ANNALES DE L'INSTITUT FOURIER

ANNALES

DE

L'INSTITUT FOURIER

Aihua FAN & Jörg SCHMELING

Everywhere divergence of one-sided ergodic Hilbert transform

Tome 68, n° 6 (2018), p. 2477-2500.

<http://aif.cedram.org/item?id=AIF_2018__68_6_2477_0>

Ann. Inst. Fourier, Grenoble
68, 6 (2018) 2477-2500

EVERYWHERE DIVERGENCE OF ONE-SIDED ERGODIC HILBERT TRANSFORM

by Aihua FAN & Jörg SCHMELING (*)

ABSTRACT. — For a given number $\alpha \in (0,1)$ and a 1-periodic function f, we study the convergence of the series $\sum_{n=1}^{\infty} \frac{f(x+n\alpha)}{n}$, called one-sided Hilbert transform relative to the rotation $x \mapsto x+\alpha \mod 1$. Among others, we prove that for any non-polynomial function of class C^2 having Taylor–Fourier series (i.e. Fourier coefficients vanish on $\mathbb{Z}_-$), there exists an irrational number α (actually a residual set of α) such that the series diverges for *all* x. We also prove that for any irrational number α, there exists a continuous function f such that the series diverges for *all* x. The convergence of general series $\sum_{n=1}^{\infty} a_n f(x+n\alpha)$ is also discussed in different cases involving the diophantine property of the number α and the regularity of the function f.

RÉSUMÉ. — Etant donné un nombre $\alpha \in (0,1)$ et une fonction 1-périodique f, nous étudions la convergence de la série $\sum_{n=1}^{\infty} \frac{f(x+n\alpha)}{n}$, appelée la transformée de Hilbert latérale relative à la rotation $x \mapsto x+\alpha \mod 1$. Entre autres, nous démontrons que pour toute fonction non-polynomiale de classe C^2 admettant une série de Taylor–Fourier (i.e. les coefficients de Fourier sont nuls sur $\mathbb{Z}_-$), il existe un α irrationnel (en réalité, un ensemble de α de deuxième catégorie au sens de Baire) tel que la série diverge pour *tous* les x. Nous démontrons aussi que pour tout α irrationnel, il existe une fonction continue f telle que la série diverge pour *tous* les x. La convergence d'une série générale $\sum_{n=1}^{\infty} a_n f(x+n\alpha)$ est aussi discutée pour divers cas où interviennent la propriété diophantienne du nombre α et la régularité de la fonction f.

1. Introduction

Let f be a Lebesgue integrable function defined on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ identified with $[0,1)$ such that $\int_{\mathbb{T}} f(x)\mathrm{d}x = 0$ and let $\alpha \in [0,1)$ be a given

Keywords: Ergodic Hilbert transform, Everywhere divergence, Irrational rotation.
2010 *Mathematics Subject Classification:* 37A30, 37A45.
(*) The authors' thanks go to M. Björklund, M. Benedicks and C. Cuny for their valuable remarks. The work was done during the visit in the Spring 2015 to Lund University of first author, who would like to thank the warm hospitality of Centre for Mathematical Sciences, Lund University and to Knuth and Alice Wallenberg Foundation for its support. This work was also partially supported by NSFC 11471132.

number. We consider the following series

$$\sum_{n=1}^{\infty} a_n f(x+n\alpha) \tag{1.1}$$

where the coefficients $\{a_n\}$ are complex numbers which are usually assumed square summable. Let T denote the rotation on $\mathbb{T}$ defined by $Tx = x+\alpha$. Then the series (1.1) takes the form $\sum_{n=1}^{\infty} a_n f(T^n x)$, which may be called an ergodic series. Such ergodic series are studied for some hyperbolic systems T in [9] and in many cases the almost everywhere (a.e.) convergence of $\sum_{n=1}^{\infty} a_n f(T^n x)$ is ensured by $\sum_n^{\infty} |a_n|^2 < \infty$ (for the study of general random series of the form $\sum a_n X_n$ see [6], which is a continuation of [9]). The method used in [9] gives nothing about (1.1). It seems a delicate problem to study the pointwise convergence and the convergence in means of (1.1) in its generality.

If $a_n = \frac{1}{n}$, the series (1.1) becomes the so-called one-sided ergodic Hilbert transform (EHT for short):

$$\sum_{n=1}^{\infty} \frac{f(x+n\alpha)}{n}. \tag{1.2}$$

More generally, for any measure-preserving map T, the one-sided EHT takes the form $\sum_{n=1}^{\infty} \frac{f(T^n x)}{n}$ and was studied in the literature. In 1939, Izumi [13] raised the question of the a.e. convergence of the one-sided EHT. In 1949, Halmos proved that for any non-atomic invariant measure μ, there exists a centered function $f \in L^2(\mu)$ such that the one-sided EHT fails to converge in L^2-norm. Later in 1959, Dowker and Erdős [8] constructed a centered function $f \in L^\infty(\mu)$ which has the following stronger divergence

$$\limsup_{N\to\infty} \left| \sum_{n=1}^{N} \frac{f(T^n x)}{n} \right| = \infty \quad \text{a.e.} \tag{1.3}$$

(see also Del Junco and Rosenblatt [15] and see [1] for additional references). In 2009, Cuny [4] proved that for any $f \in L^1(\mu)$, the L^1-convergence of the one-sided EHT implies its a.e. convergence. This answered a question of Gaposhkin [11] who, in 1996, studied the one-sided EHT associated to a general unitary operator U on $L^2(\mu)$ and he gave an example of a unitary operator U and an $f \in L^2(\mu)$ such that the one-sided EHT converges in L^2-norm, but doesn't converge a.e. ([11, p. 253-254]). It is still a question to find effective condition ensuring the a.e. convergence or the L^p-convergence for general one sided EHTs and even for (1.2).

The dynamics of the rotation $T_\alpha x = x+\alpha \mod 1$ depends strongly on the diophantine property of the number α. Consequently, as we shall see, the behavior of the associated one-sided EHTs are different for different α. We shall also see that the high order regularity (even the analyticity) of f can not ensure the convergence of the one-sided EHT for α's having very bad diophantine property (for long time, $n\alpha$ doesn't come back to the neighborhood of 0). In some cases, the divergence of (1.2) takes place everywhere:

$$\limsup_{N\to\infty}\left|\sum_{n=1}^{N}\frac{f(x+n\alpha)}{n}\right| = \infty \quad \forall\, x \in [0,1). \tag{1.4}$$

This reinforces the Dowker–Erdős' result (1.3) for some Liouville rotations.

For $f \in L^1(\mathbb{T})$ we denote by $\widehat{f}(n)$ the n-th Fourier coefficient of f defined by $\int_{\mathbb{T}} f(x)e^{-2\pi i n x}\mathrm{d}x$. We adopt the notation

$$\|f\|_{A(\mathbb{T})} = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|.$$

For $0 < \gamma \leqslant 1$, Lip_γ will denote the space of functions on $\mathbb{T}$ such that $|f(x) - f(y)| \leqslant C|x-y|^\gamma$. For $x \in \mathbb{T}$ we denote

$$\|x\| = \inf_{n\in\mathbb{Z}} |x-n|.$$

Notice that for all $x, y \in \mathbb{T}$ we have the triangle inequality $\|x+y\| \leqslant \|x\| + \|y\|$ and the estimate

$$2\|x\| \leqslant |\sin \pi x| \leqslant \pi\|x\|.$$

In this note we are concentrated on the series (1.1) and (1.2). Our results are listed below.

(1) For any non-polynomial function $f \in C^2(\mathbb{T})$ with $\widehat{f}(n) = 0$ for $n < 0$, there exists a residual set $\mathcal{R}_f$ depending on f such that for every $\alpha \in \mathcal{R}_f$ the series (1.2) diverges for every x (Theorem 2.1).

(2) For any non-polynomial function $f \in C^2(\mathbb{T})$, there exists a residual set $\mathcal{R}_f$ depending on f such that for every $\alpha \in \mathcal{R}_f$ the series (1.2) diverges for almost all x (Theorem 2.2).

(3) For any irrational number α, there exists a continuous function f such that the series (1.2) diverges for every x (Theorem 2.3).

(4) For all $f \in L^2$ and for almost all α, the series (1.2) converges for almost all x (Theorem 3.1).

(5) If $\sum_{n\in\mathbb{Z}\setminus\{0\}}^{\infty} \frac{|\widehat{f}(n)|}{\|n\alpha\|} < \infty$, the series (1.2) converges uniformly in x (Theorem 3.2).

(6) If $\sum_{n\in\mathbb{Z}\setminus\{0\}}^{\infty} \frac{|\widehat{f}(n)|^2}{\|n\alpha\|^2} < \infty$, the series (1.2) converges in L^2-norm and for almost every x (Theorem 3.5).

(7) Let $f \in L^2(\mathbb{T})$ with $\widehat{f}(0) = 0$ and $|\widehat{f}(k)| \leqslant C|k|^{-\beta}$ where $C > 0$ and $\beta > 1/2$ are two constants. Let α be an irrational number with convergents $\{p_n/q_n\}$. The series (1.2) converges in L^2-mean and a.e. if the following condition is satisfied

$$\sum_{m=1}^{\infty} \frac{\log^2 q_{m+1}}{q_m^{2\beta}} < \infty$$

(Theorem 3.6).

(8) Let α be an irrational number with convergents $\{p_n/q_n\}$. For the function f defined by the lacunary series $\sum_{m=1}^{\infty} \widehat{f}(q_m) e^{2\pi i q_m x}$ with $\sum_{m\geqslant 1} |\widehat{f}(q_m)|^2 < \infty$, the series (1.2) converges in L^2-mean if and only if

$$\sum_{m=1}^{\infty} |\widehat{f}(q_m)|^2 \log^2 q_{m+1} < \infty$$

(Proposition 3.7).

(9) Let $\alpha = p/q$ be a rational number where p, q are coprime. For any $f \in L^1(\mathbb{T})$ with $\int f(x)\mathrm{d}x = 0$, the series $\sum a_n f(x + n\alpha)$ converges almost everywhere iff for any $j = 0, 1, \ldots, q-1$, the numerical series $\sum_k a_{kq+j}$ converges (Theorem 3.8).

Notice that for any polynomial f (of cause $\widehat{f}(0) = 0$) and any number α, the series (1.2) converges everywhere. But there are analytic functions f and irrational numbers α such that the series (1.2) diverges everywhere.

The behavior of the series (1.2) depends on that of partial sums of the series $\sum_{n=1}^{\infty} n^{-1} e^{2\pi i n x}$. Notice that its real and imaginary parts are:

$$\sum_{n=1}^{\infty} \frac{1}{n} \cos 2\pi n x = \log \frac{1}{2|\sin \pi x|}, \quad \sum_{n=1}^{\infty} \frac{1}{n} \sin 2\pi n x = \pi \left(\frac{1}{2} - x\right).$$

These two series converge for all points $x \in (0,1)$. It is natural that the behavior of the series (1.1) will depend on that of partial sums of the series $\sum_{n=1}^{\infty} a_n e^{2\pi i n x}$.

Section 2 will be devoted to the divergence of the one-sided EHT (1.2). Section 3 will be devoted to different convergences of the general ergodic series (1.1).

2. Divergence of one-sided ergodic Hilbert transform

We first study the divergence of the series

$$\sum_{n=1}^{\infty} \frac{f(x + n\alpha)}{n}.$$

We say $f \in L^1(\mathbb{T})$ admits a Taylor–Fourier series if $\widehat{f}(n) = 0$ for $n \leqslant -1$. In the following, $\zeta(s)$ denotes the Riemann ζ-function $\sum_{n=1}^{\infty} n^{-s}$.

2.1. Statements on divergence

We first state three divergence statements that we will prove.

THEOREM 2.1. — *Let $f \in L^1(\mathbb{T})$ satisfy the following conditions*

(1) $\widehat{f}(k) = 0$ *if* $k \leqslant 0$*;* $\widehat{f}(k) \neq 0$ *for infinitely many* k*.*

(2) *there exists* $s > 1$ *such that* $\zeta(s) < 2$ *and* $\limsup |k|^s |\widehat{f}(k)| = 0$*.*

Then there exists a residual set $\mathcal{R} \subset [0,1]$ *of irrational numbers such that for each* $\alpha \in \mathcal{R}$*, we have*

$$\limsup_{n \to +\infty} \left| \sum_{n=1}^{N} \frac{f(x+n\alpha)}{n} \right| = +\infty, \quad \forall\, x \in [0,1).$$

The solution of $\zeta(s_0) = 2$ verifies $1 < s_0 = 1.72865\ldots < 2$. The s in the condition (2) must verify $s > s_0 > 1$. So the condition (2) implies that f admits an absolutely convergent Fourier series. All non polynomial functions of class C^2 admitting Taylor–Fourier series satisfies the conditions (1) and (2). The following analytic functions are examples

$$f(x) = \sum_{n=1}^{\infty} r^n e^{2\pi i n x} = \frac{re^{2\pi i x}}{1 - re^{2\pi i x}} = \frac{re^{2\pi i x} - r^2}{1 - 2r\cos(2\pi x) + r^2} \quad (0 < r < 1).$$

THEOREM 2.2. — *Let* $f : \mathbb{T} \to \mathbb{R}$ *be an integrable function whose Fourier coefficients verify the following conditions*

(1) $\widehat{f}(0) = 0$*,* $\widehat{f}(k) \neq 0$ *for infinitely many* k*.*

(2) *there exists* $s > 1$ *such that* $\zeta(s) < 2$ *and* $\limsup |k|^s |\widehat{f}(k)| = 0$*.*

Then there exists a residual set $\mathcal{R} \subset [0,1]$ *of irrational numbers such that for each* $\alpha \in \mathcal{R}$*, we have*

$$\liminf_{N \to +\infty} \sum_{n=1}^{N} \frac{f(x+n\alpha)}{n} = -\infty, \qquad \limsup_{N \to +\infty} \sum_{n=1}^{N} \frac{f(x+n\alpha)}{n} = +\infty,$$

for almost every x*.*

For the last theorem, we have succeeded in proving the a.e. divergence. We wonder if the everywhere divergence is still true.

THEOREM 2.3. — *For any irrational number* $\alpha \in (0,1)$, *there exists a continuous function* $f : \mathbb{T} \to \mathbb{C}$ *with* $\int_{\mathbb{T}} f(x)\mathrm{d}x = 0$ *having an absolutely convergent Fourier series such that*

$$\limsup_{N\to\infty} \left| \sum_{n=1}^{N} \frac{f(x+n\alpha)}{n} \right| = +\infty \qquad \forall\, x \in [0,1).$$

In order to prove these three theorems, we develop f into its Fourier series and we shall see that the behavior of the one-sided EHT relies heavily on that of the following trigonometric polynomials

$$G_N(x) = \sum_{n=1}^{N} \frac{e^{2\pi i n x}}{n}.$$

We shall also need a result due to Jacobsthal which concerns the biggest gap between natural numbers coprime with a given natural number. We get together such preliminaries as several lemmas before we prove the theorems.

2.2. Some lemmas

LEMMA 2.4. — *Assume* $0 < c < 1/2$. *Then*

$$\sup_{N\geqslant 1} \sup_{\|x\|\geqslant c} |G_N(x)| \leqslant \frac{\pi}{c}.$$

Proof. — Notice that $G_N(1/2) = \sum_{n=1}^{N} \frac{(-1)^n}{n}$ so that $\sup_{N\geqslant 1} |G_N(1/2)| \leqslant 1$. Also notice that

$$|G'_N(x)| = 2\pi \left| \sum_{n=1}^{N} e^{2\pi i n x} \right| \leqslant \frac{2\pi}{|\sin \pi x|} \leqslant \frac{\pi}{c}$$

if $1/2 \geqslant |x| \geqslant c$. Then, by the Newton–Leibniz formula we get

$$|G_N(x)| \leqslant |G_N(1/2)| + \left| \int_{1/2}^{x} G'_N(y)\mathrm{d}y \right| \leqslant 1 + \frac{\pi}{2c} \leqslant \frac{\pi}{c}. \qquad \square$$

LEMMA 2.5.

$$G_N(x) = \log N - 2\sum_{n=1}^{N} \frac{\sin^2 \pi n x}{n} + O(1)$$

where the constant in $O(1)$ *is uniform in* x *and in* N. *In particular, if* $|xN| \leqslant C$ *for some constant* $C > 0$, *then*

$$G_N(x) = \log N + O(1)$$

where the constant in $O(1)$ *doesn't depend on* x *and* N, *but on* C.

Proof.

$$G_N(x) - G_N(0) = \sum_{n=1}^{N} \frac{e^{2\pi inx} - 1}{n}.$$

Its imaginary part is $\sum_{n=1}^{N} \frac{\sin(2\pi nx)}{n}$ which is uniformly bounded in x and in N (see [16], p. 4). Its real part is equal to

$$\sum_{n=1}^{N} \frac{\cos(2\pi nx) - 1}{n} = -2 \sum_{n=1}^{N} \frac{\sin^2(\pi nx)}{n}.$$

We conclude for the first assertion by observing that $G_N(0) = \log N + O(1)$. Suppose $|xN| \leqslant C$. Just using $|\sin x| \leqslant |x|$, we get

$$\sum_{n=1}^{N} \frac{\sin^2 \pi nx}{n} \leqslant \pi^2 x^2 \sum_{n=1}^{N} n = \pi^2 x^2 N(N+1)/2 \leqslant \pi^2 C^2. \qquad \square$$

A corollary is that if $|Nx| \leqslant C$, then we have.

$$\sup_{m \geqslant 2} |G_N(mx)| \leqslant |G_N(x)| + O(1) = \log N + O(1).$$

LEMMA 2.6. — *Let $(\phi_k) \subset [0,1)$ be an arbitrary sequence of numbers and let $(n_k) \subset \mathbb{N}$ a sequence of increasing positive integers. For any interval $I \subset [0,1)$ of positive length, the limsup set*

$$\limsup_{k \to \infty} \{x \in [0,1) : n_k x + \phi_k \in I \mod 1\}$$

has full Lebesgue measure.

Proof. — The space $[0,1)$ identified with the circle is compact. The sequence (ϕ_k) has a limit point, say ϕ. Without loss of generality, we can assume that ϕ_k tends to ϕ as k tends to infinity. So, the intervals $-\phi_k + I$ contains a common interval I' with positive length when k is sufficiently large. We can also assume that $I' \subset I - \phi_k$ for all k. Since n_k is increasing, for almost all points x, the sequence $n_k x \pmod 1$ is uniformly distributed. So, for almost every point x, $n_k x \in I' \mod 1$ for infinitely many k. A fortiori, $n_k x + \phi_k \in I \mod 1$ for infinitely many k. $\square$

LEMMA 2.7. — *Suppose that $\{c_k\}_{k \geqslant 1}$ is a sequence of numbers such that $c_k \neq 0$ for infinitely many k's and $\limsup |k|^s |c_k| = 0$ for some $s > 1$. Then there exists a strictly increasing subsequence $\{k_\ell\}_{\ell \geqslant 1}$ of positive integers such that for any $\ell \geqslant 1$, we have*

$$(\zeta(s) - 1)|c_{k_\ell}| > \sum_{m=2}^{\infty} |c_{mk_\ell}|$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$.

Proof. — Let k_ℓ ($\ell \geqslant 1$) be defined inductively in the following way. Let A_1 be the set of maximizing points of $\max_{k\geqslant 1}|k|^s|c_k|$. Since $\limsup |k|^s|c_k| = 0$ and there are infinitely many $c_k \neq 0$, A_1 is non-empty and finite. Let

$$k_1 = \max A_1.$$

Now let A_2 be the set of maximizing points of $\max_{k>k_1} |k|^s|c_k|$, which is also non-empty and finite. Let

$$k_2 = \max A_2.$$

It is clear that $k_1 < k_2$. Inductively, we define

$$k_{\ell+1} = \max\left\{m > k_\ell : |m|^s|c_m| = \max_{k>k_\ell} |k|^s|c_k|\right\}.$$

By the definition of k_ℓ, we have

$$\forall\, m \geqslant 2, \quad k_\ell^s|c_{k_\ell}| > (mk_\ell)^s|c_{mk_\ell}|, \quad \text{i.e.} \quad m^{-s}|c_{k_\ell}| \geqslant |c_{mk_\ell}|.$$

Taking sum over $m \geqslant 2$, we get the desired result. □

Let $q_n(\alpha)$ denote the denominator of n-th convergent of α. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be an increasing function. Define

$$\mathcal{B}_\varphi(\alpha) = \{q_n(\alpha) : \varphi(q_n(\alpha)) < q_{n+1}(\alpha)\}.$$

Usually φ increases very fast. So, we asked that for $q_n(\alpha) \in \mathcal{B}_\varphi(\alpha)$ the denominator $q_{n+1}(\alpha)$ next to $q_n(\alpha)$ is much larger than $q_n(\alpha)$.

Lemma 2.8. — *Let $\Lambda \subset \mathbb{N}$ be an arbitrary infinite subset of natural numbers. For generic α, we have*

$$\#(\Lambda \cap \mathcal{B}_\varphi(\alpha)) = \infty.$$

We will apply Lemma 2.8 to $\Lambda = \{k_\ell\}$, the sequence appearing in Lemma 2.7, with $\varphi(n) = e^{\Delta n/c(n)}$ ($\Delta > 1$ being a large number and $c(n)$ being a sequence tending to 0). In order to prove Lemma 2.8 we need a result due to Jacobsthal.

2.3. An estimate on Jacobsthal's function

Let $N = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$ be the prime factorization of a natural number $N \in \mathbb{N}$. Assume that

$$1 = m_1 < m_2 < \cdots < m_i < m_{i+1} < \cdots$$

are the integers which are coprime with N. Jacobsthal's function is defined as

$$g(N) = \max_{1 \leqslant i < \infty} (m_{i+1} - m_i), \quad (N \in \mathbb{N}).$$

What we will need is $g(N) = o(N)$ as $N \to \infty$. The estimate on $g(N)$ below was known to Jacobsthal [14]. But for completeness we include a proof. There are much better estimates known (see for example [12]), but the one presented here suffices for our purpose.

THEOREM 2.9. — *Let* $N = p_1^{\alpha_1} \dots p_k^{\alpha_k}$. *Then* $g(N) \leqslant (k+1)\left(2^k - 1\right) + 1$.

Proof. — Since the definition of $g(N)$ implies that any interval of length $g(N)$ contains at least one number coprime to N, we need to find a lower bound on $m \in \mathbb{N}$ such that for any integer n the interval $I := [n, n+m-1]$ contains at least one integer coprime with N. Any such a lower bound will be an upper bound of $g(N)$.

Let $1 \leqslant j \leqslant k$ and let $1 \leqslant i_1 < \dots < i_j \leqslant k$ be given. We denote by $K(i_1, \dots, i_j)$ the number of integers $l \in I$ that are divisible by $p_{i_1} \dots p_{i_j}$. These integers l are the following ones

$$n \leqslant p_{i_1} \dots p_{i_j} < 2p_{i_1} \dots p_{i_j} < \dots < K(i_1, \dots, i_j) p_{i_1} \dots p_{i_j} \leqslant n + m - 1.$$

The number $K(i_1, \dots, i_j)$ depends on n. But it has the following bounds independent of n:

$$\left[\frac{m}{p_{i_1} \dots p_{i_j}}\right] - 1 \leqslant K(i_1, \dots, i_j) \leqslant \left[\frac{m}{p_{i_1} \dots p_{i_j}}\right] + 1.$$

By the inclusion-exclusion principle, the number L of natural numbers $l \in I$ with $\gcd(l, N) > 1$ is given by

$$L = \sum_{1 \leqslant i \leqslant k} K(i) - \sum_{1 \leqslant i_1 < i_2 \leqslant k} K(i_1, i_2) + \dots + (-1)^{k+1} K(1, \dots, k).$$

Hence the number M of natural numbers $l \in I$ that are coprime with N verifies

$$\begin{aligned} M &= m - L \\ &= m - \sum_{0 < i \leqslant k} K(i) + \sum_{0 < i_1 < i_2 \leqslant k} K(i_1, i_2) + \dots + (-1)^{k+2} K(1, \dots, k) \\ &\geqslant m \cdot \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) - \sum_{i=1}^{k} \binom{k}{i} \geqslant m \cdot \prod_{i=1}^{k} \frac{i}{i+1} - \sum_{i=1}^{k} \binom{k}{i} \\ &\geqslant m \cdot \frac{1}{k+1} - \left(2^k - 1\right). \end{aligned}$$

Therefore $M > 0$ if $m \geqslant (k+1)\left(2^k - 1\right) + 1$. □

Since $N \geqslant 2 \cdot 3^{k(N)-1}$, we have

$$k(N) \leqslant \log_3\left(\frac{3}{2}N\right) = \delta \log_2\left(\frac{3}{2}N\right)$$

where $\delta = 1/\log_2 3 < 1$. We conclude

$$\lim_{N\to\infty} \frac{g(N)}{N} \leqslant \lim_{N\to\infty} \frac{k(N)\cdot 2^{k(N)}}{N} \leqslant \lim_{N\to\infty} \frac{\delta \log_2\left(\frac{3}{2}N\right)\cdot\left(\frac{3}{2}N\right)^{\delta}}{N} = 0.$$

2.4. Proof of Lemma 2.8

Let $n_1 < n_2 < \cdots < n_k < \dots$ be the elements of Λ. For $k, l \in \mathbb{N}$ we consider the sets

$$B_{k,l} := \{\alpha \in \mathbb{R} \,:\, q_l(\alpha) = n_k \text{ and } q_{l+1}(\alpha) > \varphi(q_l(\alpha)\},$$

$$B_k := \bigcup_{l\geqslant 1} B_{k,l}.$$

These sets are open. Moreover we have

$$\mathcal{B}_\varphi := \bigcap_N \bigcup_{k\geqslant N} B_k = \{\alpha \,:\, \#(\Lambda \cap \mathcal{B}_\varphi(\alpha)) = \infty\}.$$

This set is a G_δ–set and it is left to prove that it is dense.

We observe first that if $p \in \mathbb{N}$ and $\gcd(p, n_k) = 1$, then n_k is an approximant for p/n_k. Moreover if

$$\frac{p}{n_k} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_l}}},$$

then $p = p_l(p/n_k)$, $n_k = q_l(p/n_k)$. Furthermore, for any integer $a_{\ell+1}$, let

$$\frac{p_{l+1}}{q_{l+1}} := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_l + \cfrac{1}{a_{l+1}}}}}.$$

Then we have $p_l(p_{l+1}/q_{l+1}) = p_l$ and

$$q_l(p_{l+1}/q_{l+1}) = n_k. \tag{2.1}$$

Moreover we have $\gcd(p_{l+1}, q_{l+1}) = 1$, $p_{l+1} = a_{l+1}p_l + p_{l-1}$ and $q_{l+1} = a_{l+1}q_l + q_{l-1}$. Hence if a_{l+1} is sufficiently large,

$$\frac{p_{l+1}}{q_{l+1}} \in B_k \tag{2.2}$$

Now

$$\left|\frac{p}{n_k} - \frac{p_{l+1}}{q_{l+1}}\right| = \left|\frac{p_l}{q_l} - \frac{p_{l+1}}{q_{l+1}}\right| = \frac{1}{n_k q_{l+1}} < \frac{1}{n_k^2}. \tag{2.3}$$

It follows from (2.1), (2.2) and (2.3) that it remains to show that the reduced fractions p/n_k are getting more and more dense as k increases. In fact, by Theorem 2.9, $g(n_k) = o(n_k)$. This implies that two consecutive reduced fraction of the form p/n_k have a distance of order $o(1)$ as k tends to infinity, which completes the proof of Lemma 2.8.

We finish our preliminaries with two facts on continued fractions which will be frequently used later:

$$\forall\, n \geqslant 1, \quad \frac{1}{2q_{n+1}} \leqslant \|q_n\alpha\| \leqslant \frac{1}{q_{n+1}} \tag{2.4}$$

$$\forall\, m < q_n, \quad \|m\alpha\| > \|q_n\alpha\|. \tag{2.5}$$

We refer to Khinchin ([17, Theorem 9 and Theorem 13, Theorem 16]).

2.5. Proofs of Theorem 2.1 and of Theorem 2.2

We first prove Theorem 2.2. Let $c_k = \widehat{f}(k)$. The sequence $\{c_k\}_{k\geqslant 1}$ satisfies the condition of Lemma 2.7. Take the sequence $\Lambda = \{k_\ell\}$ in Lemma 2.7. Apply Lemma 2.8 to Λ and $\varphi(n) = e^{\Delta n/c(n)}$, where the constant $\Delta > 1$ will be determined later and

$$c(n) = \min\{|c_{k_\ell}| : k_\ell \leqslant n\}, \quad (n \geqslant 1).$$

Then we get a residual set $\mathcal{R}_f$ such that for each $\alpha \in \mathcal{R}_f$ there exists a subsequence of $\{k_\ell\}$ which is a subsequence $\{q_{n_\ell}(\alpha)\}$ of $\{q_n(\alpha)\}$ (which depends on α!) such that

$$\forall\, \ell \geqslant 1,\ c(q_{n_\ell}(\alpha)) \log q_{n_\ell+1}(\alpha) \geqslant \Delta q_{n_\ell}(\alpha). \tag{2.6}$$

The number α being fixed for the discussion below, we will simply write q_{n_ℓ} and $q_{n_\ell+1}$ for $q_{n_\ell}(\alpha)$ and $q_{n_\ell+1}(\alpha)$. Recall that $q_{n_\ell}(\alpha)$ and $q_{n_\ell+1}(\alpha)$ are the denominators of two consecutive convergents of α.

The N-th partial sum of the series in question can be written as

$$S_N(x) = \sum_{n=1}^{N} \frac{f(x+n\alpha)}{n} = \sum_{k\in\mathbb{Z}\setminus\{0\}} c_k e^{2\pi i k x} G_N(k\alpha).$$

Let $0 < \epsilon < 1/4$ be a fixed small number. For any fixed ℓ, we will consider the partial sum with $N = [\epsilon q_{n_\ell+1}]$. We will cut the sum over k into four subsums:

$$S_{\epsilon q_{n_\ell+1}}(x) = S_{\ell,A}(x) + S_{\ell,B}(x) + S_{\ell,C}(x) + S_{\ell,D}(x)$$

where

$$\begin{aligned}
S_{\ell,A}(x) &= \sum_{|k|<q_{n_\ell}} c_k e^{2\pi ikx} G_{\epsilon q_{n_\ell+1}}(k\alpha)\\
S_{\ell,B}(x) &= \sum_{|k|\geqslant \epsilon q_{n_\ell+1}} c_k e^{2\pi ikx} G_{\epsilon q_{n_\ell+1}}(k\alpha)\\
S_{\ell,C}(x) &= \sum_{q_{n_\ell}<|k|<\epsilon q_{n_\ell+1}}{}' c_k e^{2\pi ikx} G_{\epsilon q_{n_\ell+1}}(k\alpha).\\
S_{\ell,D}(x) &= \sum_{1\leqslant |m|\leqslant \epsilon q_{n_\ell+1}/q_{n_\ell}} c_{mq_{n_\ell}} e^{2\pi imq_{n_\ell}x} G_{\epsilon q_{n_\ell+1}}(mq_{n_\ell}\alpha)
\end{aligned}$$

where $\sum'$ means that the sum is taken over k's which are not multiples of q_{n_ℓ}. As we shall see, $S_{\ell,D}(x)$ will be the principal term.

Since f is real, $c_{-k} = \overline{c_k}$ and consequently all the four sums above are real.

For $|k| < q_{n_\ell}$, we have $\|k\alpha\| \geqslant 1/q_{n_\ell}$. So, by Lemma 2.4, we have

$$|S_{\ell,A}(x)| \leqslant \sum_{|k|<q_{n_\ell}} |c_k| \cdot \pi q_{n_\ell} \leqslant \pi \|f\|_{A(\mathbb{T})} q_{n_\ell}. \tag{2.7}$$

Using the trivial estimate $|G_N(x)| \leqslant \log N + \gamma + o(1)$ (γ being the Euler constant) and the hypothesis $|c_k||k|^s = o(1)$, we get

$$|S_{\ell,B}(x)| \leqslant \sum_{|k|\geqslant \epsilon q_{n_\ell+1}} \frac{1}{k^s} \cdot \log(\epsilon q_{n_\ell+1}) = O\left(\frac{\log q_{n_\ell+1}}{q_{n_\ell+1}^{s-1}}\right) = O(1) \tag{2.8}$$

For any k such that $q_{n_\ell} < k < \epsilon q_{n_\ell+1}$ and $q_{n_\ell} \not| k$, we have

$$k = \ell q_{n_\ell} + r \qquad (1 \leqslant \ell \leqslant \epsilon q_{n_\ell+1}/q_{n_\ell}, \quad 1 \leqslant r < q_{n_\ell}).$$

Then

$$\|k\alpha\| \geqslant \|r\alpha\| - \|\ell q_{n_\ell}\alpha\| \geqslant \frac{1}{q_{n_\ell}} - \epsilon\frac{q_{n_\ell+1}}{q_{n_\ell}} \cdot \frac{1}{q_{n_\ell+1}} = \frac{1-\epsilon}{q_{n_\ell}}.$$

By Lemma 2.4, for such k we have

$$|G_{\epsilon q_{n_\ell+1}}(k\alpha)| \leqslant \frac{\pi}{1-\epsilon} q_{n_\ell}$$

so that

$$|S_{\ell,C}(x)| \leqslant \sum'_{q_{n_\ell}<|k|<\epsilon q_{n_\ell+1}} |c_k| \cdot \frac{\pi}{1-\epsilon} q_{n_\ell} \leqslant \frac{\pi}{1-\epsilon} \|f\|_{A(\mathbb{T})} q_{n_\ell}. \tag{2.9}$$

Since $c_{-k} = \overline{c_k}$, we have

$$|S_{\ell,D}(x)| \geqslant 2|c_{q_{n_\ell}}||G_{\epsilon q_{n_\ell+1}}(q_{n_\ell}\alpha)||\cos(2\pi q_{n_\ell}x + \phi_{q_{n_\ell}})| - 2\sum_{m=2}^{\infty} |c_{mq_{n_\ell}}||G_{\epsilon q_{n_\ell+1}}(mq_{n_\ell}\alpha)|$$

where $\phi_{q_{n_\ell}}$ is the sum of the argument of $c_{q_{n_\ell}}$ and the argument of $G_{\epsilon q_{n_\ell+1}}(q_{n_\ell}\alpha)$. Since $\|q_{n_\ell}\alpha\| \cdot \epsilon q_{n_\ell+1} \leqslant \epsilon$, by Lemma 2.5, we have

$$|G_{\epsilon q_{n_\ell+1}}(q_{n_\ell}\alpha)| = \log q_{n_\ell+1} + O(1);$$
$$|G_{\epsilon q_{n_\ell+1}}(mq_{n_\ell}\alpha)| \leqslant \log q_{n_\ell+1} + O(1) \quad (\forall\, m \geqslant 2).$$

So,

$$|S_{\ell,D}(x)| \geqslant 2\left(|c_{q_{n_\ell}}||\cos(2\pi q_{n_\ell}x + \phi_{q_{n_\ell}})| - \sum_{m=2}^{\infty} |c_{mq_{n_\ell}}|\right)(\log q_{n_\ell+1} + O(1)). \tag{2.10}$$

When $\cos(2\pi q_{n_\ell}x + \phi_{q_{n_\ell}})$ is positive and when the difference on the right hand side of (2.10) is positive, we will have $S_{\ell,D}(x) > 0$ and we can take off the absolute value on the left hand side of (2.10).

Take $\delta > 0$ such that $\zeta(s) + \delta < 2$. Apply Lemma 2.6 to a small interval $I = (-\eta, \eta)$ centered at zero such that $\cos 2\pi\eta > \zeta(s) - 1 + \delta$. For almost all x, there exist infinitely many q_{n_ℓ} depending on x such that

$$\cos(2\pi q_{n_\ell}x + \phi_{q_{n_\ell}}) \geqslant \zeta(s) - 1 + \delta.$$

For such ℓ, if we use Lemma 2.7 we get

$$S_{\ell,D}(x) \geqslant 2\delta|c_{q_{n_\ell}}|(\log q_{n_\ell+1} + O(1)). \tag{2.11}$$

Combining (2.7), (2.8), (2.9) and (2.11), we obtain that for almost every x we have

$$\limsup_{N\to+\infty} \sum_{n=1}^{N} \frac{f(x+n\alpha)}{n} = +\infty.$$

We choose

$$\Delta = \frac{\pi}{2\delta}\|f\|_{A(\mathbb{T})}\left(1 + \frac{1}{1-\epsilon}\right).$$

We can also prove that for almost every x we have

$$\liminf_{n\to\infty} \sum_{n=1}^{N} \frac{f(x+n\alpha)}{n} = -\infty.$$

The only change to do is to take a small interval centered at $1/2$ instead of $I = (-\eta, \eta)$. Thus we have proved Theorem 2.2.

The proof of Theorem 2.1 is easier. Because, in this case, f admits a Taylor–Fourier series and in the place of (2.10) we have directly the estimate

$$|S_{\ell,D}(x)| \geqslant \left(|c_{q_{n_\ell}}| - \sum_{m=2}^{\infty} |c_{mq_{n_\ell}}| \right) (\log q_{n_\ell+1} + O(1)).$$

2.6. Proof of Theorem 2.3

The idea of proof is the same as above. Take a summable sequence of positive numbers $\{c_\ell\}_{\ell \geqslant 1}$ such that

$$\forall\, \ell \geqslant 1, \quad c_\ell > \sum_{j=\ell+1}^{\infty} c_j.$$

For example, $c_\ell = r^\ell$ with $0 < r < 1/2$. Take a very sparse subsequence $\{q_{n_\ell}\}$ from the denominators $\{q_n\}$ of the convergents p_n/q_n of α such that

$$\lim_{\ell\to\infty} \frac{R_\ell \log q_{n_\ell+1}}{q_{n_{\ell-1}+1}} = +\infty, \quad \text{where} \;\; R_\ell = c_\ell - \sum_{j=\ell+1}^{\infty} c_j.$$

Then define

$$f(x) = \sum_{j=1}^{\infty} c_j e^{2\pi i q_{n_j} x}.$$

It is a continuous function with $\|f\|_{A(\mathbb{T})} < \infty$. Notice that it is a lacunary series in the sense that $\widehat{f}(n) = 0$ for $n \neq q_{n_j}$. We can write

$$\sum_{k=1}^{\epsilon q_{n_\ell+1}} \frac{f(x+k\alpha)}{k} = \sum_{j=1}^{\infty} c_j e^{2\pi i q_{n_j} x} G_{\epsilon q_{n_\ell+1}}(q_{n_j}\alpha).$$

Cut the sum into

$$\sum_{k=1}^{\epsilon q_{n_\ell+1}} \frac{f(x+k\alpha)}{k} = S_{\ell,A}(x) + S_{\ell,B}(x) + S_{\ell,D}(x)$$

where

$$S_{\ell,A}(x) = \sum_{j=1}^{\ell-1} \quad c_j e^{2\pi i q_{n_j} x} G_{\epsilon q_{n_\ell+1}}(q_{n_j}\alpha)$$

$$S_{\ell,B}(x) = \sum_{j=\ell+1}^{\infty} \quad c_j e^{2\pi i q_{n_j} x} G_{\epsilon q_{n_\ell+1}}(q_{n_j}\alpha)$$

$$S_{\ell,D}(x) = c_\ell e^{2\pi i q_{n_\ell} x} G_{\epsilon q_{n_\ell+1}}(q_{n_\ell}\alpha).$$

Then

$$\left| \sum_{k=1}^{\epsilon q_{n_\ell+1}} \frac{f(x+k\alpha)}{k} \right| \geqslant |S_{\ell,D}(x)| - |S_{\ell,A}(x)| - |S_{\ell,B}(x)|.$$

For $j < \ell$, we have $\|q_{n_j}\alpha\| \geqslant \|q_{n_{\ell-1}}\alpha\| \geqslant 1/(2(q_{n_{\ell-1}+1}))$. Then by Lemma 2.4,

$$|S_{\ell,A}(x)| \leqslant 2\pi \|f\|_{A(\mathbb{T})}(q_{n_{\ell-1}+1}).$$

By the trivial estimate $|G_N(x)| \leqslant \log N + O(1)$, we get

$$|S_{\ell,B}(x)| \leqslant \sum_{j \geqslant \ell+1} |c_j|(\log(\epsilon q_{n_\ell+1}) + O(1)).$$

Since $\|q_{n_\ell}\alpha\| \leqslant 1/q_{n_\ell+1}$, we have $\|q_{n_\ell}\alpha\| \cdot \epsilon q_{n_\ell+1} \leqslant \epsilon$. So, by Lemma 2.5,

$$|S_{\ell,D}(x)| \geqslant |c_\ell|(\log(\epsilon q_{n_\ell+1}) + O(1)).$$

Thus

$$\left| \sum_{k=1}^{\epsilon q_{n_\ell+1}} \frac{f(x+k\alpha)}{k} \right| \geqslant R_\ell(\log q_{n_\ell+1} + O(1)) - 2\pi \|f\|_{A(\mathbb{T})}(q_{n_{\ell-1}+1}).$$

The right hand side tends to infinity.

3. Convergences of $\sum_{n=1}^{\infty} a_n f(x+n\alpha)$

Now we present some results on the convergence (a.e. convergence, L^2-convergence or uniform convergence) of the series

$$\sum_{n=1}^{\infty} a_n f(x+n\alpha).$$

3.1. Almost everywhere convergence for almost all α

THEOREM 3.1. — *Assume $f \in L^2(\mathbb{T})$ with $\int f(x)\mathrm{d}x = 0$. For almost all $\alpha \in (0,1)$ and almost all $x \in (0,1)$, the series $\sum a_n f(x+n\alpha)$ converges if one of the following conditions is satisfied:*

(1) $\sum |a_n|^2 \log^2 n < \infty$;

(2) $\sum |a_n|^2 \log n < \infty$ *and* $f \in A(\mathbb{T})$ *(in particular* $f \in \mathrm{Lip}_\gamma$ *with* $\gamma > 1/2$*).*

Proof.

(1). — On the product space $\mathbb{T} \times \mathbb{T}$, the product measure $\mathrm{d}\alpha \otimes \mathrm{d}x$ is considered as a probability measure. Then we consider the random variables

$$X_n = x + n\alpha \pmod 1, \qquad (n \geqslant 0).$$

We claim that any couple X_n and X_m with $n \neq m$ are $\mathbb{P}$-independent. In fact, take any two bounded Borel functions g_1 and g_2 on $\mathbb{T}$. We can prove that

$$\mathbb{E} g_1(X_n) g_2(X_m) = \mathbb{E} g_1(X_n) \mathbb{E} g_2(X_m)$$

where $\mathbb{E}$ refers to the expectation with respect to $\mathrm{d}\alpha \otimes \mathrm{d}x$. In fact, by developing g_1 and g_2 into their Fourier series, we get

$$\begin{aligned}\mathbb{E} g_1(X_n) g_2(X_m) &= \sum \widehat{g}_1(k_1)\widehat{g}_2(k_2) \mathbb{E} e^{2\pi i (k_1+k_2)x + (nk_1+mk_2)\alpha} \\ &= \sum_{\substack{k_1+k_2=0 \\ nk_1+mk_2=0}} \widehat{g}_1(k_1)\widehat{g}_2(k_2) \\ &= \widehat{g}_1(0)\widehat{g}_2(0) = \mathbb{E} g_1(X_n)\mathbb{E} g_2(X_m).\end{aligned}$$

Since $\int f(x)\mathrm{d}x = 0$, the above independence implies the orthogonality of the system $\{f(X_n)\}$ in $L^2(\mathrm{d}\alpha \otimes \mathrm{d}x)$. Then, by the Menshov–Rademacher theorem and the hypothesis $\sum |a_n|^2 \log^2 n < \infty$, the random series $\sum a_n f(X_n)$ converges $\mathrm{d}\alpha \otimes \mathrm{d}x$-almost everywhere. Hence, we conclude by using the Fubini theorem.

(2). — Assume that $f \in A(\mathbb{T})$ and $\sum_{n=1}^\infty |a_n|^2 \log n < \infty$. By a result of Gaposhkin [10] which is a consequence of the Carleson theorem on the a.e. convergence of Fourier series, for any given x, the series $\sum a_n f(x + n\alpha)$ converges for almost every α. So, by the Fubini theorem, we conclude that for almost every α, the series $\sum a_n f(x+n\alpha)$ converges for almost every x. □

Notice that no triple X_ℓ, X_m, X_n are $\mathbb{P}$-independent.

3.2. Uniform convergence when α is diophantine

Theorem 3.2. — *Let α be an irrational number, and let $f \in L^1(\mathbb{T})$ with $\int f(x)\mathrm{d}x = 0$, and $(a_n) \subset \mathbb{C}$. Suppose*

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{|\widehat{f}(n)|}{\|n\alpha\|} < \infty; \qquad \lim_{n\to\infty} a_n = 0, \quad \sum_{n=0}^{\infty} |a_n - a_{n+1}| < \infty.$$

Then the series $\sum_{n=0}^{\infty} a_n f(x + n\alpha)$ converges everywhere, even uniformly on x.

Proof. — Under the first condition, the following cocycle equation admits a unique solution $g \in A(\mathbb{T})$:

$$g(x+\alpha) - g(x) = f(x).$$

Actually, by taking the Fourier transform, we find the solution

$$g(x) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{\widehat{f}(n)}{e^{2\pi i n\alpha} - 1} e^{2\pi i n\alpha}.$$

Thus

$$\sum_{n\geqslant 0} a_n f(x+n\alpha) = \sum_{n\geqslant 0} a_n [g(x+(n+1)\alpha) - g(x+n\alpha)].$$

Since $a_n \to 0$, by a summation by parts, we get

$$\sum_{n\geqslant 0} a_n f(x+n\alpha) = \sum_{k\geqslant 0} (a_{k-1} - a_k) g(x + k\alpha)$$

(with convention $a_{-1} = 0$). So $\sum |a_n - a_{n+1}| < \infty$ implies the uniform convergence of the series in question. □

Recall that the irrationality measure (also called Liouville–Roth constant) of an irrational number α, denoted by $\mu(\alpha)$, is defined by

$$\mu(\alpha) = \inf\left\{\mu : \exists\, A > 0, \forall\, p \in \mathbb{Z}, \forall\, q \in \mathbb{N}^*, \left|\alpha - \frac{p}{q}\right| \geqslant \frac{A}{q^\mu}\right\}.$$

It is well known that $\mu(\alpha) = 2$ for almost all irrational numbers α (Khintchine), $\mu(\alpha) = 2$ for all irrational algebraic numbers (Roth), $\mu(e) = 2$, $\mu(\pi) < 7{,}60630853$, $\mu(\log 2) < 3{,}57455391$. If $\mu(\alpha) = \infty$, α is called a Liouville number. The set of Liouville numbers is a G_δ dense set, but its Hausdorff dimension is zero.

Corollary 3.3. — *Suppose $f \in C^\beta(\mathbb{T})$ with $\beta > \mu(\alpha)$ and $\int f(x)\mathrm{d}x = 0$. Then $\sum a_n f(x+n\alpha)$ converges uniformly (on x) if*

$$\lim a_n = 0, \quad \sum |a_n - a_{n+1}| < \infty.$$

Proof. — By the hypothesis on f, we have $|\widehat{f}(n)| \leqslant B|n|^{-\beta}$. By the definition of $\mu(\alpha)$ we have $\|n\alpha\| \geqslant A|n|^{-\mu+1}$ for $\mu > \mu(\alpha)$. Thus

$$\sum \frac{|\widehat{f}(n)|}{\|n\alpha\|} \leqslant \frac{B}{A} \sum \frac{1}{|n|^{\beta-\mu+1}} < \infty. \qquad \square$$

COROLLARY 3.4. — *For almost all α, for any $f \in C^{2+\epsilon}$ with $\int f(x)\mathrm{d}x = 0$, the series $\sum a_n f(x+n\alpha)$ converges uniformly (on x) if*

$$\lim a_n = 0, \quad \sum |a_n - a_{n+1}| < \infty.$$

3.3. L^2-convergence and a.e. convergence when α is diophantine

THEOREM 3.5. — *Let $f \in L^2(\mathbb{T})$ with $\int f(x)\mathrm{d}x = 0$. The series $\sum a_n f(x+n\alpha)$ converges in L^2-norm if and only if*

$$\lim_{p,q\to\infty} \sum_{k\in\mathbb{Z}} |\widehat{f}(k)|^2 \left|\sum_{n=p}^{q} a_n e^{2\pi i n k\alpha}\right|^2 = 0. \tag{3.1}$$

The condition (3.1) *is satisfied when the series $\sum_{n=1}^{\infty} a_n e^{2\pi i n x}$ converges uniformly (on x). The condition* (3.1) *is also satisfied when*

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{|\widehat{f}(n)|^2}{\|n\alpha\|^2} < \infty; \quad \lim_{n\to\infty} a_n = 0, \ \sum_{n=1}^{\infty} |a_n - a_{n+1}| < \infty. \tag{3.2}$$

Proof. — We have

$$\int_{\mathbb{T}} f(x+n\alpha)\overline{f(x)}\mathrm{d}x = \sum_{k\in\mathbb{Z}} |\widehat{f}(k)|^2 e^{2\pi i k n\alpha}.$$

It follows that the spectral measure of f is the following discrete measure

$$\sigma_f = \sum_{k\in\mathbb{Z}} |\widehat{f}(k)|^2 \delta_{k\alpha}.$$

By the spectral lemma, we have

$$\int \left|\sum_{p}^{q} a_n f(x+n\alpha)\right|^2 \mathrm{d}x = \int \left|\sum_{p}^{q} a_n e^{2\pi i n t}\right|^2 \sigma_f(t).$$

Now we can conclude for the first assertion by the Cauchy criterion for L^2-convergence.

The second assertion is an immediate consequence.

For the third assertion, we check the condition (3.1) by an Abel summation and the fact that $\sum_{k=0}^{n} e^{2\pi ikx} = O(\|x\|^{-1})$ and obtain

$$\left|\sum_{n=p}^{q} a_n e^{2\pi inka}\right| \leqslant C\left(|a_p| + |a_q| + \frac{1}{\|k\alpha\|}\sum_{n=p}^{q-1} |a_n - a_{n+1}|\right)$$

for some constant $C > 0$. □

In particular, a sufficient condition for $\sum_{n=1}^{\infty} \frac{f(x+n\alpha)}{n}$ to converge in L^2-norm is $\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{|\widehat{f}(n)|^2}{\|n\alpha\|^2} < \infty$. By Cuny's result, $\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{|\widehat{f}(n)|^2}{\|n\alpha\|^2} < \infty$ implies the a.e. convergence of $\sum \frac{f(x+n\alpha)}{n}$.

Similarly, a sufficient condition for $\sum_{n=1} \frac{(-1)^n}{n} f(x+n\alpha)$ to converge in L^2-norm is $\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{|\widehat{f}(n)|^2}{\|n\alpha-1/2\|^2} < \infty$. We should only notice that

$$\sum_{p}^{q} \frac{(-1)^n}{n} e^{2\pi inka} = \sum_{p}^{q} \frac{1}{n} e^{2\pi in(k\alpha-1/2)}$$

and then make an Abel summation. It seems that the oscillation of $(-1)^n = e^{2\pi in\cdot 1/2}$ doesn't promote the convergence. But for a fixed α and for almost all $\beta \in (0,1)$, the series $\sum \frac{e^{2i\pi n\beta}}{n} f(x+n\alpha)$ converges in L^2 and a.e. This is a consequence of the result of Cuny [4] applied to the Dunford–Schwartz operator

$$Tf(x) = e^{2i\pi\beta} f(x+\alpha).$$

The size of the exceptional set of β was studied by Chevallier, Cohen and Conze [2]. Another oscillation sequence is the Möbius function $\mu(n)$. It can be deduced from Cuny and Weber [7] that for any $f \in L^p$ $(p > 1)$, the series $\sum \frac{\mu(n)}{n} f(x+n\alpha)$ converge in L^p and a.e.

The sufficient condition (3.2) is not so satisfactory, because $\sum \frac{|\widehat{f}(n)|^2}{\|n\alpha\|^2} < \infty$ is not so transparent. If we assume $\widehat{f}(k) = O(|k|^{-\beta})$. Then $\sum \frac{|\widehat{f}(n)|^2}{\|n\alpha\|^2} < \infty$ is ensured by $\beta > \mu(\alpha) - 1/2$ $(\geqslant 3/2)$. This can be improved to $\beta > 1/2$ in the case of the one-sided EHT (1.2). More precisely, we have the following theorem.

Theorem 3.6. — *Let $f \in L^2(\mathbb{T})$ with $\widehat{f}(0) = 0$ and $|\widehat{f}(k)| \leqslant C|k|^{-\beta}$ where $C > 0$ and $\beta > 1/2$ are two constants. Let α be an irrational number with convergents $\{p_n/q_n\}$. The one-sided EHT $\sum_{n=1}^{\infty} \frac{f(x+n\alpha)}{n}$ converges in L^2-mean and a.e. if the following condition is satisfied*

$$\sum_{m=1}^{\infty} \frac{\log^2 q_{m+1}}{q_m^{2\beta}} < \infty. \tag{3.3}$$

Proof. — The proof is based on Gaposhkin's necessary and sufficient condition for L^2- convergence ([11]):

$$\sum_{n=1}^{\infty} \frac{\log n}{n^3} \left\| \sum_{\ell=1}^{n} f(\,\cdot + \ell\alpha) \right\|_{L^2(\sigma_f)}^2 < \infty.$$

(Gaposhkin's condition holds for unitary operators. See also [1]. Cohen–Lin [3] generalized it to normal contractions. Other generalization were obtained by Cuny [5]). In the irrational rotation case, the spectral measure σ_f is the discrete measure $\sum_{k\in\mathbb{Z}} |\widehat{f}(k)|^2 \delta_{k\alpha}$ on the circle $\mathbb{T}$. Thus the above condition takes the following form

$$\sum_{k\in\mathbb{Z}} |\widehat{f}(k)|^2 \sum_{n=1}^{\infty} \frac{\log n}{n^3} |F_n(k\alpha)|^2 < \infty \tag{3.4}$$

where

$$F_n(t) = \sum_{\ell=1}^{n} e^{2\pi i\ell\, t} = e^{(n+1)\pi i} \frac{\sin \pi n t}{\sin \pi t}.$$

We are going to verify that the condition (3.4) is implied by the condition (3.3).

We cut the sum over k in (3.4) into blocks $q_m \leqslant |k| < q_{m+1}$ $(m \geqslant 1)$ and then decompose the m-th block into three parts:

$$P_{m,1} = \{k : q_m \leqslant |k| < \epsilon q_{m+1}, q_m \mid k\}$$
$$P_{m,2} = \{k : q_m \leqslant |k| < \epsilon q_{m+1}, q_m \not| \, k\}$$
$$P_{m,3} = \{k : \epsilon q_{m+1} \leqslant |k| < q_{m+1}\}$$

where $0 < \epsilon \leqslant 1/4$ is fixed. According to these three cases of k, we are going to estimate $\sum_{n=1}^{\infty} \frac{\log n}{n^3} |F_n(k\alpha)|^2$.

We make first a remark. Let $k = \ell q_m$ be a multiple of q_m with $1 \leqslant \ell \leqslant \frac{1}{2} q_{m+1}$. We have $\frac{1}{2q_{m+1}} \leqslant \|q_m\alpha\| \leqslant \frac{1}{q_{m+1}}$ which is very small and then $\|k\alpha\| = \|\ell q_m\alpha\| = \ell\|q_m\alpha\|$ and

$$\frac{\ell}{2q_{m+1}} \leqslant \|\ell q_m \alpha\| \leqslant \frac{\ell}{q_{m+1}} \leqslant \frac{1}{2}. \tag{3.5}$$

So, $q_m\alpha$ is very small and the distance of $\ell q_m\alpha$ from 0 increases with ℓ $(1 \leqslant \ell \leqslant q_{m+1}/2)$.

Assume $k \in P_{m,1}$. We have $k = \ell q_m$ for some $1 \leqslant \ell \leqslant \epsilon q_{m+1}/q_m$. By the first inequality in (3.5), we get $|\sin \pi\ell q_m\alpha| \geqslant \frac{\ell}{q_{m+1}} \geqslant \frac{1}{q_{m+1}}$ so that

$$\max_{k\in P_{m,1}} |F_n(k\alpha)| \leqslant \max\left(q_{m+1}, n\right).$$

Thus if $1 \leqslant \ell < \epsilon q_{m+1}/q_m$, we have

$$\sum_{n=1}^{\infty} \frac{\log n}{n^3} |F_n(\ell q_m \alpha)|^2 \leqslant \sum_{1\leqslant n \leqslant q_{m+1}} \frac{\log n}{n^3} \cdot n^2 + \sum_{n > q_{m+1}} \frac{\log n}{n^3} \cdot q_{m+1}^2$$
$$= O(\log^2 q_{m+1}) + O(\log q_{m+1}) = O(\log^2 q_{m+1}).$$

Here we have used the facts

$$\int_1^A \frac{\log x}{x} \mathrm{d}x \sim \frac{\log^2 A}{2}, \quad \int_A^\infty \frac{\log x}{x^3} \mathrm{d}x \sim \frac{\log A}{2A^2} \quad \text{as} \quad A \to +\infty.$$

Therefore, since $\beta > 1/2$, we have

$$\sum_{k\in P_{m,1}} |\widehat{f}(k)|^2 \sum_{n=1}^{\infty} \frac{\log n}{n^3} |F_n(k\alpha)|^2 = O\left(\frac{\log^2 q_{m+1}}{q_m^{2\beta}} \sum_{\ell=1}^{\epsilon q_{m+1}/q_m} \frac{1}{\ell^{2\beta}} \right) \tag{3.6}$$
$$= O\left(\frac{\log^2 q_{m+1}}{q_m^{2\beta}} \right).$$

Assume $k \in P_{m,2}$. Then $k = \ell q_m + r$ for some $0 \leqslant \ell \leqslant \epsilon q_{m+1}/q_m$ and $1 \leqslant r < q_m$. Then by the second inequality in (3.5), we get

$$\|k\alpha\| \geqslant \|r\alpha\| - \|\ell q_m \alpha\| \geqslant \frac{1}{2q_m} - \frac{\epsilon}{q_m}.$$

Thus we have

$$\max_{k\in P_{m,2}} |F_n(k\alpha)| \leqslant \max\left(\Big(1/2 - \epsilon\Big) q_m, n \right).$$

Just as above, but cut the sum at q_m instead of q_{m+1} we get

$$\sum_{n=1}^{\infty} \frac{\log n}{n^3} |F_n(k\alpha)|^2 \leqslant \sum_{1\leqslant n\leqslant q_m} \frac{\log n}{n^3} \cdot n^2 + \Big(1/2 - \epsilon\Big) \sum_{n > q_m} \frac{\log n}{n^3} \cdot q_m^2$$
$$= O(\log^2 q_m).$$

Therefore, again thanks to the hypothesis $\beta > 1/2$, we get

$$\sum_{k\in P_{m,2}} |\widehat{f}(k)|^2 \sum_{n=1}^{\infty} \frac{\log n}{n^3} |F_n(k\alpha)|^2 = O\left(\log^2 q_m \sum_{k\in P_{m,2}} \frac{1}{k^{2\beta}} \right) \tag{3.7}$$
$$= O\left(\frac{\log^2 q_m}{q_m^{2\beta}} \right).$$

Assume $k \in P_{m,3}$. Since $\|k\alpha\| \geqslant \|q_m \alpha\| \geqslant \frac{1}{2q_{m+1}}$, we still have

$$\max_{k\in P_{m,3}} |F_n(k\alpha)| \leqslant \max(q_{m+1}, n).$$

Then we obtain

$$\begin{aligned}\sum_{k\in P_{m,3}} |\widehat{f}(k)|^2 \sum_{n=1}^{\infty} \frac{\log n}{n^3} |F_n(k\alpha)|^2 &= O\left(\log^2 q_{m+1} \sum_{k\in P_{m,2}} \frac{1}{k^{2\beta}}\right) \\ &= O\left(\frac{\log^2 q_{m+1}}{q_{m+1}^{2\beta}}\right)\end{aligned} \tag{3.8}$$

where, following the arguments used when we deal with $P_{m,1}$, the above sum over n is also controlled by $\log^2 q_{m+1}$.

Thus, it follows from (3.6), (3.7) and (3.8) that the left hand side of (3.4) is bounded, up to a multiplicative constant, by

$$\sum \frac{\log^2 q_{m+1}}{q_m^{2\beta}} + \sum \frac{\log^2 q_m}{q_m^{2\beta}} + \sum \frac{\log^2 q_{m+1}}{q_{m+1}^{2\beta}}.$$

However, since q_m is increasing, $\sum \frac{\log^2 q_{m+1}}{q_m^{2\beta}} < \infty$ implies the finiteness of the two last sums. □

The condition (3.3) on the L^2-convergence is of Bruno type. To some extent, this condition (3.3) is optimal, as the following proposition shows.

Proposition 3.7. — *Let α be an irrational number with convergents $\{p_n/q_n\}$. Consider the function f defined by the lacunary series*

$$f(x) = \sum_{m=1}^{\infty} \widehat{f}(q_m) e^{2\pi i q_m x}, \quad \textit{with} \quad \sum_{m=1}^{\infty} |\widehat{f}(q_m)|^2 < \infty.$$

The one-sided EHT $\sum_{n=1}^{\infty} \frac{f(x+n\alpha)}{n}$ converges in L^2-mean if and only if

$$\sum_{m=1}^{\infty} |\widehat{f}(q_m)|^2 \log^2 q_{m+1} < \infty. \tag{3.9}$$

It is immediate from the following condition

$$\int_{-1/2}^{1/2} \log^2(|t|)\sigma_f(\mathrm{d}t) < \infty,$$

which is equivalent to the above mentioned Gaposhkin's condition ([3, 5]). Because $\sigma_f = \sum_{m=1}^{\infty} |\widehat{f}(q_m)|^2 \delta_{q_m\alpha}$ and $\|q_m\alpha\| \approx 1/q_{m+1}$.

3.4. Convergence when α is rational

Let $L^0(\mathbb{T})$ be the space of all Borel functions on $\mathbb{T}$.

THEOREM 3.8. — *Let $\alpha = \frac{p}{q}$ be a rational number with $(p,q)=1$. Let $(a_n) \subset \mathbb{C}$. The following propositions are equivalent:*

(1) *For any $f \in L^0(\mathbb{T})$ with $\int f(x)\mathrm{d}x = 0$, the series $\sum a_n f(x+n\alpha)$ converges almost everywhere.*
(2) *For any $f \in L^1(\mathbb{T})$ with $\int f(x)\mathrm{d}x = 0$, the series $\sum a_n f(x+n\alpha)$ converges almost everywhere.*
(3) *For any $j = 0,1,\dots,q-1$, the numerical series $\sum_k a_{kq+j}$ converges.*

Proof. — First remark that the hypothesis $(p,q)=1$ means that p is invertible in the ring $\mathbb{Z}/q\mathbb{Z}$. It follows that the sequence $\{n\alpha \pmod 1\}$ is periodic with q as minimal period.

(1) is obviously stronger than (2).

(2) $\Rightarrow$ (3): For fix $j \in \{0,1,\dots,q-1\}$, let $i \in \{0,1,\dots,q-1\}$ be such that $jp = i \pmod q$, so that $j\alpha = \frac{i}{q} \pmod 1$. Define

$$f(x) = 1_{[i/q, i/q+1/(2q))}(x) - 1_{[i/q+i/(2q),(i+1)/q)}(x).$$

This function is supported by the interval $[i/q,(i+1)/q)$, taking values 1 on the left-half interval and -1 on the right-half interval. It is clear that $\int f(x)\mathrm{d}x = 0$. For any $x_0 \in [0,1/(2q))$ such that $\sum a_n f(x_0+n\alpha)$ converges. Observe that $(kq+\ell)\alpha = \ell\alpha \pmod 1$ and that $x_0 \in [0,1/q)$ if and only if $x_0 + j\alpha \in [i/q,(i+1)/q)$, so that

$$\begin{aligned}\sum_{n\geqslant 0} a_n f(x_0+n\alpha) &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{q-1} a_{kq+\ell} f(x_0+\ell\alpha)\\ &= \sum_{k=0}^{\infty} a_{kq+j} f(x_0+j\alpha) = \sum_{k=0}^{\infty} a_{kq+j}.\end{aligned}$$

(3) $\Rightarrow$ (1): This is because $\{n\alpha \pmod 1\}$ is q-periodic and

$$\sum_{n\geqslant 0} a_n f(x_0+n\alpha) = \sum_{\ell=0}^{q-1} f(x_0+\ell\alpha)\sum_{k=0}^{\infty} a_{kq+\ell}.$$ □

BIBLIOGRAPHY

[1] I. ASSANI & M. LIN, "On the one-sided ergodic Hilbert transform", in *Ergodic theory and related fields*, Contemporary Mathematics, vol. 430, American Mathematical Society, 2007, p. 21-39.

[2] N. CHEVALLIER, G. COHEN & J.-P. CONZE, "On the convergence of the rotated one-sided ergodic Hilbert transform", *Positivity* **15** (2011), no. 2, p. 253-270.

[3] G. COHEN & M. LIN, "The one-sided ergodic Hilbert transform of normal contractions", in *Characteristic functions, scattering functions and transfer functions*, Operator Theory: Advances and Applications, vol. 197, Birkhäuser, 2010, p. 77-98.

[4] C. Cuny, "On the a.s. convergence of the one-sided ergodic Hilbert transform", *Ergodic Theory Dyn. Syst.* **29** (2009), no. 6, p. 1781-1788.

[5] ———, "Pointwise ergodic theorems with rate with applications to limit theorems for stationary processes", *Stoch. Dyn.* **11** (2011), no. 1, p. 135-155.

[6] C. Cuny & A. Fan, "Study of almost everywhere convergence of series by mean of martingale methods", *Stochastic Processes Appl.* **127** (2017), no. 8, p. 2725-2750.

[7] C. Cuny & M. Weber, "Ergodic theorems with arithmetical weights", *Isr. J. Math.* **217** (2017), no. 1, p. 139-180.

[8] Y. N. Dowker & P. Erdős, "Some examples in ergodic theory", *Proc. Lond. Math. Soc.* **9** (1959), p. 227-241.

[9] A. Fan, "Almost everywhere convergence of ergodic series", *Ergodic Theory Dyn. Syst.* **37** (2017), no. 2, p. 490-511.

[10] V. F. Gaposhkin, "On series with respect to the system $\{\varphi(nx)\}$", *Mat. Sb. (N.S.)* **69 (111)** (1966), p. 328-353.

[11] ———, "Spectral criteria for the existence of generalized ergodic transformations", *Teor. Veroyatn. Primen.* **41** (1996), no. 2, p. 251-271.

[12] H. Iwaniec, "On the problem of Jacobsthal", *Demonstr. Math.* **11** (1978), no. 1, p. 225-231.

[13] S.-i. Izumi, "A non-homogeneous ergodic theorem", *Proc. Imp. Acad., Tokyo* **15** (1939), p. 189-192.

[14] E. Jacobsthal, "Über Sequenzen ganzer Zahlen, von denen keine zu n teilerfremd ist. I, II, III", *Norske Vid. Selsk. Forhdl.* **33** (1961), p. 117-124, 125-131, 132-139.

[15] A. del Junco & J. Rosenblatt, "Counterexamples in ergodic theory and number theory", *Math. Ann.* **245** (1979), no. 3, p. 185-197.

[16] J.-P. Kahane, *Séries de Fourier absolument convergentes*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 50, Springer, 1970, viii+169 pages.

[17] A. Ya. Khinchin, *Continued fractions*, University of Chicago Press, 1964, xi+95 pages.

Manuscrit reçu le 1er octobre 2016,
révisé le 21 juillet 2017,
accepté le 13 décembre 2017.

Aihua FAN
Central China Normal University
School of Mathematics and Statistics
152 Luoyu Road
Wuhan 430079 (PRC)
Current address:
Université de Picardie Jules Verne
Faculté de Mathématiques et d'Informatique
33 rue Saint Leu
80039 Amiens CEDEX 1 (France)
ai-hua.fan@u-picardie.fr

Jörg SCHMELING
Lund University
Centre for Mathematical Sciences
Box 118, 221 00 LUND (Sweden)
joerg@maths.lth.se