ANNALES DE L'INSTITUT FOURIER

ANNALES

DE

L'INSTITUT FOURIER

Piotr POKORA, Xavier ROULLEAU & Tomasz SZEMBERG

Bounded negativity, Harbourne constants and transversal arrangements of curves

Tome 67, nº 6 (2017), p. 2719-2735.

<http://aif.cedram.org/item?id=AIF_2017__67_6_2719_0>

Ann. Inst. Fourier, Grenoble
67, 6 (2017) 2719-2735

BOUNDED NEGATIVITY, HARBOURNE CONSTANTS AND TRANSVERSAL ARRANGEMENTS OF CURVES

by Piotr POKORA,
Xavier ROULLEAU & Tomasz SZEMBERG (*)

Abstract. — The Bounded Negativity Conjecture predicts that for every complex projective surface X there exists a number $b(X)$ such that $C^2 \geqslant -b(X)$ holds for all reduced curves $C \subset X$. For birational surfaces $f : Y \to X$ there have been introduced certain invariants (Harbourne constants) relating to the effect the numbers $b(X)$, $b(Y)$ and the complexity of the map f. These invariants have been studied when f is the blowup of all singular points of an arrangement of lines in $\mathbb{P}^2$, of conics and of cubics. In the present note we extend these considerations to blowups of $\mathbb{P}^2$ at singular points of arrangements of curves of arbitrary degree d. The main result in this direction is stated in Theorem B. We also considerably generalize and modify the approach witnessed so far and study transversal arrangements of sufficiently positive curves on arbitrary surfaces with the non-negative Kodaira dimension. The main result obtained in this general setting is presented in Theorem A.

Résumé. — La conjecture de la négativité bornée prédit que pour toute surface complexe projective X, il existe un nombre $b(X)$ tel que l'inégalité $C^2 \geqslant -b(X)$ ait lieu pour toute courbe réduite $C \subset X$. Pour un morphisme birationnel $f : Y \to X$, certains invariants (les constantes de Harbourne) ont été introduits afin de relier les nombres $b(X)$ et $b(Y)$ à la complexité de f. Ces invariants ont été étudiés quand f est l'éclatement en tous les points singuliers d'un arrangement de droites, de coniques et de cubiques. Dans cette note, nous étendons ces considérations aux éclatements de $\mathbb{P}^2$ aux points singuliers d'arrangements de courbes de degré arbitraire d. Le résultat principal dans cette direction est le théorème B. Ensuite, nous généralisons considérablement et modifions l'approche usuelle afin d'étudier

---

Keywords: curve arrangements, algebraic surfaces, Miyaoka inequality, blow-ups, negative curves, bounded negativity conjecture.
2010 *Mathematics Subject Classification:* 14C20, 14J70.
(*) The first author was partially supported by National Science Centre Poland Grant 2014/15/N/ST1/02102 and the project was conduct when he was a member of SFB/TR 45 *Periods, moduli spaces and arithmetic of algebraic varieties*. The last author was partially supported by National Science Centre, Poland, grant 2014/15/B/ST1/02197. The final version of this work was written down while the last author visited the University of Mainz. A generous support of the SFB/TR 45 *Periods, moduli spaces and arithmetic of algebraic varieties* is kindly acknowledged.
Finally we would like to thank the referee for many valuable comments which led to improvements in the readability of our manuscript.

les arrangements transverses de courbes suffisamment positives sur n'importe quelle surface ayant dimension de Kodaira positive ou nulle. Le principal résulat obtenu dans ce cadre général est le théorème A.

1. Introduction

In this note we find various estimates on Harbourne constants which were introduced in [1] in order to capture and measure the bounded negativity on various birational models of an algebraic surface. Our research is motivated by Conjecture 1.2 below which is related to the following definition:

Definition 1.1 (Bounded negativity). — *Let X be a smooth projective surface. We say that X* has bounded negativity *if there exists an integer $b(X)$ such that the inequality*

$$C^2 \geqslant -b(X)$$

holds for every reduced and irreducible *curve $C \subset X$.*

The bounded negativity conjecture (BNC for short) is one of the most intriguing problems in the theory of projective surfaces and attracts currently a lot of attention, see e.g. [1, 2, 3, 5, 13].

Conjecture 1.2 (BNC). — *Every smooth* complex *projective surface has bounded negativity.*

It is well known that Conjecture 1.2 fails in positive characteristic. Hence from now on we restrict the attention to complex surfaces.

It has been showed in [2, Proposition 5.1] that no harm is done if one replaces irreducible curves in Definition 1.1 by arbitrary *reduced* divisors. It is clear that in order to obtain interesting, i.e. very negative curves on the blow up of a given surface one should study singular curves on the original surface. Whereas constructing irreducible singular curves encounters a number of obstacles (see e.g. [4]), reducible singular divisors are relatively easy to construct and control. In our set up singularities of reduced divisors arise solely as intersection points of irreducible components. In a series of papers [1, 12, 13] the authors study this situation for configurations of lines, conics and elliptic curves in $\mathbb{P}^2$. The arrangements studied so far were all modeled on arrangements of lines, in particular all curves were smooth and were assumed to intersect pairwise transversally. The technical advantage behind this assumption lies in the property that after blowing up all intersection points just once, we obtain a simple normal crossing divisor. Also working under this assumption for curves of higher degree seems to lead to the most singular divisors. Many singularities of a divisor lead to its negative arithmetic genus, which forces the divisor to split. Moreover, transversal arrangements allow to use some combinatorial identities, which fail when tangencies are allowed. For all these reasons it is reasonable to keep this assumption.

Definition 1.3 (Transversal arrangement). — *Let $D = \sum_{i=1}^{\tau} C_i$ be a reduced divisor on a smooth surface X. We say that D is a* transversal arrangement *if $\tau \geqslant 2$, all curves C_i are smooth and they intersect pairwise transversally.*

We denote by $\mathrm{Sing}(D)$ the set of all intersection points of components of D. The number of points in the set $\mathrm{Sing}(D)$ is denoted by $s(D)$ or, if D is understood, simply by s.

Furthermore we denote by $\mathrm{Esing}(D)$ the set of essential singularities *of D, i.e. those where at least 3 components meet.*

In the present note we study the bounded negativity and transversal arrangements on *fairly arbitrary* surfaces. Our main results are Theorems A and B.

Theorem A. — *Let A be a divisor on a smooth projective surface Y with Kodaira dimension $\kappa(Y) \geqslant 0$, such that for positive integers $\tau \geqslant 2$ and $d_1, \ldots, d_\tau \geqslant 1$ the following condition is satisfied:*

- ($\star$) *There exist smooth (irreducible) curves $C_1, \ldots, C_\tau$ in linear systems $|d_1 A|, \ldots, |d_\tau A|$ such that the divisor $D = \sum_{i=1}^{\tau} C_i$ is a transversal arrangement.*

Let $f : Z \to Y$ be the blow-up of Y at $\mathrm{Sing}(D)$ and denote by $\widetilde{D}$ the strict transform of D. Then

$$\widetilde{D}^2 \geqslant -\frac{9}{2}s - \left(\frac{3}{2}A^2 \sum_{i=1}^{\tau} d_i^2 + (K_Y \cdot A) \sum_{i=1}^{\tau} d_i + 2(3c_2(Y) - c_1^2(Y))\right).$$

The assumption $\kappa(Y) \geqslant 0$ guarantees that any finite branched covering of Y has also non-negative Kodaira dimension. If we can control the Kodaira dimension of a covering of Y in other way, then we can drop this assumption. This is the case in the next Theorem which addresses rational surfaces. Recently Dorfmeister [3] has announced a proof of Conjecture 1.2 for surfaces birationally equivalent to ruled surfaces (i.e. in particular for rational surfaces). This announcement has been taken back in the last days. Whereas this would be an exciting new development, it would not diminish the interest in effective bounds on Harbourne constants.

Theorem B. — *Let $D \subset \mathbb{P}^2$ be a transversal arrangement of $\tau \geqslant 4$ curves $C_1, \dots, C_\tau$ of degree $d \geqslant 3$ such that there are no points in which all τ curves meet, i.e. the linear series spanned by $C_1, \dots, C_\tau$ is base point free. Let $f : X_s \to \mathbb{P}^2$ be the blowup at* $\mathrm{Sing}(D)$ *and let $\widetilde{D}$ be the strict transform of D. Then we have*

$$\widetilde{D}^2 \geqslant \frac{9d\tau}{2} - \frac{5d^2\tau}{2} - 4s\,.$$

This result provides additional evidence for the following effective version of Conjecture 1.2 which predicts that there are uniform bounds for all blow ups of $\mathbb{P}^2$.

Conjecture 1.4 (Effective BNC for blowups of $\mathbb{P}^2$). — *Let $f : X_s \to \mathbb{P}^2$ the blow up of $\mathbb{P}^2$ in s arbitrary points. Let $D \subset \mathbb{P}^2$ be a reduced divisor and let $\widetilde{D}$ be the strict transform of D under f. Then one has $\widetilde{D}^2 \geqslant -4 \cdot s$.*

Our strategy is an extension of Hirzerbuch's results [7] for line configurations on the plane. The starting point is that (under some conditions) one can construct an abelian cover W of the studied surface branched along the chosen configurations of curves. If the singularities of these configurations are reasonable (simple crossings), the Chern numbers of that abelian cover (or rather its minimal resolution X) can be explicitly computed, and it turns out that these Chern numbers can be read off directly from combinatorics of the given configuration. Moreover, under some additional mild assumptions on multiplicities of singular points of the configuration, the surface X is of general type. The last step is made by the Miyaoka–Yau inequality $K_X^2 \leqslant 3e(X)$, which gives us the inequalities of Theorems A and B.

2. General preliminaries

We begin by introducing some invariants of transversal arrangements and pointing out their properties relevant for our purposes in this note.

Definition 2.1 (Combinatorial invariants of transversal arrangements). *Let $D = \sum_{i=1}^{\tau} C_i$ be a transversal arrangement on a smooth surface X. We say that a point P is an r-fold point of the arrangement D if there are exactly r components C_i passing through P. We say also that D has multiplicity $k_P = r$ at P.*

For $r \geqslant 2$ we set the numbers $t_r = t_r(D)$ to be the number of r-fold points in D. Thus $s(D) = \sum_{r=2}^{\tau} t_r(D)$.

These numbers are subject to the following useful equality, which follows by counting incidences in a transversal arrangement in two ways.

$$\sum_{i<j}(C_i \cdot C_j) = \sum_{r\geqslant 2}\binom{r}{2}t_r\,. \tag{2.1}$$

It is also convenient to introduce the following numbers

$$f_i = f_i(D) = \sum_{r\geqslant 2} r^i t_r\,.$$

In particular $f_0 = s(D)$ is the number of points in $\mathrm{Sing}(D)$.

Now we turn to Harbourne constants. They were first discussed at the Negative Curves on Algebraic Surfaces workshop in Oberwolfach in spring 2014 and were introduced in the literature as Hadean constants in [1]. In the present note we are interested in Harbourne constants attached to transversal arrangements. They can be viewed as a way to measure the average negativity coming from singular points in the arrangement.

Definition 2.2 (Harbourne constants of a transversal arrangement). *Let X be a smooth projective surface. Let $D = \sum_{i=1}^{\tau} C_i$ be a transversal arrangement of curves on X with $s = s(D)$. The rational number*

$$h(X;D) = h(D) = \frac{1}{s}\left(D^2 - \sum_{P\in\mathrm{Sing}(D)} k_P^2\right) \tag{2.2}$$

is the Harbourne constant of the transversal arrangement $D \subset X$.

The connection between Harbourne constants and the BNC is established by the following observation. If the Harbourne constants $h(X;D)$ (here we mean Harbourne constants for all curve configurations) on the fixed surface X are uniformly bounded from below by a number H, then BNC holds for all birational models $Y = \mathrm{Bl}_{\mathrm{Sing}(D)}X$ obtained from X by blowing up singular points of transversal arrangements D with $b(Y) = H \cdot s(D)$. The reverse implication might fail, i.e. it might happen that there is no uniform lower bound but nevertheless BNC may hold on any single model of X.

In case of the projective plane it is convenient to work with a more specific variant of Definition 2.2. In [1, Definition 3.1] the authors introduced the *linear Harbourne constant* as the infimum of quotients in (2.2), where one considers only divisors D splitting totally into lines. In [12] the conical Harbourne constant has been studied and in [13] the cubical Harbourne constant has been considered. Here we follow this line of investigation and introduce the following notion.

Definition 2.3 (Degree d Harbourne constant). — *The* degree d global Harbourne constant of $\mathbb{P}^2$ *is the infimum*

$$H_d(\mathbb{P}^2) := \inf_D h(\mathbb{P}^2; D),$$

taken over all transversal arrangements D of degree d curves in $\mathbb{P}^2$.

We will show in Section 4 bounds on the degree d Harbourne constants $H_d(\mathbb{P}^2)$ for arbitrary $d \geqslant 3$. The available bounds on the numbers $H_d(\mathbb{P}^2)$ are presented in Table 2.1.

d	lower bound on $H_d(\mathbb{P}^2)$	least known value of $H_d(\mathbb{P}^2)$
1	-4	$-225/67$
2	-4.5	$-225/68$

Table 2.1. Degree d global Harbourne constants

In the article [13] there is studied a series of configurations of smooth elliptic plane curves with Harbourne constants tending to -4. These configurations are not transversal (there are always 12 points where configuration curves are pairwise tangential). The following result is derived from Theorem B and, to the best of our knowledge, this is the first effective estimate on degree d Harbourne constants.

Corollary 2.4 (Degree d Harbourne constants). — *For any $d \geqslant 3$ we have*

$$H_d(\mathbb{P}^2) \geqslant \frac{9}{2}d - \frac{5}{2}d^2 - 4.$$

Remark 2.5. — Whereas the particular numbers appearing in Corollary 2.4 are rather high and leave space for improvements, the main interest of the Corollary lies in the conclusion that they are *finite* (which is by no means a priori obvious) and can be estimated effectively.

3. Bounded negativity and transversal arrangements on surfaces with Kodaira dimension $\kappa \geqslant 0$

In this section we will prove Theorem A. In fact, we will prove slightly more. We establish first the notation. Let Y be a smooth projective surface and let A be a semi-ample divisor on Y. We assume moreover that the following hypothesis holds for A and for integers $d_1, \ldots, d_\tau \in \mathbb{N}$, $\tau > 1$:

- There exist smooth (irreducible) curves $C_1, \ldots, C_\tau$ in linear systems $|d_1 A|, \ldots, |d_\tau A|$ such that the divisor $D = \sum_{i=1}^{\tau} C_i$ is a transversal arrangement.
- We assume moreover that either all numbers d_i are even, or there exist at least two odd numbers among them.

It is convenient to write now the equality (2.1) in the following form

$$A^2 \left(\sum d_i\right)^2 - A^2 \sum d_i^2 = f_2 - f_1 \,. \tag{3.1}$$

As a consequence we get

$$D^2 = A^2 \left(\sum d_i^2 + 2 \sum_{j<k} d_j d_k\right) = A^2 \left(\sum d_i^2 + f_2 - f_1\right). \tag{3.2}$$

Theorem 3.1. — *Let Y be a smooth projective surface with Kodaira dimension $\kappa \geqslant 0$. Let A be a divisor on Y satisfying above assumptions and let $D = \sum_{i=1}^{\tau} C_i$ be a transversal arrangement as above. Then*

$$\begin{aligned} H(Y; D) \geqslant -\frac{9}{2} + \frac{1}{f_0}\left(2t_2 + \frac{9}{8}t_3 + \frac{1}{2}t_4\right) \\ - \frac{1}{f_0}\left(\frac{3}{2}A^2 \sum d_i^2 - (K_Y \cdot A)\sum d_i - 2(3c_2(Y) - c_1^2(Y))\right). \end{aligned}$$

Our strategy for proving this statement will be to apply the refined Miyaoka inequality to a certain branched covering X, of Y. In order to prove that this branched covering does in fact exist, we need to recall some result of Namba: Let M be a manifold, let $D_1, \ldots, D_s$ be irreducible reduced divisors on M and let $n_1, \ldots, n_s$ be positive integers. We denote by D the divisor $D = \sum n_i D_i$. Let $Div(M, D)$ be the sub-group of the $\mathbb{Q}$-divisors generated by the entire divisors and:

$$\frac{1}{n_1} D_1, \ldots, \frac{1}{n_s} D_s \,.$$

Let $\sim$ be the linear equivalence in $Div(M, D)$, where $G \sim G'$ if and only if $G - G'$ is an entire principal divisor. Let $Div(M, D)/\sim$ be the quotient and let $Div^0(M, D)/\sim$ be the kernel of the Chern class map

$$\begin{array}{ccc} Div(M,D)/\sim & \longrightarrow & H^{1,1}(M, \mathbb{R}) \\ G & \longrightarrow & c_1(G)\,. \end{array}$$

Theorem 3.2 (Namba, [11, Theorem 2.3.20]). — *There exists a finite Abelian cover which branches at D with index n_i over D_i for all $i = 1, \ldots, s$ if and only if for every $j = 1, \ldots, s$ there exists an element of finite order*

$v_j = \sum \frac{a_{ij}}{n_i} D_i + E_j$ *of* $Div^0(M, D)/\sim$ *(where* E_j *is an entire divisor and* $a_{ij} \in \mathbb{Z}$*) such that* a_{jj} *is coprime to* n_j*.*

Then the subgroup in $Div^0(M, D)/\sim$ *generated by the* v_j *is isomorphic to the Galois group of such an Abelian cover.*

Let us now recall the following combination of results due to Miyaoka [9] and Sakai [14] which was formulated in this form for the first time by Hirzebruch.

Theorem 3.3 (Miyaoka–Sakai refined inequality [8, p. 144]). — *Let* X *be a smooth surface of non-negative Kodaira dimension and let* $E_1, \ldots, E_k$ *be configurations (disjoint to each other) of rational curves on* X *(arising from quotient singularities) and let* $C_1, \ldots, C_p$ *be smooth elliptic curves (disjoint to each other and disjoint to the* E_i*). Let* $c_1^2(X), c_2(X)$ *be the Chern numbers of* X*. Then*

$$3c_2(X) - c_1^2(X) \geqslant \sum_{j=1}^{p}(-C_j^2) + \sum_{i=1}^{k} m(E_i),$$

where the number $m(E_i)$ *depends on the configuration. For example, if* E_i *is a single* (-2)*-curve, then* $m(E_i) = \frac{9}{2}$ *by* [6].

Proof of Theorem 3.1. — Let be $\delta = 0$ if all the d_i's are even, and $\delta = 1$ otherwise. We apply Theorem 3.2, to the $\mathbb{Q}$-divisors $\frac{1}{2}(C_i - C_j)$ for d_i, d_j odd and $\frac{1}{2}C_j$ for d_j even) : there exists a $(\mathbb{Z}/2\mathbb{Z})^{\tau-\delta}$ abelian cover $\sigma : W \to Y$ ramified over D with order 2. We denote by $\rho : X \to W$ its minimal desingularization. We follow the ideas of Hirzebruch [7] for the computations of the Chern numbers of X.

For a singularity point P of D, let k_P be its multiplicity. Let $\pi : Z \to Y$ be the blowup at the $f_0 - t_2 = \sum_{k \geqslant 3} t_k$ singularities of D with multiplicities $k \geqslant 3$. Let $\widetilde{D} = \sum \widetilde{C}_i$ be the strict transform of D in Z and let E_P be the exceptional divisor over the point P. There exists a degree $2^{\tau-\delta}$ map

$$f : X \to Z$$

ramified over Z with the divisor $\widetilde{D}$ as the branch locus of order 2.

These constructions are summarized in the diagram in Figure 3.1.

There are $2^{\tau-\delta-k_P}$ copies of a smooth curve $F_P \subset X$ over $E_P \subset Z$. The curve F_P is a $(\mathbb{Z}/2\mathbb{Z})^{k_P-1}$-cover of E_P ramified with index 2 at k_P intersection points of E_P with $\widetilde{D}$. Thus

$$e(F_P) = 2^{k_P-1}(2 - k_P) + k_P 2^{k_P-2} = 2^{k_P-2}(4 - k_P).$$

Since the Galois group of f permutes these curves, we have $(F_P)^2 = -n^{k_P-2}$. If a singularity P of D is a double point, then X is smooth over P

$$\begin{tikzcd} X \arrow[r, "\rho"] \arrow[d, "f"'] & W \arrow[d, "\sigma"] \\ Z \arrow[r, "\pi"'] & Y \end{tikzcd}$$

Figure 3.1. Maps used in the proof of Theorem 3.1.

and the fiber of $\pi \circ f$ above P has $n^{\tau-\delta-2}$ points. Following Miyaoka [10, point G, p. 408], we define the genus $g = g(C)$ by

$$g - 1 = \sum_{i=1}^{\tau} (g_i - 1)\,, \tag{3.3}$$

where g_i is the genus of the irreducible component C_i of D, hence

$$2g_i - 2 = A^2 d_i^2 + (A \cdot K_Y) d_i\,.$$

Summing up over i we have

$$2g - 2 = A^2 \sum_{i=1}^{\tau} d_i^2 + (A \cdot K_Y) \sum_{i=1}^{\tau} d_i\,. \tag{3.4}$$

Similarly, using the additivity of the topological Euler numbers and (3.3) we have

$$e(D) = 2 - 2g + f_0 - f_1$$

and consequently

$$\begin{aligned} e(D \setminus \mathrm{Sing}(D)) &= 2 - 2g - f_1\,, \\ e(Y \setminus D) = e(Y) - e(D) &= e(Y) + 2g - 2 + f_1 - f_0\,. \end{aligned} \tag{3.5}$$

Using that if $U \to V$ is a degree n étale map one has $e(U) = ne(V)$, we obtain

$$e\left(X \setminus \bigcup_{P \in \mathrm{Esing}(D)} f^{-1} E_P\right) = 2^{\tau-\delta} e(Y \setminus D) + 2^{\tau-\delta-1} e(D \setminus \mathrm{Sing}(D)) + 2^{\tau-\delta-2} t_2\,.$$

Combining this with (3.5) we get

$$\frac{1}{2^{\tau-\delta-2}} e\left(X \setminus \bigcup_{P \in \mathrm{Esing}(D)} f^{-1} E_P\right) = 4\,(e(Y) + 2g - 2 + f_1 - f_0) + 2\,(2 - 2g - f_1) + t_2\,.$$

Since in X over each exceptional divisor E_P in Z, there are $2^{\tau-\delta-k_P}$ curves with Euler number $e(F_P)$, we get

$$\begin{aligned} e(X) &= e\left(X \setminus \bigcup_{P\in \mathrm{Esing}(D)} f^{-1}E_P\right) + \sum_{k\geqslant 3} 2^{\tau-\delta-2}(4-k)t_k \\ &= e\left(X \setminus \bigcup_{P\in \mathrm{Esing}(D)} f^{-1}E_P\right) + 2^{\tau-\delta-2}(4f_0 - f_1 - 2t_2)\,. \end{aligned}$$

Thus

$$\frac{1}{2^{\tau-\delta-2}}\cdot e(X) = 4e(Y) + 4g - 4 + f_1 - t_2\,. \tag{3.6}$$

Our purpose now is to calculate the other Chern number $c_1^2(X) = K_X^2$. The canonical divisor K_X satisfies $K_X = f^*K$ for the divisor K on Z defined as

$$\begin{aligned} K := \pi^*K_Y + \sum E_P + \frac{1}{2}\left(\sum E_P + \pi^*D - \sum k_P E_P\right) \\ = \sum \frac{3-k_P}{2}E_P + \frac{1}{2}\pi^*D + \pi^*K_Y\,, \end{aligned}$$

with the summation taken over all points $P \in \mathrm{Esing}(D)$. We have

$$K^2 = -\frac{1}{4}\sum_{k\geqslant 3}(3-k)^2 t_k + \frac{1}{4}\left(\sum d_i\right)^2 A^2 + (K_Y\cdot A)\sum d_i + K_Y^2\,.$$

Using (3.1) we get

$$K^2 = -\frac{1}{4}\left(9f_0 - 6f_1 + f_2 - t_2\right) + \frac{1}{4}\left(\sum d_i\right)^2 A^2 + (K_Y\cdot A)\sum d_i + K_Y^2\,.$$

Thus

$$\begin{aligned} \frac{1}{2^{\tau-\delta-2}}K_X^2 &= -9f_0 + 6f_1 - f_2 + t_2 \\ &\quad + \left(\sum d_i\right)^2 A^2 + 4(K_Y\cdot A)\sum d_i + 4K_Y^2\,. \end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7) we obtain

$$\begin{aligned} &\frac{1}{2^{\tau-\delta-2}}(3c_2(X) - c_1^2(X)) \\ &\qquad = 4(3c_2(Y) - c_1^2(Y)) + 12(g-1) + f_2 - 3f_1 + 9f_0 \\ &\qquad\qquad - 4t_2 - 4(K_Y\cdot A)\sum d_i - \left(\sum d_i\right)^2 A^2\,. \end{aligned}$$

The surface X contains $2^{\tau-\delta-3}t_3$ disjoint (-2)-curves (above the 3-points) and it contains $2^{\tau-\delta-4}t_4$ elliptic curves (above the 4-points), each of self-intersection -4. Since the Kodaira dimension of Y is non-negative, so is that of X. We can then apply the Miyaoka–Sakai refined inequality and we obtain that:

$$\frac{1}{2^{\tau-\delta-2}}\left(3c_2(X)-c_1^2(X)\right)\geqslant\frac{9}{4}t_3+t_4\,.$$

This gives

$$\begin{aligned}4(3c_2(Y)-K_Y^2)+12(g-1)+f_2-3f_1+9f_0&\\-4t_2-4(K_Y\cdot A)\sum d_i-A^2\left(\sum d_i\right)^2&\geqslant\frac{9}{4}t_3+t_4\,.\end{aligned}$$

Using (3.1) and (3.4) we arrive finally to the following Hirzebruch-type inequality :

$$\begin{aligned}5A^2\sum d_i^2+2(K_Y\cdot A)\sum d_i+4(3c_2(Y)-c_1^2(Y))-2f_1+9f_0&\\\geqslant 4t_2+\frac{9}{4}t_3+t_4\,.&\end{aligned}\tag{3.8}$$

Since $h(Y;D)=\frac{1}{f_0}\left(A^2(\sum d_i)^2-f_2\right)=\frac{1}{f_0}(A^2\sum d_i^2-f_1)$, we obtain:

$$\begin{aligned}&h(Y;D)\\&\quad\geqslant-\frac{9}{2}+\frac{1}{f_0}\left(2t_2+\frac{9}{8}t_3+\frac{1}{2}t_4\right)\\&\quad\quad-\frac{1}{f_0}\left(\frac{3}{2}A^2\sum d_i^2-(K_Y\cdot A)\sum d_i-2(3c_2(Y)-c_1^2(Y))\right).\end{aligned}\tag{3.9}$$ □

The statement in Theorem A is now an easy corollary. Indeed, note that $f_0=s$, $\widetilde{D}^2=s\cdot h(Y;D)$ and we can drop on the right hand side of (3.9) all summands of which we know that they are non-negative.

Sometimes it is more convenient to work with the following version of the inequality in (3.9), which we record for future reference.

Remark 3.4. — For any transversal arrangement D we have the following inequality:

$$\begin{aligned}-2f_1+9f_0&=\sum_{k\geqslant 2}(9-2k)t_k\\&\leqslant 5t_2+3t_3+t_4+\sum_{k\geqslant 5}(4-k)t_k=3t_2+2t_3+t_4+4f_0-f_1\,.\end{aligned}$$

This yields

$$\begin{aligned}
h(Y;D) &= \frac{A^2\sum d_i^2 - f_1}{f_0}\\
&\geqslant -4 + \frac{1}{f_0}\left(t_2 + \frac{1}{4}t_3\right)\\
&\quad + \frac{1}{f_0}\Big(-4A^2\sum d_i^2 - 2(K_Y\cdot A)\sum d_i - (3c_2(Y) - c_1^2(Y))\Big).
\end{aligned}$$

4. Configurations of degree d plane curves

In this part in order to abbreviate the notation it is convenient to work with the following modification of Definition 1.3.

DEFINITION 4.1. — *A d-arrangement is a transversal arrangement of smooth plane curves of degree d.*

For a d-arrangement D, the equality in (2.1) has now the following form

$$d^2\binom{\tau}{2} = \sum_{r\geqslant 2}\binom{r}{2}t_r\,. \tag{4.1}$$

where τ is the number of irreducible components of D.

Theorem B follows from the following, slightly more precise statement.

THEOREM 4.2. — *For a d-arrangement $D=\sum C_i\subset\mathbb{P}^2$ of $\tau\geqslant 4$ plane curves of degree $d\geqslant 3$ such that $t_\tau = 0$ we have*

$$h(\mathbb{P}^2, D) \geqslant -4 + \frac{-\frac{5}{2}d^2\tau + \frac{9}{2}d\tau}{s}\,.$$

Proof. — We mimic the argumentation of Hirzebruch [7]. There exists a $(\mathbb{Z}/n\mathbb{Z})^{\tau-1}$-cover W of $\mathbb{P}^2$ branched with order n along the d-arrangement D. We keep the same notations as in the proof of Theorem 3.1. In particular all maps and varieties defined in the diagram in Figure 3.1 remain the same with $Y=\mathbb{P}^2$. We compute first $c_2(X)=e(X)$. Note that

$$\begin{aligned}
&e\left(X\setminus\bigcup_{P\in\mathrm{Esing}(D)} f^{-1}E_P\right)\\
&\qquad = n^{\tau-1}\big(e(\mathbb{P}^2)-e(D)\big) + n^{\tau-2}\left(e(D)-e(\mathrm{Sing}(D)\right) + n^{\tau-3}t_2\,.
\end{aligned}$$

Simple computations lead to

$$e\left(X\setminus\bigcup_{P\in\mathrm{Esing}(D)} f^{-1}E_P\right)$$
$$= n^{\tau-1}(3+(2g-2)\tau+f_1-f_0)+n^{\tau-2}\left((2-2g)\tau-f_1\right)+n^{\tau-3}t_2\,,$$

where g denotes the genus of an irreducible component of D, i.e., $g=(d-1)(d-2)/2$. Using

$$\sum_{r\geqslant 3} n^{\tau-1-r}t_r e(F_P)=n^{\tau-2}\left(\sum_{r\geqslant 3}2t_r-\sum_{r\geqslant 3}rt_r\right)+n^{\tau-3}\sum_{r\geqslant 3}rt_r$$

we obtain

$$c_2(X)/n^{\tau-3}=n^2(3+(2g-2)\tau+f_1-f_0)+2n\left((1-g)\tau+f_0-f_1\right)+(f_1-t_2)\,.$$

Now we compute $c_1^2(X)=K_X^2$. From the diagram in Figure 3.1 with $Y=\mathbb{P}^2$ we read off that $K_X=f^*K$, where

$$K=\pi^*(K_{\mathbb{P}^2})+\sum_{P\in\mathrm{Esing}(D)}E_P + \frac{n-1}{n}\left(\sum_{P\in\mathrm{Esing}(D)}E_P+\pi^*(D)-\sum_{P\in\mathrm{Esing}(D)}k_PE_P\right). \tag{4.2}$$

We have

$$K=\pi^*(K_{\mathbb{P}^2})+\frac{n-1}{n}\pi^*(D)+\sum_{P\in\mathrm{Esing}(D)}\left(1+\frac{n-1}{n}(1-k_P)\right)E_P\,.$$

Since $K_X^2=n^{\tau-1}(K)^2$, we obtain

$$\begin{aligned}c_1^2(X)/n^{\tau-3}&=n^2(K)^2\\&=9n^2+d^2\tau^2(n-1)^2-6d\tau n(n-1)\\&\quad-\sum_{r\geqslant 3}t_r\left(n^2+(n-1)^2(1-r)^2+2n(n-1)(1-r)\right).\end{aligned}$$

We postpone the proof that X is a surface of general type until Lemma 4.4. Taking this for granted and fixing $n=3$ we apply on X the Miyaoka–Yau inequality which gives

$$36(g-1)\tau+36d\tau-4d^2\tau+16f_0-4f_1-4t_2\geqslant 0\,.$$

Here a side comment is due. Our choice of $n=3$ is a little bit ambiguous. In fact one could work with different values of n and obtain mutations of inequalities (4.3) and (4.4). These inequalities obtained with various values of n are hard to compare. Our choice seems asymptotically right and certainly sufficient in order to derive Corollary 2.4, so that we do not dwell further on this issue.

Coming back to the main course of the proof and expressing g in terms of d, we obtain the following Hirzebruch-type inequality for d-arrangements

$$\frac{9}{2}(d^2-3d)\tau+9d\tau-d^2\tau-t_2=\frac{7}{2}d^2\tau-\frac{9}{2}d\tau-t_2\geqslant\sum_{r\geqslant 2}(r-4)t_r\,. \tag{4.3}$$

For $h(\mathbb{P}^2;D)$ we have

$$h(\mathbb{P}^2;D)=\frac{d^2\tau^2-\sum_{r\geqslant 2}r^2t_r}{f_0}=\frac{d^2\tau^2-f_2}{f_0}=\frac{d^2\tau-f_1}{f_0},$$

where the last equality follows from $d^2\tau^2-d^2\tau=f_2-f_1$. From (4.3) we derive that

$$-f_1\geqslant -4f_0-\frac{7}{2}d^2\tau+\frac{9}{2}d\tau+t_2$$

and then

$$\begin{aligned} h(\mathbb{P}^2;D)&\geqslant -4+\frac{-(5/2)d^2\tau+(9/2)d\tau+t_2}{f_0}\\ &\geqslant -4+\frac{-(5/2)d^2\tau+(9/2)d\tau}{f_0}, \end{aligned} \tag{4.4}$$

which completes the proof. □

In order to pass to degree d Harbourne constants, we need to get rid of τ and f_0 in (4.4).

Lemma 4.3 (The number of singular points in a d-arrangement). — *Let $D=\sum C_i$ be a transversal arrangement of $\tau\geqslant 2$ degree d curves C_i in $\mathbb{P}^2$ such that $t_\tau=0$. Then $s=s(D)\geqslant\tau$.*

Proof. — First we claim that each curve C_i contains at least d^2+1 intersection points with other curves in the arrangement. Indeed, if not, then by the transversality assumption it contains exactly d^2 intersection points. But this implies that all τ curves C_j meet exactly in these d^2 points contradicting the assumption $t_\tau=0$. Let $f:Y\to\mathbb{P}^2$ be the blow up of all s singular points of D. Then the Picard number of Y is $s+1$. On the other hand, the proper transforms $\widetilde{C_1},\dots,\widetilde{C_\tau}$ are disjoint curves of self-intersection less or equal to $d^2-(d^2+1)=-1$ on Y. By the Hodge Index Theorem we have then $s\geqslant\tau$ as asserted. □

Now we are in the position to prove Corollary 2.4.

Proof. — It is easy to observe that in order to find a lower bound for (4.4) one needs to find an effective bound for f_0 and then by Lemma 4.3 we get the desired inequality. □

We conclude this section with the following Lemma.

Lemma 4.4 (The Kodaira dimension of the divisor K). — *For $d \geqslant 3$, $n \geqslant 2$, $\tau \geqslant 4$ and $t_\tau = 0$ the divisor K defined in* (4.2) *is big and nef.*

Proof. — We argue along the lines of [15, Section 2.3]. We want first to show that there is a way to write K as an effective $\mathbb{Q}$-divisor. From (4.2) we have

$$K = \pi^* \left(-\frac{1}{d}(C_1 + C_2 + C_3) \right) + \frac{2n-1}{n} \sum E_P + \frac{n-1}{n} \sum \widetilde{C_i}, \tag{4.5}$$

where $\widetilde{C_i} = \pi^* C_i - \sum_{P \in (C_i \cap \mathrm{Esing}(D))} E_P$ is the proper transform of C_i under π. This divisor can be written as

$$K = \sum a_i \widetilde{C_i} + \sum b_P E_P$$

with positive coefficients

$$a_i \geqslant \frac{n-1}{n} - \frac{1}{d} > 0 \quad \text{and} \quad b_P \geqslant \frac{2n-1}{n} - \frac{3}{d} > 0\,.$$

Thus in order to check that K is nef it suffices to check its intersection with curves in its support. For E_P we have from (4.5)

$$K.E_P = -\frac{2n-1}{n} + \frac{n-1}{n} k_P \geqslant \frac{n-2}{n} \geqslant 0\,.$$

For the intersection with $\widetilde{C} := \widetilde{C_i}$ for some $i \in \{1, \ldots, \tau\}$ it is more convenient to pass to the numerical equivalence classes:

$$K \equiv \left(\tau d \frac{n-1}{n} - 3 \right) H + \sum \left(\frac{2n-1}{n} - k_P \frac{n-1}{n} \right) E_P$$

and

$$\widetilde{C} \equiv dH - \sum_{P \in (C \cap \mathrm{Esing}(D))} E_P\,,$$

where $H = \pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$. We obtain

$$K.\widetilde{C} = \tau d^2 \frac{n-1}{n} - 3d + \sum_{P \in (C \cap \mathrm{Esing}(D))} \left(1 + \frac{n-1}{n}(1 - k_P) \right). \tag{4.6}$$

Now, the last summand can be written as

$$\#\left\{\mathrm{Esing}(D) \cap C\right\} - \frac{n-1}{n} \sum_{P \in (C \cap \mathrm{Esing}(D))} (k_P - 1)\,.$$

Recalling the following equality coming from counting incidences with the component C in two ways

$$\sum_{P\in(C\cap\mathrm{Sing}(D))}(k_P-1)=d^2(\tau-1) \tag{4.7}$$

and plugging it into (4.6) we obtain

$$K.\widetilde{C}=\frac{n-1}{n}d^2+\frac{n-1}{n}\#\{P\in C:\ k_P=2\}+\#\{P\in C:\ k_P\geqslant 3\}-3d.$$

Now, as in the proof of Lemma 4.3 we have $\#\{P\in C:\ k_P\geqslant 2\}\geqslant(d^2+1)$ so that the last two summand can be bounded from below by $\frac{n-1}{n}(d^2+1)$. Rearranging the terms we get finally

$$K.\widetilde{C}\geqslant\frac{2n-2}{n}d^2-3d+\frac{n-1}{n}.$$

The expression on the right is positive for $d\geqslant 3$ and $n\geqslant 2$. This finishes the proof that K is nef.

In order to show that K is also big it suffices to check that its self-intersection is positive. We omit an easy calculation. □

BIBLIOGRAPHY

[1] T. Bauer, S. Di Rocco, B. Harbourne, J. Huizenga, A. Lundman, P. Pokora & T. Szemberg, "Bounded negativity and arrangements of lines", *Int. Math. Res. Not.* (2015), no. 19, p. 9456-9471.

[2] T. Bauer, B. Harbourne, A. L. Knutsen, A. Küronya, S. Müller-Stach, X. Roulleau & T. Szemberg, "Negative curves on algebraic surfaces", *Duke Math. J.* **162** (2013), no. 10, p. 1877-1894.

[3] J. G. Dorfmeister, "Bounded Negativity and Symplectic 4-Manifolds", `https://arxiv.org/abs/1601.01202`, 2016.

[4] G.-M. Greuel, C. Lossen & E. Shustin, "Castelnuovo function, zero-dimensional schemes and singular plane curves", *J. Algebr. Geom.* **9** (2000), no. 4, p. 663-710.

[5] B. Harbourne, "Global aspects of the geometry of surfaces", *Ann. Univ. Paedagog. Crac. Stud. Math.* **9** (2010), p. 5-41.

[6] J. C. Hemperly, "The parabolic contribution to the number of linearly independent automorphic forms on a certain bounded domain", *Am. J. Math.* **94** (1972), p. 1078-1100.

[7] F. Hirzebruch, "Arrangements of lines and algebraic surfaces", in *Arithmetic and geometry, Vol. II: Geometry*, Progress in Mathematics, vol. 36, Birkhäuser, 1983, p. 113-140.

[8] ———, "Singularities of algebraic surfaces and characteristic numbers", in *The Lefschetz centennial conference, Part I (Mexico City, 1984)*, Contemporary Mathematics, vol. 58, American Mathematical Society, 1986, p. 141-155.

[9] Y. Miyaoka, "The maximal number of quotient singularities on surfaces with given numerical invariants", *Math. Ann.* **268** (1984), no. 2, p. 159-171.

[10] ———, "The orbibundle Miyaoka-Yau-Sakai inequality and an effective Bogomolov-McQuillan theorem", *Publ. Res. Inst. Math. Sci.* **44** (2008), no. 2, p. 403-417.

[11] M. Namba, *Branched coverings and algebraic functions*, Pitman Research Notes in Mathematics Series, vol. 161, Longman Scientific & Technical, Harlow; John Wiley & Sons, Inc., New York, 1987, viii+201 pages.

[12] P. Pokora & H. Tutaj-Gasińska, "Harbourne constants and conic configurations on the projective plane", *Math. Nachr.* **289** (2016), no. 7, p. 888-894.

[13] X. Roulleau, "Bounded negativity, Miyaoka-Sakai inequality and elliptic curve configurations", *Int. Math. Res. Not.* **2017** (2017), no. 8, p. 2480-2496.

[14] F. Sakai, "Semi-stable curves on algebraic surfaces and logarithmic pluricanonical maps", *Math. Ann.* **254** (1980), no. 2, p. 89-120.

[15] L. Z. Tang, "Algebraic surfaces associated to arrangements of conics", *Soochow J. Math.* **21** (1995), no. 4, p. 427-440.

Manuscrit reçu le 12 février 2016,
révisé le 30 juin 2016,
accepté le 15 mars 2017.

Piotr POKORA
Department of Mathematics
Pedagogical University of Cracow
Podchorążych 2
30-084 Kraków (Poland)
Current address:
Institut für Algebraische Geometrie
Leibniz Universität Hannover
Welfengarten 1
30167 Hannover (Germany)
piotrpkr@gmail.com
pokora@math.uni-hannover.de

Xavier ROULLEAU
Laboratoire de Mathématiques et Applications
Université de Poitiers, UMR CNRS 7348
Téléport 2 - BP 30179
86962 Futuroscope Chasseneuil (France)
Current address:
Aix Marseille Univ, CNRS
Centrale Marseille, I2M
13284 Marseille (France)
xavier.roulleau@univ-amu.fr

Tomasz SZEMBERG
Department of Mathematics
Pedagogical University of Cracow
Podchorążych 2
30-084 Kraków (Poland)
tomasz.szemberg@gmail.com