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# NODAL INTERSECTIONS FOR RANDOM WAVES ON THE 3-DIMENSIONAL TORUS

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ABSTRACT. — We investigate the number of nodal intersections of random Gaussian Laplace eigenfunctions on the standard three-dimensional flat torus with a fixed smooth reference curve, which has nowhere vanishing curvature. The expected intersection number is universally proportional to the length of the reference curve, times the wavenumber, independent of the geometry. Our main result gives a bound for the variance, if either the torsion of the curve is nowhere zero or if the curve is planar.

RÉSUMÉ. — Nous étudions le nombre d'intersections nodales des fonctions propres gaussiennes aléatoires du Laplacien sur le tore plat à trois dimensions avec une courbe régulière de référence fixée de courbure partout non-nulle. Le nombre d'intersections moyen est toujours proportionnel à la longueur de la courbe de référence, multipliée par le nombre d'onde et est indépendant de la géométrie. Notre résultat principal est une borne sur la variance, lorsque la torsion de la courbe est partout non-nulle ou lorsque la courbe est planaire.

## 1. Introduction

### 1.1. Toral nodal intersections

Let  $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$  be the standard flat  $d$ -dimensional torus and  $\mathcal{C} \subset \mathbb{T}^d$  a fixed reference curve<sup>(1)</sup>. Given a real-valued eigenfunction  $F(x)$  of the Laplacian

$$-\Delta F = 4\pi^2 E \cdot F,$$

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<sup>(1)</sup>By a curve we always mean a parameterized, compact, immersed curve.

we wish to study the number of intersections

$$\mathcal{Z}(F) := \#\{x \in \mathbb{T}^d : F(x) = 0\} \cap \mathcal{C}$$

of the nodal set of  $F$  with the reference curve  $\mathcal{C}$  as a function of the corresponding eigenvalue.

In dimension  $d = 2$ , if the curve is smooth and has nowhere vanishing curvature, then deterministically for every eigenfunction, the number of nodal intersections satisfies [3, 4]

$$\frac{E^{1/2}}{(\log E)^{5/2}} \ll \mathcal{Z}(F) \ll E^{1/2};$$

here and everywhere  $f \ll g$  (equivalently  $f = O(g)$ ) means that there exists a constant  $C > 0$  such that  $|f| \leq C|g|$ . For the upper bound we require that  $\mathcal{C}$  is real-analytic, in particular it is shown that  $\mathcal{C}$  is not contained in the nodal set for  $E$  sufficiently large. One can improve the lower bound to  $\mathcal{Z}(F) \gg E^{1/2}$  conditionally on a certain number-theoretic conjecture [4].

In this note we deal with dimension  $d = 3$ . If we consider the intersection of the nodal set with a fixed real-analytic reference surface  $\Sigma \subset \mathbb{T}^3$  with nowhere-zero Gauss–Kronecker curvature, then for  $E$  sufficiently large,  $\Sigma$  is not contained in the nodal set [2], the length of the intersection of  $\Sigma$  with the nodal set is  $\ll \sqrt{E}$  [3], and the nodal intersection is non-empty [3]. However, for the intersection of the nodal set with a fixed reference curve (where we expect only finitely many points), the following examples indicate that one cannot expect to have any deterministic bounds on the number of nodal intersections.

*Example 1.1.* — Take the eigenfunctions of the form

$$F_k(x_1, x_2, x_3) = \sin(2\pi k x_1);$$

their nodal surfaces are the planes  $\{x \in \mathbb{T}^3 : x_1 \in \frac{1}{2k}\mathbb{Z}\}$  and any curve lying on the plane  $x_1 = 1/2\pi$  does not intersect these, whereas a curve on the plane  $x_1 = 0$  is lying inside all of these nodal surfaces. We observe that the curves in this example are *planar*.

*Example 1.2.* — Let  $F_0(x, y)$  be an eigenfunction on the two-dimensional torus with eigenvalue  $4\pi^2 E_0^2$ , and  $S_0$  a curved segment contained in the nodal set, admitting an arc-length parameterization  $\gamma_0 : [0, L] \rightarrow S_0$ , with curvature  $\kappa_0(t) = |\gamma_0''(t)| > 0$ . For  $n \geq 0$  let  $F_n(x, y, z) = F_0(x, y) \cos(2\pi n z)$ , an eigenfunction on  $\mathbb{T}^3$  with eigenvalue  $4\pi^2(E_0^2 + n^2)$ . Let  $\mathcal{C}$  be the parametric curve  $\gamma(t) = \left(\gamma_0\left(\frac{t}{\sqrt{2}}\right), \frac{t}{\sqrt{2}}\right)$ . A computation shows that the curvature is  $\kappa(t) = \frac{1}{2}\kappa_0\left(\frac{t}{\sqrt{2}}\right) > 0$ , and that the torsion is  $\tau(t) =$

$\pm \frac{1}{2} \kappa_0 \left( \frac{t}{\sqrt{2}} \right) \neq 0$ , so that  $\mathcal{C}$  is non-planar. Clearly  $\mathcal{C}$  is contained in the nodal set of  $F_n$  for all  $n$ . Thus even in the non-planar case, we can have the reference curve  $\mathcal{C}$  contained in the nodal set for arbitrarily large  $E$ .

QUESTION 1.3. — *Does there exist a non-planar curve  $\mathcal{C}$  with no nodal intersections,  $E$  arbitrarily large?*

### 1.2. Arithmetic random waves

As there is no deterministic bound on  $\mathcal{Z}(F)$  in dimension 3, we investigate what happens for “typical” eigenfunctions. Let  $\mathcal{E}(E)$  be the set of lattice points lying on a sphere

$$\mathcal{E}(E) = \{ \vec{x} \in \mathbb{Z}^3 : |\vec{x}|^2 = E \},$$

and let

$$N = N_E := \# \mathcal{E}(E).$$

The Laplace spectrum on  $\mathbb{T}^3$  is of high multiplicities, with the dimension of an eigenspace corresponding to an eigenvalue  $4\pi^2 E$  being of size  $N_E \approx E^{1/2 \pm o(1)}$ . The eigenfunctions corresponding to the eigenvalue  $4\pi^2 E$  are of the form

$$(1.1) \quad F(x) = F_E(x) = \frac{1}{\sqrt{N_E}} \sum_{\mu \in \mathcal{E}(E)} a_\mu e^{2\pi i \langle \mu, x \rangle}.$$

We consider random Gaussian eigenfunctions (“arithmetic random waves” [11]) by taking the coefficients  $a_\mu$  to be standard complex Gaussian random variables, independent save for the relations  $a_{-\mu} = \overline{a_\mu}$ , making  $F$  real-valued.

### 1.3. Statement of the main results

THEOREM 1.4. — *Let  $\mathcal{C} \subset \mathbb{T}^3$  be a smooth curve of length  $L$ , with nowhere zero curvature. Assume further that one of the following holds:*

- (1)  $\mathcal{C}$  has nowhere-vanishing torsion;
- (2)  $\mathcal{C}$  is planar (so that the torsion vanishes identically).

*Then for all  $\epsilon > 0$ , as  $E \rightarrow \infty$  along integers  $E \not\equiv 0, 4, 7 \pmod{8}$ , the number of nodal intersections satisfies*

$$\lim_{\substack{E \rightarrow \infty \\ E \not\equiv 0, 4, 7 \pmod{8}}} \text{Prob} \left( \left| \frac{\mathcal{Z}(F)}{\sqrt{E}} - \frac{2}{\sqrt{3}} L \right| > \epsilon \right) \rightarrow 0.$$

Note that the condition  $E \not\equiv 0, 4, 7 \pmod{8}$  is natural, as otherwise  $E = 4^a E'$  with  $E' \not\equiv 7 \pmod{8}$  (if  $E' \equiv 7 \pmod{8}$  then  $E$  is not a sum of three squares and hence does not yield a Laplace eigenvalue); then an eigenfunction of eigenvalue  $4\pi^2 E$  is necessarily of the form  $F(x) = H(2^a x)$  with  $H$  an eigenfunction of eigenvalue  $E'$ . Hence any question on the nodal set of  $F$  reduces to the corresponding question on the nodal set of eigenfunctions with eigenvalue  $E' = E/4^a$  (which may be trivial, e.g. if  $E = 4^a$ ).

To prove Theorem 1.4 we compute the expected value of  $\mathcal{Z}$  with respect to the Gaussian measure defined on the eigenspace as above to be

$$\mathbb{E}(\mathcal{Z}) = \frac{2}{\sqrt{3}} L\sqrt{E},$$

and give an upper bound for the variance:

**THEOREM 1.5.** — *Let  $\mathcal{C} \subset \mathbb{T}^3$  be a smooth curve, with nowhere-zero curvature. Assume also that either  $\mathcal{C}$  has nowhere-vanishing torsion, or  $\mathcal{C}$  is planar. Then for  $E \not\equiv 0, 4, 7 \pmod{8}$ ,*

$$(1.2) \quad \text{Var} \left( \frac{\mathcal{Z}}{\sqrt{E}} \right) \ll \frac{1}{E^\delta}$$

for all  $\delta < 1/3$  in case  $\mathcal{C}$  has nowhere vanishing torsion, and all  $\delta < 1/4$  in case  $\mathcal{C}$  is planar.

*Remark 1.6.* — If  $\mathcal{C}$  is real-analytic and non-planar, so that the torsion is not identically zero, but may vanish at finitely many points, the result (1.2) above is valid with some  $\delta = \delta_{\mathcal{C}} > 0$ , see §5.3.

## 1.4. Outline of the paper and the key ideas

We prove the approximate Kac–Rice formula (briefly explained in §1.5) in §2, followed by a study of certain oscillatory integrals on the curve in §3. The arithmetic heart of the paper is §5, where we bound the second moment of the covariance function and its derivatives, following some background on the arithmetic of sums of three squares in §4, expanded on in Appendix A.

A similar result to Theorem 1.5 was proved in the two-dimensional case for  $\mathcal{C} \subset \mathbb{T}^2$  having nowhere-zero curvature [12]. In that case the authors found that the precise asymptotic behaviour of the nodal intersections variance is *non-universal*, namely dependent on both the angular distribution of the lattice points  $\mathcal{E}(E)$  and the geometry of  $\mathcal{C}$ . In the 3-dimensional case we were only able to obtain an upper bound (1.2) on the variance, which

implies the “almost-all” statement of Theorem 1.4. These two cases differ both in terms of analytic and arithmetic ingredients; the arithmetic of ternary quadratic forms differs significantly from that of binary quadratic forms.

### 1.5. An approximate Kac–Rice formula

We end this Introduction with a discussion of a key step in our work, the approximate Kac–Rice formula.

By restricting the arithmetic random waves (1.1) along  $\mathcal{C}$ , the problem of nodal intersections count is reduced to evaluating the number of zeros (zero crossings) of a random (nonstationary) Gaussian process. The Kac–Rice formula is a standard tool or meta-theorem for expressing the (factorial) moments of the zero crossings number of a Gaussian process precisely in terms of certain explicit integrals with integrands depending on the given process. It is then easy to evaluate the expected number of zeros precisely and explicitly via an evaluation of a standard Gaussian expectation.

For the variance, or the second factorial moment, the validity of the Kac–Rice formula is a very subtle question with a variety of sufficient conditions known in the literature (e.g. [8, 1]). While the classical treatise [8] requires the non-degeneracy of the (Gaussian) distribution of the values of the given process at two points together with their derivatives, only the non-degeneracy of the values distribution (at two points) is required for the more modern treatment [1], which, to the best knowledge of the authors, is the weakest known sufficient condition for the validity of Kac–Rice. Unfortunately, even this weaker condition may fail in our case.

In order to treat this situation (§2) we divide the interval into many small subintervals of length commensurable to the wavelength  $\frac{1}{\sqrt{E}}$  and decompose the total variance as a sum over pairs of subintervals of zero number covariances (2.14). We were able to prove the validity of Kac–Rice for *most* of the pairs of subintervals that includes all the diagonal pairs (i.e. the variance of nodal intersections along sufficiently small curves), and bound the contribution of the other pairs via a simple Cauchy–Schwartz argument. The price is that along the way we incur an error term, hence yielding an *approximate* Kac–Rice (Proposition 2.2) reducing a variance computation to an estimate for some moments of the covariance function and its derivatives; such a strategy was also used in [12, 7]. The remaining part of this paper is concerned with proving such an estimate on the second moment of the covariance function and a couple of its derivatives.

We finally record that we may omit a certain technical assumption made in our previous work on the two-dimensional case [12] by using the more general form of the Kac–Rice formula as in [1] (vs. [8]). Given an integer  $m$  expressible as a sum of 2 squares we defined the probability measure

$$\tau_m = \sum_{\|\lambda\|^2=m} \delta_{\lambda/\sqrt{m}}$$

on the unit circle  $\mathcal{S}^1 \subseteq \mathbb{R}^2$  supported on all the lattice points  $\lambda \in \mathbb{Z}^2$  lying on the centered radius- $\sqrt{m}$  circle in  $\mathbb{R}^2$  projected to the unit circle. In [12, Theorem 1.2] we evaluated the variance of the number of nodal intersections under the assumption that the Fourier coefficient  $\widehat{\tau}_m(4)$  is bounded away from  $\pm 1$  (i.e.  $\tau_m$  bounded away from the singular measures  $\frac{1}{4}(\delta_{\pm 1} + \delta_{\pm i})$  and its  $\pi/4$ -tilted version). Using [1] makes that assumption no longer necessary.

## 2. An approximate Kac–Rice formula

### 2.1. The Kac–Rice premise

Let  $d \geq 3$ , and  $\mathcal{C} \subseteq \mathbb{T}^d$  be a smooth curve. Let  $F$  be the arithmetic random wave

$$F(x) = \frac{1}{\sqrt{N_E}} \sum_{\mu \in \mathcal{E}(E)} a_\mu e^{2\pi i \langle \mu, x \rangle}$$

(with the obvious generalization of all the previous notation to higher dimensions  $d \geq 3$ ). We wish to study the number  $\mathcal{Z}(F)$  of nodal intersections of  $F$  with  $\mathcal{C}$  by restricting  $F$  to  $\mathcal{C}$  as follows.

Let  $\gamma : [0, L] \rightarrow \mathbb{R}$  be a unit speed parameterization of  $\mathcal{C}$ . We restrict  $F$  to  $\mathcal{C}$  by defining the random Gaussian process

$$(2.1) \quad f(t) = F(\gamma(t))$$

on  $[0, L]$ . It is then obvious that the nodal intersections number  $\mathcal{Z}(F)$  is equal to the number of the zeros of  $f$ . The Kac–Rice formula (see e.g. [8], [1]) is a standard tool (meta-theorem) for evaluating the expected number and higher (factorial) moments of zeros of a “generic” process: let  $X : I \rightarrow \mathbb{R}$  be a (a.s.  $C^1$ -smooth, say) random Gaussian process on an interval  $I \subseteq \mathbb{R}$ , and  $\mathcal{Z} = \mathcal{Z}_{I,X}$  the number of zeros of  $X$  on  $I$ . For  $m \geq 1$  and distinct points  $t_1, \dots, t_m \in I$  denote  $\varphi_{t_1, t_2, \dots, t_m}(u_1, \dots, u_m)$  to be the (Gaussian) probability density function of the random vector  $(X(t_1), \dots, X(t_m)) \in \mathbb{R}^m$ .

Then, under appropriate assumptions on  $X$ , the  $m$ -th factorial moment of  $\mathcal{Z}$  is given by

$$(2.2) \quad \mathbb{E} \left[ \mathcal{Z}^{[m]} \right] = \int_{I^m} K_m(t_1, \dots, t_m) dt_1 \dots dt_m,$$

where

$$\mathcal{Z}^{[m]} := \begin{cases} \mathcal{Z}(\mathcal{Z} - 1) \cdots (\mathcal{Z} - m + 1) & 1 \leq m \leq \mathcal{Z} \\ 0 & \text{otherwise,} \end{cases}$$

and  $K_m$ , given by

$$(2.3) \quad K_m(t_1, \dots, t_m) = \varphi_{t_1, \dots, t_m}(0, \dots, 0) \times \mathbb{E} \left[ |X'(t_1) \dots X'(t_m)| \mid X(t_1) = 0, \dots, X(t_m) = 0 \right],$$

is the  $m$ -th zero-intensity of  $\mathcal{Z}$ . Note that for the Gaussian case

$$(2.4) \quad \varphi_{t_1, \dots, t_m}(0, \dots, 0) = \frac{1}{(2\pi)^{m/2} \sqrt{\det A}}$$

with  $A$  the covariance matrix of the values  $(X(t_1), \dots, X(t_m))$ , provided that  $\det A \neq 0$ , or, equivalently, that the distribution of  $(X(t_1), \dots, X(t_m))$  is non-degenerate.

The validity of the meta-theorem (2.2) was established under a number of various scenarios. Originally the result (2.2) was proven to hold [8] provided that for all distinct points  $t_1, \dots, t_m \in I$  the distribution of the Gaussian vector  $(X(t_1), \dots, X(t_m), X'(t_1), \dots, X'(t_m)) \in \mathbb{R}^{2m}$  is non-degenerate. This non-degeneracy condition was relaxed<sup>(2)</sup> [1], as in the following theorem:

**THEOREM 2.1** ([1, Theorem 6.3]). — *Let  $X : I \rightarrow \mathbb{R}$  be a Gaussian process having  $C^1$  paths and  $m \geq 1$ . Assume that for every  $m$  pairwise distinct points  $t_1, \dots, t_m \in I$ , the joint distribution of  $(X(t_1), \dots, X(t_m)) \in \mathbb{R}^m$  is non-degenerate. Then (2.2) holds.*

The cases  $m = 1, 2$  are of our particular interest (see the following sections). The main problem is that for  $m = 2$  even the weaker non-degeneracy hypothesis in Theorem 2.1 may not be satisfied for our process  $f$  as in (2.1); to resolve this issue we will decompose the interval  $I = [0, L]$  into small subintervals, and apply Kac–Rice for each pair of the subintervals to develop “approximate Kac–Rice” formula following an idea from [12] in the two-dimensional case (see §2.4).

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<sup>(2)</sup> This fortunate fact simplifies our treatment of the approximate Kac–Rice formula below (though it is possible to work [12] with the more restrictive version to obtain the same results).



The covariance function of the centered Gaussian random field  $F$  reads

$$r_F(x, y) := \mathbb{E}[F(x)F(y)] = \frac{1}{N_E} \sum_{\mu \in \mathcal{E}} \cos(2\pi \langle \mu, y - x \rangle)$$

for  $x, y \in \mathbb{T}^d$ . As  $F$  is stationary ( $r_F$  depending on  $y - x$  only), we may think of  $r_F$  as a function of one variable on  $\mathbb{T}^d$ . The covariance function of  $f$  is

$$r(t_1, t_2) = r_f(t_1, t_2) := \mathbb{E}[f(t_1)f(t_2)] = r_F(\gamma(t_1) - \gamma(t_2)).$$

Therefore,  $f$  is a centered unit variance Gaussian process (non-stationary);  $r(t_1, t_2) \neq \pm 1$  if and only if the joint distribution of  $f(t_1), f(t_2)$  is non-degenerate, so the probability density  $\varphi_{t_1, t_2}$  of the Gaussian random vector  $(f(t_1), f(t_2))$  exists. Denote

$$r_1 := \frac{\partial r}{\partial t_1}, \quad r_2 := \frac{\partial r}{\partial t_2}, \quad r_{12} := \frac{\partial^2 r}{\partial t_1 \partial t_2},$$

and let

$$(2.5) \quad \mathcal{R}_2(E) := \int_{[0, L]^2} \left( r^2 + (r_1/\sqrt{E})^2 + (r_2/\sqrt{E})^2 + (r_{12}/E)^2 \right) dt_1 dt_2$$

be the sum of second moments of  $r$  and its few normalized derivatives along  $\mathcal{C}$ ; we will control the various quantities via  $\mathcal{R}_2$  (see Proposition 2.2 below). Later we will show that  $\mathcal{R}_2(E)$  is decaying with  $E$  (§5).

PROPOSITION 2.2 (Approximate Kac–Rice formula). — *We have*

$$\text{Var} \left( \frac{\mathcal{Z}}{\sqrt{E}} \right) = O(\mathcal{R}_2(E))$$

with  $\mathcal{R}_2$  is given by (2.5).

The rest of this section is dedicated to the proof of Proposition 2.2, finally given in §2.4, following some preparations.

### 2.2. Expectation

Since  $f$  is a centered unit variance Gaussian process with  $C^1$  paths (in particular, the non-degeneracy condition of Theorem 2.1 is automatically satisfied), we may use the Kac–Rice formula (2.2); for  $m = 1$  it reads

$$(2.6) \quad \mathbb{E}[\mathcal{Z}] = \int_I K_1(t) dt$$

with

$$K_1(t) = \frac{1}{\sqrt{2\pi}} \cdot \mathbb{E} \left[ |f'(t)| \mid f(t) = 0 \right],$$

the “zero density” of  $f$  (here (2.4) reads  $\varphi_t(0) = \frac{1}{\sqrt{2\pi}}$ ). Let  $\Gamma$  be the covariance matrix of  $(f(t), f'(t))$ :

$$\Gamma(t) = \begin{pmatrix} r(t, t) & r_1(t, t) \\ r_2(t, t) & r_{12}(t, t) \end{pmatrix}.$$

LEMMA 2.3. — *For a smooth curve of length  $L$ , the expectation of nodal intersections number is given by*

$$\mathbb{E}[\mathcal{Z}] = L \frac{2}{\sqrt{d}} \cdot \sqrt{E}$$

*Proof.* — Since  $f$  is unit variance, it is immediate that for every  $t \in [0, L]$  we have  $r_1(t, t) = r_2(t, t) = 0$ , so

$$\Gamma(t) = \begin{pmatrix} 1 & \\ & \alpha \end{pmatrix},$$

where

$$\alpha := r_{12}(t, t) = -\dot{\gamma}(t)^t H_{r_F}(0, 0) \dot{\gamma}(t)$$

and  $H_{r_F}$  is the Hessian of  $r_F$  (see e.g. [12]). Since  $H_{r_F}(0, 0)$  is a scalar matrix [13]

$$H_{r_F}(0, 0) = \frac{-4}{d} \pi^2 E \cdot I_d$$

it follows that

$$\alpha = \frac{4}{d} \pi^2 E.$$

The distribution of  $f'(t)$  conditional on  $f(t) = 0$  is centered Gaussian with variance  $\alpha$ . Recall that for  $X \sim N(0, \sigma^2)$  we have  $\mathbb{E}(|X|) = \sigma \sqrt{2/\pi}$ , so

$$(2.7) \quad K_1(t) = \frac{1}{\pi} \sqrt{\alpha} = \frac{2}{\sqrt{d}} \sqrt{E}$$

independent of  $x$ , and the statement of the lemma follows upon substituting (2.7) into (2.6). □

### 2.3. Variance

For  $m = 2$  the Kac–Rice formula (2.2) reads

$$(2.8) \quad \mathbb{E}[\mathcal{Z}^2 - \mathcal{Z}] = \int_{I \times I} K_2(t_1, t_2) dt_1 dt_2$$

with  $K_2$ , the “2-point correlation function”, defined for  $r(t_1, t_2) \neq \pm 1$  as (see (2.3) and (2.4))

$$K_2(t_1, t_2) = \frac{1}{2\pi\sqrt{1-r^2}} \cdot \mathbb{E} \left[ |f'(t_1)| \cdot |f'(t_2)| \mid f(t_1) = f(t_2) = 0 \right],$$

holding (Theorem 2.1) provided that for all  $t_1 \neq t_2$  we have  $r(t_1, t_2) \neq \pm 1$  (equivalently, the distribution of  $(f(t_1), f(t_2))$  is non-degenerate). Equivalently, the zero number variance is given by (cf. (2.6))

$$(2.9) \quad \text{Var}(\mathcal{Z}) = \int_{I \times I} (K_2(t_1, t_2) - K_1(t_1)K_1(t_2)) dt_1 dt_2 + \mathbb{E}[\mathcal{Z}].$$

As it was mentioned above, in our case the assumption that  $r(t_1, t_2) \neq \pm 1$  for all  $t_1 \neq t_2$  may not be satisfied, and, as explained in [12], it is easy to construct an example of a curve where the Kac–Rice formula for the second factorial moment (2.8) does not hold. To resolve this situation we will divide the interval  $I = [0, L]$  into small subintervals, and note that the proof of [1] Theorem 6.3 yields that if  $J_1, J_2 \subseteq I$  are two disjoint subintervals, then (recall that we denoted  $\mathcal{Z}_J$  to be the number of zeros of  $f$  on a subinterval  $J \subseteq I$ )

$$(2.10) \quad \mathbb{E}[\mathcal{Z}_{J_1} \cdot \mathcal{Z}_{J_2}] = \int_{J_1 \times J_2} K_2(t_1, t_2) dt_1 dt_2,$$

provided that for all  $t_1 \in J_1, t_2 \in J_2$ ,

$$r(t_1, t_2) \neq \pm 1.$$

The 2-point correlation function was evaluated [12] explicitly to be

$$(2.11) \quad K_2(t_1, t_2) = \frac{1}{\pi^2(1-r^2)^{3/2}} \cdot \mu \cdot (\sqrt{1-\rho^2} + \rho \arcsin \rho),$$

where

$$(2.12) \quad \mu = \mu_E(t_1, t_2) = \sqrt{\alpha(1-r^2) - r_1^2} \cdot \sqrt{\alpha(1-r^2) - r_2^2},$$

and

$$(2.13) \quad \rho = \rho_E(t_1, t_2) = \frac{r_{12}(1-r^2) + rr_1r_2}{\sqrt{\alpha(1-r^2) - r_1^2} \cdot \sqrt{\alpha(1-r^2) - r_2^2}}$$

(it follows from the derivation of (2.11) that  $|\rho| \leq 1$ ).

**2.4. Proof of Proposition 2.2**

Before giving the proof of Proposition 2.2 we will have to do some preparatory work. To overcome the above-mentioned obstacle we let  $c_0$  be a sufficiently small constant to be chosen below, and decompose the interval  $[0, L]$  into small intervals of length roughly  $c_0 \cdot \frac{1}{\sqrt{E}}$  so that we can apply Kac–Rice on the corresponding diagonal cubes. To be more concrete, let  $k = \lfloor L \cdot \frac{\sqrt{E}}{c_0} \rfloor + 1$  and  $\delta_0 = \frac{L}{k}$ , and divide the interval  $[0, L]$  into the subintervals  $I_i = [(i - 1)\delta_0, i\delta_0]$  where  $i = 1, \dots, k$ . Note that  $\delta_0 \asymp \frac{1}{\sqrt{E}}$ . With  $\mathcal{Z}_i$  denoting the number of zeros of  $f$  on  $I_i$  ( $i = 1, \dots, k$ ) we have

$$(2.14) \quad \text{Var}(\mathcal{Z}) = \sum_{i,j} \text{Cov}(\mathcal{Z}_i, \mathcal{Z}_j).$$

Our first goal is to give an upper bound for the individual summands in (2.14); to this end we need the following lemmas, whose proofs are postponed till §2.5:

LEMMA 2.4. — *There exists a constant  $c_0 > 0$  sufficiently small, such that*

*for all  $t_1 \neq t_2 \in [0, L]$  with  $|t_2 - t_1| < c_0/\sqrt{E}$  we have*

$$r(t_1, t_2) \neq \pm 1.$$

LEMMA 2.5 (Uniform bound on the 2-point correlation function around the diagonal). — *For all  $0 < |t_2 - t_1| < c_0/\sqrt{E}$  we have*

$$K_2(t_1, t_2) = O(E).$$

COROLLARY 2.6. — *We have*

$$\text{Cov}(\mathcal{Z}_i, \mathcal{Z}_j) = O(1),$$

*uniformly for all  $i, j$  and  $E$  (the implied constant is universal).*

*Proof of Corollary 2.6 assuming lemmas 2.4–2.5.* By Lemma 2.4,  $r(t_1, t_2) \neq \pm 1$  for all  $t_1 \neq t_2$  in every diagonal cube  $I_i^2$ . Hence (Theorem 2.1 applied on the interval  $I_i$  corresponding to a diagonal cube  $I_i^2$ ) we can apply Kac–Rice (2.9) to compute the variance of  $\mathcal{Z}_i$ :

$$\text{Var}(\mathcal{Z}_i) = \int_{I_i^2} (K_2(t_1, t_2) - K_1(t_1)K_1(t_2)) dt_1 dt_2 + \mathbb{E}[\mathcal{Z}_i].$$

By Kac–Rice we have

$$\mathbb{E}[\mathcal{Z}_i] = \int_{I_i} K_1(t) dt = \delta_0 \cdot \frac{2}{\sqrt{d}} \sqrt{E};$$

using Lemma 2.5 we conclude that

$$\text{Var}(\mathcal{Z}_i) \ll E\delta_0^2 + \sqrt{E}\delta_0 \ll 1.$$

This proves the statement of the corollary for  $i = j$ ; the result for arbitrary  $i, j$  follows from the above and the Cauchy–Schwartz inequality

$$\text{Cov}(\mathcal{Z}_i, \mathcal{Z}_j) \leq \sqrt{\text{Var}(\mathcal{Z}_i) \cdot \text{Var}(\mathcal{Z}_j)}. \quad \square$$

DEFINITION 2.7 (Singular and nonsingular cubes).

(1) *Let*

$$S_{ij} = I_i \times I_j = [i\delta_0, (i + 1)\delta_0] \times [j\delta_0, (j + 1)\delta_0]$$

*be a cube in  $[0, L]^2$ . We say that  $S_{ij}$  is a singular if it contains a point  $(t_1, t_2) \in S_{ij}$  satisfying*

$$|r(t_1, t_2)| > 1/2.$$

(2) *The union of all the singular cubes is the singular set*

$$B = B_E = \bigcup_{S_{ij} \text{ singular}} S_{ij}.$$

Note that since  $r/\sqrt{E}$  is a Lipschitz function with a universal constant (independent of  $E$ ), if  $S_{ij}$  is a singular cube, then

$$|r(t_1, t_2)| > 1/4$$

everywhere on  $S_{ij}$ , provided that  $c_0$  is chosen sufficiently small. Using the above it is easy to obtain the following bound on the number of singular cubes:

LEMMA 2.8. — *The number of singular cubes is bounded above by*

$$E \cdot \int_{[0, L]^2} r^2(t_1, t_2) dt_1 dt_2.$$

*Proof.* — Using the Chebyshev–Markov inequality, we see that

$$\text{meas}(B) \ll \int_{[0, L]^2} r^2(t_1, t_2) dt_1 dt_2.$$

The statement of this lemma follows from the fact that the volume of each cube is  $\asymp 1/E$ . □

With Lemma 2.8 together with Corollary 2.6 it is easy to bound the contribution to (2.14) of all  $(i, j)$  corresponding to singular cubes  $S_{ij}$  (i.e. “singular contribution”), see the proof of Proposition 2.2 below. Next we will deal with the nonsingular contribution. Here the Taylor expansion of  $K_2$  as a function of  $r$  and its scaled derivatives around  $r = r_1 = r_2 = r_{12} = 0$  (up to the quadratic terms) is valid; it will yield the following result, whose proof will be given postponed in §2.5.

LEMMA 2.9. — For  $(t_1, t_2)$  outside the singular set we have

$$(2.15) \quad |K_2(t_1, t_2) - K_1(t_1)K_1(t_2)| = E \cdot O\left(r^2 + \left(\frac{r_1}{\sqrt{E}}\right)^2 + \left(\frac{r_2}{\sqrt{E}}\right)^2 + \left(\frac{r_{12}}{E}\right)^2\right).$$

We are finally in a position to give a proof to the main result of this section.

*Proof of Proposition 2.2.* — First, it is easy to bound the total contribution of the singular set to (2.14) (i.e. all  $i, j$  with  $S_{ij}$  singular): Lemma 2.8 and Corollary 2.6 imply that it is bounded by

$$(2.16) \quad \sum_{(i,j): S_{ij} \subseteq B} \text{Cov}(\mathcal{Z}_i, \mathcal{Z}_j) = O\left(E \cdot \int_{[0,L]^2} r^2(t_1, t_2) dt_1 dt_2\right).$$

Next we deal with the indexes  $(i, j)$  corresponding to nonsingular  $S_{ij}$ .

We observe that, by the definition, for such a nonsingular cube  $S_{ij}$ , necessarily for every  $(t_1, t_2) \in S_{ij}$ ,

$$r(t_1, t_2) \neq \pm 1.$$

As this is a sufficient condition for the application of Kac–Rice formula (2.10) for computation of  $\text{Cov}(\mathcal{Z}_i, \mathcal{Z}_j)$ , bearing in mind (2.15) it yields that for  $S_{ij}$  nonsingular (this in particular implies  $i \neq j$ ),

$$\begin{aligned} \text{Cov}(\mathcal{Z}_i, \mathcal{Z}_j) &= \int_{S_{ij}} (K_2(t_1, t_2) - K_1(t_1)K_1(t_2)) dt_1 dt_2 \\ &= O\left(E \cdot \int_{S_{ij}} \left(r^2 + (r_1/\sqrt{E})^2 + (r_2/\sqrt{E})^2 + (r_{12}/E)^2\right) dt_1 dt_2\right). \end{aligned}$$

Hence the total contribution of the nonsingular set to (2.14) is  $O(E \cdot \mathcal{R}_2(E))$ . As the total contribution of the singular set to (2.14) was bounded in (2.16),

and obviously

$$\int_{[0,L]^2} r^2(t_1, t_2) dt_1 dt_2 \leq \mathcal{R}_2(E),$$

this concludes the proof of Proposition 2.2. □

### 2.5. Proofs of the auxiliary lemmas 2.4, 2.5 and 2.9

*Proof of Lemma 2.4.* — For  $t_1 \in [0, L]$  fixed, we compute the Taylor expansion of  $r(t_1, t_2)$  around  $t_2 = t_1$ . Recall that

$$r(t_1, t_2) = r_F(\gamma(t_1) - \gamma(t_2)) = \frac{1}{N} \sum_{\mu \in \mathcal{E}} \cos(2\pi \langle \mu, \gamma(t_1) - \gamma(t_2) \rangle).$$

Thus,  $r(t_1, t_1) = 1$ ,  $r_2(t_1, t_1) = 0$  and  $r_{22}(t_1, t_1) = \dot{\gamma}(t_1)^t H_{r_F}(0) \dot{\gamma}(t_1) = -\alpha$ . Moreover, we clearly have  $r_{222}(t_1, t_2) = O(E^{3/2})$ , and therefore

$$r(t_1, t_2) = 1 - \frac{\alpha}{2}(t_2 - t_1)^2 + O\left(\left(\sqrt{E}(t_2 - t_1)\right)^3\right).$$

Hence, for  $t_2 - t_1 \ll 1/\sqrt{E}$  we have

$$\begin{aligned} (2.17) \quad 1 - r^2(t_1, t_2) &= \alpha(t_2 - t_1)^2 + O\left(\left(\sqrt{E}(t_2 - t_1)\right)^3\right) \\ &= \alpha(t_2 - t_1)^2 \left(1 + O\left(\sqrt{E}(t_2 - t_1)\right)\right), \end{aligned}$$

so there is a constant  $c_0 > 0$  sufficiently small, such that  $1 - r^2(t_1, t_2)$  is strictly positive for  $0 < |t_2 - t_1| < c_0/\sqrt{E}$ . □

*Proof of Lemma 2.5.* — The function

$$(2.18) \quad G(\rho) := \frac{2}{\pi} \left(\sqrt{1 - \rho^2} + \rho \arcsin \rho\right)$$

satisfies  $\frac{2}{\pi} \leq G \leq 1$ . Hence, by the explicit form (2.11) of the 2-point correlation function  $K_2$  we obtain that

$$K_2(t_1, t_2) \ll \sqrt{\frac{(\alpha(1 - r^2) - r_1^2)(\alpha(1 - r^2) - r_2^2)}{(1 - r^2)^3}}.$$

For  $t_2 - t_1 \ll 1/\sqrt{E}$  we have

$$\begin{aligned} r_1(t_1, t_2) &= \alpha(t_2 - t_1) \left(1 + O\left(\sqrt{E}(t_2 - t_1)\right)\right), \\ r_2(t_1, t_2) &= -\alpha(t_2 - t_1) \left(1 + O\left(\sqrt{E}(t_2 - t_1)\right)\right). \end{aligned}$$

Using (2.17) we get that

$$\begin{aligned} \alpha(1 - r^2) - r_1^2 &= \alpha^2(t_2 - t_1)^2 \left(1 + O\left(\sqrt{E}(t_2 - t_1)\right)\right) \\ &\quad - \alpha^2(t_2 - t_1)^2 \left(1 + O\left(\sqrt{E}(t_2 - t_1)\right)\right) \\ &= O\left(E^{5/2}(t_2 - t_1)^3\right) \end{aligned}$$

and likewise

$$\alpha(1 - r^2) - r_2^2 = O\left(E^{5/2}(t_2 - t_1)^3\right),$$

so

$$K_2(t_1, t_2) \ll \frac{O\left(E^{5/2}(t_2 - t_1)^3\right)}{\alpha^{3/2}(t_2 - t_1)^3 \left(1 + O\left(\sqrt{E}(t_2 - t_1)\right)\right)} = O(E),$$

assuming  $0 < |t_2 - t_1| < c_0/\sqrt{E}$  for a sufficiently small constant  $c_0 > 0$ .  $\square$

*Proof of Lemma 2.9.* — Recall from (2.11) that

$$(2.19) \quad K_2(t_1, t_2) = \frac{1}{2\pi} \cdot \frac{1}{(1 - r^2)^{3/2}} \cdot G(\rho) \cdot \mu$$

where  $G$ ,  $\mu$  and  $\rho$  are defined respectively in (2.18), (2.12) and (2.13). Note that for every  $|\rho| \leq 1$ ,

$$G(\rho) = \frac{2}{\pi} + O(\rho^2).$$

For  $r, r_1/\sqrt{E}, r_2/\sqrt{E}, r_{12}/E$  small, we have

$$\rho = O(r + r_{12}/E).$$

Moreover, since  $|\rho| \leq 1$ , this bound holds for every  $(t_1, t_2)$ . Thus, for every  $(t_1, t_2)$  we have

$$G(\rho) = \frac{2}{\pi} + O\left(r^2 + (r_{12}/E)^2\right).$$

For every  $(t_1, t_2)$ ,

$$\mu = \alpha + E \cdot O\left(r^2 + (r_1/\sqrt{E})^2 + (r_2/\sqrt{E})^2\right),$$

and for  $r$  bounded away from  $\pm 1$  we have

$$\frac{1}{(1 - r^2)^{3/2}} = 1 + O(r^2).$$

Substituting all the expansions in (2.19), we get that

$$K_2(t_1, t_2) = \frac{\alpha}{\pi^2} + E \cdot O\left(r^2 + (r_1/\sqrt{E})^2 + (r_2/\sqrt{E})^2 + (r_{12}/E)^2\right).$$

The statement of the lemma now follows, recalling that by (2.7),  $K_1(t) = \sqrt{\alpha}/\pi$  for all  $t \in [0, L]$ .  $\square$



### 3. Oscillatory integrals and curvature

In this section we investigate certain oscillatory integrals on curves which arise in our work. A key role is played by the differential geometry of the curve.

#### 3.1. Differential geometry of 3-dimensional curves

For a smooth curve in  $\mathbb{R}^3$ , with arc-length parameterization  $\gamma : [0, L] \rightarrow \mathcal{C} \subset \mathbb{R}^3$ , so that  $T(t) = \gamma'(t)$  is the unit tangent, the curvature of  $\gamma$  at  $\gamma(t)$  is  $\kappa(t) = \|\gamma''(t)\|$ . We assume that  $\kappa(t)$  never vanishes, so that  $\gamma''(t) = \kappa(t)N(t)$  with  $N(t)$  the unit normal, and under the same assumption the torsion  $\tau(t)$  is  $B'(t) = -\tau(t)N(t)$  where  $B = T \times N$  is the binormal vector. The orthonormal basis  $(T, N, B)$  is called the Frenet–Serret frame of the curve. Recall the Frenet–Serret formulas

$$\begin{aligned} T'(t) &= \kappa(t)N(t) \\ N'(t) &= -\kappa(t)T(t) + \tau(t)B(t) \\ B'(t) &= -\tau(t)N(t) \end{aligned}$$

so in particular

$$T''(t) = \kappa'(t)N(t) - \kappa^2(t)T(t) + \kappa(t)\tau(t)B(t).$$

Let  $K_{\min}$  and  $K_{\max}$  the minimal and the maximal curvature of  $\mathcal{C}$  respectively. Since the curvature is assumed to be nowhere vanishing, we have

$$0 < K_{\min} \leq \kappa(t) \leq K_{\max}.$$

#### 3.2. Oscillatory integrals

Recall the classical form of Van der Corput Lemma: let  $[a, b]$  be a finite interval,  $\phi \in C^\infty[a, b]$  a smooth and real valued phase function, and  $A \in C^\infty[a, b]$  a smooth amplitude. For  $\lambda > 0$  define the oscillatory integral

$$I(\lambda) := \int_a^b A(t) e^{i\lambda\phi(t)} dt.$$

LEMMA 3.1 (Van der Corput). — For  $k \geq 2$ , if  $|\phi^{(k)}| \geq 1$ , then, as  $\lambda \rightarrow \infty$ ,

$$|I(\lambda)| \ll \frac{1}{\lambda^{1/k}} (\|A\|_\infty + \|A'\|_1).$$

If  $|\phi'| \geq 1$  and  $\phi'$  is monotone, then

$$|I(\lambda)| \ll \frac{1}{\lambda} (\|A\|_\infty + \|A'\|_1).$$

The implied constants are absolute.

Remark 3.2. — If  $|\phi'| \geq 1$  then, independent of the monotonicity hypothesis on  $\phi'$ ,

$$|I(\lambda)| \ll \frac{b - a + 2}{\lambda} (\|A\|_\infty + \|A'\|_1).$$

### 3.3. Curves with nowhere vanishing torsion

Assume that the curve  $\mathcal{C}$  has nowhere vanishing torsion, so that  $0 < T_{\min} \leq |\tau(t)| \leq T_{\max}$ , where  $T_{\min}$  and  $T_{\max}$  are the minimal and maximal absolute value of the torsion of  $\mathcal{C}$  respectively. Consider a unit vector  $\xi \in S^2$ , and the phase function

$$(3.1) \quad \phi_\xi(t) := \langle \xi, \gamma(t) \rangle$$

for  $t \in [0, L]$ . We define an oscillatory integral

$$I(\lambda, \xi) := \int_0^L A(t) e^{i\lambda\phi_\xi(t)} dt.$$

We apply Lemma 3.1 to give an upper bound (uniform in  $\xi$ ) for  $I(\lambda, \xi)$ :

PROPOSITION 3.3. — Let  $\mathcal{C}$  be a smooth curve with nowhere vanishing curvature and torsion. Then

$$I(\lambda, \xi) \ll_c \frac{1}{\lambda^{1/3}} (\|A\|_\infty + \|A'\|_1).$$

Proof. — We have

$$(3.2) \quad \phi'_\xi(t) = \langle \xi, T(t) \rangle,$$

$$(3.3) \quad \phi''_\xi(t) = \kappa(t) \langle \xi, N(t) \rangle,$$

and

$$\phi'''_\xi(t) = \kappa'(t) \langle \xi, N(t) \rangle - \kappa^2(t) \langle \xi, T(t) \rangle + \kappa(t) \tau(t) \langle \xi, B(t) \rangle.$$

Since  $(T, N, B)$  is an orthonormal basis for  $\mathbb{R}^3$ , we know that

$$(3.4) \quad 1 = |\xi|^2 = |\langle \xi, T(t) \rangle|^2 + |\langle \xi, N(t) \rangle|^2 + |\langle \xi, B(t) \rangle|^2.$$

Now let

$$c = \min \left( \frac{1}{3}, \frac{K_{\min}^2 T_{\min}^2}{48 \|\kappa'\|_{\infty}^2}, \frac{K_{\min}^2 T_{\min}^2}{48 K_{\max}^4} \right)$$

(if  $\|\kappa'\|_{\infty} = 0$  omit the middle term). If  $|\langle \xi, T(t) \rangle|^2 \geq c$ , then  $|\phi'_{\xi}(t)| \geq \sqrt{c}$ . If  $|\langle \xi, N(t) \rangle|^2 \geq c$ , then  $|\phi''_{\xi}(t)| \geq \sqrt{c} K_{\min}$ . Otherwise, i.e. if both  $|\langle \xi, T(t) \rangle|^2 < c$  and  $|\langle \xi, N(t) \rangle|^2 < c$ , then necessarily  $|\langle \xi, B(t) \rangle|^2 \geq \frac{1}{3}$ , so

$$|\phi'''_{\xi}(t)| \geq \frac{K_{\min} T_{\min}}{\sqrt{3}} - K_{\max}^2 \sqrt{c} - \|\kappa'\|_{\infty} \sqrt{c} \geq \frac{K_{\min} T_{\min}}{2\sqrt{3}}.$$

Note that  $\|\phi'_{\xi}\|_{\infty} \leq 1$ ,  $\|\phi''_{\xi}\|_{\infty} \leq K_{\max}$ ,

$$\|\phi'''_{\xi}\|_{\infty} \leq \left( \|\kappa'\|_{\infty}^2 + K_{\max}^4 + K_{\max}^2 T_{\max}^2 \right)^{1/2}.$$

Using again the Frenet–Serret formulas, we can also get an upper bound in the same fashion for the fourth derivative, say

$$\|\phi_{\xi}^{(4)}\|_{\infty} \leq C = C(K_{\max}, T_{\max}, \|\kappa'\|_{\infty}, \|\kappa''\|_{\infty}, \|\tau'\|_{\infty}).$$

Assume now that  $|\phi'_{\xi}(t_0)| \geq \sqrt{c}$  for some  $t_0 \in [0, L]$ . Then for every  $t$  such that  $|t - t_0| \leq \frac{\sqrt{c}}{2K_{\max}}$  we have

$$\sqrt{c} - |\phi'_{\xi}(t)| \leq |\phi'_{\xi}(t_0)| - |\phi'_{\xi}(t)| \leq |\phi'_{\xi}(t) - \phi'_{\xi}(t_0)| \leq |t - t_0| \|\phi''_{\xi}\|_{\infty} \leq \frac{\sqrt{c}}{2}$$

so  $|\phi'_{\xi}(t)| \geq \frac{\sqrt{c}}{2}$ . Similarly, if  $|\phi''_{\xi}(t_0)| \geq \sqrt{c} K_{\min}$  or  $|\phi'''_{\xi}(t_0)| \geq \frac{K_{\min} T_{\min}}{2\sqrt{3}}$ , then  $\phi''_{\xi}$  or  $\phi'''_{\xi}$  is bounded away from zero on some interval around  $t_0$ , with length independent of  $\xi$ . Hence the interval  $[0, L]$  may be divided into a finite, independent of  $\xi$ , number of subintervals, such that for every  $\xi$  either  $\phi'_{\xi}$ , or  $\phi''_{\xi}$ , or  $\phi'''_{\xi}$  is bounded away from zero on each of the subintervals. We conclude the proof of the proposition by an application of Lemma 3.1 and the remark following it.  $\square$

### 3.4. Real analytic curves

Assume now that  $\mathcal{C}$  is a real analytic, non-planar curve with nowhere zero curvature. Then the torsion of  $\mathcal{C}$  has finitely many zeros, each of them is of finite order. We have already treated the case when the torsion is

nowhere zero. Assume now, without loss of generality, that there is exactly one point  $t_0 \in [0, L]$  with zero torsion of order  $m \geq 1$ , namely

$$(3.5) \quad \tau(t_0) = \dots = \tau^{(m-1)}(t_0) = 0, \quad \tau^{(m)}(t_0) \neq 0.$$

Recall that under the notation of the previous section

$$I(\lambda, \xi) := \int_0^L A(t) e^{i\lambda\phi_\xi(t)} dt.$$

We prove the following result.

**PROPOSITION 3.4.** — *Let  $\mathcal{C}$  be a non-planar real analytic curve with nowhere zero curvature, which has exactly one point with zero torsion of order  $m \geq 1$ . Then*

$$I(\lambda, \xi) \ll c \frac{1}{\lambda^{1/(m+3)}} (\|A\|_\infty + \|A'\|_1).$$

*Proof.* — Using the Frenet–Serret formulas and (3.5), we get that

$$\begin{aligned} \phi_\xi^{(m+3)}(t_0) &= P(t_0) \langle \xi, T(t_0) \rangle + Q(t_0) \langle \xi, N(t_0) \rangle \\ &\quad + \kappa(t_0) \tau^{(m)}(t_0) \langle \xi, B(t_0) \rangle \end{aligned}$$

where  $P, Q$  are polynomials in  $\kappa, \kappa', \dots, \kappa^{(m+1)}$ .

Choose

$$c = \min \left( \frac{1}{3}, \frac{\kappa(t_0)^2 (\tau^{(m)}(t_0))^2}{48P(t_0)^2}, \frac{\kappa(t_0)^2 (\tau^{(m)}(t_0))^2}{48Q(t_0)^2} \right)$$

(if  $P(t_0) = 0$  or  $Q(t_0) = 0$ , omit the corresponding terms). As in the proof of Proposition 3.3, by the orthonormality of  $(T, N, B)$ , either  $|\phi'_\xi(t_0)| \geq \sqrt{c}$ ,  $|\phi''_\xi(t_0)| \geq \kappa(t_0) \sqrt{c}$ , or (if both  $\phi'_\xi(t_0), \phi''_\xi(t_0)$  are small)  $|\langle \xi, B(t_0) \rangle|^2 \geq \frac{1}{3}$ . Hence

$$\begin{aligned} |\phi_\xi^{(m+3)}(t_0)| &\geq \frac{1}{\sqrt{3}} \kappa(t_0) |\tau^{(m)}(t_0)| - P(t_0) \sqrt{c} - Q(t_0) \sqrt{c} \\ &\geq \frac{1}{2\sqrt{3}} \kappa(t_0) |\tau^{(m)}(t_0)|. \end{aligned}$$

Since all the derivatives of  $\phi_\xi$  are bounded from above, uniformly w.r.t.  $\xi$ , we conclude that either the first, the second or the  $(m + 3)$ -th derivative of  $\phi_\xi$  is bounded away from zero on an interval around  $t_0$  of length independent of  $\xi$ . Outside that interval the torsion doesn't vanish, so that in a neighborhood (of length independent of  $\xi$ ) around any point outside this interval, either the first, or the second, or the third derivative of  $\phi_\xi$  is bounded away from zero. Dividing the interval  $[0, L]$  to a finite number

(independent of  $\xi$ ) of subintervals, and applying Lemma 3.1 to each of the subintervals, we finally deduce the statement of Proposition 3.4.  $\square$

#### 4. Background on sums of three squares

A positive integer  $E$  is a sum of three squares if and only if  $E \neq 4^a(8b+7)$ . Let  $\mathcal{E}(E)$  be the set of solutions

$$\mathcal{E}(E) = \{\vec{x} \in \mathbb{Z}^3 : |\vec{x}|^2 = E\}$$

and denote by  $N = N_E$  the number of solutions

$$N = N_E := \#\mathcal{E}(E).$$

Gauss' formula expresses  $N_E$  in terms of class numbers. For  $E$  square-free, it says that

$$\#\mathcal{E}(E) = \frac{24h(d_E)}{w_E} \left(1 - \left(\frac{d_E}{2}\right)\right)$$

where  $d_E$ ,  $h(d_E)$  and  $w_E$  are the discriminant, class number and the number of units in the quadratic field  $\mathbb{Q}(\sqrt{-E})$ . Using Dirichlet's class number formula, one may then express  $\#\mathcal{E}(E)$  by means of the special value  $L(1, \chi_{d_E})$  of the associated quadratic  $L$ -function: If  $E \not\equiv 7 \pmod{8}$  is square-free then

$$N_E = c_E \sqrt{E} \cdot L(1, \chi_{d_E})$$

where  $c_E$  only depends on the remainder of  $E$  modulo 8. We may bound the number  $\#\mathcal{E}(E)$  of such points as

$$\#\mathcal{E}(E) \ll E^{1/2+\epsilon}$$

for all  $\epsilon > 0$ .

The existence of a *primitive* lattice point (i.e.  $\vec{x} = (x_1, x_2, x_3)$  with  $\gcd(x_1, x_2, x_3) = 1$  and  $\|x\|^2 = E$ ) is equivalent to  $E \not\equiv 0, 4, 7 \pmod{8}$ . If it is indeed the case, then Siegel's theorem yields a lower bound

$$(4.1) \quad \#\mathcal{E}(E) \gg E^{1/2-\epsilon}.$$

A fundamental result conjectured by Linnik (established by himself under the Generalized Riemann Hypothesis), is that for  $E \not\equiv 0, 4, 7 \pmod{8}$ , the points

$$\widehat{\mathcal{E}}(E) := \frac{1}{\sqrt{E}} \mathcal{E}(E) \subset S^2$$

obtained by projecting to the unit sphere, become equidistributed on the unit sphere with respect to the normalized Lebesgue measure as

$E \rightarrow \infty$ . This was proved unconditionally by Duke [9], and Golubeva and Fomenko [10].

The “Riesz  $s$ -energy” of  $N$  points  $x_1, \dots, x_N$  on  $S^2$  is defined as

$$(4.2) \quad E_s(x_1, \dots, x_N) := \sum_{i \neq j} \frac{1}{|x_i - x_j|^s}.$$

A forthcoming result of Bourgain, Rudnick and Sarnak [5] (announced in [6] for the electrostatic case  $s = 1$ ) yields a precise asymptotic expression for  $E_s(\widehat{\mathcal{E}}(E))$ : for every  $0 < s < 2$  if  $E \rightarrow \infty$  such that  $E \not\equiv 0, 4, 7 \pmod 8$ , then there exists some  $\delta > 0$  so that

$$(4.3) \quad E_s(\widehat{\mathcal{E}}(E)) = I(s)N^2 + O(N^{2-\delta})$$

with

$$I(s) = \int_{S^2} \frac{1}{|x - x_0|^s} d\sigma(x) = \frac{2^{1-s}}{2 - s},$$

with  $x_0 \in S^2$  any point on the sphere, and  $d\sigma$  the Lebesgue measure, normalized to have unit area.

As the details of (4.3) have not appeared at the time of writing, we will prove the following simple bound, which suffices for Theorem 1.5:

PROPOSITION 4.1. — *Fix  $0 < s \leq 1$ . Then for  $E \not\equiv 0, 4, 7 \pmod 8$ ,*

$$E_s(\widehat{\mathcal{E}}(E)) \ll N^2 E^\eta, \quad \forall \eta > 0.$$

The proof of Proposition 4.1 will be given in Appendix A.

### 5. The second moment of $r$ and its derivatives

We wish to bound the second moment of the covariance function  $r$  and its derivatives. It is here that we need the full arithmetic input described in §4. Recall that

$$(5.1) \quad \begin{aligned} r(t_1, t_2) &= r_F(\gamma(t_1) - \gamma(t_2)) = \frac{1}{N} \sum_{\mu \in \mathcal{E}} \cos(2\pi \langle \mu, \gamma(t_1) - \gamma(t_2) \rangle) \\ &= \frac{1}{N} \sum_{\mu \in \mathcal{E}} e(i\langle \mu, \gamma(t_1) - \gamma(t_2) \rangle). \end{aligned}$$

### 5.1. Non-planar curves

Recall that given  $E$  we defined  $\mathcal{R}_2(E)$  as in (2.5). Proposition 2.2 shows that in order to bound the nodal intersections variance from above it is sufficient to bound  $\mathcal{R}_2(E)$ , which is claimed in the following proposition for the non-planar case.

PROPOSITION 5.1. — *Assume that the curve  $\mathcal{C}$  is smooth, with nowhere zero curvature and torsion. Then for every  $\eta > 0$  we have*

$$(5.2) \quad \mathcal{R}_2(E) \ll \frac{1}{E^{1/3-\eta}}.$$

Remark 5.2. — For real-analytic non-planar curves with non-vanishing curvature, the same argument as below, invoking Proposition 3.4 instead of Proposition 3.3, yields

$$(5.3) \quad \mathcal{R}_2(E) \ll 1/E^\delta$$

for some  $\delta = \delta(\mathcal{C}) > 0$ .

Proof. — In what follows we will establish the following bounds on the 2nd moment of  $r$  and some of its normalized derivatives along  $\mathcal{C}$ : for all  $\eta > 0$  we have

$$(5.4) \quad \iint_{[0,L]^2} r(t_1, t_2)^2 dt_1 dt_2 \ll \frac{1}{E^{1/3-\eta}},$$

$$(5.5) \quad \iint_{[0,L]^2} \left( r_i(t_1, t_2)/\sqrt{E} \right)^2 dt_1 dt_2 \ll \frac{1}{E^{1/3-\eta}} \quad (i = 1, 2),$$

and

$$(5.6) \quad \iint_{[0,L]^2} (r_{12}(t_1, t_2)/E)^2 dt_1 dt_2 \ll \frac{1}{E^{1/3-\eta}}.$$

The statement (5.2) of Proposition 5.1 will follow at once upon substituting (5.4), (5.5) and (5.6) into the definition (2.5) of  $\mathcal{R}_2(E)$ .

First we show (5.4). Squaring out and integrating (5.1), we find

$$\begin{aligned} \iint_{[0,L]^2} r(t_1, t_2)^2 dt_1 dt_2 &= \frac{1}{N^2} \sum_{\mu \in \mathcal{E}} \sum_{\mu' \in \mathcal{E}} \iint_{[0,L]^2} e(\langle \mu - \mu', \gamma(t_1) - \gamma(t_2) \rangle) dt_1 dt_2 \\ &= \frac{L^2}{N} + \frac{1}{N^2} \sum_{\mu \neq \mu'} \left| \int_0^L e(\langle \mu - \mu', \gamma(t) \rangle) dt \right|^2. \end{aligned}$$

Since  $\gamma$  has nowhere vanishing curvature and torsion, we deduce from Proposition 3.3 that

$$\int_0^L e(\langle \mu - \mu', \gamma(t) \rangle) dt \ll \frac{1}{|\mu - \mu'|^{1/3}},$$

which yields

$$\iint_{[0,L]^2} r(t_1, t_2)^2 dt_1 dt_2 = \frac{L^2}{N} + O\left(\frac{1}{N^2} \sum_{\mu \neq \mu'} \frac{1}{|\mu - \mu'|^{2/3}}\right).$$

The summation inside the error term  $O(\dots)$  is  $1/E^{1/3}$  times the “Riesz 2/3-energy” of the set of projected lattice points  $\widehat{\mathcal{E}}(E) = \frac{1}{\sqrt{E}}\mathcal{E}(E) \subset S^2$ . By Proposition 4.1,

$$\sum_{\substack{\widehat{\mu}, \widehat{\mu}' \in \widehat{\mathcal{E}} \\ \widehat{\mu} \neq \widehat{\mu}'}} \frac{1}{|\widehat{\mu} - \widehat{\mu}'|^{2/3}} \ll N^2 \cdot E^\eta, \quad \forall \eta > 0,$$

and hence

$$\frac{1}{N^2} \sum_{\mu \neq \mu'} \frac{1}{|\mu - \mu'|^{2/3}} = \frac{1}{E^{1/3}} \frac{1}{N^2} \sum_{\substack{\widehat{\mu}, \widehat{\mu}' \in \widehat{\mathcal{E}} \\ \widehat{\mu} \neq \widehat{\mu}'}} \frac{1}{|\widehat{\mu} - \widehat{\mu}'|^{2/3}} \ll \frac{1}{E^{1/3-\eta}},$$

which yields (5.4).

Now we turn to proving (5.5). We have

$$\frac{1}{2\pi i \sqrt{E}} r_1(t_1, t_2) = \frac{1}{N} \sum_{\mu} \left\langle \frac{\mu}{|\mu|}, \dot{\gamma}(t_1) \right\rangle e^{2\pi i \langle \mu, \gamma(t_1) - \gamma(t_2) \rangle}.$$

Denote

$$A_{\mu, \mu'}(t) = \left\langle \frac{\mu}{|\mu|}, \dot{\gamma}(t) \right\rangle \left\langle \frac{\mu'}{|\mu'|}, \dot{\gamma}(t) \right\rangle.$$

Then

$$\begin{aligned} (5.7) \quad & \iint_{[0,L]^2} \left| \frac{1}{2\pi \sqrt{E}} r_1(t_1, t_2) \right|^2 dt_1 dt_2 = \frac{L}{N^2} \sum_{\mu} \int_0^L A_{\mu, \mu}(t_1) dt_1 \\ & + \frac{1}{N^2} \sum_{\substack{\mu, \mu' \in \mathcal{E} \\ \mu \neq \mu'}} \int_0^L A_{\mu, \mu'}(t_1) e^{2\pi i \langle \mu - \mu', \gamma(t_1) \rangle} dt_1 \int_0^L e^{2\pi i \langle \mu' - \mu, \gamma(t_2) \rangle} dt_2. \end{aligned}$$

To evaluate the main term, we use the fact (see [13, Lemma 2.3]) that for every  $v \in \mathbb{R}^d$ ,

$$\frac{1}{N} \sum_{\mu \in \mathcal{E}} \langle \mu, v \rangle^2 = \frac{E}{d} \|v\|^2.$$



Hence,

$$\frac{L}{N^2} \sum_{\mu} \int_0^L A_{\mu,\mu}(t_1) dt_1 = \frac{L^2}{dN}.$$

As for the off-diagonal terms, note that  $\|A_{\mu,\mu'}\|_{\infty} \leq 1$ ,  $\|A'_{\mu,\mu'}\|_{\infty} \leq 2K_{\max}$ , so by Proposition 3.3 each of the two last integrals in (5.7) is bounded above by  $1/|\mu - \mu'|^{1/3}$ . From here we continue as in the proof of (5.4) to obtain the estimate

$$\iint_{[0,L]^2} \left( r_1(t_1, t_2)/\sqrt{E} \right)^2 dt_1 dt_2 \ll \frac{1}{E^{1/3-\eta}}, \quad \forall \eta > 0$$

and a similar proof yields the same bound for the (normalized) second moment of  $r_2$ .

Finally we turn to proving (5.6). We have

$$\begin{aligned} & \iint_{[0,L]^2} \left| \frac{1}{4\pi^2 E} r_{12}(t_1, t_2) dt_1 dt_2 \right|^2 \\ &= \frac{1}{N^2} \sum_{\mu} \iint_{[0,L]^2} A_{\mu,\mu}(t_1) A_{\mu,\mu}(t_2) dt_1 dt_2 \\ & \quad + \frac{1}{N^2} \sum_{\substack{\mu, \mu' \in \mathcal{E} \\ \mu \neq \mu'}} \left| \int_0^L A_{\mu,\mu'}(t) e^{2\pi i \langle \mu - \mu', \gamma(t) \rangle} dt \right|^2, \end{aligned}$$

The diagonal term is bounded above by  $L^2/N$ ; by Proposition 3.3 the off-diagonal terms are bounded above by  $1/|\mu - \mu'|^{1/3}$ , so we similarly deduce (5.6). □

### 5.2. Planar curves

The goal of this section is proving the following estimate on  $\mathcal{R}_2$  for  $\mathcal{C}$  planar.

PROPOSITION 5.3. — *Assume that  $\mathcal{C}$  is a smooth planar curve, with nowhere zero curvature. Then for all  $\eta > 0$*

$$(5.8) \quad \mathcal{R}_2(E) \ll \frac{1}{E^{1/4-\eta}}.$$

We will now collect a few results needed for Proposition 5.3, whose proof is given towards the end of this section. In this section we assume that  $\mathcal{C}$  is a smooth planar curve with nowhere zero curvature, so that  $\tau \equiv 0$ ;

the binormal vector  $B$  is constant in this case. Let  $\epsilon = \epsilon(E)$  be a small parameter,  $\mu \neq \mu' \in \mathcal{E}(E)$ ,  $\lambda = \lambda(\mu, \mu') = |\mu - \mu'|$  and  $\xi = \xi(\mu, \mu') = \frac{\mu - \mu'}{|\mu - \mu'|}$ . We will reuse the definition (3.1) of  $\phi_\xi$ .

LEMMA 5.4. — For all  $\xi \in S^2$  and for all  $t \in [0, L]$  either  $|\phi'_\xi(t)| \geq \sqrt{\epsilon}$ , or  $|\phi''_\xi(t)| \geq \sqrt{\epsilon}K_{\min}$ , or otherwise

$$|\langle \xi, B \rangle|^2 > 1 - 2\epsilon.$$

Proof. — This follows from (3.4) via (3.2) and (3.3). □

LEMMA 5.5. — Let  $\mu, \mu_1, \mu_2$  be distinct points on the sphere  $\sqrt{E}S^2$ , and assume that

$$\left\langle \frac{\mu_i - \mu}{|\mu_i - \mu|}, B \right\rangle > \sqrt{1 - 2\epsilon}$$

for  $i = 1, 2$ . Then  $|\mu_2 - \mu_1| \leq 16\sqrt{\epsilon E}$ .

Proof. — Let  $v_i = \frac{\mu_i - \mu}{|\mu_i - \mu|} - B$  ( $i = 1, 2$ ), so that  $|v_i|^2 = 2 - 2\left\langle \frac{\mu_i - \mu}{|\mu_i - \mu|}, B \right\rangle \leq 4\epsilon$ . We write

$$\mu_i = \mu + |\mu_i - \mu|(B + v_i).$$

Taking norms we get

$$0 = |\mu_i - \mu|^2 + 2|\mu_i - \mu|\langle \mu, B + v_i \rangle,$$

so that

$$|\mu_i - \mu| = -2\langle \mu, B \rangle - 2\langle \mu, v_i \rangle,$$

and therefore

$$\mu_i = \mu - 2\langle \mu, B \rangle B - 2\langle \mu, v_i \rangle (B + v_i) - 2\langle \mu, B \rangle v_i.$$

By Cauchy–Schwartz  $\langle \mu, v_i \rangle \leq 2\sqrt{\epsilon E}$ , and  $\langle \mu, B \rangle \leq \sqrt{E}$ , and that implies

$$|\mu_2 - \mu_1| \leq 16\sqrt{\epsilon E}. \quad \square$$

We are now in a position to prove Proposition 5.3.

Proof of Proposition 5.3. In what follows we will establish the following bounds on the 2nd moment of  $r$  and some of its normalized derivatives along  $\mathcal{C}$  (assumed to be planar, with nowhere zero curvature):

$$(5.9) \quad \iint_{[0, L]^2} r(t_1, t_2)^2 dt_1 dt_2 = O\left(\frac{1}{E^{1/4-\eta}}\right),$$

$$(5.10) \quad \iint_{[0, L]^2} \left(r_i(t_1, t_2)/\sqrt{E}\right)^2 dt_1 dt_2 = O\left(\frac{1}{E^{1/4-\eta}}\right),$$

$i = 1, 2$ , and

$$(5.11) \quad \iint_{[0, L]^2} (r_{12}(t_1, t_2)/E)^2 dt_1 dt_2 = O\left(\frac{1}{E^{1/4-\eta}}\right).$$

First we prove (5.9). If  $B$  is the constant binormal to the curve, then

$$(5.12) \quad \iint_{[0, L]^2} r(t_1, t_2)^2 dt_1 dt_2 = \frac{L^2}{N} + \frac{1}{N^2} \sum_{\substack{\mu \neq \mu' \\ |\langle \xi(\mu, \mu'), B \rangle|^2 \leq 1-2\epsilon}} I(|\mu - \mu'|, \xi)^2 + \frac{1}{N^2} \sum_{\substack{\mu \neq \mu' \\ |\langle \xi(\mu, \mu'), B \rangle|^2 > 1-2\epsilon}} I(|\mu - \mu'|, \xi)^2$$

(here  $A$ , the amplitude involved in  $I(\lambda, \xi)$ , is  $A(t) \equiv 1$ ).

To bound the first summation in (5.12) we observe that for  $\mu, \mu'$  with

$$|\langle \xi(\mu, \mu'), B \rangle|^2 \leq 1 - 2\epsilon$$

we have, thanks to Lemma 5.4, that for every  $t \in [0, L]$  either  $|\phi'_\xi(t)| \geq \sqrt{\epsilon}$  or  $|\phi''_\xi(t)| \geq \sqrt{\epsilon} K_{\min}$ . Hence Lemma 3.1, using the same arguments as in the proof of Proposition 3.3, yields the following bound, uniform in  $\xi$ :

$$(5.13) \quad I(\lambda, \xi) = \int_0^L A(t) e^{i\lambda\phi_\xi(t)} dt \ll_c \min \left\{ \|A\|_\infty, \frac{1}{(\sqrt{\epsilon}\lambda)^{1/2}} (\|A\|_\infty + \|A'\|_1) \right\}.$$

We may then bound the first summation in (5.12) as

$$(5.14) \quad \begin{aligned} & \frac{1}{N^2} \sum_{\substack{\mu \neq \mu' \\ |\langle \xi(\mu, \mu'), B \rangle|^2 \leq 1-2\epsilon}} I(|\mu - \mu'|, \xi)^2 \\ & \ll \frac{1}{N^2} \sum_{\substack{\mu \neq \mu' \\ |\mu - \mu'| \leq 1/\sqrt{\epsilon}}} 1 + \frac{1}{N^2} \sum_{\substack{\mu \neq \mu' \\ |\mu - \mu'| > 1/\sqrt{\epsilon}}} \frac{1}{\sqrt{\epsilon} |\mu - \mu'|} \\ & \ll \frac{1}{N^2} \sum_{\mu} \#\{\mu' : |\mu - \mu'| \leq 1/\sqrt{\epsilon}\} + \frac{1}{\sqrt{\epsilon} N^2} \sum_{\mu \neq \mu'} \frac{1}{|\mu - \mu'|}. \end{aligned}$$

The second summation in the r.h.s. of (5.14) is  $1/E^{1/2}$  times the ‘‘Riesz 1-energy’’ of the set of projected lattice points  $\hat{\mathcal{E}}(E) = \frac{1}{\sqrt{E}} \mathcal{E}(E) \subset S^2$ , which

by Proposition 4.1 is bounded by

$$\sum_{\substack{\widehat{\mu}, \widehat{\mu}' \in \widehat{\mathcal{E}} \\ \widehat{\mu} \neq \widehat{\mu}'}} \frac{1}{|\widehat{\mu} - \widehat{\mu}'|} \ll N^2 \cdot E^\eta, \quad \forall \eta > 0,$$

or,

$$\frac{1}{\sqrt{\epsilon}N^2} \sum_{\mu \neq \mu'} \frac{1}{|\mu - \mu'|} \ll \frac{E^{-1/2+\eta}}{\sqrt{\epsilon}}, \quad \forall \eta > 0.$$

For every  $\eta > 0$ , the number of lattice points in a spherical cap of radius  $r$  on the sphere  $RS^2$  can be easily shown to be  $O(R^\eta (1 + r))$  (e.g. [3, Lemma 2.2]), so that the total number of solutions to  $|\mu - \mu'| \leq 1/\sqrt{\epsilon}$  is  $O(E^\eta/\sqrt{\epsilon})$ . Hence, the first summation in the r.h.s. of (5.14) is  $O(E^\eta/\sqrt{\epsilon}N)$ , and (5.14) is

$$(5.15) \quad \frac{1}{N^2} \sum_{\substack{\mu \neq \mu' \\ |\langle \xi(\mu, \mu'), B \rangle|^2 \leq 1-2\epsilon}} I(|\mu - \mu'|, \xi)^2 \ll \frac{E^{\eta-1/2}}{\sqrt{\epsilon}}.$$

For the second summation on the r.h.s. of (5.12) we bound each of the integrals  $I(|\mu - \mu'|, \xi)$  from above trivially by 1, yielding

$$(5.16) \quad \frac{1}{N^2} \sum_{\substack{\mu \neq \mu' \\ |\langle \xi(\mu, \mu'), B \rangle|^2 > 1-2\epsilon}} I(|\mu - \mu'|, \xi)^2 \ll \frac{1}{N^2} \sum_{\mu} \# \left\{ \mu' : |\langle \xi(\mu, \mu'), B \rangle|^2 > 1 - 2\epsilon \right\}.$$

Using Lemma 5.5, we see that all of the lattice points satisfying

$$|\langle \xi(\mu, \mu'), B \rangle|^2 > 1 - 2\epsilon$$

are contained in two spherical caps of radius  $\ll \sqrt{\epsilon E}$  so that the total number of solutions to  $|\langle \xi(\mu, \mu'), B \rangle|^2 > 1 - 2\epsilon$  is  $O(E^\eta (1 + \sqrt{\epsilon E}))$ . Therefore

$$(5.17) \quad \frac{1}{N^2} \sum_{\substack{\mu \neq \mu' \\ |\langle \xi(\mu, \mu'), B \rangle|^2 > 1-2\epsilon}} I(|\mu - \mu'|, \xi)^2 \ll \frac{E^\eta (1 + \sqrt{\epsilon E})}{N} \ll E^{\eta-1/2} + \sqrt{\epsilon}E^\eta.$$

Substituting (5.15) and (5.17) into (5.12) yields the inequality

$$(5.18) \quad \iint_{[0,L]^2} r(t_1, t_2)^2 dt_1 dt_2 = \frac{L^2}{N} + O\left(\frac{E^{\eta-1/2}}{\sqrt{\epsilon}} + \sqrt{\epsilon} E^\eta\right).$$

The estimate (5.9) follows by making the optimal choice  $\epsilon = E^{-1/2}$  in (5.18).

The proofs of (5.10) and (5.11) are very similar to the above (cf. the proofs of (5.5) and (5.6) within the proof of Proposition 5.1); we omit the details.  $\square$

*Remark 5.6.* — It is possible to slightly improve the exponent of the bound (5.8) in Proposition 5.3 by using a better estimate for the number of lattice points in spherical caps ([3, Proposition 1.4]).

### 5.3. Concluding the proof of Theorem 1.5

*Proof.* — Use Proposition 2.2 together with either Proposition 5.1 for the nowhere vanishing torsion case or Proposition 5.3 for the planar case.  $\square$

We finally record that the proof above will give the result described in Remark 1.6 for  $\mathcal{C}$  non-planar and analytic, by invoking the bound (5.3) instead of Proposition 5.1.

## Appendix A. A simple upper bound for the Riesz energy

We now prove Proposition 4.1. Recall that the Riesz energy of the projected lattice points is

$$E_s(\widehat{\mathcal{E}}(E)) := \sum_{\mu \neq \nu \in \mathcal{E}} \frac{1}{\left| \frac{\mu}{\sqrt{E}} - \frac{\nu}{\sqrt{E}} \right|^s}.$$

We use a dyadic subdivision to treat the double sum above, noting that for distinct lattice points  $\mu \neq \nu \in \mathcal{E}$ , we have  $1 < |\mu - \nu| \leq 2\sqrt{E}$ :

$$(A.1) \quad \begin{aligned} \sum_{\mu \neq \nu \in \mathcal{E}} \frac{1}{|\mu - \nu|^s} &= \sum_{1 \leq 2^k \leq 2\sqrt{E}} \sum_{2^k \leq |\mu - \nu| < 2^{k+1}} \frac{1}{|\mu - \nu|^s} \\ &\ll \sum_{2^k \leq 2\sqrt{E}} \frac{1}{2^{ks}} \sum_{\mu \in \mathcal{E}} \#\{\nu \in \mathcal{E} : |\mu - \nu| < 2^{k+1}\} \end{aligned}$$

The number of lattice points in a spherical cap of radius  $r$  on the sphere  $\sqrt{E}S^2$  is bounded by  $O(E^\eta(1+r))$  for all  $\eta > 0$  [3, Lemma 2.2], so that

$$\#\{\nu \in \mathcal{E} : |\mu - \nu| < 2^{k+1}\} \ll E^\eta 2^k, \quad \forall \eta > 0.$$

Inserting into (A.1) gives, for  $0 \leq s \leq 1$ , and all  $\eta > 0$ ,

$$\begin{aligned} \sum_{\mu \neq \nu \in \mathcal{E}} \frac{1}{|\mu - \nu|^s} &\ll \sum_{2^k \leq 2\sqrt{E}} \sum_{\mu \in \mathcal{E}} \frac{1}{2^{ks}} 2^k E^\eta \\ &= NE^\eta \sum_{2^k \leq 2\sqrt{E}} 2^{k(1-s)} \ll NE^{(1-s)/2+\eta} \log E . \end{aligned}$$

(the factor of  $\log E$  is needed if  $s = 1$ ).

Now insert Siegel’s lower bound (4.1) on the number of lattice points:  $E^{1/2-\delta} \ll N$ , for all  $\delta > 0$ , to bound the RHS above by

$$NE^{(1-s)/2+\eta} \log E = NE^{1/2-\eta} \cdot E^{-s/2+2\eta} \log E \ll N^2 E^{-s/2+3\eta}, \quad \forall \eta > 0.$$

This gives

$$\sum_{\mu \neq \nu \in \mathcal{E}} \frac{1}{|\mu - \nu|^s} \ll N^2 E^{-s/2+3\eta}, \quad \forall \eta > 0,$$

and hence (replacing  $\eta$  by  $\eta/3$ ), for  $0 < s \leq 1$ , the Riesz energy of the projected lattice points is bounded by

$$E_s(\widehat{\mathcal{E}}(E)) := \sum_{\mu \neq \nu \in \mathcal{E}} \frac{1}{\left| \frac{\mu}{\sqrt{E}} - \frac{\nu}{\sqrt{E}} \right|^s} \ll N^2 E^\eta, \quad \forall \eta > 0 .$$

This concludes the proof of Proposition 4.1.

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