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Existence of common zeros for commuting vector fields on three manifolds

EXISTENCE OF COMMON ZEROS FOR COMMUTING VECTOR FIELDS ON THREE MANIFOLDS

by Christian BONATTI & Bruno SANTIAGO (*)

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ABSTRACT. — In 1964, E. Lima proved that commuting vector fields on surfaces with non-zero Euler characteristic have common zeros. Such statement is empty in dimension 3, since all the Euler characteristics vanish. Nevertheless, C. Bonatti proposed in 1992 a local version, replacing the Euler characteristic by the Poincaré–Hopf index of a vector field X in a region U, denoted by $\mathrm{Ind}(X, U)$; he asked:
  Given commuting vector fields X, Y and a region U where

$$\mathrm{Ind}(X, U) \neq 0$$

  does U contain a common zero of X and Y?
  A positive answer was given in the case where X and Y are real analytic, in the same article where the above question was posed.
  In this paper, we prove the existence of common zeros for commuting C^1 vector fields X, Y on a 3-manifold, in any region U such that $\mathrm{Ind}(X, U) \neq 0$, assuming that the set of collinearity of X and Y is contained in a smooth surface. This is a strong indication that the results for analytic vector fields should hold in the C^1 setting.

RÉSUMÉ. — En 1964, E. Lima a montré que des champs de vecteurs qui commutent sur une surface ont un zéro commun. Cette énoncé est trivial en dimension 3 puisque les caractéristiques d'Euler sont nulles dans ce cas. Cependant, C. Bonatti a proposé 1992 une version locale, en remplaçant la caractéristique d'Euler par l'indice de Poincaré–Hopf d'un champ de vecteurs X dans une région U, qu'on denote par $\mathrm{Ind}(X, U)$. Il a proposé la question suivante :
  Étant donnés deux champs de vecteurs X et Y qui commutent et une région compacte U sur lequel

$$\mathrm{Ind}(X, U) \neq 0,$$

  est-ce que U contient un zéro commun de X et Y ?

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Une réponse positive a été donnée dans le cas où X et Y sont réels analytiques, dans le même papier où la question au-dessus a été posée.

Dans cet article on montre existence de zéros communs pour les champs de vecteurs de classe C^1 qui commutent en dimension 3, pour toute région U telle que l'indice $\mathrm{Ind}(X, U)$ est non nul et en supposent en plus que le lieu de colinéarité entre X et Y est contenu dans une surface lisse. C'est une forte indication que le résultat pour les champs de vecteurs analytiques doit être vrai en régularité C^1.

1. Introduction

One of the fundamental problems in dynamical systems is whether a given system possesses fixed points or not. A simple scenario to pose this question is for the $\mathbb{Z}$-action generated by a diffeomorphism or a homeomorphism of a manifold, or for the continuous time dynamical system generated by the flow of a vector field. In both cases, the theories of Poincaré–Hopf and Lefschetz indices relate the topology of the ambient manifold with the existence of fixed points.

Nonetheless, if one consider two commuting diffeomorphisms or two commuting vector fields i.e. vector fields X and Y whose flows satisfy:[1]

$$X_t \circ Y_s = Y_s \circ X_t, \ \ \forall (s, t) \in \mathbb{R}^2,$$

the existence of a fixed point for the action they generate is a wide open question in dimensions $\geqslant 3$.

The first result on this question is given by the works on surfaces of Lima [12], [11]. He proves that any family of commuting vector fields on a surface with non-zero Euler characteristic have a common zero. In the late eighties, [1] proved that commuting diffeomorphisms of the sphere $\mathbb{S}^2$ which are C^1-close to the identity have a common fixed point. Later [2] extended this result to any surface with non-zero Euler characteristic (see other generalizations in [5][6]). Then, Handel [9] provided a topological invariant in $\mathbb{Z}/2\mathbb{Z}$ for a pair of commuting diffeomorphisms of the sphere $\mathbb{S}^2$ whose vanishing guarantees a common fixed point. This was further generalized by Franks, Handel and Parwani [8] for any number of commuting diffeomorphisms on the sphere (see [10] and [7] for generalizations on other surfaces).

It is worth to note, however, that two commuting *continuous interval maps* may fail to have a common fixed point: an example is constructed in [4] of two continuous commuting, non-injective, maps of the closed interval which do not have a common fixed point.

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[1] This definition can be adapted for non-complete vector fields, see Section 3

In higher dimensions much less is known:

- One knows some relation between the topology of the manifold and the dimension of the orbits of $\mathbb{R}^p$-actions (see [13, 14]). The techniques introduced in these works made possible a simple proof of Lima's result, for smooth vector fields [15];
- [3] proved that two commuting real analytic vector fields on an analytic 4-manifold with non-zero Euler characteristic have a common zero. The same statement does not make sense in dimension three since every 3-manifold has zero Euler characteristic. Nevertheless, a local result remains true in dimension three.

Before stating the result of [3] on 3-manifolds, we briefly recall the notion of the Poincaré–Hopf index $\mathrm{Ind}(X, U)$ of a vector field X on a compact region U whose boundary ∂U is disjoint from the set $\mathrm{Zero}(X)$. If U is a small compact neighborhood of an isolated zero p of the vector field X, then $\mathrm{Ind}(X, U)$ is just the classical Poincaré–Hopf index $\mathrm{Ind}(X, p)$ of X at p. For a general compact region U with $\partial U \cap \mathrm{Zero}(X) = \emptyset$, one considers a small perturbation Y of X with only finitely many isolated zeros in U. Then, we define the index $\mathrm{Ind}(X, U)$ as the sum of the Poincaré–Hopf indices $\mathrm{Ind}(Y, p)$, $p \in \mathrm{Zero}(Y) \cap U$. We refer the reader to Section 3 for details (in particular for the fact that $\mathrm{Ind}(X, U)$ does not depend on the perturbation Y of X).

Then, the main theorem of [3] says that every pair X, Y of analytic commuting vector fields have a common zero in any compact region U such that $\mathrm{Ind}(X, U) \neq 0$. By contraposition, if X and Y are analytic and commute but do not have common zeros then $\mathrm{Ind}(X, U) = 0$. By reducing the compact region U so that it separates $\mathrm{Zero}(X) \cap U$ from $\mathrm{Zero}(Y)$ one obtains the following statement:

THEOREM 1.1 (Bonatti [3]). — *Let M be a real analytic 3-manifold and X and Y be two analytic commuting vector fields over M. Let U be a compact subset of M such that*

$$\mathrm{Zero}(Y) \cap U = \mathrm{Zero}(X) \cap \partial U = \emptyset.$$

Then,

$$\mathrm{Ind}(X, U) = 0.$$

This statement is also true when M has dimension 2 and the vector fields are just C^1 (see Proposition 11 in [2][(2)]). This motivates the following

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[(2)] The result stated there is for C^∞ vector fields, but the proof indicated there can be adapted for C^1 vector fields using cross-sections, in a similar way we do here in Section 4

CONJECTURE 1.2. — *Let X and Y be two C^1 commuting vector fields on a 3-manifold M. Let U be a compact subset of M such that*

$$\mathrm{Zero}(Y) \cap U = \mathrm{Zero}(X) \cap \partial U = \emptyset.$$

Then,

$$\mathrm{Ind}(X, U) = 0.$$

This conjecture was stated as a problem in [3].

The goal of the present paper is to solve, in the C^1-setting, what was the main difficulty in the analytic case in [3]. We explain now what was this difficulty in [3]. A crucial role is played by the set of points of U in which X and Y are collinear:

$$\mathrm{Col}(X, Y, U) := \{p \in U; \dim\left(\langle X(p), Y(p)\rangle\right) \leqslant 1\}.$$

In [3] the assumption that the commuting vector fields are analytic is used to say that $\mathrm{Col}(X, Y, U)$ is either equals to U or is an analytic set of dimension at most 2. The case where $\mathrm{Col}(X, Y, U) = U$ admits a direct proof. In the other case, a simple argument allows to assume that $\mathrm{Col}(X, Y, U)$ is a surface. The main difficulty in [3] consists in proving that, if $\mathrm{Col}(X, Y, U)$ is a smooth surface and X and Y are analytic, then the index of X vanishes on U.

Our result is the following

THEOREM A. — *Let M be a 3-manifold and X and Y be two C^1 commuting vector fields over M. Let U be a compact subset of M such that $\mathrm{Zero}(Y) \cap U = \mathrm{Zero}(X) \cap \partial U = \emptyset$. Assume that $\mathrm{Col}(X, Y, U)$ is contained in a C^1-surface which is a compact and boundaryless submanifold of M. Then,*

$$\mathrm{Ind}(X, U) = 0.$$

The hypothesis "$\mathrm{Col}(X, Y, U)$ *is contained in a C^1-surface*" consists in considering the simplest geometric configuration of $\mathrm{Col}(X, Y, U)$ for which the conjecture is not trivial: if (X, Y) is a counter example to the conjecture, then $\mathrm{Col}(X, Y)$ cannot be "smaller" than a surface. More precisely, if $\mathrm{Ind}(X, U) \neq 0$, and if Y commutes with X then the sets $\mathrm{Zero}(X - tY)$ for small t are not empty compact subsets of $\mathrm{Col}(X, Y, U)$, invariant by the flow of Y and therefore consist in orbits of Y. If X and Y are assumed without common zeros, every set $\mathrm{Zero}(X - tY)$ consists on regular orbits of Y, thus is a 1-dimensional lamination. Furthermore, these laminations are pairwise disjoint and vary semi-continously with t. In particular, $\mathrm{Col}(X, Y, U)$ cannot be contained in a 1-dimensional submanifold of M.

Another (too) simple configuration would be the case where X and Y are everywhere collinear. This case has been treated in [3] and the same proof holds at least in the C^2 setting.

We believe that the techniques that we introduce here will be usefull to prove the conjecture, at least for C^2 vector fields.

The proof of Theorem A is by contradiction. The intuitive idea which guides the argument is that, at one hand, the vector field X needs to turn in all directions in a non-trivial way in order to have a non-zero index. On the other hand, X commutes with Y and therefore is invariant under the tangent flow of a non-zero vector field. The combination of this two properties will lead to a contradiction.

This paper is organized as follows.

- In Section 2 we give an informal presentation of the proof, describing the main geometrical ideas.
- In Section 3 we give detailed definitions and state some classical facts that we shall use.
- In Section 4 we reduce the proof of Theorem A to the proof of a slightly more technical version of it (see Lemma 4.3), for which U is a solid torus and $\mathrm{Col}(X,Y,U)$ is an annulus foliated by periodic orbits of Y, and cutting U in two connected components U^+ and U^-.
- In Section 5 we consider the projection N, of the vector field X parallel to Y on the normal bundle of Y. We show that $\mathrm{Ind}(X,U)$ is related with the angular variations ℓ^+ and ℓ^- of N along generators of the fundamental group of each connected components U^+ and U^- of $U \setminus \mathrm{Col}(X,Y,U)$. More precisely we will show in Proposition 5.9 that

$$|\mathrm{Ind}(X,U)| = |\ell^+ - \ell^-|.$$

  Assuming that at least one of ℓ^+ and ℓ^- does not vanish, and the fact that $\mathrm{Col}(X,Y,U)$ is a C^1-surface, we deduce in Proposition 5.19 that the the first return map $\mathcal{P}$ of Y on a transversal Σ_0 is C^1-close to identity in a small neighborhood of $\mathrm{Col}(X,Y,U) \cap \Sigma_0$.

- In Section 6, still assuming that at least one of ℓ^+ and ℓ^- does not vanish, we give a description of the dynamics of the first return map $\mathcal{P}$. If for instance $\ell^+ \neq 0$ then every point in $\Sigma_0 \cap U^+$ belongs to the stable set of a fixed point of $\mathcal{P}$ (Lemma 6.4). We will then use the invariance of these stable sets under the orbits of the normal vector field N for getting a contradiction.

We end this introduction by a general comment. The accumulation of results proving the existence of common fixed points for commuting dynamical systems seems to indicate the possibility of a general phenomenon. However, our approach in Poincaré–Bendixson spirit has a difficulty which increases drastically with the ambient dimension. We hope that this results will motivate other attempts to study this phenomenon.

2. Idea of the proof

The proof is by contradiction. We assume that there exists a *counter example to the Theorem*, that is a pair of commuting C^1-vector fields X, Y on a 3-manifold M, and a compact set U so that

- the colinearity locus of X and Y in U is contained in a closed C^1 surface S.
- X is non-vanishing on the boundary ∂U and the index $\mathrm{Ind}(X, U)$ is non-zero,
- X and Y have no comon zero in U.

Simplifying the counter examples. A first step of the proof (see the whole Section 4 and more specifically Lemmas 4.1 and 4.3) consists in showing that, up to shrink the compact set U, and up to replace the vector fields X and Y by (constant) linear combinations of X and Y, one may assume futher, without loss of generality, that

- the manifold M is orientable;
- U is a solid torus $\mathbb{D}^2 \times (\mathbb{R}/\mathbb{Z})$;
- the surface S is an annulus invariant by the flow of X and Y whose intersection with the boundary ∂U is precisely its own boundary ∂S;
- Y is non-vanishing on U and transverse to every factor $\Sigma_t = \mathbb{D}^2 \times \{t\}$, $t \in \mathbb{R}/\mathbb{Z}$;
- for every $x \in S$ its Y orbit is a periodic orbit contained in S, of period $\tau(x)$.
- the map $x \in S \mapsto \tau(x) > 0$ is of class C^1, constant on the Y-orbits, and its derivative is non vanishing on S;
- as X is colinear on S to the non-vanishing vector field Y one may write $X(x) = \mu(x)Y(x)$ for $x \in S$; then the map μ is of class C^1, constant on the Y-orbits, and its derivative is non vanishing on S.

We endow the solid torus U with a basis $\mathcal{B}(x) = (e_1(x), e_2(x), e_3(x))$ of $T_x M$ dependind continuously with x and so that $e_3(x) = Y(x)$, the plane

$< e_1(x), e_2(x) >$ is tangent to the disc Σ_t through x, and for $x \in S$ the vector $e_1(x)$ is tangent to S.
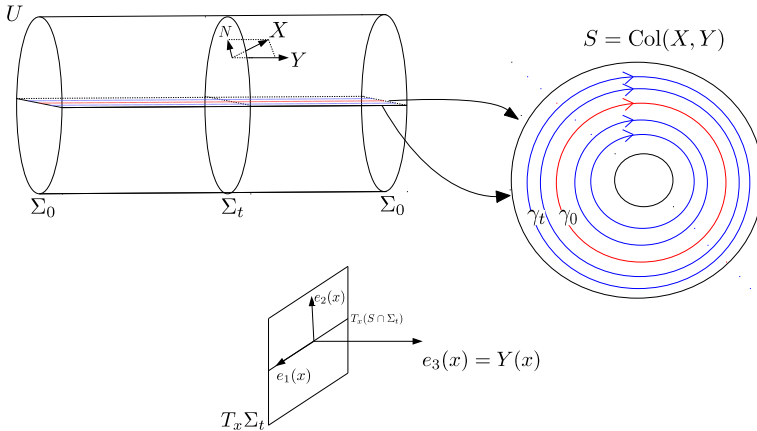

Figure 2.1. *Geometric configuration of a prepared counter example and the basis* $\mathcal{B}(x) = (e_1(x), e_2(x), e_3(x))$ *at a point* $x \in S \cap \Sigma_t$.

The pair of vector fields (X, Y) with all these extra properties and endowed with the basis $\mathcal{B}$ is called a *prepared counter example*. More formally Lemmas 4.1 and 4.3 announce that the existence of a counter example to Theorem A implies the existence of a prepared counter example. The rest of the proof consists in getting a contradiction from the existence of a prepared counter example.

Calculating the index $\mathrm{Ind}(X, U)$**.** As Y is transverse to the discs Σ_t one can write the vector field X as

$$X(x) = N(x) + \mu(x)Y(x)$$

where $N(x)$ is a vector tangent to the discs Σ_t, so that one may write $N(x) = \alpha(x)e_1(x) + \beta(x)e_2(x)$. This choice of coordinates allows us to consider the vector $N(x)$ as a vector on the plane $\mathbb{R}^2$. Notice that $N(x)$ vanishes if and only if $x \in S$.

Thus, given any closed curve $\gamma \subset U \setminus S$ which is freely homotopic to $\{(0, 0)\} \times (\mathbb{R}/\mathbb{Z})$ in U, one defines the *linking number of* N *along* γ as being the number of turns given by $N(x)$ (considered as a non-vanishing vector on $\mathbb{R}^2$) as x runs along γ. One easily checks that this linking number only depends on the connected component of $U \setminus S$ which contains γ. Since $U \setminus S$ consist in two connected components U^+ and U^-, there are only two linking

numbers which we denote by ℓ^+ and ℓ^- and we prove (Proposition 5.9)

$$| \operatorname{Ind}(X, U)| = |\ell^+ - \ell^-|.$$

The normally hyperbolic case. By assumption the annulus S is foliated by periodic orbits of Y so that the derivative associated to these periodic orbits is 1 in the direction of S.

Note that, if one of the periodic orbits of Y in S is partially hyperbolic, that is, has an eigenvalue different from 1, then it admits a neighborhood foliated by the local stable manifolds of the nearby periodic orbits. As the vector field X commutes with Y and preserves each periodic orbit in S, it preserves each leaf of this foliation. Thus the normal vector N is tangent to this foliation and one deduces that the linking numbers ℓ^+ and ℓ^- both vanish. Thus $\operatorname{Ind}(X, U)$ also vanishes leading to a contradiction (see Lemma 4.7 which formalizes this argument).

Figure 2.2. As X is tangent to the stable manifolds, it cannot turn in all directions. Its normal component N is everywhere collinear with e_2, so that both linking numbers vanish.

The first return map $\mathcal{P}$, the derivative of the return time, and the angular variation of N. The vector field N does not commute with Y but it is invariant under the holonomies of Y of the cross sections Σ_t. One deduces that N almost cannot turn along the orbits of Y. This motivate us to calculate the linking number $\ell^{\pm}$ along particular closed curves obtained in the following way: we follow the Y-orbit of a point $x \in \Sigma_0 \setminus S$ until its first return $\mathcal{P}(x)$ then we close the curve by joining a small geodesic segment in Σ_0. This allows us to prove that the angular variation of N is larger than 2π along the segment in Σ_0 joining x to $\mathcal{P}^2(x)$ (second return of x in Σ_0) (Corollary 5.14).
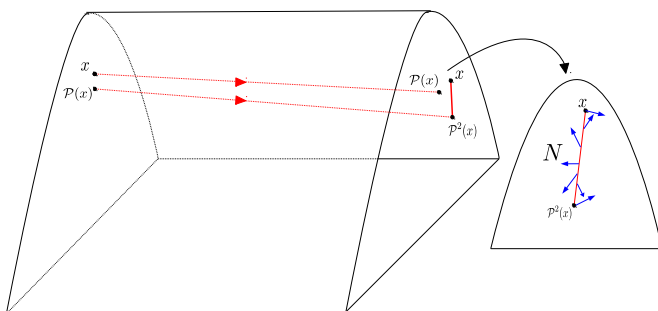
Figure 2.3. N has large angular variation along the segments $[x, \mathcal{P}^2(x)]$.

Now the contradiction we are looking for will be found in a subtle analysis of the dynamics of the first return map $\mathcal{P}$ together with the dynamics of the vector N, which commute together. In particular Corollary 5.6 links the direction of the vector field $N(x)$ (through the derivative in the direction of N of the return time of Y on Σ_0) with the variation $\mu(\mathcal{P}(x)) - \mu(x)$. The big angular variation of N along the small segments $[x, \mathcal{P}^2(x)]$ lead us to prove that

- the derivative of $\mathcal{P}$ is the identity at any point of S (Proposition 5.19)
- $\mathcal{P}$ almost preseves the levels of the map μ (see Lemmas 6.1 and 6.2): one deduces that (up to exchange $\mathcal{P}$ by $\mathcal{P}^{-1}$) every orbit of $\mathcal{P}$ converges to a point of S.

Invariant stable sets and the contradiction. The second property above allows us to obtain stable sets for the points in $S \cap \Sigma_0$ (with respect to the first return map $\mathcal{P}$) and to show that these stable sets are (as in the normally hyperbolic case) invariant under the vecter field N. Unluckely the concluison is not so straightforward as in the normally hyperbolic case. We first prove that there is an N-orbit which is invariant uner $\mathcal{P}$ (Lemma 6.8) and we prove that the angular variation of N, between x and $\mathcal{P}^2(x)$, along such a $\mathcal{P}$ invariant N-orbit is arbitrarilly small when x is close to $S \cap \Sigma_0$: that is the announced contradiction.

3. Notations and definitions

In this paper M denotes a 3-dimensional manifold. Whenever X is a vector field over M, we denote $\text{Zero}(X) = \{x \in M; X(x) = 0\}$ and

$\text{Zero}(X, U) = \text{Zero}(X) \cap U$, for any subset $U \subset M$. We shall denote its flow by X_t. A compact set $\Lambda \subset M$ is *invariant under the flow of X* if $X_t(\Lambda) = \Lambda$ for every $t \in \mathbb{R}$.

If X and Y are vector fields on M we denote by $\text{Col}(X, Y)$ the set of points p for which $X(p)$ and $Y(p)$ are collinear:

$$\text{Col}(X, Y) = \{p \in M, \dim(\langle X(p), Y(p) \rangle) \leqslant 1\}.$$

If $U \subset M$ is a compact region we denote $\text{Col}(X, Y, U) = \text{Col}(X, Y) \cap U$.

3.1. The Poincaré–Hopf index.

In this section we recall the classical definition and properties of the Poincaré–Hopf index.

Let X be a continuous vector field of a manifold M of dimension d and $x \in M$ be an isolated zero of X. The *Poincaré–Hopf index* $\text{Ind}(X, x)$ is defined as follows: consider local coordinates $\varphi \colon U \to \mathbb{R}^d$ defined in a neighbohood U of x. Up to shrink U one may assume that x is the unique zero of X in U. Thus for $y \in U \setminus \{x\}$, $X(y)$ expressed in that coordinates is a non vanishing vector of $\mathbb{R}^d$, and $\frac{1}{\|X(y)\|} X(y)$ is a unit vector hence belongs to the sphere $\mathbb{S}^{d-1}$. Consider a small ball B centred at x. The map $y \mapsto \frac{1}{\|X(y)\|} X(y)$ induces a continuous map from the boundary $S = \partial B$ to $\mathbb{S}^{d-1}$. The Poincaré–Hopf index $\text{Ind}(X, x)$ is the topological degree of this map.

Remark 3.1. — The Poincaré–Hopf index $\text{Ind}(X, x)$ of an isolated zero x does not depend on the choice of a local orientation of the manifold at x.

For instance, the Poincaré Hopf index of a hyperbolic zero x is

$$\text{Ind}(X, x) = (-1)^{dim E^s(x)},$$

where $E^s(x) \subset T_x M$ is the stable space of x.

More conceptually, a change of the local orientation of M at x:
- composes the map $x \mapsto \frac{X(x)}{\|X(x)\|}$ with a symmetry of the sphere S^{d-1}
- changes the orientation of the sphere ∂B, where B is a small ball around x.

Therefore the topological degree of the induced map from ∂B to S^{d-1} is kept unchanged.

Assume now that $U \subset M$ is a compact subset and that X does not vanishes on the boundary ∂U. The *Poincaré–Hopf index* $\text{Ind}(X, U)$ is defined as follows: consider a small perturbation Y of X so that the set of zeros of

Y in U is finite. A classical result asserts that the sum of the indices of the zeros of Y in U does not depend on the perturbation Y of X; this sum is the Poincaré–Hopf index $\mathrm{Ind}(X, U)$. Here, *small perturbation* means that Y is homotopic to X through vector fields which do not vanish on ∂U. More precisely

PROPOSITION 3.2. — *If $\{X^t\}_{t \in [0,1]}$ is a continuous family of vector fields so that $\mathrm{Zero}(X^t) \cap \partial U = \emptyset$, then $\mathrm{Ind}(X^t, U)$ does not depend on $t \in [0,1]$.*

We say that a compact subset $K \subset \mathrm{Zero}(X)$ is *isolated* if there is a compact neighborhood U of K so that $K = \mathrm{Zero}(X) \cap U$; the neighborhood U is called an *isolating neighborhood of K*. The index $\mathrm{Ind}(X, U)$ does not depend of the isolating neighborhood V of K. Thus $\mathrm{Ind}(X, U)$ is called *the index of K* and denoted $\mathrm{Ind}(X, K)$.

3.2. Calculating the Poincaré Hopf index

Next Lemma 3.3 provides a practical method for calulating the index of a vector field X in some region where it may have infinitely many zeros, without performing perturbations of X.

Assume now that ∂U is a codimension one submanifold and that U is endowed with d continous vector fields $X^1 \ldots X^d$ so that, at every point $z \in U$, $(X^1(z), \ldots, X^d(z))$ is a basis of the tangent space $T_z M$. Once again, one can write the vector field X in this basis so that the vector $X(y)$, for $y \in U$, can be considered as a vector of $\mathbb{R}^d$. One defines in such a way a map $g \colon \partial U \to \mathbb{S}^{d-1}$ by $y \mapsto g(y) = \frac{1}{\|X(y)\|} X(y)$.

As ∂U has dimension $d-1$, and is oriented as the boundary of U, this map has a topological degree. A classical result from homology theory implies the following

LEMMA 3.3. — *With the notations above the topological degre of g is $\mathrm{Ind}(X, U)$.*

In particular it does not depend on the choice of the vector fields $X^1 \ldots X^d$.

Remark 3.4. — By Lemma 3.3, if there is $j \in \{1, \ldots, d\}$ so that the vector $X(x)$ is not colinear to $X^j(x)$, for every $x \in \partial U$, then $\mathrm{Ind}(X, U) = 0$.

3.3. Topological degree of a map from $\mathbb{T}^2$ to $\mathbb{S}^2$

We consider the sphere $\mathbb{S}^2$ (unit sphere of $\mathbb{R}^3$) endowed with the north and south poles denoted $N = (0, 0, 1)$ and $S = (0, 0, -1)$ respectively.

We denote by $\mathbb{S}^1 \subset \mathbb{S}^2$ the equator, oriented as the unit circle of $\mathbb{R}^2 \times \{0\}$. For $p = (x, y, z) \in \mathbb{S}^2 \setminus \{N, S\}$ we call *projection of p on $\mathbb{S}^1$ along the meridians* the point $\frac{1}{\sqrt{x^2+y^2}}(x, y, 0)$, which is intersection of $\mathbb{S}^1$ with the unique half meridian containing p.

LEMMA 3.5. — *Let $\Phi \colon \mathbb{S}^2 \to \mathbb{S}^2$ be a continuous map so that $\Phi^{-1}(N) = \{N\}$ and $\Phi^{-1}(S) = \{S\}$.*

Let $\varphi \colon \mathbb{S}^1 \to \mathbb{S}^1$ be defined as follows: the point $\varphi(p)$, for $p \in \mathbb{S}^1$, is the projection of $\Phi(p) \in \mathbb{S}^2 \setminus \{N, S\}$ on $\mathbb{S}^1$ along the meridians of $\mathbb{S}^2$.

Then the topopological degrees of Φ and φ are equal.

As a direct consequence one gets

COROLLARY 3.6. — *Let $\Phi \colon \mathbb{S}^2 \to \mathbb{S}^2$ be a continuous map so that $\Phi^{-1}(N) = \{S\}$ and $\Phi^{-1}(S) = \{N\}$.*

Let $\varphi \colon \mathbb{S}^1 \to \mathbb{S}^1$ be defined as follows: the point $\varphi(p)$, for $p \in \mathbb{S}^1$, is the projection of $\Phi(p) \in \mathbb{S}^2 \setminus \{N, S\}$ on $\mathbb{S}^1$ along the meridians of $\mathbb{S}^2$.

Then the topopological degrees of Φ and φ are opposite.

We consider now the torus $\mathbb{T}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$. As a direct consequence of Lemma 3.5 and Corollary 3.6 one gets:

COROLLARY 3.7. — *Let $\Phi \colon \mathbb{T}^2 \to \mathbb{S}^2$ be a continuous map so that $\Phi^{-1}(N) = \{0\} \times \mathbb{R}/\mathbb{Z}$ and $\Phi^{-1}(S) = \{\frac{1}{2}\} \times \mathbb{R}/\mathbb{Z}$.*

Let $\varphi_+ \colon \{\frac{1}{4}\} \times \mathbb{R}/\mathbb{Z} \to \mathbb{S}^1$ (resp. $\varphi_- \colon \{\frac{3}{4}\} \times \mathbb{R}/\mathbb{Z} \to \mathbb{S}^1$) be defined as follows: the point $\varphi_+(p)$ (resp. $\varphi_-(p)$) is the projection of $\Phi(p) \in \mathbb{S}^2 \setminus \{N, S\}$ on $\mathbb{S}^1$ along the meridians of $\mathbb{S}^2$.

Then

$$|\deg(\Phi)| = |\deg(\varphi_+) - \deg(\varphi_-)|$$

where $\deg()$ denotes the topological degree, and $\{\frac{1}{4}\} \times \mathbb{R}/\mathbb{Z}$ and $\{\frac{3}{4}\} \times \mathbb{R}/\mathbb{Z}$ are endowed with the positive orientation of $\mathbb{R}/\mathbb{Z}$.

Proof. — Indeed, Φ is homotopic (by an homotopy preserving $\Phi^{-1}(N)$ and $\Phi^{-1}(S)$) to the map $\Phi_{d^+, d^-} \colon \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{S}^2$ defined as follows

- $\Phi_{d^+, d^-}(s, t) = \left(|\sin(2\pi s)| \cdot e^{2i\pi d^+ t}, \cos(2\pi s) \right)$ if $s \in [0, \frac{1}{2}]$,

- $\Phi_{d^+, d^-}(s, t) = \left(|\sin(2\pi s)| \cdot e^{2i\pi d^- t}, \cos(2\pi s) \right)$ if $s \in [0, \frac{1}{2}]$.

where d^+ and d^- are $\deg(\varphi_+)$ and $\deg(\varphi_-)$, respectively. $\square$

3.4. Commuting vector fields: local version

There are two usual definitions for commuting vector fields: one can require that the flows of X and Y commute; one may also require that the

Lie bracket $[X, Y]$ vanishes. These two definitions coincide for C^1 vector fields on compact manifolds. On non compact manifolds we just have the commutation of the flows for small times, as explained more precisely below.

Let M be a (not necessarily compact) manifold and X, Y be C^1-vector fields on M. The Cauchy–Lipschitz theorem asserts that the flows of X and Y are locally defined but they may not be complete.

We say that *X and Y commute* if for every point x there is $t(x) > 0$ so that for every $s, t \in [-t(x), t(x)]$ the compositions $X_t \circ Y_s(x)$ and $Y_s \circ X_t(x)$ are defined and coincide. Thus, the local diffeomorphism X_t carries integral curves of Y into integral curves of Y, and vice-versa.

Remark 3.8. — There are (non complete) commuting vector fields, a point x and $t > 0$ and $s > 0$ so that both $X_t \circ Y_s(x)$ and $Y_s \circ X_t(x)$ are defined but are different. Let us present an example.

Consider $\mathbb{C}^* = \mathbb{R}^2 \setminus \{(0,0)\}$ endowed with the two vector fields $\tilde{X} = \frac{\partial}{\partial x}$ and $\tilde{Y} = \frac{\partial}{\partial y}$. Note that $\tilde{X}$ and $\tilde{Y}$ commute. Consider the 2-cover $\phi \colon \mathbb{C}^* \to \mathbb{C}^*$ defined by $z \mapsto z^2$. Let X and Y the lifts for ϕ of $\tilde{X}$ and $\tilde{Y}$, respectively. Then X and Y commute. However, consider the point $\tilde{x} = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = e^{i\pi \frac{5}{4}}$. Consider $x = e^{i\pi \frac{5}{8}}$, so that $\phi(x) = \tilde{x}$.

Next figure illustates the fact that $X_t Y_t(x)$ and $Y_t X_t(x)$ are well defined but distinct, for $t = \sqrt{2}$:
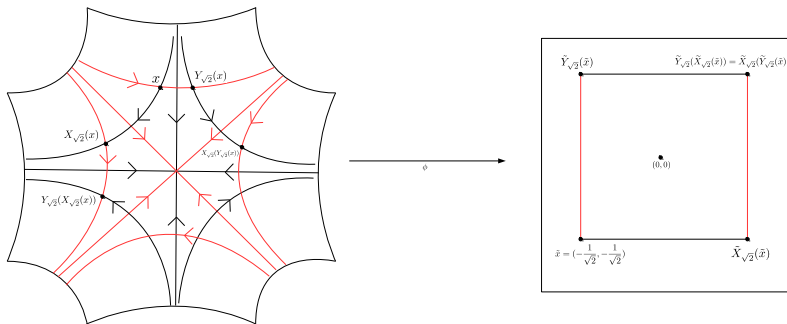

Figure 3.1. The flow of X commutes with the flow of Y until one of the composed orbits $X_t Y_s$ crosses one of the axes.

Next section states straightforward consequences of this definition.

3.5. Commuting vector fields: first properties

If X and Y are commuting vector fields then:

(1) for every $a, b, c, d \in \mathbb{R}$, $aX + bY$ commutes with $cX + dY$;
(2) for every $x \in M$, and any $t \in \mathbb{R}$ for which Y_t is defined, one has

$$DY_t(x)X(x) = X(Y_t(x))$$

(3) if $x \in \text{Zero}(X)$, Then for any $t \in \mathbb{R}$ for which Y_t is defined, $Y_t(x) \in \text{Zero}(X)$.
(4) $\text{Col}(X, Y)$ is invariant under the flow of X in the following sense: if $x \in \text{Col}(X, Y)$ and if $X_t(x)$ is defined, then $X_t(x) \in \text{Col}(X, Y)$; in the same way, $\text{Col}(X, Y)$ is invariant under the flow of $aX + bY$ for any $a, b \in \mathbb{R}$.
(5) if $\text{Zero}(Y) = \emptyset$, Then for each point x in $\text{Col}(X, Y)$ there is $\mu(x) \in \mathbb{R}$ so that $X(x) = \mu(x)Y(x)$. The map $x \mapsto \mu(x)$ is called the *ratio between X and Y at x*, is continuous on $\text{Col}(X, Y)$ and can be extended in a C^1 map on the ambient manifold.

> *Proof.* — One can extend μ to a small neighboorhod V of $\text{Col}(X, Y)$ in the following way: given some riemannien metric on M, if V is small enough then X is never orthogonal to Y in $V \setminus \text{Col}(X, Y)$. Thus, one can consider the vector field Z obtained as the orthogonal projection of X onto the direction of Y. It is clear that $Z|_{\text{Col}(X,Y)} = X$ and that there exists a C^1 function $\psi : V \to \mathbb{R}$ such that $Z(x) = \psi(x)Y(x)$. Since $\text{Zero}(Y) = \emptyset$, this implies that ψ is an extension of μ. Moreover, clearly ψ extends to M.[3]   □

(6) the ratio μ defined on $\text{Col}(X, Y)$ is invariant under the flow of $aX + bY$ for any $a, b \in \mathbb{R}$.
(7) if γ is a periodic orbit of X of period τ and if $t \in \mathbb{R}$ is such that Y_t is defined on γ, then $Y_t(\gamma)$ is a periodic orbit of X of period τ. The same occurs with the images by the flow of $cX + dY$ of periodic orbits of $aX + bY$, for $a, b, c, d \in \mathbb{R}$.
(8) as a consequence of the previous item, if γ is a periodic orbit of X of period τ and if γ is isolated among the periodic orbits of X of the same period τ, then γ is invariant under the flow of Y; as a consequence, $\gamma \subset \text{Col}(X, Y)$.

---

[3] We shall see in Section 5 a particular case of this construction.

3.6. Counter examples to Theorem A

Our proof is a long proof by *reductio ad absurdum*. To achieve this goal, we shall first show that the existence of a pair (X, Y) which do not satisfy the conclusion of Theorem A implies the existence of other pairs with simpler geometric behaviors.

For this reason, it will be convenient to define the notion of counter examples in a formal manner.

Definition 3.9. — *Let M be a 3-manifold, U a compact subset of M and X, Y be C^1 vector fields on M. We say that (U, X, Y) is a counter example to Theorem A if*

- *X and Y commute*
- $\mathrm{Zero}(Y) \cap U = \emptyset$
- $\mathrm{Zero}(X) \cap \partial U = \emptyset$
- $\mathrm{Ind}(X, U) \neq 0$.
- *the collinearity locus, $\mathrm{Col}(X, Y, U)$, is contained in a C^1 surface which is a closed submanifold of M.*

Let us illustrate our simplifying procedure by a simple argument:

Remark 3.10. — If M is a 3-manifold carrying a counter example

$$(U, X, Y)$$

to Theorem A, then there is an orientable manifold carrying a counter example to Theorem A. Indeed, consider the orientation cover $\tilde{M} \to M$ and $\tilde{U}, \tilde{X}, \tilde{Y}$ the lifts of U, X, Y on $\tilde{M}$. Then the Poincaré–Hopf index of $\tilde{X}$ on $\tilde{U}$ is twice the one of X on U, and $(\tilde{U}, \tilde{X}, \tilde{Y})$ is a counter example to Theorem A.

Thus we can assume (and we do it) without loss of generality that M is orientable.

Most of our simplifying strategy will now consist in combinations of the following remarks

Remark 3.11. — If (U, X, Y) is a counter example to Theorem A, then there is $\varepsilon > 0$ so that $(U, aX + bY, cX + dY)$ is also a counter example to Theorem A, for every a, b, c, d with $|a - 1| < \varepsilon$, $|b| < \varepsilon$, $|c| < \varepsilon$ and $|d - 1| < \varepsilon$.

Remark 3.12. — If (U, X, Y) is a counter example to Theorem A, then (V, X, Y) is also a counter example to Theorem A for any compact set $V \subset U$ containing $\mathrm{Zero}(X, U)$ in its interior.

Remark 3.13. — Let (U, X, Y) be a counter example to Theorem A and assume that $\mathrm{Zero}(X, U) = K_1 \cup \cdots \cup K_n$, where the K_i are pairwise disjoint compact sets. Let $U_i \subset U$ be compact neighborhood of K_i so that the U_i, $i = 1, \ldots, n$, are pairwise disjoint.

Then there is $i \in \{1, \ldots, n\}$ so that (U_i, X, Y) is a counter example to Theorem A.

4. Prepared counter examples (U, X, Y) to Theorem A

Simplifying $\mathrm{Col}(X, Y, U)$**.** The aim of this paragraph is to prove

LEMMA 4.1. — *If (U, X, Y) is a counter example to Theorem A then there is a counter example $(\tilde{U}, \tilde{X}, \tilde{Y})$ to Theorem A and $\mu_0 > 0$ with the following property:*

- *for any $t \in [-\mu_0, \mu_0]$, the set of zeros of $\tilde{X} - t\tilde{Y}$ in $\tilde{U}$ consists precisely in 1 periodic orbit γ_t of $\tilde{Y}$;*
- *for any $t \notin [-\mu_0, \mu_0]$, the set of zeros of $\tilde{X} - t\tilde{Y}$ in $\tilde{U}$ is empty;*
- $\mathrm{Col}(\tilde{X}, \tilde{Y}, \tilde{U})$ *is a C^1 annulus;*
- *there is a C^1-diffeomorphism $\varphi \colon \mathbb{R}/\mathbb{Z} \times [-\mu_0, \mu_0] \to \mathrm{Col}(\tilde{X}, \tilde{Y}, \tilde{U})$ so that, for every $t \in [-\mu_0, \mu_0]$, one has*

$$\varphi(\mathbb{R}/\mathbb{Z} \times \{t\}) = \gamma_t.$$

Proof. — By hypothesis $\mathrm{Col}(X, Y, U)$ is contained in a C^1-surface S. Notice that there is $\mu_1 > 0$ so that for any $t \in [-\mu_1, \mu_1]$ one has $\mathrm{Zero}(X - tY) \cap \partial U = \emptyset$ and $\mathrm{Ind}(X - tY, U) \neq 0$. In particular, we have that $\mathrm{Zero}(X - tY, U) \neq \emptyset$.

As $X - tY$ and Y commute, $\mathrm{Zero}(X - tY, U)$ is invariant under the flow of Y. Futhermore, as $\mathrm{Zero}(X - tY)$ does not intersect ∂U the Y-orbit of a point $x \in \mathrm{Zero}(X - tY, U)$ remains in the compact set U hence is complete.

Consider now the ratio function $\mu \colon \mathrm{Col}(X, Y) \to \mathbb{R}$, defined in the item (5) of Subsection 3.5. It follows that, for $x \in \mathrm{Col}(X, Y)$, $\mu(x) = t \Leftrightarrow x \in \mathrm{Zero}(X - tY)$. The map μ is invariant under the flows of X and Y (on $\mathrm{Col}(X, Y)$). As mentioned in 3.5, the map μ can be extended on M as C^1 map still denoted by μ (but no more X, Y-invariant).

Consider $\mathcal{L} = \bigcup_{t \in [-\mu_1, \mu_1]} \mathrm{Zero}(X - tY) \cap U$. Then $\mathcal{L}$ is a compact set, contained in S disjoint from the boundary of U and invariant under Y: it is a compact lamination of S.

By applying the flox box theorem and a standard compactness argument we can take $\sigma \subset S$ a union of finitely many compact segments with end

points out of $\mathcal{L}$ and so that the interior of σ cuts transversely every orbit of Y contained in $\mathcal{L}$.

CLAIM 1. — *Lebesgue almost every $t \in [-\mu_1, \mu_1]$ is a regular value of the restriction of μ to σ.*

Proof. — Recall that Sard's theorem requires a regularity $n - m + 1$ if one consider maps from an m-manifold to an n-manifold. As μ is C^1 and $\dim \sigma = 1$ we can apply Sard's theorem to the restriction of μ to σ, concluding. $\square$

Consider now a regular value $t \in (-\mu_1, \mu_1)$ of the restriction of μ to σ. Then $\mu^{-1}(t) \cap \sigma$ consists in finitely many points. Furthermore, $\mu^{-1}(t) \cap \sigma$ contains $\mathrm{Zero}(X - tY) \cap \sigma$.

CLAIM 2. — *If $t \in [-\mu_1, \mu_1]$ is regular value of the restriction of μ to σ, then the compact set $\mathrm{Zero}(X - tY) \cap U$ consists in finitely many periodic orbits γ_i, $i \in \{1, \ldots, n\}$ of Y.*

Proof. — $\mathrm{Zero}(X - tY) \cap U$ is a compact sub lamination of $\mathcal{L} \subset S$ consisting of orbits of Y, and contained in $\mu^{-1}(t)$. Now, σ cuts transversely each orbit of this lamination and $\sigma \cap \mu^{-1}(t)$ is finite. One deduces that $\mathrm{Zero}(X - tY) \cap U$ consists in finitely many compact leaves, concluding. $\square$

We now fix a regular value $t \in (-\mu_1, \mu_1)$ of the restriction of μ to σ.

Since $\mathrm{Zero}(X - tY, U) \neq \emptyset$ we have that the integer n of the above claim is positive. Moreover, notice that $\mathrm{Ind}(X - tY, U) = \sum_{i=1}^{n} \mathrm{Ind}(X - tY, \gamma_i)$. Thus there is i so that

$$\mathrm{Ind}(X - tY, \gamma_i) \neq 0.$$

CLAIM 3. — *There is a neighborhood Γ_i of γ_i in S which is contained in $\mathrm{Col}(X, Y, U)$ and which consists in periodic orbits of Y.*

Proof. — Let p be a point in $\sigma \cap \gamma_i$. As p is a regular point of the restriction of μ to σ there is a segment $I \subset \sigma$ centered at p so that the restriction of μ to I is injective and the derivative of μ does not vanish.

As γ_i has non-zero index for any s close enough to t, $\mathrm{Zero}(X - sY)$ contains an isolated compact subset K_s contained in a small neighborhood of γ_i, and hence in U, thus in $\mathrm{Col}(X, Y, U)$ and thus in a small neighborhood of γ_i in $\mathcal{L} \subset S$. This implies that each orbit of Y contained in K_s cuts I. However, μ is constant equal to s on K_s and thus $\mu^{-1}(s) \cap I$ consist in a unique point. One deduces that K_s is a compact orbit of Y.

Since this holds for any s close to t, one obtain that any point q of I close to p is the intersection point of $K_{\mu(q)} \cap I$. In other words, a neighborhood

of p in I is contained in $\mathrm{Col}(X, Y, U)$ and the corresponding leaf of $\mathcal{L}$ is a periodic orbit of Y, concluding. $\square$

Notice that Γ_i is contained in $\mathrm{Col}(X, Y, U)$ so that the function μ is invariant under Y on Γ_i. As the derivative of μ is non vanishing (by construction) on $\Gamma_i \cap I$ one gets that the derivative of the restriction of μ to Γ_i is non-vanishing. One deduces that Γ_i is diffeomorphic to an annulus: it is foliated by circles and these circles admit a transverse orientation.

For concluding the proof it remains fix $t \in (-\mu_1, \mu_1)$ regular value of μ, then one fixes $\tilde{X} = X - tY$ and $\tilde{Y} = Y$. One chooses a compact neighborhood $\tilde{U}$ of γ_i in M, which is a manifold with boundary, whose boundary is tranverse to S and so that $\tilde{U} \cap S = \Gamma_i$. By construction, $(\tilde{U}, \tilde{X}, \tilde{Y})$ satisfies all the announced properties. $\square$

4.1. Prepared counter examples to Theorem A

DEFINITION 4.2. — *We say that $(U, X, Y, \Sigma, \mathcal{B})$ is a* prepared counter example to Theorem A *if*

(1) *(U, X, Y) is a counter example to Theorem A*

(2) *There is $\mu_0 > 0$ so that (U, X, Y) satisfies the conclusion of Lemma 4.1:*
   - *for any $t \in [-\mu_0, \mu_0]$, the set of zeros of $X - tY$ in U consists precisely in 1 periodic orbit γ_t of Y;*
   - *for any $t \notin [-\mu_0, \mu_0]$, the set of zeros of $X - tY$ in U is empty;*
   - *$\mathrm{Col}(X, Y, U)$ is a C^1 annulus;*
   - *there is a C^1-diffeomorphism $\varphi \colon \mathbb{R}/\mathbb{Z} \times [-\mu_0, \mu_0] \to \mathrm{Col}(X, Y, U)$ so that, for every $t \in [-\mu_0, \mu_0]$, one has*
$$\varphi(\mathbb{R}/\mathbb{Z} \times \{t\}) = \gamma_t.$$

(3) *U is endowed with a foliation by discs; more precisely there is a smooth submersion $\Sigma \colon U \to \mathbb{R}/\mathbb{Z}$ whose fibers $\Sigma_t = \Sigma^{-1}(t)$ are discs; furthermore, the vector field Y is transverse to the fibers Σ_t.*

(4) *Each periodic orbit γ_s, $s \in [-\mu_0, \mu_0]$, of Y cuts every disc Σ_t in exactly one point. In particular the period of γ_s coincides with its return time on Σ_0 and is denoted $\tau(s)$, for $s \in [-\mu_0, \mu_0]$.*

   Thus $s \mapsto \tau(s)$ is a C^1-map on $[-\mu_0, \mu_0]$. We require that the derivative of τ does not vanish on $[-\mu_0, \mu_0]$.

(5) *$\mathcal{B}$ is a triple (e_1, e_2, e_3) of C^0 vector fields on U so that*
   - *for any $x \in U$ $\mathcal{B}(x) = (e_1(x), e_2(x), e_3(x))$ is a basis of $T_x U$.*
   - *$e_3 = Y$ everywhere*

- the vectors e_1, e_2 are tangent to the fibers Σ_t, $t \in \mathbb{R}/\mathbb{Z}$. In other words, $D\Sigma(e_1) = D\Sigma(e_2) = 0$
- The vector e_1 is tangent to $\mathrm{Col}(X, Y)$ at each point of $\mathrm{Col}(X, Y)$.

LEMMA 4.3. — *If there exists a counter example (U, X, Y) to Theorem A then there is a prepared counter example $(\tilde{U}, \tilde{X}, \tilde{Y}, \Sigma, \mathcal{B})$ to Theorem A.*

Proof. — The two first items of the definition of prepared counter example to Theorem A are given by Lemma 4.1. For getting the third item, it is enough to shrink U. For getting item (4), one replace Y by $Y + bX$ for some $b \in \mathbb{R}$, $|b|$ small enough. This does not change the orbits γ_t, as X and Y are both tangent to γ_t, but it changes its period. Thus, this allows us to change the derivative of the period τ at $s = 0$. Then one shrink again U and μ_0 so that the derivative of τ will not vanish on $\mathrm{Col}\,(X, Y, U)$.

Consider any metric on U. For any point x in the annulus $\mathrm{Col}(X, Y, U)$ and contained in the fiber Σ_t we chose $e_1(x)$ as being a unit vector tangent to the segment $\mathrm{Col}(X, Y) \cap \Sigma_t$. We extend e_1 as a continuous vector field on U tangent to the fibers Σ_t. We choose $e_2(x)$, for any x in a fiber Σ_t as being an unit vector tangent to Σ_t and orthogonal to $e_1(x)$. Now $e_3(x) = Y(x)$ is transverse to the plane spanned by $e_1(x), e_2(x)$ so that $\mathcal{B}(x) = (e_1(x), e_2(x), e_2(x))$ is a basis of $T_x M$. This provides the basis announced in item 5. $\qquad\square$

Remark 4.4. — If $(U, X, Y, \Sigma, \mathcal{B})$ is a prepared counter example to Theorem A, then for every $t \in (-\mu_0, \mu_0)$, $(U, X - tY, Y, \Sigma, \mathcal{B})$ is a prepared counter example to Theorem A.

Whenever $(U, X, Y, \Sigma, \mathcal{B})$ is a prepared counter example to Theorem A, we shall denote by $\mathcal{P}$ the *first return map*, defined on a neighborhood of $\mathrm{Col}(X, Y) \cap \Sigma_0$ in Σ_0.

Remark 4.5. — As the ambient manifold is assumed to be orientable (see Remark 3.10), the vector field Y is normally oriented so that the Poincaré map $\mathcal{P}$ preserves the orientation.

4.2. Counting the index of a prepared counter example

DEFINITION 4.6. — *Let $(U, X, Y, \Sigma, \mathcal{B})$ be a prepared counter example to Theorem A. In particular, U is a solid torus (C^1-diffeomorphic to $\mathbb{D}^2 \times$*

$\mathbb{R}/\mathbb{Z}$) and $\mathrm{Zero}(X - tY)$, $t \in (-\mu_0, \mu_0)$, is an essential simple curve γ_t isotopic to $\{0\} \times \mathbb{R}/\mathbb{Z}$. An essential torus T is the image of a continuous map from the torus $\mathbb{T}^2$ in the interior of U, disjoint from $\gamma_0 = \mathrm{Zero}(X)$ and homotopic, in $U \setminus \gamma_0$, to the boundary of a tubular neighborhood of γ_0.

In other words, $H_2(U \setminus \gamma_0, \mathbb{Z}) = \mathbb{Z}$, and T is essential if it is the generator of this second homology group.

We shall now describe how we use the basis $\mathcal{B}$, which comes with a prepared counter example, and an essential torus T to calculate the index.

For each point $x \in U$, one can write $X(x)$ as a linear combination of the vectors $e_1(x), e_2(x)$ and $e_3(x)$. Notice that, since $e_3 = Y$ everywhere, the e_3-coordinate μ of X is a C^1 extension of the ratio function, μ that we introduced on $\mathrm{Col}(X, Y)$ in item (5) of Subsection 3.5. Therefore, there exists C^1 functions $\alpha, \beta, \mu : U \to \mathbb{R}$ such that

$$(4.1) \qquad X(x) = \alpha(x)e_1(x) + \beta(x)e_2(x) + \mu(x)e_3(x).$$

For $x \notin \gamma_0$ one considers the vector

$$(4.2) \qquad \mathcal{X}(x) = \frac{1}{\sqrt{\alpha(x)^2 + \beta(x)^2 + \mu(x)^2}} \, (\alpha(x), \beta(x), \mu(x)) \in \mathbb{S}^2.$$

The map restriction $\mathcal{X}|_T \colon T \to \mathbb{S}^2$ has a topological degree, which, by Lemma 3.3, coincides with $\mathrm{Ind}(X, U)$, for some choice of an orientation on T.

4.3. The normally hyperbolic case

In this section we illustrate our procedure by giving the very simple proof of Theorem A in the case where $\mathrm{Col}(X, Y, U)$ is furthermore assumed to be normally hyperbolic for the flow of Y.

Here we shall prove

LEMMA 4.7. — *Let $(U, X, Y, \Sigma, \mathcal{B})$ be a prepared counter example to Theorem A. Then, the first return map $\mathcal{P} \colon \Sigma_0 \to \Sigma_0$ of the flow of Y satisfies: for every point x of $\mathrm{Col}(X, Y, U) \cap \Sigma_0$, the unique eigenvalue of the derivative of $\mathcal{P}$ at x is 1.*

Proof. — The argument is by contradiction. Let us denote $x_t = \gamma_t \cap \Sigma_0$, $t \in [-\mu_0, \mu_0]$, (recall $\gamma_t = \mathrm{Zero}(X - tY)$). We assume that the derivative of $\mathcal{P}$ at some point of x_{t_0} has some eigenvalue of different from 1.

Notice that the first return map $\mathcal{P}$ is the identity map in restriction to the segment $\mathrm{Col}(X, Y) \cap \Sigma_0$. In particular, the derivative of $\mathcal{P}$ at x_t admits

1 as an eigenvalue. Since $\mathcal{P}$ preserves the orientation (see Remark 4.5), the other eigenvalue is positive.

CLAIM 4. — *There exists $\tilde{U} \subset U$, $\tilde{X} = X - tY$ and a prepared counter example to Theorem A $(\tilde{U}, \tilde{X}, Y, \Sigma, \mathcal{B})$ for which the surface $\mathrm{Col}(\tilde{X}, Y, \tilde{U})$ is normally hyperbolic.*

Proof. — As the property of having an eingenvalue of mudulus different from 1 is an open condition, there exists an interval $[\mu_1, \mu_2] \subset [-\mu_0, \mu_0]$ on which the condition holds. Consider $t = \frac{\mu_1 + \mu_2}{2}$ and $\tilde{X} = X - tY$.

Then, one obtains a new prepared counter example to Theorem A by replacing X by $\tilde{X}$ (see Remark 4.4); now, by shrinking U one gets a tubular neighborhood $\tilde{U}$ of γ_t so that $\mathrm{Col}(\tilde{X}, Y, \tilde{U}) = \bigcup_{s \in [\mu_1, \mu_2]} \gamma_s$.

Moreover, the derivative of $\mathcal{P}$ at each point x_s, $s \in [\mu_1, \mu_2]$, has an eigenvalue different from 1 in a direction tranverse to $\mathrm{Col}(\tilde{X}, Y, \tilde{U}) \cap \Sigma_0$. By compactness and continuity these eigenvalues are uniformly far from 1 so that $\mathrm{Col}(\tilde{X}, Y, \tilde{U}) \cap \Sigma_0$ is normally hyperbolic for $\mathcal{P}$.

Thus $\mathrm{Col}(\tilde{X}, Y, \tilde{U})$ is an invariant normally hyperbolic annulus for the flow of Y. $\qquad\square$

By virtue of the above claim (up to change X by $\tilde{X}$ and U by $\tilde{U}$) one may assume that $\mathrm{Col}(X, Y, U)$ is normally hyperbolic, and (up change Y by $-Y$) one may assume that $\mathrm{Col}(X, Y, U)$ is normally contracting.

This implies that every periodic orbit γ_t has a local stable manifold $W_Y^s(\gamma_t)$ which is a C^1-surface depending continuously on t for the C^1-topology and the collection of these surfaces build a $\mathcal{C}^0$-foliation $\mathcal{F}_Y^s$ tangent to a continuous plane field E_Y^s, in a neighborhood of $\mathrm{Col}(X, Y, U)$. Furthermore, E_Y^s is tangent to Y, and hence is tranverse to the fibers of Σ.

Up to shrink U, one may assume that $\mathcal{F}_Y^s$ and E_Y^s are defined on U.

CLAIM 5. — *There is a basis $\tilde{\mathcal{B}} = (\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ so that $(U, X, Y, \Sigma, \tilde{\mathcal{B}})$ is a prepared counter example to Theorem A and $\tilde{e}_2$ is tangent to E_Y^s.*

Proof. — Choose $\tilde{e}_2$ as being a unit vector tangent to the intersection of E_Y^s with the tangent plane of the fibers of Σ. It remains to choose $\tilde{e}_1$ tranverse to e_2 and tangent to the fibers of Σ and tangent to $\mathrm{Col}(X, Y)$ at every point of $\mathrm{Col}(X, Y)$. $\qquad\square$

Up to change $\mathcal{B}$ by the basis $\tilde{\mathcal{B}}$ given by the claim above, we will now assume that e_2 is tangent to E_Y^s.

CLAIM 6. — *The vector field X is tangent to E_Y^s.*

Proof. — The flow of the vector field X leaves invariant the periodic orbit γ_t of Y and X commutes with Y. As a consequence, it preserves the

stable manifold $W^s(\gamma_t)$ for every t. This implies that X is tangent to the foliation $\mathcal{F}_Y^s$ and therefore to E_Y^s.   □

Therefore, for every $x \in U \setminus \mathrm{Zero}(X)$ the vector $\mathcal{X}(x) \in \mathbb{S}^2$ (see the notations in Equations 4.1 and 4.2) belongs to the circle $\{x_1 = 0\}$. In particular, for any essential torus T the map $\mathcal{X}|_T \colon T \to \mathbb{S}^2$ is not surjective, an thus has zero topological degree. This proves that $\mathrm{Ind}(X, U)$ vanishes, contradicting the fact that $(U, X, Y, \Sigma, \mathcal{B})$ is assumed to be a prepared counter example to Theorem A.   □

5. Holonomies, return time, and the normal component

In the whole section, $(U, X, Y, \Sigma, \mathcal{B})$ is a prepared counter example to Theorem A.

Definitions. Recall that $\Sigma_t = \Sigma^{-1}(t)$, $t \in \mathbb{R}/\mathbb{Z}$, is a family of cross section, each Σ_t is diffeomorphic to a disc, and we identify Σ_0 with the unit disc $\mathbb{D}^2$.

DEFINITION 5.1. — *Consider $t \in \mathbb{R}$. Consider $x \in \Sigma_0$ and $y \in \Sigma_t$. We say that y is the image by holonomy of Y over the segment $[0, t]$, and we denote $y = P_t(x)$, if there exists a continuous path $x_r \in U$, $r \in [0, t]$, so that $\Sigma(x_r) = r$, $x_0 = x$, $x_t = y$, and for every $r \in [0, t]$ the point x_r belongs to the Y-orbit of x.*

The holonomy map P_t is well defined in a neighborhood of $\mathrm{Col}(X, Y, U) \cap \Sigma_0$ and is a C^1 local diffeomorphism.

If $t = 1$ then P_1 is the first return map $\mathcal{P}$ (defined before Remark 4.5) of the flow of Y on the cross section Σ_0.

Remark 5.2. — With the notation of Definition 5.1, there is a unique continuous function $\tau_x \colon [0, t] \to \mathbb{R}$ so that $\tau_x(0) = 0$ and $x_r = Y_{\tau_x(r)}(x)$ for every $r \in [0, t]$.

We denote $\tau_t(x) = \tau_x(t)$ and we call it the *transition time from Σ_0 to Σ_t*. The map $\tau_t \colon \Sigma_0 \to \mathbb{R}$ is a C^1 map and by definition one has

$$(5.1) \qquad\qquad P_t(x) = Y_{\tau_t(x)}(x)$$

We denote $\tau = \tau_1$ and we call it the first return time of Y on Σ_0.

Remark 5.3. — In Definition 4.2 item 4 we defined $\tau(s)$ as the period of γ_s; in the notation above, it coincides with $\tau(x_s)$ where $x_s = \gamma_s \cap \Sigma_0$.

In this case, Equation 5.1 takes the special form

$$(5.2) \qquad\qquad \mathcal{P}(x) = Y_{\tau(x)}(x)$$

5.1. The normal conponent of X

DEFINITION 5.4. — *For every t and every $x \in \Sigma_t$ we define the normal component of X, which we denote by $N(x)$, the projection of $X(x)$ on $T_x\Sigma_t$ parallel to $Y(x)$.*

Thus $x \mapsto N(x)$ is a C^1-vector field tangent to the fibers of Σ and which vanishes precisely on $\mathrm{Col}(X, Y, U)$.

Moreover, in the basis $\mathcal{B}$, $N(x) = \alpha(x)e_1(x) + \beta(x)e_2(x)$ (see Equation 4.1), and we have the following formula

$$X(x) = N(x) + \mu(x)Y(x),$$

for every $x \in U$.

The first return map and the derivative of the first return time. The goal of this paragraph is the proof of Corollary 5.6 which claims the following formula

$$(5.3) \qquad - D\tau(x)N(x) = \mu((\mathcal{P}(x))) - \mu(x)$$

relating the derivative of the return time function $\tau : \Sigma_0 \to (0, +\infty)$, the normal component N and the first return map $\mathcal{P}$, at every point $x \in \Sigma_0$. This formula will be crucial for transfering informations on the normal vector field N to the first return map $\mathcal{P}$. At the end of this section, we shall use the angular variation of N (see the precise formulation in Corollary 5.14) together with formula (5.3) to prove that the derivative of $\mathcal{P}$ at any fixed point is the identity.

The geometrical idea for proving (5.3) is the following:

- at one hand, if $x \in \Sigma_0$, one has that the difference between the vectors $D\mathcal{P}(x)N(x)$ and $DY_{\tau(x)}(x)N(x)$ is parallel to $Y(\mathcal{P}(x))$, and the proportion is given exactly by $D\tau(x)N(x)$ (this is a classical fact, which we state precisely below).
- On the other hand, X is equal to $N + \mu Y$ and is invariant under DY_t, for any t.

One gets (5.3) by combining these two facts.

Let us proceed with the formal proof. The first step is the (classical) result below. The proof is an elementary and simple application of the flow box theorem, so we omit.

LEMMA 5.5. — *For every $t \in \mathbb{R}$ $x \in \Sigma_0$ and every $v \in T_x\Sigma_0$ one has,*

$$(5.4) \qquad DP_t(x)v - DY_{\tau_t(x)}(x)v = D\tau_t(x)v.Y(P_t(x)).$$
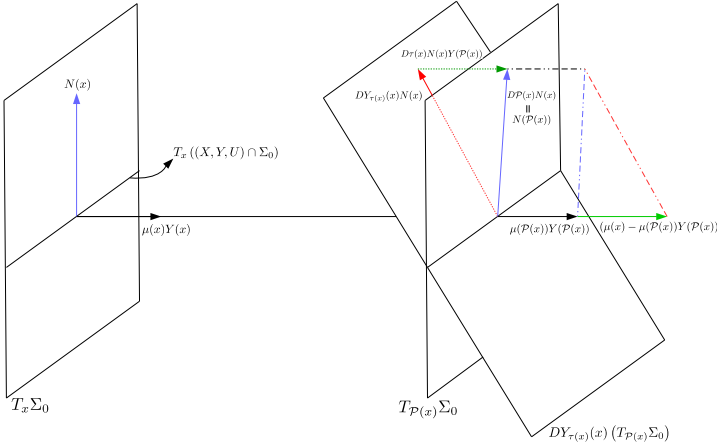
Figure 5.1. *Geometric proof of formula (5.3):*

$$N(\mathcal{P}(x)) + \mu(\mathcal{P}(x))Y(\mathcal{P}(x)) = X(\mathcal{P}(x)) = DY_{\tau(x)}(x)N(x) + \mu(x)Y(\mathcal{P}(x))$$

$$thus \quad D\tau(x)N(x)Y(\mathcal{P}(x)) = (\mu(x) - \mu(\mathcal{P}(x)))Y(\mathcal{P}(x))$$

In particular, for every $x \in \Sigma_0$ and $v \in T_x\Sigma_0$, one has

$$(5.5) \qquad D\mathcal{P}(x)v - DY_{\tau(x)}(x)v = D\tau(x)v.Y(\mathcal{P}(x)).$$

Now, recall that

$$X(x) = N(x) + \mu(x)Y(x)$$

and since

$$DY_{\tau(x)}X(x) = X(Y_{\tau(x)}) = X(\mathcal{P}(x)),$$

we obtain

$$(5.6) \qquad X(\mathcal{P}(x)) = DY_{\tau(x)}(x)N(x) + \mu(x)Y(\mathcal{P}(x)).$$

On the other hand, by Lemma 5.5

$$DY_{\tau(x)}(x)N(x) = D\mathcal{P}(x)N(x) - D\tau(x)N(x)Y(\mathcal{P}(x)).$$

As

$$X(\mathcal{P}(x)) = N(\mathcal{P}(x)) + \mu(\mathcal{P}(x))Y(\mathcal{P}(x)),$$

we conclude that

$$D\mathcal{P}(x)N(x) - D\tau(x)N(x))Y(\mathcal{P}(x)) + \mu(x)Y(\mathcal{P}(x))$$
$$= N(\mathcal{P}(x)) + \mu(\mathcal{P}(x))Y(\mathcal{P}(x)),$$

and so

$$(\mu(x) - \mu(\mathcal{P}(x)) - D\tau(x)N(x)))Y(\mathcal{P}(x)) + D\mathcal{P}(x)N(x) - N(\mathcal{P}(x)) = 0.$$

Since Y is transverse to the tangent space of Σ_0, the vectors $Y(\mathcal{P}(x))$ and $D\mathcal{P}(x)N(x) - N(\mathcal{P}(x))$ are linearly independent. As a consequence, one obtains that $D\mathcal{P}(x)N(x) = N(\mathcal{P}(x))$ and

COROLLARY 5.6. — Let $x \in \Sigma_0$. Then,

$$-D\tau(x)N(x) = \mu((\mathcal{P}(x))) - \mu(x).$$

Moreover, applying the same argument with the time t holonomy P_t in place of the first return map $\mathcal{P}$, one obtain the invariance of N under the holonomies.

LEMMA 5.7. — For $t \in \mathbb{R}$ and every $x \in \Sigma_0$ one has

$$DP_t(x)N(x) = N(P_t(x)).$$

The normal component of X and the index of X. Let $(U, X, Y, \Sigma, \mathcal{B})$ be a prepared counter example to Theorem A and N be the normal component of X.

Let us define, for $x \in U \setminus \mathrm{Col}(X, Y, U)$,

$$\mathcal{N}(x) = \frac{1}{\sqrt{\alpha(x)^2 + \beta(x)^2}}(\alpha(x), \beta(x)) \in \mathbb{S}^1 \subset \mathbb{S}^2,$$

where $\mathbb{S}^1$ is the unit circle of the plane $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$.

Recall that U is homeomorphic to the solid torus so that its first homology group $H_1(U, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$, by an isomorphism sending the class of γ_0, oriented by Y, on 1. Let U^+ and U^- be the two connected components of $U \setminus \mathrm{Col}(X, Y, U)$. These are also solid tori, and the inclusion in U induces isomorphisms of the first homology groups which allows us to identify $H_1(U^\pm, \mathbb{Z})$ with $\mathbb{Z}$.

DEFINITION 5.8. — We call linking number of X with respect to Y in U^+ (resp. in U^-) and we denote it by $\ell^+(X, Y)$ (resp. $\ell^-(X, Y)$) the integer defined as follows: the continuous map $\mathcal{N} \colon U^\pm \to \mathbb{S}^1$ induces morphisms on the homology groups $H_1(U^\pm, \mathbb{Z}) \to H_1(\mathbb{S}^1, \mathbb{Z})$. As these groups are all identified with $\mathbb{Z}$, these morphisms consist in the multiplication by an integer $\ell^\pm(X, Y)$.

In other words, consider a closed curve $\sigma \subset U^+$ homotopic in U to γ_0. Then $\ell^+(X, Y)$ is the topological degree of the restriction of $\mathcal{N}$ to σ.

PROPOSITION 5.9. — *Let* $(U, X, Y, \Sigma, \mathcal{B})$ *be a prepared counter example to Theorem A. Then*

$$|\operatorname{Ind}(X, U)| = |\ell^+(X, Y) - \ell^-(X, Y)|$$

Proof. — Consider a tubular neighborhood of γ_0 whose boundary is an essential torus T which cuts $\operatorname{Col}(X, Y, U)$ transversely and along exactly two curves σ_+ and σ_-. Then the map $\mathcal{X}$ on T takes the value $N \in \mathbb{S}^2$ (resp. $S \in \mathbb{S}^2$) exactly on σ_+ (resp. σ_-), where N and S are the points on $\mathbb{S}^2$ corresponding to $e_3 = Y$ and $-e_3$.

We identify T with $\mathbb{T}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ so that σ_- and σ_+ correspond to $\{\frac{1}{2}\} \times \mathbb{R}/\mathbb{Z}$ and $\{0\} \times \mathbb{R}/\mathbb{Z}$ respectively. It remains to apply Corollary 3.7 to $\Phi = \mathcal{X}$ and $\varphi = \mathcal{N}$, noticing that $\mathcal{N}$ is the projection of $\mathcal{X}$ on $\mathbb{S}^1$ along the meridians. This gives the announced formula. $\qquad\square$

As a direct consequence of Proposition 5.9 one gets

COROLLARY 5.10. — *Let* $(U, X, Y, \Sigma, \mathcal{B})$ *be a prepared counter example to Theorem A, then*

$$(\ell^+(X, Y), \ell^-(X, Y)) \neq (0, 0)$$

5.2. Angular variation of the normal component

Definition of the angular variation. Recall that, for any $x \in U \setminus \operatorname{Col}(X, Y)$, we defined $\mathcal{N}(x) \in \mathbb{S}^1$ as the normalisation of the expression of the normal component $N(x)$ in the basis $\mathcal{B}(x)$

Consider a path $\eta\colon [0, 1] \to U \backslash \operatorname{Col}(X, Y)$. We define the *angular variation of* $\mathcal{N}$ *along* η is as being

$$\tilde{\mathcal{N}}(\eta(1)) - \tilde{\mathcal{N}}(\eta(0))$$

where $\tilde{\mathcal{N}}(\eta(t))$ is a lift of the path $t \mapsto \mathcal{N}(\eta(t) \in \mathbb{S}^1$ on the universal cover $\mathbb{R} \mapsto \mathbb{S}^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$.

By a practical abuse of language, if there is no ambiguity on the basis $\mathcal{B}$, we will call *angular variation of N along η* the angular variation of $\mathcal{N}$ along η.

Angular variation of the normal component N along the Y-orbits. We denote $\{x_t\} = \gamma_t \cap \Sigma_0$. For every pair of points $x, y \in \Sigma_0$, we denote the segment of straight line joining x and y and contained in Σ_0 by $[x, y]$ (for some choice of coordinates on Σ_0).

LEMMA 5.11. — *For any $K > 0$ there is a neighborhood V_K of γ_0 with the following property.*

Consider $x \in V_K \cap \Sigma_0$ and $u \in T_x\Sigma_0$ a unit vector, and write $u = u_1 e_1(x) + u_2 e_2(x)$. Consider $t \in [0, K]$ and $v = DP_t(u) \in T_{P_t(x)}\Sigma_t$ the image of u by the derivative of the holonomy. Write $v = v_1 e_1(P_t(x)) + v_2 e_2(P_t(x))$.

Then

$$\left(\frac{u_1}{\sqrt{u_1^2 + u_2^2}}, \frac{u_2}{\sqrt{u_1^2 + u_2^2}} \right) \neq - \left(\frac{v_1}{\sqrt{v_1^2 + v_2^2}}, \frac{v_2}{\sqrt{v_1^2 + v_2^2}} \right)$$

Proof. — Assuming, by contradiction, that the conclusion does not hold we get $y_n \in \Sigma_0$, unit vectors $u_n \in T_{x_n}\Sigma_0$ and $t_n \in [0, K]$ so that y_n tends to $x_0 = \gamma_0 \cap \Sigma_0$, u_n tends to a unit vector u in $T_{x_0}\Sigma_0$, t_n tends to $t \in [0, K]$ and the image v_n of the vector u_n, expressed in the basis $\mathcal{B}$, is collinear to u_n with the opposite direction.

Then $DP_t(x_0)u$ is a vector that, expressed in the basis $\mathcal{B}$, is collinear to u with the opposit direction. In other words, u is an eigenvector of $DP_t(x_0)$, with a negative eigenvalue.

However, for every $t \in \mathbb{R}$ the vector e_1 is an eigenvector of $DP_t(x_0)$, with a positive eigenvalue, and $DP_t(x_0)$ preserves the orientation, leading to a contradiction. □

LEMMA 5.12. — *For every $N \in \mathbb{N}$ there exists a neighborhood $O_N \subset \Sigma_0$ of x_0 so that if $x \in O_N \setminus \mathrm{Col}(X, Y, U)$ then the segment of straight line $[x, \mathcal{P}^N(x)]$ is disjoint from $\mathrm{Col}(X, Y, U)$.*

Proof. — Assume that there is a sequence of points $y_n \to x_0$, $y_n \notin \mathrm{Col}(X, Y, U)$ so that the segment $[y_n, \mathcal{P}^N(y_n)]$ intersect $\mathrm{Col}(X, Y, U) \cap \Sigma_0$ at some point z_n. Recall that $\mathrm{Col}(X, Y, U) \cap \Sigma_0$ consists in fixed points of the first return map $\mathcal{P}$. In particular, z_n is a fixed point of $\mathcal{P}^N$.

The image of the segment $[z_n, y_n]$ is a C^1 curve joining z_n to $\mathcal{P}^N(y_n)$. Notice that the segments $[z_n, y_n]$ and $[z_n, \mathcal{P}^N(y_n)]$ are contained in the segment $[y_n, \mathcal{P}^N(y_n)]$ and oriented in opposite direction. One deduces that there is a point w_n in $[y_n, z_n]$ so that the image under the derivative $D\mathcal{P}^N$ of the unit vector u_n directing this segment is on the form $\lambda_n u_n$ with $\lambda_n < 0$.

Since y_n tends to x_0, one deduces that $D\mathcal{P}^N(x_0)$ has a negative eigenvalue. This contradicts the fact that both eigenvalues of $D\mathcal{P}^N(x_0)$ are positive and completes the proof. □

Recall that U^+ and U^- are the connected components of $U \setminus \mathrm{Col}(X, Y, U)$.

COROLLARY 5.13. — *If $x \in U^{\pm} \cap \Sigma_0 \cap O_3$ let $\theta_x \colon \mathbb{R}/\mathbb{Z} \to U$ be the curve obtained by concatenation of the Y-orbit segment from x to $\mathcal{P}^2(x)$ and the straight line segment $[\mathcal{P}^2(x), x]$, that is:*

- *for $t \in [0, \frac{1}{2}]$, $\theta_x(t) = Y_{2t.\tau_2(x)}(x)$ where τ_2 is the transition time from x to $\mathcal{P}^2(x)$*
- *for $t \in [\frac{1}{2}, 1]$, $\theta_x(t) = (2 - 2t)\mathcal{P}^2(x) + (2t - 1)x$.*

Then θ_x is a closed curved contained in $U^{\pm}$ and whose homology class in $H_1(U^{\pm}, \mathbb{Z}) = \mathbb{Z}$ is 2.

Proof. — The unique difficulty here is that the curve don't cross the colinearity locus $\mathrm{Col}\,(X, Y, U)$ and that is given by Lemma 5.12.   □

The next corollary is one of the fundamental arguments of this paper.

COROLLARY 5.14. — *Let $(U, X, Y, \Sigma, \mathcal{B})$ be a prepared counter example to Theorem A, and assume that $\ell^+(X, Y) \neq 0$.*

Consider $x \in U^+ \cap \Sigma_0 \cap O_3$. Then the angular variation of the vector $\mathcal{N}(y)$ for $y \in [x, \mathcal{P}^2(x)]$ is strictly larger than 2π in absolute value. In particular,

$$\mathcal{N}([x, \mathcal{P}^2(x)]) = \mathbb{S}^1.$$

The same statement holds in U^- if $\ell^-(X, Y) \neq 0$.

Proof. — Lemma 5.11 implies that the angular variation of $\mathcal{N}(\theta_x(t))$ is contained in $(-\pi, \pi)$ for $t \in [0, \frac{1}{2}]$. However, the topological degree of the map $\mathcal{N} \colon \theta_x \to \mathbb{S}^1$ is $2\ell^+(X, Y)$ which has absolute value at least 2. Thus the angular variation of $\mathcal{N}$ on the segment $[x, \mathcal{P}^2(x)]$ is (in absolute value) at least 3π concluding.   □

As an immediate consequence one gets

COROLLARY 5.15. — *If $\ell^+(X, Y) \neq 0$, then $\mathcal{P}^2$ has no fixed points in $O_3 \cap U^+$.*

The return map at points where N is pointing in opposite directions. We have seen in the proof of Corollary 5.14 that the vector $\mathcal{N}$ has an angular variation larger than 3π along the segment $[x, \mathcal{P}^2(x)]$, as $x \in \Sigma_0$ approaches x_0. In this section we use the large angular variation of $\mathcal{N}$ for establishing a relation between the return map $\mathcal{P}$, the return time τ, and the coordinate μ of X in the Y direction.

Recall that, for every $x \in U \setminus \mathrm{Col}(X, Y, U)$, $\mathcal{N}(x)$ is a unit vector contained in $\mathbb{S}^1$, unit circle of $\mathbb{R}^2$.

LEMMA 5.16. — *Assume that there exists sequences*

$$q_n, \bar{q}_n \in \Sigma_0 \setminus \mathrm{Col}(X, Y, U)$$

converging to x_0, such that the following two conditions are satisfied:

(1) $\mathcal{N}(q_n) \to (1, 0)$ and $\mathcal{N}(\bar{q}_n) \to (-1, 0)$, as $n \to +\infty$,

(2) $(\mu(\mathcal{P}(q_n)) - \mu(q_n))(\mu(\mathcal{P}(\bar{q}_n)) - \mu(\bar{q}_n)) \geqslant 0$, for every n.

Then, $D\tau(x_0)e_1(x_0) = 0$.

Proof. — Recall that $N(x) = \alpha(x)e_1(x) + \beta(x)e_2(x)$ and

$$\mathcal{N}(x) = \frac{1}{\sqrt{\alpha(x)^2 + \beta(x)^2}}(\alpha(x), \beta(x))$$

for $x \in U \setminus \mathrm{Col}(X, Y, U)$.

By Corollary 5.6, we have

$$(5.7) \qquad -D\tau(q_n)\frac{N(q_n)}{\sqrt{\alpha(q_n)^2 + \beta(q_n)^2}} = \frac{\mu(\mathcal{P}(q_n)) - \mu(q_n)}{\sqrt{\alpha(q_n)^2 + \beta(q_n)^2}},$$

and

$$(5.8) \qquad -D\tau(\bar{q}_n)\frac{N(\bar{q}_n)}{\sqrt{\alpha(\bar{q}_n)^2 + \beta(\bar{q}_n)^2}} = \frac{\mu(\mathcal{P}(\bar{q}_n)) - \mu(\bar{q}_n)}{\sqrt{\alpha(\bar{q}_n)^2 + \beta(\bar{q}_n)^2}}.$$

Multiplying side by side Equations 5.7 and 5.8 and using the second assumption of the lemma, we get

$$D\tau(q_n)\frac{N(q_n)}{\sqrt{\alpha(q_n)^2 + \beta(q_n)^2}}D\tau(\bar{q}_n)\frac{N(\bar{q}_n)}{\sqrt{\alpha(\bar{q}_n)^2 + \beta(\bar{q}_n)^2}} \geqslant 0.$$

Notice that the first assumption of the lemma is equivalent to

$$\frac{N(q_n)}{\sqrt{\alpha(q_n)^2 + \beta(q_n)^2}} \to e_1(x_0)$$

and

$$\frac{N(\bar{q}_n)}{\sqrt{\alpha(\bar{q}_n)^2 + \beta(\bar{q}_n)^2}} \to -e_1(x_0).$$

Since $q_n, \bar{q}_n \to x_0$, from the continuity of $D\tau$, we conclude that

$$0 \geqslant -(D\tau(x_0)(e_1(x_0)))^2 \geqslant 0,$$

which completes the proof. $\qquad\qquad\square$

Since we assumed $(D\tau(x_0)(e_1(x_0))) \neq 0$ (item (4) of the definition of a prepared counter example to Theorem A), one gets the following corollary:

COROLLARY 5.17. — *Let $(U, X, Y, \Sigma, \mathcal{B})$ is a prepared counter exam-ple to Theorem A. Assume that there exists sequences $q_n, \bar{q}_n \in \Sigma_0 \setminus \mathrm{Col}(X, Y, U)$ converging to x_0, such that $\mathcal{N}(q_n) \to (1, 0)$ and $\mathcal{N}(\bar{q}_n) \to (-1, 0)$, as n tends to $+\infty$. Then*

$$(\mu(\mathcal{P}(q_n)) - \mu(q_n))(\mu(\mathcal{P}(\bar{q}_n)) - \mu(\bar{q}_n)) < 0,$$

for every n large enough.

5.3. The case $D\mathcal{P}(x_0) \neq Id$

In this section, $(U, X, Y, \Sigma, \mathcal{B})$ is a prepared counter example to The-orem A so that the derivative of the first return map $\mathcal{P}$ at the point $x_0 = \Sigma_0 \cap \gamma_0$ is not the identity map. Recall that $D\mathcal{P}(x_0)$ admits an eigen-value equal to 1 directed by $e_1(x_0)$, has no eigenvalues different from 1 and is orientation preserving.

Recall that $\mathrm{Col}(X, Y, U)$ cuts the solid torus U in two components U^+ and U^-. Let denote $\Sigma_+ = \Sigma_0 \cap U^+$ and $\Sigma_- = \Sigma_0 \cap U^-$.

LEMMA 5.18. — *Let $(U, X, Y, \Sigma, \mathcal{B})$ be a prepared counter example to Theorem A so that the derivative of the first return map $\mathcal{P}$ at the point x_0 is not the identity map.*

There is a neighborhood W of x_0 in Σ so that the map $\mu(\mathcal{P}(x)) - \mu(x)$ restricted to W vanishes only on $\mathrm{Col}(X, Y, U)$.

More precisely,

- *$(\mu(\mathcal{P}(x)) - \mu(x))(\mu(\mathcal{P}(y)) - \mu(y)) > 0$ if x and $y \in W \cap \Sigma_+$ and if x and $y \in W \cap \Sigma_-$*
- *$(\mu(\mathcal{P}(x)) - \mu(x))(\mu(\mathcal{P}(y)) - \mu(y)) < 0$ if $x \in W \cap \Sigma_+$ and $y \in W \cap \Sigma_-$ and if $x \in W \cap \Sigma_-$ and $y \in W \cap \Sigma_+$*
- *$(\mu(\mathcal{P}(x)) - \mu(x)) = 0$ and $x \in W$ if and only if $x \in \mathrm{Col}(X, Y, U)$.*

Proof. — The derivative $D\mu(x_0)(e_1(x_0))$ do not vanish (item 2 of Defini-tion 4.2). Thus the kernel of $D\mu(x_0)$ is tranverse to $e_1(x_0)$. The derivative $D\mathcal{P}(x_0)$ admits $e_1(x_0)$ as an eigenvector (for the eigenvalue 1) and has no eigenvalue different from 1 and is not the identity. This implies that the kernel of $D\mu(x_0)$ is not an eigendirection of $D\mathcal{P}(x_0)$. Notice that the deriv-ative at x_0 of the map $\mu(\mathcal{P}(x)) - \mu(x)$ is $D\mu(x_0)D\mathcal{P}(x_0) - D\mu(x_0)$. As we have seen above the kernel of $D\mu(x_0)D\mathcal{P}(x_0)$ is different from the kernel of $D\mu(x_0)$ so that $D\mu(x_0)D\mathcal{P}(x_0) \neq D\mu(x_0)$.

As a consequence the derivative of the function $x \mapsto \mu(\mathcal{P}(x)) - \mu(x)$ does not vanish at $x = x_0$.

Thus $\{x \in \Sigma, \mu(\mathcal{P}(x)) - \mu(x)) = 0\}$ is a codimension 1 submanifold in a neighborhood of x_0 in Σ_0, and this submanifold contains $\mathrm{Col}\,(X, Y, U)$, because $\mathrm{Col}\,(X, Y, U) \subset Fix(\mathcal{P})$. Therefore these submanifolds coincide in the neighborhood of x_0, concluding. $\qquad\square$

We are now ready to prove the following proposition:

PROPOSITION 5.19. — If $(U, X, Y, \Sigma, \mathcal{B})$ is a prepared counter example to Theorem A then $D\mathcal{P}(x)$ is the identity map for every $x \in \mathrm{Col}\,(X, Y, U) \cap \Sigma_0$.

Proof. — Up to exchange $+$ by $-$, we assume that $\ell^+(X, Y) \neq 0$. Then Corollary 5.14 implies that there are sequences $q_n, \bar{q}_n \in \Sigma_+$ tending to x_0 and so that $\mathcal{N}(q_n) = (1, 0)$ and $\mathcal{N}(-q_n) = (-1, 0)$. More precisely Corollary 5.14 implies that for any $x \in \Sigma_+$ close enough to x_0 the segment $[x, \mathcal{P}^2(x)]$ contains points $q, \bar{q}$ with $\mathcal{N}(q) = (1, 0)$ and $\mathcal{N}(\bar{q}) = (-1, 0)$. Now Lemma 5.12 implies that the segment is contained in Σ_+, concluding.

Now Corollary 5.17 implies that, for n large enough, the sign of the map $(\mu(\mathcal{P}(x)) - \mu(x))$ is different on q_n and on $\bar{q}_n$. One concludes with Lemma 5.18 which says that this sign cannot change if the derivative $D\mathcal{P}(x_0)$ is not the identity. Thus we proved $D\mathcal{P}(x_0) = Id$.

Now if $x \in \mathrm{Col}\,(X, Y, U)$ then x is of the form $x = x_t = \gamma_t \cap \Sigma_0$. According to Remark 4.4 $(U, X - tY, Y, \Sigma, \mathcal{B})$ is also a prepared counter example to Theorem A, and the linking number of $\ell^+(X - tY, Y)$ is the same as the linking number of $\ell^+(X, Y)$, and therefore is not vanishing. Notice that the first return map $\mathcal{P}$ is not affected by the substitution of the pair (X, Y) by the pair $(X - tY, Y)$ but the point x_0 is now replaced by the point x_t. Therefore the argument above establishes that $D\mathcal{P}(x_t) = Id$ for every $x_t \in \mathrm{Col}\,(X, Y, U) \cap \Sigma_0$. $\qquad\square$

6. Proof of Theorem A

In the whole section, $(U, X, Y, \Sigma, \mathcal{B})$ is a prepared counter example to Theorem A. According to Corollary 5.10 one of the linking numbers $\ell^+(X, Y)$, $\ell^-(X, Y)$ does not vanish, so that, up to exchange $+$ with $-$, one may assume $\ell^+(X, Y) \neq 0$. According to Proposition 5.19 the derivative $D\mathcal{P}(x)$ is the identity map for every $x \in \mathrm{Col}\,(X, Y, U) \cap \Sigma_0$.

This means that, in a neighborhood of $\mathrm{Col}\,(X, Y, U)$, the diffeomorphism $\mathcal{P}$ is C^1 close to the identity map. The techniques introduced in [1] and [2] allow to compare the diffeomorphism $\mathcal{P}$ with the vector field $\mathcal{P}(x) - x$, and we will analyse the behavior of this vector field. We will first show that,

in the neighborhood of $\mathrm{Col}\,(X,Y,U)$ these vectors are almost tangent to the kernel of $D\mu$. As the fibers of $D\mu$ are tranverse to $\mathrm{Col}\,(X,Y,U)$ we get a topological dynamics of $\mathcal{P}$ similar to the normally hyperbolic case of Section 4.3. We will end contradicting Corollary 5.14.

Quasi invariance of the map μ by the first return map. Recall that μ is the coordinate of X in the Y direction: $X(x) = N(x) + \mu(x)Y(x)$. The aim of this section is to prove

LEMMA 6.1. — *If $x_n \in \Sigma_+$ is a sequence of points tending to $x \in \mathrm{Col}\,(X,Y,U)$ and if $v_n \in T_{x_n}\Sigma_+$ is the unit tangent vector directing the segment $[x_n, \mathcal{P}(x_n)]$ then $D\mu(x_n)(v_n)$ tends to 0.*

According to Remark 4.4, it is enough to prove Lemma 6.1 in the case $x = x_0 = \gamma_0 \cap \Sigma_0$. Lemma 6.1 is now a straighforward consequence of the following lemma

LEMMA 6.2.

$$\lim_{x \to x_0, x \in \Sigma_+} \frac{\mu(\mathcal{P}(x)) - \mu(x)}{d(\mathcal{P}(x), x)} = 0,$$

where $d(\mathcal{P}(x), x)$ denotes the distance between x and $\mathcal{P}(x)$.

Note that Lemma 5.18 implies that this estimate could not hold if the derivative $D\mathcal{P}(x_0)$ were not the identity. Here we use Proposition 5.19 which asserts that the derivative of $\mathcal{P}$ at the fixed points is the identity.

Proof. — Fix $\varepsilon > 0$ and let us prove that $\frac{|\mu(\mathcal{P}(x)) - \mu(x)|}{d(\mathcal{P}(x), x)}$ is smaller than ε for every x close to x_0 in Σ_+. Recall that, according to Lemma 5.12, there is a neighborhood O_2 of γ_0 so that if $x \in O_2^+ = O_2 \cap \Sigma_+$ then the segment of straight line $[x, \mathcal{P}^2(x)]$ is contained in Σ_+. Furthermore, Corollary 5.14 says that $\mathcal{N}|_{[x, \mathcal{P}^2(x)]}$ is surjective onto $\mathbb{S}^1$ (unit circle in $\mathbb{R}^2$). In particular, there are points $q_x, \bar{q}_x \in [x, \mathcal{P}^2(x)]$ so that $\mathcal{N}(q_x) = (1, 0)$ and $\mathcal{N}(\bar{q}_x) = (-1, 0)$.

According to Corollary 5.17 one gets

$$(6.1) \qquad (\mu(\mathcal{P}(q_x)) - \mu(q_x))(\mu(\mathcal{P}(\bar{q}_x)) - \mu(\bar{q}_x)) < 0,$$

for every $x \in O_2^+$.

The diffeomorphism $\mathcal{P}$ is C^1-close to the identity in a small neighborhood of x_0 Now [1] (see also [2]) implies that there is a neigborhood V_1 of x_0 in Σ_0 so that if $x \in V_1$ then

$$\|(\mathcal{P}(x) - x) - (\mathcal{P}(y) - y)\| < \frac{1}{2}\|\mathcal{P}(x) - x\|,$$

for every y with $d(x, y) < 3\|\mathcal{P}(x) - x\|$. In particular,

$$\|\mathcal{P}^2(x) - \mathcal{P}(x)\| < \frac{3}{2}\|\mathcal{P}(x) - x\| < 2\|\mathcal{P}(x) - x\|,$$

and thus $\|\mathcal{P}^2(x) - x\| < 3\|\mathcal{P}(x) - x\|$.

Consider the function $f(x) = \mu(\mathcal{P}(x)) - \mu(x)$. Since $D\mathcal{P}(x_0) = Id$, we have that $Df(x_0) = 0$. As a consequence, there exists a neighborhood $V_2 \subset V_1$ of x_0 such that $|Df(x)| < \frac{\varepsilon}{9}$, for every $x \in V_2$.

Since $\mathcal{P}(x_0) = x_0$, we can choose a smaller neighborhood V_3 such that $\mathcal{P}(x), \mathcal{P}^2(x) \in V_2$, for every $x \in V_3$. This ensures that

$$\frac{|f(q_x) - f(x)|}{d(x, \mathcal{P}(x))} < \frac{\varepsilon}{9} \frac{d(q_x, x)}{d(x, \mathcal{P}(x))} \leqslant \frac{\varepsilon}{3}.$$

Similar estimates hold with $\bar{q}_x$ in place of q_x and in place of x, respectively.

By Inequality (6.1) we see that $f(q_x)$ and $f(\bar{q}_x)$ have opposite signs and thus

$$\frac{|f(q_x) + f(\bar{q}_x)|}{d(x, \mathcal{P}(x))} \leqslant \frac{|f(q_x) - f(\bar{q}_x)|}{d(x, \mathcal{P}(x))} \leqslant \frac{\varepsilon}{3}.$$

We deduce

$$\left| \frac{2f(x)}{d(x, \mathcal{P}(x))} \right| = \frac{|f(x) - f(q_x) + f(q_x) + f(\bar{q}_x) + f(x) - f(\bar{q}_x)|}{d(x, \mathcal{P}(x))}$$

$$\leqslant \frac{|f(x) - f(q_x)|}{d(x, \mathcal{P}(x))} + \frac{|f(\bar{q}_x) + (q_x)|}{d(x, \mathcal{P}(x))} \frac{|f(x) - f(\bar{q}_x)|}{d(x, \mathcal{P}(x))}$$

$$\leqslant \varepsilon.$$

This establishes that $\frac{|\mu(\mathcal{P}(x)) - \mu(x)|}{d(\mathcal{P}(x), x)}$ is smaller than ε for every $x \in V_3 \cap \Sigma_+$ and completes the proof. $\qquad\square$

Remark 6.3. — The Lemmas 6.1 and 6.2 depend a priori on the choice of coordinates on Σ_0 since they are formulated in terms of segments of straight line $[x, \mathcal{P}(x)]$, and vectors $\mathcal{P}(x) - x$. Nevertheless, the choice of coordinates on Σ_0 was arbitrary (see first paragraph of Section 5.2) so that it holds indeed for any choice of C^1 coordinates on Σ_0 (on a neighborhood of x_0 depending on the choice of the coordinates).

Dynamics of the first return map $\mathcal{P}$ in the neighborhood of 0. Recall that Σ_0 is a disc endowed with an arbitrary (but fixed) choice of coordinates. Also, $\mu\colon \Sigma_0 \to \mathbb{R}$ is a C^1-map whose derivative does not vanish along $\mathrm{Col}\,(X, Y, U)$ and $\mathrm{Col}\,(X, Y, U) \cap \Sigma_0$ is a C^1-curve.

Therefore, one can choose a C^1-map $\nu\colon \Sigma_0 \to \mathbb{R}$ so that

- there is a neighborhood O of x_0 in Σ_0 so that $(\mu, \nu)\colon O \to \mathbb{R}^2$ is C^1 diffeomorphism,

- $\mathrm{Col}\,(X, Y, U) \cap O = \nu^{-1}(\{0\})$
- $\nu > 0$ on Σ_+

We denote by $(\mu(x), \nu(x))$ the image of x by (μ, ν).

Notice that $(\mu(x), \nu(x))$ are local coordinates on Σ_0 in a neighborhood of x_0. Remark 6.3 allows us to use Lemma 6.1 and Lemma 6.2 in the coordinates (μ, ν).

As a consequence, there exist $\varepsilon > 0$ so that for any point $x \in \Sigma_+$ with $(\mu(x), \nu(x)) \in [-\varepsilon, \varepsilon] \times (0, \varepsilon]$, one has

$$(6.2) \qquad |\mu(\mathcal{P}(x)) - \mu(x)| < \frac{1}{100}|\nu(\mathcal{P}(x)) - \nu(x)|.$$

In particular, since $\mathcal{P}$ has no fixed point in Σ_+ (Corollary 5.15), one gets that $\nu(\mathcal{P}(x)) - \nu(x)$ does not vanish for $(\mu(x), \nu(x)) \in [-\varepsilon, \varepsilon] \times (0, \varepsilon]$, and in particular it has a constant sign. Up to change $\mathcal{P}$ by its inverse $\mathcal{P}^{-1}$ (which is equivalent to replace Y by $-Y$), one may assume

$$\nu(\mathcal{P}(x)) - \nu(x) < 0, \text{ for } (\mu(x), \nu(x)) \in [-\varepsilon, \varepsilon] \times (0, \varepsilon]$$

Next lemma allows us to define stable sets for the points in $\mathrm{Col}\,(X, Y, U)$ and shows that every point in Σ_+ close to x_0 belongs to such a stable set.

LEMMA 6.4. — Let $x \in \Sigma_+$ be such that $(\mu(x), \nu(x)) \in [-\frac{9}{10}\varepsilon, \frac{9}{10}\varepsilon] \times (0, \varepsilon]$. Then, for any integer $n \geqslant 0$, $\mathcal{P}^n(x)$ satisfies

$$(\mu(\mathcal{P}^n(x)), \nu(\mathcal{P}^n(x))) \in [-\varepsilon, \varepsilon] \times (0, \varepsilon]$$

Furthermore the sequence $\mathcal{P}^n(x)$ converges to a point $x_\infty \in \mathrm{Col}(X, Y, U) \cap \Sigma_0$ and we have

- $\nu(x_\infty) = 0$
- $\mu(x_\infty) - \mu(x) \leqslant \frac{\nu(x)}{100} \leqslant \frac{\varepsilon}{100}$.

The map $x \mapsto x_\infty$ is continuous.

Proof. — Consider the trapezium D (in the (μ, ν) coordinates) whose vertices are $(-\varepsilon, 0)$, $(\varepsilon, 0)$, $(-\frac{9\varepsilon}{10}, \varepsilon)$, and $(\frac{9\varepsilon}{10}, \varepsilon)$. This trapezium D is contained in $[-\varepsilon, \varepsilon] \times [0, \varepsilon]$ and contains $[-\frac{9}{10}\varepsilon, \frac{9}{10}\varepsilon] \times [0, \varepsilon]$. Thus for proving the first item it is enough to check that D is invariant under $\mathcal{P}$. For that notice that, for any x with $(\mu(x), \nu(x)) \in [-\varepsilon, \varepsilon] \times [0, \varepsilon]$ one has that $(\mu(\mathcal{P}(x)), \nu(\mathcal{P}(x))$ belongs to the triangle $\delta(x)$ whose vertices are $(\mu(x), \nu(x))$, $(\mu(x) - \frac{\nu(x)}{100}, 0)$, $(\mu(x) + \frac{\nu(x)}{100}, 0)$ (according to Equation 6.2 and the fact that $\nu(\mathcal{P}(x)) < \nu(x)$); one conclude by noticing that, if $(\nu(x), \mu(x))$ belongs to D then $\delta(x) \subset D$.

Let us show that $\mathcal{P}^n(x)$ converges.

The sequence $\nu(\mathcal{P}^n(x))$ is positive and decreasing, hence converges, and

$$\sum |\nu(\mathcal{P}^{n+1}(x)) - \nu(\mathcal{P}^n(x))|$$

converges. As $|\mu(\mathcal{P}^{n+1}(x)) - \mu(\mathcal{P}^n(x))| < \frac{1}{100}|\nu(\mathcal{P}^{n+1}(x)) - \nu(\mathcal{P}^n(x))|$ one deduces that the sequence $\{\mu(\mathcal{P}^n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence, hence converges.

The continuity of $x \mapsto x_\infty$ follows from the inequality $\mu(x_\infty) - \mu(x) \leqslant \frac{\nu(x)}{100}$ applied to $\mathcal{P}^n(x)$ with n large, so that $\nu(\mathcal{P}^n(x))$ is very small, and from the continuity of $x \mapsto \mathcal{P}^n(x)$. $\qquad\square$

For any point y with $(\mu(y), \nu(y)) \in [-\frac{8}{10}\varepsilon, \frac{8}{10}\varepsilon] \times \{0\}$ the *stable set of* y, which we denote by $S(y)$, is the the union of $\{y\}$ with the set of points $x \in \Sigma_+$ with $(\mu(x), \nu(x)) \in [-\frac{9}{10}\varepsilon, \frac{9}{10}\varepsilon] \times (0, \varepsilon]$ so that $x_\infty = y$. The continuity of the map $x \mapsto x_\infty$ implies the following remark:

Remark 6.5. — For any point y with $(\mu(y), \nu(y)) \in [-\frac{8}{10}\varepsilon, \frac{8}{10}\varepsilon] \times \{0\}$, $S(y)$ is a compact set which has a non-empty intersection with the horizontal lines $\{x, \nu(x) = t\}$ for every $t \in (0, \varepsilon]$.

If E is a subset of $\mathrm{Col}\,(X, Y, U) \cap \Sigma_0$ so that $\mu(y) \in [-\frac{8}{10}\varepsilon, \frac{8}{10}\varepsilon]$ for $y \in E$ one denotes

$$S(E) = \bigcup_{y \in E} (S(y)).$$

Lemma 6.6. — *Let $I \subset \mathrm{Col}\,(X, Y, U)$ be the open interval $(\mu(x), \nu(x)) \in (-\frac{1}{2}\varepsilon, \frac{1}{2}\varepsilon) \times \{0\}$. Consider the quotient space Γ of $S(I) \setminus I$ by the dynamics. Then Γ is a C^1-connected surface diffeomorphic to a cylinder $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$.*

Proof. — Consider the compact triangle whose end points are $(-\frac{\varepsilon}{2}, 0)$, $(0, \frac{\varepsilon}{10})$ and $(+\frac{\varepsilon}{2}, 0)$. Let $\bar{\Delta}$ be its preimage by (μ, ν).

$\bar{\Delta}$ is a triangle with one side on $\mathrm{Col}(X, Y, U)$. Let $\Delta = \bar{\Delta} \setminus \mathrm{Col}(X, Y, U)$. We denote by $\partial\Delta$ the union of the two other sides.

As the vectors directing the two other sides have a first coordinated larger than the second (in other words, they are more horizontal than vertical) and as the vectors $[x, \mathcal{P}(x)]$ are almost vertical, one deduces that Δ is a trapping region for $\mathcal{P}$:

$$x \in \Delta \Longrightarrow \mathcal{P}(x) \in \Delta.$$

Now

$$\mathcal{P}(\partial\Delta)$$

is a curve contained in the interior of Δ and joining the vertex $(-\frac{\varepsilon}{2}, 0)$ to the vertex $(+\frac{\varepsilon}{2}, 0)$. Thus $\partial\Delta$ and $\mathcal{P}(\partial\Delta)$ bound a strip homeomorphic to $[0, 1] \times \mathbb{R}$ in Δ.

Let Γ be the cylinder obtained from this strip by gluing $\partial\Delta$ with $\mathcal{P}(\partial\Delta)$ along $\mathcal{P}$.

It remains to check that Γ is the quotient space of $S(I) \setminus I$ by $\mathcal{P}$. For that, one just remark that the orbit of every point in $S(I) \setminus I$ has a unique point in the strip, unless in the case where the orbits meets $\partial\Delta$: in that case the orbits meets the strip twice, the first time on $\partial\Delta$, the second on $\mathcal{P}(\partial\Delta)$.

Note that Γ is the quotient of an open region in a smooth surface by a diffeomorphism (and the action is proper and free) so that Γ is not only homeomorphic to a cylinder but diffeomorphic to a cylinder. $\square$

Let us end this section by stating important straighforward consequences of the invariance of the vector field N under $\mathcal{P}$.

Remark 6.7.

- For every $y \in \mathrm{Col}(X, Y, U)$ with $\mu(y) \in [-\frac{8}{10}\varepsilon, \frac{8}{10}\varepsilon]$, the stable set $S(y)$ is invariant under the flow of N.
- The vector field N induces a vector field, denoted by N_Γ, on the quotient space Γ. As Γ is a C^1 surface, N_Γ is only C^0. However, it defines a flow on Γ which is the quotient by $\mathcal{P}$ of the flow of N.
- The continuous map $x \mapsto x_\infty$ is invariant under $\mathcal{P}$ and therefore induces on Γ a continuous map $\Gamma \to (-\varepsilon/2, \varepsilon/2)$, and x_∞ tends to $-\varepsilon/2$ when x tends to one end of the cyclinder Γ and to $\varepsilon/2$ when x tends to the other end. This implies that, for every $t \in (-\varepsilon/2, \varepsilon/2)$ the set of points $x \in \Gamma$ for which $x_\infty = t$ is compact. Recall that this set is precisely the projection on Γ of $S(y)$ where $y \in \mathrm{Col}(X, Y, U)$ satisfies $(\mu(y), \nu(y)) = (t, 0)$.
- The vector field N_Γ on Γ leaves invariant the levels of the map $x \mapsto x_\infty$. As a consequence, every orbit of N_Γ is bounded in Γ.

One deduces

LEMMA 6.8. — *For every $y \in \mathrm{Col}(X, Y, U)$ with $\mu(y) \in [-\frac{8}{10}\varepsilon, \frac{8}{10}\varepsilon]$, the stable set $S(y) \setminus \{y\}$ contains an orbit of N which is invariant under $\mathcal{P}$. In particular, there exists x in $\Delta \cap S(x_0)$ whose orbit by N is invariant under $\mathcal{P}$.*

Proof. — A Poincaré–Bendixon argument implies that, for every flow on the cylinder $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$ without fixed points, for every bounded orbit the ω-limit set is a periodic orbit. Furthermore this periodic orbit is not homotopic to a point (otherwise it bounds a disc containing a zero).

Since N_Γ has no zeros, one just applies this argument to the flow of it restricted to the a level of the map $x \mapsto x_\infty$. The level contains a periodic

orbit which is not homotopic to 0, hence corresponds to an orbit of N joining a point in $S(y)$ to is image under $\mathcal{P}$. This orbit of N is invariant under $\mathcal{P}$, concluding. $\square$

End of the proof of Theorem A: the vector field N does not rotate along a $\mathcal{P}$-invariant orbit of N. From now on, $(U, X, Y, \Sigma, \mathcal{B})$ is a prepared counter example to Theorem A with $\ell^+(X,Y) \neq 0$. According to Proposition 5.19 the derivative $D\mathcal{P}(x)$ is the identity map for every $x \in \mathrm{Col}\,(X, Y, U) \cap \Sigma_0$.

According to Lemma 6.8, there is a point x in the stable set $S(x_0)$ whose N-orbit is invariant under $\mathcal{P}$.

LEMMA 6.9. — *If $y \in [\mathcal{P}^n(x), \mathcal{P}^{n+1}(y)]$ then the angular variation of the vector $N(y)$ tends to 0 when $n \to +\infty$.*

Before proving Lemma 6.9 let us conclude the proof of Theorem A.

Proof of Theorem A. — Lemma 6.9 is in contradiction with Corollary 5.14, which asserts that the angular variation of the vector N along any segment $[z, \mathcal{P}^2(z)]$ for $z \in \Sigma_+$ close enough to x_0 is larger than 2π. This contradiction ends the proof of Theorem A. $\square$

It remains to prove Lemma 6.9. First, notice that the angular variation of N along a segment of curve is invariant under homotopies of the curve preserving the ends points. Therefore Lemma 6.9 is a straighforward consequence of next lemma:

LEMMA 6.10. — *The angular variation of the vector $N(y)$ along the N-orbit segment joining $\mathcal{P}^n(x)$ to $\mathcal{P}^{n+1}(x)$ tends to 0 when $n \to +\infty$.*

As N is (by definition) tangent to the N-orbit segment joining $\mathcal{P}^n(x)$ to $\mathcal{P}^{n+1}(x)$, its angular variation is equal to the angular variation of the unit tangent vector to this orbit segment.

Proof of Lemma 6.10: the tangent vector to a $\mathcal{P}$-invariant embedded curve do not rotate.

Remark 6.11. — For n large enough the point $\mathcal{P}^n(x)$ belongs to the region Δ defined in the previous section, and whose quotient by the dynamics $\mathcal{P}$ is the cyclinder Γ. Then,

- any continuous curve γ in Δ joining $\mathcal{P}^n(x)$ to $\mathcal{P}^{n+1}(x)$ induces on Γ a closed curve, homotopic to the curve induced by the N-orbit segment joining $P^n(x)$ to $\mathcal{P}^{n+1}(x)$.
- The curve induced by γ is a simple curve if and only if γ is simple and disjoint from $\mathcal{P}^i(\gamma)$ for any $i > 0$.

- If the curve γ is of class C^1, the projection will be of class C^1 if and only if the image by $\mathcal{P}$ of the unit vector tangent to γ at $\mathcal{P}^n(x)$ is tangent to γ (at $\mathcal{P}^{n+1}(x)$).
- on the cylinder, any two C^1-embbedding σ_1, σ_2 of the circle, so that $\sigma_1(0) = \sigma_2(0)$ are isotopic through C^1-embeddings σ_t with $\sigma_t(0) = \sigma_i(0)$.

We consider $\mathcal{I}(\Delta, \mathcal{P})$ as being the set of C^1-immersed segment I in $\Delta \setminus \partial \Delta$, so that:

- if y, z are the initial and end points of I then $z = \mathcal{P}(y)$
- if u is a vector tangent to I at y then $\mathcal{P}_*(u)$ is tangent to I (and with the same orientation.

In other words, $I \in \mathcal{I}(\Delta, \mathcal{P})$ if the projection of I on Γ is a C^1 immersion of the circle, generating the fundamental group of the cylinder. We endow $\mathcal{I}(\Delta, \mathcal{P})$ with the C^1-topology.

We denote by $Var(I) \in \mathbb{R}$ the angular variation of the unit tangent vector to I along I. In other words, for $I \colon [0, 1] \to \Delta$, consider the unit vector

$$\mathring{I}(t) = \frac{dI(t)/dt}{\|dI(t)/dt\|} \in \mathbb{S}^1 \simeq \mathbb{R}/2\pi\mathbb{Z}.$$

One can lift $\mathring{I}$ is a continuous map $\dot{I} \colon [0, 1] \to \mathbb{R}$. Then

$$Var(I) = \dot{I}(1) - \dot{I}(0),$$

this difference does not depend on the lift.

The map $Var \colon \mathcal{I}(\Delta, \mathcal{P}) \to \mathbb{R}$ is continuous. Let $\widetilde{Var}(I) \in \mathbb{R}/2\pi\mathbb{Z}$ be the projection of $Var(I)$. In other words, $\widetilde{Var}(I)$ is the angular variation modulo 2π.

Remark 6.12. — Let $I_n \in \mathcal{I}(\Delta, \mathcal{P})$ be a sequence of immersed segments such that $I_n(0)$ tends to $x_0 \in \Sigma$. Then $\widetilde{Var}(I_n)$ tends to 0.

Indeed, since $D\mathcal{P}(I_n(0))$ tends to the identity map, the angle between the tangent vectors to I_n at $I_n(1)$ and $I_n(0)$ tends to 0.

As a consequence of Remark 6.12, we get the following lemma:

LEMMA 6.13. — *There is a neighborhood O of x_0 in Σ so that to any $I \in \mathcal{I}(\Delta, \mathcal{P})$ with $I(0) \in O$ there is a (unique) integer $[var](I) \in \mathbb{Z}$ so that*

$$Var(I) - 2\pi[var](I) \in \left[-\frac{1}{100}, \frac{1}{100}\right].$$

Furthermore, the map $I \mapsto [var](I)$ is locally constant in $\mathcal{I}(\Delta, \mathcal{P})$, hence constant under homotopies in $\mathcal{I}(\Delta, \mathcal{P})$ keeping the initial point in O.

As a consequence we get

LEMMA 6.14. — *If I and J are segments in $\mathcal{I}(\Delta, \mathcal{P})$ with the same intial point in O and whose projections on the cylinder Γ are simple closed curves, then*

$$[var](I) = [var](J).$$

Proof. — Since the projections of I and J are simple curves which are not homotopic to a point in the cylinder Γ, the projections of I and J are isotopic on Γ by an isotopy keeping the initial point. One deduces that I and J are homotopic through elements $I_t \in \mathcal{I}(\Delta, \mathcal{P})$ with the same initial point. Indeed, the isotopy on Γ between the projection of I and J can be lifted to the universal cover of Γ. This universal cover is diffeomorphic to a plane $\mathbb{R}^2$, in which $\Delta \setminus \partial\Delta$ is an half plane (bounded by two half lines). There is a diffeomorphism of $\mathbb{R}^2$ to $\Delta \setminus \partial\Delta$ which is the indentity on $I \cup J$. The image of the lifted isotopy induces the announced isotopy through elements in $\mathcal{I}(\Delta, \mathcal{P})$. Now, as $[var](I_t)$ is independent of t, one concludes that $[var](I) = [var](J)$. $\square$

Now Lemma 6.10 is a consequence of Lemma 6.14 and of the following lemma:

LEMMA 6.15. — *For any $n > 0$ large enough there is a curve $I_n \in \mathcal{I}(\Delta, \mathcal{P})$ whose initial point is $\mathcal{P}^n(x)$ and such that:*

- *the projection of I_n on Γ is a simple curve*
- *$[var](I_n) = 0$.*

End of the proof of Lemma 6.10. — The N-orbit segment J_n joining $\mathcal{P}^n(x)$ to $\mathcal{P}^{n+1}(x)$ belongs to $\mathcal{I}(\Delta, \mathcal{P})$ and its projection on Γ is a simple curve. Hence Lemma 6.14 asserts that, for n large enough, $[var](J_n) = [var](I_n) = 0$ where I_n is given by Lemma 6.15.

Now, when n tends to infinity, $Var(J_n) - 2\pi[var](J_n)$ tends to 0 (according to Remark 6.12), that is, $Var(J_n)$ tends to 0. This is precisely the statement of Lemma 6.10. $\square$

Proof of Lemma 6.15. — As n tends to $+\infty$ the derivative of $\mathcal{P}$ at $\mathcal{P}^n(x)$ tends to the identity map. Thus, the segment $[\mathcal{P}^n(x), \mathcal{P}^{n+1}(x)]$ may fail to belong to $\mathcal{I}(\Delta, \mathcal{P})$ only by a very small angle between v_n and $D\mathcal{P}(v_n)$, where v_n is the unit vector directing $[\mathcal{P}^n(x), \mathcal{P}^{n+1}(x)]$. Therefore, one easily builds an arc I_n joining $\mathcal{P}^n(x)$ to $\mathcal{P}^{n+1}(x)$, whose derivative at $\mathcal{P}^n(x)$ is v_n and its derivative at $\mathcal{P}^{n+1}(x)$ is $D\mathcal{P}(v_n)$ and whose derivative at any point of I_n belongs to an arbitrarily small neighborhood of v_n. In particular $I_n \in \mathcal{I}(\Delta, \mathcal{P})$ and $[var](I_n) = 0$ for n large. In order to complete the proof, it remains to show that:

CLAIM 7. — *For n large enough I_n projects on Γ as a simple curve.*

Proof. — We need to prove that for n large enough and for any $i > 1$, $I_n \cap \mathcal{P}^i(I_n) = \emptyset$ and $I_n \cap \mathcal{P}(I_n)$ is a singleton (the endpoint of I_n which is the image of its initial point).

Indeed, it is enough to prove that, for any $y \in I_n$ different from the initial point,

$$\nu(\mathcal{P}(y)) < \inf_{z \in I_n} \nu(z).$$

As the action of $\mathcal{P}$ consists in lowering down the value of ν, the further iterates cannot cross I_n.

For proving that, notice that the vectors tangent to I_n are very close to v_n which is uniformly (in n large) transverse to the levels of ν. As $D\mathcal{P}(y)$ tends to the identity map when y tends to 0, for n large, the vectors tangent to $\mathcal{P}(I_n)$ are also transverse to the levels of ν. Hence $\mathcal{P}(I_n)$ is a segment starting at the end point of I_n (which realizes the infimum of ν on I_n) and ν is strictly decreasing along $\mathcal{P}(I_n)$, concluding. □

This ends the proof of Lemma 6.15 (and so of Theorem A). □

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