|_{\mathcal{V}}.$$

We claim that Φ is a group isomorphism (where we view \mathcal{V}^* as a group with respect to its additive structure). Let $v \in \mathcal{V}$ and $\phi, \psi \in G(T(\mathcal{V}), \mathbb{K})$ be given, then

$$\begin{aligned} (\phi \star \psi)(v) &= m_{\mathbb{K}} \circ (\phi \otimes \psi)(\Delta(v)) = m_{\mathbb{K}}(\phi \otimes \psi)(v \otimes 1 + 1 \otimes v) \\ &= \phi(v)\psi(1) + \phi(1)\psi(v) = \phi(v) + \psi(v). \end{aligned}$$

Thus, the group $(G(T(\mathcal{V}), \mathbb{K}), \star)$ is isomorphic to the additive group $(\mathcal{V}^*, +)$. As \mathcal{V}^* is endowed with the weak*-topology, it is easy to check that Φ and Φ^{-1} are both continuous, hence Φ is also an isomorphism of topological groups. Since both Lie groups are known to be BCH–Lie groups, the the Automatic Smoothness Theorem [30, Theorem IV.1.18.] guarantees that Φ is also an analytic diffeomorphism, hence an isomorphism in the category of analytic Lie groups.

Example 4.2 (Universal enveloping algebra). — The *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$ of an abstract non-graded Lie algebra \mathfrak{g} can be constructed as a quotient of the connected graded Hopf algebra $T(\mathfrak{g})$ and hence, its character group is a Lie group by Theorem 3.4. Note that we cannot use Theorem 2.7 directly since in general $\mathcal{U}(\mathfrak{g})$ does not possess a natural connected grading (the grading of the tensor algebra induces only a filtration on $\mathcal{U}(\mathfrak{g})$, see [20, Theorem V.2.5]). If \mathfrak{g} is abelian, the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ coincides with the symmetric algebra $S(\mathfrak{g})$ (cf. [20, V.2 Example 1]). It is possible to give an explicit description of the group $G(\mathcal{U}(\mathfrak{g}), \mathbb{K})$:

Every character of $\phi \in G(\mathcal{U}(\mathfrak{g}), \mathbb{K})$ corresponds to a Lie algebra homomorphism $\phi|_{\mathfrak{g}}: \mathfrak{g} \rightarrow \mathbb{K}$ which in turn factors naturally through a linear map $\phi^{\sim}: \mathfrak{g}/(\mathfrak{g}') \rightarrow \mathbb{K}$, yielding a bijection

$$\Phi: G(\mathcal{U}(\mathfrak{g}), \mathbb{K}) \rightarrow ((\mathfrak{g}/\mathfrak{g}')^*, +), \phi \mapsto (\phi^{\sim}: v + \mathfrak{g}' \mapsto \phi(v)).$$

Like in the case of the tensor algebra and the symmetric algebra, one easily verifies that this is an isomorphism of topological groups and since both Lie groups $G(\mathcal{U}(\mathfrak{g}), \mathbb{K})$ and $((\mathfrak{g}/\mathfrak{g}')^*, +)$ are BCH–Lie groups, we use again the Automatic Smoothness Theorem [30, Theorem IV.1.18.] to see that they are also isomorphic as analytic Lie groups.

In particular, this shows that the character group of $\mathcal{U}(\mathfrak{g})$ only sees the abelian part of \mathfrak{g} and is therefore not very useful for studying the Lie algebra \mathfrak{g} .

Remark 4.3 (Universal enveloping algebra of a graded Lie algebra). — We remark that there is a notion of a *graded Lie algebra* which differs from the usual notion of a Lie algebra. Here the Lie bracket of the graded Lie algebra $\mathfrak{g} = \bigoplus_{p \in \mathbb{N}_0} \mathfrak{g}_p$ satisfies the Koszul sign convention, i.e. $[x, y] = (-1)^{pq+1}[y, x]$ for $x \in \mathfrak{g}_p$ and $y \in \mathfrak{g}_q$. Such graded Lie algebras also admit a universal enveloping algebra which inherits a grading from the Lie algebra grading. We refer to [29, 5.] for definitions and more details. (Note that the gradings encountered so far can be seen as even gradings with respect to the Koszul sign convention, cf. [10, Remark 1.24].) Then, the universal enveloping algebra becomes a graded and connected Hopf algebra.

Characters of the Hopf algebra of rooted trees

We examine the Hopf algebra of rooted trees which arises naturally in numerical analysis, renormalisation of quantum field theories and non-commutative geometry (see [4] for a survey). To construct the Hopf algebra, recall some notation first.

Notation 4.4.

- (1) A *rooted tree* is a connected *finite* graph without cycles with a distinguished node called the *root*. We identify rooted trees if they are graph isomorphic via a root preserving isomorphism.

Let \mathcal{T} be the set of all rooted trees and write $\mathcal{T}_0 := \mathcal{T} \cup \{\emptyset\}$ where \emptyset denotes the empty tree. The order $|\tau|$ of a tree $\tau \in \mathcal{T}_0$ is its number of vertices.

(2) An *ordered subtree*⁽³⁾ of $\tau \in \mathcal{T}_0$ is a subset s of all vertices of τ which satisfies

- (i) s is connected by edges of the tree τ ,
- (ii) if s is non-empty, it contains the root of τ .

The set of all ordered subtrees of τ is denoted by $\text{OST}(\tau)$. Associated to an ordered subtree $s \in \text{OST}(\tau)$ are the following objects:

- A forest (collection of rooted trees) denoted as $\tau \setminus s$. The forest $\tau \setminus s$ is obtained by removing the subtree s together with its adjacent edges from τ . We denote by $\#(\tau \setminus s)$ the number of trees in the forest $\tau \setminus s$.
- s_τ , the rooted tree given by vertices of s with root and edges induced by that of the tree τ .

Notation 4.5. — A *partition* p of a tree $\tau \in \mathcal{T}_0$ is a subset of edges of the tree. We denote by $\mathcal{P}(\tau)$ the set of all partitions of τ (including the empty partition). Associated to a partition $p \in \mathcal{P}(\tau)$ are the following objects

- A forest $\tau \setminus p$ which is defined as the forest that remains when the edges of p are removed from the tree τ . Write $\#(\tau \setminus p)$ for the number of trees in $\tau \setminus p$.
- The *skeleton* p_τ , is the tree obtained by contracting each tree of $\tau \setminus p$ to a single vertex and by re-establishing the edges of p .

Example 4.6 (The Connes–Kreimer Hopf algebra of rooted trees [7]). — Consider the algebra $\mathcal{H}_{CK}^{\mathbb{K}} := \mathbb{K}[\mathcal{T}]$ of polynomials which is generated by the trees in \mathcal{T} . We denote the structure maps of this algebra by m (multiplication) and u (unit). Indeed $\mathcal{H}_{CK}^{\mathbb{K}}$ becomes a bialgebra with the coproduct

$$\Delta: \mathcal{H}_{CK}^{\mathbb{K}} \rightarrow \mathcal{H}_{CK}^{\mathbb{K}} \otimes \mathcal{H}_{CK}^{\mathbb{K}}, \tau \mapsto \sum_{s \in \text{OST}(\tau)} (\tau \setminus s) \otimes s_\tau$$

and the counit $\varepsilon: \mathcal{H}_{CK}^{\mathbb{K}} \rightarrow \mathbb{K}$ defined via $\varepsilon(1_{\mathcal{H}_{CK}^{\mathbb{K}}}) = 1$ and $\varepsilon(\tau) = 0$ for all $\tau \in \mathcal{T}$. Furthermore, one can define an antipode S via

$$S: \mathcal{H}_{CK}^{\mathbb{K}} \rightarrow \mathcal{H}_{CK}^{\mathbb{K}}, \tau \mapsto \sum_{p \in \mathcal{P}(\tau)} (-1)^{|p_\tau|} (\tau \setminus p)$$

such that $\mathcal{H}_{CK}^{\mathbb{K}} = (\mathcal{H}_{CK}^{\mathbb{K}}, m, u, \Delta, \varepsilon, S)$ is a \mathbb{K} -Hopf algebra (see [6, 5.1] for more details and references).

⁽³⁾The term “ordered” refers to that the subtree remembers from which part of the tree it was cut.

Furthermore, the Hopf algebra $\mathcal{H}_{CK}^{\mathbb{K}}$ is graded as a Hopf algebra by the number of nodes grading: For each $n \in \mathbb{N}_0$ we define the n th degree via

$$\text{For } \tau_i \in \mathcal{T}, 1 \leq i \leq k, k \in \mathbb{N}_0 \quad \tau_1 \cdot \tau_2 \cdot \dots \cdot \tau_k \in (\mathcal{H}_{CK}^{\mathbb{K}})_n \iff \sum_{r=1}^k |\tau_r| = n$$

Clearly, $\mathcal{H}_{CK}^{\mathbb{K}}$ is connected with respect to the number of nodes grading and we identify $(\mathcal{H}_{CK}^{\mathbb{K}})_0$ with $\mathbb{K}\emptyset$. Thus we can apply Theorem 2.7 for every commutative CIA B to see that the B valued characters $G(\mathcal{H}_{CK}^{\mathbb{K}}, B)$ form an exponential BCH–Lie group.

It turns out that the \mathbb{K} -valued characters of the Connes–Kreimer Hopf algebra $\mathcal{H}_{CK}^{\mathbb{K}}$ can be identified with the Butcher group from numerical analysis.

Example 4.7 (The Butcher Group). — Let us recall the definition of the $(\mathbb{K}\text{-})$ Butcher group. As a set the $(\mathbb{K}\text{-})$ Butcher group is the set of tree maps

$$G_{\text{TM}}^{\mathbb{K}} := \{a: \mathcal{T}_0 \rightarrow \mathbb{K} \mid a(\emptyset) = 1\}$$

together with the group operation

$$a \cdot b(\tau) := \sum_{s \in \text{OST}(\tau)} b(s_\tau) a(\tau \setminus s) \quad \text{with} \quad a(\tau \setminus s) := \prod_{\theta \in \tau \setminus s} a(\theta).$$

In [3] we have constructed a Lie group structure for the $(\mathbb{K}\text{-})$ Butcher group as follows: Identify $G_{\text{TM}}^{\mathbb{K}}$ with the closed affine subspace $e + \mathbb{K}^{\mathcal{T}} = \{a \in \mathcal{T}_0 \mid a(\emptyset) = 1\}$ of $\mathbb{K}^{\mathcal{T}_0}$ with the topology of pointwise convergence. Then the subspace topology turns $G_{\text{TM}}^{\mathbb{K}}$ into a BCH–Lie group modelled on the Fréchet space $\mathbb{K}^{\mathcal{T}}$.

Note that the group operation of the Butcher group is closely related to the coproduct of the Hopf algebra of rooted trees. Indeed the obvious morphism

$$\Phi: G(\mathcal{H}_{CK}^{\mathbb{K}}, \mathbb{K}) \rightarrow G_{\text{TM}}^{\mathbb{K}}, \varphi \mapsto (\tau \mapsto \varphi(\tau))$$

is an isomorphism of (abstract) groups (see also [6, Eq. 38]). Moreover, it turns out that Φ is an isomorphism of Lie groups if we endow these groups with the Lie group structures discussed in Example 4.6 and Example 4.7.

LEMMA 4.8. — *The group isomorphism $\Phi: G(\mathcal{H}_{CK}^{\mathbb{K}}, \mathbb{K}) \rightarrow G_{\text{TM}}^{\mathbb{K}}$ is an isomorphism of \mathbb{K} -analytic Lie groups.*

Proof. — We already know that Φ is an isomorphism of abstract groups whose inverse is given by $\Phi^{-1}(a) = \varphi_a$ where φ_a is the algebra homomorphism defined via

$$\varphi_a(1_{\mathcal{H}_{CK}^{\mathbb{K}}}) = 1 \text{ and } \varphi_a(\tau) = a(\tau)$$

Recall from Lemma A.6 that the Lie groups $A^\times := \text{Hom}_K(\mathcal{H}_{CK}^\mathbb{K}, \mathbb{K})^\times$ and $G(\mathcal{H}_{CK}^\mathbb{K}, \mathbb{K}) \subseteq A^\times$ carry the subspace topology with respect to the topology of pointwise convergence on the space $\text{Hom}_K(\mathcal{H}_{CK}^\mathbb{K}, \mathbb{K})$. Furthermore, the topology on $G_{\text{TM}}^\mathbb{K}$ is subspace topology with respect to the topology of pointwise convergence on $\mathbb{K}^{\mathcal{T}^0}$. Hence a straight forward computation shows that Φ and Φ^{-1} are continuous, i.e. they are isomorphisms of topological groups. Since both $G(\mathcal{H}_{CK}^\mathbb{K}, \mathbb{K})$ and $G_{\text{TM}}^\mathbb{K}$ are BCH-Lie groups, the Automatic Smoothness Theorem [30, Theorem IV.1.18] asserts that Φ and Φ^{-1} are smooth (even real analytic). Thus Φ is an isomorphism of (\mathbb{K} -analytic) Lie groups. \square

So far we have seen that our Theorem 2.7 generalises the construction of the Lie group structure of the Butcher group from [3]. In loc. cit. we have also endowed the subgroup of symplectic tree maps with a Lie group structure. This can be seen as a special case of the construction given in Theorem 3.4 as the next example shows. Thus the results from [3] are completely subsumed in the more general framework developed in the present paper.

Example 4.9 (The subgroup of symplectic tree maps). — In [3, Theorem 5.8], it was shown that the subgroup of symplectic tree maps $S_{\text{TM}}^\mathbb{K} \subseteq G_{\text{TM}}^\mathbb{K}$ is a closed Lie subgroup of the Butcher group and that the subgroup is itself an exponential Baker–Campbell–Hausdorff Lie group.

The symplectic tree maps are defined as those $a \in G_{\text{TM}}^\mathbb{K}$ such that

$$a(\tau \circ v) + a(v \circ \tau) = a(\tau)a(v) \text{ for all } \tau, v \in \mathcal{T},$$

where $\tau \circ v$ denotes the rooted tree obtained by connecting τ and v with an edge between the roots of τ and v , and where the root of τ is the root of $\tau \circ v$ ⁽⁴⁾.

To cast [3, Theorem 5.8] in the context of Theorem 3.4, let $\mathcal{J} \subseteq \mathcal{H}_{CK}$ be the algebra ideal generated by the elements $\{\tau \circ v + v \circ \tau - \tau v\}_{\tau, v \in \mathcal{T}}$. Note that by definition of the Butcher product we have $|\tau \circ v| = |v \circ \tau| = |\tau v|$. Hence the generating elements of \mathcal{J} are homogeneous elements with respect to the number of nodes grading (see Example 4.6). Consequently \mathcal{J} is a homogeneous (algebra) ideal. It is possible to show that \mathcal{J} is also a co-ideal and stable under the antipode.

If $a \in S_{\text{TM}}^\mathbb{K}$, then $\varphi_a = \Phi^{-1}(a) \in \text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, \mathbb{K})$, since φ_a is an algebra morphism and zero on the generators of \mathcal{J} . The inverse implication also holds. Therefore, the restriction of Φ is a bijection between

⁽⁴⁾ This is known as the *Butcher product* and should not be confused with the product in the Butcher group (cf. [3, Remark 5.1]).

$S_{\text{TM}}^{\mathbb{K}}$ and $\text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, \mathbb{K})$. By Theorem 3.4, $\text{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, \mathbb{K}) \subseteq G(\mathcal{H}, \mathbb{K})$ is a closed Lie subgroup and an exponential BCH–Lie group. Using Lemma 4.8, we can show that this structure is isomorphic to the Lie group structure $S_{\text{TM}}^{\mathbb{K}} \subseteq G_{\text{TM}}^{\mathbb{K}}$ constructed in [3, Theorem 5.8].

In addition to the Lie group structure on $S_{\text{TM}}^{\mathbb{K}}$ already constructed in [3, Theorem 5.8] we derive from Corollary 3.5 that the Lie group $S_{\text{TM}}^{\mathbb{K}}$ is C^0 -regular.

Finally, let us mention certain character groups connected to Hopf ideals arising in the renormalisation of quantum field theories. Note that these Hopf ideals are not contained in the Hopf algebra of rooted trees, but instead in the larger Hopf algebra of Feynman graphs. The definition of these ideals and the ambient Hopf algebra is rather involved, whence we refer to the references given in the next remark for details.

Remark 4.10. — The Connes–Kreimer theory of renormalisation allows one to formulate renormalisation of quantum field theories in the language of (characters of) the Hopf algebra of Feynman graphs (cf. e.g. [7]). In [31] this idea is used to study the combinatorics of the renormalisation of gauge theories. Namely, loc. cit. proves that certain identities from physics, the so called “Ward–Takahashi” and “Slavnov–Taylor” identities, generate Hopf ideals in the Hopf algebra of Feynman graphs. Then in [32] character groups related to the quotient Hopf algebras associated to these Hopf ideals are studied.

Characters of Hopf algebras without connected grading

For the rest of this section let us investigate the case of a Hopf algebra \mathcal{H} without a connected grading. It will turn out that the results achieved for graded connected Hopf algebras (and quotients thereof) do not hold for Hopf algebras without grading. It should be noted however, that for scalar valued characters, we can show that they still form a so called pro-Lie group (see Theorem 5.6)

Let \mathcal{H} be a Hopf algebra without a grading then the dual space $A := \mathcal{H}^* = \text{Hom}_{\mathbb{K}}(\mathcal{H}, \mathbb{K})$ is still a locally convex algebra (see Lemma 1.4). However, in general, neither its unit group will be an open subset, nor will the group of characters $G(\mathcal{H}, \mathbb{K})$ be a Lie group modelled on a locally convex space. We give two examples for this behaviour.

Example 4.11. — Let Γ be an abstract group. Then the group algebra $\mathbb{K}\Gamma$ carries the structure of a cocommutative Hopf algebra by [20, III.3

Example 2]. The algebraic dual $A := (\mathbb{K}\Gamma)^*$ is isomorphic to the direct product algebra \mathbb{K}^Γ consisting of all functions on the group with pointwise multiplication. Its unit group $(\mathbb{K} \setminus \{0\})^\Gamma$ is a topological group (as a direct product of the topological group $\mathbb{K} \setminus \{0\}$ with itself). However, in general, it will not be open:

- (a) Let Γ be an infinite group, then the unit group $A^\times = (\mathbb{K}^\times)^\Gamma$ is not open in \mathbb{K}^Γ . Hence, A is not a CIA and in particular, the unit group A^\times does not inherit a Lie group structure from Lemma A.6 or Proposition 1.8, respectively.

Furthermore, the universal property of the group algebra $\mathbb{K}\Gamma$ (see [20, III.2 Example 2]) implies that a linear map $\phi: \mathbb{K}\Gamma \rightarrow \mathbb{K}$ is a character if and only if the map $\phi|_\Gamma: \Gamma \rightarrow \mathbb{K}^\times$ is a group homomorphism.

Thus, the group $G(\mathbb{K}\Gamma, \mathbb{K})$ is (as a topological group) isomorphic to the group of group homomorphisms from Γ to \mathbb{K}^\times with the topology of pointwise convergence.

- (b) Let $\Gamma = (\mathbb{Z}^{(I)}, +)$ be a free abelian of countable infinite rank. Then it is easy to see that $G(\mathbb{K}\Gamma, \mathbb{K})$ is topologically isomorphic to the infinite product $(\mathbb{K}^\times)^I$. This topological group is not locally contractible, hence it can not be (locally) homeomorphic to a topological vector space and thus cannot carry a locally convex manifold structure. In particular, Theorem 2.7 does not generalise to the character group of $\mathbb{K}\mathbb{Z}^{(I)}$.

If a non-graded Hopf algebra \mathcal{H} is finite-dimensional, then $A = \mathcal{H}^*$ is a finite-dimensional algebra and hence it is automatically a CIA. The group of characters will then be a (finite-dimensional) Lie group. However, by [34, 2.2] the characters are linearly independent whence this Lie group will always be finite. Hence there cannot be a bijection between the character group and the Lie algebra of infinitesimal characters (which in this case will be 0-dimensional). This shows that even when $G(\mathcal{H}, \mathbb{K})$ is a Lie group it may fail to be exponential. We consider a concrete example of this behaviour:

Example 4.12. — Take a *finite* non trivial group Γ and consider the finite-dimensional algebra $\mathcal{H} := \mathbb{K}^\Gamma$ of functions on the group with values in \mathbb{K} together with the pointwise operations. There is a suitable coalgebra structure and antipode (see [25, Example 1.5.2]) which turns algebra \mathbb{K}^Γ into a Hopf algebra. Furthermore, we can identify its dual with the group algebra $\mathbb{K}\Gamma$ of Γ (as [20, III. Example 3] shows).

With this identification we can identify $G(\mathcal{H}, \mathbb{K})$ with the group Γ (with the discrete topology). Obviously, there is no bijection between the group

of characters and the \mathbb{K} -Lie algebra of infinitesimal characters (which in this case is trivial).

5. Character groups as pro-Lie groups

In this section, we show that the group of characters of an abstract Hopf algebra (graded or not) can always be considered as a projective limit of finite-dimensional Lie groups, i.e. the group of characters is a *pro-Lie group*. The range space B has to be the ground field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ or a commutative weakly complete algebra (see Lemma 5.3).

The category of pro-Lie groups admits a very powerful structure theory which is similar to the theory of finite-dimensional Lie groups (see [18]). This should provide Lie theoretic tools to work with character groups, even in the examples where the methods of locally convex Lie groups do not apply (like Examples 4.11 and 4.12). It should be noted, however, that the concept of a pro-Lie group is of purely topological nature and involves no differential calculus. See [17] for an article dedicated to the problem of determining which pro-Lie groups do admit a locally convex differential structure and which do not.

DEFINITION 5.1 (pro-Lie group). — *A topological group G is called pro-Lie group if one of the following equivalent conditions holds:*

- (a) *G is isomorphic (as a topological group) to a closed subgroup of a product of finite-dimensional (real) Lie groups.*
- (b) *G is the projective limit of a directed system of finite-dimensional (real) Lie groups (taken in the category of topological groups)*
- (c) *G is complete and each identity neighbourhood contains a closed normal subgroup N such that G/N is a finite-dimensional (real) Lie group.*

The fact that these conditions are equivalent is surprisingly complicated to show and can be found in [14] or in [18, Theorem 3.39]. The class of pro-Lie groups contains all compact groups (see e.g. [19, Corollary 2.29]) and all connected locally compact groups (Yamabe's Theorem, see [35]). However, this does not imply that all pro-Lie groups are locally compact. In fact, the pro-Lie groups constructed in this paper will almost never be locally compact.

In absence of a differential structure we cannot define a Lie algebra as a tangent space. However, it is still possible to define a Lie functor.

DEFINITION 5.2 (The pro-Lie algebra of a pro-Lie group). — *Let G be a pro-Lie group. Consider the space $\mathcal{L}(G)$ of all continuous G -valued one-parameter subgroups, endowed with the compact-open topology.*

- (a) *The space $\mathcal{L}(G)$ is the projective limit of finite-dimensional Lie algebras and hence, carries a natural structure of a locally convex topological Lie algebra over \mathbb{R} (see [18, Definition 2.11]). As a topological vector space, $\mathcal{L}(G)$ is weakly complete, i.e. isomorphic to \mathbb{R}^I for an index set I (see also Definition C.1).*
- (b) *Assigning the pro-Lie algebra to a pro-Lie group yields a functor: Assign to a morphism of pro-Lie groups, i.e. a continuous group homomorphism $\phi: G \rightarrow H$, a morphism of topological real Lie algebras $\mathcal{L}(\phi): \mathcal{L}(G) \rightarrow \mathcal{L}(H)$, $\gamma \mapsto \gamma \circ \phi$. We thus obtain the so called pro-Lie functor.*

For more information on pro-Lie groups, pro-Lie algebras and the pro-Lie functor, see [18, Chapter 3].

Many pro-Lie groups arise as groups of invertible elements of topological algebras:

LEMMA 5.3 (Fundamental lemma of weakly complete algebras). — *For a topological \mathbb{K} -algebra A , the following are equivalent:*

- (a) *The underlying topological vector space A is (forgetting the multiplicative structure) weakly complete, i.e. isomorphic to \mathbb{K}^I for an index set I .*
- (b) *There is an abstract \mathbb{K} -coalgebra $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ such that A is isomorphic to $(\text{Hom}(\mathcal{C}, \mathbb{K}), \star)$.*
- (c) *The topological algebra A is the projective limit of a directed system of finite-dimensional \mathbb{K} -algebras (taken in the category of topological \mathbb{K} -algebras)*

A topological algebra with these properties is called weakly complete algebra.

Proof.

- (a) \Rightarrow (b) *The category of weakly complete topological vector spaces over \mathbb{K} and the category of abstract \mathbb{K} -vector spaces are dual. This implies that $A = \mathbb{K}^I$ is the algebraic dual space of the vector space $\mathcal{C} := \mathbb{K}^{(I)}$ of finite supported functions. The continuous multiplication $\mu_A: A \times A \rightarrow A$ dualises to an abstract comultiplication $\Delta_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{C}$. (see [27, Theorem 4.4] or Appendix C for details of this duality.)*

- (b) \Rightarrow (c) This is a direct consequence of the *Fundamental Theorem of Coalgebras* ([27, Theorem 4.12]) stating that \mathcal{C} is the directed union of finite-dimensional coalgebras. Dualising this, yields a projective limit of topological algebras.
- (c) \Rightarrow (a) The projective limit of finite-dimensional \mathbb{K} -vector spaces is always topologically isomorphic to \mathbb{K}^I . (see Appendix C) \square

PROPOSITION 5.4 (The group of units of a weakly complete algebra is a pro-Lie group). — *Let A be a weakly complete \mathbb{K} -algebra as in Lemma 5.3. Then the group of units A^\times is a pro-Lie group. Its pro-Lie algebra $\mathcal{L}(A)$ is (as a real Lie algebra) canonically isomorphic to $(A, [\cdot, \cdot])$ via the isomorphism*

$$A \rightarrow \mathcal{L}(A^\times), x \mapsto \gamma_x : (t \mapsto \exp(tx)),$$

where $\exp: A \rightarrow A^\times$ denotes the usual exponential series which converges on A .

Proof. — Let $A = \lim_{\leftarrow} A_\alpha$ with finite-dimensional \mathbb{K} -algebras A_α (by Lemma 5.3). Then the unit group is given by

$$A^\times = \lim_{\leftarrow} A_\alpha^\times$$

in the category of topological groups. Each group A_α^\times is a finite-dimensional (linear) real Lie group. Hence, A^\times is a pro-Lie group and in particular, inversion is continuous, which is not obvious for unit groups of topological algebras.

The exponential series converges on each algebra A_α and hence on the projective limit A . The correspondence between continuous one-parameter subgroups $\gamma \in \mathcal{L}(A^\times)$ and elements in A holds in each A_α and hence it holds on A . \square

Remark 5.5. — As Example 4.11 shows, the group of units A^\times of a weakly complete algebra need not be an open subset of A , nor will the exponential series be local homeomorphism around 0.

THEOREM 5.6 (The character group of a Hopf algebra is a pro-Lie group). — *Let \mathcal{H} be an abstract Hopf algebra and B be a commutative weakly complete \mathbb{K} -algebra (e.g. $B := \mathbb{K}$ or $B = \mathbb{K}[[X]]$). Then the group of B -valued characters $G(\mathcal{H}, B)$ endowed with the topology of pointwise convergence is pro-Lie group.*

Its pro-Lie algebra is isomorphic to the locally convex Lie algebra $\mathfrak{g}(\mathcal{H}, B)$ of infinitesimal characters via the canonical isomorphism

$$\mathfrak{g}(\mathcal{H}, B) \rightarrow \mathcal{L}(G(\mathcal{H}, B)), \phi \mapsto (t \mapsto \exp(t\phi)),$$

Remark 5.7. — The pro-Lie algebra $\mathcal{L}(G)$ of a pro-Lie group G is a priori only a real Lie algebra⁽⁵⁾ However, since we already know that $\mathfrak{g}(\mathcal{H}, B)$ is a complex Lie algebra if $\mathbb{K} = \mathbb{C}$ (Lemma 2.5), we may use the isomorphism given in the theorem above to endow the real Lie algebra structure with a complex one.

Proof of Theorem 5.6. — The space $A := \text{Hom}_{\mathbb{K}}(\mathcal{H}, B)$ is a topological algebra by Lemma 1.4. The underlying topological vector space is isomorphic to B^I by Lemma 1.6. Thus, A is a weakly complete algebra since B is weakly complete.

By Proposition 5.4, we may conclude that A^\times is a pro-Lie group. The group $G(\mathcal{H}, B)$ is a closed subgroup of this pro-Lie group by Lemma 2.3. From part (a) of Definition 5.1 it follows that closed subgroups of pro-Lie groups are pro-Lie groups. Hence, $G(\mathcal{H}, B)$ is a pro-Lie group.

It remains to show that the pro-Lie algebra $\mathcal{L}(G(\mathcal{H}, B))$ is isomorphic to $\mathfrak{g}(\mathcal{H}, B)$. Since $G(\mathcal{H}, B)$ is a closed subgroup of A^\times , every continuous 1-parameter-subgroup γ of $G(\mathcal{H}, B)$ is also a 1-parameter subgroup of A^\times and (by Proposition 5.4) of the form

$$\gamma_\phi: \mathbb{R} \rightarrow A^\times, t \mapsto \exp(t\phi)$$

for a unique element $\phi \in A$. It remains to show the following equivalence:

$$(\forall t \in \mathbb{R}: \exp(t\phi) \in G(\mathcal{H}, B)) \iff \phi \in \mathfrak{g}(\mathcal{H}, B).$$

At the end of the proof of Lemma B.10, there is a chain of equivalences. While the equivalence of the first line with the second uses the bijectivity of the exponential function which does not hold in the pro-Lie setting, the equivalence of the second line with all following lines hold by Remark B.11 also in this setting. Substituting $t\phi$ for ϕ , we obtain the following equivalence:

$$\begin{aligned} \forall t \in \mathbb{R}: \exp(t\phi) \in G(\mathcal{H}, B) \\ \iff \exp_{A_\otimes}(t\phi \circ m_{\mathcal{H}}) = \exp_{A_\otimes}(t(\phi \diamond 1_A + 1_A \diamond \phi)). \end{aligned}$$

Here the exponential function \exp_{A_\otimes} is taken in $A_\otimes := \text{Hom}_{\mathbb{K}}(\mathcal{H} \otimes \mathcal{H}, B)$.

This shows that the 1-parameter subgroups $\gamma_{\phi \circ m_{\mathcal{H}}}$ and $\gamma_{\phi \diamond 1_A + 1_A \diamond \phi}$ agree and by Proposition 5.4 applied to A_\otimes , we obtain that

$$\phi \circ m_{\mathcal{H}} = \phi \diamond 1_A + 1_A \diamond \phi$$

⁽⁵⁾This is due to the fact that the finite-dimensional real Lie groups form a full subcategory of the category of topological groups while the finite-dimensional complex Lie groups do not. In fact, there are infinitely many non-isomorphic complex Lie group structures (elliptic curves) on the torus $(\mathbb{R}/\mathbb{Z})^2$, inducing the same real Lie group structure.

which is equivalent to ϕ being an infinitesimal character. This finishes the proof. \square

Remark 5.8.

- (1) It is remarkable that Theorem 5.6 holds without any assumption on the given abstract Hopf algebra (in particular, we do not assume that it is graded or connected.)
- (2) For a weakly complete commutative algebra B (e.g. $B = \mathbb{K}[[X]]$ or B finite-dimensional) and a graded and connected Hopf algebra \mathcal{H} the results of Theorem 2.7 and Theorem 5.6 apply both to $G(\mathcal{H}, B)$.

In this case, the infinite-dimensional Lie group $G(\mathcal{H}, B)$ inherits additional structural properties as a projective limit of finite-dimensional Lie groups. In particular, the regularity of $G(\mathcal{H}, B)$ (cf. Theorem 2.11) then follows from [17].

For example, these observations apply to the Connes–Kreimer Hopf algebra $\mathcal{H}_{CK}^{\mathbb{K}}$ and the Butcher group $G(\mathcal{H}_{CK}^{\mathbb{K}}, \mathbb{K})$ (see Example 4.7). In fact, the structure as a pro-Lie group (implicitly) enabled some of the computations made in [3] to treat the Lie theoretic properties of the Butcher group.

Furthermore, in the Connes–Kreimer theory of renormalisation of quantum field theories, the pro-Lie group structure of certain character groups has been exploited. See e.g. [10, Proofs of Proposition 1.52 and Lemma 1.54] for explicit examples of results relying on this structure.

Appendix A. Locally convex differential calculus and manifolds

See [12, 21] for references on differential calculus in locally convex spaces.

DEFINITION A.1. — *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $r \in \mathbb{N} \cup \{\infty\}$ and E, F locally convex \mathbb{K} -vector spaces and $U \subseteq E$ open. Moreover we let $f: U \rightarrow F$ be a map. If it exists, we define for $(x, h) \in U \times E$ the directional derivative*

$$df(x, h) := D_h f(x) := \lim_{t \rightarrow 0} t^{-1}(f(x + th) - f(x)) \quad (\text{where } t \in \mathbb{K}^\times)$$

We say that f is $C_{\mathbb{K}}^r$ if the iterated directional derivatives

$$d^{(k)} f(x, y_1, \dots, y_k) := (D_{y_k} D_{y_{k-1}} \cdots D_{y_1} f)(x)$$

exist for all $k \in \mathbb{N}_0$ such that $k \leq r$, $x \in U$ and $y_1, \dots, y_k \in E$ and define continuous maps $d^{(k)}f: U \times E^k \rightarrow F$. If it is clear which \mathbb{K} is meant, we simply write C^r for $C_{\mathbb{K}}^r$. If f is $C_{\mathbb{K}}^\infty$ we say that f is smooth.⁽⁶⁾

DEFINITION A.2. — Let E, F be real locally convex spaces and $f: U \rightarrow F$ defined on an open subset U . We call f real analytic (or $C_{\mathbb{R}}^\omega$) if f extends to a $C_{\mathbb{C}}^\infty$ -map $\tilde{f}: \tilde{U} \rightarrow F_{\mathbb{C}}$ on an open neighbourhood \tilde{U} of U in the complexification $E_{\mathbb{C}}$.

For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$ the composition of $C_{\mathbb{K}}^r$ -maps (if possible) is again a $C_{\mathbb{K}}^r$ -map (cf. [12, Propositions 2.7 and 2.9]).

DEFINITION A.3. — Fix a Hausdorff topological space M and a locally convex space E over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. An (E) -manifold chart (U_κ, κ) on M is an open set $U_\kappa \subseteq M$ together with a homeomorphism $\kappa: U_\kappa \rightarrow V_\kappa \subseteq E$ onto an open subset of E . Two such charts are called C^r -compatible for $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$ if the change of charts map $\nu^{-1} \circ \kappa: \kappa(U_\kappa \cap U_\nu) \rightarrow \nu(U_\kappa \cap U_\nu)$ is a C^r -diffeomorphism. A $C_{\mathbb{K}}^r$ -atlas of M is a family of pairwise C^r -compatible manifold charts, whose domains cover M . Two such C^r -atlases are equivalent if their union is again a C^r -atlas.

A locally convex C^r -manifold M modelled on E is a Hausdorff space M with an equivalence class of C^r -atlases of (E) -manifold charts.

Direct products of locally convex manifolds, tangent spaces and tangent bundles as well as C^r -maps of manifolds may be defined as in the finite-dimensional setting (cf. [30]).

DEFINITION A.4. — A \mathbb{K} -analytic Lie group is a group G equipped with a $C_{\mathbb{K}}^\omega$ -manifold structure modelled on a locally convex space, such that the group operations are \mathbb{K} -analytic. For a Lie group G we denote by $\mathbf{L}(G)$ the associated Lie algebra.

DEFINITION A.5 (Baker–Campbell–Hausdorff (BCH-)Lie groups and Lie algebras).

- (a) A Lie algebra \mathfrak{g} is called Baker–Campbell–Hausdorff–Lie algebra (BCH–Lie algebra) if there exists an open 0-neighbourhood $U \subseteq \mathfrak{g}$ such that for $x, y \in U$ the BCH-series $\sum_{n=1}^\infty H_n(x, y)$ converges and defines an analytic function $U \times U \rightarrow \mathfrak{g}$. (The H_n are defined as $H_1(x, y) = x + y$, $H_2(x, y) = \frac{1}{2}[x, y]$ and for $n \geq 3$ by sums of iterated brackets, see [30, Definition IV.1.5].)

⁽⁶⁾ A map f is of class $C_{\mathbb{C}}^\infty$ if and only if it is complex analytic i.e., if f is continuous and locally given by a series of continuous homogeneous polynomials (cf. [1, Proposition 7.7]). We then also say that f is of class $C_{\mathbb{C}}^\omega$.

- (b) A locally convex Lie group G is called BCH–Lie group if it satisfies one of the following equivalent conditions (cf. [30, Theorem IV.1.8])
- (i) G is a \mathbb{K} -analytic Lie group whose Lie group exponential function is \mathbb{K} -analytic and a local diffeomorphism in 0.
 - (ii) The exponential map of G is a local diffeomorphism in 0 and $\mathbf{L}(G)$ is a BCH–Lie algebra.

LEMMA A.6 (Unit groups of CIAs are Lie groups [11, Theorem 5.6]). — Let A be a Mackey complete CIA. Then the group of units A^\times is a $C_{\mathbb{K}}^\omega$ -Lie group with the manifold structure endowed from the locally convex space A . The Lie algebra of the group A^\times is $(A, [\cdot, \cdot])$, where $[\cdot, \cdot]$ is the commutator bracket.

Moreover, the group A^\times is a Baker–Campbell–Hausdorff–Lie group, i.e. the exponential map is a local $C_{\mathbb{K}}^\omega$ -diffeomorphism around 0. This exponential map is given by the exponential series and its inverse is locally given by the logarithm series.

To establish regularity of unit groups of CIAs in [16] a sufficient criterion, called property “(*)” in *ibid.*, was introduced. We recall this now:

DEFINITION A.7 ((GN)-property). — A locally convex algebra A satisfies the (GN)-property, if for every continuous seminorm p on A , there exists a continuous seminorm q and $M \geq 0$ such that for all $n \in \mathbb{N}$, we have

$$\left\| \mu_A^{(n)} \right\|_{p,q} := \sup \{ p(\mu_A^{(n)}(x_1, \dots, x_n)) \mid q(x_i) \leq 1, 1 \leq i \leq n \} \leq M^n.$$

Here, $\mu_A^{(n)} : A \times \dots \times A \rightarrow A, (a_1, \dots, a_n) \mapsto a_1 \cdots a_n$.

LEMMA A.8 ([16]). — A locally convex algebra which is either a commutative continuous inverse algebra or locally m -convex has the (GN)-property.

LEMMA A.9 ([16, Proposition 3.4]). — Let A be a CIA with the (GN)-property.

- (a) If A is Mackey complete, then the Lie group A^\times is C^1 -regular.
- (b) If A is sequentially complete, then A^\times is C^0 -regular.

In both cases, the associated evolution map is even \mathbb{K} -analytic.

Appendix B. Graded algebra and characters

In this section we recall basic tools from abstract algebra. All results and definitions given in this appendix are well known (see for example [20, 25,

29, 33]. However, for the reader’s convenience we recall some details of the construction and proofs. We assume that the reader is familiar with the definition of algebras, coalgebras and Hopf algebras.

DEFINITION B.1 (Abstract gradings).

- (a) Let \mathcal{V} be an abstract \mathbb{K} -vector space. A family of vector subspaces $(\mathcal{V}_n)_{n \in \mathbb{N}_0}$ is called (abstract) \mathbb{N}_0 -grading (or just grading) of \mathcal{V} , if the canonical linear addition map

$$\Sigma: \bigoplus_{n \in \mathbb{N}_0} \mathcal{V}_n \rightarrow E, (v_n)_{n \in \mathbb{N}_0} \mapsto \sum_{n \in \mathbb{N}_0} v_n$$

is an isomorphism of \mathbb{K} -vector spaces, i.e. is bijective.

- (b) By a graded algebra, we mean an abstract \mathbb{K} -algebra \mathcal{A} , together with an abstract grading $(\mathcal{A}_n)_{n \in \mathbb{N}_0}$ of the underlying vector space \mathcal{A} such that

$$1_{\mathcal{A}} \in \mathcal{A}_0 \text{ and } \mathcal{A}_n \cdot \mathcal{A}_m \subseteq \mathcal{A}_{n+m} \text{ for all } n, m \in \mathbb{N}_0.$$

This implies in particular that \mathcal{A}_0 is a unital subalgebra and that the projection $\pi_0: \mathcal{A} \rightarrow \mathcal{A}_0$ onto \mathcal{A}_0 is an algebra homomorphism.

- (c) A graded coalgebra is an abstract coalgebra $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$, together with an abstract grading $(\mathcal{C}_n)_{n \in \mathbb{N}_0}$ of the underlying vector space \mathcal{C} such that for all $n \in \mathbb{N}_0$

$$\Delta_{\mathcal{C}}(\mathcal{C}_n) \subseteq \bigoplus_{i+j=n} \mathcal{C}_i \otimes \mathcal{C}_j \text{ and } \bigoplus_{n \geq 1} \mathcal{C}_n \subseteq \ker(\varepsilon_{\mathcal{C}}).$$

A graded coalgebra is called connected if \mathcal{C}_0 is one dimensional.

- (d) An abstract Hopf algebra $(\mathcal{H}, m_{\mathcal{H}}, u_{\mathcal{H}}, \Delta_{\mathcal{H}}, \varepsilon_{\mathcal{H}})$ is called graded Hopf algebra if there is an abstract grading $(\mathcal{H}_n)_{n \in \mathbb{N}_0}$ of the underlying vector space \mathcal{H} which is an algebra grading and a coalgebra grading at the same time. We call a graded Hopf algebra connected if \mathcal{H}_0 is one dimensional. An element $a \in \mathcal{A}_n$ (with $n \in \mathbb{N}_0$) is called homogeneous (of degree n).

DEFINITION B.2 (Dense Gradings).

- (a) Let E be a locally convex space. A family of vector subspaces $(E_n)_{n \in \mathbb{N}_0}$ is called a dense grading of E , if the canonical linear summation map

$$\Sigma: \bigoplus_{n \in \mathbb{N}_0} E_n \rightarrow E, (x_n)_{n \in \mathbb{N}_0} \mapsto \sum_{n \in \mathbb{N}_0} x_n$$

extends (uniquely) to an isomorphism of locally convex spaces

$$\bar{\Sigma}: \prod_{n \in \mathbb{N}_0} E_n \rightarrow E.$$

Notice that Each E_n is closed in E and E is a Fréchet space if and only if each E_n is a Fréchet space.

- (b) By a densely graded locally convex algebra, we mean a locally convex algebra A , together with a dense grading of the underlying locally convex space A such that

$$A_n \cdot A_m \subseteq A_{n+m} \text{ for all } n, m \in \mathbb{N}_0 \text{ and } 1_A \in A_0.$$

This implies that A_0 is a closed unital subalgebra and that the projection $\pi_0: A \rightarrow A_0$ is a continuous algebra homomorphism.

Denote the kernel of π_0 by $\mathcal{I}_A := \ker(\pi_0) = \overline{\bigoplus_{n \geq 1} A_n}$. The kernel \mathcal{I}_A is a closed ideal. Each element in A has a unique decomposition $a = a_0 + b$ with $a_0 \in A_0$ and $b \in \mathcal{I}_A$.

For the reader’s convenience we summarise important examples of topological algebras and some of their properties discussed in this appendix in the following chart. Here the arrows indicate that a given property is stronger than another (normal arrow) or that an example possesses the property (dashed arrow), respectively.

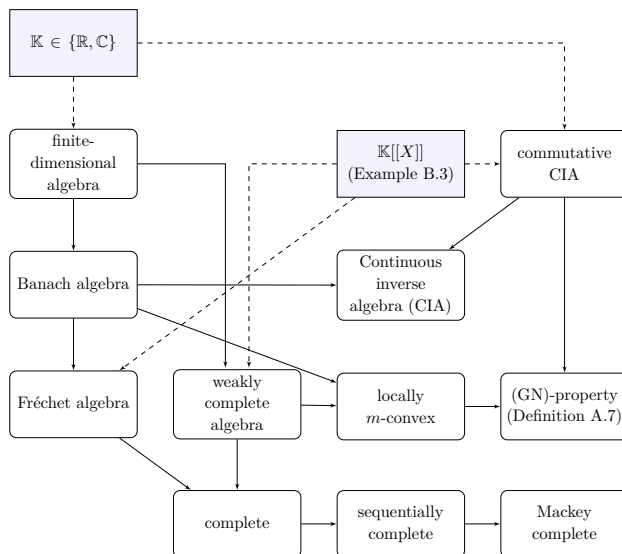


Figure B.1. Important properties and examples of locally convex algebras.

Note that *densely graded algebras* (see Definition B.2(b)) are not included in Diagram B.1 as every locally convex algebra A admits the trivial grading $A_0 = A$ and $A_n = 0$ for $n \geq 1$.

Example B.3 (Formal power series). — Let $\mathbb{K}[[X]]$ be the algebra of formal power series in one variable. We give this algebra the topology of pointwise convergence of coefficients, i.e. the initial topology with respect to the coordinate maps:

$$\kappa_n : \mathbb{K}[[X]] \rightarrow \mathbb{K}, \sum_{k=0}^{\infty} c_k X^k \mapsto c_n.$$

As a topological vector space, the algebra $\mathbb{K}[[X]]$ is isomorphic to the Fréchet space $\mathbb{K}^{\mathbb{N}_0} = \prod_{k=0}^{\infty} \mathbb{K}$. We see that $\mathbb{K}[[X]]$ is a densely graded algebra with respect to the grading $(\mathbb{K}X^n = \{cX^n \mid c \in \mathbb{K}\})_{n \in \mathbb{N}_0}$. Note that $\mathbb{K}[[X]]$ is a CIA by Lemma B.7(b).

In the following, we will identify a densely graded locally convex algebra A with the product space $\prod_{n=0}^{\infty} A_n$ such that each element $a \in A$ is a tuple $(a_n)_{n \in \mathbb{N}_0}$.

LEMMA B.4 (Functional calculus for densely graded algebras). — *Let $A = \prod_{n=0}^{\infty} A_n$ be a densely graded locally convex algebra with $\mathcal{I}_A = \ker \pi_0$. Then there exists a unique continuous map*

$$\mathbb{K}[[X]] \times \mathcal{I}_A \rightarrow A, (f, a) \mapsto f[a]$$

such that for all $a \in \mathcal{I}_A$, we have $X[a] = a$ and the map

$$\mathbb{K}[[X]] \rightarrow A, f \mapsto f[a]$$

is a morphism of unital algebras. If $f = \sum_{k=0}^{\infty} c_k X^k$ and $a = (a_n)_{n \in \mathbb{N}_0}$ with $a_0 = 0$ are given, the following explicit formula holds:

$$(B.1) \quad f[a] = \left(\sum_{k=0}^n c_k \sum_{\substack{\alpha \in \mathbb{N}^k \\ |\alpha|=n}} a_{\alpha_1} \cdot \dots \cdot a_{\alpha_k} \right)_{n \in \mathbb{N}_0}.$$

Furthermore, the map $\mathbb{K}[[X]] \times \mathcal{I}_A \rightarrow A, (f, a) \mapsto f[a]$ is a $C_{\mathbb{K}}^{\omega}$ -map (cf. Appendix A).

Proof. — First of all, the explicit formula is well-defined and continuous on $\mathbb{K}[[X]] \times \mathcal{I}_A$ since every component is a continuous polynomial in finitely many evaluations of the spaces $\mathbb{K}[[X]]$ and \mathcal{I}_A . As A is densely graded and thus isomorphic to the locally convex product of the spaces $A_n, n \in \mathbb{N}_0$, this implies that the map is continuous. In fact, this already implies that the

map is $C_{\mathbb{K}}^\omega$ for $\mathbb{K} = \mathbb{C}$. For $\mathbb{K} = \mathbb{R}$ one has to be a little bit more careful since there exist maps into products which are not $C_{\mathbb{R}}^\omega$ although every component is $C_{\mathbb{R}}^\omega$ (cf. [13, Example 3.1]). However, if each component is a continuous polynomial, the real case follows from the complex case as real polynomials complexify to complex polynomials by [2, Theorem 3].

Let $a \in \mathcal{I}_A$ be a fixed element. It remains to show that $\mathbb{K}[[X]] \rightarrow A, f \mapsto f[a]$ is an algebra homomorphism. By construction it is clear that $f \mapsto f[a]$ is linear and maps X^0 to 1_A and X^1 to a . Since f is continuous and linear, it suffices to establish the multiplicativity for series of the form X^N , i.e. it suffices to prove that

$$(X^N[a]) \cdot (X^M[a]) = X^{N+M}[a]$$

which follows from the easily verified fact that $X^N[a] = a^N$.

To establish uniqueness of the map obtained, we remark the following: A continuous map on $\mathbb{K}[[X]]$ is determined by its values on the dense space of polynomials $\mathbb{K}[X]$, and an algebra homomorphism on $\mathbb{K}[X]$ is determined by its value on the generator X . Here this value has to be a . \square

LEMMA B.5 (Exponential and logarithm). — *Consider the formal series*

$$\exp(X) := \sum_{k=0}^{\infty} \frac{X^k}{k!} \quad \text{and} \quad \log(1 + X) := \sum_{k=1}^{\infty} (-1)^{k+1} \frac{X^k}{k}.$$

Let A be a densely graded locally convex algebra. The exponential function restricted to the closed vector subspace \mathcal{I}_A

$$\exp_A : \mathcal{I}_A \rightarrow 1_A + \mathcal{I}_A, a \mapsto \exp[a]$$

is a $C_{\mathbb{K}}^\omega$ -diffeomorphism with inverse

$$\log_A : 1_A + \mathcal{I}_A \rightarrow \mathcal{I}_A, (1_A + a) \mapsto \log(1 + X)[a].$$

Proof. — The maps are $C_{\mathbb{K}}^\omega$ by Lemma B.4. As formal power series $E(X) := \exp(X) - 1$ and $L(X) := \log(X + 1)$ are inverses with respect to composition of power series. Note that $\mathbb{K}[[X]]$ is a densely graded locally convex algebra with $E(X), L(X) \in \mathcal{I}_{\mathbb{K}[[X]]}$. Apply functional calculus (Lemma B.4) to $\mathbb{K}[[X]]$ and $E(X), L(X)$. This yields for $a \in \mathcal{I}_A$ the identity

$$E[L[a]] = (E \circ L)[a] = X[a] = a.$$

Similarly $L[E[a]] = a$, whence \exp_A and \log_A are mutually inverse. \square

LEMMA B.6. — *Let $A = \prod_{n \in \mathbb{N}_0} A_n$ be a densely graded locally convex algebra with exponential map \exp_A . Then the following assertions hold:*

- (a) *For $a, b \in \mathcal{I}_A$ with $ab = ba$ we have $\exp_A(a + b) = \exp_A(a) \exp_A(b)$.*
- (b) *The derivative of \exp_A at 0 is $T_0 \exp_A = \text{id}_{\mathcal{I}_A}$.*

Proof. — By construction of \exp_A we derive from Lemma B.4 for $x \in \mathcal{I}_A$ the formula $\exp_A(x) = \lim_{N \rightarrow \infty} \sum_{k=0}^N \frac{x^k}{k!}$. The algebra A is densely graded and we have for every $n \in \mathbb{N}_0$ a continuous linear projection $\pi_n : A \rightarrow A_n$. By definition we have for $x \in \mathcal{I}_A$ that $\pi_0(x) = 0$. Hence, the definition of a densely graded algebra implies for $x, y \in \mathcal{I}_A$ that $\pi_j(x^k y^l) = 0$ if $k + l > j$.

(a) As a and b commute we can compute as follows:

$$\begin{aligned}
 \text{(B.2)} \quad \exp_A(a) \exp_A(b) &= \lim_{N \rightarrow \infty} \left(\sum_{k+l \leq N} \frac{a^k b^l}{k!l!} + \underbrace{\sum_{\substack{l+k > N, \\ l, k \leq N}} \frac{a^k b^l}{k!l!}}_{=: S_N} \right) \\
 &= \lim_{N \rightarrow \infty} \left(\sum_{n=0}^N \sum_{k+l=n} \frac{a^k b^l}{k!l!} + S_N \right) = \lim_{N \rightarrow \infty} \left(\sum_{n=0}^N \frac{(a+b)^n}{n!} + S_N \right)
 \end{aligned}$$

The first summand in the lower row converges to $\exp_A(a+b)$.

Now the definition of S_N shows that $\pi_j(S_N) = 0$ if $N \geq j$. Apply the continuous map π_j to both sides of (B.2) for $j \in \mathbb{N}_0$ to derive

$$\pi_j(\exp_A(a) \exp_A(b)) = \lim_{N \rightarrow \infty} \left(\pi_j \left(\sum_{n=0}^N \frac{(a+b)^n}{n!} \right) + \pi_j(S_N) \right)$$

On the right hand side the second term vanishes if $N > j$. Thus in passing to the limit we obtain $\pi_j(\exp_A(a) \exp_A(b)) = \pi_j(\exp_A(a+b))$ for all $j \in \mathbb{N}_0$.

(b) The image of \exp_A is the affine subspace $1_A + \mathcal{I}_A$ whose tangent space (as a submanifold of A) is \mathcal{I}_A . We can thus identify \exp_A with $F : \mathcal{I}_A \rightarrow A, a \mapsto \exp[a]$ to compute $T_0 \exp_A$ as $dF(0; \cdot)$. The projections $\pi_n, n \in \mathbb{N}_0$ are continuous linear, whence it suffices to compute $d\pi_n \circ F(0; \cdot) = \pi_n dF(0; \cdot)$ for all $n \in \mathbb{N}_0$. Now for $n \in \mathbb{N}_0$ and $a \in \mathcal{I}_A$ compute the derivative in 0:

$$\begin{aligned}
 \pi_n dF(0; a) &= d\pi_n \circ F(0; a) = \lim_{t \rightarrow 0} t^{-1} (\pi_n \circ F(ta) - \pi_n \circ F(0)) \\
 &\stackrel{\text{(B.1)}}{=} \lim_{t \rightarrow 0} \sum_{k=1}^n \frac{1}{k!} \sum_{\substack{\alpha \in \mathbb{N}^k \\ |\alpha|=n}} t^{k-1} a_{\alpha_1} \cdots a_{\alpha_k} = a_n = \pi_n \circ \text{id}_{\mathcal{I}_A}(a) \quad \square
 \end{aligned}$$

LEMMA B.7 (Unit groups of densely graded algebras). — Let $A = \prod_{n=0}^\infty A_n$ be a densely graded locally convex algebra.

(a) An element $a \in A$ with decomposition $a = a_0 + b$ is invertible in A if and only a_0 is invertible in A_0 .

- (b) *The algebra A is a CIA if and only if A_0 is a CIA. In particular, if $A_0 = \mathbb{K}$, then A is a CIA.*

Proof.

- (a) The map $\pi_0: A \rightarrow A_0$ is an algebra homomorphism. This implies that invertible elements $a \in A$ are mapped to invertible elements $a_0 \in A_0$. For the converse, take an element $a \in A$ with decomposition $a = a_0 + b$ with $b \in \mathcal{I}_A$ and a_0 is invertible in A_0 . Then we may multiply by a_0^{-1} from the left and obtain the equality

$$a_0^{-1}a = 1 + a_0^{-1}b.$$

This shows that a is invertible if we are able to prove that $1 + a_0^{-1}b$ is invertible. Apply the formal power series

$$(1 - X)^{-1} = \sum_{k=0}^{\infty} X^k$$

to the element $-a_0^{-1}b \in \mathcal{I}_A$ and obtain the inverse of $1 + a_0^{-1}b$.

- (b) We have seen in part (a) that the units in the algebra A satisfy

$$A^\times = \pi_0^{-1}(A_0^\times)$$

and hence one of the unit groups is open if and only if the other one is open. It remains to establish that continuity of inversion in A_0^\times implies continuity of inversion in A^\times . In part (a) we have seen that inversion of $a = a_0 + b$ in A is given by

$$a^{-1} = ((1 - X)^{-1}) [-a_0^{-1}b] \cdot a_0^{-1}$$

So, the continuity of inversion in A follows from the continuity of inversion in A_0 and the continuity of the functional calculus (Lemma B.4). □

LEMMA B.8. — *Let $A = \prod_{n \in \mathbb{N}_0} A_n$ be a densely graded locally convex algebra over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then $\Lambda := \mathbb{K}1_{A_0} \times \prod_{n \in \mathbb{N}} A_n \subseteq A$ is a closed subalgebra of A . Furthermore, Λ is densely graded with respect to the grading induced by $(A_n)_{n \in \mathbb{N}_0}$ and Λ is a CIA.*

Proof. — Clearly Λ is a subalgebra of A and the subspace topology turns this subalgebra into a locally convex algebra over \mathbb{K} . By definition Λ is the product of the (closed) subspaces $(\Lambda_n)_{n \in \mathbb{N}_0}$. Hence Λ is a closed subalgebra of A with dense grading $(\Lambda_n)_{n \in \mathbb{N}_0}$. Finally we have the isomorphism of locally convex algebras $\Lambda_0 = \mathbb{K}1_{A_0} \cong \mathbb{K}$. Hence Λ_0 is a CIA and thus Λ is a CIA by Lemma B.7(b). □

Auxiliary results concerning characters of Hopf algebras

Fix for this section a \mathbb{K} -Hopf algebra $\mathcal{H} = (\mathcal{H}, m_{\mathcal{H}}, u_{\mathcal{H}}, \Delta_{\mathcal{H}}, \varepsilon_{\mathcal{H}}, S_{\mathcal{H}})$ and a commutative locally convex algebra B . Furthermore, we assume that \mathcal{H} is graded and connected, i.e. $\mathcal{H} = \bigoplus_{n \in \mathbb{N}_0} \mathcal{H}_n$ and $\mathcal{H}_0 \cong \mathbb{K}$. The aim of this section is to prove that the exponential map \exp_A of $A := \text{Hom}_{\mathbb{K}}(\mathcal{H}, B)$ restricts to a bijection from the infinitesimal characters to the characters.

LEMMA B.9 (Cocomposition with Hopf multiplication). — *Let $\mathcal{H} \otimes \mathcal{H}$ be the tensor Hopf algebra (cf. [25, p. 8]). With respect to the topology of pointwise convergence and the convolution product, the algebras*

$$A := \text{Hom}_{\mathbb{K}}(\mathcal{H}, B) \quad A_{\otimes} := \text{Hom}_{\mathbb{K}}(\mathcal{H} \otimes \mathcal{H}, B)$$

become locally convex algebras (see Lemma 1.4). This structure turns

$$\cdot \circ m_{\mathcal{H}} : \text{Hom}_{\mathbb{K}}(\mathcal{H}, B) \rightarrow \text{Hom}_{\mathbb{K}}(\mathcal{H} \otimes \mathcal{H}, B), \phi \mapsto \phi \circ m_{\mathcal{H}}.$$

into a continuous algebra homomorphism.

Proof. — From the usual identities for the structure maps of Hopf algebras (cf. [25, p. 7 Fig. 1.3]) it is easy to see that $\cdot \circ m_{\mathcal{H}}$ is an algebra homomorphism. Clearly $\cdot \circ m_{\mathcal{H}}$ is continuous with respect to the topologies of pointwise convergence. □

LEMMA B.10. — *The analytic diffeomorphism $\exp_A : \mathcal{I}_A \rightarrow 1 + \mathcal{I}_A$ maps the set of infinitesimal characters $\mathfrak{g}(\mathcal{H}, B)$ bijectively onto the set of characters $G(\mathcal{H}, B)$.*⁽⁷⁾

Proof. — We regard the tensor product $\mathcal{H} \otimes \mathcal{H}$ as a graded and connected Hopf algebra with respect to the tensor grading, i.e.

$$\mathcal{H} \otimes \mathcal{H} = \bigoplus_{n \in \mathbb{N}_0} (\mathcal{H} \otimes \mathcal{H})_n \text{ with } (\mathcal{H} \otimes \mathcal{H})_n = \bigoplus_{i+j=n} \mathcal{H}_i \otimes \mathcal{H}_j \text{ for all } n \in \mathbb{N}_0.$$

The set of linear maps $A_{\otimes} := \text{Hom}_{\mathbb{K}}(\mathcal{H} \otimes \mathcal{H}, B)$ together with the convolution product $\star_{A_{\otimes}}$ forms a densely graded locally convex algebra. Let $m_B : B \otimes B \rightarrow B, b_1 \otimes b_2 \mapsto b_1 \cdot b_2$ be the multiplication in B . Define a bilinear map

$$\beta : A \times A \rightarrow A_{\otimes}, (\phi, \psi) \mapsto \phi \diamond \psi := m_B \circ (\phi \otimes \psi).$$

⁽⁷⁾It is hard to find a complete proof in the literature, whence we chose to include a proof for the reader's convenience.

We prove now that β is continuous. To this end consider a fixed element $c = \sum_{k=1}^n c_{1,k} \otimes c_{2,k} \in \mathcal{H} \otimes \mathcal{H}$. We have to prove that $\phi \diamond \psi(c)$ depends continuously on ϕ and ψ :

$$\begin{aligned} (\phi \diamond \psi)(c) &= m \circ (\phi \otimes \psi)(c) = m \circ (\phi \otimes \psi) \left(\sum_{k=1}^n c_{1,k} \otimes c_{2,k} \right) \\ &= \sum_{k=1}^n m(\phi(c_{1,k}) \otimes \psi(c_{2,k})) = \sum_{k=1}^n \phi(c_{1,k}) \cdot \psi(c_{2,k}) \end{aligned}$$

This expression is continuous in (ϕ, ψ) since point evaluations are continuous as well as multiplication in the locally convex algebra B . The convolution in A can be written as $\star_A = \beta \circ \Delta$. We obtain

$$(B.3) \quad (\phi_1 \diamond \psi_1) \star_{A_\otimes} (\phi_2 \diamond \psi_2) = (\phi_1 \star_A \phi_2) \diamond (\psi_1 \star_A \psi_2).$$

Recall, that $1_A := u_B \circ \varepsilon_{\mathcal{H}}$ is the neutral element of the algebra A . From equation (B.3), it follows at once, that the continuous linear maps

$$(B.4) \quad \begin{aligned} &\beta(\cdot, 1_A): A \rightarrow A_\otimes, \phi \mapsto \phi \diamond 1_A \\ \text{and } &\beta(1_A, \cdot): A \rightarrow A_\otimes, \phi \mapsto 1_A \diamond \phi \end{aligned}$$

are continuous algebra homomorphisms. We will now exploit \diamond to prove that the bijection $\exp_A: \mathcal{I}_A \rightarrow 1_A + \mathcal{I}_A$ (see Lemma B.5) maps the set $\mathfrak{g}(H, B)$ onto $G(H, B)$. Let $\phi \in \mathcal{I}_A$ be given and recall:

- (a) The Hopf algebra product $m_{\mathcal{H}}$ maps $\mathcal{H}_0 \otimes \mathcal{H}_0$ into \mathcal{H}_0 . Now $\mathcal{H}_0 \otimes \mathcal{H}_0 = (\mathcal{H} \otimes \mathcal{H})_0$ (tensor grading) entails for $\phi \in \mathcal{I}_A$ that $\phi \circ m_{\mathcal{H}} \in \mathcal{I}_{A_\otimes}$.
- (b) From (B.3) we derive that

$$(\phi \diamond 1_A) \star_{A_\otimes} (1_A \diamond \phi) = \phi \diamond \phi = (1_A \diamond \phi) \star_{A_\otimes} (\phi \diamond 1_A).$$

If $\phi \in A$ is an infinitesimal character then $\phi \circ m_{\mathcal{H}} = \phi \diamond 1_A + 1_A \diamond \phi$.

Combining (a) and (b) we see that Lemma B.6 is applicable and as a consequence

$$(B.5) \quad \exp_{A_\otimes}(\phi \diamond 1_A + 1_A \diamond \phi) = \exp_{A_\otimes}(\phi \diamond 1_A) \star_{A_\otimes} \exp_{A_\otimes}(1_A \diamond \phi).$$

Note that it suffices to check multiplicativity of $\exp_A(\phi)$ as $\exp_A(\phi)(1_{\mathcal{H}}) = 1_B$ is automatically satisfied. To prove the assertion we establish the following equivalences:

$$\begin{aligned}
 \phi \in \mathfrak{g}(\mathcal{H}, B) &\stackrel{\text{Def}}{\iff} \phi \circ m_{\mathcal{H}} = \phi \diamond 1_A + 1_A \diamond \phi \\
 &\stackrel{(a)}{\iff} \exp_{A_{\otimes}}(\phi \circ m_{\mathcal{H}}) = \exp_{A_{\otimes}}(\phi \diamond 1_A + 1_A \diamond \phi) \\
 &\stackrel{(B.5)}{\iff} \exp_{A_{\otimes}}(\phi \circ m_{\mathcal{H}}) = \exp_{A_{\otimes}}(\phi \diamond 1_A) \star_{A_{\otimes}} \exp_{A_{\otimes}}(1_A \diamond \phi) \\
 &\stackrel{(B.4)}{\iff} \exp_{A_{\otimes}}(\phi \circ m_{\mathcal{H}}) = (\exp_A(\phi) \diamond 1_A) \star_{A_{\otimes}} (1_A \diamond \exp_A(\phi)) \\
 &\stackrel{(B.3)}{\iff} \exp_{A_{\otimes}}(\phi \circ m_{\mathcal{H}}) = (\exp_A(\phi) \star_A 1_A) \diamond (1_A \star_A \exp_A(\phi)) \\
 &\iff \exp_{A_{\otimes}}(\phi \circ m_{\mathcal{H}}) = \exp_A(\phi) \diamond \exp_A(\phi) \\
 &\stackrel{B.9}{\iff} \exp_A(\phi) \circ m_{\mathcal{H}} = \exp_A(\phi) \diamond \exp_A(\phi) \\
 &\stackrel{\text{Def}}{\iff} \exp_A(\phi) \in G(\mathcal{H}, B) \qquad \square
 \end{aligned}$$

Remark B.11. — The chain of equivalences in the proof of Lemma B.10 uses the dense grading of $A = \text{Hom}_{\mathbb{K}}(\mathcal{H}, B)$ twice to show that the first to third lines are equivalent. However, for arbitrary \mathcal{H} and weakly complete B the second and third line are still equivalent: In this case A_{\otimes} is weakly complete and Lemma 5.3(c) allows us to embed A_{\otimes} into $P := \prod_{i \in I} A_i$ (product of Banach algebras in the category of topological algebras). This implies that the formula $\exp_{A_{\otimes}}(a + b) = \exp_{A_{\otimes}}(a) \exp_{A_{\otimes}}(b)$ used in (B.5) still holds as the power series defining \exp_A converges on P and satisfies the formula (which is component-wise true in every Banach algebra).

Appendix C. Weakly complete vector spaces and duality

The purpose of this section is to exhibit the duality between the category of abstract vector spaces and the category of weakly complete topological vector spaces. Although none of this is needed for the results of this paper, many ideas of this paper appear to be more natural in this wider setting.

Throughout this section, let \mathbb{K} be a fixed Hausdorff topological field. Although, in this paper, we are only interested in the cases $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$, the statements in this appendix hold for an arbitrary Hausdorff

field of any characteristic, including the discrete ones.⁽⁸⁾ We start with a definition.

DEFINITION C.1. — *A topological vector space E over the topological field \mathbb{K} is called weakly complete topological vector space (or weakly complete space for short) if one of the following equivalent conditions is satisfied:*

- (a) *There exists a set I such that E is topologically isomorphic to \mathbb{K}^I .*
- (b) *There exists an abstract \mathbb{K} -vector space \mathcal{V} such that E is topologically isomorphic to $\mathcal{V}^* := \text{Hom}_{\mathbb{K}}(\mathcal{V}, \mathbb{K})$ with the weak*-topology*
- (c) *The space E is the projective limit of its finite-dimensional Hausdorff quotients, with each n -dimensional quotient being topologically isomorphic to \mathbb{K}^n*

For the case $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, these conditions are also equivalent to the following conditions:

- (d) *The space E is locally convex and is complete with respect to the weak topology.*
- (e) *The space E is locally convex, it carries its weak topology and is complete with this topology.*

The proof that (a) \iff (b) \iff (c) can be found in [18, Appendix 2]). The characterisations (d) and (e) justify the name *weakly complete*.

Remark C.2. — Part (b) of the preceding definition tells us that the algebraic dual \mathcal{V}^* of an abstract vector space \mathcal{V} becomes a weakly complete topological vector space with respect to the weak*-topology, i.e. the topology of pointwise convergence.

Conversely, given a weakly complete vector space E , we can consider the topological dual E' of all continuous linear functionals. Although there are many vector space topologies on this topological dual, we will always take E' as an abstract vector space.

One of the main problems when working in infinite-dimensional linear (and multilinear) algebra is that a vector space \mathcal{V} is no longer isomorphic to its bidual $(\mathcal{V}^*)^*$. The main purpose of this section is to convince the reader that the reason for this bad behaviour of the bidual is due to the fact that the wrong definition of a bidual is used (at least for infinite-dimensional spaces).

⁽⁸⁾ In functional analysis, usually only \mathbb{R} and \mathbb{C} with their usual field topologies are considered, where in algebra usually an arbitrary field with the discrete topology is considered. Our setup includes both cases (and many more, e.g. the p -adic numbers, etc.).

If we start with an abstract vector space \mathcal{V} , then its dual is a weakly complete space \mathcal{V}^* and consequently, one should *not* take the algebraic dual $(\mathcal{V}^*)^*$ but the topological dual $(\mathcal{V}^*)'$ which is the natural choice. For a finite-dimensional space the construction coincides with the usual definition of the bidual. In the general case however, the so obtained bidual is now canonically isomorphic to the original space as the following proposition shows:

PROPOSITION C.3 (Duality and Reflexivity). — *Let E be a weakly complete space and let \mathcal{V} be an abstract vector space. There are natural isomorphisms*

$$\begin{aligned} \eta_E: E &\longrightarrow (E')^* \\ x &\longmapsto (\eta_E(x) := \phi_x: E' \rightarrow \mathbb{K}, \lambda \mapsto \lambda(x)) \end{aligned}$$

and

$$\begin{aligned} \eta_{\mathcal{V}}: \mathcal{V} &\longrightarrow (\mathcal{V}^*)' \\ v &\longmapsto (\eta_{\mathcal{V}}(v) := \lambda_v: \mathcal{V}^* \rightarrow \mathbb{K}, \phi \mapsto \phi(v)). \end{aligned}$$

Proof (Sketch). — Let E be a weakly complete vector space. We may assume that $E = \mathbb{K}^I$ for a set I . Then each projection map $\pi_i: \mathbb{K}^I \rightarrow \mathbb{K}$ on the i -th component is an element in E' . It is easy to see that $(\pi_i)_{i \in I}$ is in fact a basis of the abstract vector space E' . This means that the algebraic dual of E' is topologically isomorphic to \mathbb{K}^I . Using this identification, one can check that the map η_E is the identity.

Similarly, let \mathcal{V} be an abstract vector space. By Zorn's Lemma, pick a basis $(b_i)_{i \in I}$. Then the dual space \mathcal{V}^* is isomorphic to \mathbb{K}^I . And therefore, the dual of that one $(\mathcal{V}^*)'$ has a basis $(\pi_i)_{i \in I}$. Under this identification, the linear map $\eta_{\mathcal{V}}$ is the identity. \square

DEFINITION C.4 (The weakly complete tensor product). — *One way to understand Proposition C.3 is that every element x in a weakly complete space E can be identified with a linear functional $\phi_x = \eta_E(x) \in (E')^*$ on the abstract vector space E' . This enables us to define a tensor product of two elements $x \in E$ and $y \in F$ as the tensor product of the corresponding linear functionals*

$$\begin{aligned} x \otimes y &:= \phi_x \otimes \phi_y: E' \otimes F' \longrightarrow \mathbb{K} \\ \lambda \otimes \mu &\longmapsto \phi_x(\lambda) \cdot \phi_y(\mu) = \lambda(x) \cdot \mu(y). \end{aligned}$$

This element $x \otimes y$ is now a linear functional on the abstract vector space $E' \otimes F'$ which motivates the definition:

$$E \tilde{\otimes} F := (E' \otimes F')^*.$$

If the spaces E and F are of the form $E = \mathbb{K}^I$ and $F = \mathbb{K}^J$, it is easy to verify that the space $\mathbb{K}^I \tilde{\otimes} \mathbb{K}^J = (E' \otimes F')^*$ is canonically isomorphic to $\mathbb{K}^{I \times J}$. This could have been taken as the definition of the weakly complete tensor product in the first place. However, the definition we chose has the advantage that is independent of the choice of coordinates, i.e. the specific isomorphisms $E \cong \mathbb{K}^I$ and $F \cong \mathbb{K}^J$, respectively.

PROPOSITION C.5 (The universal property of the weakly complete tensor product). — *Let E, F, H be weakly complete spaces and let $\beta: E \times F \rightarrow H$ be a continuous bilinear map. Then there exists a unique continuous linear map $\beta^\sim: E \tilde{\otimes} F \rightarrow H$ such that the following diagram commutes:*

$$\begin{array}{ccc}
 E \times F & \xrightarrow{\beta} & H \\
 \otimes \downarrow & \nearrow \beta^\sim & \\
 E \tilde{\otimes} F & &
 \end{array}$$

For the case $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, this universal property also holds for arbitrary complete locally convex spaces H , showing that this weakly complete tensor product is just a special case of the usual projective tensor product for locally convex vector spaces.

DEFINITION C.6 (The monoidal categories $\mathbf{WCVS}_{\mathbb{K}}$ and $\mathbf{VS}_{\mathbb{K}}$).

- (i) Denote the category of weakly complete spaces and continuous linear maps by $\mathbf{WCVS}_{\mathbb{K}}$. Together with the weakly complete tensor product and the ground field \mathbb{K} as unit object, we obtain a monoidal category $(\mathbf{WCVS}_{\mathbb{K}}, \tilde{\otimes}, \mathbb{K})$.
- (ii) Denote the monoidal category of abstract vector spaces, abstract linear maps, the usual abstract tensor product and the ground field as unit object by $(\mathbf{VS}_{\mathbb{K}}, \otimes, \mathbb{K})$.

The two categories $\mathbf{WCVS}_{\mathbb{K}}$ and $\mathbf{VS}_{\mathbb{K}}$ are dual to each other. The dualities are given by the contravariant monoidal functors *algebraic dual*

$$\begin{aligned}
 (\cdot)^*: \mathbf{VS}_{\mathbb{K}} &\longrightarrow \mathbf{WCVS}_{\mathbb{K}} \\
 \mathcal{V} &\longmapsto \mathcal{V}^* \\
 (\Phi: \mathcal{V} \rightarrow \mathcal{W}) &\longmapsto (\Phi^*: \mathcal{W}^* \rightarrow \mathcal{V}^*, \phi \mapsto \phi \circ \Phi)
 \end{aligned}$$

and *topological dual*

$$\begin{aligned}
 (\cdot)': \mathbf{WCVS}_{\mathbb{K}} &\longrightarrow \mathbf{VS}_{\mathbb{K}} \\
 E &\longmapsto E' \\
 (T: E \rightarrow F) &\longmapsto (T': F' \rightarrow E', \lambda \mapsto \lambda \circ T)
 \end{aligned}$$

(cf. Proposition C.3). The duality interchanges direct sums in the abstract category with direct products in the weakly complete category, hence graded vector spaces (Definition B.1) are assigned to densely graded vector spaces (Definition B.2). For more information about this duality, we refer to [27, p. 679] and to [18, Appendix 2].

We can naturally define weakly complete algebras, weakly complete coalgebras, weakly complete bialgebras and weakly complete Hopf algebras using the weakly complete tensor product in $(\mathbf{WCVS}_{\mathbb{K}}, \tilde{\otimes}, \mathbb{K})$. By duality, we get the correspondence:

Abstract world $(\mathbf{VS}_{\mathbb{K}}, \otimes, \mathbb{K})$	Weakly complete world $(\mathbf{WCVS}_{\mathbb{K}}, \tilde{\otimes}, \mathbb{K})$
abstract vector space linear map graded vector space	weakly complete vector space continuous linear map densely graded weakly complete vector space
abstract coalgebra	weakly complete algebra
abstract algebra	weakly complete coalgebra
abstract bialgebra	weakly complete bialgebra
abstract Hopf algebra characters infinitesimal characters	weakly complete Hopf algebra group like elements primitive elements

Remark C.7. — Note that while a weakly complete algebra is an algebra with additional structure (namely a topology), a weakly complete coalgebra is in general *not* a coalgebra. This is due to the fact that the weakly complete comultiplication $\Delta: C \rightarrow C \tilde{\otimes} C$ takes values in the completion $C \tilde{\otimes} C$, while for a coalgebra it would be necessary that it takes its values in $C \otimes C$ and the canonical inclusion map $C \otimes C \mapsto C \tilde{\otimes} C$ goes into the wrong direction (see also [27, p. 680]). In particular, a Hopf algebra in the weakly complete category is not a Hopf algebra in general.

Using the duality, we may translate theorems from the abstract category to the weakly complete category, for example the Fundamental Lemma of weakly complete algebras (Lemma 5.3) follows directly from the the fundamental theorem of coalgebras, stating that every abstract coalgebra is the direct union of its finite-dimensional subcoalgebras. It should be mentioned that one of the first proofs of the fundamental theorem of coalgebras by

Larson [23, Prop. 2.5] used this duality and worked in the framework of topological algebras to show the result about abstract coalgebras.

Let \mathcal{H} be an abstract Hopf algebra and $H := \mathcal{H}^*$ the corresponding weakly complete Hopf algebra. Then the characters of \mathcal{H} are exactly the group like elements in H , while the infinitesimal characters of \mathcal{H} are exactly the primitive elements of H . This allows us to rephrase the scalar valued case of Theorem 2.7.

THEOREM C.8 (Group like elements in a weakly complete Hopf algebra). *Let H be a densely graded weakly complete Hopf algebra over \mathbb{R} or \mathbb{C} with $H_0 = \mathbb{K}$. Then the group like elements of H form a closed Lie subgroup of the open unit group H^\times . The Lie algebra of this group is the weakly complete Lie algebra of primitive elements.*

BIBLIOGRAPHY

- [1] W. BERTRAM, H. GLÖCKNER & K.-H. NEEB, “Differential calculus over general base fields and rings”, *Expo. Math.* **22** (2004), no. 3, p. 213-282.
- [2] J. BOCHNAK & J. SICIĄK, “Polynomials and multilinear mappings in topological vector spaces”, *Studia Math.* **39** (1971), p. 59-76.
- [3] G. BOGFJELLMO & A. SCHMEDING, “The Lie Group Structure of the Butcher Group”, *Foundations of Computational Mathematics* (2015), p. 1-33, <http://dx.doi.org/10.1007/s10208-015-9285-5>.
- [4] C. BROUDER, “Trees, renormalization and differential equations”, *BIT* **44** (2004), no. 3, p. 425-438.
- [5] P. CARTIER, “A primer of Hopf algebras”, in *Frontiers in number theory, physics, and geometry. II*, Springer, Berlin, 2007, p. 537-615.
- [6] P. CHARTIER, E. HAIRER & G. VILMART, “Algebraic structures of B-series”, *Found. Comput. Math.* **10** (2010), no. 4, p. 407-427.
- [7] A. CONNES & D. KREIMER, “Hopf algebras, renormalization and noncommutative geometry”, *Comm. Math. Phys.* **199** (1998), no. 1, p. 203-242.
- [8] ———, “Renormalization in quantum field theory and the Riemann-Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem”, *Comm. Math. Phys.* **210** (2000), no. 1, p. 249-273.
- [9] ———, “Renormalization in quantum field theory and the Riemann-Hilbert problem. II. The β -function, diffeomorphisms and the renormalization group”, *Comm. Math. Phys.* **216** (2001), no. 1, p. 215-241.
- [10] A. CONNES & M. MARCOLLI, *Noncommutative geometry, quantum fields and motives*, American Mathematical Society Colloquium Publications, vol. 55, American Mathematical Society, Providence, RI; Hindustan Book Agency, New Delhi, 2008, xxii+785 pages.
- [11] H. GLÖCKNER, “Algebras whose groups of units are Lie groups”, *Studia Math.* **153** (2002), no. 2, p. 147-177.
- [12] ———, “Infinite-dimensional Lie groups without completeness restrictions”, in *Geometry and analysis on finite- and infinite-dimensional Lie groups (Będlewo, 2000)*, Banach Center Publ., vol. 55, Polish Acad. Sci., Warsaw, 2002, p. 43-59.

- [13] ———, “Instructive examples of smooth, complex differentiable and complex analytic mappings into locally convex spaces”, *J. Math. Kyoto Univ.* **47** (2007), no. 3, p. 631-642.
- [14] ———, “Simplified proofs for the pro-Lie group theorem and the one-parameter subgroup lifting lemma”, *J. Lie Theory* **17** (2007), no. 4, p. 899-902.
- [15] ———, “Regularity properties of infinite-dimensional Lie groups, and semiregularity”, <http://arxiv.org/abs/1208.0715v3>, 2015.
- [16] H. GLÖCKNER & K.-H. NEEB, “When unit groups of continuous inverse algebras are regular Lie groups”, *Studia Math.* **211** (2012), no. 2, p. 95-109.
- [17] K. H. HOFMANN & K.-H. NEEB, “Pro-Lie groups which are infinite-dimensional Lie groups”, *Math. Proc. Cambridge Philos. Society* **146** (2009), no. 2, p. 351-378.
- [18] K. H. HOFMANN & S. A. MORRIS, *The Lie theory of connected pro-Lie groups*, EMS Tracts in Mathematics, vol. 2, EMS, Zürich, 2007, xvi+678 pages.
- [19] ———, *The structure of compact groups*, third ed., De Gruyter Studies in Mathematics, vol. 25, De Gruyter, Berlin, 2013, xxii+924 pages.
- [20] C. KASSEL, *Quantum groups*, Graduate Texts in Mathematics, vol. 155, Springer-Verlag, New York, 1995, xii+531 pages.
- [21] H. H. KELLER, *Differential calculus in locally convex spaces*, Lecture Notes in Mathematics, Vol. 417, Springer-Verlag, Berlin-New York, 1974, iii+143 pages.
- [22] A. KRIEGL & P. W. MICHOR, *The convenient setting of global analysis*, Mathematical Surveys and Monographs, vol. 53, American Mathematical Society, Providence, RI, 1997, x+618 pages.
- [23] R. G. LARSON, “Cocommutative Hopf algebras”, *Canad. J. Math.* **19** (1967), p. 350-360.
- [24] J.-L. LODAY & M. RONCO, “Combinatorial Hopf algebras”, in *Quanta of maths*, Clay Math. Proc., vol. 11, Amer. Math. Soc., Providence, RI, 2010, p. 347-383.
- [25] S. MAJID, *Foundations of quantum group theory*, Cambridge University Press, Cambridge, 1995, x+607 pages.
- [26] D. MANCHON, “Hopf algebras, from basics to applications to renormalization”, <http://arxiv.org/pdf/math/0408405v2>, 2006.
- [27] W. MICHAELIS, “Coassociative coalgebras”, in *Handbook of algebra*, Vol. 3, North-Holland, Amsterdam, 2003, p. 587-788.
- [28] J. W. MILNOR, “Remarks on infinite-dimensional Lie groups”, in *Relativity, groups and topology, II (Les Houches, 1983)*, North-Holland, Amsterdam, 1984, p. 1007-1057.
- [29] J. W. MILNOR & J. C. MOORE, “On the structure of Hopf algebras”, *Ann. of Math.* (2) **81** (1965), p. 211-264.
- [30] K.-H. NEEB, “Towards a Lie theory of locally convex groups”, *Jpn. J. Math.* **1** (2006), no. 2, p. 291-468.
- [31] W. D. VAN SUIJLEKOM, “Renormalization of gauge fields: a Hopf algebra approach”, *Comm. Math. Phys.* **276** (2007), no. 3, p. 773-798.
- [32] ———, “The structure of renormalization Hopf algebras for gauge theories. I. Representing Feynman graphs on BV-algebras”, *Comm. Math. Phys.* **290** (2009), no. 1, p. 291-319.
- [33] M. E. SWEEDLER, *Hopf algebras*, Mathematics Lecture Note Series, W. A. Benjamin, Inc., New York, 1969, vii+336 pages.
- [34] W. C. WATERHOUSE, *Introduction to affine group schemes*, Graduate Texts in Mathematics, vol. 66, Springer-Verlag, New York-Berlin, 1979, xi+164 pages.
- [35] H. YAMABE, “On the conjecture of Iwasawa and Gleason”, *Ann. of Math.* (2) **58** (1953), p. 48-54.

Manuscrit reçu le 19 mai 2015,
révisé le 8 février 2016,
accepté le 18 février 2016.

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