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# ABOUT JARNíK'S-TYPE RELATION IN HIGHER DIMENSION 

by Antoine MARNAT (*)


#### Abstract

Using the Parametric Geometry of Numbers introduced recently by W. M. Schmidt and L. Summerer and results by D. Roy, we show that German's transference inequalities between the two most classical exponents of uniform Diophantine approximation are optimal. Further, we establish that the $n$ uniform exponents of Diophantine approximation in dimension $n$ are algebraically independent. Thus, no Jarník's-type relation holds between them.

Résumé. - En utilisant la géométrie paramétrique des nombres introduite récemment par W. M. Schmidt et L. Summerer et des résultats de D. Roy, nous montrons que les inégalités de transfert entre les deux exposants uniformes d'approximation diophantienne les plus classiques, établies par O. German, sont optimales. De plus, nous établissons que les $n$ exposants d'approximation uniforme en dimension $n$ sont algébriquement indépendants. Ainsi en dimension supérieure à 2 , ils ne sont pas reliés par une relation de dépendance analogue à l'identité de Jarník.


## 1. Introduction

Throughout this paper, the integer $n \geqslant 1$ denotes the dimension of the ambient space, $\boldsymbol{\theta}=\left(\theta_{1}, \ldots, \theta_{n}\right)$ denotes an $n$-tuple of real numbers such that $1, \theta_{1}, \ldots, \theta_{n}$ are $\mathbb{Q}$-linearly independent.

Let $d$ be an integer with $0 \leqslant d \leqslant n-1$. We define the exponent $\omega_{d}(\boldsymbol{\theta})$ (resp. the uniform exponent $\hat{\omega}_{d}(\boldsymbol{\theta})$ ) as the supremum of the real numbers $\omega$ for which there exist rational affine subspaces $L \subset \mathbb{R}^{n}$ such that

$$
\operatorname{dim}(L)=d, \quad H(L) \leqslant H \quad \text { and } \quad H(L) d(\boldsymbol{\theta}, L) \leqslant H^{-\omega}
$$

[^0]for arbitrarily large real numbers $H$ (resp. for every sufficiently large real number $H$ ). Here $H(L)$ denotes the height of $L$ (see [15] for more details), and $d(\boldsymbol{\theta}, L)=\min _{P \in L} d(\boldsymbol{\theta}, P)$ is the minimal distance between $\boldsymbol{\theta}$ and a point of $L$.

These exponents were introduced originally by M. Laurent [11]. They interpolate between the classical exponents $\omega(\boldsymbol{\theta})=\omega_{n-1}(\boldsymbol{\theta})$ and $\lambda(\boldsymbol{\theta})=$ $\omega_{0}(\boldsymbol{\theta})\left(\right.$ resp. $\hat{\omega}(\boldsymbol{\theta})=\hat{\omega}_{n-1}(\boldsymbol{\theta})$ and $\left.\hat{\lambda}(\boldsymbol{\theta})=\hat{\omega}_{0}(\boldsymbol{\theta})\right)$ that were introduced by A. Khinchin [7, 8], V. Jarník [6] and Y. Bugeaud and M. Laurent [1, 2].

We have the relations

$$
\begin{aligned}
& \omega_{0}(\boldsymbol{\theta}) \leqslant \omega_{1}(\boldsymbol{\theta}) \leqslant \cdots \leqslant \omega_{n-1}(\boldsymbol{\theta}) \\
& \hat{\omega}_{0}(\boldsymbol{\theta}) \leqslant \hat{\omega}_{1}(\boldsymbol{\theta}) \leqslant \cdots \leqslant \hat{\omega}_{n-1}(\boldsymbol{\theta})
\end{aligned}
$$

and Minkowski's First Convex Body Theorem [12] and Mahler's compound convex bodies theory provide the lower bounds

$$
\omega_{d}(\boldsymbol{\theta}) \geqslant \hat{\omega}_{d}(\boldsymbol{\theta}) \geqslant \frac{d+1}{n-d}, \quad \text { for } 0 \leqslant d \leqslant n-1
$$

These exponents happen to be related, as was first noticed by Khinchin with his transference theorem [8]. The study of these transferences has two aspects. First, establishing transference inequalities valid for every suitable point $\boldsymbol{\theta}$. Then, there is the reverse problem, that consists in constructing points $\boldsymbol{\theta}$ to show that these inequalities are sharp. For this, one can prove that there exists points $\boldsymbol{\theta}$ whose exponents satisfy the equality in the transference inequalities. In this case, we say that the inequalities are best possible. A stronger result is to prove that given $k$ exponents $e_{1}, \ldots, e_{k}$, the transference inequalities between these $k$ exponents define a subset of $\mathbb{R}^{k}$ that is exactly the set of all $k$-uples $\left(e_{1}(\boldsymbol{\theta}), \ldots, e_{k}(\boldsymbol{\theta})\right)$ as $\boldsymbol{\theta}$ runs through all points $\boldsymbol{\theta}=\left(\theta_{1}, \ldots, \theta_{n}\right) \in \mathbb{R}^{n}$ such that $1, \theta_{1}, \ldots, \theta_{n}$ are $\mathbb{Q}$-linearly independent. The latter set is called the spectrum of the exponents $\left(e_{1}, \ldots, e_{k}\right)$.

When the dimension is $n=1$, we have the equality $\hat{\omega}_{0}(\boldsymbol{\theta})=\hat{\omega}(\boldsymbol{\theta})=$ $\hat{\lambda}(\boldsymbol{\theta})=1$. In [6], V. Jarník showed that in dimension $n=2$, we have the following algebraic relation between $\hat{\omega}_{1}(\boldsymbol{\theta})$ and $\hat{\omega}_{0}(\boldsymbol{\theta})$ :

$$
\begin{equation*}
\hat{\omega}_{0}(\boldsymbol{\theta})+\frac{1}{\hat{\omega}_{1}(\boldsymbol{\theta})}=1 \tag{*}
\end{equation*}
$$

Furthermore, V. Jarník noted that, in higher dimension $n \geqslant 3$, no algebraic relation holds anymore. He proved [6, Satz 3] that for $n \geqslant 2$, there exist two $n$-tuples of real numbers $\boldsymbol{\theta}=\left(\theta_{1}, \ldots \theta_{n}\right)$ and $\boldsymbol{\nu}=\left(\nu_{1}, \ldots, \nu_{n}\right)$ such that

$$
\hat{\omega}_{n-1}(\boldsymbol{\theta})=\hat{\omega}_{n-1}(\boldsymbol{\nu})=+\infty, \quad \hat{\omega}_{0}(\boldsymbol{\theta})=1 \quad \text { and } \quad \hat{\omega}_{0}(\boldsymbol{\nu})=\frac{1}{n-1}
$$

V. Jarník also proved the following transference theorem:

Theorem 1.1 (Jarník [6]). - Let $n \geqslant 2$. For any $n$-tuples of real number $\boldsymbol{\theta}=\left(\theta_{1}, \ldots \theta_{n}\right)$ such that $1, \theta_{1}, \ldots, \theta_{n}$ are $\mathbb{Q}$-linearly independent, we have

$$
\frac{\hat{\omega}_{n-1}(\boldsymbol{\theta})}{(n-1) \hat{\omega}_{n-1}(\boldsymbol{\theta})+n} \leqslant \hat{\omega}_{0}(\boldsymbol{\theta}) \leqslant \frac{\hat{\omega}_{n-1}(\boldsymbol{\theta})-n+1}{n} .
$$

If $\hat{\omega}_{n-1}(\boldsymbol{\theta})=n$, the interval reduces to the single point $\hat{\omega}_{0}(\boldsymbol{\theta})=\frac{1}{n}$.
Remark 1.2. - O. German [5] and A. Khinchin [9] claim that V. Jarník [6] proved the existence of $n$-tuples $\boldsymbol{\theta}=\left(\theta_{1}, \ldots, \theta_{n}\right)$ with $\hat{\omega}_{n-1}(\boldsymbol{\theta})=+\infty$ and $\hat{\omega}_{0}(\boldsymbol{\theta})$ anywhere in the interval $[1 /(n-1), 1]$. It appears to the author that this is not written explicitly in [6].

Recently, O. German [5] improved Theorem 1.1:
Theorem 1.3 (German [5]). - With the notation of Theorem 1.1, we have

$$
\begin{equation*}
\frac{\hat{\omega}_{n-1}(\boldsymbol{\theta})-1}{(n-1) \hat{\omega}_{n-1}(\boldsymbol{\theta})} \leqslant \hat{\omega}_{0}(\boldsymbol{\theta}) \leqslant \frac{\hat{\omega}_{n-1}(\boldsymbol{\theta})-(n-1)}{\hat{\omega}_{n-1}(\boldsymbol{\theta})} \tag{**}
\end{equation*}
$$

Note that the interval reduces to a single point if $n=2$, and that in this case we recover Jarník's relation $(*)$.

The first goal of this paper is to prove that German's inequalities describe the spectrum of the two exponents $\left(\hat{\omega}_{0}, \hat{\omega}_{n-1}\right)$.

Theorem 1.4. - Let $n \geqslant 2$ be an integer, let $\hat{\omega} \in[n,+\infty]$ and let

$$
\hat{\lambda} \in\left[\frac{\hat{\omega}-1}{(n-1) \hat{\omega}}, \frac{\hat{\omega}-n+1}{\hat{\omega}}\right],
$$

where we understand that the interval for $\hat{\lambda}$ is $[1 /(n-1), 1]$ when $\hat{\omega}=$ $+\infty$. Then there exist uncountably many n-tuples of real numbers $\boldsymbol{\theta}=$ $\left(\theta_{1}, \ldots, \theta_{n}\right)$, with $1, \theta_{1}, \ldots, \theta_{n} \mathbb{Q}$-linearly independent, such that $\hat{\omega}_{n-1}(\boldsymbol{\theta})=$ $\hat{\omega}$ and $\hat{\omega}_{0}(\boldsymbol{\theta})=\hat{\lambda}$.

In [19], W. Schmidt and L. Summerer obtained independently a similar result, proving that the inequalities $(* *)$ of German are best possible.

One can wonder if in higher dimension $(n \geqslant 3)$, there exists a Jarník'stype relation between the $n$ uniform exponents $\hat{\omega}_{0}, \ldots, \hat{\omega}_{n-1}$. The next theorem states that no such algebraic relation holds.

Theorem 1.5. - For every integer $n \geqslant 3$, the $n$ uniform exponents $\hat{\omega}_{0}, \ldots, \hat{\omega}_{n-1}$ are algebraically independent.

Thus, the spectrum of the $n$ uniform exponents $\hat{\omega}_{0}, \ldots, \hat{\omega}_{n-1}$ is a subset of $\mathbb{R}^{n}$ with nonempty interior.

We also know the spectrum of other families of exponents. M. Laurent [10] described the spectrum of the four exponents $\omega_{0}, \hat{\omega}_{0}, \omega_{n-1}, \hat{\omega}_{n-1}$ in dimension $n=2$. In his PhD thesis, the author gives an alternative proof of this result. However, for $n \geqslant 3$ this spectrum is still unknown.
D. Roy showed in [14] that the going-up and going-down transference inequalities of M. Laurent [11] describe the spectrum of the $n$ exponents $\omega_{0}, \ldots, \omega_{n-1}$.

In Section 2, we introduce Parametric Geometry of Numbers, which is the main tool to prove Theorem 1.4 (Section 3) and Theorem 1.5 (Section 5), and to give an alternative proof of Theorem 1.3 (Section 4).

## 2. Parametric Geometry of Numbers

The Parametric Geometry of Numbers answers a question of W. M. Schmidt [16]. Given a convex body and a lattice, we deform either of them with a one parameter diagonal map. We study the behavior of the successive minima in terms of this parameter. It was developed by W. M. Schmidt and L. Summerer [17, 18], and further by D. Roy [13, 14]. Independently, Y. Cheung $[3,4]$ also developed a similar theory.

In this paper, we use the notation introduced by D. Roy in [13, 14] which is essentially dual to the one of W. M. Schmidt and L. Summerer [17, 18]. We refer the reader to these papers for further details. Here $\boldsymbol{x} \cdot \boldsymbol{y}=x_{1} y_{1}+$ $\cdots+x_{n} y_{n}$ is the usual scalar product of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, and $\|\boldsymbol{x}\|_{2}=\sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}$ is the usual Euclidean norm.

Let $\boldsymbol{u}=\left(u_{0}, \ldots, u_{n}\right)$ be a vector in $\mathbb{R}^{n+1}$, with Euclidean norm $\|\boldsymbol{u}\|_{2}=1$. For a real parameter $Q \geqslant 1$ we consider the convex body

$$
\mathcal{C}_{\boldsymbol{u}}(Q)=\left\{\boldsymbol{x} \in \mathbb{R}^{n+1}\left|\|\boldsymbol{x}\|_{2} \leqslant 1,|\boldsymbol{x} \cdot \boldsymbol{u}| \leqslant Q^{-1}\right\} .\right.
$$

For $1 \leqslant d \leqslant n+1$ we denote by $\lambda_{d}\left(\mathcal{C}_{\boldsymbol{u}}(Q)\right)$ the $d$-th minimum of $\mathcal{C}_{\boldsymbol{u}}(Q)$ relatively to the lattice $\mathbb{Z}^{n+1}$. For $q \geqslant 0$ and $1 \leqslant d \leqslant n+1$ we set

$$
L_{\boldsymbol{u}, d}(q)=\log \lambda_{d}\left(\mathcal{C}_{\boldsymbol{u}}\left(e^{q}\right)\right)
$$

Finally, we define the following map associated with $\boldsymbol{u}$ :

$$
\begin{aligned}
\boldsymbol{L}_{\boldsymbol{u}}: \quad[0, \infty) & \rightarrow \mathbb{R}^{n+1} \\
q & \mapsto\left(L_{\boldsymbol{u}, 1}(q), \ldots, L_{\boldsymbol{u}, n+1}(q)\right)
\end{aligned}
$$

The lattice $\mathbb{Z}^{n+1}$ is invariant under permutation of coordinates. Hence, $\boldsymbol{L}_{\boldsymbol{u}}$ remains the same if we permute the coordinates in $\boldsymbol{u}$. Since $\|\boldsymbol{u}\|_{2}=1$ we can thus assume that $u_{0} \neq 0$.

The following proposition links the exponents of Diophantine approximation associated with $\boldsymbol{\theta}=\left(\frac{u_{1}}{u_{0}}, \ldots, \frac{u_{n}}{u_{0}}\right)$ to the behavior of the map $\boldsymbol{L}_{\boldsymbol{u}}$, assuming $u_{0} \neq 0$. It was first stated by W. M. Schmidt and L. Summerer in [17, Theorem 1.4]. It also appears as Relations (1.8) and (1.9) in [18]. In the notation of D. Roy [14, Proposition 3.1], it reads as follows.

Proposition 2.1 (Schmidt-Summerer [17]). - Let $\boldsymbol{u}=\left(u_{0}, \ldots, u_{n}\right) \in$ $\mathbb{R}^{n+1}$, with Euclidean norm $\|\boldsymbol{u}\|_{2}=1$ and $u_{0} \neq 0$. Set $\boldsymbol{\theta}=\left(\frac{u_{1}}{u_{0}}, \ldots, \frac{u_{n}}{u_{0}}\right)$. For $1 \leqslant k \leqslant n$, we have the following relations:

$$
\begin{aligned}
\liminf _{q \rightarrow+\infty} \frac{L_{\boldsymbol{u}, 1}(q)+\cdots+L_{\boldsymbol{u}, k}(q)}{q} & =\frac{1}{1+\omega_{n-k}(\boldsymbol{\theta})} \\
\limsup _{q \rightarrow+\infty} & \frac{L_{\boldsymbol{u}, 1}(q)+\cdots+L_{\boldsymbol{u}, k}(q)}{q}
\end{aligned}=\frac{1}{1+\hat{\omega}_{n-k}(\boldsymbol{\theta})} .
$$

Thus, if we know an explicit map $\boldsymbol{P}=\left(P_{1}, \ldots, P_{n+1}\right):[0, \infty) \rightarrow \mathbb{R}^{n+1}$, such that $\boldsymbol{L}_{\boldsymbol{u}}-\boldsymbol{P}$ is bounded, then we can compute the $2 n$ exponents $\hat{\omega}_{0}(\boldsymbol{\theta}), \ldots, \hat{\omega}_{n-1}(\boldsymbol{\theta}), \omega_{0}(\boldsymbol{\theta}), \ldots, \omega_{n-1}(\boldsymbol{\theta})$ for the above point $\boldsymbol{\theta}$ upon replac$\operatorname{ing} L_{\boldsymbol{u}, i}$ by $P_{i}$ in the above formulas for $1 \leqslant i \leqslant n$.
For this purpose, we consider the following family of maps, introduced by D. Roy in [14].

Definition 2.2 (Roy [14]). - Let $I$ be a subinterval of $[0, \infty)$ with nonempty interior. A generalized ( $n+1$ )-system on $I$ is a continuous piecewise linear map $\boldsymbol{P}=\left(P_{1}, \ldots, P_{n+1}\right): I \rightarrow \mathbb{R}^{n+1}$ with the following three properties.
(S1) For each $q \in I$, we have $0 \leqslant P_{1}(q) \leqslant \cdots \leqslant P_{n+1}(q)$ and $P_{1}(q)+$ $\cdots+P_{n+1}(q)=q$.
(S2) If $H$ is a non-empty open subinterval of $I$ on which $\boldsymbol{P}$ is differentiable, then there are integers $\underline{r}, \bar{r}$ with $1 \leqslant \underline{r} \leqslant \bar{r} \leqslant n+1$ such that $P_{\underline{r}}, P_{\underline{r}+1}, \ldots, P_{\bar{r}}$ coincide on the whole interval $H$ and have slope $1 /(\bar{r}-\underline{r}+1)$ while any other component $P_{k}$ of $\boldsymbol{P}$ is constant on $H$.
(S3) If $q$ is an interior point of $I$ at which $\boldsymbol{P}$ is not differentiable, if $\underline{r}, \bar{r}, \underline{s}, \bar{s}$ are the integers for which

$$
\begin{gathered}
P_{k}^{\prime}\left(q^{-}\right)=\frac{1}{\bar{r}-\underline{r}+1} \quad(\underline{r} \leqslant k \leqslant \bar{r}) \quad \text { and } \quad P_{k}^{\prime}\left(q^{+}\right)=\frac{1}{\bar{s}-\underline{s}+1} \quad(\underline{s} \leqslant k \leqslant \bar{s}), \\
\text { and if } \underline{r}<\bar{s}, \text { then we have } P_{\underline{r}}(q)=P_{\underline{r}+1}(q)=\cdots=P_{\bar{s}}(q) .
\end{gathered}
$$

Here $P_{k}^{\prime}\left(q^{-}\right)\left(\right.$resp. $\left.P_{k}^{\prime}\left(q^{+}\right)\right)$denotes the left (resp. right) derivative of $P_{k}$ at $q$. The next result combines Theorem 4.2 and Corollary 4.7 of [14].

Theorem 2.3 (Roy [14]). - For each non-zero point $\boldsymbol{u} \in \mathbb{R}^{n+1}$, there exists $q_{0} \geqslant 0$ and a generalized $(n+1)$-system $\boldsymbol{P}$ on $\left[q_{0}, \infty\right)$ such that $\boldsymbol{L}_{\boldsymbol{u}}-\boldsymbol{P}$ is bounded on $\left[q_{0}, \infty\right)$. Conversely, for each generalized $(n+1)$ system $\boldsymbol{P}$ on an interval $\left[q_{0}, \infty\right)$ with $q_{0} \geqslant 0$, there exists a non-zero point $\boldsymbol{u} \in \mathbb{R}^{n+1}$ such that $\boldsymbol{L}_{\boldsymbol{u}}-\boldsymbol{P}$ is bounded on $\left[q_{0}, \infty\right)$.

In view of the remark following Proposition 2.1, this result reduces the determination of the joint spectrum of Diophantine approximation exponents to a combinatorial study of generalized $(n+1)$-systems.

Although the definition of a generalized $(n+1)$-system $\boldsymbol{P}=\left(P_{1}, \ldots, P_{n+1}\right)$ may look complicated, it is easy to understand in terms of the combined graph of $\boldsymbol{P}$, that is the union of the graphs of $P_{1}, \ldots, P_{n+1}$ over the interval of definition $I$ of $\boldsymbol{P}$. We explain this below.

A division point of $\boldsymbol{P}$ is an endpoint of $I$ contained in $I$ or an interior point of $I$ at which $\boldsymbol{P}$ is not differentiable. Such points form a discrete subset of $I$. Between two consecutive division points $q^{*}<q$ of $I$, the graph of each component of $\boldsymbol{P}$ is a line segment. All these line segments have slope 0 except for one line segment of positive slope $1 / t$ where $t$ is the number of components of $\boldsymbol{P}$ whose graph over $\left[q^{*}, q\right]$ is that line segment. In view of the condition $P_{1} \leqslant P_{2} \leqslant \cdots \leqslant P_{n+1}$, there must be consecutive components $P_{\underline{r}}, \ldots, P_{\bar{r}}$ of $\boldsymbol{P}$ with $\bar{r}-\underline{r}+1=t$. If $q$ is also an interior point of $I$ and if $P_{\underline{s}}, \ldots, P_{\bar{s}}$ are the components of $\boldsymbol{P}$ whose graph has positive slope $\frac{1}{\bar{s}-\underline{s}+1}$ to the right of $q$, then there are two cases.
(1) If $\underline{r}<\bar{s}$, we say that $q$ is an ordinary division point. In this case, we have $P_{\underline{r}}(q)=\cdots=P_{\bar{s}}(q)$ according to (S3). This implies that $\underline{r} \leqslant \underline{s}$ and $\bar{r} \leqslant \bar{s}$. Among $P_{\underline{r}}, \ldots, P_{\bar{s}}$, the components $P_{j}$ with $\underline{s} \leqslant j \leqslant \bar{r}$ (if any) change slope from $\frac{1}{\bar{r}-\underline{r}+1}$ to $\frac{1}{\bar{s}-\underline{s}+1}$. Those with $j \leqslant \min (\bar{r}, \underline{s}-1)$ change slope from $\frac{1}{\bar{r}-\underline{r}+1}$ to 0 . The remaining components $P_{j}$ with $\bar{r}+1 \leqslant j \leqslant \underline{s}-1$ (if any) have constant slope 0 in a neighborhood of $q$. The reader is invited to draw a picture for himself or to look at those in $[14, \S 4]$.
(2) Otherwise, we have $\underline{r}>\bar{s}$ because it cannot happen that $\underline{r}=\bar{s}$ (or $\boldsymbol{P}$ is differentiable at $q$ ). Then, we say that $q$ is a switch point. In this case, we have $P_{\underline{r}}(q)=\cdots=P_{\bar{r}}(q)>P_{\underline{s}}(q)=\cdots=P_{\bar{s}}(q)$ which mean that the end point of the line segment of slope $\frac{1}{\bar{r}-\underline{r}+1}$ at the left of $q$ lies above the initial point of the line segment of slope $\frac{1}{\bar{s}-\underline{s}+1}$ at the right of $q$.

It can be shown that the combined graph of a generalized $(n+1)$-system $\boldsymbol{P}$ uniquely determines the map $\boldsymbol{P}$ provided that we know the value of $\boldsymbol{P}$ at one point of its interval of definition. An example of this is shown in [14, $\S 4]$. We will see two other examples in the Sections 3 and 5.

In [17, 18] W. M. Schmidt and L. Summerer introduce the following exponents for an integer $1 \leqslant d \leqslant n+1$ :

$$
\begin{aligned}
& \underline{\varphi}_{d}=\liminf _{q \rightarrow \infty} \frac{L_{\boldsymbol{u}, d}(q)}{q}, \\
& \bar{\varphi}_{d}=\limsup _{q \rightarrow \infty} \frac{L_{\boldsymbol{u}, d}(q)}{q} .
\end{aligned}
$$

For these exponents, we have the following analogue of Theorem 1.5:
Theorem 2.4. - For every integer $n \geqslant 3$, the exponents $\bar{\varphi}_{1}, \ldots, \bar{\varphi}_{n}$ are algebraically independent.

## 3. Proof of Theorem 1.4

In this section, we construct a family of generalized $(n+1)$-systems. Then, via Theorem 2.3, we get a family of $n$-tuples having the requested properties stated in Theorem 1.4. We first treat the case where $\hat{\omega}_{n-1}$ is finite and $n \geqslant 3$. We will explain later how to adapt the construction if $n=2$ or $\hat{\omega}_{n-1}$ is infinite.

First, note that a generalized $(n+1)$-system with all components equal to $q /(n+1)$ provides via Theorem 2.3 a point $\boldsymbol{\theta}$ with $\hat{\omega}_{n-1}(\boldsymbol{\theta})=n$ and $\hat{\omega}_{0}(\boldsymbol{\theta})=1 / n$. Thus, we can exclude this case in the next construction.

Let $q_{0}$ be a positive real number, fix a real number $\hat{\omega}>n \geqslant 2$ and set a parameter $a$ with $\frac{1}{n-1} \leqslant a \leqslant 1$. We define the sequence $\left(q_{6 m}\right)_{m \geqslant 0}$ by:

$$
q_{6 m}=(1+a(\hat{\omega}-n)) q_{6(m-1)}, \text { for } m \geqslant 1 .
$$

Since $\hat{\omega}>n$, the term $q_{6 m}$ goes to infinity as $m$ does.
We construct a generalized $(n+1)$-system $\boldsymbol{P}$ whose graph is invariant under the dilation of factor $(1+a(\hat{\omega}-n))>1$ on the interval $\left[q_{0},+\infty\right)$. Thus, we only need to define $\boldsymbol{P}$ on a generic interval $\left[q_{6 m}, q_{6(m+1)}\right]$. Figure 3.1 shows the pattern of the combined graph of $\boldsymbol{P}$.


Figure 3.1. Combined graph of $\boldsymbol{P}$ on a generic interval $\left[q_{6 m}, q_{6(m+1)}\right]$

For every integer $m \geqslant 0$, we define $\boldsymbol{P}$ at $q_{6 m}$ as follows:

$$
\begin{aligned}
P_{1}\left(q_{6 m}\right) & =P_{2}\left(q_{6 m}\right)=\frac{q_{6 m}}{\hat{\omega}+1}, \\
P_{3}\left(q_{6 m}\right) & =\cdots=P_{n}\left(q_{6 m}\right)=\frac{1+\frac{1-a}{n-2}(\hat{\omega}-n)}{\hat{\omega}+1} q_{6 m}, \\
P_{n+1}\left(q_{6 m}\right) & =\frac{1+a(\hat{\omega}-n)}{\hat{\omega}+1} q_{6 m} .
\end{aligned}
$$

Here the parameter $a$ says how large $P_{n+1}$ is at each point $q_{6 m}$. The condition $a \geqslant 1 /(n-1)$ imposes the condition $P_{n+1}\left(q_{6 m}\right) \geqslant P_{n}\left(q_{6 m}\right)$, and the condition $a \leqslant 1$ imposes that $P_{3}\left(q_{6 m}\right) \geqslant P_{2}\left(q_{6 m}\right)$. We have the dilation condition $\boldsymbol{P}\left(q_{6(m+1)}\right)=\boldsymbol{P}\left((1+a(\hat{\omega}-n)) q_{6 m}\right)=(1+a(\hat{\omega}-n)) \boldsymbol{P}\left(q_{6 m}\right)$ by the definition of the sequence $\left(q_{6 m}\right)_{m \geqslant 0}$.

For $k=0, \ldots, 5$ the graph has only one line segment of positive slope on the interval $\left[q_{6 m+k}, q_{6 m+k+1}\right]$. The graph is clearly the combined graph of a generalized $(n+1)$-system with seven division points $q_{6 m}, \ldots, q_{6 m+6}$. The points $q_{6 m+3}$ and $q_{6 m+5}$ are switch points while the others are ordinary division points. Furthermore it is uniquely defined since we know the value of $\boldsymbol{P}$ at the point $q_{6 m}$, where as requested

$$
P_{1}\left(q_{6 m}\right)+\cdots+P_{n+1}\left(q_{6 m}\right)=q_{6 m} .
$$

Easy computation gives

$$
\left\{\begin{array}{l}
q_{6 m}=(1+a(\hat{\omega}-n)) q_{6(m-1)} \\
q_{6 m+1}=\frac{(n-2)(\hat{\omega}+1)+(1-a)(\hat{\omega}-n)}{(n-2)(\hat{\omega}+1)} q_{6 m} \\
q_{6 m+2}=\frac{(n+1)+(1+a)(\hat{\omega}-n)}{\hat{\omega}+1} q_{6 m} \\
q_{6 m+3}=\frac{\hat{\omega}+(1+a(\hat{\omega}-n))^{2}}{\hat{\omega}+1} q_{6 m} \\
q_{6 m+4}=\frac{1+(1+a(\hat{\omega}-n))(n+a(\hat{\omega}-n)}{\hat{\omega}+1} q_{6 m} \\
q_{6 m+5}=\frac{1+2 a(\hat{\omega}-n)+\hat{\omega}(1+a(\hat{\omega}-n))}{\hat{\omega}+1} q_{6 m}
\end{array}\right.
$$

We now compute its associated exponents with Proposition 2.1. One can notice that the local extrema of the functions $q \rightarrow q^{-1} P_{k}(q), 1 \leqslant k \leqslant n+1$ are located at division points where $P_{k}$ changes slope.

Since $\boldsymbol{P}$ is invariant under dilation of factor $C=(1+a(\hat{\omega}-n))$ we have for every $m \geqslant 0$, every $1 \leqslant k \leqslant n+1$, and every $q$ in $\left[q_{6 m}, q_{6 m+6}\right)$ the relation

$$
q^{-1} P_{k}(q)=q^{-1} C^{m} P_{k}\left(q C^{-m}\right)
$$

where $C^{-m} q$ lies in the fundamental interval $\left[q_{0}, q_{6}\right]$.
Thus,

$$
\begin{aligned}
\limsup _{q \rightarrow+\infty} \frac{P_{1}(q)}{q} & =\max _{q_{0} \leqslant q \leqslant q_{6}} \frac{P_{1}(q)}{q}=\frac{P_{1}\left(q_{0}\right)}{q_{0}}=\frac{1}{\hat{\omega}+1}, \\
\liminf _{q \rightarrow+\infty} \frac{P_{n+1}(q)}{q} & =\min _{q_{0} \leqslant q \leqslant q_{6}} \frac{P_{n+1}(q)}{q}=\frac{P_{n+1}\left(q_{2}\right)}{q_{2}}=\frac{1+a(\hat{\omega}-n)}{n+1+(1+a)(\hat{\omega}-n)},
\end{aligned}
$$

because the component $P_{n+1}$ changes slope from zero to some positive value only at $q_{6 m+2}$.

Then, according to Proposition 2.1, Theorem 2.3 provides an $n$-tuple $\boldsymbol{\theta}=\left(\theta_{1}, \ldots, \theta_{n}\right)$ such that

$$
\begin{aligned}
\frac{1}{\hat{\omega}_{n-1}(\boldsymbol{\theta})+1} & =\limsup _{q \rightarrow+\infty} \frac{P_{1}(q)}{q}=\frac{1}{\hat{\omega}+1} \\
\frac{\hat{\omega}_{0}(\boldsymbol{\theta})}{\hat{\omega}_{0}(\boldsymbol{\theta})+1} & =\liminf _{q \rightarrow+\infty} \frac{P_{n+1}(q)}{q}=\frac{1+a(\hat{\omega}-n)}{n+1+(1+a)(\hat{\omega}-n)}
\end{aligned}
$$

Thus, this $\boldsymbol{\theta}$ satisfies

$$
\hat{\omega}_{n-1}(\boldsymbol{\theta})=\hat{\omega} \quad \text { and } \quad \hat{\omega}_{0}(\boldsymbol{\theta})=\frac{1+a(\hat{\omega}-n)}{\hat{\omega}}
$$

When $a$ runs through the interval $[1 /(n-1), 1]$, then $\hat{\omega}_{0}(\boldsymbol{\theta})$ runs through the interval

$$
\left[\frac{\hat{\omega}-1}{(n-1) \hat{\omega}}, \frac{\hat{\omega}-(n-1)}{\hat{\omega}}\right] .
$$

If $n=2$, we remove the line $P_{3}=\cdots=P_{n}$ and the interval $\left[q_{6 m+3}, q_{6 m+5}\right.$ ] from the generic graph on the interval $\left[q_{6 m}, q_{6(m+1)}\right]$, the parameter $a$ is then forced to be equal to 1 . Thus, we construct $\boldsymbol{\theta}$ with

$$
\hat{\omega}_{1}(\boldsymbol{\theta})=\hat{\omega} \quad \text { and } \quad \hat{\omega}_{0}(\boldsymbol{\theta})=1-\frac{1}{\hat{\omega}}
$$

which agrees with Jarník's relation $(*)$.
If $\hat{\omega}$ is infinite, we replace $\hat{\omega}$ by $m+n+1$ in our construction. For a given real number $q_{0}$ we consider the sequence $\left(q_{6 m}\right)_{m \geqslant 1}$ defined by

$$
q_{6 m}=(m+1) q_{6(m-1)}
$$

Figure 3.1 still represents the combined graph from $\boldsymbol{P}$ on a generic interval [ $q_{6 m}, q_{6 m+6}$ ], with the following settings at $q_{6 m}$ :

$$
\begin{aligned}
P_{1}\left(q_{6 m}\right) & =P_{2}\left(q_{6 m}\right)=\frac{q_{6 m}}{m+n+2}, \\
P_{3}\left(q_{6 m}\right) & =\cdots=P_{n}\left(q_{6 m}\right)=\frac{1+\frac{1-a}{n-2}(m+1)}{m+n+2} q_{6 m}, \\
P_{n+1}\left(q_{6 m}\right) & =\frac{1+a(m+1)}{m+n+2} q_{6 m} .
\end{aligned}
$$

Note that the combined graph is not invariant under dilation anymore. We have

$$
\begin{aligned}
\limsup _{q \rightarrow+\infty} \frac{P_{1}(q)}{q} & =\limsup _{m \rightarrow+\infty} \max _{q_{6 m} \leqslant q \leqslant q_{6(m+1)}} \frac{P_{1}(q)}{q}=\limsup _{m \rightarrow+\infty} \frac{P_{1}\left(q_{6 m}\right)}{q_{6 m}} \\
& =\limsup _{m \rightarrow+\infty} \frac{1}{m+n+2}=0, \\
\liminf _{q \rightarrow+\infty} \frac{P_{n+1}(q)}{q} & =\liminf _{m \rightarrow+\infty} \min _{q_{6 m} \leqslant q \leqslant q_{6(m+1)}} \frac{P_{n+1}(q)}{q}=\liminf _{m \rightarrow+\infty} \frac{P_{n+1}\left(q_{6 m+2}\right)}{q_{6 m+2}} \\
& =\liminf _{m \rightarrow+\infty} \frac{1+a(m+1)}{n+1+(1+a)(m+1)}=\frac{a}{a+1} .
\end{aligned}
$$

Again, Theorem 2.3 provides us with an $n$-tuple $\boldsymbol{\theta}=\left(\theta_{1}, \ldots, \theta_{n}\right)$ such that

$$
\hat{\omega}_{n-1}(\boldsymbol{\theta})=+\infty \quad \text { and } \quad \hat{\omega}_{0}(\boldsymbol{\theta})=a
$$

where $a$ runs through the interval $[1 /(n-1), 1]$.
Note that if $1, \theta_{1}, \ldots, \theta_{n}$ are $\mathbb{Q}$-linearly dependent, then there exists an integer point $\boldsymbol{x} \in \mathbb{Z}^{n}$ such that $|\boldsymbol{x} \cdot \boldsymbol{u}|=0$. This implies that $L_{\boldsymbol{u}, 1}(q)$ is
bounded above by $\log \left(\|x\|_{2}\right)$. In our construction by dilatation $P_{1}$ is not bounded, hence the independence by contradiction.

To complete the proof of Theorem 1.4, we have to check that we can construct uncountably many $n$-tuples with given exponents. Let $\hat{\omega}$ and $\hat{\lambda}$ as in Theorem 1.4, and $a$ the parameter such that Theorem 2.3 provides an $n$-tuple $\boldsymbol{\theta}$ whose exponents satisfy

$$
\hat{\omega}_{n-1}(\boldsymbol{\theta})=\hat{\omega} \quad \text { and } \quad \hat{\omega}_{0}(\boldsymbol{\theta})=\hat{\lambda}=\frac{1+a(\hat{\omega}-n)}{\hat{\omega}}
$$

Fix $q_{0}$ a real number to start the construction from $\boldsymbol{P}$ as above with parameter $a$. For every $\rho_{1}$ and $\rho_{2}$ such that $q_{0} \leqslant \rho_{1}<\rho_{2} \leqslant q_{5}$, we denote by $\boldsymbol{P}_{\rho_{1}}$ and $\boldsymbol{P}_{\rho_{2}}$ the $(n+1)$-generalized system with parameter $a$ starting in $\rho_{1}$ and $\rho_{2}$. We have $\boldsymbol{P}_{\rho_{1}}\left(q_{6}\right) \neq \boldsymbol{P}_{\rho_{2}}\left(q_{6}\right)$ and

$$
\left\|\boldsymbol{P}_{\rho_{1}}\left(q_{6 m}\right)-\boldsymbol{P}_{\rho_{2}}\left(q_{6 m}\right)\right\|_{\infty}=\frac{q_{6 m}}{q_{6}}\left\|\boldsymbol{P}_{\rho_{1}}\left(q_{6}\right)-\boldsymbol{P}_{\rho_{2}}\left(q_{6}\right)\right\|_{\infty} \rightarrow_{n \rightarrow \infty} \infty
$$

where $\left\|\left(x_{1}, \ldots, x_{n}\right)\right\|_{\infty}=\max _{1 \leqslant k \leqslant n}\left|x_{k}\right|$.
Thus, their difference is unbounded, and they cannot correspond to the same $\boldsymbol{\theta}$ via Theorem 2.3.

## 4. An alternative proof of Theorem 1.3

In this section, we give an alternative proof of Theorem 1.3 using arguments from Parametric Geometry of Numbers. As in previous section, we reduce the study of Diophantine properties of a $n$-tuples of real numbers $\boldsymbol{\theta}$ to the study of generalized $(n+1)$-systems. If $\boldsymbol{\theta}=\left(\theta_{1}, \ldots, \theta_{n}\right) \in \mathbb{R}^{n}$ is such that $1, \theta_{1}, \ldots, \theta_{n}$ are linearly independent over $\mathbb{Q}$, by Theorem 2.3 there exist $q_{0}>0$ and a generalized $(n+1)$-system $\boldsymbol{P}=\left(P_{1}, \ldots, P_{n+1}\right)$ on $\left[q_{0}, \infty\right)$ such that $\boldsymbol{P}-\boldsymbol{L}_{\boldsymbol{u}}$ is bounded where $\boldsymbol{u}=\left(1, \theta_{1}, \ldots, \theta_{n}\right)$. Since $\boldsymbol{u}$ has linearly independent coordinates, the first component $P_{1}$ of $\boldsymbol{P}$ is unbounded. For simplicity, we set $\hat{\omega}=\hat{\omega}_{n-1}(\boldsymbol{\theta})$ and $\hat{\lambda}=\hat{\omega}_{0}(\boldsymbol{\theta})$. Then according to Proposition 2.1, we have

$$
\begin{equation*}
\limsup _{q \rightarrow+\infty} \frac{P_{1}(q)}{q}=\frac{1}{\hat{\omega}+1} \quad \text { and } \quad \liminf _{q \rightarrow+\infty} \frac{P_{n+1}(q)}{q}=\frac{\hat{\lambda}}{\hat{\lambda}+1} \tag{4.1}
\end{equation*}
$$

where we understand that, if $\hat{\omega}=+\infty$, then the limsup is zero.
One can notice that the extremal values of the components of $\boldsymbol{P}$ are reached at the division points. The condition (S3) translates into the fact that for every division point $q$, the right endpoint of the segment with nonzero slope ending at $q$ lies above the left endpoint of the one starting at
$q$. A first consequence is that when $P_{1}$ is non constant, it increases until reaching $P_{2}(q)$. A second consequence is the following proposition.

Proposition 4.1. - For every $1 \leqslant k<m \leqslant n+1$, if $p_{0}$ is a point such that $P_{k}^{\prime}\left(p_{0}^{+}\right)>0$, then for every $p>p_{0}$

$$
P_{m}(p) \leqslant \max \left(P_{m}\left(p_{0}\right), P_{k}\left(p_{0}\right)+p-p_{0}\right)
$$

In particular, $P_{m}$ is constant on the interval $\left[p_{0}, p_{0}+P_{m}\left(p_{0}\right)-P_{k}\left(p_{0}\right)\right]$.
The reason is that, if $p_{1}$ is the largest real number such that $P_{m}$ is constant on $\left[p_{0}, p_{1}\right]$, then the combined graph of $\boldsymbol{P}$ contains a polygonal line joining the points $\left(p_{0}, P_{k}\left(p_{0}\right)\right)$ and $\left(p_{1}, P_{m}\left(p_{1}\right)\right)$. Since the line segments composing such a polygonal line have slope in $[0,1]$, we must have $p_{1} \geqslant p_{0}+P_{m}\left(p_{0}\right)-P_{k}\left(p_{0}\right)$. The conclusion follows since $P_{m}(p) \leqslant$ $\max \left\{P_{m}\left(p_{0}\right), P_{m}\left(p_{0}\right)+p-p_{1}\right\}$ for any $p \geqslant p_{0}$. This is illustrated on the picture below.


Upper bound. - Suppose first that $\hat{\omega}$ is finite. Let $\varepsilon>0$. By (4.1), there exist arbitrarily large division points $p_{0}$ where $q^{-1} P_{1}(q)$ has a local maximum and

$$
\frac{1-\varepsilon}{\hat{\omega}+1} \leqslant \frac{P_{1}\left(p_{0}\right)}{p_{0}} \leqslant \frac{1+\varepsilon}{\hat{\omega}+1}
$$

Since $p_{0}$ is a local maximum, we have $P_{1}\left(p_{0}\right)=P_{2}\left(p_{0}\right)$. Furthermore, $P_{1}(q) \leqslant P_{2}(q) \leqslant \cdots \leqslant P_{n+1}(q)$ and $P_{1}(q)+\cdots+P_{n+1}(q)=q$ provide

$$
P_{n+1}\left(p_{0}\right) \leqslant p_{0}-n P_{1}\left(p_{0}\right) \leqslant \frac{\hat{\omega}+1-n-n \varepsilon}{\hat{\omega}+1} p_{0} .
$$

At the point $p=p_{0}+\frac{\hat{\omega}-n-n \varepsilon}{\hat{\omega}+1} p_{0}$, according to Proposition 4.1, we have the upper bound

$$
P_{n+1}(p) \leqslant \max \left(P_{n+1}\left(p_{0}\right), P_{1}\left(p_{0}\right)+p-p_{0}\right) \leqslant \frac{1+\varepsilon+\hat{\omega}-n-n \varepsilon}{\hat{\omega}+1} p_{0}
$$

Note that equality case corresponds to $\boldsymbol{P}$ with a polygonal line of maximal slope 1 joining the points $\left(p_{0}, P_{1}\left(p_{0}\right)\right)$ and $\left(p, P_{n+1}(p)\right)$. We deduce that

$$
\frac{P_{n+1}(p)}{p} \leqslant \frac{\hat{\omega}+1-n-(n-1) \varepsilon}{2 \hat{\omega}-n+1-n \varepsilon}
$$

Since $p$ can be made arbitrarily large, we conclude that

$$
\frac{\hat{\lambda}}{\hat{\lambda}+1}=\liminf _{q \rightarrow+\infty} \frac{P_{n+1}(q)}{q} \leqslant \frac{\hat{\omega}+1-n}{2 \hat{\omega}-n+1}
$$

giving that

$$
\hat{\lambda} \leqslant \frac{\hat{\omega}-(n-1)}{\hat{\omega}} .
$$

Suppose now that $\hat{\omega}$ is infinite. Let $\varepsilon>0$. Since $P_{1}$ is unbounded, there are arbitrarily large values of $p_{0}$ at which $P_{1}\left(p_{0}\right)=P_{2}\left(p_{0}\right)$. At such a point, we have $P_{2}^{\prime}\left(p_{0}\right)>0$. If $p_{0}$ is large enough, by (4.1) we also have

$$
0 \leqslant \frac{P_{1}\left(p_{0}\right)}{p_{0}} \leqslant \varepsilon
$$

Then, Proposition 4.1 applied at the point $p=(2-n \varepsilon) p_{0}$ provides

$$
P_{n+1}(p) \leqslant p_{0}(1-(n-1) \varepsilon) .
$$

Thus, we get the upper bound

$$
\frac{P_{n+1}(p)}{p} \leqslant \frac{p_{0}(1-(n-1) \varepsilon)}{p_{0}(2-n \varepsilon)}
$$

Since $p$ can be made arbitrarily large, we conclude that

$$
\hat{\lambda} \leqslant 1
$$

Hence, we have proved the upper bound in Theorem 1.3.
Lower bound. - If $P_{1}(q)=P_{n+1}(q)$ for arbitrarily large $q$, then $\hat{\omega}=n$ and $\hat{\lambda}=1 / n$, and the inequalities of Theorem 1.4 are satisfied. So, we may assume that $P_{1}(q)<P_{n+1}(q)$ for any sufficiently large $q$.

Suppose first that $\hat{\omega}$ is finite. Let $\varepsilon_{1}>0$. By (4.1), there exists a real number $q_{0}$ such that $q \geqslant q_{0}$ implies

$$
\begin{equation*}
\frac{P_{1}(q)}{q} \leqslant \frac{1+\varepsilon_{1}}{\hat{\omega}+1} \quad \text { and } \quad P_{1}(q) \neq P_{n+1}(q) \tag{4.2}
\end{equation*}
$$

Let $\varepsilon_{2}>0$. There exist arbitrarily large division points $p \geqslant q_{0}$ where $q^{-1} P_{n-1}(q)$ has a local minimum and

$$
\left|\frac{P_{n+1}(p)}{p}-\frac{\hat{\lambda}}{\hat{\lambda}+1}\right| \leqslant \varepsilon_{2} .
$$

Let $p_{0}=\max \left\{q \leqslant p \mid P_{1}(q)=P_{2}(q)\right\}$. At the point $p_{0}$ we have

$$
P_{1}\left(p_{0}\right)=P_{2}\left(p_{0}\right) \leqslant \frac{1+\varepsilon_{1}}{\hat{\omega}+1} p_{0} \quad \text { and } \quad P_{n+1}\left(p_{0}\right) \geqslant \frac{p_{0}-2 P_{1}\left(p_{0}\right)}{n-1}
$$

since $p_{0}=P_{1}\left(p_{0}\right)+\cdots+P_{n+1}\left(p_{0}\right) \leqslant 2 P_{1}\left(p_{0}\right)+(n-1) P_{n+1}\left(p_{0}\right)$.
We first show that $q \rightarrow P_{1}(q)$ is constant on the interval $\left[p_{0}, p\right]$. If not, there exists a real number $p_{0}<p_{1}<p$ where $P_{1}$ has slope $>0$. Since $p$ is a local minimum from $q^{-1} P_{n+1}(q)$, then $P_{n+1}$ changes slope at $p$. Then, $P_{1}(p) \neq P_{n+1}(p)$ and condition (S3) imply that $P_{1}^{\prime}\left(p^{-}\right)=0$. Thus, there exists a point in the interval $\left(p_{1}, p\right)$ where $P_{1}$ changes slope from $>0$ to 0 . At this point $P_{1}=P_{2}$, which contradicts the definition of $p_{0}$. Thus,

$$
P_{1}\left(p_{0}\right)=P_{1}(p)
$$

We can write

$$
\begin{equation*}
p=\sum_{k=1}^{n+1} P_{k}(p) \leqslant n P_{n+1}(p)+P_{1}\left(p_{0}\right) . \tag{4.3}
\end{equation*}
$$

Note that equality provides that all components except $P_{1}$ are equal. In this case, we have a polygonal line joining $\left(p_{0}, P_{1}\left(p_{0}\right)\right.$ and $\left(p, P_{n+1}(p)\right)$ growing as slowly as possible.

We deduce the lower bound

$$
\frac{P_{n+1}(p)}{p} \geqslant \frac{P_{n+1}(p)}{n P_{n+1}(p)+P_{1}\left(p_{0}\right)}
$$

where the right hand side is an increasing function of $P_{n+1}(p)$. Since

$$
P_{n+1}(p) \geqslant P_{n+1}\left(p_{0}\right) \geqslant \frac{p_{0}-2 P_{1}\left(p_{0}\right)}{n-1}
$$

we have

$$
\frac{P_{n+1}(p)}{p} \geqslant \frac{p_{0}-2 P_{1}\left(p_{0}\right)}{n p_{0}-(n+1) P_{1}\left(p_{0}\right)}
$$

where the right hand side is a decreasing function of $P_{1}\left(p_{0}\right)$. Since

$$
P_{1}\left(p_{0}\right) \leqslant \frac{1+\varepsilon_{1}}{\hat{\omega}+1} p_{0}
$$

we have

$$
\frac{P_{n+1}(p)}{p} \geqslant \frac{\hat{\omega}-1-2 \varepsilon_{1}}{n \hat{\omega}-1-(n+1) \varepsilon_{1}}
$$

Finally,

$$
\frac{\hat{\lambda}}{\hat{\lambda}+1} \geqslant \frac{\hat{\omega}-1-2 \varepsilon_{1}}{n \hat{\omega}-1-(n+1) \varepsilon_{1}}-\varepsilon_{2} .
$$

This gives the expected bound

$$
\hat{\lambda} \geqslant \frac{\hat{\omega}-1}{(n-1) \hat{\omega}} .
$$

Suppose now that $\hat{\omega}$ is infinite. Choose $q_{0}$ so that $q \geqslant q_{0}$ implies

$$
\begin{equation*}
0 \leqslant \frac{P_{1}(q)}{q} \leqslant \varepsilon_{1} \quad \text { and } \quad P_{1}(q) \neq P_{n+1}(q) \tag{4.4}
\end{equation*}
$$

Following the same steps as in the finite case, with the same choice of $p$ we obtain :

$$
\frac{P_{n+1}(p)}{p} \geqslant \frac{1-2 \varepsilon_{1}}{n-(n+1) \varepsilon_{1}}
$$

Thus, we get

$$
\frac{\hat{\lambda}}{\hat{\lambda}+1} \geqslant \frac{1-2 \varepsilon_{1}}{n-(n+1) \varepsilon_{1}}-\varepsilon_{2}
$$

This gives the expected lower bound

$$
\hat{\lambda} \geqslant \frac{1}{n-1}
$$

## 5. Proof of Theorems 1.5 and 2.4

In this section, we construct a family of generalized $(n+1)$-systems depending on $n$ parameters which via Theorem 2.3 provides us with a family of $n$-tuples $\boldsymbol{\theta}$ whose uniform exponents are expressed as a function of these $n$ parameters. Then, we show that these functions are algebraically independent.

Fix the dimension $n \geqslant 3$. Choose $n+2$ parameters $A_{1}, A_{2}, \ldots, A_{n+1}, C$ satisfying

$$
\begin{align*}
0<A_{1} & =A_{2}<A_{3}<A_{4}<\cdots<A_{n+1} \\
1 & =A_{1}+A_{2}+\cdots+A_{n+1} \\
\frac{A_{k+1}}{A_{k}} & <C<\frac{A_{k+2}}{A_{k}} \text { for } 2 \leqslant k \leqslant n-1,  \tag{0}\\
1 & <\frac{A_{n+1}}{A_{n}}<C
\end{align*}
$$

We consider the generalized $(n+1)$-system $\boldsymbol{P}$ on the interval $[1, C]$ whose combined graph is given by Figure 5.1, where

$$
P_{k}(1)=A_{k} \quad \text { and } \quad P_{k}(C)=C A_{k} \quad \text { for } \quad 1 \leqslant k \leqslant n+1
$$

On each interval between two consecutive division points, there is only one line segment with nonzero slope. This line segment has slope 1 on the intervals $\left[1, \delta_{2,1}\right],\left[\delta_{n+1,1}, C\right]$ and $\left[\delta_{k-1,2}, \delta_{k, 1}\right]$ for $3 \leqslant k \leqslant n+1$, and has slope $1 / 2$ on the interval $\left[\delta_{k, 1}, \delta_{k, 2}\right]$, for $3 \leqslant k \leqslant n$, where the two components $P_{k}$ and $P_{k+1}$ coincide. We have $2 n+1$ division points 1, $C$,


Figure 5.1. Pattern of the combined graph of $\boldsymbol{P}$ on the fundamental interval $[1, C]$.
$\delta_{k, 1}$ and $\delta_{l, 2}$ for $2 \leqslant k \leqslant n+1$ and $2 \leqslant l \leqslant n$. They are all ordinary division points except $\delta_{n+1,1}$ which is a switch point. Note that the conditions (0) are consistent with the graph. The points which will be most relevant for the proofs are labeled with black dots.

We extend $\boldsymbol{P}$ to the interval $[1, \infty)$ by self-similarity, that is $\boldsymbol{P}(q)=$ $C^{m} \boldsymbol{P}\left(q C^{-m}\right)$ for every positive integer $m$. In view of the value of $\boldsymbol{P}$ and its derivative at 1 and $C$, one sees that this extension provides a generalized $(n+1)$-system on $[1, \infty)$.

Proposition 2.1 suggests to define quantities $\hat{W}_{n-1}, \ldots, \hat{W}_{0}$ by

$$
\begin{equation*}
\frac{1}{1+\hat{W}_{n-k}}:=\limsup _{q \rightarrow+\infty} \frac{P_{1}(q)+\cdots+P_{k}(q)}{q}, \quad 1 \leqslant k \leqslant n . \tag{5.1}
\end{equation*}
$$

Since $\boldsymbol{P}$ is invariant under dilation of factor $C$, we can replace $\lim _{\sup }^{q \rightarrow \infty}$ by $\max _{[1, C]}$ in the above formulae.

We observe that for $1 \leqslant k \leqslant n$, the function $P_{1}+\cdots+P_{k}$ has slope 1 on the intervals $\left[1, \delta_{k, 1}\right]$ and $\left[\delta_{n+1,1}, C\right]$, slope $1 / 2$ on the interval $\left[\delta_{k, 1}, \delta_{k, 2}\right]$ and is constant on the interval $\left[\delta_{k, 2}, \delta_{n+1,1}\right]$. Thus the maximum on $[1, C]$ of the function $q \rightarrow q^{-1}\left(P_{1}(q)+\cdots+P_{k}(q)\right)$ is reached either at $\delta_{k, 1}$ or at $\delta_{k, 2}$, when slope changes from 1 to $1 / 2$ or from $1 / 2$ to 0 . Namely, the
maximum is reached at $\delta_{k, 1}$ if

$$
\begin{equation*}
\frac{P_{1}\left(\delta_{k, 1}\right)+\cdots+P_{k}\left(\delta_{k, 1}\right)}{\delta_{k, 1}} \geqslant \frac{1}{2} \tag{5.2}
\end{equation*}
$$

and at $\delta_{k, 2}$ if the lefthand side is $\leqslant 1 / 2$. We deduce that for $1 \leqslant k \leqslant n$,

$$
\hat{W}_{n-k}=\frac{P_{k+1}(q)+\cdots+P_{n+1}(q)}{P_{1}(q)+\cdots+P_{k}(q)} \text { where } q= \begin{cases}\delta_{k, 1} & \text { if }(5.2) \text { is satisfied } \\ \delta_{k, 2} & \text { otherwise }\end{cases}
$$

For $2 \leqslant k \leqslant n+1$, we have the following values at $\delta_{k, 1}$ and $\delta_{k, 2}$ :

$$
\begin{aligned}
& P_{i}\left(\delta_{k, 1}\right)= \begin{cases}A_{1} & \text { if } i=1 \\
C A_{i} & \text { if } 2 \leqslant i \leqslant k-1 \\
A_{k+1} & \text { if } i=k \\
A_{i} & \text { if } k+1 \leqslant i \leqslant n+1\end{cases} \\
& P_{i}\left(\delta_{k, 2}\right)= \begin{cases}A_{1} & \text { if } i=1 \\
C A_{i} & \text { if } 2 \leqslant i \leqslant k \\
C A_{k} & \text { if } i=k+1 \\
A_{i} & \text { if } k+2 \leqslant i \leqslant n+1\end{cases}
\end{aligned}
$$

It is easy to check that the parameters

$$
\begin{equation*}
C=3, \quad A_{1}=A_{2}=2^{-n}, \quad A_{k}=2^{-n+k-2} \quad \text { for } \quad 3 \leqslant k \leqslant n+1 \tag{5.3}
\end{equation*}
$$

satisfy the conditions (0). For this choice of parameters, the lefthand side of inequality (5.2) is $>1 / 2$ for $1 \leqslant k \leqslant n-1$ and $<1 / 2$ for $k=n$. This property remains true for $\left(C, A_{2}, \ldots, A_{n}\right)$ in an open neighborhood of $\left(3,2^{-n}, \ldots, 2^{-2}\right)$ provided that we set $A_{1}=A_{2}$ and $A_{n+1}=1-\left(A_{1}+\right.$ $\cdots+A_{n}$ ). In this neighborhood, the quantities $\hat{W}_{0}, \ldots, \hat{W}_{n-1}$ are given by the following rational fractions in $\mathbb{Q}\left(C, A_{2}, A_{3}, \ldots, A_{n}\right)$ :

$$
\begin{align*}
\hat{W}_{n-1} & =\frac{1}{A_{2}}-1 \\
\hat{W}_{n-k} & =\frac{1-\left(2 A_{2}+A_{3}+A_{4}+\cdots+A_{k+1}\right)+C A_{k}}{A_{2}+C\left(A_{2}+\cdots+A_{k}\right)}, 2 \leqslant k \leqslant n-1  \tag{5.4}\\
\hat{W}_{0} & =\frac{1-\left(2 A_{2}+A_{3}+A_{4}+\cdots+A_{n}\right)}{A_{2}+C\left(A_{2}+\cdots+A_{n-1}\right)} .
\end{align*}
$$

Since $\hat{W}_{0}, \ldots, \hat{W}_{n-1}$ come from a generalized $(n+1)$-system $\boldsymbol{P}$, Theorem 2.3 provides a point $\boldsymbol{\theta}$ in $\mathbb{R}^{n}$ such that $\hat{\omega}_{k}(\boldsymbol{\theta})=\hat{W}_{k}$ for every $0 \leqslant$ $k \leqslant n-1$. Thus, to prove Theorem 1.5, it is sufficient to show that the rational fractions $\hat{W}_{0}, \ldots, \hat{W}_{n-1} \in \mathbb{Q}\left(C, A_{2}, A_{3}, \ldots, A_{n}\right)$ are algebraically independent.

Suppose on the contrary that there exists an irreducible polynomial $R \in$ $\mathbb{Q}\left(X_{1}, \ldots, X_{n}\right)$ such that

$$
R\left(\hat{W}_{0}, \hat{W}_{1}, \ldots, \hat{W}_{n-1}\right)=0
$$

Specializing $C$ in 0 , we obtain

$$
\begin{aligned}
R\left(\frac{1-A_{2}-A_{2}-\cdots-A_{n}}{A_{2}},\right. & \frac{1-A_{2}-A_{2}-\cdots-A_{n}}{A_{2}}, \ldots \\
& \left.\ldots, \frac{1-A_{2}-A_{2}-A_{3}}{A_{2}}, \frac{1-A_{2}}{A_{2}}\right)=0
\end{aligned}
$$

Here, the first two rational fractions are the same, and the last $n-1$ rational fractions generate the field $\mathbb{Q}\left(A_{2}, A_{3}, \ldots, A_{n}\right)$. Therefore the latter are algebraically independent, and $R=\alpha\left(X_{2}-X_{1}\right)$ for a nonzero constant $\alpha \in \mathbb{Q}$. This is impossible since $\hat{W}_{0} \neq \hat{W}_{1}$.

Proof of Theorem 2.4. - We consider the same generalized $(n+1)$ system as above. Notice that for $1 \leqslant k \leqslant n$ we have $P_{k} \leqslant P_{n+1}$ and therefore

$$
0 \leqslant \frac{P_{k}(q)}{q} \leqslant 1 / 2
$$

Since all nonzero slopes of the combined graph are at least $1 / 2$, the maxima of the functions $q \mapsto q^{-1} P_{k}(q)$ are reached at points where $P_{k}$ changes slope from 1 or $1 / 2$ to 0 . It happens that for each component there is only one such point on the interval $[1, C[$.
The definition of the exponents $\bar{\varphi}_{k}$ leads to define quantities $F_{k}$ by

$$
\begin{aligned}
F_{k}:= & \limsup _{q \rightarrow \infty} \frac{P_{k}(q)}{q}=\max _{[1, C]} \frac{P_{k}(q)}{q}=\frac{P_{k}(p)}{p} \\
& \text { where } \quad p= \begin{cases}1 & \text { if } k=1, \\
\delta_{k, 2} & \text { if } 2 \leqslant k \leqslant n .\end{cases}
\end{aligned}
$$

We express the quantities $F_{1}, \ldots F_{n}$ as rational fractions in $\mathbb{Q}\left(C, A_{2}, \ldots, A_{n}\right)$, using the relations $A_{1}=A_{2}$ and $A_{n+1}=1-A_{1}-A_{2}-$ $\cdots-A_{n}$ :

$$
\begin{aligned}
& F_{1}=A_{1} \\
& F_{k}=\frac{C A_{k}}{A_{1}+C\left(A_{2}+\cdots+A_{k}\right)+C A_{k}+1-\left(2 A_{2}+A_{3}+\cdots+A_{k+1}\right)}
\end{aligned}
$$

Since $F_{1}, \ldots, F_{n}$ come from a generalized $(n+1)$-system $\boldsymbol{P}$, by Theorem 2.3 there exists a point $\boldsymbol{\theta}$ in $\mathbb{R}^{n}$ such that $\bar{\varphi}_{k}(\boldsymbol{\theta})=F_{k}$ for every $1 \leqslant k \leqslant n$. To prove Theorem 2.4 it is sufficient to show that the rational fractions $F_{1}, \ldots, F_{n} \in \mathbb{Q}\left(C, A_{2}, A_{3}, \ldots, A_{n}\right)$ are algebraically independent.

Suppose that there exists an irreducible polynomial $R \in \mathbb{Q}\left(X_{1}, \ldots, X_{n}\right)$ such that

$$
R\left(F_{1}, \ldots, F_{n}\right)=0
$$

Specializing $C$ in infinity, we obtain

$$
R\left(A_{2}, \frac{1}{2}, \frac{A_{3}}{\left(A_{2}+A_{3}\right)+A_{3}}, \ldots, \frac{A_{n}}{\left(A_{2}+\ldots+A_{n}\right)+A_{n}}\right)=0
$$

where all coordinates except $1 / 2$ are algebraically independent. Thus, $R$ is a constant multiple of $2 X_{2}-1$, which contradicts $F_{2} \neq 1 / 2$.

We are not able to prove Theorem 2.4 for the $n+1$ exponents $\bar{\varphi}_{1}, \ldots, \bar{\varphi}_{n+1}$ with this construction. However with some extra work, we can show that the theorem holds for any $n$ exponents among them.

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