ANNALES

DE

L'INSTITUT FOURIER

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Quasicircle boundaries and exotic almost-isometries

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QUASICIRCLE BOUNDARIES AND EXOTIC ALMOST-ISOMETRIES

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ABSTRACT. — We show that the limit set of an isometric and convex cocompact action of a surface group on a proper geodesic hyperbolic metric space, when equipped with a visual metric, is a Falconer–Marsh (weak) quasicircle. As a consequence, the Hausdorff dimension of such a limit set determines its bi-Lipschitz class. We give applications, including the existence of almost-isometries between periodic negatively curved metrics on $\mathbb{H}^2$ that cannot be realized equivariantly.

RÉSUMÉ. — Nous démontrons que l'ensemble limite d'une action isométrique et convexe co-compacte d'un groupe de surface sur un espace hyperbolique, équipé de sa métrique visuelle canonique, est un (faible) quasi-cercle au sens de Falconer et Marsh. Ceci implique que ces métriques visuelles sur ces ensembles limites sont classifiées, à équivalence bi-Lipschitz près, par leur dimension de Hausdorff. Nous donnons plusiers consequences, y compris l'existence de presque-isométries entre des paires de métriques périodiques sur $\mathbb{H}^2$ (malgré le faite qu'en général, il n'existe pas de presque-isométrie équivariante).

1. Introduction

Consider a closed Riemannian manifold (M, g) with sectional curvatures $\leqslant -1$. The metric g lifts to a $\pi_1(M)$-invariant metric $\tilde{g}$ on the universal covering X, inducing an action of $\pi_1(M)$ on the boundary at infinity ∂X. This boundary, equipped with a visual metric d_∂ (see Definition 2.3), exhibits fractal-like behavior as a consequence of this action. More generally, this phenomenon occurs if $\pi_1(M)$ acts properly discontinuously and convex cocompactly on a hyperbolic metric space X. Our first result follows this general philosophy:

THEOREM 1.1. — *Let Γ be a surface group acting isometrically, properly discontinuously, and convex cocompactly on a proper geodesic hyperbolic metric space (X, d). Then the limit set of this action, with a visual metric d_∂, is a (weak) quasicircle in the sense of Falconer–Marsh.*

The following results use the special case of Theorem 1.1 where (X, d) is the universal cover of a closed surface equipped with a locally CAT(-1) metric, and Γ acts on X by deck transformations.

COROLLARY 1.2. — *Let $(M_1, d_1), (M_2, d_2)$ be any pair of closed surfaces equipped with locally CAT(-1) metrics, and let $(X_i, \tilde{d}_i)$ be their universal covers. Then one can find real numbers $0 < \lambda_i \leqslant 1$, with $\max\{\lambda_1, \lambda_2\} = 1$, having the property that $(X_1, \lambda_1 \tilde{d}_1)$ is almost-isometric to $(X_2, \lambda_2 \tilde{d}_2)$.*

Recall that an *almost-isometry* is a quasi-isometry with multiplicative constant $= 1$. When M_1, M_2 are locally CAT(-1) manifolds, the existence of an almost-isometry $X_1 \to X_2$ sometimes forces the universal covers to be isometric [12]. On the other hand, Corollary 1.2 yields examples of almost-isometric universal covers that are not isometric – take for example d_1 to be Riemannian and d_2 to be non-Riemannian.

COROLLARY 1.3. — *Let M be a closed surface with Riemannian metrics g_1, g_2 having curvatures $\leqslant -1$. Equip ∂X with the corresponding canonical visual metrics d_1 and d_2. Then the following three statements are equivalent:*

  (1) *The topological entropies of the two geodesic flows on $T^1 M$ are equal.*
  (2) *The boundaries $(\partial X, d_1)$ and $(\partial X, d_2)$ are bi-Lipschitz equivalent.*
  (3) *The universal covers $(X, \tilde{g}_1)$ and $(X, \tilde{g}_2)$ are almost-isometric.*

By Corollary 1.2, given two negatively curved Riemannian metrics g_1 and g_2 on a closed surface M, after possibly scaling one of the metrics (and also its area), the universal covers are almost-isometric. Our next result shows that one can arrange for equal area examples with almost-isometric universal covers.

THEOREM 1.4. — *Let M be a closed surface of genus $\geqslant 2$, and $k \geqslant 1$ an integer. One can find a k-dimensional family $\mathcal{F}_k$ of Riemannian metrics on M, all of curvature $\leqslant -1$, with the following property. If g, h are any two distinct metrics in $\mathcal{F}_k$, then*

  • $Area(M, g) = Area(M, h)$.
  • *the lifted metrics $\tilde{g}, \tilde{h}$ on the universal cover X are almost-isometric.*

- *the lifted metrics $\tilde{g}, \tilde{h}$ on the universal cover X are not isometric.*

The almost-isometries between the lifted metrics in Theorem 1.4 are *exotic* in the sense that they cannot be realized equivariantly with respect to the $\pi_1(M)$-actions on X. Indeed, a $\pi_1(M)$-equivariant almost-isometry between the two lifted metrics on X implies that the metrics on M have equal marked length spectra (see [12], for example), and are therefore isometric by [8, 16].

As a final application, we exhibit a *gap phenomenon* for the optimal multiplicative constant for quasi-isometries between certain periodic metrics.

COROLLARY 1.5. — *Let $(M_1, d_1), (M_2, d_2)$ be any pair of closed surfaces equipped with locally CAT(-1) metrics, and assume that their universal covers $(X_i, \tilde{d}_i)$ are not almost-isometric. Then there exists a constant $\epsilon > 0$ with the property that any (C, K)-quasi-isometry from (X_1, d_1) to (X_2, d_2) must satisfy $C \geqslant 1 + \epsilon$.*

Acknowledgments

2. Background material

This section reviews the definitions and existing results used in our proofs.

2.1. Convex cocompact actions

We refer the reader to [2, Section 1.8] for more details concerning this subsection. Let a finitely generated group Γ act properly discontinuously and isometrically on a proper geodesic hyperbolic metric space X with limit set $\Lambda_\Gamma \subset \partial X$.

DEFINITION 2.1. — *The Γ-action on X is* convex cocompact *if it satisfies any of the following equivalent conditions:*

(1) *the map $\Phi : \Gamma \to X$ given by $\Phi(g) := g(x)$ is quasi-isometric.*
(2) *the orbit of a point $\Gamma \cdot p$ is a quasi-convex subset of X.*
(3) *Γ acts cocompactly on the Gromov hull $Q(\Lambda_\Gamma)$ of its limit set (consisting of all geodesics joining pairs of points in Λ_Γ).*

These conditions imply that Γ is δ-hyperbolic, and therefore has a Gromov boundary $\partial \Gamma$. An important consequence of the action being convex cocompact is that the limit set Λ_Γ is homeomorphic to $\partial \Gamma$. If Γ is a surface group, then $\partial \Gamma$ is a topological circle.

2.2. Metrics on the boundary

We refer the reader to [5, Chapter III.H.3] for more details concerning this subsection. Let X be a proper geodesic hyperbolic metric space with boundary ∂X. Fix a basepoint $w \in X$. The Gromov product $(\cdot|\cdot)_w :$ $X \times X \to \mathbb{R}$ is defined by $(p|q)_w := \frac{1}{2} \left(d(w,p) + d(w,q) - d(p,q) \right)$ for each $p, q \in X$, and extends to $\partial X \times \partial X$ by

$$(x,y)_w := \sup \varliminf_{n,m\to\infty} (x_n|y_m)_w$$

where the supremum is over all sequences $\{x_n\}$ and $\{y_m\}$ in X such that $x_n \to x$ and $y_m \to y$. Given a basepoint $w \in X$ and $\epsilon > 0$, we set $\rho_\epsilon(x,y) := e^{-\epsilon(x|y)_w}$. In general ρ_ϵ may not define a metric on ∂X, so we define

$$d_\epsilon(x,y) := \inf \sum_{i=1}^{n} \rho_\epsilon(\xi_i, \xi_{i-1}),$$

where the infimum is taken over all finite chains $x = \xi_0, \ldots \xi_n = y$.

PROPOSITION 2.2 (See [5, III.H.3.21]). — *Fix $w \in X$. Then for $\epsilon > 0$ sufficiently small, d_ϵ is a metric on ∂X. Further, d_ϵ is bi-Lipschitz equivalent to ρ_ϵ, i.e. there exists $C > 0$ such that for every x, y, we have*

$$\frac{1}{C}\rho_\epsilon(x,y) \leqslant d_\epsilon(x,y) \leqslant C\rho_\epsilon(x,y).$$

Further the bi-Lipschitz class of d_ϵ is independent of w.

DEFINITION 2.3. — *With a slight abuse of language, we refer to the family of bi-Lipschitz equivalent metrics d_ϵ obtained in Proposition 2.2 (with w varying in X) as the* visual metric *on ∂X with parameter ϵ.*

If X is CAT(-1), then Bourdon showed that we can choose $\epsilon = 1$ [3]. In this case we refer to d_1 as the *canonical visual metric* on ∂X.

Let Γ be a properly discontinuous group of isometries of X acting convex cocompactly with limit set $\Lambda \subseteq \partial X$. Let $s = \mathrm{Hdim}(\partial X)$ be the Hausdorff dimension of a visual metric d_ϵ on ∂X, and H^s the corresponding Hausdorff measure. Then H^s is a finite measure, fully supported on Λ. The Hausdorff dimension and measure can be estimated using the following result (see Patterson–Sullivan [19] and its verbatim extension to the Gromov hyperbolic setting by Coornaert [7, Section 7]).

THEOREM 2.4 (Patterson–Sullivan, Coornaert).

(i) $s = \epsilon \, \overline{\lim_{n \to \infty}} \, \dfrac{1}{n} \log \#\{\gamma \in \Gamma \mid d(w, \gamma w) \leqslant n\}$

(ii) *There exists a constant $C \geqslant 1$ such that for each metric ball $B(x, r)$ in ∂X (with center x and radius r), $C^{-1} r^s \leqslant H^s\left(B(x, r)\right) \leqslant C r^s$.*

Remark 2.5. — When (M, g) is a closed Riemannian manifold with sectional curvatures $\leqslant -1$, its universal Riemannian covering X is a CAT(-1) space equipped with a geometric action of $\Gamma = \pi_1(M)$ by deck transformations. Hence we can equip ∂X with the canonical visual metric (i.e. $\epsilon = 1$). In this case, Sullivan (for Kleinian groups) and Otal and Peigné [17] proved that $s = h(g)$, where the latter denotes the topological entropy of the geodesic flow on the unit tangent bundle $T^1(M)$.

2.3. (Weak) Quasicircles according to Falconer–Marsh

Next let us briefly review some notions and results of Falconer and Marsh [9].

DEFINITION 2.6 (Falconer–Marsh). — *A metric space (C, d) is a quasicircle if*

(i) *C is homeomorphic to S^1,*
(ii) *(expanding similarities) There exist $a, b, r_0 > 0$ with the following property. For any $r < r_0$ and $N \subseteq C$ with $\mathrm{diam}(N) = r$, there exists an expanding map $f : N \to C$ with expansion coefficient between $\frac{a}{r}$ and $\frac{b}{r}$, i.e. for all distinct $x, y \in N$, we have the estimate*

$$\frac{a}{r} \leqslant \frac{d(f(x), f(y))}{d(x, y)} \leqslant \frac{b}{r}$$

Note that a, b are independent of the size of N, and of the choice of points in N.

(iii) *(contracting similarities) There exist $c, r_1 > 0$ with the following property. For any $r < r_1$ and ball $B \subseteq C$ with radius r, there exists a map $f : C \to B \cap C$ contracting no more than a factor cr.*

Let $s = \mathrm{Hdim}(C)$ be the Hausdorff dimension of C, and H^s the corresponding Hausdorff measure. The following alternate property to (iii) is implied by (ii) and (iii):

(iiia) For any open $U \subseteq C$ one has $0 < H^s(U) < \infty$, and $H^s(U) \to 0$ as $\mathrm{diam}(U) \to 0$.

The main result of Falconer–Marsh [9] is that two quasicircles C_1 and C_2 are bi-Lipschitz equivalent if and only if their Hausdorff dimensions are equal. While stated this way, their proof only uses conditions (i), (ii), and (iiia). For this reason, we will say that a metric space (C, d) is a *weak quasicircle* when conditions (i), (ii), and (iiia) are satisfied.

THEOREM 2.7 (Falconer–Marsh). — *Two weak quasicircles C_1 and C_2 are bi-Lipschitz equivalent if and only if their Hausdorff dimensions are equal.*

Remark 2.8. — In geometric function theory, the term quasicircle is used for the image of S^1 under a quasisymmetric map f. We will call these *quasisymmetric circles*. Falconer–Marsh quasicircles form a strict subset of the quasisymmetric circles. Indeed, a Falconer–Marsh quasicircle X has the same Hausdorff dimension as the visual boundary Y of some negatively curved surface. By Theorems 1.1 and 2.7, X and Y are bi-Lipschitz equivalent. Since Y is a quasisymmetric circle, so is X. On the other hand, Property (ii) of Falconer–Marsh quasicircles implies that any nonempty open subset U of a Falconer–Marsh quasicircle X has the same Hausdorff dimension as X, a property that quasisymmetric circles need not have (see [18]).

2.4. Almost-isometries and the work of Bonk–Schramm

Now let us recall some results of Bonk and Schramm [1] that we will need.

DEFINITION 2.9. — *A map $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is* quasi-isometric *if there exists constants C, K such that*

$$\frac{1}{C} d_X(x, y) - K \leqslant d_Y(f(x), f(y)) \leqslant C d_X(x, y) + K.$$

A map $f : X \to Y$ is coarsely onto *if Y lies in a bounded neighborhood of $f(X)$. If the quasi-isometric map f is coarsely onto, then we call it a* quasi-isometry, *and we say that X, Y are* quasi-isometric. *A map is* almost-isometric *if it is quasi-isometric with multiplicative constant $C = 1$. An almost-isometric map $f : X \to Y$ which is coarsely onto is called an* almost-isometry, *in which case we say that X, Y are* almost-isometric.

Special cases of the results in [1] relate the existence of almost-isometries with metric properties of ∂X, as follows.

THEOREM 2.10 (Bonk–Schramm). — *Let X, Y be a pair of CAT(-1) spaces. Then X, Y are almost-isometric if and only if the canonical visual metrics on the boundaries ∂X, ∂Y are bi-Lipschitz homeomorphic to each other.*

The fact that an almost-isometry between X and Y induces a bi-Lipschitz homeomorphism between the boundaries ∂X and ∂Y appears in [1, proof of Theorem 6.5] – where it should be noted that, in the notation of their proof, our more restrictive context corresponds to $\epsilon = \epsilon' = 1$ and $\lambda = 1$. As for the converse, the interested reader should consult [1, Theorem 7.4] to see that a bi-Lipschitz map between boundaries ∂X and ∂Y induces an almost-isometry between the metric spaces $\text{Con}(X)$ and $\text{Con}(Y)$. The comment following [1, Theorem 8.2] applies in our context where $a = 1$, so that $\text{Con}(X)$ and $\text{Con}(Y)$ are almost isometric to X and Y, respectively, whence X and Y are almost-isometric.

3. Surface boundaries are (weak) quasicircles

Having covered the preliminaries, let us now give a proof of Theorem 1.1, as well as the various corollaries.

Proof of Theorem 1.1. — Looking at Definition 2.6, there are three properties to establish. Property (i) is obvious – see the discussion in Section 2.1. If Γ acts cocompactly on X, then Property (ii) is a consequence of the more general result of Bourdon and Kleiner – see [4, Section 3.1]. In the convex cocompact case, let Y be the Gromov hull (in X) of the limit set of Γ. Then the Bourdon–Kleiner argument with minor modifications (using quasi-convexity of Y in X) applies to the Γ action on Y, and this proves Property (ii) in general (an earlier version of this paper, available on the arXiv [15], contains a more detailed discussion of this point).

Finally, property (iiia) is an immediate consequence of Theorem 2.4. We conclude that the limit set is indeed a (weak) quasicircle in the sense of Falconer–Marsh. □

The proof of all three corollaries are now completely straightforward.

Proof of Corollary 1.3. — Theorem 1.1 gives us that $(\partial X, d_1)$ and $(\partial X, d_2)$ are weak quasicircles. Then Falconer and Marsh's Theorem 2.7 and Remark 2.5 gives the equivalence of statements (1) and (2), while Bonk and Schramm's Theorem 2.10 gives the equivalence of statements (2) and (3). □

Proof of Corollary 1.2. — When two metrics on X are related by a scale factor λ, it easily follows from the formula for the canonical visual metric that the Hausdorff dimension of the boundary at infinity scales by $1/\lambda$. Note also that if (X, d) is CAT(-1) and $0 < \lambda \leqslant 1$, then the rescaled metric $(X, \lambda d)$ is also CAT(-1). Corollary 1.2 then immediately follows by combining our Theorem 1.1, Falconer and Marsh's Theorem 2.7, and Bonk and Schramm's Theorem 2.10. □

Proof of Corollary 1.5. — Combining our Theorem 1.1, Falconer and Marsh's Theorem 2.7, and Bonk and Schramm's Theorem 2.10, we see that the boundaries at infinity $(\partial X_i, d_i)$ must have distinct Hausdorff dimensions. Without loss of generality, let us assume $\mathrm{Hdim}(\partial X_1, d_1) < \mathrm{Hdim}(\partial X_2, d_2)$. If $\phi : X_1 \to X_2$ is a (C, K)-quasi-isometry, we want to obtain a lower bound on C. But the quasi-isometry induces a homeomorphism $\partial\phi : \partial X_1 \to \partial X_2$ which, from the definition of the Gromov product, has the property that

$$\rho_2\left(\partial\phi(x), \partial\phi(y)\right) \leqslant e^K \cdot \rho_1(x, y)^C$$

for all $x, y \in \partial X_1$. Because d_i and ρ_i are bi-Lipschitz equivalent, we obtain the desired inequality

$$1 < \frac{\mathrm{Hdim}(\partial X_2, d_2)}{\mathrm{Hdim}(\partial X_1, d_1)} \leqslant C. \qquad \qquad \square$$

4. Constructing exotic almost-isometries

This section is devoted to the proof of Theorem 1.4. In view of Corollary 1.3, we want to produce a k-dimensional family $\mathcal{F}_k$ of equal area metrics on a higher genus surface M, which all have the same topological entropy, but whose lifts to the universal cover are not isometric to each other.

4.1. Perturbations of metrics

Start with a fixed reference hyperbolic metric g_0 on M, normalized to have constant curvature -2. Pick $(k+2)$ distinct points $p_1, \ldots p_{k+2} \in M$, and choose r_2 smaller than the injectivity radius of M and satisfying $2r_2 < \inf_{i \neq j}\{d(p_i, p_j)\}$. Let U_i denote the open metric ball of radius r_2 centered at p_i – note that the U_i are all isometric to each other, and are pairwise disjoint. Now choose $r_1 < r_2$ so that the area of the ball of radius r_1 is at least $4/5$ the area of the ball of radius r_2. Denote by $V_i \subset U_i$ the ball of radius r_1 centered at each p_i.

We will vary the metric g_0 by introducing a perturbation on each of the U_i in the following manner. Let us choose a smooth bump function $\rho : [0, \infty) \to [0, 1]$ with the property that $\rho|_{[0,r_1]} \equiv 1$ and $\rho|_{[r_2,\infty)} \equiv 0$. Next define $u_i : M \to [0, 1]$ via $u_i(x) := \rho\left(d(x, p_i)\right)$. Given a parameter $\vec{t} := (t_1, \ldots, t_{k+2}) \in \mathbb{R}^{k+2}$, define the function $u_{\vec{t}} : M \to [0, \infty)$ by setting $u_{\vec{t}} := t_1 u_1 + \cdots + t_{k+2} u_{k+2}$. Finally, we define the metric $g_{\vec{t}} := e^{2u_{\vec{t}}} g_0$ (and note the identification $g_{\vec{0}} = g_0$). The family $\mathcal{F}_k$ will be obtained by choosing suitable values of $\vec{t}$ close to $\vec{0}$.

Since the metric $g_{\vec{t}}$ is obtained by making a conformal change on each U_i, and since the U_i are pairwise disjoint, we first analyze the behavior of such a change on an individual U_i. To simplify notation, denote by $V \subset U$ open balls of radius $r_1 < r_2$ centered at a point p in the hyperbolic plane $\mathbb{H}^2_{-2}$ of curvature -2, and set $g_t := e^{2tu} g_0$ where g_0 is the hyperbolic metric of curvature -2, and $u : \mathbb{H}^2_{-2} \to [0, \infty)$ is given by $u(x) := \rho\left(d(p, x)\right)$. We start with the easy:

LEMMA 4.1. — *As $t \to 0$, we have the following estimates:*

(1) *the curvatures $K(g_t)$ tend uniformly to -2.*
(2) *the area $Area(U; g_t)$ of the ball U tends to $Area(U; g_0)$.*
(3) *the area $Area(V; g_t)$ of the ball V tends to $Area(V; g_0)$.*

Proof. — This is straightforward from the formulas expressing how curvature and area change when one makes a conformal change of metric. We have that the new curvature $K(g_t)$ is related to the old curvature $K(g_0)$ via the formula

$$K(g_t) = (e^{-2u})^t K(g_0) - t(e^{-2u})^t \Delta u$$

where Δu denotes the Laplacian of the function u in the hyperbolic metric g_0. As t tends to zero, it is clear that the expression on the right converges to $K(g_0)$ uniformly, giving (1). Similarly, the area form dg_t for the new

metric is related to the area form dg_0 for the original metric via the formula $dg_t = (e^{2u})^t dg_0$ giving us (2) and (3). □

4.2. Lifted metrics are almost-isometric

Next we establish that, for suitable choices of the parameter $\vec{t}$, we can arrange for the lifted metrics to be almost-isometric. By Lemma 4.1, we can take the parameters $\vec{t}$ close enough to $\vec{0}$ to ensure that all the metrics we consider have sectional curvatures $\leqslant -1$. Then from Corollary 1.3, it suffices to consider values of the parameter $\vec{t}$ for which the corresponding metrics have the same topological entropy for the geodesic flow on $T^1 M$. Notice that varying $\vec{t}$ near $\vec{0}$ gives a C^∞ family of perturbations of the metric g_0. Work of Katok, Knieper, Pollicott and Weiss [13, Theorem 2] then implies that the topological entropy map h, when restricted to any line $l(s)$ through the origin $\vec{0}$ in the $\vec{t}$-space, is a C^∞ map. Moreover the derivative of h along the line is given by (see [14, Theorem 3])

$$\frac{\partial}{\partial s}\Big|_{s=0} h\left(g_{l(s)}\right) = -\frac{h(g_0)}{2} \int_{T^1 M} \frac{\partial}{\partial s}\Big|_{s=0} g_{l(s)}(v,v) d\mu_0$$

where $T^1 M$ denotes the unit tangent bundle of M with respect to the g_0-metric, and μ_0 denotes the Margulis measure of g_0 (the unique measure of maximal entropy for the g_0-geodesic flow on $T^1 M$).

Consider the map $F: \mathbb{R}^{k+2} \to \mathbb{R}$ given by $F(t_1, \ldots, t_{k+2}) := h(g_{(t_1,\ldots,t_{k+2})})$, where h denotes the topological entropy of (the geodesic flow associated to) a metric. Let us compute the directional derivative in the direction $\frac{\partial}{\partial t_1}$:

$$\frac{\partial F}{\partial t_1}(0,\ldots,0) = \frac{d}{dt}\Big|_{t=0} h\left(g_{(t,0,\ldots,0)}\right)$$

$$= -\frac{h(g_0)}{2} \int_{T^1 M} \frac{d}{dt}\Big|_{t=0} g_{(t,0,\ldots,0)}(v,v) d\mu_0$$

$$= -\frac{h(g_0)}{2} \int_{T^1 M} \frac{d}{dt}\Big|_{t=0} e^{2tu_1}\left(\pi(v)\right) d\mu_0$$

$$= -\frac{h(g_0)}{2} \int_{T^1 M} 2u_1\left(\pi(v)\right) d\mu_0$$

where $\pi : T^1 M \to M$ is the projection from the unit tangent bundle onto the surface M. Finally, we observe that by construction u_1 is a non-negative function, which is identically zero on the complement of U_1, and identically one on the set V_1. Hence the integral above is positive, and we obtain $\frac{\partial F}{\partial t_1}(\vec{0}) < 0$.

Now, a similar calculation applied to each of the other coordinates gives us the general formula for the directional derivative of F. The gradient of F is given by the non-vanishing vector:

$$\nabla F = -h(g_0) \int_{T^1 M} \langle u_1\big(\pi(v)\big), \ldots, u_{k+2}\big(\pi(v)\big) \rangle d\mu_0.$$

In fact, since each u_i is supported solely on U_i, and each u_i is defined as $\rho\big(d(p_i, x)\big)$ on the U_i, each of the integrals in the expression for ∇F has the same value. So ∇F is just a nonzero multiple of the vector $\langle 1, \ldots, 1 \rangle$.

The implicit function theorem locally gives us an embedded codimension one submanifold $\sigma(z)$ (where $z \in \mathbb{R}^{k+1}$, $||z|| < \epsilon$) in the $(t_1, \ldots, t_{k+2})$-space, with normal vector $\langle 1, \ldots, 1 \rangle$ at the point $\sigma(\vec{0}) = \vec{0}$, on which the topological entropy functional is constant. From Corollary 1.3, we see that the lifts of these metrics to the universal cover are all pairwise almost-isometric.

4.3. Lifted metrics are not isometric

LEMMA 4.2. — *There is an $\epsilon > 0$ so that if the parameters $\vec{s} = (s_1, \ldots, s_{k+2})$ and $\vec{t} = (t_1, \ldots, t_{k+2})$ satisfy $0 < |s_i| < \epsilon$ and $0 < |t_i| < \epsilon$ and the lifted metrics $(\tilde{M}, \tilde{g}_{\vec{s}})$ and $(\tilde{M}, \tilde{g}_{\vec{t}})$ are isometric to each other, then we must have an equality of sets $\{s_1, \ldots, s_{k+2}\} = \{t_1, \ldots, t_{k+2}\}$.*

Proof. — By Lemma 4.1, it is possible to pick ϵ small enough so that, for all parameters $\vec{s}, \vec{t}$ within the ϵ-ball around $\vec{0}$, we have that

$$\text{Area}(V_i; g_{\vec{t}}) \geqslant \frac{3}{4} \text{Area}(U_j; g_{\vec{s}})$$

for every $1 \leqslant i, j \leqslant k + 2$.

Now let us assume that there is an isometry $\Phi : (\tilde{M}, \tilde{g}_{\vec{s}}) \to (\tilde{M}, \tilde{g}_{\vec{t}})$. Observe that the lifted metrics have the following properties:

(i) on the complement of the lifts of the U_i, both metrics have curvature identically -2.

(ii) on any lift of the set V_1, the metric $\tilde{g}_{\vec{s}}$ has curvature identically $-2e^{-2s_1}$.

(iii) on any lift of the set V_i, the metric $\tilde{g}_{\vec{t}}$ has curvature identically $-2e^{-2t_i}$.

Take a lift $\tilde{V}_1$ of V_1 in the source, and consider its image under Φ. The metric in the source has curvature identically $-2e^{-2s_1}$ on this lift $\tilde{V}_1$, and since Φ is an isometry, the image set $\Phi(\tilde{V}_1)$ must have the same curvature. From property (i), we see that $\Phi(\tilde{V}_1)$ must lie, as a set, inside the union

of lifts of the U_i. Since $\Phi(\tilde{V}_1)$ is path-connected, it must lie inside a single connected lift $\tilde{U}_i$ of one of the U_i. But from the area estimate, we see that for the $\tilde{V}_i \subset \tilde{U}_i$ inside the lift, one has that the intersection $\Phi(\tilde{V}_1) \cap \tilde{V}_i$ is non-empty. Looking at the curvature of a point in the intersection, we see that

$$-2e^{-2s_1} = -2e^{-2t_i}$$

and hence that $s_1 = t_i$ for some i. Applying the same argument to each of s_i, t_i completes the proof. □

Now pick a vector $\vec{v} = \langle v_1, \ldots, v_{k+2} \rangle$ with the property that $v_1 + \cdots + v_{k+2} = 0$, and such that $v_i \neq v_j$ for each $i \neq j$. Notice that the first constraint just means that $\vec{v} \cdot \nabla F = 0$, and hence that $\vec{v}$ is tangent to the $(k+1)$-dimensional submanifold σ. So there exists a curve $\gamma \subset \sigma$ satisfying $\gamma(0) = \vec{0}$, and $\gamma'(0) = \vec{v}$. Notice that, from our second condition, when $t \approx 0$ we have $\gamma(t) \approx (v_1 t, \ldots, v_{k+2} t)$, and hence the point $\gamma(t)$ has *all coordinates distinct*. It follows from Lemma 4.2 that, for any $t \approx 0$ $(t \neq 0)$, one can find a small enough connected neighborhood W_t of $\gamma(t)$ with the property that all the metrics in that neighborhood have lifts to the universal cover that are pairwise non-isometric.

4.4. Metrics with equal area

Now consider the smooth function

$$A : \sigma \to \mathbb{R}$$

defined by $A(z) := \text{Area}(g_{\sigma(z)})$ for each $z \in \sigma$. The change of area formula for a conformal change of metric (see the proof of Lemma 4.1) implies that A is nonconstant on W_t. By Sard's theorem, there is a regular value r of A in the interval $A(W_t)$. Then $\tau := A^{-1}(r)$ is a smooth k-dimensional submanifold of the $(k+1)$-dimensional manifold σ consisting of parameters for area r metrics. A connected component $\mathcal{F}_k$ of $W_t \cap A^{-1}(r)$ satisfies all of the constraints of Theorem 1.4.

5. Concluding remarks

As the reader undoubtedly noticed, our results rely heavily on the surprising result of Falconer and Marsh. As such, it is very specific to the case of circle boundaries – which essentially restricts us to surface groups (see

Gabai [11]). In higher dimensions, we would not expect the bi-Lipschitz class of a self-similar metric on a sphere to be classified by its Hausdorff dimension. Thus, the following problem seems substantially more difficult.

CONJECTURE 5.1. — *Let M be a smooth closed manifold of dimension $\geqslant 3$, and assume that M supports a negatively curved Riemannian metric. Then M supports a pair of equal volume Riemannian metrics g_1, g_2 with curvatures $\leqslant -1$, and having the property that the Riemannian universal covers $(\tilde{M}, \tilde{g}_i)$ are almost-isometric, but are not isometric.*

In another direction, if one were to *drop* the dimension, then there are many examples of 0-dimensional spaces having analogous self-similarity properties (i.e. properties (ii), (iii) in Definition 2.6). The metrics on these boundaries turn them into Cantor sets – and the classification of (metric) Cantor sets up to bi-Lipschitz equivalence seems much more complex than in the circle case (for some foundational results on this problem, see for instance Falconer and Marsh [10] and Cooper and Pignataro [6]). Of course, from the viewpoint of boundaries, such spaces would typically arise as the boundary at infinity of a metric tree T. This suggests the following:

PROBLEM. — *Study periodic metrics on trees up to the relation of almost-isometry.*

In particular, invariance of the metric under a cocompact group action translates to additional constraints on the canonical visual metric on ∂T, e.g. the existence of a large (convergence) group action via conformal automorphisms (compare with the main theorem in [6]). It would be interesting to see if this makes the bi-Lipschitz classification problem any easier.

Finally, given a pair of quasi-isometric spaces, we can consider the collection of *all* quasi-isometries between them, and try to find the quasi-isometry which has smallest multiplicative constant. More precisely, given a pair of quasi-isometric metric spaces X_1, X_2, define the real number $\mu(X_1, X_2)$ to be the infimum of the real numbers C with the property that there exists some (C, K)-quasi-isometry from X_1 to X_2. We can now formulate the:

PROBLEM. — *Given a pair of quasi-isometric metric spaces X_1, X_2, can one estimate $\mu(X_1, X_2)$? Can one find a (C, K)-quasi-isometry from X_1 to X_2, where $C = \mu(X_1, X_2)$? In particular, can one find a pair of quasi-isometric spaces X_1, X_2 which are not almost-isometric, but which nevertheless satisfy $\mu(X_1, X_2) = 1$?*

Our Corollary 1.5 gives a complete answer in the case where the X_i are universal covers of locally CAT(-1) metrics on surfaces – the real number

$\mu(X_1, X_2)$ is exactly the ratio of the Hausdorff dimensions of the canonical visual metrics on the boundary, and one can always find a quasi-isometry with multiplicative constant $\mu(X_1, X_2)$. It is unclear what to expect in the more general setting.

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