ANNALES

DE

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On the varieties of the second row of the split Freudenthal–Tits Magic Square

ON THE VARIETIES OF THE SECOND ROW OF THE SPLIT FREUDENTHAL–TITS MAGIC SQUARE

by Jeroen SCHILLEWAERT & Hendrik VAN MALDEGHEM (*)

ABSTRACT. — Our main aim is to provide a uniform geometric characterization of the analogues over arbitrary fields of the four complex Severi varieties, i.e. the quadric Veronese varieties in 5-dimensional projective spaces, the Segre varieties in 8-dimensional projective spaces, the line Grassmannians in 14-dimensional projective spaces, and the exceptional varieties of type E_6 in 26-dimensional projective space. Our theorem can be regarded as a far-reaching generalization of Mazzocca and Melone's approach to finite quadric Veronesean varieties. This approach uses combinatorial analogues of smoothness properties of complex Severi varieties as axioms.

RÉSUMÉ. — Le but principal de cet article est de fournir une caractérisation géométrique des analogues sur les corps quelconques des quatre variétés complexes de Severi, c'est-à-dire la surface de Veronese, la variété de Segre $S(2,2)$, la grassmannienne $G(2,6)$ et la variété exceptionnelle de type E_6. Notre théorème peut être vu comme une généralisation considérable de l'approche de Mazzocca et Melone pour les surfaces de Veronese sur les corps finis. Cette approche utilise des analogues combinatoires de certaines propriétés, qui expriment que les variétés de Severi complexes sont lisses, comme axiomes.

1. Introduction

1.1. General context of the results

In the mid fifties, Jacques Tits [29] found a new way to approach complex Lie groups by attaching a canonically defined abstract geometry to it. The

main observation was that the Dynkin diagram related to the simple complex Lie algebra could be interpreted as an identity card for this geometry. This led Tits to define such geometries over arbitrary fields giving birth to the theory of *buildings*. In the same work [29], Jacques Tits introduced for the first time what Freudenthal would call much later the *Magic Square*, which was in [29] a (4×4)-table of Dynkin diagrams, each symbolizing a precise geometry that was constructed before, except for the very last entry (the E_8-entry), for which Tits only made some conjectures regarding various dimensions. All these geometries were defined over the complex numbers, but one could easily extend the construction to arbitrary fields, although small characteristics would give some problems. It is essential to note that these geometries were constructed as subgeometries of a projective geometry, and not as abstract geometries. As such, they can be regarded as a kind of realization of the corresponding building (which is the abstract geometry). The geometries of the Magic Square, later better known as the *Freudenthal–Tits Magic Square* [31], were intensively investigated with tools from algebraic geometry, since they define smooth varieties in complex projective space. One prominent example of this, which is directly related to the present paper, is the classification of the complex Severi varieties by Zak [35], see also Chaput [5] and Lazarsfeld & Van de Ven [16]. It turns out that the complex Severi varieties correspond exactly to the split geometries of the second row of the Freudenthal–Tits Magic Square (FTMS), namely the quadric Veronese varieties in 5-dimensional projective spaces, the Segre varieties in 8-dimensional projective spaces, the line Grassmannians in 14-dimensional projective spaces, and the exceptional varieties of type E_6 in 26-dimensional projective spaces. For a recent approach with K-theory, see Nash [20]. In the present paper we present a way to approach the geometries of the second row of the FTMS *over any field*. The main idea can be explained by both a bottom-up approach and a top-down approach.

- **Bottom-up.** The smallest complex Severi variety or, equivalently, the geometry of the first cell of the second row of the complex FTMS, is the Veronesean of all conics of a complex projective plane. This object can be defined over any field, and a characterization of the finite case by Mazzocca–Melone [18] was achieved in the mid-eighties. This characterization was generalized to arbitrary fields by the authors [23]. Here the question is whether a further generalization is possible by considering "quadrics" in the Mazzocca–Melone axioms instead of conics. A first step was made in [22, 25], where all

quadrics in 3-space which are not the union of two planes are considered, and the corresponding objects are classified. The axioms below generalize this further to all dimensions, using split quadrics. Hence, our results can be seen as a far-reaching generalization of Mazzocca & Melone's characterization of the quadric Veronesean in 5-dimensional finite projective space.

- **Top-down.** Here, the question is how to include the (split) geometries of the second row of the FTMS, defined over arbitrary fields in Zak's result. Thus, one would like to have a characteristic property of these point sets in projective space close to the requirement of being a Severi variety (which, roughly, just means that the secant variety of the smooth non-degenerate complex variety is not the whole space, and the dimension of the space is minimal with respect to this property). The naturalness of the Mazzocca–Melone axioms alluded to above is witnessed by the fact that they reflect the properties of complex Severi varieties used by Chaput [5] and by Lazarsfeld & Van de Ven [16] to give an alternative proof of Zak's result. Note that, indeed, Zak proves (see [35, Thm. IV.2.4]) that every pair of points of a $2n$-dimensional complex Severi variety is contained in a non-degenerate n-dimensional quadric (and no more points of the variety are contained in the space spanned by that quadric). Also, the spaces generated by two of these quadrics intersect in a space entirely contained in both quadrics. These are the (Mazzocca–Melone) properties that we take as axioms, together with the in the smooth complex case obvious fact that the tangent space at a point is (at most) $2n$-dimensional. The latter is achieved by requiring that the tangent spaces to the quadrics through a fixed point are contained in a $2n$-dimensional space. Remarkably, we show in the present paper that these requirements suffice to classify these "point sets".

So it is interesting to see how these two points of view meet in our work. It remains to explain the choice of which kind of quadrics. We consider the same class as Zak was dealing with in the complex case: split quadrics (as the complex numbers are algebraically closed). Hence we consider so-called *hyperbolic* quadrics in odd-dimensional projective space, and so-called *parabolic* quadrics in even-dimensional projective space. This way, Zak's result follows entirely from ours, once the immediate consequences of the definition of a Severi variety are accepted.

The complexity and generality of the characterization (remember we work over arbitrary fields) does not allow a uniform proof dealing with all dimensions at the same time (where dimension d means that $d + 1$ is the dimension of the ambient projective space of the quadrics; we do not assume anything on the dimension of the whole space). Some small cases for d, however, have already been dealt with in the literature in other contexts: The case $d = 1$, leading to the quadric Veronesean varieties in [23], the case $d = 2$, leading to the Segre varieties in [25], and the case $d = 3$, leading to nonexistence, in [26]. We consider it the main achievement of this paper to present a general and systematic treatment of the remaining cases, including the intrinsically more involved line-Grassmannians and exceptional varieties of type E_6.

We obtain more than merely a characterization of all geometries of the second row of the FTMS. Indeed, in contrast to Zak's original theorem, we do not fix the dimension of the space we are working in. This implies that we obtain some more "varieties" in our conclusion. Essentially, we obtain all subvarieties of the varieties of the second row of the split Freudenthal–Tits Magic Square that are controlled by the diagram of the corresponding building.

Note that we do not attempt to generalize Zak's result to arbitrary fields in the sense of being secant defective algebraic varieties. Such a result would require scheme theory and does not fit into our general approach to the Freudenthal–Tits Magic Square.

Let us mention that there is a non-split version of the second row of the FTMS, which consists of the varieties corresponding to the projective planes defined over quadratic alternative *division* algebras. These objects satisfy the same Mazzocca–Melone axioms which we will introduce below, except that the quadrics are not split anymore, but on the contrary have minimal Witt index, namely Witt index 1. It is indeed shown in [15] that point sets satisfying the Mazzocca–Melone axioms below, but for quadrics of Witt index 1, are precisely the Veronesean representation of the projective planes over composition division algebras (and this includes the rather special case of inseparable field extensions in characteristic 2). In fact, this leads to an even more daring conjecture which, loosely, is the following.

(∗)   *The split and non-split varieties related to the second row of the Freudenthal–Tits Magic Square are characterized by the Mazzocca–Melone axioms, using arbitrary nondegenerate quadrics.*

Conjecture (∗) thus consists of three parts, the *split* case, which uses split quadrics, the *non-split* case, using quadrics of Witt index 1, and the

intermediate case, where one uses quadrics which have neither maximal nor minimal Witt index. If Conjecture (∗) is true, then the intermediate case does not occur. The other cases are now proved.

1.2. The varieties of the second row of the split FTMS

We now discuss the different geometries that are captured by the second row of the FTMS. The split case of the second row of the Freudenthal–Tits Magic Square (FTMS) can be seen as the family of "projective planes" coordinatized by the standard split composition algebras $\mathbb{A}$ over an arbitrary field $\mathbb{K}$. These algebras are $\mathbb{K}$, $\mathbb{K} \times \mathbb{K}$, $\mathsf{M}_{2\times 2}(\mathbb{K})$ and $\mathbb{O}'(\mathbb{K})$, which are the field $\mathbb{K}$ itself, the split quadratic extension of $\mathbb{K}$ (direct product with component-wise addition and multiplication), the split quaternions (or the algebra of 2×2 matrices over $\mathbb{K}$), and the split octonions, respectively. Each of these cases corresponds with a different cell in the second row of the FTMS.

The first cell corresponds with $\mathbb{A} = \mathbb{K}$ and contains the ordinary quadric Veronese variety of the standard projective plane over $\mathbb{K}$, as already mentioned above. Mazzocca and Melone [18] characterized this variety in the finite case for fields of odd characteristic using three simple axioms, which we referred to as the *Mazzocca–Melone axioms* above. They can be formulated as follows. Let X be a spanning point set of a projective space, and Ξ a family of planes such that $X \cap \xi$ is a conic for each $\xi \in \Xi$. Then Axiom 1 says that every pair of points of X is contained in a member of Ξ; Axiom 2 says that the intersection of two members of Ξ is entirely contained in X; Axiom 3 says that for given $x \in X$, the tangents at x to the conics $X \cap \xi$ for which $x \in \xi$ are all contained in a common plane. This characterization has been generalized step-by-step by various authors [10, 28], until the present authors proved it in full generality for Veronese surfaces over arbitrary fields [23] (where in the above axioms "conics" can be substituted with "ovals").

The second cell corresponds with $\mathbb{A} = \mathbb{K} \times \mathbb{K}$ and contains the ordinary Segre variety $\mathcal{S}_{2,2}(\mathbb{K})$ of two projective planes. In [25], the authors characterize this variety using the above Mazzocca–Melone axioms substituting "conic" with "hyperbolic quadric in 3-dimensional projective space", "family of planes" with "family of 3-spaces", and in Axiom 3 "common plane" with "common 4-space", and using a dimension requirement to exclude $\mathcal{S}_{1,2}(\mathbb{K})$ and $\mathcal{S}_{1,3}(\mathbb{K})$.

In the present paper, we consider the natural extension of these axioms using "quadric of maximal Witt index (or "split" quadric) in $(d+1)$-dimensional space", "family of $(d+1)$-dimensional spaces", and "common $2d$-space", instead of "conic", "family of planes" and "common 4-space", respectively. The first cell corresponds with $d = 1$, the second with $d = 2$. Sets satisfying these axioms will be referred to as *Mazzocca–Melone sets of split type d*. It will turn out that the third cell corresponds with $d = 4$ and the fourth with $d = 8$.

The third cell corresponds with $\mathbb{A} = \mathsf{M}_{2 \times 2}(\mathbb{K})$ and contains the line Grassmannian variety of projective 5-space over $\mathbb{K}$. The fourth cell corresponds with a split octonion algebra. The corresponding geometry is the 1-shadow of the building of type E_6 over $\mathbb{K}$. In order to handle the latter case, we need an auxiliary result on varieties related to the half-spin geometries of buildings of type D_5. In fact, that result corresponds with $d = 6$. In short, our first Main Result states that Mazzocca–Melone sets of split type d only exist for $d \in \{1, 2, 4, 6, 8\}$, and a precise classification will be given in these cases (containing the varieties mentioned above). As a corollary, we can single out the varieties of the second row of the FTMS by a condition on the dimension (namely, the same dimensions used by Zak).

2. Mazzocca–Melone sets of split type

2.1. The axioms

We now present the precise axioms and in the next subsection the main examples. Then, in the next section, we can state our main results.

In the definition of our main central object, the notion of a quadric plays an important role. A quadric in a projective space is the null set of a quadratic form. This is an analytic definition. However, we insist on using only projective properties of quadrics as subsets of points of a projective space. One of the crucial notions we use is the tangent subspace $T_x(Q)$ at a point x of a quadric Q. This can be defined analytically, leaving the possibility that different quadratic forms (different up to a scalar multiple) could define the same point set, or that the tangent space would depend on the chosen basis, which would imply that the tangent space is not well defined. However, it is well known, and very easy to prove, that a point $y \neq x$ belongs to the analytically defined tangent space at x to Q if and only if the line $\langle x, y \rangle$ either meets Q in exactly one point (namely, x), or fully belongs to Q. Alternatively, a point z does not belong to the tangent

space at x if and only if the line $\langle x, y \rangle$ intersects Q in precisely 2 points. So $T_x(Q)$ is defined by the point set Q.

Now let X be a spanning point set of $\mathbb{P}^N(\mathbb{K})$, $N \in \mathbb{N} \cup \{\infty\}$, with $\mathbb{K}$ any field, and let Ξ be a collection of $(d + 1)$-spaces of $\mathbb{P}^N(\mathbb{K})$, $d \geqslant 1$, such that, for any $\xi \in \Xi$, the intersection $\xi \cap X =: X(\xi)$ is a non-singular split quadric (which we will call a *symp*, inspired by the theory of parapolar spaces, see [27]; for $d = 1$, we sometimes say *conic*) in ξ (and then, for $x \in X(\xi)$, we denote the tangent space at x to $X(\xi)$ by $T_x(X(\xi))$ or sometimes simply by $T_x(\xi)$). We call (X, Ξ) a *Mazzocca–Melone set (of split type d)* if (MM1), (MM2) and (MM3) below hold.

The condition $d \geqslant 1$ stems from the observation that, if we allowed $d \in \{-1, 0\}$, then we would only obtain trivial objects (for $d = -1$, a single point; for $d = 0$ a set of points no 3 of which collinear and no 4 of which co-planar).

A Mazzocca–Melone set is called *proper* if $|\Xi| > 1$. Non-proper Mazzocca–Melone sets of split type are just the split quadrics themselves (we use the expression "of split type" to leave the exact number d undetermined, but still to indicate that the quadrics we use are split). Also, the members of Ξ are sometimes called *the quadratic spaces*.

(MM1) Any pair of points x and y of X lies in at least one element of Ξ, denoted by $[x, y]$ if unique.

(MM2) If $\xi_1, \xi_2 \in \Xi$, with $\xi_1 \neq \xi_2$, then $\xi_1 \cap \xi_2 \subset X$.

(MM3) If $x \in X$, then all d-spaces $T_x(\xi)$, $x \in \xi \in \Xi$, generate a subspace T_x of $\mathbb{P}^N(\mathbb{K})$ of dimension at most $2d$.

The central problem of this paper is to classify all Mazzocca–Melone sets of split type d, for all $d \geqslant 1$. We will state this classification in Section 3, after we have introduced the examples in the next section.

We conclude by mentioning that the first case of non-existence, namely $d = 3$, was proved in [26] in the context of the first cell of the third row of the FTMS. So we can state the following proposition.

PROPOSITION 2.1. — *A proper Mazzocca–Melone set of split type 3 does not exist.*

Proof. — This is basically Proposition 5.4 of [26]. That proposition states that no Lagrangian set of diameter 2 exists. A Lagrangian set (X, Ξ) satisfies the same axioms as a Mazzocca–Melone set of split type 3, except that (MM1) is replaced by (MM1″) below. The diameter of a Lagrangian set is defined as the diameter of the following graph Γ. The vertices are

the elements of X and two vertices are adjacent if they are contained in a singular line of X (a line fully contained in X).

(MM1$''$) Any pair of points x and y of X at distance at most 2 in the graph Γ is contained in at least one element of Ξ.

By Axiom (MM1), any Mazzocca–Melone set of split type 3 has diameter 2 and hence is a Lagrangian set of diameter 2, which does not exist by Proposition 5.4 of [26]. $\qquad\square$

As a side remark, we note that a similar conjecture as above for the third row can be stated, supported by a recent result of De Bruyn and the second author [8]. Some other work that is related to our axiomatic approach, but still with a restriction on the underlying field, has been carried out by Russo [21] and by Ionescu & Russo [12, 13].

2.2. Examples of Mazzocca–Melone sets of split type

We define some classes of varieties over the arbitrary field $\mathbb{K}$. Each class contains Mazzocca–Melone sets of split type. This section is meant to introduce the notation used in the statements of our results.

Quadric Veronese varieties. The *quadric Veronese variety* $\mathcal{V}_n(\mathbb{K})$, $n \geqslant 1$, is the set of points in $\mathbb{P}^{\binom{n+2}{2}-1}(\mathbb{K})$ obtained by taking the images of all points of $\mathbb{P}^n(\mathbb{K})$ under the Veronese mapping, which maps the point $(x_0, \ldots, x_n)$ of $\mathbb{P}^n(\mathbb{K})$ to the point $(x_i x_j)_{0 \leqslant i \leqslant j \leqslant n}$ of $\mathbb{P}^{\binom{n+2}{2}-1}(\mathbb{K})$. If $\mathbb{K} = \mathbb{C}$, then this is a smooth non-degenerate complex algebraic variety of dimension n.

Line Grassmannian varieties. The *line Grassmannian variety*, denoted by $\mathcal{G}_{m,1}(\mathbb{K})$, $m \geqslant 2$, of $\mathbb{P}^m(\mathbb{K})$ is the set of points of $\mathbb{P}^{\frac{m^2+m-2}{2}}(\mathbb{K})$ obtained by taking the images of all lines of $\mathbb{P}^m(\mathbb{K})$ under the Plücker map

$$\rho(\langle (x_0, x_1, \ldots, x_m), (y_0, y_1, \ldots, y_m)\rangle) = \left(\left| \begin{array}{cc} x_i & x_j \\ y_i & y_j \end{array} \right|\right)_{0 \leqslant i < j \leqslant m}.$$

If $\mathbb{K} = \mathbb{C}$, then this is a smooth non-degenerate complex algebraic variety of dimension $2m - 2$.

Segre varieties. The *Segre variety* $\mathcal{S}_{k,\ell}(\mathbb{K})$ of $\mathbb{P}^k(\mathbb{K})$ and $\mathbb{P}^\ell(\mathbb{K})$ is the set of points of $\mathbb{P}^{k\ell+k+\ell}(\mathbb{K})$ obtained by taking the images of all pairs of points, one in $\mathbb{P}^k(\mathbb{K})$ and one in $\mathbb{P}^\ell(\mathbb{K})$, under the Segre map

$$\sigma(\langle (x_0, x_1, \ldots, x_k), (y_0, y_1, \ldots, y_\ell)\rangle) = (x_i y_j)_{0 \leqslant i \leqslant k; 0 \leqslant j \leqslant \ell}.$$

If $\mathbb{K} = \mathbb{C}$, then this is a smooth non-degenerate complex variety of dimension $k + \ell$.

Varieties from split quadrics. Every split quadric is by definition a variety, which we will refer to as the *variety corresponding to a split quadric*. The ones corresponding to a *parabolic quadric* in $\mathbb{P}^{2n}(\mathbb{K})$ will be denoted by $\mathcal{B}_{n,1}(\mathbb{K})$; the ones corresponding to a *hyperbolic quadric* in $\mathbb{P}^{2n-1}(\mathbb{K})$ by $\mathcal{D}_{n,1}(\mathbb{K})$.

Half-spin varieties. The exposition below is largely based on [11], but the results are due to Chevalley [6], see also the recent reference [17].

Let V be a vector space of dimension $2n$ over a field $\mathbb{K}$ with a nonsingular quadratic form q of maximal Witt index giving rise to a hyperbolic quadric and $(\,\cdot\,,\cdot\,)$ the associated bilinear form. Then the maximal singular subspaces of V with respect to q have dimension n and fall into two classes Σ^+ and Σ^- so that two subspaces have an intersection of even codimension if and only if they are of the same type.

Then we define the half spin geometry $\mathcal{D}_{n,n}(\mathbb{K})$ as follows. The point set $\mathcal{P}$ is the set of maximal totally singular subspaces of V of one particular type, say $+$. For each totally singular $(n-2)$-space U, form a line by taking all the points containing U. The line set $\mathcal{L}$ is the collection of sets of this form.

Fix a pair of maximal isotropic subspaces U_0 and U_∞ such that $V = U_0 \oplus U_\infty$. Let $S = \bigwedge U_\infty$ be the exterior algebra of U_∞, called the *spinor space of (V, q)* and the even and odd parts of S are called the *half-spinor spaces*, $S^+ = \bigwedge^{\text{even}} U_\infty$, $S^- = \bigwedge^{\text{odd}} U_\infty$.

To each maximal isotropic subspace $U \in \Sigma^+ \cup \Sigma^-$ one can associate a unique, up to proportionality, nonzero half-spinor $s_U \in S^+ \cup S^-$ such that $\phi_u(s_U) = 0$, for all $u \in U$, where $\phi_u \in \text{End}(S)$ is the Clifford automorphism of S associated to U:

$$\phi_u(v_1 \wedge \cdots \wedge v_k)$$
$$= \sum_i (-1)^{i-1}(u_0, v_i)v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_k + u_\infty \wedge v_1 \wedge \cdots \wedge v_k ,$$

where $u = u_0 + u_\infty$, with $u_o \in U_0, u_\infty \in U_\infty$.

One can also obtain an explicit coordinate description, as well as a set of defining quadratic equations for the point set, see e.g. [6, 11, 17, 19].

We shall refer to these varieties as the *half-spin varieties $\mathcal{D}_{n,n}(\mathbb{K})$*.

The variety related to a building of type E_6. Let V be a 27-dimensional vector space over a field $\mathbb{K}$ consisting of all ordered triples $x = [x^{(1)}, x^{(2)}, x^{(3)}]$ of 3×3 matrices $x^{(i)}, 1 \leqslant i \leqslant 3$, where addition and scalar multiplication are defined entrywise. Moreover the vector space is equipped with the cubic form $d : V \to \mathbb{K}$ given by

$$d(x) = \det x^{(1)} + \det x^{(2)} + \det x^{(3)} - \text{Tr}(x^{(1)} x^{(2)} x^{(3)})$$

for $x \in V$, see for instance Aschbacher's article [1].

Let $x \in V$. The map $\partial_x d : V \to \mathbb{K}$ such that $\partial_x d(y)$ is the coefficient in the expansion of $d(x + ty) \in \mathbb{K}[t]$ as a polynomial in t is called the *derivative* of d at x. Define the adjoint square $x^\sharp$ of x by $\partial_x(d)(y) = \text{Tr}(x^\sharp y)$. The *variety* $\mathcal{E}_{6,1}(\mathbb{K})$ consists of the set of projective points $\langle x \rangle$ of V for which $x^\sharp = 0$.

3. Main result

We can now state our main result.

MAIN RESULT. — *A proper Mazzocca–Melone set of split type $d \geqslant 1$ in $\mathbb{P}^N(\mathbb{K})$ is projectively equivalent to one of the following:*

$d = 1$ – *the quadric Veronese variety $\mathcal{V}_2(\mathbb{K})$, and then $N = 5$;*
$d = 2$ – *the Segre variety $\mathcal{S}_{1,2}(\mathbb{K})$, and then $N = 5$;*
        – *the Segre variety $\mathcal{S}_{1,3}(\mathbb{K})$, and then $N = 7$;*
        – *the Segre variety $\mathcal{S}_{2,2}(\mathbb{K})$, and then $N = 8$;*
$d = 4$ – *the line Grassmannian variety $\mathcal{G}_{4,1}(\mathbb{K})$, and then $N = 9$;*
        – *the line Grassmannian variety $\mathcal{G}_{5,1}(\mathbb{K})$, and then $N = 14$;*
$d = 6$ – *the half-spin variety $\mathcal{D}_{5,5}(\mathbb{K})$, and then $N = 15$;*
$d = 8$ – *the variety $\mathcal{E}_{6,1}(\mathbb{K})$, and then $N = 26$;*

Remark 3.1. —   If one includes the non-proper cases in the previous statement, then a striking similarity between these and the proper cases becomes apparent (and note that each symp of any proper example in the list is isomorphic to the non-proper Mazzocca–Melone set of the same split type). Indeed, for $d = 1$, a conic is a Veronesean variety $\mathcal{V}_1(\mathbb{K})$; for $d = 2$, a hyperbolic quadric in 3-space is a Segre variety $\mathcal{S}_{1,1}(\mathbb{K})$; for $d = 4$, a hyperbolic quadric in 5-space is the line Grassmannian variety $\mathcal{G}_{3,1}(\mathbb{K})$; for $d = 6$, the hyperbolic quadric in 7-space is, by triality, also the half-spin variety $\mathcal{D}_{4,4}(\mathbb{K})$; finally for $d = 8$, the hyperbolic quadric in 9-space is a variety $\mathcal{D}_{5,1}$, sometimes denoted $\mathcal{E}_{5,1}$ to emphasize the similarity between objects of type D_5 and the ones of type $\mathsf{E}_6, \mathsf{E}_7$ and E_8.

The following corollary precisely characterizes the varieties related to the second row of the FTMS. It is obtained from our Main Result by either adding a restriction on the global dimension, or a restriction on the *local* dimension, i.e., the dimension of at least one tangent space.

MAIN COROLLARY. — *A Mazzocca–Melone set of split type d, with $d \geqslant 1$, in $\mathbb{P}^N(\mathbb{K})$, with either $N \geqslant 3d+2$, or $\dim T_x = 2d$ for at least one $x \in X$, is projectively equivalent to one of the following:*

- *the quadric Veronese variety $\mathcal{V}_2(\mathbb{K})$, and then $d = 1$;*
- *the Segre variety $\mathcal{S}_{2,2}(\mathbb{K})$, and then $d = 2$;*
- *the line Grassmannian variety $\mathcal{G}_{5,1}(\mathbb{K})$, and then $d = 4$;*
- *the variety $\mathcal{E}_{6,1}(\mathbb{K})$, and then $d = 8$.*

In all cases $N = 3d + 2$ and $\dim T_x = 2d$ for all $x \in X$.

Concerning the proof, the cases $d = 1$, $d = 2$ and $d = 3$ of the Main Result are proved in [23], [25] and Proposition 2.1, respectively. So we may suppose $d \geqslant 4$. However, our proof is inductive, and in order to be able to use the cases $d \in \{1, 2, 3\}$, we will be forced to prove some results about sets only satisfying (MM1) and (MM2), which we will call *pre-Mazzocca–Melone sets (of split type)*, and this will include $d = 1, 2, 3$.

The rest of the paper is organized as follows. In the next section, we prove some general results about (pre-)Mazzocca–Melone sets, and use these in Section 5 to finish the proof of the Main Result for all the cases corresponding with the varieties in the conclusion of the Main Result. In Section 6, we prove the non-existence of proper Mazzocca–Melone sets for $d \notin \{1, 2, 4, 6, 8\}$, and in the last section, we verify the axioms.

4. General preliminary results for pre-Mazzocca–Melone sets

We introduce some notation. Let (X, Ξ) be a pre-Mazzocca–Melone set of split type d in $\mathbb{P}^N(\mathbb{K})$, $d \geqslant 1$. Axiom (MM1) implies that, for a given line L of $\mathbb{P}^N(\mathbb{K})$, either 0, or 1, or 2 or all points of L belong to X, and in the latter case L is contained in a symp. In this case, we call L a *singular line*. More generally, if all points of a k-space of $\mathbb{P}^N(\mathbb{K})$ belong to X, then we call this k-space *singular*. Two points of X contained in a common singular line will be called *X-collinear*, or simply *collinear*, when there is no confusion. Note that there is a unique symp through a pair of non-collinear points (existence follows from (MM1) and uniqueness from (MM2)). A *maximal singular subspace* is one that is not properly contained in another.

The linear span of a set S of points in $\mathbb{P}^N(\mathbb{K})$ will be denoted by $\langle S \rangle$. We now have the following lemmas.

LEMMA 4.1 (The Quadrangle Lemma). — *Let L_1, L_2, L_3, L_4 be four (not necessarily pairwise distinct) singular lines such that L_i and L_{i+1} (where $L_5 = L_1$) share a (not necessarily unique) point p_i, $i = 1, 2, 3, 4$, and suppose that p_1 and p_3 are not X-collinear. Then L_1, L_2, L_3, L_4 are contained in a unique common symp.*

Proof. — Since $\langle p_1, p_3 \rangle$ is not singular, we can pick a point $p \in \langle p_1, p_3 \rangle$ which does not belong to X. Let ξ be the unique symp containing p_1 and p_3. We choose two arbitrary but distinct lines M_2, M_3 through p inside the plane $\langle L_2, L_3 \rangle$. Denote $M_i \cap L_j = \{p_{ij}\}$, $\{i, j\} \subseteq \{2, 3\}$. By (MM1) there is a symp ξ_i containing p_{i2} and p_{i3}, $i = 2, 3$. If $\xi_2 \neq \xi_3$, then (MM2) implies that p, which is contained in $\xi_2 \cap \xi_3$, belongs to X, a contradiction. Hence $\xi_2 = \xi_3 = \xi$ and contains L_2, L_3. We conclude ξ contains L_2, L_3, and similarly also L_4, L_1. $\square$

Note that, if in the previous lemma all lines are different, then they span a 3-space; this just follows from the lemma a posteriori.

LEMMA 4.2. — *Let $p \in X$ and let H be a symp not containing p. Then the set of points of H collinear with p is either empty or constitutes a singular subspace of H.*

Proof. — Suppose first that p is X-collinear with two non-collinear points $x_1, x_2 \in H$, and let $x \in H$ be collinear with both x_1 and x_2. Then the Quadrangle Lemma applied to p, x_1, x, x_2 implies that $p \in H$, a contradiction. Hence all points of H collinear with p are contained in a singular subspace. Suppose now that p is collinear with two collinear points $y_1, y_2 \in H$ and let y be a point on the line $\langle y_1, y_2 \rangle$. If we assume that p and y are not collinear, then the Quadrangle Lemma applied to p, y_1, y, y_2 yields that the plane $\langle p, y_1, y_2 \rangle$ is entirely contained in a symp, contradicting the non-collinearity of p and y. Hence p and y are collinear and the lemma is proved. $\square$

Remark 4.3. — Note that, up to now, we did not use the part of (MM1) that says that two *collinear* points of X are contained in some symp. In fact, this follows from the previous lemmas. Indeed, let x, y be two X-collinear points and suppose that they are not contained in any common symp. By considering an arbitrary symp, we can, using Lemma 4.2, find a point not collinear to y and hence a symp H containing y (but not x, by assumption). Let z be a point of H collinear with y but not collinear with x (and z exists

by Lemma 4.2). The Quadrangle Lemma applied to x, y, z, y now implies that the (unique) symp $X([x, z])$ contains y. Hence we can relax (MM1) by restricting to pairs of points that are not X-collinear. For ease of reference, we have not included this minor reduction in the axioms.

LEMMA 4.4. — *A pair of singular k-spaces, $k \geqslant 0$, that intersect in a singular $(k-1)$-space is either contained in a symp, or in a singular $(k+1)$-space. In particular, if $k > \lfloor \frac{d}{2} \rfloor$, then such a pair is always contained in a singular $(k+1)$-space.*

Proof. — If $k = 0$, then this follows from (MM1). If $k = 1$, this follows from the Quadrangle Lemma. Now let $k \geqslant 2$. Suppose A, B are singular k-spaces intersecting in a singular $(k-1)$-space and suppose that $C := \langle A, B \rangle$ is not singular. Then there is a point $p \in C$ which does not belong to X. Choose a point $p_A \in A \setminus B$ and let p_B be the intersection of $\langle p, p_A \rangle$ and B. Then there is a unique symp ξ containing p_A and p_B. Let q_A be any other point of $A \setminus B$ and put $q = \langle p_A, q_A \rangle \cap B$. Then the Quadrangle Lemma (with $L_1 = L_2$ and $L_3 = L_4$) implies that $q_A \in \xi$. Hence $A \subseteq \xi$ and similarly $B \subseteq \xi$. □

LEMMA 4.5. — *Every singular k-space, $k \leqslant \lfloor \frac{d}{2} \rfloor$, which is contained in a finite-dimensional maximal singular subspace, is contained in a symp.*

Proof. — Clearly, the lemma is true for $k = 0, 1$. Now let $k \geqslant 2$.

Let A be a singular k-space contained in the finite-dimensional maximal singular subspace M. Note that, in principle, the dimension of M should not necessarily be larger than $\lfloor \frac{d}{2} \rfloor$. Let S be a subspace of A of maximal dimension with the property that it is contained in some symp H. We may assume $S \neq A$, as otherwise we are done. In H, the subspace S is not a maximal subspace as this would imply that $\lfloor \frac{d}{2} \rfloor = \dim S < \dim A = k \leqslant \lfloor \frac{d}{2} \rfloor$, a contradiction. Hence, since H is a polar space, we can find two singular subspaces B_1 and B_1' of H, containing S, having dimension $\dim S + 1$ and such that B_1 and B_1' are not contained in a common singular subspace of H. It then follows that B_1 and B_1' are not contained in a common singular subspace of X, as $\langle B_1, B_1' \rangle \subseteq \langle H \rangle$ and since $H = \langle H \rangle \cap X$, this would imply that $\langle B_1, B_1' \rangle$ is a singular subspace of H, a contradiction. Hence at most one of B_1, B_1' belongs to A, and so we may assume that B_1 is not contained in A. Consequently, B_1 is a singular subspace with dimension $\dim(S) + 1$ contained in H, containing S and not contained in M. Put $\ell = \dim(M) - \dim(A) + 1$ and let $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_\ell$ be a family of nested subspaces, with $\dim(A_i) = \dim(S) + i$, for all $i \in \{1, 2, \ldots, \ell\}$, with $A_\ell = M$ and $S \subseteq A_1 \subseteq A$. Put $B_i = \langle A_{i-1}, B_1 \rangle$, $i = 2, 3, \ldots, \ell+1$.

By the maximality of S, Lemma 4.4 implies that B_2 is a singular subspace. Let $2 \leqslant i \leqslant \ell$. If B_i is singular, then, since $B_i \cap A_i = A_{i-1} \supseteq A_1$ and A_1 is not contained in a symp, Lemma 4.4 again implies that $\langle A_i, B_i \rangle = \langle A_i, B_1 \rangle = B_{i+1}$ is a singular subspace. Inductively, this implies that $B_{\ell+1}$, which properly contains M, is a singular subspace. This contradicts the maximality of M. Hence $S = A$ and the assertion follows. $\qquad \square$

The next lemma suggests an inductive approach.

LEMMA 4.6. — *Suppose (X, Ξ) is a pre-Mazzocca–Melone set of split type d, $d \geqslant 4$. Let $x \in X$ be arbitrary and assume that T_x is finite-dimensional. Let C_x be a subspace of T_x of dimension $\dim(T_x) - 1$ not containing x. Consider the set X_x of points of C_x which are contained in a singular line of X through x. Let Ξ_x be the collection of subspaces of C_x obtained by intersecting C_x with all $T_x(\xi)$, for ξ running through all symps of X containing x. Then (X_x, Ξ_x) is a pre-Mazzocca–Melone set of split type $d - 2$.*

Proof. — We start by noting that the dimension of a maximal singular subspace in (X_x, Ξ_x) is bounded above by $\dim T_x - 1$. Hence we may use Lemma 4.5 freely.

Now, Axiom (MM1) follows from Lemma 4.4 (two lines through x not in a singular plane are contained in a symp) and Lemma 4.5 (a singular plane is contained in a symp).

Axiom (MM2) is a direct consequence of the validity of Axiom (MM2) for (X, Ξ). $\qquad \square$

The pair (X_x, Ξ_x) will be called the *residue at x*. This can also be defined for split type 3 in the obvious way, but we do not necessarily obtain a pre-Mazzocca–Melone set of split type 1, as the following shows (the proof is the same as the proof of the previous lemma, except that we do not need Lemma 4.5 anymore by the weakening of Axiom (MM1)).

LEMMA 4.7. — *Suppose (X, Ξ) is a pre-Mazzocca–Melone set of split type 3. Let $x \in X$ be arbitrary and let (X_x, Ξ_x) be the residue at x. Then (X_x, Ξ_x) satisfies Axiom (MM2) for split type 1 (so for each $\xi \in \Xi_x$ we have that $X(\xi)$ is a plane conic), and the following weakened version of Axiom (MM1):*

(MM1$'$) *Any pair of points x and y of X either lies on a singular line, or is contained in a unique member of Ξ_x.*

If in particular (X, Ξ) has no singular planes, then (X_x, Ξ_x) is a pre-Mazzocca–Melone set of split type 1. $\qquad \square$

In general, we also want the residue to be a proper pre-Mazzocca–Melone set whenever the original is proper. This follows from the next lemma, but we state a slightly stronger property, which we will also use in other situations.

LEMMA 4.8. — Let (X, Ξ) be a proper Mazzocca–Melone set of split type $d \geqslant 1$, and let $x, y \in X$ be collinear. Then there is a symp containing x and not containing y. In particular, the residue at any point of a proper Mazzocca–Melone set of split type $d \geqslant 4$ is a proper pre-Mazzocca–Melone set of split type $d-2$. If $d = 3$, then the residue contains at least two conics.

Proof. — It is easy to see that there are at least two different symps H, H' containing x (this follows straight from the properness of (X, Ξ), and, in fact, this already implies that the residue at x is proper). Suppose both contain y. Select a point $p \in H \setminus H'$ collinear to x but not collinear to y, and a point $p' \in H' \setminus H$ collinear to x but not collinear to p (these points exist by Lemma 4.2 and since $H \cap H'$ is a singular subspace). By Lemma 4.4 the unique symp H'' containing p and p' also contains x. It does not contain y, however, because the intersection $H \cap H''$ would otherwise not be a singular subspace, as it would contain the non-collinear points p and y. $\qquad\square$

In order to be able to use such an inductive strategy, one should also have that Axiom (MM3) is inductive. To obtain this, one has to apply different techniques for the different small values of d (for $d = 1$ and $d = 2$, there was not even such an induction possible). Roughly, the arguments are different for $d = 4, 5$, there is a general approach for $6 \leqslant d \leqslant 9$ and another one again for $d \geqslant 10$.

From now on, we also assume (MM3). The basic idea for the induction for larger d is to prove that the tangent space T_x does not contain any point $y \in X$ with $\langle x, y \rangle$ not singular. Such a point y will be called a *wrinkle (of x)*, and if x does not have any wrinkles, then x is called *smooth*. Once we can prove that all points are smooth, we get control over the dimension of the tangent spaces in the residues at the points of X. For $d = 4, 5$, this fails, however, and we have to use additional arguments to achieve this. The case $d = 3$ is "extra special", since we cannot even apply Lemma 4.6. This case is the most technical of all (but is done in [26], see Proposition 2.1), but also the case $d = 2$ is rather technical, see [25].

Concerning wrinkles, we prove below a general lemma that helps in ruling them out. Before that, we need two facts about hyperbolic and parabolic quadrics. In passing, we also prove a third result which we will need later. We freely use the well known fact already mentioned above that, if for any

quadric Q and any point $x \in Q$, some line L through x does not belong to the tangent space of Q at x, then L intersects Q in precisely two points, one of which is x.

LEMMA 4.9. — *Let H be a hyperbolic quadric in $\mathbb{P}^{2n+1}(\mathbb{K})$, $n \geqslant 1$. Then every $(n+1)$-space of $\mathbb{P}^{2n+1}(\mathbb{K})$ contains a pair of non-collinear points of H.*

Proof. — Let, by way of contradiction, U be a subspace of dimension $n + 1$ intersecting H precisely in a singular subspace W. Since U meets every subspace of dimension n that is completely contained in H in at least one point, we deduce that W is nonempty. Let $x \in W$ be an arbitrary point. Since no line in U through x intersects H in precisely two points, each such line is contained in the tangent space $T_x(H)$ of H at x. Now, $T_x(H) \cap H$ is a cone over x of a hyperbolic quadric H' in some $\mathbb{P}^{2n-1}(\mathbb{K})$, and U can be considered as a cone over x of an n-space in $\mathbb{P}^{2n-1}(\mathbb{K})$ intersecting H' in some subspace. An obvious induction argument reduces the lemma now to the case $n = 1$, which leads to a contradiction, proving the lemma. $\square$

LEMMA 4.10. — *Let P be a parabolic quadric in $\mathbb{P}^{2n}(\mathbb{K})$, $n \geqslant 1$. Then through every singular $(n-1)$-space there exists exactly one n-space containing no further points of P. Also, every $(n+1)$-space of $\mathbb{P}^{2n}(\mathbb{K})$ contains a pair of non-collinear points of P.*

Proof. — To prove the first assertion, let W be a singular $(n-1)$-space of P and let $x \in W$. Then, as in the previous proof, $T_x(P) \cap P$ is a cone over x of a parabolic quadric P' in some $\mathbb{P}^{2n-2}(\mathbb{K})$, and every n-space of $\mathbb{P}^{2n}(\mathbb{K})$ intersecting P in exactly W corresponds to an $(n-1)$ space of $\mathbb{P}^{2n-2}(\mathbb{K})$ intersecting P' in exactly W', where W' corresponds to W. An obvious induction argument reduces the assertion to the case $n = 1$, where the statement is obvious (there is a unique tangent in every point).

The second assertion is proved in exactly the same way as Lemma 4.9, again reducing the problem to $n = 1$, where the result is again obvious (since $n + 1 = 2n$ in this case). $\square$

LEMMA 4.11. — *Let Q be a hyperbolic or parabolic quadric in $\mathbb{P}^n(\mathbb{K})$, $n \geqslant 3$, and let S be a subspace of dimension $n - 2$ of $\mathbb{P}^n(\mathbb{K})$. Then some line of Q does not intersect S.*

Proof. — Pick two disjoint maximal singular subspaces M_1, M_2 of Q. We may assume that S intersects every maximal singular subspace M in either a hyperplane of M, or in M itself, as otherwise we easily find a line inside M not intersecting S. If Q is hyperbolic, then, as $\langle M_1, M_2 \rangle = \mathbb{P}^n(\mathbb{K})$, this easily implies that $S = \langle S \cap M_1, S \cap M_2 \rangle$ and $\dim(S \cap M_i) = \frac{n-3}{2}$. Since

collinearity induces a duality between disjoint maximal singular subspaces, there is a unique point in M_1 not collinear to any point of $M_2 \setminus S$. It follows that we find a point $x_1 \in M_1 \setminus S$ collinear in Q to some point $x_2 \in M_2 \setminus S$ and that the line L joining x_1, x_2 does not intersect S. If Q is parabolic, then the same argument and conclusion holds if $\dim(S \cap \langle M_1, M_2 \rangle) = n-3$. If not, then $S \subseteq \langle M_1, M_2 \rangle$, and we find a point $x \in M_1 \cup M_2 \setminus S$ (otherwise $S = \langle M_1, M_2 \rangle$, which is $(n-1)$-dimensional, a contradiction). Since not all points of M_1 and M_2 are collinear to x, and since the points collinear to x in Q span a hyperplane of $\mathbb{P}^n(\mathbb{K})$, we find a point $y \in Q$ outside $\langle M_1, M_2 \rangle$ collinear to x. The line $\langle x, y \rangle$ belongs to Q and does not meet S. $\qquad \square$

We are now ready to prove a result that restricts the possible occurrences of wrinkles. It is one of the fundamental observations in our proof.

LEMMA 4.12. — Let (X, Ξ) be a Mazzocca–Melone set of split type $d \geqslant 1$. Let $x \in X$. Then no wrinkle y of x is contained in the span of two tangent spaces $T_x(\xi_1)$ and $T_x(\xi_2)$, for two different $\xi_1, \xi_2 \in \Xi$.

Proof. — Suppose, by way of contradiction, that T_x contains the wrinkle y, and that there are two symps H_1 and H_2 such that $x \in H_1 \cap H_2$ and $y \in \langle T_x(H_1), T_x(H_2) \rangle$. Then there are points $a_i \in T_x(H_i)$, $i \in \{1,2\}$, such that $y \in a_1 a_2$. Our aim is to show that we can (re)choose the wrinkle y and the point a_1 in such a way that $a_1 \in X$. Considering a symp through y and a_1 and using Axiom (MM2) then implies that $a_2 \in X$, and so the plane $\langle x, a_1, a_2 \rangle$, containing two singular lines and an extra point $y \in X$, must be singular, contradicting the fact that y is a wrinkle of x.

Set $U = T_x(H_1) \cap T_x(H_2)$ and $\dim U = \ell$, $0 \leqslant \ell \leqslant \lfloor \frac{d}{2} \rfloor$.

By assumption $\dim \langle T_x(H_1), T_x(H_2) \rangle = 2d - \ell$. Hence

$$\dim([x, y] \cap \langle T_x(H_1), T_x(H_2) \rangle) \geqslant d + 1 - \ell$$

and

$$\dim(T_x([x, y]) \cap \langle T_x(H_1), T_x(H_2) \rangle) \geqslant d - \ell.$$

The latter implies that we can find a subspace W of dimension $\lceil \frac{d}{2} \rceil - \ell$ through x contained in $T_x([x, y]) \cap \langle T_x(H_1), T_x(H_2) \rangle$, but intersecting $T_x(H_1) \cup T_x(H_2)$ exactly in $\{x\}$ (using the fact that $T_x([x, y])$ intersects $T_x(H_i)$ in a subspace of dimension at most $\lfloor \frac{d}{2} \rfloor$). Note that W is not necessarily a singular subspace. We consider the space $\Pi = \langle W, y \rangle$ of dimension $\lceil \frac{d}{2} \rceil + 1 - \ell$. Every line in Π through x outside W contains a unique wrinkle. It follows that $\Pi \cap \langle H_i \rangle = \{x\}$, $i \in \{1,2\}$. This is already a contradiction for $d = 1$ and $\ell = 0$, since this implies $3 = 2 + 1 = \dim \Pi + \dim \langle T_x(H_1) \rangle \leqslant 2d - \ell = 2$. So we assume $d > 1$. Using straightforward dimension arguments, we deduce that there are unique $(\lceil \frac{d}{2} \rceil + 1)$-spaces $U_i \subseteq \langle H_i \rangle$, $i = 1, 2$,

containing U such that $\Pi \subseteq \langle U_1, U_2 \rangle$ (and $\dim\langle U_1, U_2 \rangle = 2\lceil\frac{d}{2}\rceil + 2 - \ell$). Let U_1' be the $\lceil\frac{d}{2}\rceil$-space obtained by intersecting $\langle U_2, W \rangle$ with U_1. Then, by Lemmas 4.9 and 4.10, we can pick a point $a_1 \in (X \cap U_1) \setminus U_1'$. Since U_2 and Π meet in only x, and $a_1 \in \langle U_2, \Pi \rangle$, there is a unique plane π containing x, a_1 and intersecting both Π and U_2 in (distinct) lines. By our choice of a_1 outside U_1', the line $\pi \cap \Pi$ is not contained in W, hence contains a wrinkle, which we may assume without loss of generality to be y. Inside the plane π, the line $\langle a_1, y \rangle$ intersects $\pi \cap U_2$ in a point $a_2 \in \langle H_2 \rangle$. This completes the proof of the lemma. $\qquad\square$

We can show the power of the previous lemma by the following lemma, which establishes the non-existence of wrinkles for small values of d.

LEMMA 4.13. — *If $2 \leqslant d \leqslant 9$, then for every non-smooth point $x \in X$, there exist at least one wrinkle y of x and two symps H_1 and H_2 through x such that $y \in \langle T_x(H_1), T_x(H_2) \rangle$.*

Proof. — Let z be a wrinkle of x. Let $H = X([x, z])$. Since $\langle T_x(H), z \rangle = \langle H \rangle$, we know that $H \subseteq T_x$. Let H' be a symp through x which intersects H in a subspace of minimal dimension. Since $H, T_x(H') \subseteq T_x$, this dimension is at least 1. Since $d \leqslant 9$, there are four cases.

We first observe that

$$H \cap T_x(H') \subseteq \langle H \rangle \cap \langle H' \rangle \subseteq H \cap H' \subseteq H \cap T_x(H'),$$

hence $H \cap T_x(H') = H \cap H'$.

Case 1: H intersects H' in a line L. — In this case $\langle H, T_x(H') \rangle = T_x$ by a simple dimension argument, which also shows $\dim(T_x) = 2d$. Since L is a singular line, $\dim\langle T_x(H), T_x(H') \rangle = 2d - 1$. Hence we can pick a point $u \in (X \cap T_x) \setminus \langle T_x(H), T_x(H') \rangle$, with $\langle x, u \rangle$ singular. Let H'' be a symp through x and u. Since $\dim\langle u, H \rangle = d + 2$, we see that $\dim(\langle u, H \rangle \cap T_x(H')) = 2$. Consequently, there is a point $v \in (\langle u, H \rangle \cap \langle T_x(H') \rangle) \setminus \langle H \rangle$. The line $\langle u, v \rangle$ inside the $(d+2)$-space $\langle u, H \rangle$ intersects the $(d+1)$-space $\langle H \rangle$ in a point y'. If y' were contained in $T_x(H)$, then the line $\langle u, v \rangle = \langle v, y' \rangle$ would be contained in $\langle T_x(H), T_x(H') \rangle$, contradicting the choice of u. Hence $y' \in \langle H \rangle \setminus T_x(H)$. Then there is a unique point $y \in H \setminus T_x(H)$ on the line $\langle x, y' \rangle$ (and y is a wrinkle of x). By replacing u with an appropriate point on $\langle x, u \rangle$, we may assume that $y' = y$. Hence $y \in \langle u, v \rangle \subseteq \langle T_x(H''), T_x(H') \rangle$ and the lemma is proved.

Case 2: H intersects H' in a plane π. — In this case $\dim\langle H, T_x(H') \rangle = 2d - 1$. If $\dim(T_x) = 2d - 1$, then the lemma follows from an argument completely similar to the one of Case 1. So we may assume $\dim(T_x) = 2d$. Note

that $\dim\langle T_x(H), T_x(H')\rangle = 2d - 2$. Hence there exist two points $v_1, v_2 \in X$ such that $\langle x, v_1 \rangle$ and $\langle x, v_2 \rangle$ are singular lines, $v_1 \notin \langle T_x(H), T_x(H')\rangle$, and $v_2 \notin \langle v_1, T_x(H), T_x(H')\rangle$. This implies that $\langle v_1, v_2 \rangle$ does not intersect $\langle T_x(H), T_x(H')\rangle$.

If $v_1 v_2$ intersects $\langle H \rangle$ in a point w then Axiom (MM2) implies $w \in X$. Hence $\langle v_1, v_2, x \rangle$ is a singular plane. Since $\langle v_1, v_2 \rangle \cap T_x(H) = \emptyset$, $w \in H \setminus T_x(H)$, contradicting the fact that $\langle x, w \rangle$ is a singular line.

Hence $\dim(\langle v_1, v_2, H \rangle) = d + 3$, and so, putting $U = \langle v_1, v_2, H \rangle$ we have $\dim(U \cap T_x(H')) \geqslant 3$. Consequently we find a point u in $T_x(H') \setminus \langle H \rangle$, with $u \in U$. In U, the plane $\langle u, v_1, v_2 \rangle$ intersects the space $\langle H \rangle$ in some point y. If $y \in T_x(H)$, then the point $\langle u, y \rangle \cap \langle v_1, v_2 \rangle$ belongs to $\langle T_x(H), T_x(H')\rangle$, contradicting the choices of v_1, v_2. Hence $y \in \langle H \rangle \setminus T_x(H)$. As in Case 1, we can rechoose u on $\langle x, u \rangle$ so that $y \in H$, and so y is a wrinkle of x. It follows that $y \in \langle T_x(H'), T_x(H'')\rangle$, concluding Case 2.

Case 3: H intersects H' in a 3-space Σ. — Here again, the case where $\dim(T_x) = 2d - 2$ is completely similar to Case 1, and if $\dim(T_x) = 2d - 1$, then we can copy the proof of Case 2, adjusting some dimensions.

So assume T_x is $2d$-dimensional. Since $\dim\langle T_x(H), T_x(H')\rangle = 2d - 3$, we can find points $v_1, v_2, v_3 \in X$ such that $\langle x, v_i \rangle$ is a singular line, $i = 1, 2, 3$, and $\langle v_1, v_2, v_3, T_x(H), T_x(H')\rangle = T_x$. By Lemma 4.4 and Lemma 4.5, there is a symp H_{12} containing x, v_1, v_2 (note $d \geqslant 6$ by the definition of Σ). Since $\langle v_1, v_2 \rangle$, which is contained in $T_x(H_{12})$, does not meet $\langle T_x(H), T_x(H')\rangle$, the latter intersects $T_x(H_{12})$ in a subspace of dimension at most $d - 2$. Lemma 4.11 implies that we can find a singular line L in $T_x(H_{12})$ skew to $\langle T_x(H), T_x(H')\rangle$. We may now assume that $L = \langle v_1, v_2 \rangle$, and possibly rechoose v_3 so that we still have $\langle v_1, v_2, v_3, T_x(H), T_x(H')\rangle = T_x$. Note that every point v of the plane $\pi = \langle v_1, v_2, v_3 \rangle$ is contained in a quadratic space ξ, since v_3 together with any point of $\langle v_1, v_2 \rangle$ is contained in one. Hence, as in Case 2 above, π does not meet $\langle H \rangle$. Note also that v in fact belongs to $T_x(\xi)$.

So, the space $U = \langle \pi, H \rangle$ is $(d + 4)$-dimensional and intersects $T_x(H')$ in a subspace of dimension at least 4. Similarly as above in Case 2, we find a point $u \in U$ in $T_x(H') \setminus \langle H \rangle$, and the 3-space $\langle u, \pi \rangle$ intersects $\langle H \rangle$ in some point y, which does not belong to $T_x(H)$, and which we may assume to belong to X (by rechoosing u on $\langle x, u \rangle$). As we noted above, the point $\langle u, y \rangle \cap \pi$ is contained in $T_x(\xi)$, for some quadratic space ξ, and so $y \in \langle T_x(\xi), T_x(H')\rangle$, completing Case 3.

Case 4: H intersects H′ in a 4-space α. — As before, we may again assume that $\dim(T_x) = 2d$, as otherwise the proofs are similar to the previous cases. Note that $\dim\langle T_x(H), T_x(H')\rangle = 2d - 4$.

Now, similarly to Case 3, we first find points $v_1, v_2 \in X$ such that $\langle x, v_1, v_2 \rangle$ is a singular plane and $\dim\langle v_1, v_2, T_x(H), T_x(H')\rangle = 2d - 2$, and then we find, by the same token, points $v_3, v_4 \in X$ such that $\langle x, v_3, v_4 \rangle$ is a singular plane and $\langle v_1, v_2, v_3, v_4, T_x(H), T_x(H')\rangle = T_x$. Also as in Case 3, every point of the 3-space $\Sigma = \langle v_1, v_2, v_3, v_4 \rangle$ is contained in a quadratic space. But then we can copy the rest of the proof of Case 3 to complete Case 4.

This completes the proof of the lemma. □

Combining Lemma 4.13 with Lemma 4.12, we obtain the following consequence.

LEMMA 4.14. — *If $2 \leqslant d \leqslant 9$, then every point $x \in X$ is smooth.*

This implies the following inductive result.

COROLLARY 4.15. — *If $6 \leqslant d \leqslant 9$, for all $p \in X$, the residue (X_p, Ξ_p) is a Mazzocca–Melone set of split type $d - 2$.*

Proof. — In view of Lemma 4.6, it suffices to show that (X_p, Ξ_p) satisfies Axiom (MM3). Let $x \in X$ be collinear with p. By Lemma 4.8 there exists a symp H through p not containing x. Put $W_x = T_x \cap \langle H \rangle$. Since the points of H collinear to x are contained in a maximal singular subspace of H, Lemmas 4.9, 4.10 and 4.14, imply that $\dim(W_x) \leqslant \lfloor \frac{d+1}{2} \rfloor$. Hence we can select a subspace W'_x in $T_p(H)$ complementary to W_x (which is indeed contained in $T_p(H)$). Since $\dim(W'_x) \geqslant \lceil \frac{d-1}{2} \rceil - 1$, we see that $\dim(T_x \cap T_p) \leqslant \lfloor \frac{3d+1}{2} \rfloor$. Hence, if $x' \in X_p$ corresponds to x, then $T_{x'}$ has dimension at most $\lfloor \frac{3d+1}{2} \rfloor - 1$. For $d = 6, 7, 8, 9$ this yields $8, 10, 11, 13$, respectively, which finishes the proof. □

The strategy of the proof of the Main Result for $d \in \{4, 5\}$ will also be to show that the residue satisfies Axiom (MM3). However, this will need some very particular arguments. Concerning the cases $d \geqslant 10$, the arguments to prove Lemma 4.13 cannot be pushed further to include higher dimensions. Hence we will need different arguments, which, curiously, will not be applicable to any of the cases $d \leqslant 9$; see Proposition 6.9.

Remark 4.16. — The varieties in the conclusion of the Main Corollary are analogues of complex Severi varieties, as already mentioned. The varieties in the conclusion of the Main Result that are not in the conclusion of the Main Corollary have, in the complex case, as secant variety the whole

projective space. It can be shown that this remains true over an arbitrary field; this is rather easy for the Segre and line Grassmannian varieties, and slightly more involved for the half-spin variety $\mathcal{D}_{5,5}$, although the calculations are elementary, but tedious, using [34]. Likewise, the secant variety of the secant variety of a complex Severi variety is the whole projective space, and this remains true for our analogues over an arbitrary field. This now has the following consequence.

PROPOSITION 4.17. — *Let X be one of the varieties in the Main Result for $d \geqslant 2$, and regard X as a set of points. Let $\Gamma = (X, \mathcal{L})$ be the point-line geometry obtained from X by collecting (in the set $\mathcal{L}$) all projective lines entirely contained in X, and considering the natural incidence relation. Let Ξ' be the family of all maximal split quadrics contained in X. Then there is a unique projective embedding of Γ with the property that any two projective subspaces generated by members of Ξ' intersect in a subspace all points of which belong to X.*

Proof. — Every geometry Γ has a natural embedding as given in Section 2.2. This embedding is always the absolute universal one, meaning that every other embedding is a projection of the absolute universal one. This follows by results of Zanella ($d = 2$), see Theorem 3 of [36], Havlicek ($d = 4$), see [9], Wells ($d = 6$), see Theorem 5 of [34], and combining Theorem 6.1 of Cooperstein & Shult [7] (which bounds the dimension of any embedding of $\mathcal{E}_{6,1}$ to 26) and Paragraph 4.11.1 of Kasikova & Shult [14] (which shows the existence of the absolute universal embedding for $d = 8$). Now either the secant variety is the whole projective space (in which case there are no proper projections), or the secant variety of the secant variety is the whole projective space (in which case every projection reduces the dimension of the space spanned by some pair of maximal split quadrics, contradicting Axiom (MM2)). This shows the assertion. □

In view of the previous proposition, is suffices to show that the points and singular lines of a Mazzocca–Melone set of split type constitute the geometries related to one of the varieties in the Main Result, and only those. The geometries Γ defined above related to a line Grassmannian variety, half-spin variety or variety $\mathcal{E}_{6,1}(\mathbb{K})$ will be referred to as a *line Grassmannian geometry, half-spin geometry* or *geometry $\mathcal{E}_{6,1}(\mathbb{K})$*, respectively. These geometries are related to (Tits-)buildings by defining additional objects in the geometry; it is in this way that we will recognize the varieties of our Main Result. This is the content of the next section.

5. Recognizing the line Grassmannian, half-spin and $\mathcal{E}_{6,1}$ geometries

We start by recognizing the line Grassmannian geometries. Let (X, Ξ) be a proper Mazzocca–Melone set of split type 4 in $\mathbb{P}^N(\mathbb{K})$. For each point $p \in X$, we denote the residue at p by (X_p, Ξ_p), and we also denote the subspace spanned by X_p by C_p, as before (see Lemma 4.6). We will first recognize these residues, and then the whole geometry. To that aim, we need one more lemma.

LEMMA 5.1. — *Let $p \in X$. If $k \geqslant 4$, then there are no singular k-spaces in the residue (X_p, Ξ_p).*

Proof. — Suppose otherwise and let M be a maximal singular k-space in (X_p, Ξ_p), with $k \geqslant 4$. Note that, by repeated use of Lemma 4.4, since, by Lemma 4.6, (X_p, Ξ_p) is a proper pre-Mazzocca–Melone set of split type 2, no point of X_p outside M is collinear with at least two points of M. Hence every singular line meeting M in a point x is contained, together with any point of M distinct from x, in a unique symp. We will use this observation freely.

First we claim that every point r of M is contained in at least two singular lines not contained in M. Indeed, by considering any symp through r, we see that r is contained in at least one such singular line, say L_r. Suppose now that some point $s \in M$ is contained in at least two lines L_s, L_s' not contained in M. Then the symps defined by $\langle r, s \rangle$ and L_s, and by $\langle r, s \rangle$ and L_s' both contain $\langle r, s \rangle$, so they cannot both contain L_r. Consequently there exists a second singular line through r not contained in M. Hence we may assume that through every point r of M there is a unique singular line L_r not contained in M.

Consider three points x, y and z spanning a plane π in M. Consider a point x' on L_x different from x. Using a symp through L_x and L_y, we deduce that there is a unique point y' on L_y collinear with x', and similarly, there is a unique point z' on L_z collinear with x'. If x', y' and z' were contained in a common line L, then the symps $X([x, y'])$ and $X([y', z])$ would share the lines L and L_y, and hence coincide, a contradiction since this symp then contains the plane π. If y' and z' are not collinear then, by the Quadrangle Lemma, the symp $X([y', z'])$ contains x', $\langle y, y' \rangle$ and $\langle z, z' \rangle$, and hence it contains three singular lines through y', a contradiction. Hence the plane $\pi' := \langle x', y', z' \rangle$ is singular. Consider next a 3-space Π in M containing π. This leads with similar reasonings to a singular 3-space Π' containing π'. But then, since $\dim(T_p) \leqslant 8$, and so $\dim(C_p) \leqslant 7$, $\Pi' \cap M$ is non-empty,

contradicting our assumption of the previous paragraph. This completes the proof of our claim.

Hence there exist lines L_x and L_y through x and y, respectively, such that the symps H_x and H_y containing L_x, y and L_y, x, respectively, are distinct.

As $\dim T_p \leqslant 8$, the $(k+2)$-spaces $\langle H_x, M \rangle$ and $\langle H_y, M \rangle$ intersect at least in a $(2(k+2) - 7)$-space. Since $2k - 3 \geqslant k + 1$ (because $k \geqslant 4$), those spaces share a $(k+1)$-space S containing M. Now S intersects $\langle H_x \rangle$ and $\langle H_y \rangle$ in planes containing $\langle x, y \rangle$ and so S contains singular lines R_x and R_y of H_x and H_y meeting $\langle x, y \rangle$ in the single points p_x and p_y, respectively. Now pick two points r_x and r_y on R_x and R_y, respectively, and not contained in M. Then $R = \langle r_x, r_y \rangle \subset S$ intersects M in a point, which necessarily belongs to X_p since $M \subseteq X_p$, and hence is a singular line (this also implies $p_x \neq p_y$). The Quadrangle Lemma applied to $\langle x, y \rangle, R_x, R, R_y$ now implies that $H_x = H_y$, a contradiction. The lemma is proved. □

PROPOSITION 5.2. — *Exactly one of the following holds*

(1) *For all* p, (X_p, Ξ_p) *is isomorphic to the Segre variety* $\mathcal{S}_{1,3}(\mathbb{K})$.
(2) *For all* p, (X_p, Ξ_p) *is isomorphic to the Segre variety* $\mathcal{S}_{1,2}(\mathbb{K})$.

Proof. — First we prove that (X_p, Ξ_p) is a Mazzocca–Melone set of split type 2 in the space C_p of dimension at most 7. By Lemma 4.6, we only have to show that Axiom (MM3) holds.

Suppose that (MM3) is not satisfied. Then there exist a point x contained in a symp H and three singular lines L_1, L_2 and L_3 containing x and spanning a 3-space Π which intersects $\langle H \rangle$ only in x. We distinguish two cases.

Case 1. — Suppose that at least two of the three planes determined by L_1, L_2 and L_3, say $\langle L_1, L_2 \rangle$ and $\langle L_2, L_3 \rangle$, are not singular. Then the symps $H_{1,2}$ (through L_1 and L_2) and $H_{2,3}$ (through L_2 and L_3) intersect H only in x. Consequently the subspaces $\langle H, H_{1,2} \rangle$ and $\langle H, H_{2,3} \rangle$ are 6-dimensional and hence have a 5-space Σ containing H and L_2 in common, which meets each of $\langle H_{1,2} \rangle$ and $\langle H_{2,3} \rangle$ in a plane π_1 and π_3, respectively. These planes contain L_2 and hence contain further singular lines of $H_{1,2}$ and $H_{2,3}$, say R_1 and R_3, respectively.

By assumption we have $(R_1, R_3) \neq (L_1, L_3)$ (the latter pair is not contained in a 5-space together with H and L_2). Suppose now that $\langle R_1, R_3 \rangle$ is a 3-space. Then the latter meets $\langle H \rangle$ in a line R, every point of which is on a (necessarily singular) line meeting R_1 and R_3. Let r be any point of R. Then, since R_1, R_3 are two non-intersecting lines and r belongs to the

3-space spanned by R_1 and R_3, there is a line T_r containing r and intersecting R_1 and R_3 in two points, say r_1 and r_3, respectively. Any symp through r_1 and r_3 must meet the symp H in at least the point r (by (MM2)) and so T_r is a singular line. Now, we can choose r such that T_r does not intersect L_2, and then we have the quadrangle T_r, R_1, L_2, R_3. The Quadrangle Lemma implies that there is a unique (because the lines are distinct) symp H^* containing this quadrangle. But H^* contains R_1 and L_2, hence must coincide with $H_{1,2}$. Similarly, $H^* = H_{2,3}$. Consequently, $H_{1,2} = H_{2,3}$, a contradiction.

So we may assume that R_1 and R_3 meet in a point y of L_2 not belonging to H. Then the plane $\langle R_1, R_3 \rangle$ meets $\langle H \rangle$ in a point z, and this point is easily seen to belong to $H \setminus L_2$. If $\langle x, z \rangle$ is a singular line, then we have the singular planes $\langle R_1, R_3 \rangle$ and $\langle L_2, z \rangle$, which have the singular line $\langle y, z \rangle$ in common. Lemma 4.4 implies that $\langle R_1, L_2, R_3 \rangle$ is a singular 3-space containing the non-singular plane $\langle R_1, L_2 \rangle \subseteq \langle H_{1,2} \rangle$, a contradiction. Suppose now that the line $\langle x, z \rangle$ is not singular. Since $\langle R_1, R_3 \rangle$ is a singular plane, the line $\langle y, z \rangle$ is singular. So we can apply the Quadrangle Lemma to the lines $L_2, \langle y, z \rangle$ and two lines of H connecting z with x, obtaining that there is a symp through x, z and L_2. Hence H contains L_2, a contradiction.

Case 2. — Suppose that both $\langle L_1, L_2 \rangle$ and $\langle L_1, L_3 \rangle$ are singular planes. Then Π is a singular 3-space by Lemma 4.4. Let N_1 and N_2 be the singular lines of H meeting L_1. If any of the planes $\langle L_i, N_j \rangle$, $i \in \{1, 2, 3\}$, $j \in \{1, 2\}$, is singular then by Lemma 4.4 we obtain the singular 4-space $\langle \Pi, N_j \rangle$, a contradiction to Lemma 5.1. Hence we jump back to Case 1 using the symp defined by L_1, N_1 instead of H, and the lines L_2, N_2, L_3 instead of L_1, L_2, L_3, respectively.

Hence (MM3) is satisfied and (X_p, Σ_p) is a Mazzocca–Melone set of split type 2. Since $\dim C_p \leqslant 7$, we know by [25] that X_p is either $\mathcal{S}_{1,2}(\mathbb{K})$ or $\mathcal{S}_{1,3}(\mathbb{K})$. Suppose (X_p, Ξ_p) is not always isomorphic to $\mathcal{S}_{1,2}$. Then there exists a point r such that X_r is $\mathcal{S}_{1,3}(\mathbb{K})$. For every point q collinear with r, the residue X_q is also isomorphic to $\mathcal{S}_{1,3}(\mathbb{K})$ as there is a 4-space containing q and r. Now consider a point s not collinear with r. Then in $X([r, s])$ there are points collinear with both r and s. Repeating the above argument twice yields that X_s is projectively equivalent to $\mathcal{S}_{1,3}(\mathbb{K})$. $\square$

We now define an incidence geometry $\mathcal{G}(X)$ with elements of Type 1, 2 and 3, which we will prove to be a projective space of dimension 5 or 4, corresponding to the conclusion (1) or (2), respectively, of Proposition 5.2. We will treat both cases at once, so let f be in $\{3, 4\}$. The elements of Type 1 are the singular f-spaces of X, the elements of Type 2 are the points of

X, and the elements of Type 3 are the singular planes of X not contained in f-spaces. The incidence relation is inclusion made symmetric except for elements of Type 1 and 3, which we declare incident if the corresponding singular f-space intersects the corresponding singular plane in a line.

We now show that the Type 1, 2 and 3 elements of $\mathcal{G}(X)$, with the given incidence, are the points, lines and planes of a projective space over $\mathbb{K}$ (but not of dimension 3). Notice that collinear points of X correspond to "concurrent" elements of Type 2 in $\mathcal{G}(X)$.

LEMMA 5.3. — *Every pair of distinct elements of $\mathcal{G}(X)$ of Type 1 is incident with a unique element of Type 2.*

Proof. — We have to show that two singular f-spaces intersect in a unique point of X. First note that two such f-spaces intersect in at most a point, since, if they would have a line in common, we can consider the residue in a point of that line and obtain a contradiction in view of Proposition 5.2 (two $(f-1)$-spaces always belong to the same family of non-intersecting subspaces of the corresponding Segre variety).

Now consider two singular f-spaces Π_1 and Π_2 and let $p_1 \in \Pi_1$ and $p_2 \in \Pi_2$ be two arbitrary points. Consider a symp H containing p_1, p_2. Then H intersects both Π_1 and Π_2 in planes π_1, π_2, respectively (this follows from Proposition 5.2 by considering the residues in p_1 and p_2 and noting that in the Segre variety $\mathcal{S}_{1,f-1}(\mathbb{K})$, viewed as Mazzocca–Melone set, every symp intersects every singular $(f-1)$-space in a line). We claim that π_1 intersects π_2 in a point.

Indeed, if not, then they are disjoint and we can find a singular plane π intersecting π_1 in a line L and π_2 in a point x. Looking in the residue of a point on L, we conclude with the aid of Proposition 5.2 that π is a maximal singular subspace. But looking in the residue of x, we see that π corresponds to a line L in a Segre variety $\mathcal{S}_{1,f-1}(\mathbb{K})$ disjoint from the $(f-1)$-dimensional subspace corresponding to Π_2 (maximal singular subspaces of different dimension in a Segre variety always interest in a unique point). Hence L itself is contained in an $(f-1)$-dimensional subspace of $\mathcal{S}_{1,3}(\mathbb{K})$. Consequently π is contained in a singular f-space, a contradiction. The claim and the lemma are proved. $\qquad\square$

LEMMA 5.4. — *Every pair of distinct elements of Type 2 of $\mathcal{G}(X)$ which are incident with a common element of Type 1, are incident with a unique element of type 3.*

Proof. — Translated back to (X, Ξ), it suffices to prove that, if p_1, p_2 are two points of a singular k-space U, then the line $L := \langle p_1, p_2 \rangle$ is contained in

a unique singular plane π which is a maximal singular subspace. Existence follows from considering a symp through $\langle p_1, p_2 \rangle$; uniqueness follows from Lemma 4.4 (one could also use Proposition 5.2 here). □

LEMMA 5.5. — *The elements of Type 1 and 2 incident with a given element of Type 3 induce the structure of a projective plane over $\mathbb{K}$.*

Proof. — This follows immediately from the definition of $\mathcal{G}(X)$, and the fact that every singular line is contained in a unique singular f-space, by Proposition 5.2. □

It follows from Theorem 2.3 of [4] that $\mathcal{G}(X)$ is a projective space. We determine the dimension.

LEMMA 5.6. — *The dimension of $\mathcal{G}(X)$, as a projective space, equals $f + 1$.*

Proof. — Fix an element of Type 1, which is a singular f-space Π. The elements of Type 2 incident with it are the points $x \in \Pi$, and the elements of Type 3 incident with Π are the singular planes not contained in Π intersecting Π in a line. Since there is only one such singular plane for a given line of Π, we may identify each such plane with that line. Incidence between the Type 2 and Type 3 elements is the natural incidence in Π, and so we see that Type 2 and 3 elements incident with Π form a projective space of dimension f. Hence the dimension of $\mathcal{G}(X)$ as a projective space is equal to $1 + f$. □

We can now prove the case $d = 4$ of the Main Result.

PROPOSITION 5.7. — *A Mazzocca–Melone set of split type 4 in $\mathbb{P}^N(\mathbb{K})$ is projectively equivalent to one of the following:*

- *the line Grassmannian variety $\mathcal{G}_{3,1}(\mathbb{K})$, and then $N = 5$;*
- *the line Grassmannian variety $\mathcal{G}_{4,1}(\mathbb{K})$, and then $N = 9$;*
- *the line Grassmannian variety $\mathcal{G}_{5,1}(\mathbb{K})$, and then $N = 14$;*

The first case is not proper and corresponds to a hyperbolic quadric of Witt index 3.

Proof. — By the previous lemmas, if (X, Ξ) is proper, then $\mathcal{G}(X)$ is a projective space of dimension 4 or 5 over $\mathbb{K}$. Now, the lines of $\mathcal{G}(X)$ are the points of X, and the planar line pencils are the points in the non-trivial intersection of a singular 4-space and a singular 2-space not contained in a singular 4-space (this follows from Proposition 5.2 by picking a point in the intersection).
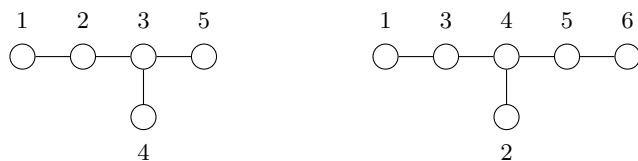
Figure 5.1. Dynkin diagrams for D_5 (on the left) and E_6 (on the right)

Consequently X, endowed with the singular lines, is an embedding of the line Grassmannian geometry $\mathcal{G}_{4,1}(\mathbb{K})$ or $\mathcal{G}_{5,1}(\mathbb{K})$. Proposition 4.17 implies the uniqueness of this embedding, and the assertions follow. □

Now we assume $d \in \{6,8\}$.

In order to identify the half-spin geometry variety $\mathcal{D}_{5,5}(\mathbb{K})$ and the variety $\mathcal{E}_{6,1}(\mathbb{K})$, we will use some basic theory of *diagrams*, as developed by Tits, culminating in the beautiful characterizations of building geometries by their diagrams in [33]. This works as follows.

A *building* is in fact a numbered simplicial chamber complex satisfying some axioms. One of the axioms says that every pair of chambers (i.e., every pair of simplices containing a vertex of every type) is contained in a thin subcomplex (this means that every panel — a panel is a simplex obtained from a chamber by removing one vertex — is contained in exactly two chambers). The other axioms then ensure that all these thin subcomplexes (called *apartments*) are isomorphic and when they are finite, then we talk about a *spherical* building. The type preserving automorphism group of each apartment is a Coxeter group. The type of the Coxeter group is also the *type* of the building. Attached to each Coxeter group is a Coxeter diagram. In most spherical types, this Coxeter diagram is just a Dynkin diagram with the arrow removed. In our current case, types $\mathsf{D_5}$ and $\mathsf{E_6}$, we have the diagrams of Figure 5.1.

The diagram now acts as an identity card for the geometry of the building. Each node stands for a type of vertices, and each (simple) edge stands for a family of residues isomorphic to the flag complex of a projective plane. Namely, let the nodes of an edge be labeled i and j (so these are types of vertices of the building); then, for any simplex F obtained from a chamber by removing the vertices of types i and j, the graph formed by the vertices of types i and j completing F to a panel, two vertices being adjacent if they form a simplex (this graph is a residue of type $\{i,j\}$), is the incidence graph of a projective plane. If the nodes labeled i and j are not on an edge in the diagram, then the similarly defined graph is complete bipartite.

Likewise, one defines the residue of any simplex. The rank of a residue is the difference between the size of a chamber and the size of the simplex in question.

Now, given a numbered chamber complex $\mathcal{C}$, we can define residues as in the previous paragraph. Given a graph Γ, we say that $\mathcal{C}$ *conforms to* Γ if the nodes of Γ can be identified with the types of vertices in such a way that, for each pair of types i, j, a residue of type $\{i, j\}$ is the incidence graph of a projective plane if $\{i, j\}$ is an edge of Γ, and a complete bipartite graph otherwise.

Now it follows from the main result in [33] and from work of Brouwer & Cohen [3] that a numbered chamber complex is isomorphic to a building of type D_5 or E_6 if and only if it conforms to the Coxeter diagram of type D_5 or E_6, respectively, and if all residues of rank at least 2 are *connected*, i.e., in each residue of rank at least 2, the graph on the vertices where two vertices are adjacent if they form a simplex, is connected (we say that the complex is *residually connected*). Moreover, by Theorem 6.13 of [32], given the field $\mathbb{K}$, there exists a unique building of each such type such that at least one residue isomorphic to a projective plane is the Pappian projective plane over $\mathbb{K}$. This means that, in order to prove that the Mazzocca–Melone sets in question are really the half-spin variety $\mathcal{D}_{5,5}(\mathbb{K})$ and the variety $\mathcal{E}_{6,1}(\mathbb{K})$, respectively, it suffices to define a residually connected numbered simplicial chamber complex where the points of the Mazzocca–Melone set are the vertices of type 5 and 1, respectively, and the singular lines are the vertices of type 3 (we use standard Bourbaki labelling, see [2]), and to show that it conforms to the diagram D_5 and E_6, respectively. That is exactly how we are going to proceed.

PROPOSITION 5.8. — *A Mazzocca–Melone set (X, Ξ) of split type 6 in $\mathbb{P}^N(\mathbb{K})$ is projectively equivalent to one of the following:*

- *the half-spin variety $\mathcal{D}_{4,4}(\mathbb{K})$, and then $N = 7$;*
- *the half-spin variety $\mathcal{D}_{5,5}(\mathbb{K})$, and then $N = 15$.*

The first case is not proper and corresponds to a hyperbolic quadric of Witt index 4.

Proof. — We may assume that (X, Ξ) is proper, as otherwise we have the half-spin variety $\mathcal{D}_{4,4}(\mathbb{K})$, which, by triality, is isomorphic to $\mathcal{D}_{4,1}$. By Lemma 4.15 we know that every residue is a Mazzocca–Melone set of split type 4, which is proper by Lemma 4.8. Since the ambient space of the residue has dimension at most 12 by (MM3), Proposition 5.7 implies that the residue corresponds to a line Grassmannian variety $\mathcal{G}_{4,1}(\mathbb{K})$.

Define the following numbered simplicial complex $\mathcal{G}(X)$ with vertices of types 1 up to 5. The vertices of type 1 are the symps of (X, Ξ), the ones of type 2 are the singular 3-spaces not contained in a singular 4-space, the type 3 vertices are the singular lines, the type 4 ones are the singular 4-spaces and, finally, the vertices of type 5 are the points belonging to X. We define an adjacency relation between two vertices of distinct type as containment made symmetric, except that a vertex of type 1 is incident with a vertex of type 4 if the corresponding 4-space intersects the corresponding hyperbolic quadric in a 3-space, and also that a vertex of type 2 is incident with one of type 4 if the corresponding 3-space intersects the corresponding 4-space in a plane. The simplices of $\mathcal{G}(X)$ are the cliques of the corresponding graph.

It is straightforward to verify that this simplicial complex is a chamber complex. We show that it conforms to the diagram of type D_5 (where we have chosen the types above so that they conform to the Bourbaki labeling [2]), which, by Tits [33], implies that $\mathcal{G}(X)$ is the building of type D_5 over $\mathbb{K}$. It follows easily from the definition of incidence in $\mathcal{G}(X)$ that any residue of type $\{1, 2, 3, 4\}$ of $\mathcal{G}(X)$ (say, of an element p of type 5) corresponds to the residue (X_p, Ξ_p) of (X, Ξ) as defined in the present paper. Hence all rank 2 residues of type $\{i, j\}$, with $\{i, j\} \subseteq \{1, 2, 3, 4\}$, are correct. Also, in the same way, the residues of type $\{2, 3, 4, 5\}$ of $\mathcal{G}(X)$ correspond to the geometry of the symps, which establishes the correctness of all rank 2 residues of type $\{i, j\}$, for all $\{i, j\} \subseteq \{2, 3, 4, 5\}$. It remains to check the residues of type $\{1, 5\}$. But these all are trivially complete bipartite graphs.

With the information of the previous paragraph, it is not difficult to see that $\mathcal{G}(X)$ is residually connected. Since type 5 vertices of $\mathcal{G}(X)$ correspond to elements of X, and type 3 to the singular lines, X is an embedding of the half-spin geometry $\mathcal{D}_{5,5}(\mathbb{K})$. Proposition 4.17 completes the proof of the proposition. □

PROPOSITION 5.9. — *A Mazzocca–Melone set (X, Ξ) of split type 8 in $\mathbb{P}^N(\mathbb{K})$ is projectively equivalent to one of the following:*

- *the variety $\mathcal{D}_{5,1}(\mathbb{K})$, and then $N = 9$;*
- *the variety $\mathcal{E}_{6,1}(\mathbb{K})$, and then $N = 26$.*

The first case is not proper and corresponds to a hyperbolic quadric of Witt index 5.

Proof. — We may assume that (X, Ξ) is proper. By Lemma 4.15 we know that every residue is a Mazzocca–Melone set of split type 6, which is proper

by Lemma 4.8. Hence it follows from Proposition 5.8 that every residue is a half-spin variety $\mathcal{D}_{5,5}(\mathbb{K})$.

Define the following numbered simplicial complex $\mathcal{G}(X)$ with vertices of types 1 up to 6. The *vertices of type* $1, \ldots, 6$ are the points of X, singular 5-spaces, singular lines, singular planes, singular 4-spaces which are not contained in a singular 5-space, and symps, respectively. First defining a graph on these vertices, adjacency is containment made symmetric, except in the following two cases. A singular 5-space and a singular 4-space which is not contained in a singular 5-space are adjacent if they intersect in a 3-space, and a singular 5-space and a symp are adjacent if their intersection is a 4-dimensional singular space. The simplices of $\mathcal{G}(X)$ are again the cliques of the graph just defined. It is routine to check that this is a chamber complex.

We claim that $\mathcal{G}(X)$ is a building of type E_6 over the field $\mathbb{K}$, with standard Bourbaki [2] labeling of the types.

We now show that $\mathcal{G}(X)$ conforms to the diagram of type E_6. Since the residue in $\mathcal{G}(X)$ of an element of type 6 is a symp, we already know that the residues of types $\{i, j\} \subseteq \{1, 2, 3, 4, 5\}$ are the correct ones. Moreover, it follows directly from the definition of $\mathcal{G}(X)$ that the residue of an element p of type 1 in $\mathcal{G}(X)$ corresponds to the residue (X_p, Ξ_p) in (X, Ξ). Since the latter is the half-spin variety $\mathcal{D}_{5,5}(\mathbb{K})$, the residues of type $\{i, j\} \subseteq \{2, 3, 4, 5, 6\}$ are also the correct ones. It remains to check that the residues of type $\{1, 6\}$ are complete bipartite graphs, which is straightforward.

With the information of the previous paragraph, it is not difficult to see that $\mathcal{G}(X)$ is residually connected. Since type 1 elements of $\mathcal{G}(X)$ correspond to elements of X, and type 3 to the singular lines, X is an embedding of the geometry $\mathcal{E}_{6,1}(\mathbb{K})$. Proposition 4.17 completes the proof of the proposition. $\qquad\square$

6. The cases of non-existence

In this section, we show the nonexistence of proper Mazzocca–Melone sets of split type d for $d \in \{5, 7, 9\}$ and $d \geqslant 10$. There are essentially two arguments for that: one ruling out $d = 5$ (and then use Corollary 4.15 to get rid of $d = 7, 9$ as well), and one for $d \geqslant 10$. Interestingly, the argument to rule out the general case $d \geqslant 10$ can not be used for smaller values of d, whereas the argument in Corollary 4.15 cannot be pushed further to include higher values of d.

6.1. Mazzocca–Melone sets of split type 5, 7 and 9

By Corollary 4.15, a proper Mazzocca–Melone set of split type 7 or 9 does not exist as soon as we show that no such set of split type 5 exists. The latter is the content of this subsection.

So let (X, Ξ) be a proper Mazzocca–Melone set of split type 5. Select a point $p \in X$ and consider the residue (X_p, Ξ_p), which is a pre-Mazzocca–Melone set of split type 3 in $\mathbb{P}^k(\mathbb{K})$, for $k \leqslant 9$. We note that the proof of Corollary 4.15 implies that the tangent space at each point of (X_p, Ξ_p) has dimension at most 7. If the dimension is 6, for every point, then we have a proper Mazzocca–Melone set and have reached our desired contradiction, in view of Proposition 2.1. So we may assume that the dimension is 7, for at least one point.

Our argument is based on the following observation.

LEMMA 6.1. — Let $x \in X_p$ be such that $\dim T_x = 7$, and let H_1, H_2, H_3 be three different symps of (X_p, Ξ_p) containing x. Suppose $\dim\langle H_1, H_2\rangle = 8$, suppose also $\dim(\langle H_2, H_3\rangle) = 7$ and $\dim\langle T_x(H_1), T_x(H_2), T_x(H_3)\rangle = 7$. Then $\dim\langle H_1, H_3\rangle < 8$.

Proof. — Suppose, by way of contradiction, that $\dim\langle H_1, H_3\rangle = 8$.

Let Σ be a subspace of C_p $(= \langle X_p\rangle)$ complementary to $\langle H_1\rangle$ and consider the projection of $X_p \setminus H_1$ with center $\langle H_1\rangle$ onto Σ. By assumption, $\langle H_2\rangle$ and $\langle H_3\rangle$ are projected onto 3-spaces of the $(k-5)$-space Σ. Suppose, by way of contradiction, that some point y_2 of H_2 not collinear to x is projected onto the same point as some point y_3 of H_3 not collinear to x. Then the line $\langle y_2, y_3\rangle$ is a singular line intersecting H_1 in some point y_1. The Quadrangle Lemma immediately implies that y_1 is not collinear with x. Now, by assumption, H_2 and H_3 share a line L containing x. There is a unique point z_i on L collinear to y_i in H_i, for $i \in \{2, 3\}$. If $z_2 \neq z_3$, then the Quadrangle Lemma applied to y_2, z_2, z_3, y_3 leads to $H_2 = H_3$, a contradiction. If $z_2 = z_3$, then the Quadrangle Lemma implies that $\langle y_2, y_3, z_2\rangle$ is a singular plane, which contains y_1, implying that $\langle y_1, z_2\rangle$ is a singular line. If u is any point of H_1 collinear with both x and y_1, then the Quadrangle Lemma applied to x, u, y_1, z_2 implies $z_2 \in H_1$, so the only possibility is $z_2 = x$. But then x is collinear with y_2, a contradiction. We conclude that no point of H_2 not collinear to x is projected onto the same point as some point of H_3 not collinear to x.

If $k \leqslant 8$, then $k = 8$ and since the set of points of H_i, $i = 2, 3$, not collinear to x is projected surjectively onto some affine part of Σ, these two affine parts intersect nontrivially, contradicting the previous paragraph.

Hence $k = 9$. Let U_i be the span of the projection of H_i, $i = 2, 3$. Then $U_2 \cap U_3$ is a plane π, which contains the projection x_L of L. Let α_i be the projection of $T_x(H_i) \setminus \{x\}$, $i = 2, 3$. Then every point of $U_i \setminus \alpha_i$ is the projection of a unique point of H_i not collinear with x. The first paragraph of this proof implies that π cannot intersect both α_2 and α_3 in a line. Hence we may assume that $\alpha_2 = \pi$. The condition $\dim\langle T_x(H_1), T_x(H_2), T_x(H_3)\rangle = 7$ forces $\alpha_3 \neq \pi$. So it is easy to see that we can find a point in π which is the projection of a point $w \in H_3$ not collinear to x, and also the projection of a singular line K of H_2 through x. In the 5-space generated by H_1 and K, the plane $\langle w, K \rangle$ intersects $\langle H_1 \rangle$ in a line, which must, by Axiom (MM2), completely belong to X. Now it is easy to see that this forces $\langle w, K \rangle$ to be a singular plane, contradicting the fact that w and x are not X_p-collinear. $\square$

We also need another lemma in case the field $\mathbb{K}$ is finite.

LEMMA 6.2. — *Let $|\mathbb{K}| = q$ be finite and consider two non-X_p-collinear points x, y in X_p. If there are v singular lines through y inside X_p, and there are no singular planes through y intersecting $X_p([x, y])$ in a line, then there are at least $(v - q - 1)q + 1$ different symps in X_p containing x.*

Proof. — Exactly $v - q - 1$ singular lines through y are not contained in the symp ξ determined by x and y. On these lines lie in total $(v - q - 1)q + 1$ points (including y). The Quadrangle Lemma implies that no two symps defined by these points and x coincide. The lemma follows. $\square$

We now consider the residue $((X_p)_{p'}, (\Xi_p)_{p'})$ of a point p' of X_p and assume that it spans a space of dimension 6. We denote this residue briefly by $(X', \Xi') \subseteq \mathbb{P}^6(\mathbb{K})$. Its structure is explained by Lemma 4.7, from which we in particular recall that, if X' does not contain singular lines, then (X', Ξ') is a pre-Mazzocca–Melone set of split type 1.

LEMMA 6.3. — *Let C be a conic of (X', Ξ') and $x \in X' \setminus C$. Then there exists at most one conic containing x and disjoint from C.*

Proof. — Suppose that there are at least two conics D_1, D_2 containing x and disjoint from C. We claim that $\dim(\langle C, D_1, D_2 \rangle) = 5$. Indeed, this dimension is at least 5, by Axiom (MM2) and the fact that D_1 and C are disjoint, hence span a 5-space. So suppose the dimension is 6 (this is the maximum). Denote by H_1 the symp in X_p corresponding to C, likewise for H_2 corresponding to D_1 and H_3 to D_2. Then, since the intersection of $\langle H_i \rangle$ and $\langle H_j \rangle$ is contained in a singular subspace through p', $i, j = 1, 2, 3$, $i \neq j$, we see that $\dim\langle H_1, H_2 \rangle = \dim\langle H_1, H_3 \rangle = 8$ and $\dim\langle H_2, H_3 \rangle = 7$. Moreover, by our assumption, $\dim\langle T_{p'}(H_1), T_{p'}(H_2), T_{p'}(H_3)\rangle = 6 + 1 = 7$. This contradicts Lemma 6.1. The claim is proved.

Let $y \in X'$ be a point not in $\langle C, D_1, D_2 \rangle$ and let z_1 be a point of $D_1 \setminus \{x\}$. Since there is at most one singular line through y meeting D_1, we may choose z_1 in such a way that there is a conic $E \subseteq X'$ containing z_1 and y. Now $\langle C, D_1, E \rangle$ coincides with $\mathbb{P}^6(\mathbb{K})$ and hence by Lemma 6.1 the conic E intersects C, say in the point u. By the same token, now observing that $\langle E, C, D_2 \rangle$ coincides with $\mathbb{P}^6(\mathbb{K})$, the conic E intersects D_2, say in z_2. Now the points u, z_1, z_2 of E generate $\langle E \rangle$ and hence $E \subseteq \langle C, D_1, D_2 \rangle$, contradicting $y \notin \langle C, D_1, D_2 \rangle$. This contradiction proves the lemma. □

For the rest of this section, we assume that $\dim \langle X' \rangle = 6$. Our intention is to show, in a series of lemmas, that this can only happen when $|\mathbb{K}| = 2$.

LEMMA 6.4. — *The set X' does not contain singular 3-spaces.*

Proof. — Let, for a contradiction, U be a singular ℓ-space with $\ell \geqslant 3$ and ℓ maximal. Since X' contains at least one plane π such that $\pi \cap X'$ is a conic, we see that $\ell \leqslant 4$. If $\ell = 4$, then we consider a point $x \in X'$ outside U. For any $u \in U$, the line $\langle u, x \rangle$ is non-singular, as repeated use of Lemma 4.4 would otherwise lead to a singular subspace of dimension 5. Pick two distinct points $u, v \in U$. So we have a conic $C \subseteq X$ through x and u, and for each point y of $C \setminus \{u\}$, we have a conic C_y containing v and y. Let y_1, y_2 be two distinct points of $C \setminus \{u\}$. An arbitrary 5-space W through U not containing the tangent lines at v to the conics C_{y_1} and C_{y_2}, respectively, intersects C_{y_i} in a point z_i, $i = 1, 2$. The line $\langle z_1, z_2 \rangle$ intersects U and so is singular, a contradiction.

Next suppose $\ell = 3$. Let π be a plane in $\mathbb{P}^6(\mathbb{K})$ skew to U. If π contains a conic, then, as before, the projection with center π onto U is injective, implying that $X' = U \cup (\pi \cap X')$, a contradiction. Hence every conic intersects U. Also, the projection of $X' \setminus U$ with center U onto π is injective, as the line joining two points with same image must meet U and hence is singular, a contradiction as above. If $|\mathbb{K}| = q$ is finite, then there are $q^3 + q^2 + q + 1$ conics through a fixed point of X' outside U and some point of U, giving rise to $q^4 > q^2 + q + 1 = |\pi|$ points of X' outside U, a contradiction. So we may assume that $\mathbb{K}$ is infinite. We claim that no 4-space through U contains a tangent T to a conic C at a point of U and a point v of $X' \setminus U$. Indeed, let $u \in C \setminus U$. Then the conic D containing u and v is contained in $\langle U, C \rangle$ (as D contains a point of U), contradicting the injectivity of the projection with center U onto π (C and D project into the same line). This implies that all conics in X' through the same point u of U project onto ("affine") lines of π sharing the same point p_u ("at infinity") corresponding to the tangents to these conics at u. For different u, the points p_u are also different as otherwise, by injectivity of the projection, we find two conics

through a common point of $X_p \setminus U$ intersecting in all points but the ones in U, a contradiction. This now implies that two different conics containing a (possibly different) point of U meet in a unique point of X. We choose a plane $\alpha \subseteq U$ and project $X' \setminus \alpha$ from α onto some skew 3-space Σ. Let u_i, $i = 1, 2, 3$, be three distinct points in α. The conics through these points project onto three families of ("affine") lines such that lines from different families intersect in a unique point. Considering two families, we see that, since $|\mathbb{K}| > 2$, these lie either on a hyperbolic quadric, and the third family cannot exist, or in a plane. In the latter case, we easily see that all points of $X' \setminus U$ are contained in a 5-space together with α, a contradiction considering a conic through some point of $U \setminus \alpha$. $\qquad\square$

LEMMA 6.5. — *The field $\mathbb{K}$ is finite.*

Proof. — Assume $\mathbb{K}$ is infinite. If there is a non-trivial singular subspace in X', then we let U be a singular line; if there is no (non-trivial) singular subspace, then we let U be a quadratic plane. Consider a point $x \in U \cap X'$ and let $\mathcal{F}$ be the family of all conics in X' containing x. Pick a point $x' \in (U \cap X') \setminus \{x\}$ and let $C_1, \ldots, C_4$ be four distinct conics containing x'. By Lemma 6.3, there are at most four members of $\mathcal{F}$ that do not intersect all of $C_1, \ldots, C_4$. Hence we can find three members $D_1, D_2, D_3 \in \mathcal{F}$ intersecting all of $C_1, \ldots, C_4$. We now project $X' \setminus U$ with center U onto a complementary space of U in $\langle X' \rangle$. Using Axiom (MM2), this projection is injective on the set of points not X'-collinear with a point of U. Hence we see that the projections $C'_1, \ldots, C'_4$ of $C_1 \setminus \{x\}, \ldots, C_4 \setminus \{x\}$ are contained in lines which meet the three lines spanned by the projections of $D_1 \setminus \{x'\}, D_2 \setminus \{x'\}, D_3 \setminus \{x'\}$ in distinct points, and hence these lines are contained in either a plane π, or a hyperbolic quadric $\mathcal{H}$ in some 3-space. Now, every member of $\mathcal{F}$ intersects at least three of $C_1, \ldots, C_4$. Consequently, the projections of the members of $\mathcal{F}$ are contained in π or in $\mathcal{H}$, and in the latter case, they are contained in lines belonging to one system of generators, while the projections of the conics through x' are contained in lines of the other system. Considering a third point x'' in $U \cap X'$, we obtain a third set of lines in π or in $\mathcal{H}$. In the latter case, all these lines intersect infinitely many generators from each system, a contradiction. Hence all points of X' not X'-collinear with a point of U are contained, together with U, in either a 4-space (if U is singular), or a 5-space (if U is a quadratic plane). This is a contradiction as soon as U is not singular, or U is singular and not contained in a singular subspace of dimension at least 3 (as otherwise X' does not span a 6-space). But there are no singular

3-spaces by the previous lemma. This contradiction completes the proof of the lemma. □

LEMMA 6.6. — *There are no singular planes.*

Proof. — Suppose there is a singular plane U. By Lemma 6.5, we may assume that $|\mathbb{K}| = q$, a prime power. Consider a point $x \in X' \setminus U$. By joining with points from U, we obtain a set of $q^2 + q + 1$ conics through x, and, in view of Lemma 6.3, at least $q^2 + q$ of them must meet an arbitrary conic in X' not containing x, contradicting $q^2 + q > q + 1$. □

LEMMA 6.7. — *The field $\mathbb{K}$ has only two elements; moreover there are no singular lines in X' and $|X'| = 7$ (hence every pair of conics intersects).*

Proof. — By Lemma 6.6 and Lemma 4.4, X' cannot contain two intersecting singular lines. Let $|\mathbb{K}| = q$ (which is allowed by Lemma 6.5). Suppose first that there is some singular line L (and remember that there is no singular plane by the previous lemma). We project $X' \setminus L$ with center L onto a suitable 4-space Θ. We consider a conic C containing a point x of L. If $x \neq x' \in L$, then Lemma 6.3 assures that all points of X' are obtained by considering all conics through x' and a point of C, except possibly for the points on one more conic through x'. Hence either there are $q^2 + q + 1$ or $(q + 1)^2$ points in X'. In the former case, the projections of the conics through x and x' form the systems of generators (except for one generator of each system) of a hyperbolic quadric in a hyperplane of Θ, or are contained in a plane, hence $\dim \langle X' \rangle \leqslant 5$. In the latter case, there are exactly $(q+1)^2$ conics meeting L. Through a point z of $X' \setminus L$, there are $q+1$ conics meeting L, taking account of $q^2 + q + 1$ points of X'; hence there is room for either one more conic, or a singular line L' through z. In the latter case, varying z on L', we see that every conic that intersects L also intersects L'. It follows that, if there were at least three singular lines, then all points of X' are contained in the span of these lines, which is 5-dimensional. Hence we may assume that there are at most two singular lines. But then, there is a conic not intersecting any singular line. We now first look at the case in which there are no singular lines.

If there are no singular lines, then we consider a point x and a conic $C \not\ni x$. The number of conics through x is either $q+1$ (and then $|X'| = q^2+q+1$ and there are $q^2 + q + 1$ conics), or $q + 2$ (there are no other possibilities by applying Lemma 6.3 to x and C), in which case $|X'| = (q + 1)^2$ and $|\Xi'| = (q + 1)(q + 2)$.

Assume now that $q > 2$. By the foregoing, we have at most $(q+1)(q+2)$ conics, at least $q^2 + q + 1$ points and at least one conic intersecting no

singular line. As this holds for every residue, Lemma 6.2 implies $q^3 + 1 \leqslant (q+1)(q+2)$, which is only possible when $q = 2$.

If $q = 2$ and if we do have singular lines, then we also have a conic not intersecting any singular line. In this case, we have nine points. Note that this is the maximum number of points possible, implying that we have at most twelve conics. But since we have nine points in X', we have nine singular lines in X_p through the point p'. The conic in X' not intersecting any singular line corresponds to a symp H of X_p through p'. Take a point x on H not X_p-collinear with p'. Putting $p' = y$ in Lemma 6.2 (and x remains x), the conditions are satisfied with $v = 9$ and $q = 2$, and so the residue through x contains $(v - q - 1)q + 1 = 13$ conics, exceeding the maximum number possible, a contradiction. The same argument works if we have nine points without singular lines, of course. Hence we only have seven points and hence every pair of conics intersects.   □

Hence it follows that, if $|\mathbb{K}| > 2$, then (X_p, Ξ_p) is a proper Mazzocca–Melone set of split type 3, which does not exist by Proposition 2.1. But if $|\mathbb{K}| = 2$, then either $\dim\langle X'\rangle \leqslant 5$, or we have seven points and every pair of conics intersects. Now (X_p, Ξ_p) cannot exist by [26, Remark 5.5].

Hence we have shown that proper Mazzocca–Melone sets of split type 5 cannot exist. Taking account of Corollary 4.15, this implies:

PROPOSITION 6.8. — *Proper Mazzocca–Melone sets of split type* $d \in \{5, 7, 9\}$ *do not exist.*

6.2. Mazzocca–Melone sets of split type at least 10

Here, we assume that (X, Ξ) is a Mazzocca–Melone set of split type $d \geqslant 10$. We show that it necessarily is a non-proper one. Note that, this time, we cannot use Lemma 4.15 anymore to apply induction. The idea is now to use Lemma 4.12 in a completely different way, namely, to use wrinkles as centers to project the residues in order to get the dimensions small enough so that (MM3) is satisfied. Lemma 4.12 assures that (MM2) is preserved, which is not the case for arbitrary projections.

PROPOSITION 6.9. — *Proper Mazzocca–Melone set of split type* $d \geqslant 10$ *do not exist.*

Proof. — We use induction to show this, including the cases $d = 8, 9$, which were already handled. For $d = 8$, we moreover assume that the ambient projective space has dimension at most 19. Then the cases $d = 8, 9$

do not occur by Propositions 5.9 and 6.8. The cases $d = 8, 9$ are the base of our inductive argument.

Now let $d \geqslant 10$. Let $p \in X$ be arbitrary, and let $x \in X$ be collinear to p. Our principal aim is to show that $T_p \cap T_x$ has dimension at most $2d - 3$.

Consider an arbitrary symp H through x which does not contain p, and which exists by Lemma 4.8. Suppose first that T_p intersects $\langle H \rangle$ in a subspace U_p of dimension at most $d-3$. Then, as $T_x(H) \cap U_p$ has dimension at most $d - 3$, we can find a plane π in $T_x(H)$ disjoint from U_p. Since the plane π has empty intersection with T_p, we have $\dim(T_p \cap T_x) \leqslant 2d - 3$.

Now suppose that U_p has dimension at least $d - 2$. By Lemma 4.2, the set of points of H collinear with p is contained in a maximal singular subspace M_1 of H. We can now take a second maximal singular subspace M_2 of H disjoint from M_1. The intersection $M_2 \cap U_p$ has dimension at least $\lfloor \frac{d}{2} \rfloor + (d-2) - (d+1) = \lfloor \frac{d}{2} \rfloor - 3 \geqslant 2$. Hence we can find a plane $\pi \subseteq M_2 \cap U_p$. All points of π are wrinkles of p.

Let X'_p be the set of points of X collinear with p. Lemma 4.12 implies that the projection Y'_p of X'_p with center π onto a subspace U of T_p complementary to π (and of dimension at most $2d - 3$) and containing p is injective and induces an isomorphism on the span of any two subspaces $T_p(H_1), T_p(H_2)$. Let Ξ'_p be the family of projections of subspaces $T_p(\xi)$, with ξ ranging through all quadratic spaces containing p. Now we consider a hyperplane V of U not containing p and put $Y_{p,\pi} = U' \cap Y'_p$ and $\Xi_{p,\pi} = \{\xi' \cap U' : \xi' \in \Xi'_p\}$. Clearly, $Y_{p.\pi} \cap \xi^*$ is a split quadric and every member of $\Xi_{p,\pi}$ is a $(d-1)$-dimensional space. Moreover, the choice of π implies that the pair $(Y_{p,\pi}, \Xi_{p,\pi})$ satisfies (MM1) and (MM2). Now U' has dimension at most $2d - 4$, so (MM3) is satisfied trivially. By induction, such a pair does not exist.

Hence our principal aim is proved. Hence we know that, for all x collinear with p, the space $T_p \cap T_x$ has dimension at most $2d - 3$. This implies that the residue (X_p, Ξ_p) is a proper Mazzocca–Melone set of split type $d - 2$ in a projective space of dimension at most $2d - 1$. By induction, such a set does not exist. $\qquad\square$

7. Verification of the axioms

In this section we verify that the Mazzocca–Melone axioms hold in all examples listed in the Main Result. Our approach is almost completely geometric. We only have to know that an embedding in a projective space with given dimension exists. For that, we can refer to the literature (note these

embeddings are always established in an algebraic way). The advantage is that we do not have to introduce the rather long algebraic formulae leading to every construction. It also shows that the Mazzocca–Melone axioms are really natural, and we can see geometrically why they have to hold.

So let (X, Ξ) be either a Segre variety $\mathcal{S}_{p,q}(\mathbb{K})$, $p, q \geqslant 1$, $p+q \leqslant 4$, where Ξ is the family of 3-spaces spanned by the subvarieties isomorphic to $\mathcal{S}_{1,1}(\mathbb{K})$; or a line Grassmannian variety $\mathcal{G}_{p,1}(\mathbb{K})$, $p \in \{4, 5\}$, where Ξ is the family of 5-spaces spanned by the subvarieties isomorphic to $\mathcal{G}_{3,1}(\mathbb{K})$; or the half-spin variety $\mathcal{D}_{5,5}(\mathbb{K})$, where Ξ is the family of 7-spaces spanned by subvarieties isomorphic to a half-spin variety $\mathcal{D}_{4,4}(\mathbb{K})$; or the variety $\mathcal{E}_{6,1}(\mathbb{K})$, where Ξ is the family of 9-spaces spanned by subvarieties isomorphic to the variety $\mathcal{D}_{5,1}(\mathbb{K})$. Suppose X spans $\mathbb{P}^N(\mathbb{K})$.

We first show that for every member ξ of Ξ, the intersection $X \cap \xi$ is the desired subvariety (which is always isomorphic to some hyperbolic quadric).

This is clear for the Segre varieties and the line Grassmannian varieties since their definition implies immediately that every Segre or line Grassmannian, respectively, subvariety induced by a suitable subspace of the underlying projective space is also induced by a suitable subspace of the ambient projective space. Since the half-spin variety $\mathcal{D}_{5,5}(\mathbb{K})$ appears as residue in the variety $\mathcal{E}_{6,1}(\mathbb{K})$, it suffices to show the result for the latter.

We know that $N = 26$ in this case and that the residue in a point $p \in X$ is isomorphic to a half-spin variety $\mathcal{D}_{5,5}(\mathbb{K})$, which is embedded in a space of dimension at most 15 (note that we ignore Remark 4.16 here, which says that this dimension is precisely 15); hence the space T_p generated by all singular lines through p has dimension at most 16 (which already shows (MM3)). Now consider any symp H which is opposite p in the corresponding building of type E_6. Then no point of H is collinear with p in X. We claim that T_p and H generate the whole space $\mathbb{P}^{26}(\mathbb{K})$. Indeed, let $x \in X$ be an arbitrary point, which does not belong to H and which is not collinear with p. By 3.7 of [30], there is a unique symp H' containing p and x. Then, by the same reference, $H \cap H'$ is some point x'. Then x' and the points of H' collinear with p generate $\langle H' \rangle$, hence x belongs to that space and the claim is proved. It follows that all points are contained in the space generated by T_p and H. Since the former has dimension at most 16 and the latter dimension 9, and since the whole space has dimension 26, we see that T_p and $\langle H \rangle$ are disjoint. In particular, $\langle H \rangle$ does not contain p. Since p and H are essentially arbitrary, this shows that the space generated by any symp does not contain any point "opposite" that symp. Now suppose $\langle H \rangle$ contains a point $x \in X \setminus H$ not opposite H. Then, by 3.5.4 and 3.9

of [30], there is a 4-space U in H contained in a singular 5-space U' together with x. But Lemma 4.9 implies that U' contains points of H outside U, a contradiction. Our assertion is proved.

Remark 7.1. — In the previous argument, the two symps H and H' can be regarded as an arbitrary pair of symps meeting in one point. Since $T_p(H')$ belongs to T_p and is hence disjoint from $\langle H \rangle$, we see that $\langle H \rangle \cap \langle H' \rangle = \{x'\}$, proving (MM2) in this case.

We now verify the axioms for (X, Ξ).

Axiom (MM1) follows in each case directly from the (geometric) definition of the variety.

It is readily seen that the validity of (MM2) is inherited by the residues. Hence we only need to check (MM2) for the three varieties $\mathcal{S}_{2,2}(\mathbb{K}), \mathcal{G}_{5,1}(\mathbb{K})$ and $\mathsf{E}_{6,1}(\mathbb{K})$. Let us concentrate on the latter; for the former the proofs are similar.

Let H and H' be two symps. There are two possibilities. If $H \cap H'$ is a singleton, then (MM2) follows from Remark 7.1. If $H \cap H'$ is a 4-space, then (MM2) follows from Lemma 4.9 and the fact that $\langle H \rangle$ does not contain points of $H' \setminus H$.

Axiom (MM3) follows immediately from the dimensions of the universal embeddings of the residues.

This completes the proof of the Main Result.

Remark 7.2. — The assumptions of Theorem 5.1 of [24] present a variation of Axiom (MM3) involving the dimension of the space generated by the tangent spaces at a point x to the symps intersecting a given singular line none of whose points is collinear to x. In that case, one characterizes the same varieties as in the Main Result with $d \geqslant 2$, but additionally all Segre varieties $\mathcal{S}_{p,q}(\mathbb{K})$ for arbitrary $p, q \geqslant 1$, and all line Grassmannian varieties $\mathcal{G}_{p,1}(\mathbb{K})$, for $p \geqslant 3$. The proofs rely on the Main Result of the present paper. This result shows that our approach can also include the higher dimensional FTMS (the North-West 3×3 square for higher dimensions).

Acknowledgement

BIBLIOGRAPHY

[1] M. ASCHBACHER, "The 27-dimensional module for E_6, I.", *Invent. Math.* **89** (1987), p. 159-195.

[2] N. BOURBAKI, *Groupes et Algèbres de Lie, Chapitres IV, V and VI*, Actualités Scientifiques et Industrielles, vol. 1337, Hermann, 1968.

[3] A. E. BROUWER & A. M. COHEN, "Some remarks on Tits geometries.", *Indag. Math.* **45** (1983), p. 393-402.

[4] F. BUEKENHOUT & P. CAMERON, "Projective and Affine Geometry over Division Rings", in *Handbook of Incidence Geometry*, North-Holland, 1995, p. 27-62.

[5] P.-E. CHAPUT, "Severi varieties", *Math. Z.* **240** (2002), no. 2, p. 451-459.

[6] C. CHEVALLEY, *The Algebraic Theory of Spinors*, Columbia University Press, 1954, viii+131 pages.

[7] B. N. COOPERSTEIN & E. E. SHULT, "Frames and bases of Lie incidence geometries", *J. Geom.* **60** (1997), no. 1-2, p. 17-46.

[8] B. DE BRUYN & H. VAN MALDEGHEM, "Dual polar spaces of rank 3 defined over quadratic alternative division algebras", *J. Reine Angew. Math.* **715** (2016), p. 39-74.

[9] H. HAVLICEK, "Zur Theorie linearer Abbildungen I, II.", *J. Geom.* **16** (1981), p. 152-180.

[10] J. W. P. HIRSCHFELD & J. A. THAS, *General Galois Geometries*, Oxford Mathematical Monographs, Clarendon Press, 1991, xii+407 pages.

[11] A. ILIEV & D. MARKUSHEVICH, "Elliptic curves and rank-2 vector bundles on the prime Fano threefold of genus 7", *Adv. Geom.* **4** (2004), no. 3, p. 287-318.

[12] P. IONESCU & F. RUSSO, "Varieties with quadratic entry locus II.", *Compos. Math.* **144** (2008), no. 4, p. 949-962.

[13] ———, "Conic-connected manifolds", *J. Reine Angew. Math.* **644** (2010), p. 145-158.

[14] A. KASIKOVA & E. E. SHULT, "Absolute embeddings of point-line geometries", *J. Algebra* **238** (2001), no. 1, p. 265-291.

[15] O. KRAUSS, J. SCHILLEWAERT & H. VAN MALDEGHEM, "Veronesean representations of Moufang planes", *Mich. Math. J.* **64** (2015), p. 819-847.

[16] R. LAZARSFELD & A. VAN DE VEN, *Topics in the Geometry of Projective Space. Recent work of F. L. Zak*, DMV Seminar, vol. 4, Birkhäuser, 1984.

[17] L. MANIVEL, "On spinor varieties and their secants", *SIGMA Symmetry Integrability Geom. Methods Appl.* **5** (2009), Paper 078, 22 pp.

[18] F. MAZZOCCA & N. MELONE, "Caps and Veronese varieties in projective Galois spaces", *Discrete Math.* **48** (1984), p. 243-252.

[19] S. MUKAI, "Curves and symmetric spaces, I.", *Am. J. Math.* **117** (1995), no. 6, p. 1627-1644.

[20] O. NASH, "K-theory, LQEL manifolds and Severi varieties", *Geom. Topol.* **18** (2014), no. 3, p. 1245-1260.

[21] F. RUSSO, "Varieties with quadratic entry locus I.", *Math. Ann.* **344** (2009), no. 3, p. 597-617.

[22] J. SCHILLEWAERT & H. VAN MALDEGHEM, "Hermitian Veronesean caps", in *Buildings, finite geometries and groups (ICM, Hyderabad, India, 2010)*, Springer Proceedings in Mathematics, vol. 10, Springer, 2012, p. 175-191.

[23] ———, "Quadric Veronesean caps", *Bull. Belg. Math. Soc. Simon Stevin* **20** (2013), no. 1, p. 19-25.

[24] ———, "Imbrex geometries", *J. Combin. Theory Ser. A* **127** (2014), p. 286-302.

[25] ———, "Projective planes over quadratic 2-dimensional algebras", *Adv. Math.* **262** (2014), p. 784-822.

[26] ———, "A combinatorial characterization of the Lagrangian Grassmannian LG(3, 6)", *Glasg. Math. J.* **58** (2016), no. 2, p. 293-311.

[27] E. E. Shult, *Points and Lines. Characterizing the Classical Geometries*, Universitext, Springer, 2011, xxii+676 pages.

[28] J. A. Thas & H. Van Maldeghem, "Characterizations of the finite quadric Veroneseans $\mathcal{V}_n^{2^n}$", *Q. J. Math.* **55** (2004), p. 99-113.

[29] J. Tits, *Sur certaines classes d'espaces homogènes de groupes de Lie*, Acad. Roy. Belg. Cl. Sci. Mém. Collect., 8°, vol. 29(3), Palais des académies, 1955, 268 pages.

[30] ———, "Sur la géometrie des *R*-espaces", *J. Math. Pures Appl.* **36** (1957), p. 17-38.

[31] ———, "Algèbres alternatives, algèbres de Jordan et algèbres de Lie exceptionnelles", *Nederl. Akad. Wet., Proc.* **28** (1966), p. 223-237.

[32] ———, *Buildings of Spherical Type and Finite BN-Pairs*, Lecture Notes in Mathematics, vol. 386, Springer, 1974, x+299 pages.

[33] ———, "A local approach to buildings", in *The Geometric Vein. The Coxeter Festschrift*, Springer, 1982, p. 519-547.

[34] A. L. Wells, "Universal projective embeddings of the Grassmannian, half spinor, and dual orthogonal geometries", *Q. J. Math., Oxf.* **34** (1983), p. 375-386.

[35] F. L. Zak, *Tangents and Secants of Algebraic Varieties*, Translation of Mathematical Monographs, vol. 127, American Mathematical Society, 1983, vii+164 pages.

[36] C. Zanella, "Universal properties of the Corrado-Segre embedding", *Bull. Belg. Math. Soc. Simon Stevin* **3** (1996), no. 1, p. 65-79.

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