ANNALES

DE

L'INSTITUT FOURIER

Xiuxiong CHEN & Yuanqi WANG

On the regularity problem of complex Monge–Ampere equations with conical singularities

ON THE REGULARITY PROBLEM OF COMPLEX MONGE–AMPERE EQUATIONS WITH CONICAL SINGULARITIES

by Xiuxiong CHEN & Yuanqi WANG

————

ABSTRACT. — In the category of metrics with conical singularities along a smooth divisor with angle in $(0, 2\pi)$, we show that locally defined weak solutions ($C^{1,1}$−solutions) to the Kähler–Einstein equations actually possess maximum regularity, which means the metrics are actually Hölder continuous in the singular polar coordinates. This shows the weak Kähler–Einstein metrics constructed by Guenancia–Păun, and independently by Yao, are all actually strong-conical Kähler–Einstein metrics. The key step is to establish a Liouville-type theorem for weak-conical Kähler–Ricci flat metrics defined over $\mathbb{C}^n$, which depends on a Calderon–Zygmund theory in the conical setting. The regularity of globally defined weak-conical Kähler–Einstein metrics is already proved by Guenancia–Paun using a different method.

RÉSUMÉ. — Dans la catégorie des métriques à singularités coniques autour d'un diviseur lisse avec angle strictement compris entre 0 et 2π, on montre que les solutions faibles localement définies (au sens $C^{(1,1)}$) des équations de Kähler–Einstein sont de régularité maximale, ce qui implique que les métriques sont Höldériennes dans les coordonnées polaires singulières. Ceci montre que les métriques de Kähler–Einstein faibles construites par Guénancia-Păun, et Yao indépendamment, sont en fait des métriques de Kähler–Einstein coniques au sens fort. Le point clé est d'établir un théorème de type Liouville pour les métriques Ricci-plates au sens faible sur $\mathbb{C}^n$, ce qui découle d'une théorie de Calderon–Zygmund dans un contexte conique. La régularité des métriques de Kähler–Einstein coniques globalement définies avait déjà été obtenue par Guénancia-Păun par une autre méthode.

1. Introduction

Consider the singular space $(\mathbb{C} \times \mathbb{C}^{n-1}, \omega_\beta)(\beta \in (0, 1))$, where ω_β is the standard flat background metric with conical singularities along $\{0\} \times \mathbb{C}^{n-1}$,

————

written as

$$\omega_\beta = \frac{\beta^2}{|z|^{2-2\beta}} dz \otimes d\bar{z} + \sum_{j=1}^{n-1} dv_j \otimes d\bar{v}_j,$$

where $z \in \mathbb{C}$ and v_j are tangential variables to $\{0\} \times \mathbb{C}^{n-1}$. Geometrically, this is a product of a flat two-dimension cone with Euclidean $\mathbb{C}^{n-1}$. From now on, we denote the singular divisor $\{0\} \times \mathbb{C}^{n-1}$ as D. In this introduction, we take the balls to be centered at the origin, with respect to ω_β. For more detailed notations, please refer to Section 2 of this article.

We want to understand the PDE theory in this space, using intrinsic metric. For any domain $\Omega \in \mathbb{C} \times \mathbb{C}^{n-1}$, the complex Monge Ampere equation take a simpler form

$$\text{(1.1)} \qquad \det(\phi_{i\bar{j}}) = \frac{f}{|z|^{2-2\beta}},$$

where

$$\text{(1.2)} \qquad \omega_\phi = \sqrt{-1}\partial\bar{\partial}\phi$$

gives a Kähler metric in Ω with conical angle $2\pi\beta$ along D. The Laplacian operator of ω_β is

$$\triangle_\beta = \frac{|z|^{2-2\beta}}{\beta^2} \frac{\partial^2}{\partial z \partial \bar{z}} + \sum_{j=1}^{n-1} \frac{\partial^2}{\partial v_j \partial \bar{v}_j}.$$

Sometimes we also use the real laplacian of ω_β, denoted as Δ. Notice that $\Delta = 4\triangle_\beta$.

DEFINITION 1.1. — *For any constant $\lambda > 0$, suppose ϕ solves (1.1) with*

$$f = e^{\lambda\phi+h}, \ h \in C^\infty(\Omega) \text{ and } \sqrt{-1}\partial\bar{\partial}h = 0,$$

then ω_ϕ is a conical Kähler–Einstein metric with scalar curvature $-n\lambda$. When $\lambda = 0$, ω_ϕ is a conical Kähler–Ricci flat metric.

Remark 1.2. — Notice that the conical Kähler–Einstein metrics (along smooth divisors) considered in all the references we know (including [1], [2], [4], [6], [10], [14], [17], [18], [19], [20], [24], ...), can be written as in Definition 1.1 near the D, under holomorphic coordinates.

To state our main results, we define the following.

DEFINITION 1.3 (Weak-conical Kähler metrics). — *A function u defined in Ω is called a $C^{1,1,\beta}(\Omega)$-function if it satisfies*

- *$u \in C^{2,\alpha}(\Omega \setminus D) \cap C^\alpha(\Omega)$, for some $1 > \alpha > 0$;*

- $-K\omega_\beta \leqslant \sqrt{-1}\partial\bar\partial u \leqslant K\omega_\beta$ over $\Omega \setminus D$. The minimum of all such K is defined as our $C^{1,1,\beta}(\Omega)$-seminorm and denoted as $[\,\cdot\,]_{C^{1,1,\beta}(\Omega)}$.

A closed positive $(1,1)$-currrent ω defined in Ω is called a weak-conical Kähler metric if ω admits a plurisubharmonic $C^{1,1,\beta}$−potential (in the sense of (1.2)) near any $p \in \Omega$, and

$$\frac{\omega_\beta}{K} \leqslant \omega \leqslant K\omega_\beta \text{ over } \Omega \setminus D, \text{ for some } K \geqslant 1.$$

Sometimes we call such metrics as $L^{\infty,\beta}$-metrics, with norm defined as the $C^{1,1,\beta}(\Omega)$-seminorm in the previous paragraph with respect to the potentials.

Remark 1.4. — Notice that for a function, being $C^{1,1,\beta}$ is stronger (away from D) than being $C^{1,1}$ in the usual sense, even in the smooth case (when $\beta = 1$). Namely, we require the function to be $C^{2,\alpha}$ away from the singularity. The $C^{1,1,\beta}$ and $L^{\infty,\beta}$ spaces are really adapted to the conic case only.

The above definition is the same as in [23] and [8], we just formulate it here to include the definition of $C^{1,1,\beta}$ functions.

Definition 1.5 (CKS operators). — Similar to Definition 1.3, we say L is a Conelike Kähler Second-order operator over a ball B, if $L = A^{i\bar j}\frac{\partial^2}{\partial z_i \partial \bar z_j}$ such that

(1) $A^{i\bar j} \in C^\alpha(B \setminus D)$ is a Hermitian matrix valued function.
(2) $-K\omega_\beta^{i\bar j} \leqslant A^{i\bar j} \leqslant K\omega_\beta^{i\bar j}$ as Hermitian matrix functions over B, for some constant $K \geqslant 1$.

We define the $L_O^{\infty,\beta}$-norm of a CKS operator L as the infimum of the constant K in the item 2 above. The laplace operator of any weak conical-Kähler metric is an elliptic CKS operator, but in general a CKS operator does not have to be elliptic.

According to [6], if a conical Kähler–Einstein metric is in $C^{\alpha,\beta}$ for some $\alpha > 0$, then it is necessarily in $C^{\alpha',\beta}$ for all $\alpha' \in (0, \min(1, \frac{1}{\beta} - 1))$. The fundamental problem is when $\alpha = 0$, in other words, when the metric tensor is barely $L^{\infty,\beta}$, does the metric actually possess higher regularity? This is of course a core problem in the study of conical Kähler geometry. In this paper, we prove

Theorem 1.6. — Let Ω be an open set in $\mathbb{C}^n$. Suppose $f \in C^{1,1,\beta}(\Omega)$, $f > 0$. For any solution $\phi \in C^{1,1,\beta}(\Omega)$ to equation (1.1) such that $\sqrt{-1}\partial\bar\partial\phi$ is a weak conical metric, ϕ is actually in $C^{2,\alpha,\beta}(\Omega)$, for all α such that $0 < \alpha < \min(\frac{1}{\beta} - 1, 1)$.

Remark 1.7. — Theorem 1.6 does not give any $C^{2,\alpha,\beta}$−norm bound on ϕ, it only says ϕ has $C^{2,\alpha,\beta}$−regularity in the open set Ω. Actually, the norm bounds and the apriori estimate are already proved in [8], from page 13 to page 19. The point of Theorem 1.6 is the regularity, but not the norm bounds.

Theorem 1.6 has an immediate corollary. For the sake of accuracy, we prefer to state it in a more geometric way.

COROLLARY 1.8. — *Any weak-conical Kähler–Einstein metric in a domain* $\Omega \subset \mathbb{C} \times \mathbb{C}^{n-1}$ *must be a* $C^{\alpha,\beta}$ *conical Kähler–Einstein metric, for any* $0 < \alpha < \min(\frac{1}{\beta} - 1, 1)$.

Remark 1.9. — Using Yau-type Schwartz lemmas and some tricky oberservations, Guenancia–Păun constructed weak-conical Kähler–Einstein metrics in [14]. Yao also independently constructed weak-conical Kähler–Einstein metrics in [24], using interesting tricks. Corollary 1.8 implies when the divisor D is smooth, both Guenancia–Păun and Yao's weak-conical Kähler–Einstein metrics are (strong) conical metrics i.e. they are all Hölder continous metrics.

Remark 1.10. — Very recently, we learned that the regularity of globally defined weak-conical Kähler–Einstein metrics is already proved by Guenancia–Păun in [14] using a different method. Corollary 1.8 is a local regularity result. Jeffres–Mazzeo–Rubinstein also have an edge calculus approach to the local regularity in [17].

In Theorem 1.6 and Corollary 1.8, we only assume the underlying metric tensor is $L^{\infty,\beta}$. A crucial step is to prove the following Liouville type theorem:

THEOREM 1.11 (Strong Liouville Theorem). — *Suppose* ω *is a weak-conical Kähler metric over* $\mathbb{C}^n$, *and* ω *satisfies*

$$(1.3) \qquad \omega^n = \omega_\beta^n, \ \frac{\omega_\beta}{K} \leqslant \omega \leqslant K\omega_\beta \ \text{ over } \ \mathbb{C}^n \setminus D,$$

for some $K \geqslant 1$. *Then, there is a linear transformation* L *which preserves* $\{z = 0\}$ *and* $\omega = L^\star \omega_\beta$.

Remark 1.12. — This strong Liouville Theorem is first proved by Chen–Donaldson–Sun in [6], with the additional assumption that ω is a metric cone. Later, assuming ω has $C^{\alpha,\beta}$-regularity for some $\alpha > 0$ instead of being a metric cone, the Liouville Theorem is proved by the authors in Theorem 1.14 in [8].

This strong Liouville theorem is much harder, since we assume the underlying metric tensor is only $L^{\infty,\beta}$. In particular, we can not take any more derivatives to the Einstein equation (1.3) globally, so existing methods are not sufficient anymore. For this purpose, we need to develop $W^{2,p}$−estimate in the conical settings. In [10], Donaldson developed the Schauder theory for conical Laplace operator, and used that to deform the cone angle of conical Kähler–Einstein metrics. In this paper, we establish the corresponding conical $W^{2,p}$-theory. The definition of $W^{k,p,\beta}(k = 1, 2)$ is given in Section 2. To prove the $W^{2,p,\beta}$−estimate, it suffices to consider the following set of second order operators of non-purely normal $(1, 1)$−derivatives as in [10].

$$(1.4) \quad \mathbb{T} = \left\{ \frac{\partial^2}{\partial w_i \partial r}, \ 1 \leqslant i \leqslant 2n - 2; \quad \frac{\partial^2}{\partial w_i \partial w_j}, \ 1 \leqslant i, j \leqslant 2n - 2; \right.$$
$$\left. \frac{1}{r} \frac{\partial^2}{\partial w_i \partial \theta}, \ 1 \leqslant i \leqslant 2n - 2 \right\},$$

where $r = |z|^\beta$, and the θ is the angle of z. There will be more detailed definition in Section 2.

Following Chap. 9 of [12], we define a class of operators T as

$$(1.5) \qquad\qquad Tf = \mathfrak{D} N_{\beta,B} f, \ \mathfrak{D} \in \mathbb{T},$$

where $N_{\beta,B} f$ is the Newtonian potential of f, defined by convolution with the Green's function as in Definition 2.5.

Actually, the operator T and its dual $T^\star$ are both very similar to the singular integral operators considered by Calderon–Zygmund in [3], and by Stein [21] (see Theorem 1 in [21, Section 2.2]). Though our conical case is different from the classical cases on several aspects, the really surprising thing is: the proof of Theorem 9.9 in [12] proceeds well in our case, after overcoming several analytical difficulties. Namely, the following $W^{2,p,\beta}$-estimate is true.

THEOREM 1.13. — *Suppose L is an elliptic CKS operator defined over $B(2)$. Suppose there is a sufficiently small constant δ_0 such that*

$$|L - \Delta_\beta|_{L_O^{\infty,\beta}} \leqslant \delta_0, \ over \ B(2).$$

Suppose $u \in C^2(B(2) \setminus D) \cap W^{2,p,\beta}[B(2)]$ is a classical-solution to

$$Lu = f \ in \ B(2) \setminus D, \ f \in L^p[B(2)], \ \infty > p \geqslant 2.$$

Then

$$[u]_{W^{2,p,\beta},B(1)} \leqslant C(|f|_{L^p,B(2)} + |u|_{W^{1,p},B(2)}),$$

where C only depends on n, β, p. In particular, we have

$$[u]_{W^{2,p,\beta},B(1)} \leqslant C(|\Delta_\beta u|_{L^p,B(2)} + |u|_{W^{1,p},B(2)}).$$

The following Sobolev-imbedding Theorem in the conical category is also crucially needed in the proof of Theorem 1.6.

THEOREM 1.14. — *Let $u \in W^{1,2}(B(2)) \cap C^2(B(2) \setminus D)$ be a weak solution to $\Delta_\beta u = f$ in $B(2)$, $f \in L^p$, $p > 2n$. Then for all $\alpha < \min\left\{1 - \frac{2n}{p}, \frac{1}{\beta} - 1\right\}$, we have $u \in C^{1,\alpha,\beta}(B(1))$ and*

$$|u|_{1,\alpha,\beta,B(1)} \leqslant C(|u|_{W^{1,2},B(2)} + |f|_{L^p,B(2)}).$$

Convention of the constants

The "C"'s in the estimates mean constants independent of the object estimated, suppose the object satisfies the conditions and bounds in the correponding statement. In some cases we say explicitly what does the "C" depend on. When we don't say anything to the "C", we mean it can depend on the conditions and bounds in the corresponding statement, for example, like the $C^{1,1,\beta}$-bound on the given potential, or the C^α−bound on the given metric given away from the divisor, or the $C^{1,1,\beta}$-bound on the given volume form f, or the quasi-isometric constant of ω with respect to ω_β, ... and so on.

Distances and Balls

In most of the cases, we use distance and balls defined by the model cone metric ω_β, unless otherwise specified. The balls are usually centered at the origin or some point on the divisor. In this case, the conic balls are exactly the balls with respect to the Euclidean distance in polar coordinates.

The model cone metric ω_β is exactly the usual Euclidean metric in the following coordinate (see Remark 2.3 in [9]).

$$\xi = z^\beta = re^{iv}, \ v = \beta\theta \in (-\beta\pi, \beta\pi].$$

The tangential coordinates are as usual.

This coordinate is only defined on $\mathfrak{W}_\beta \times \mathbb{C}^{n-1}$, where $\mathfrak{W}_\beta$ is the wedge of angle $2\pi\beta$ in $\mathbb{C}$. Thus in this coordinate, the conic balls are exactly the Euclidean balls.

The only big exception is in Section 3, where we use the Euclidean metric ω_E in the polar coordinates. The reason is that it's super convenient for using the cube decomposition which is necessary in Calderon–Zygmund theory. ω_E and ω_β are quasi-isometric to each other i.e.

$$\beta\omega_\beta \leqslant \omega_E \leqslant \frac{\omega_\beta}{\beta}.$$

Thus the distances defined by them are equivalent.

Acknowledgement

We would like to thank Mihai Păun for pointing out that the global version of Corollary 1.8 is already proved in the newest version of [14] using a different method.

2. The L^2-estimate

In this section, we fix the necessary notations and prove the L^2-estimate of the conical Laplace equation in Lemma 2.7. This is the first step toward a full $W^{2,p,\beta}$−theory for all $p \in (1, \infty)$.

Let $r = |z|^\beta$ and θ be just the angle of z from the positive real axis. In the polar coordinates $r, \theta, w_i, \ 1 \leqslant i \leqslant 2n - 2$, ω_β can be written as

$$\omega_\beta = dr^2 + \beta^2 r^2 d\theta^2 + \sum_{i=1}^{2n-2} dw_i \otimes dw_i,$$

where r is the distance to the divisor $D = \{0\} \times \mathbb{C}^{n-1}$, θ is the usual angle of the variable z, and w_i are the tangential variables.

Notice in the polar coordinates we have $\beta^2 g_E \leqslant \omega_\beta \leqslant \frac{1}{\beta^2} g_E$, where g_{Euc} is Euclidean metric in the polar coordinates i.e.

$$(2.1) \qquad\qquad g_E = dr^2 + r^2 d\theta^2 + + \sum_{i=1}^{2n-2} dw_i \otimes dw_i.$$

We denote ω_E as the Kähler form of g_E. We will be frequently using the polar coordinates in most of the following content, as in this nice coordinates, the conical metrics are quasi-isometric to the Euclidean metric g_E. We first define the space $W^{1,p,\beta}(B)$ as usual $W^{1,p}$-space in the polar coordinates, $\infty > p \geqslant 2$.

DEFINITION 2.1 ($W^{2,p,\beta}$−space). — *Given $p \geqslant 2$, and a ball B, a function u is said to be in the space $W^{2,p,\beta}(B)$ if the following holds. We can understand the polar coordinates as the intrinsic coordinates of ω_β.*

- *$u \in W^{2,p}_{loc}[B \setminus D]$ where $W^{2,p}_{loc}[B \setminus D]$ is the usual Sobolev space in the holomorphic coordinates.*
- *$|z|^{2-2\beta} \frac{\partial^2 u}{\partial z \partial \bar{z}} \in L^p(B)$, the $L^p(B)$−space is the usual L^p−space in the polar coordinates (so are the "$L^p(B)$"'s in the following);*
- *$|z|^{1-\beta} \frac{\partial^2 u}{\partial z \partial w_j} \in L^p(B)$, for all $0 \leqslant j \leqslant 2n-2$;*
- *$\frac{\partial^2 u}{\partial w_i \partial w_j} \in L^p(B)$, for all $0 \leqslant i,j \leqslant 2n-2$;*
- *$u \in W^{1,p,\beta}(B)$.*

The norm and seminorm are written as

$$[u]_{W^{2,p,\beta}(B)} = \left\| |z|^{2-2\beta} \frac{\partial^2 u}{\partial z \partial \bar{z}} \right\|_{L^p(B)} + \sum_{j=1}^{2n-2} \left\| |z|^{1-\beta} \frac{\partial^2 u}{\partial z \partial w_j} \right\|_{L^p(B)}$$

(2.2)
$$+ \sum_{i,j=1}^{2n-2} \left\| \frac{\partial^2 u}{\partial w_i \partial w_j} \right\|_{L^p(B)}.$$

$$|u|_{W^{2,p,\beta}(B)} = [u]_{W^{2,p,\beta}(B)} + |u|_{W^{1,p,\beta}(B)}.$$

LEMMA 2.2. — *For any ball B, $W^{2,p,B}(B)$ is a (complete) Banach space.*

Remark 2.3. — This completeness lemma is used in the definition of $N_\beta f$ for $f \in L^p$, $p \geqslant 2$, as in Lemma 2.7. We present a full proof for the convenience of the readers, though it's straightforward.

Proof of Lemma 2.2. — Without loss of generality, we assume B is the unit ball (centered at the origin). We only consider the case $W^{2,2,\beta}(B)$, the proof for all p is exactly the same. It suffices to construct a limit. Suppose $\{u_k\}$ is a Cauchy-Sequence of $W^{2,2,\beta}(B)$, then in the sense of $W^{1,2,\beta}(B)$, u_k admits a limit denoted as $\underline{u}$. Then it remains to show $\underline{u}$ is actually in $W^{2,2,\beta}(B)$.

Denote B_R and the radius of R, and $T_R(D)$ as the turbular neighborhood of D with radius R (as in Definition 2.1). Over $B_{1-\frac{\epsilon}{2}} \setminus T_{\frac{\epsilon}{2}}(D)$, we deduce that $\{\Delta_\beta u_k\}$ is a Cauchy-Sequence in $L^2[B_{1-\frac{\epsilon}{2}} \setminus T_{\frac{\epsilon}{2}}(D)]$. Then we apply the interior elliptic estimate to the pair of domains

$$B_{1-\frac{\epsilon}{2}} \setminus T_{\frac{\epsilon}{2}}(D), \ B_{1-\epsilon} \setminus T_\epsilon(D).$$

By Theorem 8.8 in [12], we deduce

$$|u_k - u_l|_{2,2,B_{1-\epsilon} \setminus T_\epsilon(D))}$$
$$\leqslant C(\epsilon) \left[|u_k - u_l|_{1,2,B_{1-\frac{\epsilon}{2}} \setminus T_{\frac{\epsilon}{2}}(D)} + |\Delta_\beta u_k - \Delta_\beta u_l|_{0,2,B_{1-\frac{\epsilon}{2}} \setminus T_{\frac{\epsilon}{2}}(D)} \right].$$

Thus, $\{u_k\}$ is a Cauchy-Sequence in the usual Sobolev space $W^{2,2}(B_{1-\epsilon} \setminus T_\epsilon(D))$. Then, by the completeness of the usual Sobolev spaces, and the diagonal sequence trick, there exists a limit function in $W_{loc}^{2,2}(B \setminus D)$, which can be nothing else than $\underline{u}$, with the following property.

$$\lim_{k \to \infty} |u_k - \underline{u}|_{2,2,B_{1-\epsilon} \setminus T_\epsilon(D)} = 0, \text{ for any } \epsilon > 0.$$

Since over $B_{1-\epsilon} \setminus T_\epsilon(D)$, the $W^{2,2,\beta}(B_{1-\epsilon} \setminus T_\epsilon(D))$-norm is weaker than the usual $W^{2,2}(B_{1-\epsilon} \setminus T_\epsilon(D))-$norm, and $\{u_k\}$ is a Cauchy-Sequence in $W^{2,2,\beta}(B)$, we deduce the following by the Minkovski inequality

$$(2.3) \quad |\underline{u}|_{W^{2,2,\beta}(B_{1-\epsilon} \setminus T_\epsilon(D))}$$
$$\leqslant \limsup_k \{|\underline{u} - u_k|_{W^{2,2,\beta}(B_{1-\epsilon} \setminus T_\epsilon(D))} + |u_k|_{W^{2,2,\beta}(B_{1-\epsilon} \setminus T_\epsilon(D))}\}$$
$$= \limsup_k |u_k|_{W^{2,2,\beta}(B_{1-\epsilon} \setminus T_\epsilon(D))} \leqslant C,$$

where C does not depend on ϵ. Since ϵ is arbitrary, (2.3) implies

$$(2.4) \quad |\underline{u}|_{W^{2,2,\beta}(B)} \leqslant C < \infty, \text{ then } \underline{u} \in W^{2,2,\beta}(B).$$

The proof of Lemma 2.2 is complete. $\qquad \square$

To study the $W^{2,p,\beta}$-estimate, we quote the heat kernel formula in [9]. Denote $x = (r, \theta, \underline{x})$ and $y = (r', \theta', \underline{x}')$, where $\underline{x}$ is the tangential projection of x. Denote $R = |\underline{x} - \underline{x}'|$. The heat kernel is

$$(2.5) \quad H(x, y, t)$$
$$= \frac{1}{(4\pi t)^n} e^{-\frac{r^2 + \hat{r}^2 + R^2}{4t}} \left\{ \sum_{k, -\pi < \beta[\theta - \theta'] + 2k\beta\pi < \pi} e^{\frac{rr' \cos(\beta[\theta - \theta'] + 2k\beta\pi)}{2t}} \right.$$
$$\left. + K\left(\frac{rr'}{2t}, \beta[\theta - \theta']\right) + \frac{1}{2} \sum_{k, \beta[\theta - \theta'] + 2k\beta\pi = \pm\pi} e^{-\frac{rr'}{2t}} \right\},$$

where

$$(2.6) \quad K(\frac{rr'}{2t}, \beta[\theta - \theta'])$$
$$= \frac{\sin \frac{\pi}{\beta}}{\pi\beta} \int_0^\infty e^{-\frac{rr'}{2t} \cosh y} \frac{[\cos \frac{\pi}{\beta} - \cos[\theta - \theta'] \cosh \frac{y}{\beta}]}{[\cosh \frac{y}{\beta} - \cos \frac{\beta[\theta - \theta'] - \pi}{\beta}][\cosh \frac{y}{\beta} - \cos \frac{\beta[\theta - \theta'] + \pi}{\beta}]} dy.$$

In the above formula, we actually abused the notation a little bit, as in [9]. To be precise, the "$\theta - \theta'$" means the unique angle in $(-\pi, \pi]$ which is mod 2π equivalent to $\theta - \theta'$.

We define the Green function of ω_β as

$$\Gamma(x, y) = -\int_0^\infty H(x, y, t)dt.$$

The following lemma is true.

LEMMA 2.4. — *For any $x \notin D$, we have*

$$\lim_{\epsilon \to 0} \int_{\partial B_x(\epsilon)} \frac{\partial \Gamma(x, y)}{\partial \nu_y} dy = 1.$$

Proof of Lemma 2.4. — It suffices to notice that, by the assumption that $x \notin D$, we have $r_x > 0$ ($r_x = r$, we just add the sub x to emphasize its dependence on x). Then, when $k \neq 0$, we deduce

$$(2.7) \qquad e^{-\frac{r^2+r'^2}{4t}} e^{\frac{rr'\cos(\beta[\theta-\theta']+2k\beta\pi)}{2t}} \leqslant e^{-\frac{(r-r')^2+2(1-\cos\beta\pi)rr'}{4t}} \leqslant e^{-\frac{a}{t}},$$

where a is a positive constant depending on x, especially r_x (the distance from x to D). Then, by defining

$(2.8) \quad \Gamma_E$

$$= -\int_0^\infty \frac{1}{(4\pi t)^n} e^{-\frac{r^2+\acute{r}^2+R^2}{4t}} \left\{ \sum_{\substack{k \neq 0, \\ -\pi < \beta[\theta-\theta']+2k\beta\pi < \pi}} e^{\frac{rr'\cos(\beta[\theta-\theta']+2k\beta\pi)}{2t}} \right.$$

$$\left. + K\left(\frac{rr'}{2t}, \beta[\theta-\theta']\right) + \frac{1}{2} \sum_{k, \beta[\theta-\theta']+2k\beta\pi=\pm\pi} e^{-\frac{rr'}{2t}} \right\} dt,$$

we obtain when $y \in B_x(\frac{r_x\sin\beta\pi}{2})$ and $\beta[\theta - \theta'] \neq \pm\pi \mod 2\beta\pi$ that

$(2.9) \quad |\nabla_y \Gamma_E|$

$$\leqslant \int_0^\infty \left| \nabla_y \left\{ \frac{1}{(4\pi t)^n} e^{-\frac{r^2+\acute{r}^2+R^2}{4t}} \left[\sum_{\substack{k \neq 0 \\ -\pi < \beta[\theta-\theta']+2k\beta\pi < \pi}} e^{\frac{rr'\cos(\beta[\theta-\theta']+2k\beta\pi)}{2t}} \right. \right. \right.$$

$$\left. \left. \left. + K\left(\frac{rr'}{2t}, \beta[\theta-\theta']\right) \right\} \right| dt$$

$$\leqslant C \int_0^\infty \left(\frac{1}{t^n} + \frac{1}{t^{n+\frac{1}{2}}} + \frac{1}{t^{n+1}} \right) e^{-\frac{a}{t}} dt \leqslant C_a.$$

By continuity, we have for all $x \notin D$ and $y \in B_x(\frac{r_x\sin\beta\pi}{2}) \subset \mathfrak{W}_\beta \times \mathbb{C}^{n-1}$ that

$$(2.10) \qquad\qquad |\nabla_y \Gamma_E(x, y)| \leqslant C_a.$$

Notice that

$$\Gamma(x,y) = -\int_0^\infty \frac{1}{(4\pi t)^n} e^{-\frac{r^2+\hat{r}^2+R^2}{4t}} e^{\frac{rr'\cos(\beta[\theta-\theta'])}{2t}} dt + \Gamma_E.$$

(2.11)
$$= -\int_0^\infty \frac{1}{(4\pi t)^n} e^{-\frac{|x-y|^2}{4t}} dt + \Gamma_E.$$

$$= -\frac{1}{4\pi^n \rho^{2n-2}} \Gamma(n-1) + \Gamma_E,$$

where $\rho = |x-y|$ and $\Gamma(n)$ is the Gamma-function. Using (2.10), we deduce

$$\lim_{\epsilon\to 0} \int_{\partial B_x(\epsilon)} \frac{\partial \Gamma_E(x,y)}{\partial \rho} dy = 0.$$

Moreover, we have $\frac{\Gamma(n)S(2n-1)}{2\pi^n} = 1$, where $S(2n-1)$ is the area of $(2n-1)$-dimensional unit sphere. Then we compute

(2.12) $\displaystyle \lim_{\epsilon\to 0} \int_{\partial B_x(\epsilon)} \frac{\partial \Gamma(x,y)}{\partial \rho} dy$

$$= \lim_{\epsilon\to 0} \int_{\partial B_x(\epsilon)} \frac{\partial \Gamma_E(x,y)}{\partial \rho} dy + \frac{1}{2\pi^n \epsilon^{2n-1}} \Gamma(n)S(2n-1)\epsilon^{2n-1}$$

$$= 1. \qquad \qquad \square$$

DEFINITION 2.5. — *We denote $N_{\beta,\Omega}f$ as the Newtonian potential of f over Ω i.e.*

$$N_{\beta,\Omega}f = \int_\Omega \Gamma(x,y)f(y)dy.$$

LEMMA 2.6 (Green Representation). — *Suppose $u \in W_c^{2,2,\beta}(\mathbb{C}^n) \cap C^2(\mathbb{C}^n \setminus D)$, then the Green's representation formula holds for u i.e. for all $x \notin D$, we have*

(2.13) $$u(x) = N_{\beta,\mathbb{C}^n}(\Delta_\beta u)(x).$$

Proof of Lemma 2.6. — First, since $x \notin D$ and $u \in C^2(\mathbb{C}^n \setminus D)$, then when ϵ_0 such that $B_x(\epsilon_0) \cap D = \emptyset$, the following

$$N_{\beta,\mathbb{C}^n}\Delta_\beta u = \int_{B_x(\epsilon_0)} \Gamma(x,y)\Delta_\beta u(y)dy + \int_{\mathbb{C}^n \setminus B_x(\epsilon_0)} \Gamma(x,y)\Delta_\beta u(y)dy$$

is well-defined pointwisely for all $x \in \mathbb{C}^n \setminus D$.

Lemma C.1 implies for $u \in W_c^{2,2,\beta}(\mathbb{C}^n) \cap C^2(\mathbb{C}^n \setminus D)$ that

$$\int_{\mathbb{C}^n \setminus D} v\Delta_\omega u\, \omega^n = -\int_{\mathbb{C}^n \setminus D} (\nabla_\omega v \cdot \nabla_\omega u)\, \omega^n \text{ for any conical metric } \omega,$$

then (2.13) follows from the well known derivation of formula (2.17) in page 18 of [12], and Lemma 2.4. $\qquad \square$

In the conical case, the operator T (as defined in 1.5) might not be self adjoint because there is one special direction. Nevertheless, this could compensated by the good properties of $T^\star$. For any $f, g \in C_c^{\alpha,\beta}(B)$, we have

$$(2.14) \qquad \int_B (Tf)g dx = \int_B f T^\star g dy.$$

It's easy to show that

$$T^\star g = -D_{y_j} \int_B D_x \Gamma(x, y) g(x) dx,$$

where y_j is a tangential varible in the y-component, and D_x is an order 1 differential operator in the x-component.

Notice D_x can not be integrated by parts in general, since $div \frac{\partial}{\partial r} \neq 0$ and $div\{\frac{1}{r}\frac{\partial}{\partial \theta}\} \neq 0$. Nevertheless, Lemma 3.3 guarantees that $T^\star$ is densely defined in $L^2(B)$, which leads to our necessary L^2-estimate. The proof of the following crucial L^2-estimate is almost the same as proof i of Theorem 9.9 in [12]. Nevertheless, since it concerns the correct choice of the Hessian operator to integrate by parts toward, we still present a detailed proof. The Hessian operator we choose here leads to the necessary $W^{2,p,\beta}$-estimate when p is large.

LEMMA 2.7. — *Given a ball B, suppose $f \in L^2(B)$, then $N_{\beta,B}f$ is well-defined. Moreover,*

$$|Tf|_{L^2(B)}^2 \leqslant C \int_B f^2,$$

and

$$|T^\star f|_{L^2(B)}^2 \leqslant C \int_B f^2.$$

Proof of Lemma 2.7. — For any sequence $\epsilon_k > 0$ such that $\epsilon_k \to 0$, we consider the smoothing of cutoffs of f with parameter ϵ_k, denoted as f_{ϵ_k}. The point is that the smoothing and cutoffs work well in the conical case. Namely, the approximation functions f_{ϵ_k} are in $C_c^\infty(B)$, and

$$\lim_{k \to \infty} |f_{\epsilon_k} - f|_{L^2(B)} = 0.$$

The space $C^\infty(B)$ is of compact supported smooth functions in the polar coordinates, not holomorphic coordinates.

Step 1. — Then we consider $w_{\epsilon_k} = N_{\beta,B}f_{\epsilon_k}$. Then, by the work in Donaldson (also see [9]), $N_{\beta,B}f_{\epsilon_k} \in C^{2,\alpha,\beta}(B)$, thus it makes sense to consider Hessian of ω_{ϵ_k} in some sense. It sufficies to prove

$$(2.15) \qquad \int_B f_{\epsilon_k}^2 \omega_\beta^n = \int_{\mathbb{C}^n} |\Delta w_{\epsilon_k}|^2 \omega_\beta^n = \int_{\mathbb{C}^n} |\nabla^{1,1,\beta} w_{\epsilon_k}|^2 \omega_\beta^n,$$

where the $\nabla^{1,1,\beta}$ is the Hessian operator whose components are exactly those in the seminorm (2.2). This choice integrates well with Definition 1.3.

Then, the integration by parts proceeds line to line as in proof (i) of Theorem 9.9 in [12]. For the sake of a self-contained proof, and of emphasizing the operator $\nabla^{1,1,\beta}$ we choose, we include the detail here. Denote

$$\Delta_{0,\beta} = \frac{|z|^{2-2\beta}}{\beta^2} \frac{\partial^2}{\partial z \partial \bar{z}}, \quad \omega_{0,\beta} = \frac{\beta^2}{|z|^{2-2\beta}} \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}.$$

Then

$$(2.16) \qquad \Delta = 4\Delta_{0,\beta} + \sum_{j=1}^{2n-2} \frac{\partial^2}{\partial y_j^2}.$$

Denote A_R as the polycylinder of radius R. To be precise, we define

$$(2.17) \qquad A_R = D_R \times B_R,$$

where $D(R)$ is the disk with radius R in the z-component of $\mathbb{C}^n$ (centered at the origin), and B_R is the ball with radius R in $D = \{0\} \times \mathbb{C}^{n-1}$ (also centered at the origin). Let R be large enough such that $A(R) \supset \operatorname{supp} f$, then

$$(2.18) \quad \int_{\underline{B}(R)} \int_{\underline{D}(R)} (\Delta w_{\epsilon_k})^2 \omega_{0,\beta} \wedge dy_1 \ldots \wedge dy_{2n-2}$$

$$= 16 \int_{\underline{B}(R)} \int_{\underline{D}(R)} (\Delta_{0,\beta} w_{\epsilon_k})^2 \omega_{0,\beta} \wedge dy_1 \ldots \wedge dy_{2n-2}$$

$$+ 8 \sum_{j=1}^{2n-2} \int_{\underline{B}(R)} \int_{\underline{D}(R)} (\Delta_{0,\beta} w_{\epsilon_k}) w_{\epsilon_k,jj} \omega_{0,\beta} \wedge dy_1 \ldots \wedge dy_{2n-2}$$

$$+ \sum_{i,j=1}^{2n-2} \int_{\underline{B}(R)} \int_{\underline{D}(R)} w_{\epsilon_k,ii} w_{\epsilon_k,jj} \omega_{0,\beta} \wedge dy_1 \ldots \wedge dy_{2n-2}.$$

Ignoring the constant coefficient temporarily, it suffices to deal with the second term above. Since $f \in C_c^\infty(B)$, and Donaldson's Schauder estimate in [10] is smooth in the tangential directions, we have

$$(2.19) \qquad \frac{\partial}{\partial x_j} \Delta_{0,\beta} w_{\epsilon_k} \in C^0[A(R)].$$

We will show below that it's convenient to do integration by parts over these polycylinders, in our case.

Using the condition (2.19) and Lemma 2.5 in [22] (see Appendix C), the tangential derivatives $\frac{\partial}{\partial y_j}$ can be integrated by parts. Hence

$$(2.20) \quad \sum_{j=1}^{2n-2} \int_{\underline{B}(R)} \int_{\underline{D}(R)} (\Delta_{0,\beta} w_{\epsilon_k}) w_{\epsilon_k,jj} \omega_{0,\beta} \wedge dy_1 \ldots \wedge dy_{2n-2}$$

$$= - \sum_{j=1}^{2n-2} \int_{\underline{B}(R)} \int_{\underline{D}(R)} (\Delta_{0,\beta} w_{\epsilon_k,j}) w_{\epsilon_k,j} \omega_{0,\beta} \wedge dy_1 \ldots \wedge dy_{2n-2}$$

$$+ \sum_{j=1}^{2n-2} \int_{\partial(\underline{B}(R) \times \underline{D}(R))} (\Delta_{0,\beta} w_{\epsilon_k}) w_{\epsilon_k,j} n_j \omega_{0,\beta} \wedge dy_1 \ldots \wedge dy_{2n-2}.$$

$w_{\epsilon_k} \in C^{2,\alpha,\beta}[A(R)]$ implies the tangential-normal mixed derivatives $\nabla_{0,\beta} w_{\epsilon_k,j}$ are in $L^\infty[A(R)]$. Moreover, ω_β is a product metric of $\omega_{0,\beta}$ with the Euclidean metric in the tangential directions along D. Then, again, Lemma 2.5 in [22] and Fubini's Theorem imply we can integrate the $\Delta_{0,\beta}$ on the first $\mathbb{C}$-slice by parts to obtain

$$(2.21) \quad -\sum_{j=1}^{2n-2} \int_{\underline{B}(R)} \int_{\underline{D}(R)} (\Delta_{0,\beta} w_{\epsilon_k,j}) w_{\epsilon_k,j} \omega_{0,\beta} \wedge dy_1 \ldots \wedge dy_{2n-2}$$

$$= \sum_{j=1}^{2n-2} \int_{\underline{B}(R)} dy_1 \ldots \wedge dy_{2n-2} \int_{\underline{D}(R)} |\nabla_{0,\beta} w_{\epsilon_k,j}|^2 \omega_{0,\beta}$$

$$- \sum_{j=1}^{2n-2} \int_{\underline{B}(R)} dy_1 \ldots \wedge dy_{2n-2} \int_{\partial \underline{D}(R)} <\nu, \nabla_{0,\beta} w_{\epsilon_k,j}>_\beta w_{\epsilon_k,j} \omega_{0,\beta},$$

where ν is the outer-normal of $\partial \underline{D}(R)$ with respect to $\omega_{0,\beta}$. Theorem 1.11 in [9] and the compactly supported property of f implies

$$(2.22) \quad |\nabla_{0,\beta} w_{\epsilon_k,j}| \in O(|x|^{-2n}), \quad w_{\epsilon_k,j} \in O(|x|^{-2n+1}),$$

$$|\Delta_{0,\beta} w_{\epsilon_k}| \in O(|x|^{-2n}).$$

Thus, combing (2.20) and (2.21), using (2.22), let $R \to \infty$, then the boundary terms all tend to 0, and we obtain the following as in proof (i) of Theorem 9.9 in [12].

$$(2.23) \quad \sum_{j=1}^{2n-2} \int_{\mathbb{C}^n} (\Delta_{0,\beta} w_{\epsilon_k}) w_{\epsilon_k,jj} \omega_{0,\beta} \wedge dy_1 \ldots \wedge dy_{2n-2}$$

$$= \sum_{j=1}^{2n-2} \int_{\mathbb{C}^n} |\nabla_{0,\beta} w_{\epsilon_k,j}|^2 \omega_{0,\beta} \wedge dy_1 \ldots \wedge dy_{2n-2}.$$

Handling the term $\sum_{i,j=1}^{2n-2} \int_{\underline{B}(R)} \int_{\underline{D}(R)} w_{\epsilon_k,ii} w_{\epsilon_k,jj} \omega_{0,\beta} \wedge dy_1 \dots \wedge dy_{2n-2}$ in (2.18) in the similar and easier way, let $R \to \infty$, then the boundary terms all tend to 0, and we deduce from (2.18) and (2.23) that

$$(2.24) \quad \int_{\mathbb{C}^n} (\Delta w_{\epsilon_k})^2 \omega_{0,\beta} \wedge dy_1 \dots \wedge dy_{2n-2}$$

$$= 16 \int_{\mathbb{C}^n} (\Delta_{0,\beta} w_{\epsilon_k})^2 \omega_{0,\beta} \wedge dy_1 \dots \wedge dy_{2n-2}$$

$$+ 8 \sum_{j=1}^{2n-2} \int_{\mathbb{C}^n} |\nabla_{0,\beta} w_{\epsilon_k,j}|^2 \omega_{0,\beta} \wedge dy_1 \dots \wedge dy_{2n-2}$$

$$+ \sum_{i,j=1}^{2n-2} \int_{\mathbb{C}^n} |w_{\epsilon_k,ij}|^2 \omega_{0,\beta} \wedge dy_1 \dots \wedge dy_{2n-2}.$$

Thus identity (2.15) is true for Newtonian potentials of compactly supported smooth functions.

Step 2. — By Young's inequality, since $\Gamma(x,y), \nabla\Gamma(x,y) \in L^1(B)$, (by Donaldson's work [10], also see [9]), we conclude

$$(2.25) \quad |w_{\epsilon_{k_1}} - w_{\epsilon_{k_2}}|_{L^2(B)} \leqslant \left| \int_B \Gamma(x,y)(f_{\epsilon_{k_1}}(y) - f_{\epsilon_{k_2}}(y)) dy \right|_{L^2(B)}$$

$$\leqslant |f_{\epsilon_{k_1}} - f_{\epsilon_{k_2}}|_{L^2(B)},$$

and

$$(2.26) \quad |\nabla w_{\epsilon_{k_1}} - \nabla w_{\epsilon_{k_2}}|_{L^2(B)} \leqslant \left| \int_B [\nabla_x \Gamma(x,y)](f_{\epsilon_{k_1}}(y) - f_{\epsilon_{k_2}}(y)) dy \right|_{L^2(B)}$$

$$\leqslant |f_{\epsilon_{k_1}} - f_{\epsilon_{k_2}}|_{L^2(B)}.$$

Then, since $f_{\epsilon_k} \to f$ in the $L^2(B)$-sense, then w_{ϵ_k} is a Cauchy-Sequence in $W^{2,2,\beta}(B)$−space. Thus, by the completeness of the $W^{2,2,\beta}(B)$−space, there exists a $w \in W^{2,2,\beta}(B)$ such that

$$\lim_{k \to \infty} |w_{\epsilon_k} - w|_{W^{2,2,\beta}(B)} = 0.$$

Then, we define $w = N_{\beta,B} f$. By (2.26), (2.15), and (2.25), the proof of Lemma 2.7 for $|Tf|_{L^2(B)}$ is complete.

The proof for $|T^\star f|_{L^2(B)}$ is simply by using (4.3) (let $p = 2$) with T and $T^\star$ interchanged.   □

Remark 2.8. — The feature of the $\nabla^{1,1,\beta}$ operator we consider in (2.15) is: it is just the usual real Hessian in the tangential direction of D, it contains all the mixed derivatives. But, in the normal direction of D, it only contains the complex $(1,1)$ derivative. This is the one of the main points of

this article: with this slightly weaker Hessian, we obtain $W^{2,p,\beta}-$estimates for all $p \in (1, \infty)$. We don't think any $W^{2,p}-$theory for the real Hessian ∇^2 (of ω_β) could be true when p is large.

3. The Calderon–Zygmund inequalities

In this section, we use the Euclidean metric ω_E in the polar coordinates to define the distances and balls, for the sake of the cube-decomposition. We show that with the help of Theorem 1.11 in [9], the Calderon–Zygmund theory in [3] works suprisingly well in the conical setting, after overcoming a technical difficulty. Namely, the main technical difficulty is that T is not selfadjoint. However, as presented below, this difficulty can be easily overcomed, by observing that $T^\star$ (the dual of T) also possess similar good properties as the Calderon–Zygmund singular integral operators. We follow the proof of Theorem 9.9 in [12], and define $\mu_{Tf}(t) \triangleq m\{x \in K_0 | f(x) > t\}$.

LEMMA 3.1. — *Let B be a ball with finite radius. The operator T is weakly-$(1,1)$ bounded i.e. for any $f \in L^2(B)$, we have*

$$(3.1) \qquad\qquad \mu_{Tf}(t) \leqslant \frac{C}{t}|f|_{L^1(B)},$$

and

$$(3.2) \qquad\qquad \mu_{T^\star f}(t) \leqslant \frac{C}{t}|f|_{L^1(B)},$$

where C only depend on β and n.

Proof of Lemma 3.1. — In the polar coordinates, with respect to the Euclidean metric, we consider a cube K_0 (with respect to the Euclidean metric) big enough so that the following holds. For every K in the first $[\frac{(10n)^{10n}}{\beta}]$ (the smallest integer bigger than $\frac{(10n)^{10n}}{\beta}$) dyadic cut of K, $\int_K |f| \leqslant t|K|$. Exactly as in Theorem 9.9 in [12], we consider the dyadic cuts of K_0 subject to f and t. Then we obtain cubes K_l, $l = 1, 2 \ldots$ such that

$$(3.3) \qquad\qquad t|K_l| \leqslant \int_{K_l} |f| \leqslant 2^{2n}t|K_l|, \text{ for all } l,$$

and

$$(3.4) \qquad\qquad f \leqslant t \text{ almost everywhere over } G = K_0 \setminus F,$$

where $F = \cup_l K_l$.

Then we consider the "good" and "bad" decomposition of f as $f = g + b$ such that

$$g = \begin{cases} f, & \text{over } G; \\ \frac{1}{|K_l|} \int_{K_l} |f|, & \text{over } K_l. \end{cases}$$

and

$$b = \begin{cases} 0, & \text{over } G; \\ b_l, & \frac{1}{|K_l|} \int_{K_l} b_l = 0. \end{cases}$$

Thus, (3.3) and (3.4) imply

(3.5) $$|g|_{L^\infty(K_0)} \leqslant 2^{2n} t.$$

We have

(3.6) $$\mu_{Tf}(t) \leqslant \mu_{Tg}\left(\frac{t}{2}\right) + \mu_{Tb}\left(\frac{t}{2}\right).$$

As in the proof Theorem 9.9 in [12], by Lemma 2.7, we estimate $\mu_{Tg}\left(\frac{t}{2}\right)$ as

(3.7) $$\mu_{Tg} \leqslant \frac{4}{t^2} \int_{K_0} g^2 \leqslant \frac{2^{2n+2}}{t} \int_{K_0} g \leqslant \frac{2^{2n+2}}{t} \int_{K_0} |f|.$$

(3.8) $$Tb_l = \int_{K_l} \mathfrak{D}\Gamma(x, y) b_l(y) dy$$

is well-defined when $x \notin K_l$. At this stage, actually for any $\mathfrak{D} \in \mathbb{T}$, there exists a $\mathfrak{D}' \in \mathfrak{M}$ (see Definition 3.4) such that

(3.9) $$\mathfrak{D}\Gamma(x, y) = \mathfrak{D}'\Gamma(x, y).$$

This is by the translation invariance of the model metric ω_β in the directions tangential to D. To see this, for example, take $\mathfrak{D} = \frac{\partial^2}{\partial r_x \partial x_j} \in \mathbb{T}$, where both the derivatives act on x. Notice that in (2.5), in the D-tangential directions, the heat kernel only depends on $|\underline{x} - \underline{y}|$, which means

(3.10) $$\frac{\partial^2}{\partial r_x \partial x_j} \Gamma(x, y) = -\frac{\partial^2}{\partial r_x \partial y_j} \Gamma(x, y).$$

Notice that the biggest feature of $\mathfrak{D}'$ is that the two derivatives are distributed to different variables, and Lemma 3.5 holds for them.

Hence, using $\frac{1}{|K_l|} \int_{K_l} b_l = 0$, we have

(3.11) $$Tb_l(x) = \int_{K_l} [\mathfrak{D}'\Gamma(x, y) - \mathfrak{D}'\Gamma(x, \bar{y})] b_l(y) dy,$$

where $\bar{y}$ is the center of K_l, and $x \notin B_{\bar{y}}(\frac{10^{10} D_l}{\beta})$, D_l is the diameter of K_l.

Case 1. — Suppose $dist(\bar{y}, \{z = 0\}) < \frac{\beta^2 dist(\bar{y},x)}{1000}$, then using Lemma 3.5, we have

$$(3.12) \quad |Tb_l|(x) \leqslant \int_{K_l} \frac{C|y - \bar{y}|^{\rho_0}}{|x - \bar{y}|^{2n+\rho_0}} |b_l| dy \leqslant CD_l^{\rho_0} \int_{K_l} |x - \bar{y}|^{-(2n+\rho_0)} |b_l| dy.$$

Case 2. — Suppose $dist(\bar{y}, \{z = 0\}) > \frac{\beta^2 dist(\bar{y},x)}{1000}$, then using Lemma 7.5 in [9] (with the condition $P(y) \geqslant \frac{\beta^4 |x-y|}{100^{100}}$), we have

$$
\begin{aligned}
(3.13) \quad |Tb_l|(x) &\leqslant \int_{K_l} \frac{C|y - \bar{y}|}{|x - \widehat{y}_x|^{2n+1}} |b_l| dy \leqslant CD_l \int_{K_l} |x - \widehat{y}_x|^{-(2n+1)} |b_l| dy \\
&\leqslant CD_l \int_{K_l} |x - \bar{y}|^{-(2n+1)} |b_l| dy,
\end{aligned}
$$

where $\widehat{y}_x$ is a point in the line segment connecting y and $\bar{y}$. Thus,

$$
\begin{aligned}
(3.14) \quad &\int_{K_0 \setminus B_{\bar{y}}(\frac{10^{10} D_l}{\beta})} |Tb_l(x)| dx \\
&= \int_{\{K_0 \setminus B_{\bar{y}}(\frac{10^{10} D_l}{\beta})\} \cap \{dist(\bar{y},\{z=0\}) < \frac{\beta^2 dist(\bar{y},x)}{1000}\}} |Tb_l(x)| dx \\
&\quad + \int_{\{K_0 \setminus B_{\bar{y}}(\frac{10^{10} D_l}{\beta})\} \cap \{dist(\bar{y},\{z=0\}) \geqslant \frac{\beta^2 dist(\bar{y},x)}{1000}\}} |Tb_l(x)| dx \\
&\leqslant CD_l^{\rho_0} \int_{K_l} |b_l| dy \int_{\substack{\{K_0 \setminus B_{\bar{y}}(\frac{10^{10} D_l}{\beta})\} \\ \cap \{dist(\bar{y},\{z=0\}) < \frac{\beta^2 dist(\bar{y},x)}{1000}\}}} |x - \bar{y}|^{-(2n+\rho_0)} dx \\
&\quad + CD_l \int_{K_l} |b_l| dy \int_{\substack{\{K_0 \setminus B_{\bar{y}}(\frac{10^{10} D_l}{\beta})\} \\ \cap \{dist(\bar{y},\{z=0\}) \geqslant \frac{\beta^2 dist(\bar{y},x)}{1000}\}}} |x - \bar{y}|^{-(2n+1)} dx \\
&\leqslant CD_l^{\rho_0} \int_{K_l} |b_l| dy \int_{\{K_0 \setminus B_{\bar{y}}(\frac{10^{10} D_l}{\beta})\}} |x - \bar{y}|^{-(2n+\rho_0)} dx \\
&\quad + CD_l \int_{K_l} |b_l| dy \int_{\{K_0 \setminus B_{\bar{y}}(\frac{10^{10} D_l}{\beta})\}} |x - \bar{y}|^{-(2n+1)} dx \\
&\leqslant C \int_{K_l} |b_l| dy \left\{ D_l^{\rho_0} \int_{|a| \geqslant D_l} |a|^{-(2n+\rho_0)} da \right\} \\
&\quad + C \int_{K_l} |b_l| dy \left\{ D_l \int_{|a| \geqslant D_l} |a|^{-(2n+1)} da \right\} \\
&\leqslant C \int_{K_l} |b_l| dy.
\end{aligned}
$$

Remark 3.2. — The reason we have so many β's in the proof is that in this section we use the distance and cubes with respect to the Euclidean metrics ω_E in the polar coordinates, but in [9] we use the cone distances with respect to ω_β. Their relation is

$$\beta\omega_\beta \leqslant \omega_E \leqslant \frac{\omega_\beta}{\beta}.$$

We can only consider cubes with respect to the reference metric ω_E.

Thus, by the argument in page 234 of [12], we deduce

$$(3.15) \qquad \mu_{Tb}(f) \leqslant \frac{C}{t}|f|_{L^1(B)}.$$

Combining (3.15) and (3.7), the proof of Lemma 3.1 on the opeartors $T \in \mathbb{T}$ is complete. Using exactly the proof above, the estimate on the adjoint operator $T^\star$ follows. Actually we have a slightly shorter proof for T (which does not require dividing the situation into 2 cases). However, since we want a single proof to work for both T and $T^\star$, we only present the longer proof above. $\qquad\square$

The following lemma is only needed in proving $T^\star f$ is densely defined in L^p, combined with the other results of this article. Though not fully needed in the proof of the results in the introduction, we think it has its own interest.

LEMMA 3.3. — $T^\star$ *is bounded linear map from $C^{\alpha,\beta}$ to itself, for all* $\alpha < \min\{\frac{1}{\beta} - 1, 1\}$.

Proof of Lemma 3.3. — Using Theorem 1.11 in [9] and Lemma 3.5, the proof is exactly as in Proposition 5.3 in [9]. $\qquad\square$

DEFINITION 3.4. — *Similar to the definition in (1.4), we define the mixed derivative operators as*

$$\mathfrak{M} = \{D_x D_{y_j},\ y_j \text{ is a tangential variable to } D\,;D_y D_{x_j},$$

$$x_j \text{ is a tangential variable to } D.\}$$

The feature of this set of operators is that the two derivatives are distributed to different variables.

LEMMA 3.5. — *For any second order spatial derivative operator $\mathfrak{D} \in \mathfrak{M}$. Suppose $\rho_0 = \min(\frac{1}{\beta} - 1,\ 1)$, $|x| = 1$ and $|v_1|$, $|v_2| < \frac{1}{8}$, we have*

$$|\mathfrak{D}\Gamma(x, v_1) - \mathfrak{D}\Gamma(x, v_2)| \leqslant C|v_1 - v_2|^{\rho_0}.$$

Proof of Lemma 3.5. — It's an easier version of the arguments in Section 8 of [9]. $\qquad\square$

4. $W^{2,p}$ and $C^{1,\alpha,\beta}$ estimates with L^p-right hand side

In this section, we prove Theorem 1.13 by proving Theorem 4.1, and we also prove Theorem 1.14. These 3 theorems are the main technical building blocks of the local regularity results in Theorem 1.6 and Corollary 1.8.

Proof of Theorem 1.13. — Suppose $L = a^{i\bar{j}}\frac{\partial^2}{\partial z_i \bar{z}_j}$, by multiplying a cutoff function η^2, we compute

$$\Delta_\beta(\eta^2 u) = (\Delta_\beta - L)(\eta^2 u) + L(\eta^2 u)$$
$$= (\Delta_\beta - L)(\eta^2 u) + 2Rea^{i\bar{j}}(\eta^2)_i(u)_{\bar{j}} + a^{i\bar{j}}(\eta^2)_{i\bar{j}}u + \eta^2 f.$$

Using Theorem 4.1, we deduce

$$[\eta^2 u]_{W^{2,p,\beta},B(2)} \leqslant C|\Delta_\beta(\eta^2 u)|_{L^p,B(2)}$$
$$\leqslant C|(\Delta_\beta - L)(\eta^2 u)|_{L^p,B(2)} + C|2Rea^{i\bar{j}}(\eta^2)_i(u)_{\bar{j}}|_{L^p,B(2)}$$
$$+ C|a^{i\bar{j}}(\eta^2)_{i\bar{j}}u|_{L^p,B(2)} + C|\eta^2 f|_{L^p,B(2)}$$
$$\leqslant C\delta_0[\eta^2 u]_{W^{2,p,\beta},B(2)} + C|u|_{W^{1,p,\beta},B(2)} + |\eta^2 f|_{L^p,B(2)}.$$

Choosing $\delta_0 \leqslant \frac{1}{2C}$, and η be the cutoff function such that

$$\eta \equiv 1 \text{ over } B(1); \ \eta = 0 \text{ over } B(2) \setminus B\left(\frac{3}{2}\right),$$

the desired conclusion in Theorem 1.13 follows. $\square$

THEOREM 4.1. — *Let B be a ball in C^n. Then T is a bounded linear map from $L^p(B)$ to itself, for $p \in (1,\infty)$. i.e. for all $f \in L^p(B)$, we have*

$$|Tf|_{L^p(B)} \leqslant |f|_{L^p(B)}.$$

Consequently, let $u \in W_c^{2,p,\beta}(B) \cap C^2(B \setminus D)$, $p \in [2,\infty)$, then

$$[u]_{W^{2,p,\beta}(B)} \leqslant C|\Delta_\beta u|_{L^p(B)},$$

where C only depend on n, β, p.

Proof of Theorem 4.1. — Lemma 2.6 says when u is compactly supported, u is equal to the Newtonian potential of its Laplacian. Then for any $\mathfrak{D} \in \mathbb{T}$, $\mathfrak{D}u = T(\Delta_\beta u)$ (for the T of $\mathfrak{D}$ in (1.5)). Hence the conic version of Calderon–Zygmund inequality in Lemma 3.1 is directly applicable.

By Marcinkiewicz-intepolation in Theorem 9.8 in [12], Lemma 2.7, and Lemma 3.1, we deduce both T and $T^\star$ are bounded linear map from L^p to L^p, $1 < p \leqslant 2$ i.e.

(4.1) $|Tf|_{p,B} \leqslant C|f|_{p,B}$, for all $1 < p \leqslant 2$,

and

(4.2) $$|T^\star f|_{p,B} \leqslant C|f|_{p,B}, \text{ for all } 1 < p \leqslant 2.$$

Then for $p > 2$, we conclude for all $f, g \in C_c^\infty(B)$ that

$$
\begin{aligned}
|Tf|_{p,B} &= \sup_{|g|_{p',B}=1} \int_B (Tf)g dx = \sup_{|g|_{p',B}=1} \int_B f(T^\star g) dy \\
&\leqslant \sup_{|g|_{p',B}=1} |f|_{p,B} |T^\star g|_{p',B} \quad (1 < p' < 2) \\
&\leqslant C \sup_{|g|_{p',B}=1} |f|_{p,B} |g|_{p',B} \\
&= C|f|_{p,B}.
\end{aligned}
$$

Notice that $u = N_{\beta,\mathbb{C}^n}(\Delta_\beta u)$, by Lemma 2.6. Then, combining (4.3), (4.1), the fact that $C_c^\infty(B)$ is dense in $L^p(B)$, and the Laplace equation

$$\frac{|z|^{2-2\beta}}{\beta^2} \frac{\partial^2 u}{\partial z \partial \bar{z}} = \Delta_\beta u - \sum_{i=1}^{n-1} \frac{\partial^2 u}{\partial w_i \partial \bar{w}_i},$$

we obtain the estimates in all directions are obtained. The proof of Theorem 4.1 is complete. Moreover, we've shown

$$|T^\star g|_{q,B} \leqslant C|g|_{q,B}, \text{ for all } 1 < q < \infty. \qquad \square$$

COROLLARY 4.2. — *Suppose* $u \in W^{1,2,\beta}[B(2)]$ *is a weak-solution to*

$$\Delta_\beta u = f, \ f \in L^p[B(1)], \ \infty > p \geqslant 2.$$

Then u *is actually in* $W^{2,p,\beta}[B(\frac{1}{2})]$ *and is therefore a strong-solution to the above equation in* $B(\frac{1}{2})$.

Proof of Corollary 4.2. — The proof is quite straight forward. Just notice $N_{\beta,B(1)}f \in W^{2,p,\beta}[B(1)]$, and $v = u - N_{\beta,B(1)}f \in W^{1,2,\beta}[B(1)]$ is a weak solution to the harmonic equation

$$\Delta_\beta v = 0, \text{ over } B(1).$$

Thus by Lemma C.3 (Lemma 2.1 in [22]), $v \in C^{2,\alpha,\beta}[B(\frac{2}{3})]$. Thus $u = v + N_{\beta,B(1)}f \in W^{2,p,\beta}[B(\frac{1}{2})]$. $\qquad \square$

Proof of Theorem 1.14. — This is an easier version of the work in [9]. By Corollary 4.2 and Donaldson's Schauder-estimate in [10], it suffices to estimate the Newtonian potential of f:

$$N_\beta f = \int_{B(1)} \Gamma(x, y) f(y) dy.$$

By Lemma 7.2 in [9], we estimate

$$
\begin{aligned}
|\nabla(\Gamma \star f)| &= \left| \int_{B(1)} (\nabla \Gamma(x,y)) f(y) dy \right| \\
(4.3) \qquad &\leqslant C \int_{B(1)} \frac{1}{|x-y|^{2n-1}} |f(y)| dy \\
&\leqslant C \left| \frac{1}{|x-y|^{2n-1}} \right|_{L^{p'},B(1)} |f|_{L^p,B(1)}.
\end{aligned}
$$

Since $p > 2n$, we have $p' < \frac{2n}{2n-1}$. Then

$$
\left| \frac{1}{|x-y|^{2n-1}} \right|_{L^{p'},B(1)} \leqslant C,
$$

and

$$
|\nabla(\Gamma \star f)|_{C^0[B(\frac{1}{2})]} \leqslant C|f|_{L^p,B(1)}.
$$

Next we estimate the Hölder norm of $\nabla(\Gamma \star f)$. Without loss of generality, we assume $|x_1| = \delta$ and $x_2 = 0$, which is the main issue. The Hölder estimate for all general x_1, x_2 follows from the proof of Proposition 5.3 in [9]. We compute

$$
\begin{aligned}
(4.4) \quad |\nabla(\Gamma \star f)(x_1) - \nabla(\Gamma \star f)(0)| &= \left| \int_{B(1)} [\nabla \Gamma(x_1,y) - \nabla \Gamma(0,y)] f(y) dy \right| \\
&\leqslant I_1 + I_2,
\end{aligned}
$$

where

$$
I_1 = \left| \int_{B(1) \cap \{|y| > 10\delta\}} [\nabla \Gamma(x_1,y) - \nabla \Gamma(0,y)] f(y) dy \right|
$$

and

$$
I_2 = \left| \int_{B(1) \cap \{|y| \leqslant 10\delta\}} [\nabla \Gamma(x_1,y) - \nabla \Gamma(0,y)] f(y) dy \right|.
$$

Then it's obvious that

$$
\begin{aligned}
I_2 &\leqslant \left| \int_{\{|y| \leqslant 10\delta\}} \nabla \Gamma(x_1,y) f(y) dy \right| + \left| \int_{\{|y| \leqslant 10\delta\}} \nabla \Gamma(0,y) f(y) dy \right| \\
(4.5) \quad &\leqslant C \int_{\{|y| \leqslant 10\delta\}} \frac{1}{|x_1-y|^{2n-1}} |f(y)| dy + C \int_{\{|y| \leqslant 10\delta\}} \frac{1}{|y|^{2n-1}} |f(y)| dy \\
&\leqslant C \delta^{\alpha_0} |f|_p,
\end{aligned}
$$

where $\alpha_0 = 1 - \frac{2n}{p}$. For the estimate of I_1, we should assume

$$
2n < p < \frac{2n}{1 - \min\{\frac{1}{\beta} - 1; 1\}} \quad \text{(if } \beta \leqslant \frac{1}{2} \text{ we just assume } 2n < p < \infty\text{)}.
$$

Thus $1 - \frac{2n}{p} < \min\{\frac{1}{\beta} - 1, 1\}$. This does not change the conclusion of Theorem 1.14, because what we assume is an additional upper bound on p. Next, we estimate I_1. By Lemma 8.2 in [9], we compute

$$
\begin{aligned}
I_1 &\leqslant \int_{\{|y| \geqslant 10\delta\}} \frac{1}{|y|^{2n-1}} \left| \nabla\Gamma\left(\frac{x_1}{|y|}, \frac{y}{|y|}\right) - \nabla\Gamma\left(0, \frac{y}{|y|}\right) \right| |f(y)| dy \\
&\leqslant C\delta^{\alpha_0 + \epsilon} \int_{\{|y| \geqslant 10\delta\}} \frac{1}{|y|^{2n-1+\alpha_0+\epsilon}} |f(y)| dy \\
&\leqslant C\delta^{\alpha_0} |f|_p
\end{aligned}
$$

(4.6)

where $\alpha_0 = 1 - \frac{2n}{p}$, and ϵ is chosen such that $\alpha_0 + \epsilon < \min\{\frac{1}{\beta} - 1, 1\}$. $\square$

5. KRF metrics with small ossilations

In $\mathbb{C} \times \mathbb{C}^{n-1}$, consider the standard conical Kähler–Ricci flat metric with cone angle $\beta \in (0, 1)$ along the divisor $\{0\} \times \mathbb{C}^{n-1}$.

$$
\omega_\beta = \frac{\beta^2}{|z|^{2-2\beta}} |dz|^2 + |dw|^2,
$$

where $z \in \mathbb{C}$ and $w \in \mathbb{C}^{n-1}$. We say a complex linear transformation L splits along D, if the first component $\mathbb{C} \times \{0\}$ in $\mathbb{C} \times \mathbb{C}^{n-1}$ is an invariant space of L, and the tangential component $\{0\} \times \mathbb{C}^{n-1}$ is also an invariant space of L. In this section, we prove the following regularity proposition, which is crucial to establish Theorem 1.6.

PROPOSITION 5.1. — Suppose L is a linear transformation which splits along D, and $(L^\star \omega_\beta)^n = \omega_\beta^n$. Then there exists a constant Q_0 depending on β, n, and the supremum of eigenvalues of $LL^\star$ with the following properties. Suppose ϕ is a pluri-subharmonic function which satisfies

- $\omega_\phi^n = e^f \omega_\beta^n$, $f \in C^{1,1,\beta}[B(100)]$;
- $(1-\delta)L^\star \omega_\beta \leqslant \omega_\phi \leqslant (1+\delta)L^\star \omega_\beta$, where $\delta \ll 1$ is sufficiently small with respect to the supremum of eigenvalues of $LL^\star$.

Then $\phi \in C^{2,\alpha,\beta}[B(\frac{1}{Q_0})]$, for all $\alpha < \min\{\frac{1}{\beta} - 1, 1\}$.

In particular, suppose

(5.1) $\omega_\phi^n = \omega_\beta^n$, $(1-\delta)\omega_\beta \leqslant \omega_\phi \leqslant (1+\delta)\omega_\beta$ over $B(100)$

for $\delta \ll 1$ small enough, then $\phi \in C^{2,\alpha,\beta}[B(\frac{1}{2})]$, for all $\alpha < \min\{\frac{1}{\beta} - 1, 1\}$.

Proof of Proposition 5.1. — We only prove the second part on the special case (5.1), the proof of the general case is the same.

Using (5.1) and Proposition 5.2, over $B(10)$, we can choose a potential, still denoted as ϕ, such that

$$(5.2) \qquad |\phi|_{0,B(10)} \leqslant C(n,\beta),$$

where $C(n,\beta)$ is a constant which only depends on the dimension n and angle β.

For any unit vector $v \in \{0\} \times \mathbb{C}^{n-1}$. tangential to the divisor, and for any small positive constant $\epsilon > 0$, define difference quotient as

$$(D_{\epsilon,v}\phi)(z,w) = \frac{\phi(z, w + \epsilon \cdot v) - \phi(z,w)}{\epsilon}.$$

Let $\epsilon \to 0$, we have

$$\lim_{\epsilon \to 0} D_{\epsilon,v}\phi = (\nabla \phi, v).$$

By (5.1), we end up with a trivial but important fact

$$(5.3) \qquad |\Delta_\beta \phi| \leqslant C \text{ in } B(10).$$

Using Theorem 1.13, (5.2), Corollary 4.2 and intepolations in the appendix of [9], we obtain

$$|\phi|_{W^{2,p,\beta}(B(5))} \leqslant C, \text{ for all } 1 < p < \infty.$$

This implies

$$\left| \frac{\partial \phi}{\partial v} \right|_{W^{1,p}(B(5))} \leqslant C, \text{ for all } 1 < p < \infty.$$

Then, by Lemma 7.23 in [12], we conclude the following estimate on the tangential difference quotients

$$(5.4) \qquad |D_{\epsilon,v}\phi|_{W^{1,p}B(4)} \leqslant C.$$

Therefore, take $D_{\epsilon,v}$ to both hand sides of the Ricci-flat equation

$$(5.5) \qquad \omega_\phi^n = \omega_\beta^n,$$

we obtain

$$\triangle_{\epsilon,v} D_{\epsilon,v}\phi = 0, \text{ over } B(9),$$

where

$$\triangle_{\epsilon,v}(x) = \int_0^1 [s\phi(x + \epsilon v) + (1-s)\phi(x)]^{i\bar{j}} ds \frac{\partial^2}{\partial z_i \partial \bar{z}_j}.$$

(5.1) implies directly the following.

$$|\triangle_\beta - \triangle_{\epsilon,v}| \leqslant \delta(\omega_\beta)^{-1}.$$

By Evans–Krylov Theorem away from D, we have $\phi \in C^2(\mathbb{C} \times \mathbb{C}^{n-1} \setminus D)$ (actually ϕ is smooth away from D, but C^2 of ϕ is all we need here).

Then the aprori estimate in Theorem 4.1 and 1.13 are directly applicable. Applying Theorem 1.13 and (5.4), we have

$$[D_{\epsilon,v}\phi]_{W^{2,p,\beta},B(2)} \leqslant C(|D_{\epsilon,v}\phi|_{W^{1,p},B(4)}) \leqslant C.$$

Since the above holds for all p, then applying Theorem (1.14), and again the lower order estimate (5.4), we obtain the crucial estimate

$$|D_{\epsilon,v}\phi|_{1,\alpha,\beta,B(1)} \leqslant C, \text{ for all } \alpha < \min\left\{\frac{1}{\beta} - 1, 1\right\}.$$

Now, let $\epsilon \to 0$, since ϕ is smooth away from D, we have

$$\frac{\partial \phi}{\partial v} \in C^{1,\alpha,\beta}\left[B\left(\frac{1}{2}\right)\right]$$

and

(5.6) $$\left|\frac{\partial \phi}{\partial v}\right|_{1,\alpha,\beta,B(\frac{1}{2})} \leqslant C, \text{ for all } \alpha < \min\left\{\frac{1}{\beta} - 1, 1\right\}.$$

This means the mixed derivatives $\frac{\partial^2 \phi}{\partial r \partial w_i}$, $\frac{1}{r}\frac{\partial^2 \phi}{\partial \theta \partial w_i}$, and the pure tangential derivatives $\frac{\partial^2 \phi}{\partial w_i \partial w_j}$, are all bounded in $C^{1,\alpha,\beta}[B(\frac{1}{2})]$-norm by C whose dependence is as in Proposition 5.1.

Using the equation (1.3) and the quasi-isometric condition (5.1), exactly as in the proof of Theorem 10.1 in [8], we deduce the crucial normal-$(1,1)$ derivative.

$$\left|\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{\beta^2 r^2}\frac{\partial^2}{\partial \theta^2}\right)\phi\right|_{\alpha,\beta,B(\frac{1}{2})} \leqslant C.$$

The above implies our final conclusion

$$\phi \in C^{2,\alpha,\beta}\left[B\left(\frac{1}{2}\right)\right] \text{ and } |\phi|_{2,\alpha,\beta,B(\frac{1}{2})} \leqslant C, \text{ for all } \alpha < \min\left\{\frac{1}{\beta} - 1, 1\right\}. \quad \square$$

The following Proposition is important.

PROPOSITION 5.2. — *There exists a constant C depending on β and n with the following properties. Given the equation*

(5.7) $$\sqrt{-1}\partial\bar{\partial}v = \eta \text{ over } A_{1000},$$

where η is a closed (1,1)-form such that $\eta = \sqrt{-1}\partial\bar{\partial}\phi_\eta$ for some $\phi_\eta \in C^{1,1,\beta}$. Then there exists a solution v in $W^{2,p,\beta}$ (for any p) such that

$$|v|_{W^{2,p,\beta},A_1} + |v|_{0,A_1} \leqslant C|\eta|_{L^{\beta,\infty},A_{1000}}.$$

Proof of Proposition 5.2. — The proof is exactly as in Proposition 4.1 in [8]. Just notice when η is merely in $L^{\infty,\beta}$, the orbifold trick in Lemma 4.3 of [8] works the same. Then pulling back upstairs we still obtain a solution by Lemma A.1. Hence, take average of this upstairs solution over the

discrete orbit of the monodromy group, and push this average down as in Lemma 4.4 in [8], we obtained the solution v we want.   □

6. Proof of the Main Theorems

In this section we prove Theorem 1.11 and 1.6. These proofs summarizes the work done in this article. Corollary 1.8 is directly implied by Theorem 1.6, by Definition 1.1.

Proof of Theorem 1.11. — It suffices to show that (1.3) already implies ω is $C^{\alpha,\beta}$, then Theorem 1.14 in [8] implies Theorem 1.11. The $C^{\alpha,\beta}$-regularity of the weak conical metric ω in Theorem 1.11 is the main work of this article. This can be divided into 2 steps.

Step 1. — 7 important results in [8] directly work in the our weak conical case. These 7 results are

- Lemma 6.1 on bounded weakly subharmonic functions in [8] (directly works when ω is merely a weak conical metric);
- Theorem 6.2 on weak-maximal principles in [8](directly works when ω is merely a weak conical metric);
- Theorem 7.3 and 7.4 on solvability of Dirichilet boundary value problems in [8] (directly works when ω is merely a weak conical metric);
- Theorem 8.1 on strong-maximal principles in [8] (directly works when ω is merely a weak conical metric);
- Lemma 13.1 on Trudinger's harnack inequality in [8] (directly works when ω is merely a weak conical metric);
- Proposition 4.1 in [8] on solvability of Poincare–Lelong equation with $C^{\alpha,\beta}$ right hand side. This is substituted by Propsition 5.2 on solvability of Poincare–Lelong equation with $L^{\infty,\beta}$-right hand side, with almost the same proof.

The above 7 results imply any weak conical metric ω satisfying the conditions in Theorem 1.11 is either linearly-isometric to ω_β or admits a tangent cone which is linearly-isometric to ω_β.

Step 2. — The last paragraph in Step 1 means the second assumption in Theorem 5.1 in [8] is fulfilled. Then Theorem 5.1 in [8] implies Theorem 1.11, provided we can show ω is in $C^{\alpha,\beta}$. This is precisely what Proposition 5.1 says. Actually, Proposition 5.1 is really the main technical result of this article.   □

Proof of Theorem 1.6. — This theorem is an interior regularity result, and away from D the regularity automatically follows from Proposition 16 of [7]. Thus, without loss of generality, we assume $\Omega = B_0(1)$ (the unit ball centered at the origin). This proof is a simple combination of Proposition 5.1, Theorem 1.11, and the Chen–Donaldson–Sun's trick in the proof of Proposition 26 in [6].

We consider the rescaling of the metrics and potential as

$$(6.1) \qquad \phi_\lambda = \lambda^2 \phi, \ \omega_\lambda = \lambda^2 \omega, \ \widehat{\omega}_\beta = \lambda^2 \omega_\beta,$$

and the rescaling of the coordinates as

$$(6.2) \qquad \widehat{z} = \lambda^{\frac{1}{\beta}} z, \ \widehat{y}_j = \lambda y_j, \ 1 \leqslant j \leqslant 2n - 2.$$

Then the $\widehat{\omega}_\beta$ is the model cone metric in the coordinates in (6.2). Then equation (1.1) is rescaled to the following geometric equation

$$(6.3) \qquad \omega_\lambda^n = \frac{\widehat{f}}{\beta^2} \widehat{\omega}_\beta^n$$

in the coordinates of (6.2), where $\widehat{f}$ is the pulled back function under the coordinate change. Since $\widehat{f} \in C^{1,1,\beta} B_0(\lambda)$, by the usual Evans–Krylov Theorem away from D, we deduce

$$|\omega_\lambda|_{C^\alpha[B(R) \backslash T_\epsilon(D)]} \leqslant C(R, \epsilon), \ \text{for all } R \leqslant \frac{\lambda}{2}.$$

Since $f \in C^\alpha[B(1)]$ before rescaling, then $\lim_{\lambda \to \infty} \widehat{f} = Const$ in the sense of $C^{\widehat{\alpha}}$, for all $0 < \widehat{\alpha} < \alpha$. Without loss of generality we can assume

$$\lim_{\lambda \to \infty} \frac{\widehat{f}}{\beta^2} = 1.$$

Then, ω_λ converges to ω_∞ uniformly over any fixed $B(R) \backslash T_\epsilon(D)$ such that

$$(6.4) \qquad \omega_\infty^n = \widehat{\omega}_\beta^n \ \text{over } \mathbb{C}^n \backslash D,$$

and

$$(6.5) \qquad \frac{\widehat{\omega}_\beta}{C} \leqslant \omega_\infty \leqslant C\widehat{\omega}_\beta.$$

To prove ω_∞ is a weak conical metric in the sense of Definition 1.3, it suffices to show ω_∞ admits a C^α-potential near any $p \in D$. By the proof of the Harnack inequality in item 2 of Lemma 6.1 in [8], and the quasi-isometric condition (6.5), it suffices to show ω_∞ admits a L^∞-potential

near any $p \in D$. This is done simply by applying Proposition 5.2 to ω_λ. Namely, using the quasi-isometric condition

$$(6.6) \qquad \frac{\widehat{\omega}_\beta}{C} \leqslant \omega_\lambda \leqslant C\widehat{\omega}_\beta,$$

Proposition 5.2 and Theorem 1.14 imply for any $p \in D$ in the rescaled coordinates (6.2), when λ is sufficiently large, there exists a potential $\phi_{p,\lambda}$ defined in $B_p(10^{10})$ such that

$$(6.7) \qquad |\phi_{p,\lambda}|_{C^\alpha[B(1)]} \leqslant C, \ \omega_\lambda = \sqrt{-1}\partial\bar{\partial}\phi_{p,\lambda}.$$

Thus, for any $\widehat{\alpha} < \alpha$, $\phi_{p,\lambda}$ converges in $C^{\widehat{\alpha}}[B(\frac{1}{2})]$-topology to $\phi_{p,\infty} \in C^{\widehat{\alpha}}[B(\frac{1}{2})]$ such that

$$(6.8) \qquad \omega_\infty = \sqrt{-1}\partial\bar{\partial}\phi_{p,\infty} \text{ over } B_p\left(\frac{1}{2}\right), \text{ in the sense of current.}$$

Then ω_∞ is a weak conical metric in the sense of Definition 1.3.

Therefore, by Theorem 1.11, we deduce

$$\omega_\infty = L^\star\widehat{\omega}_\beta,$$

where L is a linear transformation preserving D. By the uniform convergence of ω_∞ over any fixed $B(R) \setminus T_\epsilon(D)$, and the the proof of Proposition 2.5 in [5], we deduce

$$(6.9) \qquad \lim_{\lambda\to\infty} |\omega_\lambda - L^\star\widehat{\omega}_\beta|_{L^2(B_0(1))} = 0.$$

To modify the convergence in (6.9) to pointwise convergence, we use the assumption that $f \in C^{1,1,\beta}(B)$.

Since $(L^\star\widehat{\omega}_\beta)^n = \widehat{\omega}_\beta^n$, we translate equation (6.3) to be

$$(6.10) \qquad \omega_\lambda^n = \frac{\widehat{f}}{\beta^2}(L^\star\widehat{\omega}_\beta)^n.$$

By Yau's Bochner technique and $h \in C^{1,1,\beta}[B(1)]$, we deduce for any $\delta > 0$ that

$$(6.11) \qquad \Delta_{L^\star\widehat{\omega}_\beta}\left(tr_{L^\star\widehat{\omega}_\beta}\omega_\lambda - n + \delta\right) \geqslant -\frac{[h]_{C^{1,1,\beta}[B(1)]}}{\lambda^2} \to 0 \text{ in } B(1).$$

Then, (6.11), (6.9) and the Moser's iteration (as in the proof of Proposition 26 in [6]) imply

$$\lim_\lambda |\omega_\lambda - L^\star\widehat{\omega}_\beta|_{L^{\infty,\beta},(B_0(\frac{1}{2}))} = 0.$$

Let δ_0 be small enough with respect to the δ in Proposition 5.1 and the quasi-isometric constant of ω_ϕ with respect to ω_β in the original coordinates, there exists a λ_0 such that for all $\lambda \geqslant \lambda_0$, we have

$$|\omega_\lambda - L^\star \widehat{\omega}_\beta|_{L^{\infty,\beta}(B_0(\frac{1}{2}))} < \delta_0. \tag{6.12}$$

Since (6.12) implies the following crucial small ossilation estimate before rescaling,

$$|\omega_\phi - L^\star \omega_\beta|_{L^{\infty,\beta}(B_0(\frac{1}{2\lambda_0}))} < \delta_0, \tag{6.13}$$

then Proposition 5.1 implies $\omega_\phi \in C^{\alpha,\beta}(\frac{1}{2Q\lambda_0}))$, where Q is a constant which only depends on the quasi-isometric constant of ω_ϕ with respect to ω_β in the original coordinates. The proof of Theorem 1.6 is complete. $\square$

Appendix A. Poincare–Lelong equation in the smooth case

The following lemma is necessary for the results in [8] and also in this article (in the proof of of Proposition 5.2). We believe it's well known to experts, but for the sake of being self-contained we still would like to give a proof here. The proof is actually a simple combination of the proof of the Lemma in page 387 of [13], and Hormander's results.

LEMMA A.1. — There exists a constant C depending on n and p with the following properties. Suppose $\sigma \in L^2(B_{10})$ is a closed (1,1)-form such that

- $\sigma = \sqrt{-1}\partial\bar{\partial}\phi_\sigma$ for some $\phi_\sigma \in C^\alpha$.
- $\sigma \in L^\infty(B) \cap C^\alpha(B_{10} \setminus D)$.

Then there exists a solution φ in $W^{2,p}$ (for any $0 < p < \infty$) to

$$\sqrt{-1}\partial\bar{\partial}\varphi = \sigma \quad \text{over } B_1, \tag{A.1}$$

such that

$$|\varphi|_{W^{2,p},B_1} + |\varphi|_{0,B_1} \leqslant C|\sigma|_{L^\infty,B_{10}}.$$

Proof of Lemma A.1. — The two conditions of σ imply $\sigma \in L^\infty(B)$ as a distribution.

By Hormander's $\bar{\partial}$-solvability in Theorem 2.2.1 [15], there exists a (1,0)-form $\eta \in L^2(B(9))$ such that

$$\sigma = -\sqrt{-1}\bar{\partial}\eta. \tag{A.2}$$

Then, since

$$\sqrt{-1}\bar{\partial}(\partial\eta) = \partial\sigma = 0,$$

then $\partial\eta$ is a closed holomorphic $(2,0)$-current. Thus, by the regularity of closed holomorphic $(2,0)-$forms, $\partial\eta$ is actually a smooth holomorphic $(2,0)$-form. By the d-Poincare Lemma for smooth holomorphic $(p,0)$-forms, there exists a holomorphic $(1,0)$-form ξ such that

$$\partial\eta = \partial\xi.$$

Thus, we deduce

(A.3) $$\partial(\eta - \xi) = 0.$$

By the conjugate of $\bar{\partial}$-solvability in Theorem 4.2.5 in page 86 of Hormander's book [16], we end up with ∂-solvability and therefore a form γ such that

(A.4) $$\partial\gamma = \eta - \xi.$$

Then $\sqrt{-1}\partial\bar{\partial}\gamma = \sigma$. Let

$$\varphi = \frac{1}{2}(\gamma + \bar{\gamma}),$$

then φ is real and $\sqrt{-1}\partial\bar{\partial}\varphi = \sigma$. Since $\varphi \in W^{1,2}$, then φ is a weak solution to

$$\Delta\varphi = tr\sigma \in L^{\infty}(B_5).$$

Then, φ is a strong solution to the above equation in the sense of Chap. 9 in [12]. Then, the estimate in Lemma A.1 follows from Theorem 9.11 in [12] and the Moser's iteration. $\square$

Appendix B. An alternative approach to Corollary 1.8 by the conical Kähler–Ricci flow

In this section, we present a short proof of Corollary 1.8 when the weak conical Kähler–Einstein metric lives on a closed Kähler manifold. This proof, while lives on a closed manifold, does not require the $W^{2,p,\beta}$-estimate established in Sections 3 and 4.

Let $(M, [\omega_0])$ be a smooth Kähler manifold, D be a smooth divisor, S be the defining section of D, and $|\cdot|$ be a smooth metric of the line bundle associated to D, we consider the Monge–Ampère equation as

(B.1) $$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\phi)^n = \frac{e^h}{|S|^{2-2\beta}}\omega_0^n, \ h \in C^{1,1,\beta}(M).$$

LEMMA B.1. — *Suppose both ϕ_1 and ϕ_2 are $C^{1,1,\beta}(M)$ solutions to equation (B.1), such that both $\omega_0 + \sqrt{-1}\partial\bar{\partial}\phi_1$ and $\omega_0 + \sqrt{-1}\partial\bar{\partial}\phi_2$ are weak conical Kähler-metrics over M. Then*

$$\phi_1 - \phi_2 \equiv Constant \ over \ M \setminus D.$$

Proof of Lemma B.1. — Substracting (B.1) with $\phi = \phi_2$ from (B.1) with $\phi = \phi_1$, we end up with

(B.2) $$\Delta_s(\phi_1 - \phi_2) = 0 \ over \ M \setminus D,$$

where

$$\Delta_s = \int_0^1 g_{s\phi_1 + (1-s)\phi_2}^{i\bar{j}} ds \frac{\partial^2}{\partial z_i \partial \bar{z}_j}.$$

Thus, Lemma B.1 follows from equation (B.2) and the strong maximal-principle in Theorem 8.1 in [8]. $\qquad\square$

THEOREM B.2. — *Suppose $h \in C^{1,1,\beta}(M)$. Then any weak solution to (B.1) is strong i.e. in $C^{2,\alpha,\beta}$, for any $0 < \alpha < \min\{\frac{1}{\beta} - 1, 1\}$.*

In particular, any weak-conical Kähler–Einstein metric of $[M, (1-\beta)D]$ $(0 < \beta < 1)$ must be a $C^{\alpha,\beta}$ conical Kähler–Einstein metric, for any $0 < \alpha < \min(\frac{1}{\beta} - 1, 1)$.

Proof. — Define

$$K(\phi) = \int_M \log \frac{|S|^{2-2\beta}\omega_\phi^n}{e^h \omega_0^n} \frac{\omega_\phi^n}{n!}.$$

Then, along the corresponding conical Kähler–Ricci flow,

(B.3) $$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\phi)^n = \frac{e^{h + \frac{\partial\phi}{\partial t}}}{|S|^{2-2\beta}}\omega_0^n,$$

we deduce

$$K(\phi(t)) = \int_M \frac{\partial\phi}{\partial t} \frac{\omega_\phi^n}{n!}.$$

Then, along the flow (B.3), we obtain the following monotonicity by direct computation.

(B.4) $$\frac{dK(\phi(t))}{dt} = -\int_M |\nabla_\phi \frac{\partial\phi}{\partial t}|^2 \frac{\omega_\phi^n}{n!}.$$

Then, applying the proof of Theorem 1.7 in [8], with modifications in the C^2-estimate part (which we will specify later), together with the monotonicity of the K-energy (B.4) in the convergence argument in Section 11 of [8], we deduce that the flow (B.3) converges to a $\phi_\infty \in C^{2,\alpha,\beta}(M)$ which solves equation (B.1). The point is, the solution ϕ_∞ produced by the conical Kähler–Ricci flow in [8] is in $C^{2,\alpha,\beta}(M)$ (strong conical)!

Then, both $\underline{\phi}$ and ϕ_∞ solve equation (B.1). By the uniqueness of $C^{1,1,\beta}$ solutions in Lemma B.1, we obtain

$$\underline{\phi} = \phi_\infty + Const \in C^{2,\alpha,\beta}.$$

The proof of Theorem B.2 is complete.

The modification on the C^2-estimate is that, in the setting of Theorem B.2, it's super easy to apply the Guenancia–Păun type C^2-estimate as in [14], while we surely believe the Chern–Lu inequality as in [6] and [17], and the trick in [24] all work equally well. Namely, using the assumption that $\Delta_\beta h \geqslant -C$, formula (22) in [23] (for $\epsilon = 0$) says

$$(B.5) \quad (\Delta_\phi - \frac{\partial}{\partial t})\{\log tr_{\omega_D}\omega_\phi + B|S|^{2\beta} - A\phi\} \geqslant tr_{\omega_\phi}\omega_D + A\frac{\partial\phi}{\partial t} - C.$$

By using the barrier function in the proof of Theorem 6.2 in [8], the rest of the proof of the C^2-estimate goes exactly as the proof of Lemma 3.1 in [23], with $\epsilon = 0$. □

Appendix C. Integration by parts and regularity of harmonic function

LEMMA C.1 (Lemma 2.5 in [22]). — *Let B be a ball in the polar coordinate, and ω be a conic metric defined in $2B$. Suppose X is a C^1-vector field away from D and $X \in L^2(B) \cap W^{1,1,\beta}(B)$ (this says $\nabla X \in L^1(B)$), then*

$$(C.1) \qquad \int_{B\backslash D} div X\, \omega^n = \int_{\partial B\backslash D} < X, v > \omega^n.$$

Proof of Lemma C.1. — There are 2 approaches, we use the one involving a very nice cutoff function due to Berdtsson. Let r be the distance function to D (near D with respect to ω_β). Let $\psi_\epsilon = \psi(\epsilon \log(-\log r))$, ψ is the standard cutoff function such that $\psi(x) \equiv 1$ when $x \leqslant \frac{1}{2}$, and $\psi(x) \equiv 0$ when $x \geqslant \frac{4}{5}$. Then

$$(C.2) \qquad \psi_\epsilon \equiv 0 \text{ when } r \leqslant e^{-e^{\frac{4}{5\epsilon}}};\ \psi_\epsilon \equiv 1 \text{ when } r \geqslant e^{-e^{\frac{1}{2\epsilon}}}.$$

The following claim is true by elementary calculation.

CLAIM C.2.

$$(C.3) \qquad \lim_{\epsilon \to 0} |\nabla \psi_\epsilon|_{L^2(B)} = 0.$$

By applying Theorem 1 in page 271 of Evans [11] to $\psi_\epsilon X$, and the monotone convergence theorem, we deduce $X \in L^1(\partial B)$. By definition we have $\lim_{\epsilon \to 0} \psi_\epsilon = 1$ everywhere except on $suppD$. Since $\psi_\epsilon X$ is supported away from D, we compute for any $\epsilon > 0$ that

$$\int_{B \setminus D} div(\psi_\epsilon X) \omega^n = \int_{\partial B \setminus D} <\psi_\epsilon X, v> \omega^n$$

$$= \int_{B \setminus D} <\nabla \psi_\epsilon, X> \omega^n + \int_{B \setminus D} \psi_\epsilon div X \omega^n$$

Let $\epsilon \to 0$, the condition $X \in L^2(B)$, Cauchy–Schwartz inequality, and Claim C.2 imply $|\int_{B \setminus D} <\nabla \psi_\epsilon, X> \omega^n| \to 0$, then the proof is complete by the above identity.   □

LEMMA C.3 (Elliptic version of Lemma 2.1 in [22]). — Suppose $u \in W^{1,2,\beta}(B)$ is a weak solution to $\Delta_\beta u = 0$ in B. Then $u \in C^{2,\alpha,\beta}(\frac{B}{2})$ for any $0 < \alpha < \frac{1}{\beta} - 1$.

Proof of Lemma C.3. — It suffices to prove the regularity for polycylinders defined in (2.17), because $\frac{B}{2}$ can be covered by finitely many cylinders. Suppose the conditions of the lemma hold in A_R.

Step 1. — Notice that $A_{\frac{3R}{4}}$ is piecewisely smooth. Differentiating the harmonic equation with respect to w_i, we get

(C.4) $$\Delta_\beta \frac{\partial u}{\partial w_i} = 0.$$

By exacly the same argument in Theorem 8.8 in [12] involving difference quotients, we have

(C.5) $$\frac{\partial u}{\partial w_i} \in W^{1,2,\beta}(A_{\frac{3R}{4}}).$$

Lemma C.1, (C.4), and (C.5) imply that $\frac{\partial u}{\partial w_i}$ is a weak solution to (C.4) in $A_{\frac{3R}{4}}$. Trudinger's Harnack inequality (Lemma 6.1 in [8]) implies $\frac{\partial u}{\partial w_i} \in C^{\alpha,\beta}$. Let n be the outer normal vector of $\partial A_{\frac{3R}{4}}$ over $D_{\frac{3R}{4}} \times \partial B_{\frac{3R}{4}}$, the above implies $\frac{\partial u}{\partial n} \in C^{\alpha,\beta}$ therein.

Step 2. — Lemma C.1 implies (c.f. Lemma 2.6)

(C.6) $$u(x) = \int_{\partial A_{\frac{2R}{3}}} [<\nabla_y \Gamma(x,y), n> u(y) - <\nabla u(y), n> \Gamma(x,y)] dy,$$

for all $x \in int A_{\frac{2R}{3}}$. We note that over $\{r = \frac{3R}{4}\}$, $n = \frac{\partial}{\partial r}$, then $\frac{\partial \Gamma(x,y)}{\partial r}$ and $\frac{\partial u}{\partial r}$ are both smooth in y. Moreover, $<\nabla_y \Gamma(x,y), n>$ and $\Gamma(x,y)$ are both

in $C^{2,\alpha,\beta}(A_{\frac{R}{2}})$ (with respect to x) when $y \in \partial A_{\frac{2R}{3}}$. Thus the above integral is actually regular when $x \in A_{\frac{R}{2}}$, and $u \in C^{2,\alpha,\beta}(A_{\frac{R}{2}})$. □

□

BIBLIOGRAPHY

[1] R. J. BERMAN, "A thermodynamic formalism for Monge-Ampère equations, Moser-Trudinger inequalities and Kähler-Einstein metrics", *Adv. Math.* **248** (2013), p. 1254-1297.

[2] S. BRENDLE, "Ricci flat Kähler metrics with edge singularities", *Int. Math. Res. Not.* **2013** (2013), no. 24, p. 5727-5766.

[3] A. P. CALDERÓN & A. ZYGMUND, "On the existence of certain singular integrals", *Acta Math.* **88** (1952), p. 85-139.

[4] F. CAMPANA, H. GUENANCIA & M. PĂUN, "Metrics with cone singularities along normal crossing divisors and holomorphic tensor fields", *Ann. Sci. Éc. Norm. Supér.* **46** (2013), no. 6, p. 879-916.

[5] X. CHEN, S. DONALDSON & S. SUN, "Kähler-Einstein metric on Fano manifolds, I: approximation of metrics with cone singularities", *J. Amer. Math. Soc.* **28** (2015), p. 183-197.

[6] ———, "Kähler-Einstein metric on Fano manifolds, II: limits with cone angle less than 2π", *J. Amer. Math. Soc.* **28** (2015), p. 199-234.

[7] ———, "Kähler-Einstein metric on Fano manifolds, III: limits with cone angle approaches 2π and completion of the main proof", *J. Amer. Math. Soc.* **28** (2015), p. 235-278.

[8] X. CHEN & Y. WANG, "On the long time behaviour of the Conical Kähler-Ricci flows", https://arxiv.org/abs/1402.6689, to appear in *J. Reine. Angew. Math.*

[9] ———, "Bessel functions, heat kernel and the Conical Kähler-Ricci flow", *J. Funct. Anal.* **269** (2015), no. 2, p. 551-362.

[10] S. DONALDSON, "Kähler metrics with cone singularities along a divisor", in *Essays in mathematics and its applications*, Springer, 2012, p. 49-79.

[11] L. C. EVANS, *Partial differential equations*, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, 1998, xvii+662 pages.

[12] D. GILBARG & N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Grundlehren der mathematischen Wissenschaften., vol. 224, Springer-Verlag, 1977, x+401 pages.

[13] P. GRIFFITHS & J. HARRIS, *Principles of Algebraic Geometry*, 2nd ed., Wiley Classics Library, John Wiley & Sons Ltd., 1994, xii+813 pages.

[14] H. GUENANCI & M. PĂUN, "Conic singularities metrics with perscribed Ricci curvature: the case of general cone angles along normal crossing divisors", *J. Differ. Geom.* **103** (2016), no. 1, p. 15-57.

[15] L. HÖRMANDER, "L^2-estimates and existence theorems for the $\bar{\partial}$-operators", *Acta Math.* **113** (1965), p. 89-152.

[16] ———, *An introduction to complex analysis in several variables*, 3rd revised ed., North-Holland Mathematical Library, vol. 7, North-Holland, 1990, xii+254 pages.

[17] T. JEFFRES, R. MAZZEO & Y. A. RUBINSTEIN, "Kähler-Einstein metrics with edge singularities", https://arxiv.org/abs/1105.5216, to appear in *Ann. Math.*

[18] C. LI & S. SUN, "Conical Kähler-Einstein metrics revisited", *Commun. Math. Phys.* **331** (2014), no. 3, p. 927-973.

[19] J. LIU & X. ZHANG, "The conical Kähler-Ricci flow on Fano manifolds", `https://arxiv.org/abs/1402.1832`.

[20] J. SONG & X. WANG, "The greatest Ricci lower bound, conical Einstein metrics and the Chern number inequality", *Geom. Topol.* **20** (2016), no. 1, p. 49-102.

[21] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970, xiv+287 pages.

[22] Y. WANG, "Notes on the L^2-estimates and regularity of parabolic equations over conical manifolds", unpublished work.

[23] ———, "Smooth approximations of the Conical Kähler-Ricci flows", *Math. Ann.* **365** (2016), no. 1-2, p. 835-856.

[24] C. YAO, "Existence of Weak Conical Kähler-Einstein Metrics Along Smooth Hypersurfaces", *Math. Ann.* **362** (2015), no. 3-4, p. 1287-1304.

Xiuxiong CHEN
Department of Mathematics
Stony Brook University
Stony Brook, NY 11794 (USA)
xiu@math.sunysb.edu

Yuanqi WANG
Department of Mathematics
University of California at Santa Barbara
Santa Barbara, CA 93106 (USA)
wangyuanqi@math.ucsb.edu
ywang@scgp.stonybrook.edu