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TOPOLOGICAL RIGIDITY OF GENERIC UNFOLDINGS OF TANGENT TO THE IDENTITY DIFFEOMORPHISMS

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Abstract. — We prove that a homeomorphism conjugating two generic 1-parameter unfoldings, of local 1-variable tangent to the identity biholomorphisms with a double fixed point at the origin, is real analytic outside the origin by restriction to the unperturbed parameter. Moreover if one of the unfoldings has a restriction to the unperturbed parameter that is not analytically trivial, meaning that is not the time 1 flow of a holomorphic vector field, then the restriction of the conjugating map to the unperturbed parameter is holomorphic or anti-holomorphic. We provide examples that show that the non-analytically trivial hypothesis is necessary. Moreover we characterize the possible behavior of conjugacies for the unperturbed parameter in the analytically trivial case.

We describe the structure of the limits of orbits when we approach the unperturbed parameter. The proof of the rigidity results is based on the study of the action of a topological conjugacy on the limits of orbits.

Résumé. — On considère des germes de biholomorphisme ϕ_0 et η_0 tangents à l'identité et avec un point fixe double. On montre qu'un homéomorphisme qui conjugue deux déploiements génériques à un paramètre de ϕ_0 et η_0 est analytique réel si l'on se restreint au paramètre initial (sauf peut-être à l'origine). De plus si ϕ_0 ou η_0 n'est pas analytiquement trivial, i.e. n'est pas contenu dans un group à un paramètre, la conjugaison induite sur le paramètre initial est holomorphe ou anti-holomorphe. L'hypothèse de non-trivialité est nécessaire. On détermine aussi la nature des conjugaisons sur le paramètre initial se ϕ_0 ou η_0 ne sont pas analytiquement triviaux.

On décrit la structure des limites d'orbites quand on approche le paramètre initial. Les resultats de rigidité sont conséquences de l'étude de l'action d'une conjugaison topologique sur les limites d'orbites.

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1. Introduction

Consider an arc of diffeomorphisms $(\phi_x(y))$ defined in some neighborhood of 0 in $\mathbb{R}$ where the parameter x also varies in a neighborhood of 0 in $\mathbb{R}$. We can express the arc in the form $\phi(x,y) = (x, \phi_x(y))$. Suppose

$$\phi_0(0) = 0,\ \frac{\partial \phi_0}{\partial y}(0) = 1,\ \frac{\partial^2 \phi_0}{\partial y^2}(0) \neq 0 \ \text{ and } \ \frac{\partial \phi_x(y)}{\partial x}(0,0) \neq 0.$$

The first three conditions guarantee that ϕ_0 is a local tangent to the identity diffeomorphism that has a double fixed point at the origin. The last property is a genericity condition for the arc of diffeomorphisms. Arcs of diffeomorphisms of the above form, where $\phi_x(y)$ is C^6 as a function of x and y and ϕ_0 is C^∞ as a function of y, are called *saddle-node arcs* by Newhouse, Palis and Takens [14]. They study the properties of saddle-node arcs for the topological conjugacy.

The diffeomorphism ϕ_0 has a unique C^∞ *infinitesimal generator* such that ϕ_0 is the time 1 flow $\exp(X_0)$ of X_0 [24]. This vector field is also called the *Szekeres vector field* [23]. Newhouse, Palis and Takens show the following results:

Theorem 1.1 ([14, Theorem 3.1]). — *Two saddle-node arcs are locally conjugated.*

Theorem 1.2 ([14, Theorem 3.2]). — *Let (ϕ_x) and (η_x) be two saddle-node arcs conjugated by a local homeomorphism σ. Then σ_0 conjugates the infinitesimal generators of ϕ_0 and η_0. In particular σ_0 is C^∞ outside the origin.*

We are studying conjugacies of unfoldings (or families of diffeomorphisms). Thus all the conjugacies in this paper are of the form $\sigma(x,y) = (\sigma_1(x), \sigma_2(x,y))$. In particular the restriction σ_0 of σ to the unperturbed line $x=0$ is well-defined and we have $\sigma_0(y) = \sigma_2(0,y)$.

Our goal is studying this kind of properties in the holomorphic case. Analogously as for saddle-node arcs we say that a local biholomorphism $\phi(x,y) = (x, f(x,y))$, defined in a neighborhood of the origin in $\mathbb{C}^2$, is a *saddle-node unfolding* if

$$f(0,0) = 0,\ \frac{\partial f}{\partial y}(0,0) = 1,\ \frac{\partial^2 f}{\partial y^2}(0,0) \neq 0, \ \text{ and } \ \frac{\partial f}{\partial x}(0,0) \neq 0.$$

In other words ϕ is a generic unfolding of a tangent to the identity diffeomorphism ϕ_0 that has a double fixed point at the origin. We prove the following rigidity theorem:

THEOREM 1.3 (Main Theorem). — *Let $\phi, \eta \in \mathrm{Diff}(\mathbb{C}^2, 0)$ be saddle-node unfoldings that are topologically conjugated by a local homeomorphism σ. Suppose that either ϕ_0 or η_0 is non-analytically trivial. Then σ_0 is holomorphic or anti-holomorphic.*

By definition a tangent to the identity local biholomorphism in one complex variable ϕ is analytically trivial if it is embedded in an analytic flow, i.e. ϕ is the time 1 flow $\exp(X)$ of an analytic singular local vector field $X = g(y)\partial/\partial y$. Analytic triviality is non-generic by the Ecalle–Voronin analytic classification of tangent to the identity diffeomorphisms [5, 11, 25].

The Main Theorem implies that Theorem 1.1 does not hold true in the holomorphic case. Consider the saddle-node unfoldings

$$\phi(x, y) = \exp\left((y^2 - x)\frac{\partial}{\partial y}\right) \quad \text{and} \quad \eta(x, y) = (x, y + y^2 - x).$$

The diffeomorphism $\phi_0 = \exp(y^2\partial/\partial y)$ is analytically trivial whereas $\eta_0(y) = y + y^2$ is not analytically trivial since it is a non-trivial polynomial tangent to the identity local biholomorphism (cf. [1]). Since analytic triviality is obviously invariant by analytic conjugacy, it follows that ϕ and η are not topologically conjugated. Analogously η is not topologically conjugated to the time 1 map of any local holomorphic vector field of the form $g(x, y)\partial/\partial y$.

Anyway, the rigidity result provided by the Main Theorem can be interpreted as a first analogue of Theorem 1.2. Indeed under generic conditions a topological conjugacy σ between analytic unfoldings also conjugates the complex flows of the infinitesimal generators of ϕ_0 and η_0 in their attracting (resp. repelling) petals (if σ_0 is orientation-preserving). This is far from trivial since a topological class of conjugacy of a tangent to the identity diffeomorphism in one variable contains a continuous infinitely dimensional moduli of analytic classes of conjugacy.

A natural problem is determining the classes of conjugacy of unfoldings up to topological, formal or analytic equivalence. The study of the analytic properties of unfoldings is an active field of research. We denote by $\mathrm{Diff}(\mathbb{C}^n, 0)$ and $\mathrm{Diff}_1(\mathbb{C}^n, 0)$ the group of complex analytic biholomorphisms defined in a neighborhood of the origin of $\mathbb{C}^n$ and their subgroup of tangent to the identity diffeomorphisms respectively. A natural idea to study an unfolding $\phi(x, y) = (x, f(x, y)) \in \mathrm{Diff}(\mathbb{C}^2, 0)$ is comparing the dynamics of ϕ and $\exp(Y)$ where $Y = g(x, y)\partial/\partial y$ is a vector field with $\mathrm{Fix}(\phi) = \mathrm{Sing}(Y)$ whose time 1 flow "approximates" ϕ. This point of view has been developed by Glutsyuk [6]. In this way extensions of the Ecalle–Voronin invariants [5, 25] to some sectors in the parameter space are obtained. The extensions are uniquely defined. The sectors of definition have to avoid a finite set of directions of instability, typically associated (but not exclusively) to small divisors phenomena. The rich dynamics of ϕ around the directions of instability prevents the extension of the Ecalle–Voronin invariants to be defined in the neighborhood of the instability directions. Interestingly the study of the dynamics around instability directions is one of the key elements of the proof of the Main Theorem.

A different point of view was introduced by Shishikura for codimension 1 unfoldings [22]. The idea is constructing appropriate fundamental domains bounded by two curves with common ends at singular points: one curve is the image of the other one. Pasting the boundary curves by the dynamics yields (by quasiconformal surgery) a Riemann surface that is conformally equivalent to the Riemann sphere. The logarithm of an appropriate affine complex coordinate on the sphere induces a Fatou coordinate for ϕ. These ideas were generalized to higher codimension unfoldings by Oudkerk [15]. In this approach the first curve is a phase curve of an appropriate vector field transversal to the real flow of X. In both cases the Fatou coordinates provide Lavaurs vector fields X^{ϕ} such that $\phi = \exp(X^{\phi})$ [9]. The Shishikura's approach was used by Mardesic, Roussarie and Rousseau to provide a complete system of invariants for unfoldings of codimension 1 tangent to the identity diffeomorphisms [12]. Rousseau and Christopher classified the generic unfoldings of codimension 1 resonant diffeomorphisms [21]. The analytic classification for the unfoldings of finite codimension resonant diffeomorphisms was completed in [17] by using the Oudkerk's point of view.

We described the formal invariants for n-parameter unfoldings of 1-variable tangent to the identity biholomorphisms in [16] for any $n \in \mathbb{N}$.

1.1. Topological classification

In contrast with the analytic and formal cases there is no topological classification of unfoldings of tangent to the identity diffeomorphisms. One of the obstacles is the absence of a complete system of topological invariants for elements of $\mathrm{Diff}(\mathbb{C}, 0)$. More precisely the problem is associated with small divisors; it is not known the topological classification of elements $\phi(z) = \lambda z + O(z^2) \in \mathrm{Diff}(\mathbb{C}, 0)$ such that $\lambda \in \mathbb{S}^1$ is not a root of unity and ϕ is not analytically linearizable.

A key point in the proof of Theorems 1.1 and 1.2 is a shadowing property. Indeed given a saddle-node arc ϕ there exists an adapted vector field $Y = g(x,y)\partial/\partial y$ with $\mathrm{Fix}(\phi) = \mathrm{Sing}(Y)$ such that every orbit of ϕ can be approximated by an orbit of $\exp(Y)$ [14, Lemma 3.7]. As a consequence the continuous dynamical system defined by the real flow $\mathrm{Re}(Y)$ of Y is a good model of the topological behavior of ϕ. In spite of this, generically there is no shadowing for unfoldings of tangent to the identity biholomorphisms, establishing a difference between the real and the complex cases. Indeed if a saddle-node unfolding has a shadowing property then it is embedded in an analytic flow [18]. Our strategy in this paper includes, as in [14], approximating a saddle-node unfolding ϕ with $\exp(Y)$ for some local vector field $Y = g(x,y)\partial/\partial y$ and then profiting of the properties of Y to obtain interesting dynamical phenomena associated to ϕ. Since there is no shadowing property for all orbits of ϕ we have to show that the dynamics of $\mathrm{Re}(Y)$ that we are trying to replicate for ϕ takes place in regions in which the orbits of $\exp(Y)$ and ϕ remain close. We will see that these regions are domains of definition of extensions of Fatou coordinates of ϕ_0 to the nearby parameters. Moreover such Fatou coordinates can be used to compare ϕ and $\exp(Y)$.

The main tool in this paper is the study of Long Orbits of saddle-node unfoldings. These concepts were introduced in [14] (even if they do not have an explicit name) and we used them in [19] to classify topologically multi-parabolic unfoldings, i.e. unfoldings $\phi(x,y) = (x, f(x,y)) \in \mathrm{Diff}(\mathbb{C}^2, 0)$ such that $(\frac{\partial f}{\partial y})_{|\,\mathrm{Fix}(\phi)} \equiv 1$. They are analogous to the concept of homoclinic trajectories for polynomial vector fields introduced by Douady, Estrada and Sentenac [4]. Let us focus on vector fields since the concepts are analogous and the presentation is a little simpler. Consider a local vector field $Y = g(x,y)\partial/\partial y$ with $g(0) = 0$, $(\partial g/\partial y)(0) = 0$, $(\partial^2 g/\partial y^2)(0) \neq 0$ and $(\partial g/\partial x)(0) \neq 0$. In particular $\exp(tY)$ is a saddle-node unfolding for any $t \in \mathbb{C}^*$. Roughly speaking a Long Trajectory is given by the choice of a point $y_+ \neq 0$, a curve β in the parameter space and a continuous function $T : \beta \to \mathbb{R}^+$ such that

$$(0, y_-) \stackrel{\mathrm{def}}{=} \lim_{x \in \beta, x \to 0} \exp(T(x)Y)(x, y_+)$$

exists and $\lim_{x \in \beta, x \to 0} T(x) = \infty$. In general $(0, y_-)$ does not belong to the trajectory of $\mathrm{Re}(Y)$ through $(0, y_+)$. We go from $(0, y_+)$ to $(0, y_-)$ by following the real flow of Y an infinite time. Denote $\phi = \exp(Y)$. The point $(0, y_-)$ is in the limit of the orbits of ϕ passing through points (x, y_+) with $x \in T^{-1}(\mathbb{N})$ when $x \to 0$. We say that (ϕ, y_+, β, T) is a *Long Orbit* containing $(0, y_-)$. By replacing T with $T+s$ for $s \in \mathbb{R}$ we obtain that $\exp(sY)(0, y_-)$ is in the Long Orbit generated by $(\phi, y_+, \beta, T+s)$. The rest of the points in a neighborhood of $(0, y_-)$ in $x=0$ are also in Long Orbits of ϕ through $(0, y_+)$. They are obtained by varying the curve β. In particular the complex flow of the infinitesimal generator of ϕ_0 in the repelling petal containing $(0, y_-)$ can be retrieved from Long Orbits through $(0, y_+)$. In other words such complex flow is in the topological closure of the pseudogroup generated by ϕ.

Long Orbits are related to instable behavior in the unfolding. For instance consider the saddle-node unfolding $\phi = \exp((y^2 - x)\partial/\partial y)$. The multiplicators of the vector field at the singular points $(x, \sqrt{x})$ and $(x, -\sqrt{x})$ are $2\sqrt{x}$ and $-2\sqrt{x}$ respectively. These points are hyperbolic outside the direction $\mathbb{R}^-$ in the parameter space where the character of the fixed points change (from attractor to repeller or vice versa). The direction $\mathbb{R}^-$ is the direction of instability of the unfolding and all other directions are stable. Indeed given a point y_+ in the attracting petal of $\mathrm{Re}(y^2\partial/\partial y)$ and a direction $\lambda\mathbb{R}^+$ different than $\mathbb{R}^-$ ($\lambda \in \mathbb{S}^1 \setminus \{-1\}$), we have that $\exp(tY)(x, y_+)$ is well-defined for any $t \in \mathbb{R}^+$ and for any x in a neighborhood of 0 such that $x/|x|$ is in a neighborhood of λ. Moreover the set $K_x := \overline{\exp([0,\infty)Y)(x, y_+)}$ converges to K_0 (in the Hausdorff topology for compact sets) when x tends to 0 and $x/|x|$ is in a neighborhood of λ (These properties are consequences of Lemma 6.13 of [17]). So the limit of trajectories through the points (x, y_+) is the trajectory through the limit point $(0, y_+)$ and hence Long Orbits are not possible in the neighborhood of any direction $\lambda\mathbb{R}^+$ in the parameter space different than $\mathbb{R}^-$.

In spite of being scarce Long Orbits somehow vary continuously. For instance the function T in the definition can be calculated by applying conveniently the residue theorem. The residue formula allows to describe the evolution of the Long Orbits when we replace β with nearby curves. On the one hand Long Orbits appear in the regions of instability of the unfolding and generically together with small divisors phenomena. On the other hand they have a (rich) regular structure. The main technical difficulty regarding Long Orbits is proving their existence and properties. Once the setup is established the Main Theorem is obtained by a relatively simple description of the action of topological conjugacies on Long Orbits.

1.2. Rigidity of unfoldings

Now we introduce the most general rigidity result.

Theorem 1.4 (General Theorem). — *Let $\phi, \eta \in \mathrm{Diff}(\mathbb{C}^2, 0)$ be saddle-node unfoldings that are topologically conjugated by a local homeomorphism σ. Then σ_0 is affine in Fatou coordinates. Moreover σ_0 is orientation-preserving if and only if the action of σ on the parameter space is orientation-preserving. In particular σ is orientation-preserving.*

Topological conjugacies are of the form $\sigma(x, y) = (\sigma_1(x), \sigma_2(x, y))$. We say that the action of σ in the parameter space is orientation-preserving if σ_1 is. Analogously we define the concept of holomorphic action on the parameter space.

The definition of affine in Fatou coordinates is provided in Subsection 2.5. Affine in Fatou coordinates implies real analytic outside the origin. In order to compare the Main Theorem and Theorem 1.4 let us point out that holomorphic conjugacies between elements of $\mathrm{Diff}_1(\mathbb{C}, 0) \setminus \{\mathrm{Id}\}$ are translations in Fatou coordinates. The Main Theorem is a consequence of Theorem 1.4. Indeed we show that affine in Fatou coordinates implies holomorphic or anti-holomorphic in the non-analytically trivial case.

Theorem 1.4 can also be interpreted as an analogue of Theorem 1.2 for the complex case. Indeed the affine nature of σ_0 implies that the real flows of the infinitesimal generators (Lavaurs vector fields) of ϕ_0 and η_0 (in the corresponding attracting and repelling petals) are conjugated. Anyway the analogy is not perfect, since the complex flows of the Lavaurs vector fields are not conjugated in general.

How to strengthen the General Theorem? A first approach is provided by the Main Theorem by considering generic classes of analytic conjugacy. Another possibility is trying to impose conditions on the action of conjugacies on the parameter space. Finally we notice that for analytically trivial elements of $\mathrm{Diff}_1(\mathbb{C}, 0)$ the formal and analytic conjugacy classes coincide. So it is interesting to study the action of σ_0 on formal invariants. The next propositions establish a relation between the topological, formal and analytic classifications.

Proposition 1.5. — *Let $\phi, \eta \in \mathrm{Diff}(\mathbb{C}^2, 0)$ be saddle-node unfoldings. Let σ be a local homeomorphism such that $\sigma \circ \phi = \eta \circ \sigma$. Suppose that the action of σ on the parameter space is holomorphic (resp. anti-holomorphic). Then σ_0 is holomorphic (resp. anti-holomorphic).*

It is simple to reduce the setting of Proposition 1.5 to the case where σ is of the form $\sigma(x, y) = (x, f(x, y))$ and satisfies $\sigma_{|\,\mathrm{Fix}(\phi)} \equiv \mathrm{Id}$. The analogue of Proposition 1.5 in absence of small divisors (multi-parabolic case) has been studied in [19]. A complete system of topological invariants is presented in [19] for the classification of multi-parabolic diffeomorphisms under the assumption that the conjugating map $\sigma(x,y) = (x, f(x,y))$ satisfies $\sigma_{|\,\mathrm{Fix}(\phi)} \equiv \mathrm{Id}$. In the non-trivial case, where the fixed point set $\{f(x,y) = y\}$ is not of the form $\{y = a(x)\}$ for some $a \in \mathbb{C}\{x\}$, one of the topological invariants is the analytic class of conjugacy of the unperturbed diffeomorphism of the unfolding. As in Proposition 1.5 the map σ_0 is a local biholomorphism.

Let $\phi(y) = y + cy^{\nu+1} + h.o.t. \in \mathrm{Diff}(\mathbb{C}, 0)$ with $\nu \in \mathbb{N}$ and $c \in \mathbb{C}^*$. The number ν determines the class of topological conjugacy of ϕ. The diffeomorphism ϕ is formally conjugated to a unique diffeomorphism $y + y^{\nu+1} + ((\nu+1)/2 - \lambda)y^{2\nu+1}$ for some $\lambda \in \mathbb{C}$. The pair (ν, λ) provides a complete system of formal invariants. We define $\mathrm{Res}_\phi = \lambda$.

Proposition 1.6. — *Let $\phi, \eta \in \mathrm{Diff}(\mathbb{C}^2, 0)$ be saddle-node unfoldings that are topologically conjugated by a local homeomorphism σ. Then $\mathrm{Re}(\mathrm{Res}_{\phi_0})$ and $\mathrm{Re}(\mathrm{Res}_{\eta_0})$ have the same sign.*

The sign of a real number can be positive, negative or zero by convention. The above proposition implies that the analogue of Theorem 1.1 for unfoldings of saddle-node vector fields [14, Theorem 3.5] does not hold in the complex setting. More precisely, consider

$$Y = (y^2 - x)\frac{\partial}{\partial y} \quad \text{and} \quad Z = \frac{y^2 - x}{1+y}\frac{\partial}{\partial y}.$$

We have that $\exp(tY)$ and $\exp(tZ)$ are saddle-node unfoldings for any $t \in \mathbb{C}^*$. Denote $\phi = \exp(Y)$ and $\eta = \exp(Z)$. We have $\mathrm{Res}_{\phi_0} = 0$ and $\mathrm{Res}_{\eta_0} = 1$ by a simple calculation. Hence ϕ and η are not topologically conjugated by Proposition 1.6. The existence of a conjugacy σ_x between ϕ_x and $\eta_{\sigma_1(x)}$, for any $x \in \mathbb{C}$ in a neighborhood of 0, provides a monodromy condition when we turn around $x = 0$ that does not appear for saddle-node arcs.

Proposition 1.7. — *Let $\phi, \eta \in \mathrm{Diff}(\mathbb{C}^2, 0)$ be saddle-node unfoldings that are topologically conjugated by a local homeomorphism σ. Suppose that either ϕ_0 or η_0 is analytically trivial. Suppose that either $\mathrm{Res}_{\phi_0} \not\in i\mathbb{R}$ or $\mathrm{Res}_{\eta_0} \not\in i\mathbb{R}$. Then*

- *If σ_0 is orientation-preserving then σ_0 is holomorphic iff $\mathrm{Res}_{\phi_0} = \mathrm{Res}_{\eta_0}$.*
- *If σ_0 is orientation-reversing then σ_0 is anti-holomorphic if and only if $\mathrm{Res}_{\phi_0} = \overline{\mathrm{Res}_{\eta_0}}$.*

On the one hand it is possible to construct examples of diffeomorphisms ϕ, η satisfying the hypotheses of the previous proposition such that ϕ_0 and η_0 are neither holomorphically nor anti-holomorphically conjugated (Section 6). On the other hand if they are holomorphically conjugated (in the orientation-preserving case) then σ_0 is also holomorphic. In other words given a saddle-node unfolding ϕ as in Proposition 1.7 the analytic class of η_0 is not determined for η in the class of topological conjugacy of ϕ but the conjugacy σ_0 is determined up to composition with a holomorphic diffeomorphism (see Proposition 2.39). The condition $\mathrm{Res}_{\phi_0} \notin \mathrm{i}\,\mathbb{R}$ on formal invariants implies flexibility in the analytic classes of η_0 but once they are fixed there is rigidity of the conjugating maps.

Next we consider the case of purely imaginary formal invariants.

Proposition 1.8. — *Let $\phi, \eta \in \mathrm{Diff}(\mathbb{C}^2, 0)$ be saddle-node unfoldings that are topologically conjugated by a local homeomorphism σ. Suppose that either ϕ_0 or η_0 is analytically trivial. Suppose that either $\mathrm{Res}_{\phi_0} \in \mathrm{i}\,\mathbb{R}$ or $\mathrm{Res}_{\eta_0} \in \mathrm{i}\,\mathbb{R}$. Then ϕ_0 and η_0 are holomorphically conjugated (resp. anti-holomorphically conjugated) if σ is orientation-preserving (resp. orientation-reversing) on the parameter space.*

The roles of analytic classes and conjugacies are reversed with respect to Proposition 1.7. Indeed there are at most 2 classes of analytic conjugacy of η_0 in the set consisting of the saddle-node unfoldings η in the topological class of ϕ. In spite of the rigidity of analytic classes, conjugacies are not rigid. Even if ϕ_0 and η_0 are analytically conjugated the map σ_0 is not necessarily holomorphic. Examples of this behavior are presented in Section 6.

Proposition 1.8 provides new counterexamples for the analogue of Theorem 1.1 for saddle-node vector fields. Indeed consider

$$Y = \frac{y^2 - x}{1 + \mathrm{i}\, y} \frac{\partial}{\partial y}, \; Z = \frac{y^2 - x}{1 + 2\,\mathrm{i}\, y} \frac{\partial}{\partial y}, \; \phi = \exp(Y) \;\text{ and }\; \eta = \exp(Z).$$

Since $\mathrm{Res}_{\phi_0} = \mathrm{i}$ and $\mathrm{Res}_{\eta_0} = 2\,\mathrm{i}$, it follows that $\mathrm{Res}_{\eta_0} \notin \{\mathrm{Res}_{\phi_0}, \overline{\mathrm{Res}_{\phi_0}}\}$. Therefore ϕ_0 and η_0 are neither holomorphically nor anti-holomorphically conjugated and then ϕ and η are not topologically conjugated.

A very simple consequence of our results is that a homeomorphism conjugating two generic unfolding of saddle-nodes of the form

$$x^2 \mathrm{d}y - f(x, y)\mathrm{d}x = 0$$

is either transversaly conformal or transversaly anti-conformal by restriction to the unperturbed parameter.

2. Fatou affine conjugacy

The goal of this section is introducing a new notion of conjugacy between local tangent to the identity diffeomorphisms in dimension 1, the so called *Fatou affine conjugacy*. This concept is interesting because the restriction to the unperturbed line of a topological conjugacy of unfoldings is Fatou affine by Theorem 1.4. Generically Fatou affine conjugacies are holomorphic or anti-holomorphic (Proposition 2.34). Moreover we will describe the action of Fatou affine conjugacies on formal invariants (Proposition 2.36). Along the way we classify local tangent to the identity diffeomorphisms modulo Fatou affine conjugacy (Proposition 2.34, Corollary 2.38 and Proposition 2.40).

2.1. Basic definitions

Let us introduce some well-known concepts.

Definition 2.1. — *We denote by* $\mathrm{Diff}\,(\mathbb{C}^n,0)$ *the group of local biholomorphisms defined in a neighborhood of* 0 *in* $\mathbb{C}^n$*. Their elements are of the form*

$$\phi(y_1,\dots,y_n)=(\phi_1(y_1,\dots,y_n),\dots,\phi_n(y_1,\dots,y_n))$$

where $\phi_1,\dots,\phi_n$ *belong to the maximal ideal of the ring of convergent complex power series* $\mathbb{C}\{y_1,\dots,y_n\}$ *and the Jacobian at the origin is invertible. Analogously we define the group of formal diffeomorphisms as the set of expressions*

$$\phi(y_1,\dots,y_n) = \left(\sum_{j_1+\dots+j_n\geqslant 1} a_{j_1\cdots j_n;1}y_1^{j_1}\cdots y_n^{j_n},\dots,\sum_{j_1+\dots+j_n\geqslant 1} a_{j_1\cdots j_n;n}y_1^{j_1}\cdots y_n^{j_n}\right)$$

in the ring of formal complex power series $\mathbb{C}[\![y_1,\cdots,y_n]\!]^n$ *such that its linear part*

$$j^1\phi(\underline{y})=\left(\sum_{j_1+\dots+j_n=1} a_{j_1\dots j_n;1}y_1^{j_1}\dots y_n^{j_n},\dots,\sum_{j_1+\dots+j_n=1} a_{j_1\cdots j_n;n}y_1^{j_1}\dots y_n^{j_n}\right)$$

at the origin is invertible. The group operation is the composition defined in the natural way.

Definition 2.2. — *Let* ϕ *be an element of the group* $\mathrm{Diff}\,(\mathbb{C},0)$ *of local biholomorphisms defined in a neighborhood of* 0 *in* $\mathbb{C}$. *We say that* ϕ *is* tangent to the identity *if* $\phi'(0)=1$. *We denote by* $\mathrm{Diff}_1(\mathbb{C},0)$ *the subgroup of* $\mathrm{Diff}\,(\mathbb{C},0)$ *of tangent to the identity diffeomorphisms.*

Definition 2.3. — *Let* $\phi \in \mathrm{Diff}_1(\mathbb{C},0)\setminus\{\mathrm{Id}\}$. *The diffeomorphism* ϕ *is of the form*

$$\phi(y) = y + \sum_{j=\nu(\phi)}^{\infty} a_{j+1} y^{j+1} \tag{2.1}$$

for some power series $\sum_{j=\nu(\phi)}^{\infty} a_{j+1}y^{j+1} \in \mathbb{C}\{y\}$ *with* $a_{\nu(\phi)+1}\neq 0$. *We say that* $\nu(\phi)$ *is the* order of tangency *of* ϕ *with the identity map.*

Definition 2.4. — *Let* $\phi \in \mathrm{Diff}_1(\mathbb{C},0)\setminus\{\mathrm{Id}\}$. *Consider a local vector field* $Y = f(y)\partial/\partial y$ *singular at* 0, *i.e.* $f \in \mathbb{C}\{y\}$ *and* $f(0)=0$. *We say* Y *is an* adapted vector field *of* ϕ *if* $\phi(y) - \exp(Y)(y)$ *belongs to the ideal* $(y^{2(\nu(\phi)+1)})$ *of* $\mathbb{C}\{y\}$.

We can consider the ideal $(y^{\nu(\phi)+1})$ as the ideal of fixed points since it is generated by the equation $\phi(y)-y$ of the fixed point set. The ideal $(y^{2(\nu(\phi)+1)})$ is then the square of the ideal of fixed points.

Remark 2.5. — Let ϕ be an element of $\mathrm{Diff}_1(\mathbb{C},0)\setminus\{\mathrm{Id}\}$. The formal classification of tangent to the identity diffeomorphisms in dimension 1 implies that there exists a unique

$$Y' = \frac{y^{\nu(\phi)+1}}{1+\lambda y^{\nu(\phi)}}\frac{\partial}{\partial y},$$

where $\lambda \in \mathbb{C}$, such that there exists $\sigma' \in \widehat{\mathrm{Diff}}\,(\mathbb{C},0)$ with $\sigma' \circ \exp(Y') = \phi \circ \sigma'$ (cf. [7, Theorem 4.26] and [10]). Now consider a local diffeomorphism $\sigma \in \mathrm{Diff}\,(\mathbb{C},0)$ such that the $2(\nu(\phi)+1)$-jet of σ and σ' at 0 coincide. Then $\sigma_* Y'$ is an adapted vector field of ϕ.

Definition 2.6. — *We denote* $\mathrm{Res}_\phi(0)=\lambda$ *or* $\mathrm{Res}_\phi=\lambda$.

Remark 2.7. — The order $\nu(\phi)$ is a complete topological invariant in $\mathrm{Diff}_1(\mathbb{C},0)$ [2]. The pair $(\nu(\phi),\mathrm{Res}_\phi)$ is a complete system of formal invariants in $\mathrm{Diff}_1(\mathbb{C},0)$ (cf. [7, Theorem 4.26] and [10]).

Remark 2.8. — Given an adapted vector field $Y = a(y)\partial/\partial y$ of $\phi \in \mathrm{Diff}_1(\mathbb{C},0)\setminus\{\mathrm{Id}\}$ we can consider the dual form ω of Y, i.e. the unique meromorphic 1-form such that $\omega(Y)\equiv 1$. We have $\omega = \mathrm{d}y/a(y)$. Its residue at 0 coincides with $\mathrm{Res}_\phi(0)$, thus justifying the notation.

Definition 2.9. — *Let*
$$\phi(y) = y + \sum_{j=\nu(\phi)}^{\infty} a_{j+1} y^{j+1} \in \mathrm{Diff}_1(\mathbb{C},0) \setminus \{\mathrm{Id}\}.$$
We denote
$$D_+ = \{\lambda \in \mathbb{S}^1 : a_{\nu(\phi)+1}\lambda^{\nu(\phi)} \in \mathbb{R}^-\}, \ D_- = \{\lambda \in \mathbb{S}^1 : a_{\nu(\phi)+1}\lambda^{\nu(\phi)} \in \mathbb{R}^+\}$$
and $D = D_+ \cup D_-$. *Fix a small open neihborhood* U *of* 0 *in* $\mathbb{C}$. *Given* $\lambda \in D_\pm$ *the set*
$$\mathcal{P}_\lambda = \left\{ y_0 \in U \setminus \{0\} : \phi^{\pm j}(y_0) \in U \textit{ for all } j \in \mathbb{N} \ \textit{ and } \lim_{j \to \infty} \frac{\phi^{\pm j}(y_0)}{|\phi^{\pm j}(y_0)|} = \lambda \right\}$$
is a petal *of* ϕ *in* U *(or* $\phi_{|U}$*) bisected by the direction* λ. *If* $\lambda \in D_+$ *then* $\mathcal{P}_\lambda$ *is an* attracting petal, *otherwise it is a* repelling petal.

Remark 2.10. — Two petals $\mathcal{P}_\lambda$ and $\mathcal{P}_\mu$ have non-empty intersection if and only if they are consecutive, i.e. $\mu \in \{\lambda\, \mathrm{e}^{-\mathrm{i}\,\pi/\nu(\phi)}, \lambda\, \mathrm{e}^{\mathrm{i}\,\pi/\nu(\phi)}\}$.

Remark 2.11. — Let $U = B(0,\epsilon)$ for a sufficiently small $\epsilon > 0$, where $B(0,\epsilon)$ is the open ball in $\mathbb{C}$ of center 0 and radius ϵ. It can be proved that $\mathcal{P}_\lambda$ is simply connected for any $\lambda \in D$. Moreover U is equal to the union $\cup_{\lambda \in D} \mathcal{P}_\lambda$.

2.2. Fatou coordinates

Next, we define Fatou coordinates both for vector fields and diffeomorphisms.

Definition 2.12. — *Let* Y *be a holomorphic vector field defined in an open set* U *of* $\mathbb{C}^n$. *We say that a holomorphic* $\psi : U \to \mathbb{C}$ *is a* Fatou coordinate *of* Y *if* $Y(\psi) \equiv 1$ *in* U.

Remark 2.13. — If $n = 1$, U is simply connected and Y has no singular points then the Fatou coordinate is well-defined up to an additive complex constant.

Definition 2.14. — *Let* ϕ *be a biholomorphism defined in an open set* U *of* $\mathbb{C}^n$. *We say that a holomorphic* $\psi : U \to \mathbb{C}$ *is a* Fatou coordinate *of* ϕ *if* $\psi \circ \phi = \psi + 1$ *where both sides of the equality are well-defined.*

Definition 2.15. — *Let* $\phi \in \mathrm{Diff}_1(\mathbb{C},0) \setminus \{\mathrm{Id}\}$. *Consider a petal* $\mathcal{P}$ *of* ϕ *(in some neighborhood of* 0*) and an adapted vector field* Y *of* ϕ. *We denote by* $\psi_{\mathcal{P}}^{Y}$ *a Fatou coordinate of* Y *in* $\mathcal{P}$.

Definition 2.16. — *Let* $\phi \in \mathrm{Diff}_1(\mathbb{C}, 0) \setminus \{\mathrm{Id}\}$. *Consider a petal* $\mathcal{P}$ *of* ϕ *(in some neighborhood of* 0*). In this case the Fatou coordinates of* ϕ *in* $\mathcal{P}$ *are always supposed to be injective. We denote by* $\psi_{\mathcal{P}}^{\phi}$ *a Fatou coordinate of* ϕ *in* $\mathcal{P}$.

Remark 2.17. — Fatou coordinates in Definition 2.15 are well-defined up to an additive complex constant by Remark 2.13. The analogous property for the Fatou coordinates in Definition 2.16 also holds true [10] (here the injective hypothesis is necessary).

Definition 2.18. — *Let* $\phi \in \mathrm{Diff}_1(\mathbb{C}, 0) \setminus \{\mathrm{Id}\}$. *Consider a petal* $\mathcal{P}$ *of* ϕ. *There is a unique holomorphic vector field* $Y_{\mathcal{P}}^{\phi} = g_{\mathcal{P}}^{\phi}(y)\partial/\partial y$ *such that* $Y_{\mathcal{P}}^{\phi}(\psi_{\mathcal{P}}^{\phi}) \equiv 1$ *for some (and then for every) Fatou coordinate* $\psi_{\mathcal{P}}^{\phi}$ *of* ϕ *in* $\mathcal{P}$. *It is called the* Lavaurs vector field *of* ϕ *in* $\mathcal{P}$ [9].

2.3. Analytically trivial diffeomorphisms

Definition 2.19. — *Let* $\phi \in \mathrm{Diff}_1(\mathbb{C}, 0) \setminus \{\mathrm{Id}\}$. *We say that* ϕ *is analytically trivial if there exists a local holomorphic singular vector field* $Y = f(y)\partial/\partial y$ *such that* $\phi = \exp(Y)$. *This condition is equivalent to* $\psi_{\mathcal{P}}^{\phi} - \psi_{\mathcal{Q}}^{\phi}$ *being locally constant for all petals* $\mathcal{P}$, $\mathcal{Q}$ *of* ϕ *(or also* $Y_{\mathcal{P}}^{\phi} \equiv Y_{\mathcal{Q}}^{\phi}$ *in* $\mathcal{P} \cap \mathcal{Q}$*) such that* $\mathcal{P} \cap \mathcal{Q} \neq \emptyset$.

Remark 2.20. — Notice that if ϕ is analytically trivial then $Y \equiv Y_{\mathcal{P}}^{\phi}$ in every petal $\mathcal{P}$ of ϕ.

The condition of being non-analytically trivial is generic among the tangent to the identity local diffeomorphisms in one variable. More precisely, every formal class of conjugacy (i.e. a class of equivalence for the relation given by the formal conjugacy) contains a continuous moduli of analytic classes of conjugacy and a unique analytically trivial class. These properties are a consequence of the analytic classification of tangent to the identity diffeomorphisms (cf. [10]).

Remark 2.21. — The pair $(\nu(\phi), \mathrm{Res}_{\phi})$ is a complete system of analytic invariants for analytically trivial diffeomorphisms in $\mathrm{Diff}_1(\mathbb{C}, 0)$ (cf. [7, Theorem 5.25] and [10]).

2.4. Adapted vector fields

Let us consider the holomorphic local vector field

$$Y = \sum_{j=\nu+1}^{\infty} a_j y^j \frac{\partial}{\partial y}$$

where $\nu \geqslant 1$ and $a_{\nu+1} \neq 0$. Fix a small domain of definition $B(0,\epsilon)$ for $\epsilon > 0$. Given $\mu \in \mathbb{S}^1$ we denote by $\mathrm{Re}(\mu Y)$ the real flow of μY, i.e. the flow of the vector field μY restricted to real times.

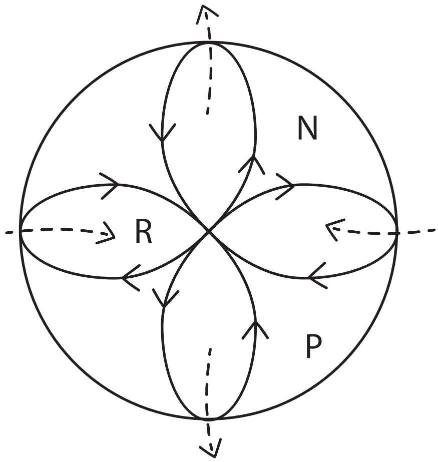

Figure 2.1. Dynamics of $\mathrm{Re}(\mu Y)$

We want to describe the dynamics of $\mathrm{Re}(Y)_{|B(0,\epsilon)}$. We divide $B(0,\epsilon)$ in connected sets (or regions in the following discussion) whose $\mathrm{Re}(\mu Y)_{|B(0,\epsilon)}$ trajectories have similar behavior. There are 2ν regions that are invariant by $\mathrm{Re}(\mu Y)_{|B(0,\epsilon)}$ and are shaped as petals (cf. Figure 2.1 where one of these regions is denoted by R). There are ν regions that are invariant by the positive flow of $\mathrm{Re}(\mu Y)$ but not for the negative flow (P is one of such regions in Figure 2.1) and ν regions that are invariant by the negative flow of $\mathrm{Re}(\mu Y)$ but not for the positive flow (N is an example in Figure 2.1). Consider now $\mu \in \mathbb{S}^1 \setminus \{-1,1\}$. Then ν of the $\mathrm{Re}(\mu Y)$-invariant regions are positively invariant by $\mathrm{Re}(Y)$ (the dashed lines in Figure 2.1 correspond to trajectories of $\mathrm{Re}(Y)$ and R is positively invariant). Each of these regions is contained exactly in one attracting petal of $\exp(Y)$ and every petal of $\exp(Y)$ contains exactly one of these regions. Moreover if R is a $\mathrm{Re}(\mu Y)_{|B(0,\epsilon)}$-invariant region contained in the attracting petal $\mathcal{P}$ then every point $y_0 \in \mathcal{P}$ satisfies $\phi^j(y_0) \in R$ for any $j \geqslant j_0$ and some $j_0 \geqslant 0$. Thus every (injective) Fatou coordinate ψ^ϕ of ϕ defined in R can be extended to $\mathcal{P}$ by using the equation $\psi^\phi \circ \phi \equiv \psi^\phi + 1$. The other ν invariant regions by $\mathrm{Re}(\mu Y)$ are in bijective correspondence with the repelling petals of $\exp(Y)$

and have analogous properties. Such $\mathrm{Re}(\mu Y)$-invariant regions can be generalized for unfoldings of tangent to the identity maps and thus they can be used to extend Fatou coordinates to the perturbed parameters $x_0 \neq 0$ [17].

The Fatou coordinates of $\phi \in \mathrm{Diff}_1(\mathbb{C},0)$ and an adapted vector field Y are very similar.

LEMMA 2.22 (cf. [10]). — *Let $\phi \in \mathrm{Diff}(\mathbb{C},0) \setminus \{\mathrm{Id}\}$. Consider a petal $\mathcal{P}$ of ϕ. Let Y be a local holomorphic vector field adapted to ϕ. Then we obtain*

$$\lim_{y \in \mathcal{P},\ |\mathrm{Im}(\psi_{\mathcal{P}}^{Y}(y))| \to \infty} (\psi_{\mathcal{P}}^{\phi} - \psi_{\mathcal{P}}^{Y})(y) = c \tag{2.2}$$

for some $c \in \mathbb{C}$.

Remark 2.23. — We have $\lim_{y \in \mathcal{P},\ y \to 0} \psi_{\mathcal{P}}^{Y}(y) y^{\nu} = -1/(\nu a_{\nu+1})$ for any $\lambda \in D$ where $a_{\nu+1}$ is given by Expression (2.1). Thus $y \in \mathcal{P}$ tends to 0 if and only if $\psi_{\mathcal{P}}^{Y}(y)$ tends to ∞. The property $\mathrm{Im}(\psi_{\mathcal{P}}^{\phi} \circ \phi) \equiv \mathrm{Im}(\psi_{\mathcal{P}}^{\phi})$ and (2.2) imply that given $y \in \mathcal{P}$ the condition $|\mathrm{Im}(\psi_{\mathcal{P}}^{Y}(y))| \to \infty$ is equivalent to the point $\phi^{j}(y)$ being well-defined for any $j \in \mathbb{Z}$ and $\inf_{j \in \mathbb{Z}} |\psi_{\mathcal{P}}^{Y}(\phi^{j}(y))| \to \infty$. Therefore $|\mathrm{Im}(\psi_{\mathcal{P}}^{Y}(y))| \to \infty$ is equivalent to the orbit $(\phi^{j}(y))_{j \in \mathbb{Z}}$ tending uniformly to the origin. More precisely given $\epsilon > 0$ there exists $C > 0$ such that if $y \in \mathcal{P}$ and $|\mathrm{Im}(\psi_{\mathcal{P}}^{Y}(y))| > C$ then $\phi^{j}(y)$ is well-defined and belong to $B(0,\epsilon) \cap \mathcal{P}$ for any $j \in \mathbb{Z}$. Reciprocally given $\epsilon > 0$ there exists $M > 0$ such that if the orbit $(\phi^{j}(y))_{j \in \mathbb{Z}}$ is well-defined and contained in $B(0,\epsilon) \cap \mathcal{P}$ then $|\mathrm{Im}(\psi_{\mathcal{P}}^{Y}(y))| > M$.

The previous discussion implies that the property $|\mathrm{Im}(\psi_{\mathcal{P}}^{Y}(y))| \to \infty$ is invariant by topological conjugacy. Indeed if ϕ is conjugated to $\eta \in \mathrm{Diff}_1(\mathbb{C},0)$ (with adapted vector field Z) by a local homeomorphism σ we obtain

$$\lim_{y \in \mathcal{P},\ |\mathrm{Im}(\psi_{\mathcal{P}}^{Y}(y)| \to \infty} |\mathrm{Im}(\psi_{\sigma(\mathcal{P})}^{Z}(\sigma(y))| = \infty$$

where $\psi_{\sigma(\mathcal{P})}^{Z}$ is the Fatou coordinate of Z in the petal of η containing $\sigma(\mathcal{P})$.

2.5. Fatou affine conjugacies

We are ready to introduce the Fatou affine conjugacy and then discussing its main properties.

DEFINITION 2.24. — *Let $\phi, \eta \in \mathrm{Diff}_1(\mathbb{C},0) \setminus \{\mathrm{Id}\}$. Consider a local homeomorphism σ conjugating ϕ and η. We say that σ is* affine in Fatou coordinates *(of ϕ and η) in a petal $\mathcal{P}$ of $\phi_{|B(0,\epsilon)}$ if there exists a $\mathbb{R}$-linear isomorphism $\mathfrak{h}_{\mathcal{P}} : \mathbb{C} \to \mathbb{C}$ such that*

$$\psi^{\eta}_{\sigma(\mathcal{P})}(\sigma(y')) - \psi^{\eta}_{\sigma(\mathcal{P})}(\sigma(y)) = \mathfrak{h}_{\mathcal{P}}(\psi^{\phi}_{\mathcal{P}}(y') - \psi^{\phi}_{\mathcal{P}}(y)) \tag{2.3}$$

for all $y, y' \in \mathcal{P}$. We say that σ is a Fatou affine conjugacy *if it is affine in Fatou coordinates for every petal of $\phi_{|B(0,\epsilon)}$ and some $\epsilon > 0$.*

Remark 2.25. — Since $\psi^{\phi}_{\mathcal{P}}(y') - \psi^{\phi}_{\mathcal{P}}(y)$ does not depend on the choice of the Fatou coordinate $\psi^{\phi}_{\mathcal{P}}$ the definition does not depend on the choices of $\psi^{\phi}_{\mathcal{P}}$ and $\psi^{\eta}_{\sigma(\mathcal{P})}$.

Remark 2.26. — The properties $\psi^{\phi}_{\mathcal{P}} \circ \phi \equiv \psi^{\phi}_{\mathcal{P}} + 1$, $\psi^{\eta}_{\sigma(\mathcal{P})} \circ \eta \equiv \psi^{\eta}_{\sigma(\mathcal{P})} + 1$ and $\sigma \circ \phi \equiv \eta \circ \sigma$ imply $\mathfrak{h}_{\mathcal{P}}(1) = 1$ for any petal $\mathcal{P}$ of ϕ.

Remark 2.27. — We claim that the function $\mathfrak{h}_{\mathcal{P}}$ in Definition 2.24 is uniquely defined in each petal of ϕ. By fixing $y \in \mathcal{P}$ and considering $y' \in \mathcal{P}$ in a neighborhood of y we obtain that $\mathfrak{h}_{\mathcal{P}}$ is uniquely defined in a neighborhood of 0. Since $\mathfrak{h}_{\mathcal{P}}$ it $\mathbb{R}$-linear, it is uniquely defined.

Remark 2.28. — Fatou affine conjugacies are real analytic in a pointed neighborhood of 0.

Next we see that the linear map that describes a Fatou affine conjugacy is canonically determined and does not depend on the petal.

LEMMA 2.29. — *Let $\phi_1, \phi_2 \in \mathrm{Diff}_1(\mathbb{C}, 0) \setminus \{\mathrm{Id}\}$ that are conjugated by a Fatou affine conjugacy σ. Then we obtain $\mathfrak{h}_{\mathcal{P}} \equiv \mathfrak{h}_{\mathcal{Q}}$ for all petals $\mathcal{P}$ and $\mathcal{Q}$ of ϕ_1.*

Proof. — We denote by Y_j a vector field adapted to ϕ_j for $j \in \{1, 2\}$. It suffices to prove $\mathfrak{h}_{\mathcal{P}} \equiv \mathfrak{h}_{\mathcal{Q}}$ for two consecutive petals $\mathcal{P}$ and $\mathcal{Q}$ of ϕ_1. We denote by ψ_1 a Fatou coordinate of Y_1 defined in $\mathcal{P} \cup \mathcal{Q}$. Analogously we denote by ψ_2 a Fatou coordinate of Y_2 defined in $\sigma(\mathcal{P} \cup \mathcal{Q})$. We have

$$\begin{aligned} &\lim_{y \in \mathcal{P},\ |\mathrm{Im}(\psi_1(y))| \to \infty} (\psi^{\phi_1}_{\mathcal{P}} - \psi_1)(y) = 0, \\ &\lim_{y \in \mathcal{Q},\ |\mathrm{Im}(\psi_1(y))| \to \infty} (\psi^{\phi_1}_{\mathcal{Q}} - \psi_1)(y) = 0 \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} &\lim_{y \in \sigma(\mathcal{P}), |\mathrm{Im}(\psi_2(y))| \to \infty} (\psi^{\phi_2}_{\sigma(\mathcal{P})} - \psi_2)(y) = 0, \\ &\lim_{y \in \sigma(\mathcal{Q}), |\mathrm{Im}(\psi_2(y))| \to \infty} (\psi^{\phi_2}_{\sigma(\mathcal{Q})} - \psi_2)(y) = 0 \end{aligned} \tag{2.5}$$

by Lemma 2.22 up to add additive constants to $\psi_{\mathcal{P}}^{\phi_1}$, $\psi_{\mathcal{Q}}^{\phi_1}$, $\psi_{\sigma(\mathcal{P})}^{\phi_2}$ and $\psi_{\sigma(\mathcal{Q})}^{\phi_2}$. We deduce

$$\lim_{y \in \mathcal{P} \cap \mathcal{Q},\ |\mathrm{Im}(\psi_1(y))| \to \infty} (\psi_{\mathcal{P}}^{\phi_1} - \psi_{\mathcal{Q}}^{\phi_1})(y) = 0 \tag{2.6}$$

and

$$\lim_{y \in \sigma(\mathcal{P} \cap \mathcal{Q}),\ |\mathrm{Im}(\psi_2(y))| \to \infty} (\psi_{\sigma(\mathcal{P})}^{\phi_2} - \psi_{\sigma(\mathcal{Q})}^{\phi_2})(y) = 0. \tag{2.7}$$

Consider a sequence y_n in $\mathcal{P} \cap \mathcal{Q}$ such that $|\mathrm{Im}(\psi_1(y_n))| \to \infty$. Fix $z \in \mathbb{C}$. We denote by y_n' the point in $\mathcal{P}$ such that $\psi_{\mathcal{P}}^{\phi_1}(y_n') - \psi_{\mathcal{P}}^{\phi_1}(y_n) = z$ for $n \gg 1$. We have $\lim_{n\to\infty} |\mathrm{Im}(\psi_1(y_n')| = \infty$ by (2.4). We denote

$$z_n = \psi_{\mathcal{Q}}^{\phi_1}(y_n') - \psi_{\mathcal{Q}}^{\phi_1}(y_n)$$

for $n \gg 1$. The sequence $(z_n)_{n \geqslant 1}$ satisfies $\lim_{n\to\infty} z_n = z$ by (2.6). We have

$$\mathfrak{h}_{\mathcal{P}}(z) = \psi_{\sigma(\mathcal{P})}^{\phi_2}(\sigma(y_n')) - \psi_{\sigma(\mathcal{P})}^{\phi_2}(\sigma(y_n))$$

and

$$\mathfrak{h}_{\mathcal{Q}}(z_n) = \psi_{\sigma(\mathcal{Q})}^{\phi_2}(\sigma(y_n')) - \psi_{\sigma(\mathcal{Q})}^{\phi_2}(\sigma(y_n))$$

by definition. Since $\lim_{n\to\infty} |\mathrm{Im}(\psi_1(y_n))| = \infty$ and $\lim_{n\to\infty} |\mathrm{Im}(\psi_1(y_n'))| = \infty$, we obtain

$$\lim_{n\to\infty} |\mathrm{Im}(\psi_2(\sigma(y_n)))| = \infty \quad \text{and} \quad \lim_{n\to\infty} |\mathrm{Im}(\psi_2(\sigma(y_n')))| = \infty$$

by Remark 2.23. (2.7) implies $\lim_{n\to\infty}(\mathfrak{h}_{\mathcal{P}}(z) - \mathfrak{h}_{\mathcal{Q}}(z_n)) = 0$. Since $\lim_{n\to\infty} z_n = z$ and $\mathfrak{h}_{\mathcal{Q}}$ is continuous, we obtain $\mathfrak{h}_{\mathcal{P}}(z) = \mathfrak{h}_{\mathcal{Q}}(z)$ for any $z \in \mathbb{C}$. $\square$

Definition 2.30. — *Let $\phi, \eta \in \mathrm{Diff}_1(\mathbb{C}, 0) \setminus \{\mathrm{Id}\}$ that are conjugated by a Fatou affine conjugacy σ. We denote by $\mathfrak{h}_{\phi,\eta,\sigma}$ any of the functions $\mathfrak{h}_{\mathcal{P}}$ in Definition 2.24. We denote $\mathfrak{h} = \mathfrak{h}_{\phi,\eta,\sigma}$ if the data are implicit.*

Remark 2.31. — Let $\phi, \eta \in \mathrm{Diff}_1(\mathbb{C}, 0) \setminus \{\mathrm{Id}\}$ that are conjugated by a Fatou affine conjugacy σ. It is clear that σ is orientation-preserving if and only if $\mathfrak{h}_{\phi,\eta,\sigma}$ is orientation-preserving.

Remark 2.32. — Let $\phi, \eta, \rho \in \mathrm{Diff}_1(\mathbb{C}, 0) \setminus \{\mathrm{Id}\}$. Consider a Fatou affine map σ conjugating ϕ and η and a Fatou affine map σ' conjugating η and ρ. It is immediate to prove that $\sigma' \circ \sigma$ is a Fatou affine map conjugating ϕ and ρ. Moreover we have

$$\mathfrak{h}_{\phi,\rho,\sigma'\circ\sigma} = \mathfrak{h}_{\eta,\rho,\sigma'} \circ \mathfrak{h}_{\phi,\eta,\sigma} \quad \text{and} \quad \mathfrak{h}_{\eta,\phi,\sigma^{-1}} = \mathfrak{h}_{\phi,\eta,\sigma}^{-1}.$$

Fatou affine conjugacies are well-behaved for compositions.

Next, we see that holomorphic and anti-holomorphic conjugacies are Fatou affine. Moreover such conjugacies can be characterized in terms of the function $\mathfrak{h}$.

Lemma 2.33. — *Let $\phi, \eta \in \mathrm{Diff}_1(\mathbb{C}, 0) \setminus \{\mathrm{Id}\}$ that are conjugated by a local homeomorphism σ. Then σ is holomorphic if and only if σ is Fatou affine and $\mathfrak{h}_{\phi,\eta,\sigma} \equiv z$. Moreover σ is anti-holomorphic if and only if σ is Fatou affine and $\mathfrak{h}_{\phi,\eta,\sigma} \equiv \overline{z}$.*

Proof. — Suppose that σ is holomorphic. Given a petal $\mathcal{P}$ of ϕ the map $\psi_{\mathcal{P}}^{\phi} \circ \sigma^{-1}$ is a Fatou coordinate of η in $\sigma(\mathcal{P})$. As a consequence we obtain $\mathfrak{h}_{\phi,\eta,\sigma} \equiv \mathrm{Id}$.

Suppose that σ is anti-holomorphic. Then the complex conjugate $\overline{\psi_{\mathcal{P}}^{\phi} \circ \sigma^{-1}}$ of $\psi_{\mathcal{P}}^{\phi} \circ \sigma^{-1}$ is a Fatou coordinate of η in $\sigma(\mathcal{P})$. Hence we get $\mathfrak{h}_{\phi,\eta,\sigma} \equiv \overline{z}$.

Suppose $\mathfrak{h}_{\phi,\eta,\sigma} \equiv z$. Hence σ is holomorphic in every petal of ϕ and it follows that σ is holomorphic in a pointed neighborhood of the origin. We obtain that σ is holomorphic by Riemann's removable singularity theorem. The anti-holomorphic nature of σ if $\mathfrak{h}_{\phi,\eta,\sigma} \equiv \overline{z}$ is proved analogously. $\square$

Fatou affine conjugacies are interesting because of their connection with unfoldings. Anyway it is natural to compare this conjugacy with the formal and analytic conjugacies. It is possible to construct Fatou affine conjugacies that are not holomorphic or anti-holomorphic if both ϕ and η are embedded in analytic flows (see Section 6). They are essentially the only examples.

Proposition 2.34. — *Let $\phi, \eta \in \mathrm{Diff}_1(\mathbb{C}, 0) \setminus \{\mathrm{Id}\}$ that are conjugated by a Fatou affine conjugacy σ. Suppose that either ϕ or η is non-analytically trivial. Then either σ is holomorphic or anti-holomorphic.*

Proof. — The isomorphism $\mathfrak{h}_{\eta,\phi,\sigma^{-1}}$ is the inverse of $\mathfrak{h}_{\phi,\eta,\sigma}$ (Remark 2.32). Thus we can suppose that ϕ is non-analytically trivial.

Let $\mathcal{P}$ be a petal of ϕ. Fix $y_0 \in \mathcal{P}$. We have

$$\psi_{\sigma(\mathcal{P})}^{\eta}(\sigma(y)) = \psi_{\sigma(\mathcal{P})}^{\eta}(\sigma(y_0)) + \mathfrak{h}(\psi_{\mathcal{P}}^{\phi}(y) - \psi_{\mathcal{P}}^{\phi}(y_0))$$

and then

$$(\psi_{\sigma(\mathcal{P})}^{\eta} \circ \sigma)(y) = ((z + c_{\mathcal{P}}) \circ \mathfrak{h} \circ \psi_{\mathcal{P}}^{\phi})(y)$$

for some $c_{\mathcal{P}} \in \mathbb{C}$ and any $y \in \mathcal{P}$. Consider two consecutive petals $\mathcal{P}$ and $\mathcal{Q}$ of ϕ. We consider the change of charts

$$\begin{aligned}\psi_{\sigma(\mathcal{Q})}^{\eta} \circ (\psi_{\sigma(\mathcal{P})}^{\eta})^{-1} &= (\psi_{\sigma(\mathcal{Q})}^{\eta} \circ \sigma) \circ (\psi_{\sigma(\mathcal{P})}^{\eta} \circ \sigma)^{-1} \\ &= (z + c_{\mathcal{Q}}) \circ \mathfrak{h} \circ \psi_{\mathcal{Q}}^{\phi} \circ (\psi_{\mathcal{P}}^{\phi})^{-1} \circ \mathfrak{h}^{-1} \circ (z - c_{\mathcal{P}}).\end{aligned}$$

Since the left hand side is holomorphic, $\mathfrak{h} \circ \psi^{\phi}_{\mathcal{Q}'} \circ (\psi^{\phi}_{\mathcal{P}'})^{-1} \circ \mathfrak{h}^{-1}$ is also holomorphic. We denote $H = \psi^{\phi}_{\mathcal{Q}'} \circ (\psi^{\phi}_{\mathcal{P}'})^{-1}$, it is holomorphic. The isomorphisms $\mathfrak{h}$ and $\mathfrak{h}^{-1}$ are of the form $\mathfrak{h}(z) = \varsigma_0 z + \varsigma_1 \overline{z}$ and $\mathfrak{h}^{-1}(z) = \varrho_0 z + \varrho_1 \overline{z}$ where $\varrho_0 = \overline{\varsigma_0}/(|\varsigma_0|^2 - |\varsigma_1|^2)$ and $\varrho_1 = -\varsigma_1/(|\varsigma_0|^2 - |\varsigma_1|^2)$. We have

$$\begin{aligned}\frac{\partial(\mathfrak{h} \circ H \circ \mathfrak{h}^{-1})}{\partial \overline{z}} &= \frac{\partial \mathfrak{h}}{\partial z} \frac{\partial (H \circ \mathfrak{h}^{-1})}{\partial \overline{z}} + \frac{\partial \mathfrak{h}}{\partial \overline{z}} \frac{\partial (\overline{H \circ \mathfrak{h}^{-1}})}{\partial \overline{z}} \\ &= \varsigma_0 \frac{\partial H}{\partial z} \varrho_1 + \varsigma_1 \overline{\frac{\partial H}{\partial z}}\, \overline{\varrho_0} = \frac{\varsigma_0 \varsigma_1}{|\varsigma_0|^2 - |\varsigma_1|^2} \left(\overline{\frac{\partial H}{\partial z}} - \frac{\partial H}{\partial z} \right).\end{aligned}$$

Suppose $\varsigma_1 = 0$. Since $\mathfrak{h}(1) = 1$ we deduce $\mathfrak{h} \equiv z$. Hence σ is holomorphic by Lemma 2.33. Suppose $\varsigma_0 = 0$. Since $\mathfrak{h}(1) = 1$ we deduce $\mathfrak{h} \equiv \overline{z}$. Hence σ is anti-holomorphic by Lemma 2.33. So we can suppose $\varsigma_0 \varsigma_1 \neq 0$ from now on. Since $\mathfrak{h} \circ H \circ \mathfrak{h}^{-1}$ is holomorphic, we obtain $\overline{\partial H/\partial z} \equiv \partial H/\partial z$. The function $\partial H/\partial z$ is real and then locally constant by the open map theorem. Therefore H is of the form $az + b$ for some constants $a, b \in \mathbb{C}$ in every connected component of $\mathcal{P} \cap \mathcal{Q}$. The constant a is equal to 1 since $H(z+1) \equiv H(z) + 1$. We obtain $\frac{\partial(\psi^{\phi}_{\mathcal{Q}} - \psi^{\phi}_{\mathcal{P}})}{\partial y} \equiv 0$. The last property does not hold true for every pair of consecutive petals of ϕ by hypothesis. Thus σ is either holomorphic or anti-holomorphic. □

Corollary 2.35. — *Let $\phi, \eta \in \mathrm{Diff}_1(\mathbb{C}, 0) \setminus \{\mathrm{Id}\}$ that are Fatou affine conjugated. Then ϕ is analytically trivial if and only if η is analytically trivial.*

Proposition 2.34 describes the generic invariance of the analytic classes of the diffeomorphisms by Fatou affine conjugacy in the orientation-preserving case. The next result describes the action on formal invariants.

Proposition 2.36. — *Let $\phi, \eta \in \mathrm{Diff}_1(\mathbb{C}, 0) \setminus \{\mathrm{Id}\}$ that are conjugated by a Fatou affine conjugacy σ. Then we have*

$$\mathfrak{h}_{\phi,\eta,\sigma}(2\pi \,\mathrm{i}\, \mathrm{Res}_\phi) = 2\pi \,\mathrm{i}\, \mathrm{Res}_\eta \quad \textit{or} \quad \mathfrak{h}_{\phi,\eta,\sigma}(2\pi \,\mathrm{i}\, \mathrm{Res}_\phi) = -2\pi \,\mathrm{i}\, \mathrm{Res}_\eta$$

(see Definition 2.6) depending on whether or not σ is orientation-preserving. In particular $\mathrm{Re}(\mathrm{Res}_\phi)$ and $\mathrm{Re}(\mathrm{Res}_\eta)$ have the same sign.

Proof. — By convention the sign of a real number can be positive, negative or 0. We denote $R_\phi = \mathrm{Res}_\phi$ and $R_\eta = \mathrm{Res}_\eta$. Suppose that either ϕ or η is non-analytically trivial. If σ is orientation-preserving then $\mathfrak{h} \equiv z$ and σ is holomorphic by Proposition 2.34. The result is a consequence of the residues being analytic invariants. If σ is orientation-reversing then σ is anti-holomorphic and $\mathfrak{h} \equiv \overline{z}$ by Proposition 2.34. The equation $R_\eta = \overline{R_\phi}$ implies $\mathfrak{h}(2\pi \,\mathrm{i}\, R_\phi) = -2\pi \,\mathrm{i}\, R_\eta$.

Suppose that both ϕ and η are analytically trivial. Let Y and Z be the local vector fields such that $\phi = \exp(Y)$ and $\eta = \exp(Z)$. Consider Fatou coordinates ψ^Y and ψ^Z of Y and Z respectively. The complex number $2\pi\,\mathrm{i}\,R_\phi$ is the additive monodromy of ψ^Y along a path turning once around the origin in counter clock wise sense. Since we have

$$(\psi^Z \circ \sigma)(y) \equiv (\mathfrak{h} \circ \psi^Y)(y) + c$$

for some $c \in \mathbb{C}$ then $\mathfrak{h}(2\pi\,\mathrm{i}\,R_\phi) = \pm 2\pi\,\mathrm{i}\,R_\eta$ depending on whether $\mathfrak{h}$ is orientation-preserving or orientation-reversing.

Suppose that $\mathfrak{h}$ is orientation-preserving. We obtain

$$\mathrm{sign}(\mathrm{Re}(R_\eta)) = \mathrm{sign}(\mathrm{Im}(\mathfrak{h}(2\pi\,\mathrm{i}\,R_\phi))) = \mathrm{sign}(\mathrm{Im}(2\pi\,\mathrm{i}\,R_\phi)) = \mathrm{sign}(\mathrm{Re}(R_\phi)).$$

We have

$$\mathrm{sign}(\mathrm{Re}(R_\eta)) = -\,\mathrm{sign}(\mathrm{Im}(\mathfrak{h}(2\pi\,\mathrm{i}\,R_\phi))) = \mathrm{sign}(\mathrm{Im}(2\pi\,\mathrm{i}\,R_\phi)) = \mathrm{sign}(\mathrm{Re}(R_\phi))$$

when $\mathfrak{h}$ is orientation-reversing. □

2.6. Analytically trivial case

In this section we further study the properties of Fatou affine conjugacies in the analytically trivial case. Examples of the specific kind of behavior described in next propositions for unfoldings are presented in Section 6.

Consider $\phi, \eta \in \mathrm{Diff}_1(\mathbb{C}, 0) \setminus \{\mathrm{Id}\}$ and a Fatou affine topological conjugacy σ. Suppose that ϕ is analytically trivial. The $\mathbb{R}$-linear map $\mathfrak{h}_{\phi,\eta,\sigma}$ satisfies two conditions, namely

$$\mathfrak{h}(1) = 1 \quad \text{and} \quad \mathfrak{h}(2\pi\,\mathrm{i}\,\mathrm{Res}_\phi) = \pm 2\pi\,\mathrm{i}\,\mathrm{Res}_\eta\,. \tag{2.8}$$

The sign in the second equation depends on whether σ is orientation-preserving or orientation-reversing. These equations are independent if and only if $\{1, 2\pi\,\mathrm{i}\,\mathrm{Res}_\phi\}$ is a base of $\mathbb{C}$ as a $\mathbb{R}$-vector space. Indeed if $\mathrm{Res}_\phi \in \mathrm{i}\,\mathbb{R}$ then (2.8) does not impose any condition on $\mathfrak{h}(\mathrm{i})$. As a consequence there are plenty of Fatou affine maps conjugating ϕ and η but ϕ and η are always analytically or anti-analytically conjugated. On the other hand if $\mathrm{Res}_\phi \notin \mathrm{i}\,\mathbb{R}$ then (2.8) determines $\mathfrak{h}$ and there is rigidity of conjugacies. In fact there are at most two conjugacies (up to precomposition with elements of the center of ϕ in $\mathrm{Diff}(\mathbb{C}, 0)$) and in general none of them is holomorphic or anti-holomorphic.

Proposition 2.37. — *Let $\phi, \eta \in \mathrm{Diff}_1(\mathbb{C},0) \setminus \{\mathrm{Id}\}$ that are conjugated by a Fatou affine conjugacy σ. Suppose that either ϕ or η is analytically trivial. Suppose that either $\mathrm{Res}_\phi \in \mathrm{i}\,\mathbb{R}$ or $\mathrm{Res}_\eta \in \mathrm{i}\,\mathbb{R}$. Then ϕ and η are analytically conjugated (resp. anti-analytically conjugated) if σ is orientation-preserving (resp. orientation-reversing).*

Proof. — Notice that $\nu(\phi) = \nu(\eta)$ by Remark 2.7. Both residues Res_ϕ and Res_η belong to $\mathrm{i}\,\mathbb{R}$ by Proposition 2.36. Suppose that σ is orientation-preserving. The map $\mathfrak{h}$ satisfies $\mathfrak{h}_{|\mathbb{R}} \equiv \mathrm{Id}$ since $\mathfrak{h}$ is $\mathbb{R}$-linear and $\mathfrak{h}(1) = 1$ by Remark 2.26. Then we have

$$2\pi\,\mathrm{i}\,\mathrm{Res}_\phi = \mathfrak{h}(2\pi\,\mathrm{i}\,\mathrm{Res}_\phi) = 2\pi\,\mathrm{i}\,\mathrm{Res}_\eta$$

by Proposition 2.36. Since $\nu(\phi) = \nu(\eta)$, $\mathrm{Res}_\phi = \mathrm{Res}_\eta$ and ϕ, η are analytically trivial, we obtain that ϕ and η are analytically conjugated by Remark 2.21.

Suppose that σ is orientation-reversing on the parameter space. We have

$$2\pi\,\mathrm{i}\,\mathrm{Res}_\phi = \mathfrak{h}(2\pi\,\mathrm{i}\,\mathrm{Res}_\phi) = -2\pi\,\mathrm{i}\,\mathrm{Res}_\eta = 2\pi\,\mathrm{i}\,\overline{\mathrm{Res}_\eta}$$

by Proposition 2.36. Since $\nu(\phi) = \nu(\eta)$, $\mathrm{Res}_\phi = \overline{\mathrm{Res}_\eta}$ and ϕ, η are analytically trivial, ϕ, η are anti-holomorphically conjugated. □

Corollary 2.38. — *Let ϕ, η be analytically trivial elements of $\mathrm{Diff}_1(\mathbb{C},0) \setminus \{\mathrm{Id}\}$ such that either $\mathrm{Res}_\phi \in \mathrm{i}\,\mathbb{R}$ or $\mathrm{Res}_\eta \in \mathrm{i}\,\mathbb{R}$. Then ϕ and η are Fatou affine conjugated if and only if $\nu(\phi) = \nu(\eta)$ and $\mathrm{Res}_\eta \in \{\mathrm{Res}_\phi, \overline{\mathrm{Res}_\phi}\}$.*

Now we deal with the case in which we have rigidity of conjugating maps.

Proposition 2.39. — *Let $\phi, \eta \in \mathrm{Diff}_1(\mathbb{C},0) \setminus \{\mathrm{Id}\}$ that are conjugated by a Fatou affine conjugacy σ. Suppose that either ϕ or η is analytically trivial. Suppose that either $\mathrm{Res}_\phi \notin \mathrm{i}\,\mathbb{R}$ or $\mathrm{Res}_\eta \notin \mathrm{i}\,\mathbb{R}$. Then*

- *If σ is orientation-preserving then σ is holomorphic iff $\mathrm{Res}_\phi = \mathrm{Res}_\eta$.*
- *If σ is orientation-reversing then σ is anti-holomorphic iff $\mathrm{Res}_\phi = \overline{\mathrm{Res}_\eta}$.*
- *If $\mathrm{Res}_\phi = \mathrm{Res}_\eta \in \mathbb{R}^*$ then σ is holomorphic or anti-holomorphic.*
- *If $\mathrm{Res}_\phi \notin \{\mathrm{Res}_\eta, \overline{\mathrm{Res}_\eta}\}$ then ϕ and η are neither holomorphically nor anti-holomorphically conjugated. In particular σ is neither holomorphic nor anti-holomorphic.*

Consider a pair of homeomorphisms σ, $\tilde{\sigma}$ conjugating ϕ, η and such that both are orientation-preserving or orientation-reversing. Then we obtain $\tilde{\sigma} = \sigma \circ \upsilon$ for some holomorphic $\upsilon \in \mathrm{Diff}(\mathbb{C},0)$ commuting with ϕ.

Proof. — We denote $R_\phi = \mathrm{Res}_\phi$ and $R_\eta = \mathrm{Res}_\eta$. The isomorphism $\mathfrak{h}$ satisfies $\mathfrak{h}(2\pi\,\mathrm{i}\,R_\phi) = \pm 2\pi\,\mathrm{i}\,R_\eta$ (Proposition 2.36) and $\mathfrak{h}_{|\mathbb{R}} \equiv \mathrm{Id}$ (Remark 2.26).

Thus $\mathfrak{h}$ depends only on whether or not σ is orientation-preserving. We deduce

$$\mathfrak{h}_{\phi,\phi,\sigma^{-1}\circ\tilde{\sigma}} = \mathfrak{h}_{\phi,\eta,\sigma}^{-1} \circ \mathfrak{h}_{\phi,\eta,\tilde{\sigma}} = \mathrm{Id}$$

if σ, $\tilde{\sigma}$ have the same orientation by Remark 2.32. The map $\sigma^{-1} \circ \tilde{\sigma}$ is holomorphic (Lemma 2.33) and commutes with ϕ.

Suppose that σ is orientation-preserving. The equation $\mathfrak{h}(2\pi \mathrm{i}\, R_\phi) = 2\pi \mathrm{i}\, R_\eta$ in Proposition 2.36 implies that $\mathfrak{h} \equiv z$ is equivalent to $R_\phi = R_\eta$. Hence σ is holomorphic if and only if $R_\phi = R_\eta$ by Lemma 2.33.

Suppose that σ is orientation-reversing. The equation $\mathfrak{h}(2\pi \mathrm{i}\, R_\phi) = -2\pi \mathrm{i}\, R_\eta$ in Proposition 2.36 implies that $\mathfrak{h} \equiv \overline{z}$ is equivalent to $R_\phi = \overline{R_\eta}$. Hence σ is anti-holomorphic if and only if $R_\phi = \overline{R_\eta}$ by Lemma 2.33.

The third item is a consequence of the previous ones. Suppose $R_\phi \notin \{R_\eta, \overline{R_\eta}\}$. Then ϕ and η are neither holomorphically nor anti-holomorphically conjugated. □

In the cases that we described so far we have that $\phi, \eta \in \mathrm{Diff}_1(\mathbb{C}, 0)$ are conjugated by an orientation-preserving (resp. orientation-reversing) local Fatou affine conjugacy if and only if they are holomorphically (resp. anti-holomorphically) conjugated. This is not always the case by next proposition.

Proposition 2.40. — *Let ϕ, η be analytically trivial elements of $\mathrm{Diff}_1(\mathbb{C}, 0) \setminus \{\mathrm{Id}\}$ such that either $\mathrm{Res}_\phi \notin \mathrm{i}\mathbb{R}$ or $\mathrm{Res}_\eta \notin \mathrm{i}\mathbb{R}$. Then ϕ and η are Fatou affine conjugated if and only if $\nu(\phi) = \nu(\eta)$ and $\mathrm{Re}(\mathrm{Res}_\phi)$ and $\mathrm{Re}(\mathrm{Res}_\eta)$ have the same sign.*

Proof. — The sufficient part is a consequence of Remark 2.7 and Proposition 2.36. Let us show the necessary part. We have $\phi = \exp(Y)$ and $\eta = \exp(Z)$ where Y, Z are local holomorphic vector fields defined in a neighborhood of 0 such that $Y(0) = Z(0) = 0$ by hypothesis. We denote by ψ^Y and ψ^Z Fatou coordinates of Y and Z respectively.

We denote $R_\phi = \mathrm{Res}_\phi$ and $R_\eta = \mathrm{Res}_\eta$. Since $\mathrm{Re}(R_\phi)\,\mathrm{Re}(R_\eta) > 0$, there exists an orientation-preserving $\mathbb{R}$-linear isomorphism $\mathfrak{h}$ such that

$$\mathfrak{h}(1) = 1 \quad \text{and} \quad \mathfrak{h}(2\pi \mathrm{i}\, R_\phi) = 2\pi \mathrm{i}\, R_\eta.$$

Given a petal $\mathcal{P}$ of ϕ we consider the function $\mathfrak{h}\circ\psi^Y$. It is a Fatou coordinate since

$$(\mathfrak{h} \circ \psi^Y)(\phi(y)) = \mathfrak{h}(\psi^Y(y) + 1) = (\mathfrak{h} \circ \psi^Y)(y) + 1$$

except for the fact that $\mathfrak{h}\circ\psi^Y$ is not holomorphic. Then $\sigma := (\psi^Z)^{-1}\circ\mathfrak{h}\circ\psi^Y$ is defined in every petal of ϕ and conjugates ϕ and η. The unique problem to obtain a conjugacy defined in a neighborhood of 0 is that the Fatou coordinates ψ^Y and ψ^Z are not univalued. The monodromy of ψ^Y and ψ^Z is $2\pi \mathrm{i}\, R_\phi$ and $2\pi \mathrm{i}\, R_\eta$ respectively. It suffices to show that

$$(\psi^Z)^{-1} \circ \mathfrak{h} \circ \psi^Y = [(z + 2\pi \mathrm{i}\, R_\eta) \circ \psi^Z]^{-1} \circ \mathfrak{h} \circ [(z + 2\pi \mathrm{i}\, R_\phi) \circ \psi^Y]$$

in order to show that σ is univalued. But such property holds since we have

$$(z + 2\pi \mathrm{i}\, R_\eta)^{-1} \circ \mathfrak{h} \circ (z + 2\pi \mathrm{i}\, R_\phi) \equiv \mathfrak{h}$$

by construction of $\mathfrak{h}$. $\square$

Remark 2.41. — Let

$$\phi = \exp\left(\frac{y^{\nu+1}}{1+y^\nu}\frac{\partial}{\partial y}\right) \quad \text{and} \quad \eta = \exp\left(\frac{y^{\nu+1}}{1+2y^\nu}\frac{\partial}{\partial y}\right)$$

for some fixed $\nu \geqslant 1$. Since $\mathrm{Res}_\phi = 1$ and $\mathrm{Res}_\eta = 2$, they are conjugated by an orientation-preserving Fatou affine local conjugacy σ by the proof of Proposition 2.40. Moreover $\overline{y} \circ \eta \circ \overline{y} = \eta$ implies that $\overline{y} \circ \sigma$ is an orientation-reversing local homeomorphism conjugating ϕ and η and indeed it is a Fatou affine conjugacy. Anyway ϕ and η are neither holomorphically nor anti-holomorphically conjugated since $\mathrm{Res}_\phi \notin \{\mathrm{Res}_\eta, \overline{\mathrm{Res}_\eta}\}$.

3. Unfoldings of tangent to the identity diffeomorphisms

We study the properties of the topological conjugacy of unfoldings of tangent to the identity diffeomorphisms in one variable. In order to treat this problem the main tools are the so called Long Orbits (Section 4). In this section we prepare the unfoldings to make the description of Long Orbits simpler.

3.1. Preparation results for unfoldings

Next, we introduce the unfoldings that are the object of this paper.

Definition 3.1. — *Given un unfolding $\phi(x, y) = (x, f(x, y))$ we denote by ϕ_{x_0} the restriction to ϕ to the line $x = x_0$, i.e. we have $\phi_{x_0}(y) = f(x_0, y)$.*

Definition 3.2. — *Let $\phi_0 \in \mathrm{Diff}(\mathbb{C}, 0)$ be a local diffeomorphism that has a double fixed point at the origin. We say that an unfolding $\phi(x, y) = (x, f(x, y))$ of ϕ_0 is* generic *if $\frac{\partial f}{\partial x}(0, 0) \neq 0$. In this case we also say that ϕ is a* saddle-node unfolding.

We will establish at several points the parallelisms and differences between this setting and the case of saddle-node arcs treated in the introduction. For instance in [14] the authors compare the dynamics of a saddle-node arc ϕ with the time 1 flow of a suitable vector field Y, the so called *adapted* vector field. The dynamics of $\exp(Y)$ is simpler than the dynamics of ϕ but anyway is a "good" approximation. We also provide a definition of adapted vector field in our context.

Definition 3.3. — *Let $\phi \in \mathrm{Diff}(\mathbb{C}^2,0)$ be a saddle-node unfolding. Let*

$$Y = g(x,y)\frac{\partial}{\partial y}$$

be a local vector field singular at the origin (i.e. $g \in \mathbb{C}\{x,y\}$ and $g(0)=0$). We say that Y is adapted *to ϕ if $y\circ\phi - y\circ\exp(Y)$ belongs to the ideal $(f(x,y)-y)^2$.*

The definition is analogous to the Definition 2.4 for dimension 1. In both cases we require that the components of $\phi - \exp(Y)$ belong to the square of the ideal of fixed points. Our condition is not the same as in [14]; ours involves the Taylor expansion along $\mathrm{Fix}(\phi)$ whereas theirs is about the Taylor expansion at 0 (plus the property $\phi_0 = \exp(Y)_0$ that can not happen in our setting whenever ϕ_0 is not analytically trivial). Our definition implies that ϕ and $\exp(Y)$ are formally conjugated by a map $\sigma(x,y) = (x,\hat{g}(x,y)) \in \widehat{\mathrm{Diff}}(\mathbb{C}^2,0)$ such that $\sigma_{|\,\mathrm{Fix}(\phi)} \equiv \mathrm{Id}$, i.e. $\hat{g}(x,y)-y$ belongs to the radical ideal of the ideal $(y\circ\phi - y)$ of the ring $\mathbb{C}[[x,y]]$ [16, Propositions 1.3 and 5.12].

Let us prepare a saddle-node unfolding such that it has a simple adapted vector field.

Proposition 3.4. — *Let $\phi(x,y) = (x,f(x,y)) \in \mathrm{Diff}(\mathbb{C}^2,0)$ be an unfolding of a local diffeomorphism $\phi_0(y)$ that has a double fixed point at the origin. Then up to a holomorphic change of coordinates of the form $(x,g_2(x,y))$ we can suppose that ϕ is of the form*

$$\phi(x,y) = \exp\left(\frac{y^2-a(x)}{1+b(x)y}\frac{\partial}{\partial y}\right) + \left(0, O((y^2-a(x))^2)\right). \tag{3.1}$$

Proof. — We denote $h(x,y) = f(x,y) - y$. The diffeomorphism ϕ is of the form

$$\phi(x,y) = \exp\left(\hat{u}h\frac{\partial}{\partial y}\right) = \left(x, \sum_{j=0}^{\infty}\left(\hat{u}h\frac{\partial}{\partial y}\right)^j(y)\right)$$

where $\hat{u} \in \mathbb{C}[\![x,y]\!]$ is a formal unit [16, Proposition 3.3]. Let us clarify that $(\hat{u}h\partial/\partial y)^j(y)$ is the power series obtained by applying j times the derivation $\hat{u}h\partial/\partial y$ to y where $(\hat{u}h\partial/\partial y)^0(y) \equiv y$. If $\hat{u}$ belongs to $\mathbb{C}\{x,y\}$ then ϕ is just the time 1 flow of $\hat{u}h\partial/\partial y$. Moreover there exists a unit $u \in \mathbb{C}\{x,y\}$ such that $\hat{u}-u$ belongs to the ideal (h) of $\mathbb{C}[\![x,y]\!]$ [16, Proposition 4.4 and Lemma 4.7]. We denote $Y = uh\partial/\partial y$ and $\eta = \exp(Y)$. Since $y \circ \phi - y \circ \eta$ belongs to the ideal (h^2), η_0 has a double fixed point.

Kostov's theorem on versal deformations of vector fields implies that up to a conjugacy of the form $(x, \mathrm{d}(x,y))$ the vector field Y is of the form $\frac{y^2-a(x)}{1+b(x)y}\frac{\partial}{\partial y}$ ([8], cf. [16, Proposition 5.14]). Since $h=0$ is the equation of the fixed point set of the unfolding and $y\circ\phi - y\circ\eta \in (h^2)$, the conjugacy that transforms η into $\exp\left(\frac{y^2-a(x)}{1+b(x)y}\frac{\partial}{\partial y}\right)$ it also transforms ϕ into Expression (3.1). □

Proposition 3.5. — *Let ϕ be a saddle-node unfolding. Then up to a holomorphic change of coordinates of the form $(g_1(x), g_2(x,y))$ we can suppose that ϕ is of the form*

$$\phi(x,y) = \exp(Y)(x,y) + \big(0, O((y^2-x)^2)\big) \tag{3.2}$$

where $Y = \frac{y^2-x}{1+b(x)y}\frac{\partial}{\partial y}$. In particular the vector field Y is adapted to ϕ.

This result is analogous to Theorem 2.1 of [3].

Proof. — We consider the notations in Proposition 3.4. Since $y\circ\phi - y\circ\eta$ belongs to the ideal (h^2), we have $\frac{\partial(y\circ\eta)}{\partial x}(0,0) \neq 0$. This implies $\frac{\partial a(x)}{\partial x}(0,0) \neq 0$. The vector field Y is of the form $\frac{y^2-a(x)}{1+c(x)y}\frac{\partial}{\partial y}$ by the proof of Proposition 3.4. It follows that Y is of the form $\frac{y^2-x}{1+b(x)y}\frac{\partial}{\partial y}$ up to a change of coordinates in the x variable. Since $h=0$ is the equation of the fixed point set of the unfolding and $y\circ\phi - y\circ\eta \in (h^2)$, the conjugacy that transforms η into $\exp\left(\frac{y^2-x}{1+b(x)y}\frac{\partial}{\partial y}\right)$ also transforms ϕ into Expression (3.2). □

Definition 3.6. — *Given an unfolding ϕ in prepared form* (3.2) *we denote the function $b(x)$ by $b_\phi(x)$.*

3.2. Topological invariance of genericity

Consider an unfolding $\phi(x,y) = (x, f(x,y)) \in \mathrm{Diff}(\mathbb{C}^2, 0)$ that is conjugated to an unfolding η of a local diffeomorphism η_0 that has a double fixed point at the origin. Then obviously ϕ_0 has a double fixed point at the origin. Moreover we will see that ϕ is a saddle-node unfolding when η is a saddle-node unfolding.

Let $\phi(x,y) = (x, f(x,y)) \in \mathrm{Diff}(\mathbb{C}^2, 0)$ be an unfolding of a local diffeomorphism $\phi_0(y)$ that has a double fixed point at the origin in prepared form (3.1).

Definition 3.7. — *We say that a parameter x_0 is* neutral *if it supports a neutral fixed point of ϕ. More precisely, the set N_ϕ of neutral parameters is defined by*

$$N_\phi = \left\{ x \in \mathbb{C} : \left| \frac{\partial f}{\partial y}(x, \sqrt{a(x)}) \right| = 1 \text{ or } \left| \frac{\partial f}{\partial y}(x, -\sqrt{a(x)}) \right| = 1 \right\}.$$

We say that x_0 is neutral of multiplicity 1 if just one of the fixed points $(x_0, \sqrt{a(x_0)})$, $(x_0, -\sqrt{a(x_0)})$ is neutral. If both points $(x_0, \sqrt{a(x_0)})$, $(x_0, -\sqrt{a(x_0)})$ are neutral then we say that x_0 is neutral of multiplicity 2.

The set of neutral parameters is a topological invariant. Roughly speaking it depends uniquely on the curve of fixed points.

Lemma 3.8. — *Let ϕ be an unfolding in prepared form (3.1) with $a \not\equiv 0$. Then up to ramification $x = w^2$ the set of neutral parameters is a union of finitely many branches of real analytic curves. Moreover there are 2ν such branches of neutral parameters counted with multiplicity where ν is the vanishing order of $a(x)$ at $x = 0$.*

Proof. — The set N_ϕ of neutral parameters satisfies

$$N_\phi = \left\{ x \in \mathbb{C} : \mathrm{Re}\left(\frac{2\sqrt{a(x)}}{1 + b(x)\sqrt{a(x)}} \right) = 0 \ \text{ or } \ \mathrm{Re}\left(-\frac{2\sqrt{a(x)}}{1 - b(x)\sqrt{a(x)}} \right) = 0 \right\}. \tag{3.3}$$

by (3.1). Consider the ramification $x = w^2$. We denote

$$h_\pm(x) = \frac{\pm 2\sqrt{a(x)}}{1 \pm b(x)\sqrt{a(x)}} \quad \text{and} \quad E_\pm = \{ w \in \mathbb{C}^* : \mathrm{Re}(h_\pm(w^2)) = 0 \}.$$

The function $h_\pm(w^2)$ is holomorphic in a neighborhood of 0 and its vanishing order at $w = 0$ is equal to ν. Up to a holomorphic change of coordinates $w = \eta(w')$, it is of the form $(w')^\nu$. We deduce that $\mathrm{Re}(h_+(w^2)) = 0$ consists of 2ν branches of real analytic curves. Analogously $\mathrm{Re}(h_-(w^2)) = 0$ consists of 2ν branches of real analytic curves. Notice that a branch of $E_+ \cup E_-$ has multiplicity 2 if and only if it is contained in $E_+ \cap E_-$. As a consequence $E_+ \cup E_-$ consists of 4ν branches counted with multiplicity. By undoing the ramification we obtain that N_ϕ has 2ν branches counted with multiplicity. $\square$

Remark 3.9. — In general an unfolding has no multiplicity 2 branches of neutral parameters. Consider an unfolding in prepared form (3.1). A neutral parameter x_0 of multiplicity 2 satisfies

$$\mathrm{Re}\left(\frac{1+b(x_0)\sqrt{a(x_0)}}{2\sqrt{a(x_0)}}\right)=0 \text{ and } \mathrm{Re}\left(\frac{-1+b(x_0)\sqrt{a(x_0)}}{2\sqrt{a(x_0)}}\right)=0.$$

By adding the previous expressions we obtain that $b(x_0)$ is a purely imaginary number. Moreover since $(2\sqrt{a(x_0)})^{-1}+b(x_0)/2$ is purely imaginary, it follows that $a(x_0)$ is a negative real number. Therefore

$$N'=\{x\in\mathbb{C}^*: a(x)\in\mathbb{R}^- \text{ and } \mathrm{Re}(b(x))=0\}$$

is the set of neutral parameters of multiplicity 2. In general N' is empty, for instance if $b(0)\in\mathbb{R}^*$.

Next we show the main result of this subsection.

Proposition 3.10. — *Let ϕ be a saddle-node unfolding. Suppose that ϕ is topologically conjugated to a holomorphic unfolding*

$$\eta(x,y)=(x,g(x,y))\in \mathrm{Diff}\,(\mathbb{C}^2,0).$$

Then η is a saddle-node unfolding.

Proof. — We can suppose

$$\eta(x,y)=\exp\left(\frac{y^2-a(x)}{1+b_\eta(x)y}\frac{\partial}{\partial y}\right)+\left(0,O((y^2-a(x))^2)\right)$$

by Proposition 3.4. There are two fixed points of η in every line $x=x_0$ with $x_0\neq 0$ except if $a\equiv 0$. In particular any unfolding topologically conjugated to a generic one satisfies $a\not\equiv 0$. We denote by ν the vanishing order of $a(x)$ at $x=0$. There are 2ν branches of neutral parameters counted with multiplicity by Lemma 3.8. Analogously a generic unfolding has 2 branches of neutral parameters counted with multiplicity. Since neutral fixed points are topological invariants, we deduce that ϕ and η have the same number of branches of neutral parameters counted with multiplicity. We obtain $\nu=1$ and as a consequence η is a generic unfolding. □

Remark 3.11. — In all results in the introduction it suffices to suppose that at least one of the unfoldings ϕ,η is a saddle-node unfolding by Proposition 3.10

3.3. Example

Let $\phi(x,y) = (x, f(x,y)) \in \mathrm{Diff}(\mathbb{C}^2, 0)$ be a saddle-node unfolding. Suppose $f \in \mathbb{R}\{x,y\}$. We consider the following questions: is it possible to have neutral parameters of multiplicity 2? When? What is the relation between the properties of ϕ and the saddle-node arc $(\phi_x(y))_{x\in\mathbb{R}}$?

The multiplicators of ϕ_x at its fixed points are of the form

$$\lambda_+(x) := \exp\left(\frac{2\sqrt{\alpha(x)}}{1+\beta(x)\sqrt{\alpha(x)}}\right) \ \text{ and } \ \lambda_-(x) := \exp\left(-\frac{2\sqrt{\alpha(x)}}{1-\beta(x)\sqrt{\alpha(x)}}\right)$$

for some $\alpha, \beta \in \mathbb{C}\{x\}$ such that $(\partial\alpha/\partial x)(0) \neq 0$. Indeed given a prepared unfolding η topologically conjugated to ϕ of the form (3.1) the multiplicators of η_x a its fixed points are

$$\exp\left(\frac{2\sqrt{a(x)}}{1+b(x)\sqrt{a(x)}}\right) \quad \text{and} \quad \exp\left(-\frac{2\sqrt{a(x)}}{1-b(x)\sqrt{a(x)}}\right).$$

As a consequence we obtain $\alpha = a \circ \sigma$ and $\beta = b \circ \sigma$ for some $\sigma \in \mathrm{Diff}(\mathbb{C}, 0)$. Analogously as in Remark 3.9 we have that

$$N' := \{x \in \mathbb{C}^* : \alpha(x) \in \mathbb{R}^- \ \text{ and } \ \mathrm{Re}(\beta(x)) = 0\} \tag{3.4}$$

is the set of neutral parameters of multiplicity 2.

We define

$$\Sigma(x) = \frac{1}{\log\lambda_+(x)} + \frac{1}{\log\lambda_-(x)} \ \text{ and } D(x) = \left(\frac{1}{\log\lambda_+(x)} - \frac{1}{\log\lambda_-(x)}\right)^2.$$

The function Σ is holomorphic in a neighborhood of the origin since $\Sigma \equiv \beta$. The function $D(x)$ is meromorphic with a pole of order 1 at 0 since $D \equiv \alpha^{-1}$.

The condition $f \in \mathbb{R}\{x,y\}$ implies $\{\lambda_+(\overline{x}), \lambda_-(\overline{x})\} = \{\overline{\lambda_+(x)}, \overline{\lambda_-(x)}\}$ for any $x \in \mathbb{C}^*$ in a neighborhood of 0. Therefore we obtain $\Sigma(\overline{x}) \equiv \overline{\Sigma(x)}$ and $D(\overline{x}) \equiv \overline{D(x)}$. As a consequence α and β belong to $\mathbb{R}\{x\}$. Since $(\partial\alpha/\partial x)(0) \neq 0$ and $\alpha \in \mathbb{R}\{x\}$, the set $\{x \in \mathbb{C}^* : \alpha(x) \in \mathbb{R}^-\}$ is contained in $\mathbb{R}$. Since the image of $\beta_{|\mathbb{R}}$ is contained in $\mathbb{R}$, we get $N' \subset \{x \in \mathbb{C}^* : \beta(x) = 0\}$. In particular if N' is non-empty then $\beta \equiv 0$ by the isolated zeros principle. The reciprocal also holds by (3.4). Notice that the property $\beta \equiv 0$ is equivalent to $b \equiv 0$. We proved

Proposition 3.12. — *Let $\phi(x,y) = (x, f(x,y)) \in \mathrm{Diff}(\mathbb{C}^2, 0)$ be a saddle-node unfolding. Suppose $f \in \mathbb{R}\{x,y\}$. Then there exists a branch of* *neutral parameters of multiplicity* 2 *if and only if the prepared form of* ϕ *is of the form*

$$\phi(x,y) = \exp\left((y^2-x)\frac{\partial}{\partial y}\right) + \left(0, O((y^2-x)^2)\right).$$

Remark 3.13. — Let $\phi(x,y) = (x, f(x,y)) \in \mathrm{Diff}(\mathbb{C}^2, 0)$ be a saddle-node unfolding with $f \in \mathbb{R}\{x,y\}$. Since $\phi(\mathbb{R}^2) \subset \mathbb{R}^2$, ϕ induces a saddle-node arc and we can apply the ideas in [14]. In particular the orbits through points of the form (x, y_0) where $x < 0$ is close to 0 and $y_0 < 0$ accumulate to any point of the form $(0, y_1)$ with $y_1 > 0$. As an example consider $\phi = \exp((y^2-x)\partial/\partial y)$. Given $y_0 < 0$ and $y_1 > 0$ let $(x_n)_{n \geqslant 1}$ be a sequence of negative real numbers such that $\lim_{n\to\infty} x_n = 0$ and

$$T_n := \frac{-1}{y_1} + \frac{1}{y_0} + \frac{\pi}{\sqrt{|x_n|}} \in \mathbb{N}$$

for any $n \in \mathbb{N}$. We have $\lim_{n\to\infty} \phi^{T_n}(x_n, y_0) = (0, y_1)$. We do not make here the calculations. The property is a consequence of (4.3) in Proposition 4.18.

The orbits through points of the form (x, y_0) with $x \in \mathbb{R}$ can not accumulate in points of the form $(0, y)$ with $y \notin \mathbb{R}$ since $\phi(\mathbb{R}^2) \subset \mathbb{R}^2$. The value of a topological conjugacy σ at points $(0, y)$ with $y > 0$ is related to the value at the point $(0, y_0)$. This is a key property in showing the rigidity results in [14]. In our case the petal $\mathcal{P}$ of ϕ_0 containing $\mathbb{R}^+$ is an open set and then we are missing almost the whole petal. We are forced to leave the setting in [14] and consider orbits outside of $\mathbb{R}^2$ to relate $\sigma(0, y_0)$ with the value of σ at every point of $\mathcal{P}$, namely ϕ-orbits through points of the form (x, y_0) where x lies in an open set β and x is close to 0. As a consequence of using non-real parameters, we can not suppose anymore that ϕ_x is a 1-dimensional real dynamical system and we are in a pure 2-dimensional case for $x \in \beta$.

4. Long Orbits

Consider a saddle-node unfolding ϕ in prepared form (3.2). Now our goal is introducing Long Orbits, showing their existence, properties and connections with [14]. Roughly speaking, we consider sections $(x, \upsilon'(x)) : \beta \cup \{0\} \to \mathbb{C} \times B(0, \epsilon)$, where $\upsilon'(0)$ belongs to the attracting petal of ϕ_0 and β is a connected set with $0 \in \overline{\beta}$, such that the limit of the orbits of ϕ through $(x, \upsilon'(x))$ splits in two orbits in the limit. One of the orbits is obviously the orbit through $\upsilon'(0)$ whereas the other orbit consists of points y_- in the repelling petal of ϕ_0 such that to go from $(x, \upsilon'(x))$ to a neighborhood of $(0, y_-)$ we have to iterate ϕ a number of times $T(x)$ that tends to ∞ when $x \to 0$. This is a non-generic phenomenon that happens in the neighborhood of the negative real line in the parameter space. An example is provided in Remark 3.13.

In [14] the dynamics of a saddle-node arc is compared to the dynamics of the time 1 flow of its adapted vector field. We will develop the same strategy but we do not need to work with estimates of the deviation of the dynamics of a saddle-node unfolding ϕ with respect to $\exp(Y)$ where Y is an adapted vector field. The comparison result that we need can be deduced immediately of the properties of the extension of the Fatou coordinates of ϕ_0 to the nearby parameters that we obtained in [17] and more precisely of Proposition 7.3.

4.1. The polynomial vector field $(t^2 + 1)\partial/\partial t$.

Let us recap briefly some results of [17]. Let ϕ be a saddle-node unfolding in prepared form (3.2). We consider unfoldings in [17] whose fixed point set is of the form $\{f(x, y) = 0\}$ where $f(x, y)$ is of the form $\prod_{j=1}^{a}(y - g_j(x))^{n_j}$. In order to bring our case to such a setting, we make a ramification $x = -w^2$. The vector field Y and the diffeomorphism ϕ are transformed into

$$\frac{y^2 + w^2}{1 + b(-w^2)y}\frac{\partial}{\partial y}$$

and

$$\phi(w, y) = \exp\left(\frac{y^2 + w^2}{1 + b(-w^2)y}\frac{\partial}{\partial y}\right) + \big(0, O((y^2 + w^2)^2)\big).$$

Next we associate to ϕ a polynomial vector field Z whose properties determine the domains of definition of extensions of Fatou coordinates (of the petals) of ϕ_0 to the nearby parameters. We consider the blow-up of the origin of $\mathbb{C}^2$ and coordinates $w = w$, $y = wt$ in the first chart of the blow-up. The transform of Y in the new coordinates (w, t) is equal to

$$Y = w^2 \frac{t^2 + 1}{1 + b(-w^2)wt}\frac{1}{w}\frac{\partial}{\partial t} = w\frac{t^2 + 1}{1 + b(-w^2)wt}\frac{\partial}{\partial t}.$$

We define the vector field

$$Z' = (t^2 + 1)\frac{\partial}{\partial t};$$

it is a polynomial vector field in the variable t. The vector field

$$Z = w(t^2 + 1)\frac{\partial}{\partial t}.$$

is the *polynomial vector field associated* to Y. Let us explain the idea behind the definition of Z. The real flows of $\operatorname{Re}(Y)$ and $\operatorname{Re}(|w|^{-1}Y)$ have the same trajectories outside $w = 0$ up to a reparametrization of time. Given a sequence of points $(w_n)_{n\geqslant 1}$ in $\mathbb{C}^*$ such that $\lim_{n\to 0} w_n = 0$ and $\lim_{n\to\infty} w_n/|w_n| = \lambda \in \mathbb{S}^1$, the vector field $\operatorname{Re}(|w|^{-1}Y)_{|w=w_n}$ tends to $\operatorname{Re}(\lambda Z')$ when $n \to \infty$ in any domain of the form

$$(4.1)\quad \{(w,t) \in B(0,\delta) \times B(0,\epsilon_0)\} = \{(w,y) \in B(0,\delta) \times \mathbb{C} : |y| < \epsilon_0|w|\}.$$

This is also the limit of $\operatorname{Re}(|w|^{-1}Z)_{|w=w_n}$ when $n \to \infty$. So it makes sense to consider the simpler vector field Z when studying the real flow of Y in "infinitesimal domains" of the form (4.1). In order to obtain extensions of Fatou coordinates of $\phi_0 \in \operatorname{Diff}_1(\mathbb{C},0)$ for a general unfolding ϕ, further blow-ups are considered (and hence more polynomial vector fields are associated to ϕ) until all the irreducible components of $f = 0$ separate. In our case the curve $t^2+1=0$ has two irreducible components $t+\mathrm{i}=0$ and $t-\mathrm{i}=0$ that pass through different points of $w=0$. So they are separated in the first step and we do not need to continue the process.

Notice that given w_0 and w_1 such that $w_1/w_0 \in \mathbb{R}^+$ the dynamics of $\operatorname{Re}(Z)_{|w=w_0}$ and $\operatorname{Re}(Z)_{|w=w_1}$ are orbitally equivalent.

Definition 4.1. — *A direction $\lambda\mathbb{R}^+$ in the w-parameter space is a stable direction of* $\operatorname{Re}(Y)$ *if* $\operatorname{Re}(\mu\lambda Z')$ *is orbitally equivalent to* $\operatorname{Re}(\lambda Z')$ *for any $\mu \in \mathbb{S}^1$ in a neighborhood of* 1.

Remark 4.2. — It is easy to see that in our case $\lambda\mathbb{R}^+$ is an unstable direction if and only if the multiplicator of $\lambda Z'$ at both singular points is an imaginary number. Indeed it is known that $\lambda\mathbb{R}^+$ is unstable if and only if $\operatorname{Re}(\lambda Z')$ has homoclinic trajectories, i.e. trajectories $\gamma : (a,b) \to \mathbb{C}$ of $\operatorname{Re}(\lambda Z')$ such that $\lim_{s\to a}\gamma(s) = \infty = \lim_{s\to b}\gamma(s)$ [4]. Moreover, if $\sharp\operatorname{Sing}(Z') = 2$ then $\operatorname{Re}(\lambda Z')$ has a homoclinic trajectory if and only if the multiplicators in the singular points are imaginary (cf. [17, Lemma 6.7]). Since $-\mathrm{i}$ and i are the singular points of Z and the multiplicator of Z at these points is $-2\,\mathrm{i}\,w$ and $2\,\mathrm{i}\,w$ respectively, we obtain that

$$\mathcal{U} := \{\mathbb{R}^+, \mathbb{R}^-\}$$

is the set of unstable directions of $\operatorname{Re}(Y)$. Since $x = -w^2$, it follows that these directions correspond to the direction $\mathbb{R}^-$ in the x-parameter. Notice that the directions in $\mathcal{U}$ are stable for $\operatorname{Re}(\mathrm{i}\,Z)$. The vector field $\operatorname{Re}(\mathrm{i}\,Y)$ can be used to construct regions that support Fatou coordinates [17].

Remark 4.3. — The existence of Long Orbits is related to the existence of unstable directions. Indeed the Long Orbits will be constructed in the neighborhood of the unique direction of instability $\mathbb{R}^-$ in the x-parameter space.

Remark 4.4. — Even if there are no multiplicity 2 branches of neutral parameters in general, somehow they can be found in the infinitesimal setting. Notice that both singular points of $\mathrm{Re}((t^2+1)\partial/\partial t)$ are neutral. Roughly speaking, we can say that the direction $\mathbb{R}^-$ (in the x-parameter) is infinitesimally a direction of neutral parameters of multiplicity 2.

4.2. Construction of extensions of Fatou coordinates

The unperturbed diffeomorphism ϕ_0 (of the saddle-unfolding ϕ) has two petals.

DEFINITION 4.5. — *We denote by $\mathcal{P}_+$ (resp. $\mathcal{P}_-$) the attracting (resp. repelling) petal of $(\phi_0)_{|B(0,\epsilon)}$ for some $\epsilon > 0$ small enough.*

We are going to extend the Fatou coordinates of ϕ_0 in $\mathcal{P}_+$ and $\mathcal{P}_-$ to the nearby parameters in the neighborhood of the direction $\mathbb{R}^-$ of the parameter space. We will follow the approach in [17] (see also [12]).

4.2.1. Domains of definition of Fatou coordinates

Given any $\epsilon > 0$ sufficiently small consider the set $T_Y^\epsilon(x_0)$ of tangency points between $\{x_0\} \times \partial B(0,\epsilon)$ and $\mathrm{Re}(\mathrm{i}\, Y)$; it has exactly two points. Moreover we have $T_Y^\epsilon(x_0) = \{T_+(x_0), T_-(x_0)\}$ where $T_\pm$ is continuous and defined in a neighborhood of $x_0 = 0$ [17, Remark 6.1].

We can suppose up to renaming the tangent points that $\mathrm{Re}(Y)$ points towards the interior (resp. exterior) of $\{x_0\} \times B(0,\epsilon)$ at $T_+(x_0)$ (resp. $T_-(x_0)$) for any x_0 in a neighborhood of 0 in $\mathbb{C}$ (see Figure 4.1 where the continuous lines are trajectories of $\mathrm{Re}(\mathrm{i}\, Y)$ and the dashed lines are trajectories of $\mathrm{Re}(Y)$). Both points $T_+(x_0)$ and $T_-(x_0)$ are convex, meaning that for any x_0 close to 0, there exists $a > 0$ such that $\exp(\mathrm{i}\, tY)(T_\pm(x_0))$ belongs to $\{x_0\} \times B(0,\epsilon)$ for any $t \in (-a,0) \cup (0,a)$ [17, Lemma 6.1].

Let us study the dynamics of $\Im(Y) = \mathrm{Re}(\mathrm{i}\, Y)$ for the x-parameters in the neighborhood of the direction $\mathbb{R}^-$. It depends on the polynomial vector field Z. All vector fields of the form $\mathrm{Re}(\mathrm{e}^{\mathrm{i}\,\theta}\, \mathrm{i}\, Z')$ are pairwise orbitally conjugated for any $\theta \in (-\pi/2, \pi/2)$. This is key to define Fatou coordinates of ϕ over sectors of the form $\{r\,\mathrm{e}^{\mathrm{i}\,\theta} : r \in \mathbb{R}^+,\ \theta \in (-\iota,\iota)\}$ in the w-coordinate where $0 < \iota < \pi/2$, or equivalently sectors of the form

$$S_\iota := \{-r\,\mathrm{e}^{\mathrm{i}\,\theta} : r \in \mathbb{R}^+ \cup \{0\},\ \theta \in [-\iota,\iota]\}$$

in the x-variable where $\iota \in (0,\pi)$ (cf. Definition 4.6). The stability of $\mathrm{Re}(e^{i\theta} i Z')$ for $\theta \in (-\pi/2, \pi/2)$ implies the stability of $\Im(Y)$ over S_ι. In particular we obtain

$$\{\exp(i tY)(T_+(-r e^{i\theta})), \exp(i tY)(T_-(-r e^{i\theta}))\} \subset \mathbb{C} \times B(0,\epsilon)$$

for all $r \in \mathbb{R}^+ \cup \{0\}$ in a neighborhood of 0, $\theta \in [-\iota, \iota]$ and $t \in \mathbb{R} \setminus \{0\}$ [17, Lemma 6.13]. Consider a determination $\sqrt{x}$ of the square root in S_ι such that $\sqrt{-1} = i$. We have

$$\lim_{t\to\infty} \exp(i tY)(T_+(x)) = \lim_{t\to\infty} \exp(i tY)(T_-(x)) = (x, \sqrt{x})$$

and

$$\lim_{t\to-\infty} \exp(i tY)(T_+(x)) = \lim_{t\to-\infty} \exp(i tY)(T_-(x)) = (x, -\sqrt{x})$$

for all $x \in S_\iota$ in a neighborhood of 0 [17, Lemma 6.13]. Notice that the point $y = \sqrt{x}$ (resp. $y = -\sqrt{x}$) is an attractor (resp. a repeller) of $\Im(Y)(x, \cdot)$ for any x close to 0 in $S_\iota \setminus \{0\}$.

Let $x_0 \in S_\iota$ in a neighborhood of 0. Consider the support $\gamma_\pm(x_0)$ of the trajectory of $\Im(Y)$ through $T_\pm(x_0)$. We denote

$$\gamma(x_0) = \gamma_+(x_0) \cup \gamma_-(x_0) \cup \{\sqrt{x_0}, -\sqrt{x_0}\},$$

it is a curve contained in the line $x = x_0$. Moreover $\gamma(x_0)$ is a closed simple curve for any x_0 in $S_\iota \setminus \{0\}$ close to 0.

Definition 4.6. — *We denote by $H_+(x_0)$ (resp. $H_-(x_0)$) the bounded connected of the complementary of $\gamma(x_0)$ in $x = x_0$ whose closure contains $T_+(x_0)$ (resp. $T_-(x_0)$). We denote*

$$H(x_0) = H_+(x_0) \cup H_-(x_0).$$

We have $H_+(x_0) = H_-(x_0)$ if $x_0 \neq 0$ and $H_+(x_0) \cap H_-(x_0) = \emptyset$ if $x_0 = 0$. We denote

$$H_+ = \cup_{x \in S_\iota, |x|<\delta} H_+(x), \ \ H_- = \cup_{x \in S_\iota, |x|<\delta} H_-(x)$$

and

$$H = \cup_{x \in S_\iota, |x|<\delta} H(x)$$

for some $\delta > 0$ sufficiently small.

Notice that Figure 4.1 for $x = 0$ corresponds to the dynamics in Figure 2.1 for the case of just two petals.

By construction the vector field $\mathrm{Re}(Y)$ points towards the interior (resp. exterior) of $H(x_0)$ at the point $T_+(x_0)$ (resp. $T_-(x_0)$) for any $x_0 \in S_\iota$ close to 0.

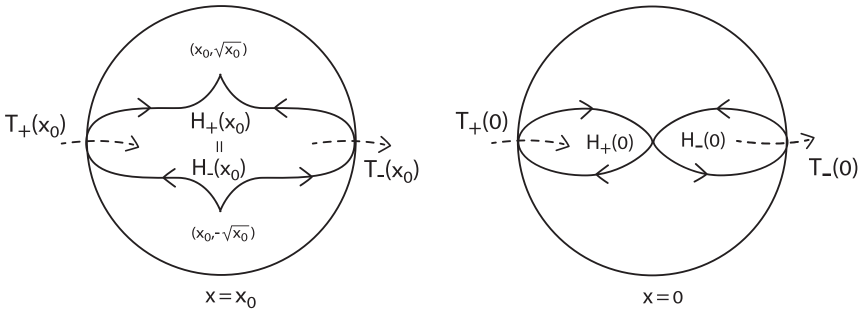

Figure 4.1. Dynamics of $\mathrm{Re}(\mathrm{i}\, Y)$

4.2.2. Construction of Fatou coordinates

Next we define a Fatou coordinate ψ_+^Y of the vector field Y in H_+ Since $H_+(x_0)$ is simply connected for any $x_0 \in S_\iota$, it suffices to define ψ_+^Y in a neighborhood of $T_+(0)$ and then extending ψ_+^Y to H_+ by analytic continuation. We can define ψ_-^Y in a neighborhood of $T_-(0)$ by making analytic continuation of ψ_+^Y along a simple path in $\{0\} \times \partial B(0, \epsilon)$ going from $T_+(0)$ to $T_-(0)$ in counterclockwise sense. Analogously as for ψ_+^Y we can extend analytically ψ_-^Y to H_-.

The dual form of Y is equal to

$$\omega := \frac{1 + b_\phi(x) y}{y^2 - x} \mathrm{d}y,$$

i.e. we have $\omega(Y) \equiv 1$. Since Fatou coordinates are primitives of ω in the lines $x = x_0$, the difference $\psi_-^Y(x_0, \cdot\,) - \psi_+^Y(x_0, \cdot\,)$ in $H(x_0)$ is equal to the integral of ω along a simple closed path turning once around the point $-\sqrt{x_0}$ once in counterclockwise sense in the line $x = x_0$. We obtain

$$\psi_-^Y(x, y) - \psi_+^Y(x, y) = 2\pi\,\mathrm{i} \left(\frac{-1}{2\sqrt{x}} + \frac{b_\phi(x)}{2} \right) = -\frac{\pi\,\mathrm{i}}{\sqrt{x}} + \pi\,\mathrm{i}\, b_\phi(x) \tag{4.2}$$

by the residue formula.

The following results can be found in [17].

Lemma 4.7 ([17, Lemma 7.8]). — *There exists a Fatou coordinate ψ_+^ϕ (resp. ψ_-^ϕ) of ϕ defined in H_+ (resp. H_-). Moreover we can choose ψ_+^ϕ and ψ_-^ϕ such that*

$$\psi_+^\phi - \psi_+^Y \equiv \psi_-^\phi - \psi_-^Y.$$

Proposition 4.8 ([17, Proposition 7.2]). — *The map $(x, \psi_\pm^\phi)$ is continuous and injective in $H_\pm$ and holomorphic in the interior of H.*

Definition 4.9. — *We define* $\psi^{\phi} - \psi^{Y}$ *as either* $\psi_{+}^{\phi} - \psi_{+}^{Y}$ *or* $\psi_{-}^{\phi} - \psi_{-}^{Y}$ *in* H.

Proposition 4.10 ([17, Proposition 7.3]). — *The function* $\psi^{\phi} - \psi^{Y}$ *extends continuously to* $\overline{H}$. *Moreover we can define* ψ_{+}^{ϕ} *and* ψ_{-}^{ϕ} *such that* $(\psi^{\phi} - \psi^{Y})(x, \sqrt{x}) \equiv 0$.

Remark 4.11. — Let us remind the reader that we can extend ψ_{+}^{Y} and ψ_{+}^{ϕ} to a neighborhood of the attracting petal $\mathcal{P}_{+}$ of ϕ_0 by using the equations $Y(\psi_{+}^{Y}) \equiv 1$ and $\psi_{+}^{\phi} \circ \phi \equiv \psi_{+}^{\phi} + 1$. Analogously we can extend ψ_{-}^{Y} and ψ_{-}^{ϕ} to a neighborhood of the repelling petal $\mathcal{P}_{-}$ of ϕ_0.

4.3. Existence of Long Orbits

We are ready to introduce Long Orbits and prove their existence via extensions of Fatou coordinates. We consider points of the form $\phi^{T(x)}(x, y_{+})$ for some $y_{+} \in \mathcal{P}_{+}$ and their accumulation points when $x \in S_{\iota}$ tends to 0. What is the function T that determines how much we must iterate? We will see that it is basically the additive inverse of the residue function in the right hand side of (4.2).

Definition 4.12. — *We define*

$$\mathcal{T}(x) = \mathrm{Re}\left(\frac{\pi \, \mathrm{i}}{\sqrt{x}}\right)$$

where $\sqrt{-1} = \mathrm{i}$.

Next, we define sets in the parameter space in which the structure of the Long Orbits is particularly simple.

Definition 4.13. — *We define the wedge*

$$\beta_M = \left\{ x \in \mathbb{C}^* : \left| \mathrm{Im}\left(\frac{\pi \, \mathrm{i}}{\sqrt{x}}\right) \right| \leqslant M \right\} = \left\{ x \in \mathbb{C}^* : \left| \mathrm{Re}\left(\frac{\pi}{\sqrt{x}}\right) \right| \leqslant M \right\}$$

for any $M \in \mathbb{R}^{+}$.

Remark 4.14. — Notice that any $x \in \beta_M$ near 0 is contained in S_{ι}.

Lemma 4.15. — *Let* $\phi \in \mathrm{Diff}(\mathbb{C}^2, 0)$ *be a saddle-node unfolding in prepared form* (3.2). *The curves of neutral parameters are contained in* β_M *for* $M > 0$ *sufficiently big.*

Proof. — We have to show that the curves given by the equations

$$\operatorname{Re}\left(\frac{1+b_\phi(x)\sqrt{x}}{2\sqrt{x}}\right)=0 \quad \text{and} \quad \operatorname{Re}\left(\frac{-1+b_\phi(x)\sqrt{x}}{2\sqrt{x}}\right)=0$$

respectively, are contained in β_M for $M \gg 1$. It is clear that in both cases $\operatorname{Re}(\pi/\sqrt{x})$ is bounded in a neighborhood of 0. Thus both curves are contained in β_M for $M \gg 1$. □

Next, we define Long Orbits. Given a real number s its ceiling $\lceil s \rceil$ is the smallest integer greater or equal than s. Given a set S in the parameter space we say that $\upsilon : S \to \mathbb{C}^2$ is a section if $\upsilon(x_0)$ belongs to $x = x_0$ for any $x_0 \in S$.

Definition 4.16. — *Let $\phi \in \operatorname{Diff}(\mathbb{C}^2, 0)$ be a saddle-node unfolding in prepared form* (3.2). *Let β be a connected set in the x-parameter space such that $0 \in \overline{\beta}$, β contains the curves of neutral parameters and there exists $M \in \mathbb{R}_+$ with $\beta \subset \beta_M$. Let $T : \beta \to \mathbb{R}^+$ be a continuous function such that $\lim_{x\in\beta,\ x\to 0} T(x) = \infty$. We say that $\mathcal{O} = (\phi, y_+, \beta, T)$ is a* Long Orbit *if $(0, y_+)$ belongs to the attracting petal of $\phi_{|\{0\}\times B(0,\epsilon)}$ and there exists a continuous section $\upsilon_{\mathcal{O}} : \beta \cup \{0\} \to \mathbb{C} \times B(0,\epsilon)$ with $\upsilon_{\mathcal{O}}(0) = (0, y_+)$ such that*

- *$\phi^j(\upsilon_{\mathcal{O}}(x))$ is well-defined and belongs to $\mathbb{C} \times B(0,\epsilon)$ for any $0 \leqslant j \leqslant [T(x)] + 1$ and any $x \in \beta$ in a neighborhood of* 0.
- *Given any $\epsilon' > 0$ there exists $M'' \in \mathbb{N}$ such that*

$$\{\phi^{M''}(\upsilon_{\mathcal{O}}(x)), \dots, \phi^{\lceil T(x)\rceil - M''}(\upsilon_{\mathcal{O}}(x))\}$$

  is contained in $\mathbb{C} \times B(0,\epsilon')$ for any $x \in \beta$ in a neighborhood of 0.

Moreover we require the existence of a continuous function $\vartheta_{\mathcal{O}} : \beta \to \mathbb{R}$ such that $\mathcal{S}_{\mathcal{O}} := \vartheta_{\mathcal{O}}(\beta)$ is a connected compact set containing 0 *and there exists a continuous map $\chi_{\mathcal{O}} : [0,1] + \mathrm{i}\,\mathcal{S}_{\mathcal{O}} \to \{0\} \times (B(0,\epsilon) \setminus \{0\})$ such that*

- *$\chi_{\mathcal{O}}(1 + \mathrm{i}\,u) = \phi(\chi_{\mathcal{O}}(\mathrm{i}\,u))$ for any $u \in \mathcal{S}_{\mathcal{O}}$.*
- *Given any $z = s + \mathrm{i}\,u \in [0,1] + \mathrm{i}\,\mathcal{S}_{\mathcal{O}}$ and a sequence $\{x_n\}$ in β with $x_n \to 0$ and*

$$s = \lim_{n\to\infty} (\lceil T(x_n)\rceil - T(x_n)), \quad \lim_{n\to\infty} \vartheta_{\mathcal{O}}(x_n) = u$$

  we obtain $\lim_{n\to\infty} \phi^{\lceil T(x_n)\rceil}(\upsilon_{\mathcal{O}}(x_n)) = \chi_{\mathcal{O}}(z)$.

Remark 4.17. — We skip the subindex $\mathcal{O}$ in $\upsilon_{\mathcal{O}}$, $\vartheta_{\mathcal{O}}$, $\mathcal{S}_{\mathcal{O}}$ and $\chi_{\mathcal{O}}$ when the Long Orbit $\mathcal{O}$ is implicit.

In order to understand better the concept let us consider a saddle-node unfolding of the form $\exp(Y)$ where Y is in prepared form (3.2). Suppose

$\mathcal{O} = (\exp(Y), y_+, \beta, T)$ is a Long Orbit. We denote by $\Gamma_{(x,y)}$ the trajectory of $\mathrm{Re}(Y)$ with initial condition $\Gamma_{(x,y)}(0) = (x, y)$. Fix $u \in \mathcal{S}_{\mathcal{O}}$. Then the set $\Gamma_{(x,y_+)}[0, T(x)]$ converges to

$$\Gamma_{(0,y_+)}[0, \infty) \cup \{(0, 0)\} \cup \Gamma_{\chi_{\mathcal{O}}(\mathrm{i}\,u)}(-\infty, 0]$$

when $x \in \vartheta_{\mathcal{O}}^{-1}(u)$ tends to 0 and we consider the Hausdorff topology for compact sets. Outside of the origin such limit is the union of two trajectories of $\mathrm{Re}(Y)$ by the second bullet property of Definition 4.16. The limit depends on the value of the function $\vartheta_{\mathcal{O}}$. If we replace T with $T + 1$ then we have to replace $\chi_{\mathcal{O}}(\mathrm{i}\,u)$ with $\phi(\chi_{\mathcal{O}}(\mathrm{i}\,u))$. This makes natural the third bullet property. In the case of general saddle-node unfoldings the properties in Definition 4.16 are adapted from the case of vector fields. For instance we have to work with the ceiling function $\lceil T(x) \rceil$ in the fourth bullet point since we can only consider integer iterates.

The definition of Long Orbit was introduced in [19]. There the section $\upsilon_{\mathcal{O}}$ is defined in a curve denoted by $\beta \cup \{0\}$ in [19] and whose analogue in this paper is $\vartheta_{\mathcal{O}}^{-1}(0) \cup \{0\}$. Then the evolution of the Long Trajectories is studied when that curve varies in a family as $\{\vartheta_{\mathcal{O}}^{-1}(s)\}_{s \in \mathcal{S}_{\mathcal{O}}}$. In this paper we can deal with all the curves simultaneously as a consequence of the properties of the extensions of Fatou coordinates.

The main goal of this section is proving next proposition.

Proposition 4.18. — *Let $\phi \in \mathrm{Diff}(\mathbb{C}^2, 0)$ be a saddle-node unfolding in prepared form (3.2). Let $(0, y_+')$ be a point in $\mathcal{P}_+$ and $\upsilon : B(0, \delta) \to \mathbb{C}^2$ be a section such that $\upsilon(0) = (0, y_+')$. There exists $M' \in \mathbb{N}$ such that $\mathcal{O}' := (\phi, y_+', \beta_M, \mathcal{T} - M')$ is a Long Orbit for any $M \geqslant M_0$ such that $\upsilon_{\mathcal{O}'} = \upsilon_{|\beta \cup \{0\}}$. Moreover any Long Orbit $\mathcal{O} := (\phi, y_+, \beta, T)$ satisfies the residue formula, i.e. we have*

$$\psi_-^{\phi}(\phi^{\lceil T(x) \rceil}(\upsilon_{\mathcal{O}}(x))) - \psi_+^{\phi}(\upsilon_{\mathcal{O}}(x)) = \lceil T(x) \rceil - \frac{\pi\,\mathrm{i}}{\sqrt{x}} + \pi\,\mathrm{i}\, b_\phi(x) \tag{4.3}$$

for any $x \in \beta$ in a neighborhood of 0. In particular we obtain

$$\psi_-^{\phi}(\chi_{\mathcal{O}}(s + \mathrm{i}\,u)) - \psi_+^{\phi}(0, y_+) = \lim_{\substack{\lceil T(x) \rceil - T(x) \to s \\ \vartheta_{\mathcal{O}}(x) \to u,\ x \to 0}} \left(\lceil T(x) \rceil - \frac{\pi\,\mathrm{i}}{\sqrt{x}} + \pi\,\mathrm{i}\, b_\phi(x) \right) \tag{4.4}$$

for any $s + \mathrm{i}\,u \in [0, 1] + \mathrm{i}\,\mathcal{S}_{\mathcal{O}}$.

We will see in the proof that the functions $\vartheta_{\mathcal{O}'}$ and $\chi_{\mathcal{O}'}$ do not depend on M. Hence we do not include the subindex M in $\mathcal{O}'$.

Proof. — Given any point $x \in \beta_M$ close to 0, x belongs to S_ι. So we can apply the techniques and to use the tools in Subsection 4.1. Up to replace $x \mapsto \upsilon(x)$ with $x \mapsto \phi^{k_0}(\upsilon(x))$ for some $k_0 \in \mathbb{N}$ we can suppose that $\upsilon(0)$ belongs to H_+.

The value $|\mathrm{Re}(\psi_-^Y(x,y_1) - \psi_+^Y(x,y_0))|$, where $(x,y_0) \in \gamma_+(x)$ and $(x,y_1) \in \gamma_-(x)$, is the width of the strip $H(x)$ in Fatou coordinates. It is of the form $\mathrm{Re}(\pi\,\mathrm{i}/\sqrt{x}) + O(1)$ by (4.2). Since $\psi_+^\phi - \psi_+^Y$ is bounded by Proposition 4.10, it follows that there exists $M' \in \mathbb{N}$ such that $\phi^j(\upsilon(x))$ is well-defined and belongs to $H(x)$ for any $0 \leqslant j \leqslant [\mathcal{T}(x)] + 1 - M'$ and any $x \in \beta_M$ in a neighborhood of 0. We denote $T' = \mathcal{T} - M'$. We have

$$\psi_+^\phi(\phi^j(\upsilon(x))) = j + \psi_+^\phi(\upsilon(x))$$

and

$$\psi_-^\phi(\phi^j(\upsilon(x))) = \psi_+^\phi(\upsilon(x)) + j - \frac{\pi\,\mathrm{i}}{\sqrt{x}} + \pi\,\mathrm{i}\,b_\phi(x).$$

The second equation follow from $\psi_-^Y - \psi_+^Y \equiv \psi_-^\phi - \psi_+^\phi$ (Lemma 4.7) and (4.2). Thus given $C > 0$ there exists $M'' \in \mathbb{N}$ such that

$$\mathrm{Re}(\psi_+^Y(\phi^j(\upsilon(x)))) > C$$

for all $x \in S_\iota$ close to 0 and $M'' \leqslant j \leqslant \lceil T'(x) \rceil$. Analogously we get

$$\mathrm{Re}(\psi_-^Y(\phi^j(\upsilon(x)))) < -C$$

for all $x \in S_\iota$ close to 0 and $0 \leqslant j \leqslant \lceil T'(x) \rceil - M''$ by considering if necessary a greater M''. Since the values of ψ_+^Y in $H_+ \setminus \mathbb{C} \times B(0,\epsilon')$ (resp. ψ_-^Y in $H_- \setminus \mathbb{C} \times B(0,\epsilon')$) are bounded for any $\epsilon' > 0$ we deduce the second property of a Long Orbit.

We have

$$\psi_+^\phi(\phi^{\lceil T'(x) \rceil}(\upsilon(x))) - \psi_+^\phi(\upsilon(x)) = \lceil T'(x) \rceil.$$

The equality $\psi_-^Y - \psi_+^Y \equiv \psi_-^\phi - \psi_+^\phi$ and (4.2) imply (4.3) for $\mathcal{O}'$. We define the function $\vartheta_{\mathcal{O}'}$ by $\vartheta_{\mathcal{O}'}(x) = -\mathrm{Im}\left(\frac{\pi\,\mathrm{i}}{\sqrt{x}}\right)$. We denote $\mathcal{S}_{\mathcal{O}'} = [-M, M]$. We have

$$\lceil T'(x) \rceil - \frac{\pi\,\mathrm{i}}{\sqrt{x}} + \pi\,\mathrm{i}\,b_\phi(x) = s + \mathrm{i}\,\vartheta_{\mathcal{O}'}(x) - M' + \pi\,\mathrm{i}\,b_\phi(x)$$

if $\lceil T'(x) \rceil - T'(x) = s$. Given $z = s + \mathrm{i}\,u \in [0,1] + \mathrm{i}\,\mathcal{S}_{\mathcal{O}'}$ we have that whenever $x \in \beta_M$ tends to 0, $\lceil T'(x) \rceil - T'(x)$ tends to s and $\vartheta_{\mathcal{O}'}(x)$ tends to u, the point $\phi^{\lceil T'(x) \rceil}(x, y_+)$ tends to a point $\chi_{\mathcal{O}'}(z) \in \{0\} \times (B(0,\epsilon) \setminus \{0\})$ such that

$$\psi_-^\phi(\chi_{\mathcal{O}'}(z)) = (\psi_+^\phi(0, y_+) - M' + \pi\,\mathrm{i}\,b_\phi(0)) + z \tag{4.5}$$

as we want to prove. In particular $\chi_{\mathcal{O}'}$ is continuous and $\chi_{\mathcal{O}'}(1 + \mathrm{i}\, u) = \phi(\chi_{\mathcal{O}'}(\mathrm{i}\, u))$ for any $u \in \mathcal{S}$. We found a Long Orbit $(\phi, \phi_0^{k_0}(y'_+), \beta_M, T')$. It induces a Long Orbit $(\phi, y'_+, \beta_M, T' + k_0)$ that still satisfies (4.3) and (4.4).

Let us consider now any Long Orbit (ϕ, y_+, β, T). The set β is contained by definition in some set β_M. There exists a Long Orbit $(\phi, y_+, \beta_M, \mathcal{T} - M')$ for some $M' \in \mathbb{N}$ by the first part of the proof. The second bullet point of the definition of a Long Orbit implies that $T - (\mathcal{T} - M')$ is bounded in a neighborhood of 0 in β. We can suppose $T - (\mathcal{T} - M') \geqslant 0$ in a neighborhood of 0 in β by considering a greater $M' \in \mathbb{N}$ if necessary. (4.3) for the Long Orbit $(\phi, y_+, \beta_M, \mathcal{T} - M')$ implies (4.3) for (ϕ, y_+, β, T) since we just have to add $\lceil T(x) \rceil - (\lceil \mathcal{T}(x) - M' \rceil)$ to both terms. We also obtain (4.4) for the Long Orbit (ϕ, y_+, β, T) since such a equation is a consequence of (4.3). □

The residue formulas (4.3) and (4.4) are fundamental in the sequel since they allow to describe the quantitative properties of Long Orbits.

Remark 4.19. — The Long Orbit $\mathcal{O}' := (\phi, y'_+, \beta_M, T')$ (for $M \geqslant M_0$) constructed in Proposition 4.18 satisfies

$$\psi_-^\phi(\chi_{\mathcal{O}'}(z)) - \psi_-^\phi(\chi_{\mathcal{O}'}(0)) = z \tag{4.6}$$

for any $z \in [0, 1] + \mathrm{i}\,\mathbb{R}$ by (4.5). This property does not necessarily hold for any Long Orbit $\mathcal{O} := (\phi, y_+, \beta, T)$. The reason is that the functions $\vartheta_{\mathcal{O}'}$ and $\vartheta_{\mathcal{O}}$ can be different. For instance if $h \circ \vartheta_{\mathcal{O}} = \vartheta_{\mathcal{O}'}$ then we obtain

$$\chi_{\mathcal{O}}(s + \mathrm{i}\, u) \equiv \chi_{\mathcal{O}'}(s + \mathrm{i}\, h(u))$$

and then

$$\psi_-^\phi(\chi_{\mathcal{O}}(s + \mathrm{i}\, u)) - \psi_-^\phi(\chi_{\mathcal{O}}(0)) = s + \mathrm{i}(h(u) - h(0)).$$

For a general function, for instance $h(u) = 2u$, we do not get (4.6).

Remark 4.20. — Suppose that we have the setting of a Long Orbit as in Definition 4.16 but we consider that the first, second and fourth bullet points hold for a subset β' of β instead of the whole set β. The last paragraph in the proof of Proposition 4.18 shows that (4.3) holds for the points of β'.

We want to use Long Orbits to study topological conjugacies of unfoldings. Hence we have to prove that Long Orbits are topological invariants.

PROPOSITION 4.21. — *Let $\phi, \eta \in \mathrm{Diff}(\mathbb{C}^2, 0)$ be saddle-node unfoldings in prepared form (3.2). Consider a local homeomorphism $\sigma(x, y) = (\sigma_1(x), \sigma_2(x, y))$ conjugating ϕ and η. Then the image of a Long Orbit* *of ϕ by σ is a Long Orbit of η. More precisely consider $\epsilon, \epsilon' > 0$ sufficiently small such that $\sigma(B(0,\delta) \times B(0,\epsilon)) \subset \mathbb{C} \times B(0,\epsilon')$ and suppose that $\mathcal{O} := (\phi, y_+, \beta, T)$ is a Long Orbit in $\mathbb{C} \times B(0,\epsilon)$. Then $\mathcal{O}' := (\eta, \sigma_0(y_+), \sigma_1(\beta), T \circ \sigma_1^{-1})$ is a Long Orbit of η.*

We have $\sigma \circ \upsilon_{\mathcal{O}} \equiv \upsilon_{\mathcal{O}'}$, $\vartheta_{\mathcal{O}} \circ \sigma_1^{-1} \equiv \vartheta_{\mathcal{O}'}$, $\mathcal{S}_{\mathcal{O}} = \mathcal{S}_{\mathcal{O}'}$ and $\sigma_0 \circ \chi_{\mathcal{O}} \equiv \chi_{\mathcal{O}'}$.

Proof. — All the conditions of the definition of Long Orbit are automatically verified except that $\sigma_1(\beta) \subset \beta_M$ for some $M \in \mathbb{R}^+$. We consider $\beta' = \sigma_1(\beta) \cap \beta_M$. Since β contains the curves of neutral parameters of ϕ, the set $\sigma_1(\beta)$ contains the curves of neutral parameters of η. Moreover such curves are contained in β_M for $M \gg 1$ by Lemma 4.15. Therefore β' contains the curves of neutral parameters of η for $M \gg 1$.

We denote $T' = T \circ \sigma_1^{-1}$. Next we apply Remark 4.20 to β'. We obtain

$$\psi_-^{\eta}(\eta^{\lceil T'(x) \rceil}(\sigma(\upsilon_{\mathcal{O}}(x)))) - \psi_+^{\eta}(\sigma(\upsilon_{\mathcal{O}}(x))) = \lceil T'(x) \rceil - \frac{\pi \mathrm{i}}{\sqrt{x}} + \pi \mathrm{i}\, b_\eta(x) \tag{4.7}$$

for any $x \in \beta'$ in a neighborhood of 0. The points of the form $\eta^{\lceil T'(x) \rceil}(\sigma(\upsilon(x)))$ are in the neighborhood of the compact subset $(\sigma_0 \circ \chi_{\mathcal{O}})([0,1] + \mathrm{i}\,\mathcal{S}_{\mathcal{O}})$ of $\{0\} \times (B(0,\epsilon) \setminus \{0\})$ when $x \in \beta'$ is close to 0. As a consequence the left hand side of (4.7) is bounded for $x \in \beta'$ close to 0 independently of $M > 0$. Hence $|\mathrm{Im}(\pi \mathrm{i}/\sqrt{x})| < C$ for any $x \in \beta'$ close to 0 and C does not depend on M. By considering $M > C$ we obtain $\sigma_1(\beta) \subset \beta_M$. □

Long Orbits are invariant under translations in Fatou coordinates.

Proposition 4.22. — *Let $\phi \in \mathrm{Diff}(\mathbb{C}^2, 0)$ be a saddle-node unfolding in prepared form* (3.2). *Consider a Long Orbit $\mathcal{O} = (\phi, y_+, \beta, T)$. Let y_+' be a point in $\mathcal{P}_+$ such that there exists $\chi' : [0,1] + \mathrm{i}\,\mathcal{S} \to \mathcal{P}_-$ satisfying*

$$\psi_-^{\phi}(\chi'(z)) - \psi_+^{\phi}(0, y_+') = \psi_-^{\phi}(\chi_{\mathcal{O}}(z)) - \psi_+^{\phi}(0, y_+)$$

for any $z \in [0,1] + \mathrm{i}\,\mathcal{S}_{\mathcal{O}}$. Then $\mathcal{O}' = (\phi, y_+', \beta, T)$ is a Long Orbit such that $\chi_{\mathcal{O}'} \equiv \chi'$.

The analogue propery for saddle-node arcs is described in [14]. As a consequence they show that any topological conjugacy also conjugates the infinitesimal generators of the restriction of the saddle-node arcs to 0. Analogously we will obtain that a topological conjugacy σ between saddle-node unfoldings ϕ, η, also conjugates translations in Fatou coordinates of ϕ_0 and η_0. We will deduce that σ_0 is a Fatou affine conjugacy (Proposition 5.1).

Proof. — We have

$$\psi_+^{\phi}(\phi^{\lceil T(x) \rceil}(\upsilon_{\mathcal{O}}(x))) - \psi_+^{\phi}(\upsilon_{\mathcal{O}}(x)) = \lceil T(x) \rceil$$

for any $x \in \beta$ in a neighborhood of 0. We denote $d = \psi_+^\phi(0, y_+') - \psi_+^\phi(0, y_+)$. We are going to translate the orbit of $\upsilon_{\mathcal{O}}(x)$ by d in the coordinate ψ_+^ϕ. We define continuous sections $\upsilon_{\mathcal{O}'} : \beta \cup \{0\} \to \mathbb{C}^2$ and $\alpha : \beta \to \mathbb{C}^2$ such that $\upsilon_{\mathcal{O}'}(0) = (0, y_+')$, $\alpha(x)$ is in the neighborhood of $\mathcal{P}_-$ and

$$\psi_+^\phi(\upsilon_{\mathcal{O}'}(x)) - \psi_+^\phi(\upsilon_{\mathcal{O}}(x)) = d \ \text{ and } \ \psi_-^\phi(\alpha(x)) - \psi_-^\phi(\phi^{\lceil T(x) \rceil}(\upsilon_{\mathcal{O}}(x))) = d$$

for any $x \in \beta$ in a neighborhood of 0. Notice that in the left hand side of the previous equations we can replace ψ_-^ϕ with ψ_+^ϕ or vice versa since $\psi_-^\phi - \psi_+^\phi$ depends only on x. The hypothesis on $\chi_{\mathcal{O}}$ implies that α is well-defined and satisfies

$$\lim_{\substack{\lceil T(x) \rceil - T(x) \to s \\ \vartheta(x) \to u, \ x \to 0}} \alpha(x) = \chi'(s + \mathrm{i}\, u)$$

for all $s \in [0,1]$ and $u \in \mathcal{S}_{\mathcal{O}}$. Our choice of α and $\upsilon_{\mathcal{O}'}$ implies

$$\psi_+^\phi(\alpha(x)) - \psi_+^\phi(\upsilon_{\mathcal{O}'}(x)) = \psi_+^\phi(\phi^{\lceil T(x) \rceil}(\upsilon_{\mathcal{O}}(x))) - \psi_+^\phi(\upsilon_{\mathcal{O}}(x)) = \lceil T(x) \rceil$$

for any $x \in \beta$ in a neighborhood of 0. It follows $\alpha(x) = \phi^{\lceil T(x) \rceil}(\upsilon_{\mathcal{O}'}(x))$ for any $x \in \beta$ in a neighborhood of 0. Since $\psi_+^\phi(\phi^j(\upsilon_{\mathcal{O}'}(x)) = \psi_+^\phi(\phi^j(\upsilon_{\mathcal{O}}(x)) + d$ for all $x \in \beta$ close to 0 and $0 \leqslant j \leqslant \lceil T(x) \rceil + 1$ and $\psi^\phi - \psi^Y$ is bounded, we deduce that there exists $C > 0$ such that

$$|\psi_+^Y(\phi^j(\upsilon_{\mathcal{O}'}(x)) - \psi_+^Y(\phi^j(\upsilon_{\mathcal{O}}(x))| < C.$$

More precisely there exists $\epsilon_0 > 0$ such that if $x \in \beta \cap B(0, \epsilon_0)$, $0 \leqslant j \leqslant \lceil T(x) \rceil + 1$ and $|(y \circ \phi^j)(\upsilon_{\mathcal{O}}(x))| < \epsilon_0$ then $\phi^j(\upsilon_{\mathcal{O}'}(x)) = \exp(tY)(\phi^j(\upsilon_{\mathcal{O}}(x)))$ holds for some $t \in B(0, C)$. Since $(0,0)$ is a singular point of Y, the second condition defining a Long Orbit is satisfied. □

5. Topological conjugacies

We present some consequences of the existence of Long Orbits and their properties for saddle-node unfoldings. We are interested in the study of the rigidity properties of topological conjugacies at the unperturbed line $x = 0$. More precisely we want to describe the behavior of σ_0 where σ is a homeomorphism conjugating saddle-node unfoldings $\phi, \eta \in \mathrm{Diff}(\mathbb{C}^2, 0)$. Let us remark that we always consider that the topological conjugacy σ preserves the fibration $dx = 0$. In other words σ is of the form $\sigma(x,y) = (\sigma_1(x), \sigma_2(x,y))$. The dynamics of ϕ can be very rich, containing for instance small divisors phenomena. It is then natural to think that conjugating all the dynamics of ϕ, η for the lines $x = cte$ should impose heavy restrictions on the conjugacy at $x = 0$. We prove that σ_0 is a Fatou affine conjugacy and generically holomorphic or anti-holomorphic.

5.1. Fatou affine conjugacies

In the next proposition we analyze the behavior of topological conjugacies in a petal of the unperturbed line $x = 0$.

Proposition 5.1. — *Let $\phi, \eta \in \mathrm{Diff}(\mathbb{C}^2, 0)$ be saddle-node unfoldings. Let $\sigma(x,y) = (\sigma_1(x), \sigma_2(x,y))$ be a local homeomorphism such that $\sigma \circ \phi = \eta \circ \sigma$. Then σ_0 is a local Fatou affine conjugacy.*

Proof. — Up to replace ϕ and η by saddle-unfoldings analytically conjugated to them we can suppose that both are in prepared form (3.2). Consider a petal $\mathcal{P}$ of ϕ_0. We denote by $\psi_{\mathcal{P}}^{\phi}$ a Fatou coordinate of ϕ_0 in $\mathcal{P}$. Given $z \in \mathbb{C}$ and $y \in \mathcal{P}$ there exists at most one point $y' \in \mathcal{P}$ such that $\psi_{\mathcal{P}}^{\phi}(y') - \psi_{\mathcal{P}}^{\phi}(y) = z$. If such a point exists we denote it by $E_{\mathcal{P}}(z,y)$. Let $\psi_{\sigma_0(\mathcal{P})}^{\eta}$ be a Fatou coordinate of η_0 in the petal $\sigma_0(\mathcal{P})$. We define the function

$$\mathfrak{h}_{\mathcal{P}}(y,z) = \psi_{\sigma_0(\mathcal{P})}^{\eta}(\sigma_0(E_{\mathcal{P}}(z,y))) - \psi_{\sigma_0(\mathcal{P})}^{\eta}(\sigma_0(y))$$

whenever it is defined.

Consider the Long Orbit $\mathcal{O}' := (\phi, y'_+, \beta_M, T')$ provided by Proposition 4.18 for $M \geqslant M_0$ and defined in $\mathbb{C} \times B(0,\epsilon)$ where $T' := \mathcal{T} - M'$ for some $M' \in \mathbb{N}$. The equality

$$\psi_{-}^{\phi}(\chi_{\mathcal{O}'}(z)) - \psi_{-}^{\phi}(\chi_{\mathcal{O}'}(0)) = z \tag{5.1}$$

holds for any $z \in [0,1] + \mathrm{i}\,\mathbb{R}$ ((4.6) in Remark 4.19). Given $w \in \mathbb{C}$ close to 0, $\mathcal{O}'_w := (\phi, E_{\mathcal{P}_+}(w, y'_+), \beta_M, T')$ is a Long Orbit in $\mathbb{C} \times B(0,\epsilon)$ for any $M \geqslant M_0$ where $\chi_{\mathcal{O}'_w}(z) = E_{\mathcal{P}_-}(w, \chi_{\mathcal{O}'}(z))$ for any $z \in [0,1] + \mathrm{i}\,\mathbb{R}$ by Proposition 4.22. As a consequence $\tilde{\mathcal{O}}_w := (\eta, \sigma_0(E_{\mathcal{P}_+}(w, y'_+)), \sigma_1(\beta_M), T' \circ \sigma_1^{-1})$ is a Long Orbit for any $M \geqslant M_0$ by Proposition 4.21. Proposition 4.22 implies

$$\mathfrak{h}_{\mathcal{P}_+}(y'_+, w) = \mathfrak{h}_{\mathcal{P}_-}(\chi_{\mathcal{O}'}(z), w) \tag{5.2}$$

for all $w \in \mathbb{C}$ close to 0 and $z \in [0,1] + \mathrm{i}\,\mathbb{R}$. Since $\mathfrak{h}_{\mathcal{P}_-}(\phi_0(y), w) \equiv \mathfrak{h}_{\mathcal{P}_-}(y, w)$ and every orbit of ϕ in $\mathcal{P}_-$ intersects $\chi_{\mathcal{O}'}([0,1] + \mathrm{i}\,\mathbb{R})$ by (5.1), we obtain that $\mathfrak{h}_{\mathcal{P}_-}(y, w)$ does not depend on $y \in \mathcal{P}_-$ for any w in a neighborhood $B(0,c)$ of 0 in $\mathbb{C}$. Consider $z \in \mathbb{C}$, it satisfies $|z/m| < c$ for some $m \in \mathbb{N}$. Since

$$\mathfrak{h}_{\mathcal{P}_-}(y,z) = \sum_{j=1}^{m} \mathfrak{h}_{\mathcal{P}_-}\left(E_{\mathcal{P}_-}\left((j-1)\frac{z}{m}, y\right), \frac{z}{m}\right)$$

holds, $\mathfrak{h}_{\mathcal{P}_-}(y,z)$ does not depend on $y \in \mathcal{P}_-$ for any $z \in \mathbb{C}$. We have

$$\begin{aligned}\mathfrak{h}_{\mathcal{P}_-}(z_1+z_2) = \mathfrak{h}_{\mathcal{P}_-}(y, z_1+z_2) &= \mathfrak{h}_{\mathcal{P}_-}(E_{\mathcal{P}_-}(z_2,y), z_1) + \mathfrak{h}_{\mathcal{P}_-}(y, z_2) \\ &= \mathfrak{h}_{\mathcal{P}_-}(z_1) + \mathfrak{h}_{\mathcal{P}_-}(z_2)\end{aligned}$$

for any $z_1, z_2 \in \mathbb{C}$. Since $\mathfrak{h}_{\mathcal{P}_-}$ is continuous by definition, $\mathfrak{h}_{\mathcal{P}_-}$ is $\mathbb{R}$-linear. The function $\mathfrak{h}_{\mathcal{P}_-}$ is injective in a neighborhood of 0, hence $\mathfrak{h}_{\mathcal{P}_-}$ is an isomorphism. Thus $\sigma_{0|\mathcal{P}_-}$ is Fatou affine. Since σ conjugates ϕ^{-1} and η^{-1} and $\mathcal{P}_+$ is the repelling petal of ϕ_0^{-1}, we obtain that $\sigma_{0|\mathcal{P}_+}$ is Fatou affine analogously. □

Remark 5.2. — We obtain $\mathfrak{h}_{\mathcal{P}_-} = \mathfrak{h}_{\mathcal{P}_+}$ by Lemma 2.29. It could also be deduced from (5.2).

Remark 5.3. — Analogously as in [14] we use the property in Proposition 4.22 to show that σ_0 conjugates translations in Fatou coordinates (this is equivalent to $\mathfrak{h}$ not depending on y) and a consequence $\mathfrak{h}$ is a $\mathbb{R}$-linear map such that $\mathfrak{h}(1) = 1$. In the case of saddle-node arcs where $\mathfrak{h} : \mathbb{R} \to \mathbb{R}$ this implies $\mathfrak{h} \equiv \mathrm{Id}$ and hence σ_0 conjugates the infinitesimal generators of ϕ_0 and η_0 [14]. Since in our case we still have $\mathfrak{h}(s) = s$ for any $s \in \mathbb{R}$, it follows that σ_0 conjugates the real flows of the Lavaurs vector fields $Y^{\phi_0}_{\mathcal{P}_\pm}$, $Y^{\eta_0}_{\sigma_0(\mathcal{P}_\pm)}$ of ϕ_0 and η_0 in the attracting and repelling petals. Anyway, in general it does not conjugate the complex flows since $\mathfrak{h}(1) = 1$ does not imply $\mathfrak{h} \equiv \mathrm{Id}$. For instance we could have $\mathfrak{h}(z) = 2z - \overline{z}$.

Definition 5.4. — *Let $\phi, \eta \in \mathrm{Diff}(\mathbb{C}^2, 0)$ be saddle-node unfoldings. Let $\sigma(x,y) = (\sigma_1(x), \sigma_2(x,y))$ be a local homeomorphism such that $\sigma \circ \phi = \eta \circ \sigma$. We say that the action of σ on the parameter space is holomorphic (resp. anti-holomorphic, orientation-preserving) if σ_1 is holomorphic (resp. anti-holomorphic, orientation-preserving).*

The orientation properties of the restriction of the conjugating map to $x = 0$ and of its action on the parameter space are the same.

Lemma 5.5. — *Let $\phi, \eta \in \mathrm{Diff}(\mathbb{C}^2, 0)$ be saddle-node unfoldings. Let σ be a local homeomorphism such that $\sigma \circ \phi = \eta \circ \sigma$. Then the following properties are equivalent:*

(1) *σ_0 is orientation-preserving,*
(2) *$\mathfrak{h}_{\phi_0,\eta_0,\sigma_0}$ is orientation-preserving,*
(3) *the action of σ on the parameter space is orientation-preserving.*

In particular σ is orientation-preserving.

Proof. — The map σ_0 is a Fatou affine conjugacy by Proposition 5.1. Thus the first two properties are equivalent (Remark 2.31). Up to replace

ϕ and η by saddle-node unfoldings analytically conjugated to them we can suppose that both are in prepared form (3.2). Suppose that σ does not preserve orientation in the parameter space. We denote

$$\zeta(x,y) = (\overline{x}, \overline{y}), \ \ \tilde{\eta} = \zeta \circ \eta \circ \zeta \ \text{ and } \ \tilde{\sigma} = \zeta \circ \sigma$$

where $\overline{x}$ is the complex conjugation. The diffeomorphism $\tilde{\eta}$ is a saddle-node unfolding and $\tilde{\sigma} \circ \phi = \tilde{\eta} \circ \tilde{\sigma}$. The action of $\tilde{\sigma}$ on the parameter space is orientation-preserving. We have

$$\mathfrak{h}_{\phi_0, \tilde{\eta}_0, \tilde{\sigma}_0}(z) = \overline{\mathfrak{h}_{\phi_0, \eta_0, \sigma_0}(z)}$$

for any $z \in \mathbb{C}$ by Remark 2.32 and Lemma 2.33. Therefore it suffices to prove that if the action of σ in the parameter space is orientation-preserving so is $\mathfrak{h} = \mathfrak{h}_{\phi_0, \eta_0, \sigma_0}$.

Consider the notations in Proposition 5.1. We have

$$\begin{aligned}\psi^{\eta}_{\sigma_0(\mathcal{P}_-)}(\chi_{\tilde{\mathcal{O}}_0}(z)) - \psi^{\eta}_{\sigma_0(\mathcal{P}_+)}(\sigma(0, y'_+)) \\ = \lim_{\substack{\lceil T'(x) \rceil - T'(x) \to s \\ \vartheta_{\mathcal{O}'}(x) \to u, \ x \to 0}} \left(\lceil T'(x) \rceil - \frac{\pi \mathrm{i}}{\sqrt{\sigma_1(x)}} + \pi \mathrm{i}\, b_\eta(0) \right)\end{aligned}$$

for any $z = s + \mathrm{i}\, u \in [0,1] + \mathrm{i}\mathbb{R}$ by Proposition 4.18. Since $\mathfrak{h}_{|\mathbb{R}} \equiv \mathrm{Id}$, $\chi_{\tilde{\mathcal{O}}_0} \equiv \sigma \circ \chi_{\mathcal{O}'}$ and $\psi^{\phi}_{\mathcal{P}_-}(\chi_{\mathcal{O}'}(z)) - \psi^{\phi}_{\mathcal{P}_-}(\chi_{\mathcal{O}'}(0)) \equiv z$ we deduce that

$$\lim_{x \in \vartheta^{-1}_{\mathcal{O}'}(u), \ x \to 0} -\mathrm{Im}\left(\frac{\pi \mathrm{i}}{\sqrt{\sigma_1(x)}} \right) = c_u \tag{5.3}$$

$$\text{and} \quad \lim_{x \in \vartheta^{-1}_{\mathcal{O}'}(u), \ x \to 0} \mathrm{Re}\left(\frac{\pi \mathrm{i}}{\sqrt{\sigma_1(x)}} \right) = \infty.$$

where $c_u = \mathrm{Im}(\psi^{\eta}_{\sigma_0(\mathcal{P}_-)}(\sigma(\chi_{\mathcal{O}'}(\mathrm{i}\, u))) - \psi^{\eta}_{\sigma_0(\mathcal{P}_+)}(\sigma(0, y'_+)) - \pi \mathrm{i}\, b_\eta(0))$. We obtain

$$\lim_{x \in \vartheta^{-1}_{\mathcal{O}'}(u), \ x \to 0} -\mathrm{Im}\left(\frac{\pi \mathrm{i}}{\sqrt{\sigma_1(x)}} \right) - \lim_{x \in \vartheta^{-1}_{\mathcal{O}'}(0), \ x \to 0} -\mathrm{Im}\left(\frac{\pi \mathrm{i}}{\sqrt{\sigma_1(x)}} \right) = \mathrm{Im}(\mathfrak{h}(\mathrm{i}\, u))$$

for any $u \in \mathbb{R}$.

The curves

$$\vartheta^{-1}_{\mathcal{O}'}(u) = \left\{ -\mathrm{Im}\left(\frac{\pi \mathrm{i}}{\sqrt{x}} \right) = u \quad \text{and} \quad \mathrm{Re}\left(\frac{\pi \mathrm{i}}{\sqrt{x}} \right) \in \mathbb{R}^+ \right\}$$

move in counterclockwise sense when we increase $u \in \mathbb{R}$. Since σ_1 is orientation-preserving, $\sigma_1(\vartheta^{-1}_{\mathcal{O}'}(u))$ move in counterclockwise sense when we increase u. (5.3) implies that the curves $\vartheta^{-1}_{\mathcal{O}'}(c_u)$ move in counterclockwise sense when u increases. Hence c_u is an increasing function of u. We obtain

$$\mathrm{Im}(\mathfrak{h}(\mathrm{i})) = c_1 - c_0 > 0.$$

As a consequence $\mathfrak{h}$ is orientation-preserving. □

Proof of Theorem 1.4. — The map σ_0 is a Fatou affine conjugacy by Proposition 5.1. Lemma 5.5 implies that σ_0 is orientation-preserving if and only if the action of σ in the parameter space is orientation-preserving. In particular σ is orientation-preserving. □

Proof of the Main Theorem (Theorem 1.3). — The map σ_0 is a Fatou affine conjugacy by the General Theorem 1.4. Hence σ_0 is holomorphic or anti-holomorphic by Proposition 2.34. □

Proof of Proposition 1.5. — Suppose that the action of σ on the parameter space is holomorphic. Consider the notations in Proposition 5.1. We have

$$\psi^{\phi}_{\mathcal{P}_-}(\chi_{\mathcal{O}'}(z)) - \psi^{\phi}_{\mathcal{P}_+}(0, y'_+) = \lim_{\substack{\lceil T'(x)\rceil - T'(x) \to s \\ \vartheta_{\mathcal{O}'}(x)\to u,\ x\to 0}} \left(\lceil T'(x) \rceil - \frac{\pi\,\mathrm{i}}{\sqrt{x}} + \pi\,\mathrm{i}\, b_{\phi}(0) \right)$$

and

$$\begin{aligned} &\psi^{\eta}_{\sigma_0(\mathcal{P}_-)}(\chi_{\tilde{\mathcal{O}}_0}(z)) - \psi^{\eta}_{\sigma_0(\mathcal{P}_+)}(\sigma(0, y'_+)) \\ &\qquad = \lim_{\substack{\lceil T'(x)\rceil - T'(x) \to s \\ \vartheta_{\mathcal{O}'}(x)\to u,\ x\to 0}} \left(\lceil T'(x) \rceil - \frac{\pi\,\mathrm{i}}{\sqrt{\sigma_1(x)}} + \pi\,\mathrm{i}\, b_{\eta}(0) \right) \end{aligned}$$

for any $z = s + \mathrm{i}\, u \in [0,1] + \mathrm{i}\,\mathbb{R}$ by Proposition 4.18. Thus the function

$$G(x) := \frac{\pi\,\mathrm{i}}{\sqrt{\sigma_1(x)}} - \frac{\pi\,\mathrm{i}}{\sqrt{x}}$$

is bounded in β_M for any $M \geqslant M_0$. Since $G(x^2)$ is meromorphic, the limit $\lim_{x\to 0} G(x)$ exists and belongs to $\mathbb{C}$. Hence there exists a constant $C \in \mathbb{C}$ such that

$$\psi^{\eta}_{\sigma_0(\mathcal{P}_-)}(\sigma(\chi_{\mathcal{O}'}(z))) - \psi^{\phi}_{\mathcal{P}_-}(\chi_{\mathcal{O}'}(z)) = C$$

for any $z \in [0,1] + \mathrm{i}\,\mathbb{R}$. We obtain

$$\psi^{\eta}_{\sigma_0(\mathcal{P}_-)}(\sigma(\chi_{\mathcal{O}'}(z))) - \psi^{\eta}_{\sigma_0(\mathcal{P}_-)}(\sigma(\chi_{\mathcal{O}'}(0))) = \psi^{\phi}_{\mathcal{P}_-}(\chi_{\mathcal{O}'}(z)) - \psi^{\phi}_{\mathcal{P}_-}(\chi_{\mathcal{O}'}(0)) = z$$

for $z \in [0,1] + \mathrm{i}\,\mathbb{R}$. Since $\psi^{\phi}_{\mathcal{P}_-}(\chi_{\mathcal{O}'}(z)) - \psi^{\phi}_{\mathcal{P}_-}(\chi_{\mathcal{O}'}(0)) = z$, it follows $\mathfrak{h}(z) = z$ for any $z \in [0,1] + \mathrm{i}\,\mathbb{R}$. Since $\mathfrak{h}(z+1) \equiv \mathfrak{h}(z) + 1$, we get $\mathfrak{h}(z) = z$ for any $z \in \mathbb{C}$. Hence σ_0 is holomorphic by Lemma 2.33.

Suppose that the action of σ on the parameter space is anti-holomorphic. Denote $\zeta(x, y) = (\overline{x}, \overline{y})$, $\tilde{\eta} = \zeta \circ \eta \circ \zeta$ and $\tilde{\sigma} = \zeta \circ \sigma$. We have $\tilde{\sigma} \circ \phi = \tilde{\eta} \circ \tilde{\sigma}$.

The action of $\tilde{\sigma}$ on the parameter space is holomorphic. Therefore $\tilde{\sigma}_0$ is holomorphic and σ_0 is anti-holomorphic. □

Proof of Proposition 1.6. — The map σ_0 is a Fatou affine conjugacy by Theorem 1.4. Hence $\mathrm{Re}(\mathrm{Res}_{\phi_0})$ and $\mathrm{Re}(\mathrm{Res}_{\eta_0})$ have the same sign by Proposition 2.36. □

Proof of Proposition 1.7. — The map σ_0 is a Fatou affine conjugacy by the General Theorem. The result is a consequence of Proposition 2.39. □

Proof of Proposition 1.8. — The map σ_0 is a Fatou affine conjugacy by the General Theorem. The result is a consequence of Proposition 2.37 and Lemma 5.5. □

6. Building examples

Let us construct topological conjugacies between real flows of vector fields of the form

$$\frac{y^2 - x}{1 + a(x)y} \frac{\partial}{\partial y}$$

whose restrictions to $x = 0$ are neither holomorphic nor anti-holomorphic. This section provides examples for the exceptional cases covered by Propositions 1.7 and 1.8.

6.1. Description of the construction

Consider complex numbers $a, b \in \mathbb{C}$ and a $\mathbb{R}$-linear orientation-preserving isomorphism $\mathfrak{h} : \mathbb{C} \to \mathbb{C}$ such that $\mathfrak{h}(1) = 1$ and $\mathfrak{h}(2\pi\,\mathrm{i}\,a) = 2\pi\,\mathrm{i}\,b$. In particular $\mathrm{Re}(a)$ and $\mathrm{Re}(b)$ have the same sign. Denote

$$Y = \frac{y^2 - x}{1 + ay} \frac{\partial}{\partial y} \quad \text{and} \quad Z = \frac{y^2 - x}{1 + by} \frac{\partial}{\partial y}.$$

We define

$$\mathrm{Res}_Y(x, \sqrt{x}) = \frac{1}{2}\left(\frac{1}{\sqrt{x}} + a\right), \quad \mathrm{Res}_Y(x, -\sqrt{x}) = \frac{1}{2}\left(\frac{-1}{\sqrt{x}} + a\right).$$

Notice that $\mathrm{Res}_Y(x_0, \pm\sqrt{x_0})$ is the residue of the dual form

$$\omega_Y = \frac{1 + ay}{y^2 - x} dy$$

restricted to $x = x_0$ at the point $\pm\sqrt{x_0}$. Analogously we can define

$$\mathrm{Res}_Z(x, \sqrt{x}) = \frac{1}{2}\left(\frac{1}{\sqrt{x}} + b\right), \quad \mathrm{Res}_Z(x, -\sqrt{x}) = \frac{1}{2}\left(\frac{-1}{\sqrt{x}} + b\right).$$

Consider a germ of homeomorphism σ_1 defined in a neighborhood of 0 in $\mathbb{C}$ such that $\mathfrak{h}(\pi \mathrm{i}/\sqrt{x}) = \pi \mathrm{i}/\sqrt{\sigma_1(x)}$ for any x defined in a neighborhood of 0. In this way we obtain

$$\mathfrak{h}(2\pi \mathrm{i} \operatorname{Res}_Y(x, \pm\sqrt{x})) = 2\pi \mathrm{i} \operatorname{Res}_Z(\sigma_1(x), \pm\sqrt{\sigma_1(x)}) \tag{6.1}$$

for any x in a neighborhood of 0. The isomorphism $\mathfrak{h}$ is of the form $\mathfrak{h}(z) = \varsigma_0 z + \varsigma_1 \overline{z}$ with $|\varsigma_0| > |\varsigma_1|$. We obtain

$$\frac{\varsigma_0}{\sqrt{x}} - \frac{\varsigma_1}{\overline{\sqrt{x}}} \equiv \frac{1}{\sqrt{\sigma_1(x)}} \implies \frac{\sqrt{x}}{\sqrt{\sigma_1(x)}} \equiv \varsigma_0 - \varsigma_1 \frac{\sqrt{x}}{\overline{\sqrt{x}}}. \tag{6.2}$$

Fix a point $y_0 \in B(0,\epsilon) \setminus \{0\}$ close to 0. Let ψ_Y, ψ_Z be Fatou coordinates of Y, Z such that $\psi_Y(x, y_0) \equiv \psi_Z(x, y_0) \equiv 0$. We want to define a homeomorphism σ conjugating $\operatorname{Re}(Y)$ and $\operatorname{Re}(Z)$ such that

- σ is of the form $\sigma(x,y) = (\sigma_1(x), \sigma_2(x,y))$.
- $\sigma(x, y_0) = (\sigma_1(x), y_0)$ for any x in a neighborhood of 0.
- $\psi_Z \circ \sigma \circ \exp(zY) - \psi_Z \circ \sigma \equiv \mathfrak{h}(z)$ for any $z \in \mathbb{C}$.

Let us remark that the monodromies of $(\mathfrak{h} \circ \psi_Y)(x_0, y)$ and $\psi_Z(\sigma_1(x_0), y)$ around $(x_0, \pm\sqrt{x_0})$ and $(\sigma_1(x_0), \pm\sqrt{\sigma_1(x_0)})$ respectively coincide by (6.1). The natural way of defining σ is by using the equation $\mathfrak{h} \circ \psi_Y \equiv \psi_Z \circ \sigma$.

6.2. The method of the path

In order to prove the existence of σ satisfying $\mathfrak{h} \circ \psi_Y \equiv \psi_Z \circ \sigma$ we apply the method of the path (cf. [13, 20]). First we relocate the points in $\operatorname{Sing}(Z)$ by considering the change of coordinates $y = y\sqrt{\sigma_1(x)/x}$ for the parameter $\sigma_1(x)$. This corresponds to the change of coordinates $\tilde{\sigma}(x,y) = (x, y\sqrt{x/\sigma_1^{-1}(x)})$. We obtain

$$\begin{aligned} Z(\sigma_1(x), y) &= \frac{\left(\frac{\sqrt{\sigma_1(x)}}{\sqrt{x}} y\right)^2 - \sigma_1(x)}{1 + by\sqrt{\sigma_1(x)}/\sqrt{x}} \frac{\sqrt{x}}{\sqrt{\sigma_1(x)}} \frac{\partial}{\partial y} \\ &= \frac{\sqrt{\sigma_1(x)}}{\sqrt{x}} \frac{y^2 - x}{1 + by\sqrt{\sigma_1(x)}/\sqrt{x}} \frac{\partial}{\partial y} \end{aligned}$$

in the new coordinates. Let us point out that the change of coordinates $\tilde{\sigma}$ is not well-defined at $x = 0$. It is not very pathological either since $\sqrt{x/\sigma_1^{-1}(x)}$ is bounded away from 0 and ∞ for x in a neighborhood of 0 (cf. (6.2)).

Let us consider the function

$$\Psi = (1-s)(\mathfrak{h} \circ \psi_Y)(x,y) + s\psi_Z\left(\sigma_1(x), \frac{\sqrt{\sigma_1(x)}}{\sqrt{x}}y\right) = \Psi_1 + \mathrm{i}\,\Psi_2.$$

In general Ψ is neither holomorphic nor anti-holomorphic. Roughly speaking all the functions $\Psi(s_0,x,y)$ have the same poles and the monodromy around those poles does not depend on s_0. We are trying to conjugate $\mathrm{Re}(Y)$ and $\tilde{\sigma}^*(\mathrm{Re}(Z))$. In order to do this we want to find a continuous vector field W defined in coordinates (x,y,s) in a neighborhood of $\{(0,0)\} \times [0,1]$ in $((\mathbb{C}^* \times \mathbb{C}) \cup \{(0,0)\}) \times \mathbb{C}$ of the form

$$W = \frac{\partial}{\partial s} + c_1(x,y,s)\frac{\partial}{\partial y_1} + c_2(x,y,s)\frac{\partial}{\partial y_2}$$

where $y = y_1 + \mathrm{i}\,y_2$. Moreover we require W to satisfy

- $W(\Psi) \equiv 0$.
- c_j is a continuous function defined in a neighborhood of $\{(0,0)\} \times [0,1]$ in $((\mathbb{C}^* \times \mathbb{C}) \cup \{(0,0)\}) \times \mathbb{C}$ and vanishing at $y^2 - x = 0$ for any $j \in \{1,2\}$.

The idea is that for such W the map $\exp(W)(x,y,0)$ conjugates $\mathrm{Re}(Y)$ and $\tilde{\sigma}^*(\mathrm{Re}(Z))$ in a neighborhood of $(0,0)$ in $(\mathbb{C}^* \times \mathbb{C}) \cup \{(0,0)\}$. We define

$$\rho = (\mathfrak{h} \circ \psi_Y)(x,y) - \psi_Z\left(\sigma_1(x), \frac{\sqrt{\sigma_1(x)}}{\sqrt{x}}y\right).$$

The equation $W(\Psi) \equiv 0$ is equivalent to

$$\begin{cases} c_1\frac{\partial \Psi_1}{\partial y_1} + c_2\frac{\partial \Psi_1}{\partial y_2} = \mathrm{Re}(\rho) \\ c_1\frac{\partial \Psi_2}{\partial y_1} + c_2\frac{\partial \Psi_2}{\partial y_2} = \mathrm{Im}(\rho). \end{cases}$$

The solutions are

$$c_1 = \frac{\det\begin{pmatrix} \mathrm{Re}(\rho) & \frac{\partial \Psi_1}{\partial y_2} \\ \mathrm{Im}(\rho) & \frac{\partial \Psi_2}{\partial y_2} \end{pmatrix}}{\det\begin{pmatrix} \frac{\partial \Psi_1}{\partial y_1} & \frac{\partial \Psi_1}{\partial y_2} \\ \frac{\partial \Psi_2}{\partial y_1} & \frac{\partial \Psi_2}{\partial y_2} \end{pmatrix}}, \quad c_2 = \frac{\det\begin{pmatrix} \frac{\partial \Psi_1}{\partial y_1} & \mathrm{Re}(\rho) \\ \frac{\partial \Psi_2}{\partial y_1} & \mathrm{Im}(\rho) \end{pmatrix}}{\det\begin{pmatrix} \frac{\partial \Psi_1}{\partial y_1} & \frac{\partial \Psi_1}{\partial y_2} \\ \frac{\partial \Psi_2}{\partial y_1} & \frac{\partial \Psi_2}{\partial y_2} \end{pmatrix}}. \tag{6.3}$$

The denominator D of the previous expressions satisfies $D = |\partial\Psi/\partial y|^2 - |\partial\Psi/\partial\overline{y}|^2$. We have

$$\begin{cases} \frac{\partial \Psi}{\partial y} = (1-s)\varsigma_0\frac{1+ay}{y^2-x} + s\frac{\sqrt{x}}{\sqrt{\sigma_1(x)}}\frac{1+by\sqrt{\sigma_1(x)}/\sqrt{x}}{y^2-x} \\ \frac{\partial \Psi}{\partial \overline{y}} = (1-s)\varsigma_1\overline{\left(\frac{1+ay}{y^2-x}\right)}. \end{cases}$$

Lemma 6.1. — *The function $D|y^2-x|^2$ is bounded away from 0 and ∞.*

Proof. — Consider the function $\kappa = (1-s)\varsigma_0 + s\sqrt{x}/\sqrt{\sigma_1(x)}$. It suffices to prove that κ is bounded by below for x in a neighborhood of 0 and s in a neighborhood of $[0,1]$ and that we have $|\kappa(x,s)|/|(1-s)\varsigma_1| \geqslant C > 1$ for some constant $C \in \mathbb{R}^+$. Notice that κ is bounded by above. The property $\sqrt{x}/\sqrt{\sigma_1(x)} \equiv \varsigma_0 - \varsigma_1\sqrt{x}/\overline{\sqrt{x}}$ implies

$$|\kappa(x,s)| = \left|\varsigma_0 - s\varsigma_1\frac{\sqrt{x}}{\overline{\sqrt{x}}}\right| \geqslant |\varsigma_0| - |\varsigma_1| > 0$$

for all $s \in [0,1]$ and $x \in \mathbb{C}^*$ in a neighborhood of 0. We have

$$|\kappa(x,s)|^2 = (1-s)^2|\varsigma_0|^2 + s^2\left|\frac{\sqrt{x}}{\sqrt{\sigma_1(x)}}\right|^2 + 2s(1-s)\operatorname{Re}\left(\varsigma_0\overline{\frac{\sqrt{x}}{\sqrt{\sigma_1(x)}}}\right)$$

and

$$\varsigma_0\overline{\frac{\sqrt{x}}{\sqrt{\sigma_1(x)}}} = \varsigma_0\left(\overline{\varsigma_0} - \overline{\varsigma_1}\frac{\overline{\sqrt{x}}}{\sqrt{x}}\right) \implies \operatorname{Re}\left(\varsigma_0\overline{\frac{\sqrt{x}}{\sqrt{\sigma_1(x)}}}\right) \geqslant |\varsigma_0|^2 - |\varsigma_0||\varsigma_1| > 0.$$

We deduce that $|\kappa| \geqslant |(1-s)\varsigma_0| \geqslant C|(1-s)\varsigma_1|$ for some constant $C > 1$. $\square$

The next step of the proof is showing that $(y^2-x)\rho$ can be extended as a continuous function vanishing at $y^2 - x = 0$. We define the auxiliary functions

$$R_{\pm} = \frac{\pm 1}{2\sqrt{x}}\left(\varsigma_0 - \frac{\sqrt{x}}{\sqrt{\sigma_1(x)}}\right) + \frac{1}{2}(\varsigma_0 a - b)$$

and

$$\tilde{\rho} = R_+(x)\ln|y-\sqrt{x}|^2 + R_-(x)\ln|y+\sqrt{x}|^2.$$

Notice that $\mathfrak{h}(2\pi\,\mathrm{i}\,a) = 2\pi\,\mathrm{i}\,b$ implies $b = \varsigma_0 a - \varsigma_1\overline{a}$. We have

$$\frac{\partial\tilde{\rho}}{\partial y} = \frac{R_+(x)(y+\sqrt{x}) + R_-(x)(y-\sqrt{x})}{y^2-x} = \frac{\partial\rho}{\partial y}$$

and

$$\frac{\partial\tilde{\rho}}{\partial\overline{y}} = \frac{(\varsigma_0 a - b)\overline{y} + \varsigma_1}{\overline{y^2-x}} = \varsigma_1\overline{\left(\frac{1+ay}{y^2-x}\right)} = \frac{\partial\rho}{\partial\overline{y}}$$

by (6.2).

Lemma 6.2. — *The function $\rho - \tilde{\rho}$ is a bounded function of x.*

Proof. — It is obvious that $\rho - \tilde{\rho}$ is constant in each line $x = x_0$. Since $\rho(x,y_0)$ is bounded by construction, it suffices to show that $\tilde{\rho}(x,y_0)$ is bounded in the neighborhood of 0. We have

$$\tilde{\rho}(x,y) = \frac{1}{2\sqrt{x}}\left(\varsigma_0 - \frac{\sqrt{x}}{\sqrt{\sigma_1(x)}}\right)\ln\left|\frac{y-\sqrt{x}}{y+\sqrt{x}}\right|^2 + (\varsigma_0 a - b)\ln|y^2-x|.$$

It suffices to prove that given

$$\hat{\rho}(x,y) := \frac{1}{2\sqrt{x}} \left(\varsigma_0 - \frac{\sqrt{x}}{\sqrt{\sigma_1(x)}} \right) \ln \left| \frac{y-\sqrt{x}}{y+\sqrt{x}} \right|^2$$

the function $\hat{\rho}(x,y_0)$ is bounded. We have

$$\hat{\rho}(x,y_0) = \frac{1}{2\sqrt{x}} \left(\varsigma_0 - \frac{\sqrt{x}}{\sqrt{\sigma_1(x)}} \right) \ln \left(1 - 2 \frac{\sqrt{x}\overline{y_0} + \overline{\sqrt{x}}y_0}{|y_0|^2 + \sqrt{x}\overline{y_0} + \overline{\sqrt{x}}y_0 + |x|} \right)$$

and then

$$\hat{\rho}(x,y_0) \sim - \left(\varsigma_0 - \frac{\sqrt{x}}{\sqrt{\sigma_1(x)}} \right) \left(\frac{1}{y_0} + \frac{1}{\overline{y_0}} \frac{\overline{\sqrt{x}}}{\sqrt{x}} \right)$$

is bounded. □

Lemma 6.3. — *We have* $\lim_{(x,y)\to(x_0,y_0)} \rho(x,y)(y^2-x) = 0$ *for any* (x_0,y_0) *in the curve* $y^2 - x = 0$.

Proof. — It suffices to show the result for $\hat{\rho}$, see the proof of Lemma 6.2. Suppose $(x_0,y_0) = (0,0)$, otherwise the proof is straightforward.

Suppose that $|y/\sqrt{x}| \leqslant 8$. We have

$$|\hat{\rho}(x,y)(y^2-x)| \leqslant \frac{9}{2} \left| \varsigma_0 - \frac{\sqrt{x}}{\sqrt{\sigma_1(x)}} \right| \left(|y-\sqrt{x}| \, |\ln |y-\sqrt{x}|^2| + |y+\sqrt{x}| \, |\ln |y+\sqrt{x}|^2| \right).$$

We deduce $\lim_{|y/\sqrt{x}|\leqslant 8, \ (x,y)\to(0,0)} \rho(x,y)(y^2-x) = 0$.

Suppose that $|y/\sqrt{x}| \geqslant 8$. We have

$$|\hat{\rho}(x,y)(y^2-x)| \leqslant \frac{1}{2\sqrt{|x|}} \left| \varsigma_0 - \frac{\sqrt{x}}{\sqrt{\sigma_1(x)}} \right| |y^2 - x| \left| \ln \left| \frac{y-\sqrt{x}}{y+\sqrt{x}} \right|^2 \right|$$

and then

$$|\hat{\rho}(x,y)(y^2-x)| \leqslant \frac{1}{2\sqrt{|x|}} \left| \varsigma_0 - \frac{\sqrt{x}}{\sqrt{\sigma_1(x)}} \right| |y^2 - x| C \frac{\sqrt{|x|}}{|y|}$$

for some $C \in \mathbb{R}^+$. We obtain

$$|\hat{\rho}(x,y)(y^2-x)| \leqslant \frac{C}{2} \left| \varsigma_0 - \frac{\sqrt{x}}{\sqrt{\sigma_1(x)}} \right| |y| \left| 1 - \frac{x}{y^2} \right| \leqslant \frac{65C}{128} |y| \left| \varsigma_0 - \frac{\sqrt{x}}{\sqrt{\sigma_1(x)}} \right|.$$

We deduce $\lim_{|y/\sqrt{x}|\geqslant 8, \ (x,y)\to(0,0)} \rho(x,y)(y^2-x) = 0$. □

Lemma 6.4. — *The functions* c_1 *and* c_2 *are continuous. They are defined in a neighborhood of* $\{(0,0)\} \times [0,1]$ *in* $((\mathbb{C}^* \times \mathbb{C}) \cup \{(0,0)\}) \times \mathbb{C}$ *and vanish at* $y^2 - x = 0$.

Proof. — Let us prove the result for c_1 without lack of generality. (6.3) and Lemma 6.1 imply that it suffices to show that

$$|y^2 - x|^2 \det \begin{pmatrix} \mathrm{Re}(\rho) & \frac{\partial \Psi_1}{\partial y_2} \\ \mathrm{Im}(\rho) & \frac{\partial \Psi_2}{\partial y_2} \end{pmatrix}$$

tends to 0 when we approach a point of $y^2 - x = 0$. Since $\partial \Psi_j / \partial y_k = O(1/(y^2 - x))$ for all $j, k \in \{1, 2\}$, the property is a consequence of Lemma 6.3. □

6.3. Consequences of the construction

We have all the ingredients required to provide examples of exceptional conjugating maps that are not holomorphic or anti-holomorphic. Hence the condition requiring the unperturbed diffeomorphisms to be non-analytically trivial in the Main Theorem can not be removed.

PROPOSITION 6.5. — *The map $\sigma_\flat(x, y) = \exp(W)(x, y, 0)$ conjugates $\mathrm{Re}(Y)$ and $\tilde{\sigma}^*(\mathrm{Re}(Z))$ in a neighborhood of $(0, 0)$ in $(\mathbb{C}^* \times \mathbb{C}) \cup \{(0, 0)\}$. Moreover the map $\sigma = \tilde{\sigma} \circ \sigma_\flat$ is a homeomorphism conjugating $\mathrm{Re}(Y)$ and $\mathrm{Re}(Z)$ defined in a neighborhood of $(0, 0)$. Moreover σ and σ^{-1} are real analytic outside $x(y^2 - x) = 0$.*

Proof. — It is clear that $\sigma_\flat$ and its inverse $\exp(-W)(x, y, 1)$ satisfy the properties by construction. The map σ conjugates $\mathrm{Re}(Y)$ and $\mathrm{Re}(Z)$. The remaining issue is the study of the properties of σ in the neighborhood of the line $x = 0$. Since $\tilde{\sigma}$, $\tilde{\sigma}^{-1}$, $\sigma_\flat$ and $\sigma_\flat^{-1}$ are continuous in a neighborhood of $(0, 0)$ in $(\mathbb{C}^* \times \mathbb{C}) \cup \{(0, 0)\}$, it follows that the same property is shared by σ and σ^{-1}.

Suppose $a \in i\mathbb{R}$. Then we have $a = b$ and we define $\mathfrak{h}_u(z) = (1 - u)z + u\mathfrak{h}(z)$ and $a_u = a$ for $u \in [0, 1]$. If $a \notin i\mathbb{R}$ the real parts of a and b have the same sign since $\mathfrak{h}$ is orientation preserving. We define $a_u = (1 - u)a + ub$ and $\mathfrak{h}_u : \mathbb{C} \to \mathbb{C}$ as the $\mathbb{R}$-linear map such that $\mathfrak{h}_u(1) = 1$ and $\mathfrak{h}_u(2\pi i a) = 2\pi i a_u$ for $u \in [0, 1]$. We define $Y_u = [(y^2 - x)/(1 + a_u y)]\partial/\partial y$ and a Fatou coordinate ψ_u of Y_u such that $\psi_u(x, y_0) \equiv 0$ for $u \in [0, 1]$. We denote by τ_u the conjugating map obtained by applying the previous method to Y, Y_u and $\mathfrak{h}_u$. We obtain $\tau_0 = \mathrm{Id}$ and $\tau_1 = \sigma$. Since τ_u depends continuously on u and τ_u conjugates the functions $\mathfrak{h}_u \circ \psi_Y$ and ψ_u, we obtain $\tau_u(x, y_0) = (\tau_{u,1}(x), y_0)$ for all $u \in [0, 1]$ and x in a neighborhood of 0. In particular we deduce $\sigma(x, y_0) = (\sigma_1(x), y_0)$ for any x in a neighborhood of 0. Analogously as in the proof of Proposition 2.40 the equation $\mathfrak{h} \circ \psi_Y \equiv \psi_Z \circ \sigma$ implies that σ_0 and σ_0^{-1} are local homeomorphisms. We obtain that σ and σ^{-1} are homeomorphisms that are real analytic outside $\{y^2 - x = 0\} \cup \{x = 0\}$ by another application of the equation $\mathfrak{h} \circ \psi_Y \equiv \psi_Z \circ \sigma$. $\square$

Example 6.6. — Consider $b = a \in \mathrm{i}\,\mathbb{R}$ and a $\mathbb{R}$-linear isomorphism such that $\mathfrak{h}(1) = 1$.

Suppose $\mathfrak{h}$ is orientation-preserving. Then the map σ provided by Proposition 6.5 satisfies $\sigma^*(\mathrm{Re}(Y)) = \mathrm{Re}(Y)$ and $\sigma \circ \exp(Y) = \exp(Y) \circ \sigma$. Moreover σ_0 is holomorphic if and only if $\mathfrak{h} \equiv z$.

Suppose $\mathfrak{h}$ is orientation-reversing. Consider the conjugacy σ associated to $(Y, Y, \overline{z} \circ \mathfrak{h})$. Now $\zeta \circ \sigma$ conjugates $\mathrm{Re}(Y)$ and $\mathrm{Re}(Z)$ where $Z = [(y^2 - x)/(1 + \overline{a}y)]\partial/\partial y$ and $\zeta(x, y) = (\overline{x}, \overline{y})$. The map $\mathfrak{h}_{\exp(Y_0), \exp(Z_0), (\zeta \circ \sigma)_0}$ is equal to $\mathfrak{h}$. We described both situations in Proposition 1.8 where the diffeomorphisms by restriction to $x = 0$ are holomorphically (resp. anti-holomorphically) conjugated but the conjugacy is not holomorphic (resp. anti-holomorphic) in general.

Example 6.7. — Consider $a, b \notin \mathrm{i}\,\mathbb{R}$ such that $\mathrm{Re}(a)\,\mathrm{Re}(b) > 0$. Let $\mathfrak{h}$ be the $\mathbb{R}$-linear map such that $\mathfrak{h}(1) = 1$ and $\mathfrak{h}(2\pi\,\mathrm{i}\,a) = 2\pi\,\mathrm{i}\,b$. Then the map σ provided by Proposition 6.5 satisfies $\sigma^*(\mathrm{Re}(Z)) = \mathrm{Re}(Y)$ and $\sigma \circ \exp(Y) = \exp(Z) \circ \sigma$. Moreover σ_0 is holomorphic if and only if $a = b$.

Let $\mathfrak{h}$ be the $\mathbb{R}$-linear map such that $\mathfrak{h}(1) = 1$ and $\mathfrak{h}(2\pi\,\mathrm{i}\,a) = -2\pi\,\mathrm{i}\,b$. We obtain $(\overline{z} \circ \mathfrak{h})(2\pi\,\mathrm{i}\,a) = 2\pi\,\mathrm{i}\,\overline{b}$. Then the map σ provided by Proposition 6.5 and associated to Y, $\tilde{Z} = [(y^2 - x)/(1 + \overline{b}y)]\partial/\partial y$ and $\overline{z} \circ \mathfrak{h}$ satisfies $\sigma^*(\mathrm{Re}(\tilde{Z})) = \mathrm{Re}(Y)$ and $\sigma \circ \exp(Y) = \exp(\tilde{Z}) \circ \sigma$. The homeomorphism $\zeta \circ \sigma$ conjugates $\mathrm{Re}(Y)$ and $\mathrm{Re}(Z)$. Moreover $(\zeta \circ \sigma)_0$ is anti-holomorphic if and only if $a = \overline{b}$. We described these situations in Proposition 1.7.

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