ANNALES

DE

L'INSTITUT FOURIER

Nikolai NIKOLSKI

A correction to "In a shadow of the RH: Cyclic vectors of Hardy spaces on the Hilbert multidisc"

Tome 68, nº 2 (2018), p. 563-567.

<http://aif.cedram.org/item?id=AIF_2018__68_2_563_0>

Ann. Inst. Fourier, Grenoble
68, 2 (2018) 563-567

A CORRECTION TO "IN A SHADOW OF THE RH: CYCLIC VECTORS OF HARDY SPACES ON THE HILBERT MULTIDISC"

Annales de l'institut Fourier, vol. 62 (2012), n°5, 1601–1626

by Nikolai NIKOLSKI

Abstract. — This note corrects some inaccuracies remarked in the paper mentioned in the title. It contains also a few references to recent developments on the dilations $f(nx)$ completeness problem and points out some old statements of V. Ya. Kozlov from early 1950's still unproved.

Résumé. — Quelques inexactitudes dans l'article mentionné dans le titre sont corrigées, et une information sur les progrès récents dans le sujet (sur la complétude des dilatations $f(nx)$) est ajoutée. On discute également quelques assertions sur la complétude (de V. Ya. Kozlov, 1950) qu'on n'a pas réussi à démontrer jusqu'à nos jours.

In this note I correct a few vague points of my presentation and report on some additional information on the subject of which I was unaware when writing my paper in 2012.

I would like to thank Boris Mityagin of Ohio State University for indicating me some incomplete and/or mistaken arguments in the paper, as well as for interesting discussions.

I also indebted to the referee indicating a few misprints.

Keywords: dilations completeness, Hilbert's multidisc, Bohr transform, cyclic vectors, Riemann hypothesis.
2010 *Mathematics Subject Classification:* 32A35, 32A60, 42B30, 42C30, 47A16.

1. How to define real powers of a function $F \in H^2(\mathbb{D}_2^\infty)$, $F(\zeta) \neq 0$ ($\zeta \in \mathbb{D}_2^\infty$), point 2.1 (12)-i) of the paper in question.

At this point of the paper, I have used the inequality $|a^t - b^t| \leqslant C|a-b|^t$ (which is true for real positive a and b but is inapplicable to arbitrary $a, b \in \mathbb{C}$) instead of applying property 2.1 (2), as follows.

By 2.1 (2), a sequence $(F_n)_{n\geqslant 1}$, $F_n \in H^2(\mathbb{D}^n)$ with a chain property $F_{n+1}(\zeta, 0) = F_n(\zeta)$ ($\zeta \in \mathbb{D}^n$) and $\sup_n \|F_n\|_2 < \infty$, uniquely defines a function $F \in H^2(\mathbb{D}_2^\infty)$ such that $F_{(n)} = F_n$ ($F_{(n)} =: F(\zeta, 0, \dots)$, $\zeta \in \mathbb{D}^n$ are defined in 2.1 (2)). Now, given $F \in H^2(\mathbb{D}_2^\infty)$, $F(\zeta) \neq 0$ ($\zeta \in \mathbb{D}_2^\infty$) and $F(0) = 1$, we have $F_{(n)}(\zeta) \neq 0$ ($\zeta \in \mathbb{D}^n$), and hence there exists a holomorphic $\log F_{(n)}$, which is uniquely defined by the choice $\log 1 = 0$. Setting $F^t_{(n)}(\zeta) = \exp(t \log F_{(n)}(\zeta))$ ($\zeta \in \mathbb{D}^n$) we obtain a family of functions satisfying $F^t_{(n)} F^s_{(n)} = F^{t+s}_{(n)}$ for all $0 < t, s < 1$. Since $|F^t_{(n)}(\zeta)| = |F_{(n)}(\zeta)|^t$ for all $\zeta \in \mathbb{D}^n$, we have $\|F^t_{(n)}\|_{2/t} = \|F_{(n)}\|_2^t \leqslant \|F\|_2^t$ for every n. Since $F_{(n)}(0) = 1$, we also get $F^t_{(n+1)}(\zeta, 0) = F^t_{(n)}(\zeta)$ ($\zeta \in \mathbb{D}^n$), and hence obtain functions $F^t \in H^2(\mathbb{D}_2^\infty)$ such that $F^t F^s = F^{t+s}$. By the previous estimates for $\|F^t_{(n)}\|_{2/t}$, it is clear that $F^t \in H^{2/t}(\mathbb{D}_2^\infty)$. □

2. How to prove that polynomials are dense in $H^p(\mathbb{D}_2^\infty)$, $2 \leqslant p < \infty$, point 2.1 (12)-iii).

In my article, such a proof is also based on the mentioned above inequality inapplicable to complex a and b. Now, we give an alternative proof.

(a) *First, we show that the union of* $\bigcup_{n\geqslant 1} H^p(\mathbb{D}^n)$ *is dense in* $H^p(\mathbb{D}_2^\infty)$ (as usual, a function $F \in H^p(\mathbb{D}^n)$ is also considered in $H^p(\mathbb{D}_2^\infty)$ as $F(\zeta) = F(\zeta_1, \dots, \zeta_n)$, $\zeta \in \mathbb{D}_2^\infty$). Indeed, if $F \in H^p(\mathbb{D}_2^\infty)$, we have $\lim_n F_{(n)}(\zeta) = F(\zeta)$ for every $\zeta \in \mathbb{D}_2^\infty$ (it is clear that, as in 2.1 (5), this property is true for all $F \in H^2(\mathbb{D}_2^\infty)$) and $\|F_{(n)}\|_p \leqslant \|F\|_p$. In a moment, we will verify that the linear hull $\mathcal{L} =: \mathcal{L}in(\varphi_\zeta : \zeta \in \mathbb{D}_2^\infty)$ of functionals $\varphi_\zeta(f) =: f(\zeta)$, is norm dense in the dual space $(H^p(\mathbb{D}_2^\infty))^*$, which implies that $\lim_n F_{(n)} = F$ weakly in $H^p(\mathbb{D}_2^\infty)$. Then, by a known functional analysis argument we conclude that there exist convex combinations f_k of functions $F_{(n)}$ such that $\lim_k \|F - f_k\|_p = 0$.

In order to check the above density of $\mathcal{L}$, we mention that $H^p(\mathbb{D}_2^\infty)$ is a uniformly convex (and so, reflexive) Banach space and the family $(\varphi_\zeta : \zeta \in \mathbb{D}_2^\infty)$ is total ($\varphi_\zeta(f) = f(\zeta) = 0$, $\forall \zeta \in \mathbb{D}_2^\infty \Rightarrow f = 0$), hence $\mathcal{L}$ is norm dense (Hahn–Banach). The completeness and uniform convexity of $H^p(\mathbb{D}_2^\infty)$ follow, as above, from a limiting reasoning with $F_{(n)}$ and the corresponding properties of $H^p(\mathbb{D}^n)$; for example, given $F, G \in H^p(\mathbb{D}_2^\infty)$, such that $\|F\|_p \leqslant 1$, $\|G\|_p \leqslant 1$ and $\|F - G\|_p > \varepsilon$, we get $\|F_{(n)}\|_p \leqslant 1$, $\|G_{(n)}\|_p \leqslant 1$ and $\|F_{(n)} - G_{(n)}\|_p > \varepsilon$ (for sufficiently large n), which imply $\left\|\frac{1}{2}(F_{(n)} + G_{(n)})\right\|_p \leqslant 1 - \frac{1}{p}(\varepsilon/2)^p$ (following [2] for every $L^p(\mu)$ space); it remains to use $\lim_n \left\|\frac{1}{2}(F + G)_{(n)}\right\|_p = \left\|\frac{1}{2}(F + G)\right\|_p$. □

(b) *Since the polynomials are dense in each* $H^p(\mathbb{D}^n)$ (see [7], §3.4), the claimed density in $H^p(\mathbb{D}_2^\infty)$ follows. □

3. The statements of point (4) of Theorem 3.4 and Lemma 3.5

Unfortunately, the formula for an invariant subspace E in point (4) of Theorem 3.4 and Lemma 3.5 are stated unproperly.

In fact, *the last sentence of point (4)* should say:

Moreover, if E is a (M_ζ)-invariant subspace of $H^2(\mathbb{D}_2^\infty)$ generated by $F = Uf$, then

$$E = (I \circ \zeta^{\alpha(n)}) \cdot H^2(\mathbb{D}_2^\infty),$$

where I is the inner part of φ.

Indeed, since $F = \varphi \circ \zeta^{\alpha(n)}$, and the multiplication by $I \circ \zeta^{\alpha(n)}$, $h \mapsto (I \circ \zeta^{\alpha(n)})h$, is isometric on $H^2(\mathbb{D}_2^\infty)$, we have $E \subset (I \circ \zeta^{\alpha(n)}) \cdot H^2(\mathbb{D}_2^\infty)$. By Beurling invariant subspace theorem, there exist polynomials p_m (in one variable) such that $\|p_m\varphi - I\|_{H^2(\mathbb{D})} = \left\|(p_m \circ \zeta^{\alpha(n)})F - I \circ \zeta^{\alpha(n)}\right\|_{H^2(\mathbb{D}_2^\infty)}$ tends to zero (as $m \to \infty$), which shows that $(I \circ \zeta^{\alpha(n)}) \in E$, and hence $(I \circ \zeta^{\alpha(n)}) \cdot H^2(\mathbb{D}_2^\infty) \subset E$. □

For *Lemma 3.5*, what is really proved (and what is a correct statement of the Lemma) is the following:

$Z(F)$ is the union $\bigcup_\sigma (A_{\sigma'} \times \mathbb{D}^\sigma)$ (σ runs over all proper subsets of $\{1, 2, \ldots, n\}$), where $A_{\sigma'}$ are the same quality subsets of $\mathbb{T}^{\sigma'}$ as in Lemma (finite unions of real analytic varieties of real dimension less than card(σ')*).*

This corrected form of the Lemma (a finite union of sets $A_{\sigma'} \times \mathbb{D}^\sigma$ instead of just one of them) can be used in the proof of Theorem 3.4 (points (2) and (3)) just in the same way as it is done in the text. □

4. Comments to Example 4.1 (4)

The *second statement of Example 4.1 (4)*, i.e.

$$f = z(\lambda - z)^N,\ 1 < |\lambda| < N \quad \Longrightarrow \quad f \text{ is non-cyclic in } H_0^2,$$

is proved *for real positive numbers* λ *only* (in fact, the proof is valid for all λ, $0 \leqslant \lambda \leqslant N$).

For complex λ, for the moment, we can prove some weaker non-cyclicity statements (see [6]), for example: (a) if q is a prime such that $\sqrt{N+1} < q \leqslant N+1$ and $|\lambda| < (C_N^{q-1})^{1/q-1}$, then f is non-cyclic in H_0^2; (b) if $N = q^2 - 1$ and $|\lambda| \leqslant q - 1$ then f is non-cyclic; (c) given $\varepsilon > 0$, there exists $N(\varepsilon)$ such that $N > N(\varepsilon)$, $|\lambda| < N^{1-\varepsilon} \Rightarrow f_{\lambda,N}$ is non-cyclic.

Notice that *the first (unchanged) statement* of the same Example can be read as follows: *if* $|\lambda| > N/\log 2 \approx 1,45 \cdot N$ *then* f *is cyclic.*

5. V. Ya. Kozlov's enigmatic statements

In 1948–1950, V. Ya. Kozlov published a few notes in Doklady AN SSSR where he claimed several beautiful results on the completeness (cyclicity) of the dilates systems $(h(nx))_{n\geqslant 1}$ in $L^2_{\text{odd}}(-1,1)$, [3, 4] (see also D. Bourgin [1], of the same period). It is easy to see that this system is complete in $L^2_{\text{odd}}(-1,1)$ if and only if $(f(z^n))_{n\geqslant 1}$ is complete in H_0^2, where $f = \sum_{k\geqslant 1} \hat{h}(k) z^k$. In particular, Kozlov stated the following astonishing results:

> *Let* θ, $0 < \theta \leqslant 1$, *and* $h = h_\theta$ *be an odd and 2-periodic function defining for* $0 < x < 1$ *by*
>
> $$h_\theta(x) = \chi_{(0,\theta)}(x);$$
>
> *then,* h_θ *is cyclic for* $\theta = 1$, $\theta = \frac{1}{2}$ *and* $\theta = \frac{2}{3}$, *and is not cyclic for* $\theta = \frac{1}{3}$ *(and for* θ *in a neighborhood of* $\frac{1}{3}$*); the same non-cyclicity is claimed for* $\theta = \frac{q}{p}$ *where* $p > 2$ *is a prime and* q *an odd integer such that* $\tan^2 \frac{q\pi}{2p} < \frac{1}{p}$.

(By the way, the latter non-cyclicity condition is NOT fulfilled for values $q = 1$, $p = 3$ corresponding to the previous claim (for $\theta = \frac{1}{3}$)!)

Until now, no proofs of these claims were published (their author died in 2007). We can quite easily treat the cases $\theta = 1$ and $\theta = \frac{1}{2}$, when the Bohr transforms Uf of the functions f corresponding to h_1 and $h_{1/2}$ are, respectively, a reproducing kernel of $H^2(\mathbb{D}_2^\infty)$ at a point or a product of a reproducing kernel with a linear function, and hence are cyclic ([5, Corollary 3.7]). It would be much interesting to recover other Kozlov's claims (maybe, also with the use of the Bohr lift U).

BIBLIOGRAPHY

[1] D. G. Bourgin, "A class of sequences of functions", *Trans. Am. Math. Soc.* **60** (1946), p. 478-518.

[2] O. Hanner, "On the uniform convexity of L^p and l^p", *Ark. Mat.* **3** (1955), no. 19, p. 239-244.

[3] V. Ya. Kozlov, "On the completeness of systems of functions $\{\varphi(nx)\}$ in the space $L^2(0, 2\pi)$", *Doklady Akad. Nauk SSSR* **61** (1948), p. 977-980.

[4] ———, "On the completeness of a system of functions of type $\{\varphi(nx)\}$ in the space L^2", *Doklady Akad. Nauk SSSR* **73** (1950), p. 441-444.

[5] N. Nikolski, "In a shadow of the RH: Cyclic vectors of Hardy spaces on the Hilbert multidisc", *Ann. Inst. Fourier* **62** (2012), no. 2, p. 1601-1626.

[6] ———, "Binomials whose dilations generate $H^2(\mathbb{D})$", *Algebra and Analysis* **29** (2017), no. 6, p. 159-177.

[7] W. Rudin, *Function theory in polydiscs*, Mathematics Lecture Note Series, W. A. Benjamin, Inc, 1969, 188 pages.

Manuscrit reçu le 22 août 2017,
révisé le 11 décembre 2017,
accepté le 13 décembre 2017.

Nikolai NIKOLSKI
Université de Bordeaux
UFR de Mathématiques et Informatique
351 cours de la Libération
33405 Talence (France)
nikolski@math.u-bordeaux.fr