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Milton Espinoza *&* Eduardo Friedman

Twisters and signed fundamental domains for number fields

TWISTERS AND SIGNED FUNDAMENTAL DOMAINS
FOR NUMBER FIELDS

by Milton ESPINOZA & Eduardo FRIEDMAN (*)

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ABSTRACT. — We give a signed fundamental domain for the action on $\mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2}$ of the totally positive units E_+ of a number field k of degree $n = r_1 + 2r_2$ which we assume is not totally complex. Here r_1 and r_2 denote the number of real and complex places of k and $\mathbb{R}_+$ denotes the positive real numbers. The signed fundamental domain consists of n-dimensional k-rational cones C_α, each equipped with a sign $\mu_\alpha = \pm 1$, with the property that the net number of intersections of the cones with any E_+-orbit is 1.

The cones C_α and the signs μ_α are explicitly constructed from any set of fundamental totally positive units and a set of 3^{r_2} "twisters", i.e. elements of k whose arguments at the r_2 complex places of k are sufficiently varied. Introducing twisters gives us the right number of generators for the cones C_α and allows us to make the C_α turn in a controlled way around the origin at each complex embedding.

RÉSUMÉ. — Soit k un corps de nombres de degré $n = r_1 + 2r_2$ admettant au moins une place réelle (i.e. $r_1 > 0$). Nous présentons un domaine fondamental signé pour l'action des unités totalement positives E_+ de k sur $\mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2}$, où $\mathbb{R}_+$ désigne l'ensemble des nombres réels positifs. Le domaine fondamental signé est formé de cônes k-rationnels C_α de dimension n, chacun muni d'un signe $\mu_\alpha = \pm 1$ avec la propriété suivante : pour chaque E_+-orbite, la somme signée des intersections des cônes avec l'orbite est égale à 1.

Nous construisons explicitement les cônes C_α et les signes μ_α à partir d'un ensemble arbitraire d'unités fondamentales totalement positives et d'un ensemble de 3^{r_2} « twisters ». Ces derniers sont des éléments de k dont les arguments aux r_2 places complexes de k sont suffisamment bien distribués. L'introduction des « twisters » fournit le nombre exact de générateurs pour les cônes C_α et permet de les faire tourner autour de l'origine de façon contrôlée dans chaque plongement complexe.

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1. Introduction

The usual embedding of a number field k into a Euclidean space V gives rise to an action of the units of k on V. Good fundamental domains for this action are important in the study of abelian L-functions. For totally real fields, Shintani [8] showed in 1976 the existence of fundamental domains consisting of a finite number of k-rational polyhedral cones, now known as Shintani cones. A few years later, in a posthumous and rarely cited work [9], Shintani extended this to all number fields.

Shintani's papers gave no practical procedure to construct his cones, or even to estimate how many cones were needed. In the late 1980's Colmez [2] showed for totally real fields the existence of certain special subgroups of the units for which he could explicitly construct Shintani-type fundamental domains for the action of this subgroup on V. This was a significant theoretical advance, but it was not effective since no practical procedure is known for producing Colmez's special units, except in the quadratic or cubic case [3].

A few years ago Charollois, Dasgupta and Greenberg [1], and independently Diaz y Diaz and Friedman [4], found a way around this non-effectiveness for totally real fields by introducing signed fundamental domains. Espinoza [6] then found effective signed fundamental domains for number fields having exactly one complex place.

Signed fundamental domains can be naturally defined if one defines fundamental domains using characteristic functions. Indeed, a set of subsets $\{C_\alpha\}_{\alpha \in J}$ of a space Y, on which a countable group G acts, is a fundamental domain if

$$\sum_{\alpha \in J} \sum_{\varepsilon \in G} \chi_{C_\alpha}(\varepsilon \cdot y) = 1 \qquad (\forall\, y \in Y),$$

where χ_{C_α} is the characteristic function of C_α. A signed fundamental domain is a finite list $\{C_\alpha\}_{\alpha \in J}$ of subsets of Y and a corresponding list of signs $\mu_\alpha = \pm 1$ such that

$$\sum_{\alpha \in J} \mu_\alpha \sum_{\varepsilon \in G} \chi_{C_\alpha}(\varepsilon \cdot y) = 1 \qquad (\forall\, y \in Y).$$

For technical reasons [4, Lemma 5], we also require that the cardinality of $C_\alpha \cap (G \cdot y)$ be bounded independently of $y \in Y$.

Before going into details, we outline the difficulties that appear with Shintani cones when k is not totally real. Let k be a number field of degree $n = r_1 + 2r_2$ having r_1 real places and r_2 pairs of complex conjugate embeddings. The most obvious problem is that we no longer have n natural

generators for the n-dimensional Shintani cones. In the totally real case, following Colmez, we can use $1 \in k$ and $n-1$ independent units of k to generate the cones, but each complex case leaves us a unit short of the n generators that we need. Thus, we must introduce one new generator for each complex place. An additional difficulty is that $\mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2}$, while it is a cone, is not convex if $r_2 > 0$. This means that n independent vectors are liable to generate a cone including (nonzero) points outside $\mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2}$, for example where one complex coordinate vanishes. We choose the new generators for the cones so as to force all of them inside convex subsets of $\mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2}$. These generators we call twisters, as their arguments at the complex places are chosen so as to twist the generators into a convex conical sector.

The approach to signed fundamental domains in [4, 6] and here can be summarized in the following commutative diagram of topological spaces

(1.1)
$$
\begin{array}{ccc}
\mathfrak{X} \times \mathbb{R} & \xrightarrow[f]{g} & \mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2} \\
\downarrow{\scriptstyle\widehat{\pi}} & & \downarrow{\scriptstyle\pi} \\
\widehat{T} & \xrightarrow[F]{G} & T.
\end{array}
$$

We begin by explaining the vertical arrow on the right. Suppose we are given totally positive independent units $\varepsilon_1, \dots, \varepsilon_r$ generating a subgroup $\mathcal{E}$ of finite index in the units of k. Thus $\mathcal{E}$ acts on $\mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2}$, where $\mathbb{R}_+$ denotes the multiplicative group of positive real numbers. Let $T = (\mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2})/\mathcal{E}$ be the quotient manifold and $\pi : \mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2} \to T$ the natural map.

It is well-known that T is homeomorphic to the product of an $(n-1)$-torus with $\mathbb{R}$. This is made explicit by the left vertical arrow in (1.1) and the maps g and G, in the following sense. The standard (additive) model for T is $\widehat{T} := (\mathbb{R}^{n-1}/\Lambda) \times \mathbb{R}$, where $\Lambda \subset \mathbb{R}^{n-1}$ is a lattice of dimension $n-1$. We let $\widehat{\pi} : \mathbb{R}^{n-1} \times \mathbb{R} \to \widehat{T}$ be the natural quotient map and $g : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2}$ the group homomorphism ("exponential") which induces a homeomorphism $G : \widehat{T} \to T$ of the quotient manifolds, i.e. $G \circ \widehat{\pi} = \pi \circ g$. Now let $\mathfrak{X}$ be a fundamental domain for the action of Λ on $\mathbb{R}^{n-1}$ and restrict g and $\widehat{\pi}$ to $\mathfrak{X} \times \mathbb{R}$. This defines everything in (1.1), except the interesting part, namely f and F.

We obtain a classical fundamental domain $\mathcal{F}$ (used by Hecke and Landau, for example) for the action of $\mathcal{E}$ on $\mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2}$ by letting $\mathcal{F} := g(\mathfrak{X} \times \mathbb{R})$. Unfortunately, $\mathcal{F}$ is of limited use because of its complicated geometry. In particular, it is difficult to describe the intersection of $\mathcal{F}$ with fractional ideals of k, a vital step in Shintani's treatment of L-functions.

To remedy this, for totally real fields Colmez deformed g to a new and simpler function f so that its image $f(\mathfrak{X} \times \mathbb{R})$ is a union of k-rational polyhedral cones. From our present standpoint, the work of Colmez [2] can be described as follows. Take the fundamental domain $\mathfrak{X} = [0,1]^{n-1}$ for the lattice $\Lambda := \mathbb{Z}^{n-1} \subset \mathbb{R}^{n-1}$. There is a well-known decomposition, parametrized by the symmetric group S_{n-1},

$$\mathfrak{X} = \bigcup_{\alpha \in S_{n-1}} \mathfrak{X}_\alpha$$

of the hypercube into simplices $\mathfrak{X}_\alpha$ [4, (19)]. For each $\alpha \in S_{n-1}$, let $A_\alpha : \mathfrak{X}_\alpha \to \mathbb{R}^n_+$ be the unique affine function which on each vertex κ of $\mathfrak{X}_\alpha$ takes the value $g(\kappa \times 0) \in \mathbb{R}^n_+$ (recall that Colmez only dealt with the totally real case). Then we can unambiguously define $f : \mathfrak{X} \times \mathbb{R} \to \mathbb{R}^n_+$ by $f(x \times t) := \mathrm{e}^t A_\alpha(x)$ where $x \in \mathfrak{X}_\alpha$, $t \in \mathbb{R}$. As $\mathbb{R}^n_+$ is convex, f takes values in $\mathbb{R}^n_+$ and $f(\mathfrak{X}_\alpha \times \mathbb{R})$ is the cone $\overline{C}_\alpha$ generated by the $g(\kappa \times 0)$ as κ ranges over the vertices of $\mathfrak{X}_\alpha$. Colmez [2] proved that if the cones $\overline{C}_\alpha$ meet only along common faces, then $\bigcup_\alpha C_\alpha$ is a fundamental domain for the action of $\mathcal{E}$ on $\mathbb{R}^n_+$. Here C_α is $\overline{C}_\alpha$ minus some boundary faces.

Colmez's proof is rather complicated, but it can be greatly simplified [4] by observing that the decomposition $\mathfrak{X} = \bigcup_\alpha \mathfrak{X}_\alpha$ is good enough that f induces a map $F : \widehat{T} \to T$ of the quotient manifolds in (1.1). One can then use topological degree theory to show that $\bigcup_\alpha C_\alpha$ is a signed fundamental domain with weights μ_α given by the degree of f restricted to $\mathfrak{X}_\alpha \times \mathbb{R}$. If we add Colmez's hypothesis that the $\overline{C}_\alpha$ meet only along common faces, then $\mu_\alpha = 1$ for all $\alpha \in S_{n-1}$ and the signed fundamental domain is a true one.

When k is not totally real we would like to have a similar construction. The topology is unchanged, as the manifolds $\widehat{T}$ and T in (1.1) are still homeomorphic to the product of an $(n-1)$-torus with $\mathbb{R}$. It is also easy to write down the homomorphism g inducing the homeomorphism G in (1.1). However, as we remarked above, the cone $\mathbb{R}^{r_1}_+ \times \mathbb{C}^{*r_2}$ is not convex, so to define f we must ensure that the generators of the cones lie in convex subsets of $\mathbb{R}^{r_1}_+ \times \mathbb{C}^{*r_2}$. Moreover, the decomposition $\mathfrak{X} = \bigcup_\alpha \mathfrak{X}_\alpha$ must again allow f to descend to a map F of the quotient manifolds.

An additional difficulty is that the natural choice of g does not give cone generators in k, as in general $g(\kappa \times 0) \notin k$ for κ a vertex of $\mathfrak{X}_\alpha$. We fix this by modifying g slightly. Fortunately, the homotopy involved in formalizing this approximation does not alter topological degrees, and so this turns out to be a minor difficulty.

Espinoza [6] obtained a signed fundamental domain for fields k with exactly one pair of complex embeddings. In this paper we extend the ideas there to all non-totally complex number fields. We exclude totally complex fields solely because in this case we have not been able to give a satisfactory description of the boundary faces that should be included in the signed fundamental domain. The reader will find in the next section a detailed description of the signed fundamental domain obtained.

As mentioned at the beginning of this paper, signed fundamental domains are useful for working with abelian L-functions. More precisely, as in [4, Cor. 6] and [6, Cor. 3], using a signed fundamental domain one can explicitly write any abelian L-function as a finite linear combination of Shintani zeta functions [8]. The Shintani functions do not in general satisfy a functional equation of the usual kind, but they do satisfy a ladder of difference equations and are normalized by vanishing integrals [7, eqs. 1.4 and 1.7]. In a more geometric vein, in a forthcoming doctoral thesis Alex Capuñay gives a practical algorithm producing a true k-rational fundamental cone domain starting from a signed one.

2. Signed fundamental domain

We fix a number field k of degree $[k : \mathbb{Q}] = n$, having r_1 real embeddings $\tau_1, \ldots, \tau_{r_1}$ and r_2 pairs of conjugate complex embeddings

$$\tau_{r_1+1}, \overline{\tau}_{r_1+1}, \ldots, \tau_{r_1+r_2}, \overline{\tau}_{r_1+r_2},$$

which we use to embed k in $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ by $\gamma \to (\gamma^{(i)})_i$ where $\gamma^{(i)} := \tau_i(\gamma)$ $(1 \leqslant i \leqslant r_1 + r_2)$. To save notation, we identify $\gamma \in k$ with its image $(\gamma^{(i)})_i \in \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. We also fix independent totally positive units $\varepsilon_1, \ldots, \varepsilon_r$ of k, where $r = r_1 + r_2 - 1$, and assume we have chosen an integer $N_j \geqslant 3$ $(1 \leqslant j \leqslant r_2)$ for each complex embedding. To have the smallest number of cones we should take $N_j = 3$ for all j.

2.1. Raising the dimension of a complex

An ordered p-complex X for us will be a decomposition

$$(2.1) \qquad X := \bigcup_{\alpha \in \mathfrak{I}} X_{\alpha, \prec_\alpha},$$

where the $X_{\alpha, \prec_\alpha} \subset V$ are ordered p-dimensional simplices in a p-dimensional real vector space V. Thus, for each α in the (possibly infinite) index set $\mathfrak{I}$

we are given the set of vertices $\mathrm{Vt}(X_\alpha)$ of the p-simplex X_α and a total ordering $\prec_\alpha$ on $\mathrm{Vt}(X_\alpha)$. We will later want our complexes to satisfy quite a few properties, but the above definition will do to identify a complex. We shall be loose with the notation and denote by X both the underlying point set and the complex, i.e. its decomposition (2.1). Similarly, we shall often write X_α for both the simplex and the ordered simplex $X_{\alpha, \prec_\alpha}$.

Our purpose in this subsection is to construct a new ordered $(p+1)$-complex

$$(2.2) \qquad Y = Y(X, \omega, [M_1, M_2]) := \bigcup_{\gamma \in J} Y_{\gamma, \prec_\gamma} \subset V \times \mathbb{R}$$

from a given p-complex $X = \bigcup_{\alpha \in \mathfrak{I}} X_{\alpha, \prec_\alpha} \subset V$, a linear function $\omega : V \to \mathbb{R}$ and an interval $[M_1, M_2]$. Here the M_i are integers and $M_1 < M_2$. In our application ω will either vanish identically or be determined by the arguments of the units ε_i at some complex embedding. The new ambient vector space is $V \times \mathbb{R}$, and the new index set is

$$(2.3) \qquad J := \big\{ (\alpha, v, \ell) \,\big|\, \alpha \in \mathfrak{I}, \ v \in \mathrm{Vt}(X_\alpha), \ \ell \in \mathbb{Z}, \ M_1 \leqslant \ell < M_2 \big\}.$$

If $\mathfrak{I}$ is a finite set, then so is J and its cardinality is

$$(2.4) \qquad |J| = (M_2 - M_1)(p+1)|\mathfrak{I}|.$$

To define the simplex Y_γ for $\gamma = (\alpha, v, \ell) \in J$ requires some preliminaries. Let $A : V \to (0, 1]$ be the upper fractional part of ω, i.e. $A(u) \in \mathbb{R}$ is uniquely determined by

$$(2.5) \qquad A(u) - \omega(u) \in \mathbb{Z}, \qquad 0 < A(u) \leqslant 1 \qquad\qquad (u \in V).$$

We use A to define a new total order $\prec_\alpha^A$ on $\mathrm{Vt}(X_\alpha)$ by

$$(2.6) \quad u \prec_\alpha^A u' \iff \Big[\big[A(u) < A(u') \big] \ \text{ or } \ \big[A(u) = A(u') \text{ and } u \prec_\alpha u' \big] \Big]$$

for $u, u' \in \mathrm{Vt}(X_\alpha)$. Note that if $\omega = 0$ identically, then $\prec_\alpha^A = \prec_\alpha$.

We now define Y_γ in (2.2) for $\gamma = (\alpha, v, \ell) \in J$. Order the vertices $\{v_0, v_1, \ldots, v_p\}$ of X_α so that $v_0 \prec_\alpha^A v_1 \prec_\alpha^A \cdots \prec_\alpha^A v_p$. Since $v \in \mathrm{Vt}(X_\alpha)$ by definition (2.3), there is a unique $j = j_\gamma$ such that $v = v_j$. Note that j depends on the order $\prec_\alpha^A$. Let $[s_1, \ldots, s_t]$ denote the convex hull of $s_1, \ldots, s_t$. Then $Y_\gamma \subset V \times \mathbb{R}$ is defined as

$$(2.7) \ \ Y_\gamma := \big[v_0 \times (A(v_0) - \omega(v_0) + \ell), \ v_1 \times (A(v_1) - \omega(v_1) + \ell), \ldots,$$
$$v_j \times (A(v_j) - \omega(v_j) + \ell), v_j \times (A(v_j) - \omega(v_j) + \ell - 1),$$
$$v_{j+1} \times (A(v_{j+1}) - \omega(v_{j+1}) + \ell - 1), \ldots, v_p \times (A(v_p) - \omega(v_p) + \ell - 1) \big].$$

Note that vertices of Y_γ map to vertices of X_α by the projection $\pi_V :$ $V \times \mathbb{R} \to V$, and that the coordinate we have added to each vertex is an integer (see (2.5).

Finally, the new ordering $\prec_\gamma$ on the vertices of the new simplex Y_γ is defined by

(2.8) $\quad \rho \prec_\gamma \rho'$

$$\iff \Big[\big[\pi_V(\rho) \prec_\alpha \pi_V(\rho') \big] \ \text{ or } \ \big[\pi_V(\rho) = \pi_V(\rho') \text{ and } \pi_\mathbb{R}(\rho') < \pi_\mathbb{R}(\rho) \big] \Big],$$

where $\pi_\mathbb{R} : V \times \mathbb{R} \to \mathbb{R}$ is the projection onto the second factor. Note that $\prec_\gamma$ is defined using the original order $\prec_\alpha$ on the vertices of X_α, even though we used the modified order $\prec_\alpha^A$ to construct Y_γ.

2.2. The $(n-1)$-complex $\mathfrak{X}$

We start from the trivial 0-complex

(2.9) $$X_0 := X_{O, \prec_O} \subset \mathbb{R}^0,$$

where $\mathbb{R}^0 := \{0\}$ is the trivial vector space, the index set is $\{O\}$ (or any one-element set), and the ordered 0-simplex $X_{O, \prec_O} = \{0\}$ is equipped with the trivial (empty) ordering $\prec_O$ on the single vertex 0. For $1 \leqslant j \leqslant r :=$ $r_1 + r_2 - 1$, let $\omega_{j-1} : \mathbb{R}^{j-1} \to \mathbb{R}$ be the trivial linear function $\omega_{j-1}(x) := 0$, and define

$$X_j := Y(X_{j-1}, \omega_{j-1}, [0,1]) \subset \mathbb{R}^j \qquad (1 \leqslant j \leqslant r),$$

where Y was defined in (2.2).[1] If k is totally real, so $r = n - 1$, we are done constructing $\mathfrak{X} := X_{n-1}$. Otherwise, we must carry out further steps, one for each complex place of k. For $1 \leqslant j \leqslant r_2$, let $\omega_{r+j-1} : \mathbb{R}^{r+j-1} \to \mathbb{R}$ be given by

$$\omega_{r+j-1}(x) := \frac{N_j}{2\pi} \sum_{\ell=1}^r x[\ell] \arg\left(\varepsilon_\ell^{(r_1+j)} \right) \quad (x = (x[1], x[2], \dots, x[r+j-1])),$$

where we recall that for each $1 \leqslant j \leqslant r_2$ we have fixed an integer $N_j \geqslant 3$. For the sake of definiteness, the arguments above are taken so that $-\pi < \arg(z) \leqslant \pi$, but any branch would do as well. Define

(2.10) $\quad X_{r+j} := Y(X_{r+j-1}, \omega_{r+j-1}, [0, N_j]) \subset \mathbb{R}^{r+j} \qquad (1 \leqslant j \leqslant r_2),$

with Y as in Section 2.1.

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[1] Although we will not need this, X_r is the well-known decomposition of the r-cube $[0,1]^r$ into $r!$ simplices determined by the order of the coordinates [4, (19)].

We have thus constructed an $(n-1)$ ordered complex

$$(2.11) \quad \mathfrak{X} = \bigcup_{\alpha \in \mathfrak{I}} \mathfrak{X}_\alpha := X_{n-1} \subset \mathbb{R}^{n-1} \qquad \left(|\mathfrak{I}| = (n-1)! \prod_{j=1}^{r_2} N_j\right),$$

corresponding to $j = r_2$ above. We have dropped the order $\prec_\alpha$ on $\mathrm{Vt}(\mathfrak{X}_\alpha)$ from our notation since, once $\mathfrak{X}$ is constructed we will have no further interest in ordering the set of vertices

$$(2.12) \qquad \mathrm{Vt}(\mathfrak{X}_\alpha) := \{v_{0,\alpha}, v_{1,\alpha}, \ldots, v_{n-1,\alpha}\} \subset \mathbb{Z}^{n-1}.$$

However, we will use the fact that the vertices $v_{\ell,\alpha}$ have integral coordinates. This is clear since we started from the single vertex 0 of X_0. As we noted after (2.7), in passing from X_{j-1} to X_j the new coordinate appended to a vertex is always integral.

2.3. The twister function β

Recall that we have fixed $r := r_1 + r_2 - 1$ independent totally positive units $\varepsilon_1, \ldots, \varepsilon_r$ in the number field k and integers $N_j \geqslant 3$ for $1 \leqslant j \leqslant r_2$. We must now also fix "twisters" $\beta(x)$, which are totally positive elements of k^* whose arguments at the various complex places are sufficiently close to those of certain roots of unity determined by x. More precisely, choose a function $\beta : \mathbb{Z}^{n-1} \to k^*$ with the following properties:

- $\beta(x)$ is a totally positive element of k.
- For $x = (x[1], \ldots, x[n-1]) \in \mathbb{Z}^{n-1}$ and $1 \leqslant j \leqslant r_2$, there is a $t_j(x) \in \mathbb{R}$ such that

$$(2.13) \quad \frac{|t_j(x)|}{N_j} < \frac{1}{4} - \frac{1}{2N_j},$$

$$\frac{\beta(x)^{(j+r_1)}}{|\beta(x)^{(j+r_1)}|} = \exp\left(2\pi\, \mathrm{i}\, (x[j+r] + t_j(x))\,/N_j\right).$$

For example, if $N_j = 3$ this requires that, at the $(j + r_1)$-th embedding, the argument of $\beta(x)$ be no farther than $\pi/6$ from that of $\exp(2\pi\, \mathrm{i}\, x[j+r]/3)$.

- We have

$$(2.14) \quad \beta(x + \lambda) = \beta(x) \quad (x \in \mathbb{Z}^{n-1}, \; \lambda \in \Lambda := \mathbb{Z}^r \times N_1\mathbb{Z} \times \cdots \times N_{r_2}\mathbb{Z}).$$

Thus, the twister β amounts to a function on the finite set $\mathbb{Z}^{n-1}/\Lambda$. Note that in (2.13) we can take $t_j(x + \lambda) = t_j(x)$. Twister functions exist by our assumption $N_j \geqslant 3$ and the density of k in $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$.[(2)]

2.4. Signs

We need to compute three determinants to define the sign

$$(2.15) \qquad \mu_\alpha := (-1)^{r_2(r_2-1)/2} \operatorname{sign}\left(\det(R)\det(V_\alpha)\det(W_\alpha)\right).$$

Although R and V_α will prove to be invertible matrices, W_α may not be. Thus, μ_α may take the values ± 1 or 0.

Recall that we have listed the embeddings τ_i of k so that they are real for $1 \leqslant i \leqslant r_1$ and are complex conjugate pairs $\tau_i, \bar{\tau}_i$ for $r_1 < i \leqslant r_1 + r_2$. The $(r_1 + r_2) \times (r_1 + r_2)$ matrix $R = (r_{ij})$ is defined by

$$(2.16) \quad r_{ij} := \begin{cases} 1 & \text{if } i = 1, \\ \log |\varepsilon_{i-1}^{(j)}| & \text{if } 2 \leqslant i \leqslant r_1 + r_2, \end{cases} \qquad (1 \leqslant i, j \leqslant r_1 + r_2).$$

Actually, $|\det(R)| = 2^{-r_2} n \operatorname{Reg}(\varepsilon_1, \dots, \varepsilon_r)$, where Reg is the regulator of the units ε_i, but we care here only for the sign of $\det(R)$.

Define V_α as the $(n-1) \times (n-1)$ matrix whose i-th column is $v_{i,\alpha} - v_{0,\alpha} \in \mathbb{Z}^{n-1}$ $(1 \leqslant i \leqslant n-1)$. Here $\operatorname{Vt}(\mathfrak{X}_\alpha) = \{v_{0,\alpha}, v_{1,\alpha}, \dots, v_{n-1,\alpha}\} \subset \mathbb{Z}^{n-1}$ was given in (2.11) with the complex $\mathfrak{X}$. To define the matrix W_α in (2.15), let

$$(2.17) \qquad w_\ell = w_{\ell,\alpha} := \beta(v_{\ell,\alpha}) \prod_{j=1}^{r} \varepsilon_j^{v_{\ell,\alpha}[j]} \qquad (0 \leqslant \ell \leqslant n-1).$$

Thus $w_\ell \in k^*$. Define W_α as the real $n \times n$ matrix with $(\ell+1)$-th row

$$\tau_1(w_\ell), \tau_2(w_\ell), \dots, \tau_{r_1}(w_\ell), \operatorname{Re}\left(\tau_{r_1+1}(w_\ell)\right), \operatorname{Im}\left(\tau_{r_1+1}(w_\ell)\right), \dots,$$
$$\operatorname{Re}\left(\tau_{r_1+r_2}(w_\ell)\right), \operatorname{Im}\left(\tau_{r_1+r_2}(w_\ell)\right)$$

for $0 \leqslant \ell \leqslant n-1$.

[(2)] $\beta(x)$ can be computed by taking a $\mathbb{Q}$-basis $\gamma_1, \dots, \gamma_n$ of $k \subset \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, finding for $x \in \mathbb{Z}^{n-1}$ with $0 \leqslant x[j+r] < N_j$ $(1 \leqslant j \leqslant r_2)$, rational solutions $a_\ell = a_\ell(x)$ $(1 \leqslant \ell \leqslant n)$ to the approximate equality

$$(1, \dots, 1, \exp(2\pi \mathrm{i}\, x[r+1]/N_1), \dots, \exp(2\pi \mathrm{i}\, x[n-1]/N_{r_2})) \approx \sum_{\ell=1}^{n} a_\ell \gamma_\ell,$$

and letting $\beta(x) := \sum_{\ell=1}^{n} a_\ell \gamma_\ell$. Then extend β to all $x \in \mathbb{Z}^{n-1}$ by periodicity (2.14).

2.5. Cones

For $\alpha \in \mathfrak{I}$ as in (2.11), let

$$(2.18) \qquad \overline{C}_\alpha := \left\{ \sum_{\ell=0}^{n-1} x_\ell w_{\ell,\alpha} \,\middle|\, x_\ell \geqslant 0 \text{ for } 0 \leqslant \ell \leqslant n-1 \right\} - \{0\},$$

where $w_{\ell,\alpha} \in k \subset \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ was defined in (2.17).[3] If $w_{0,\alpha}, \ldots, w_{n-1,\alpha}$ is not an $\mathbb{R}$-basis of the real vector space $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, i.e. if $\mu_\alpha = 0$, we will not define the subset C_α of $\overline{C}_\alpha$. If $\mu_\alpha \neq 0$, we can write $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ uniquely as $e_1 = \sum_{\ell=0}^{n-1} y_{\ell,\alpha} w_{\ell,\alpha}$, where $y_{\ell,\alpha} \in \mathbb{R}$. We shall prove (see Lemma 4.12) that $y_{\ell,\alpha} \neq 0$ for $0 \leqslant \ell \leqslant n-1$. Define the cones

$$(2.19) \quad C_\alpha := \left\{ \sum_{\ell=0}^{n-1} x_\ell w_{\ell,\alpha} \,\middle|\, x_\ell \in R_{\ell,\alpha} \right\}, \quad R_{\ell,\alpha} := \begin{cases} [0,\infty) & \text{if } y_{\ell,\alpha} > 0, \\ (0,\infty) & \text{if } y_{\ell,\alpha} < 0. \end{cases}$$

Note that C_α is the open n-dimensional cone generated by the $w_{\ell,\alpha}$, together with some of its boundary faces.

We can now state our main result.

THEOREM 2.1. — *Let independent totally positive units $\varepsilon_1, \ldots, \varepsilon_r$ and a twister funtion β be given as in Section 2.3 for a number field k, let $E := \langle \varepsilon_1, \ldots, \varepsilon_r \rangle$ be the subgroup of the units of k generated by the ε_ℓ, and assume that k is not totally complex. Then $\{C_\alpha, \mu_\alpha\}_{\alpha \in \mathfrak{I}, \mu_\alpha \neq 0}$, defined in (2.19), (2.15) and (2.11), is a signed fundamental domain for the action of E on $\mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2}$ consisting of k-rational signed cones.*

Thus, the cones C_α have generators $w_{\ell,\alpha} \in k$ defined in (2.17), and for any $x \in \mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2}$ we have

$$(2.20) \qquad \sum_{\substack{\alpha \in \mathfrak{I} \\ \mu_\alpha \neq 0}} \mu_\alpha \sum_{\varepsilon \in E} \chi_{C_\alpha}(\varepsilon x) = 1,$$

where χ_{C_α} is the characteristic function of C_α. Furthermore, $\chi_{C_\alpha}(\varepsilon x) = 0$ except for ε in a finite set of cardinality bounded independently of x.

We recall that in (2.11) there is a free choice of integers $N_j \geqslant 3$ for $1 \leqslant j \leqslant r_2$ and that the number of cones C_α is at most $(n-1)! \prod_j N_j$, where $n = [k : \mathbb{Q}]$. If we pick all $N_j = 3$, then there are at most $3^{r_2} \cdot (n-1)!$ cones.

---

[3] Note that we removed the origin in (2.18). A great part of our efforts will be directed to showing that $\overline{C}_\alpha$ is contained in $\mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2}$. Here the difficulty is in ensuring that the complex components do not vanish.

If k is totally complex, we still prove (2.20), but only for x outside the E-orbit of the boundary of all $\overline{C}_\alpha$. The excluded set has Lebesgue measure 0, but is still unfortunate for the calculation of abelian L-functions.

There are two very different parts to the proof of Theorem 2.1. The first (see Section 3) consists of showing that if a complex X has a number of properties with respect to a lattice Λ, then so does the complex $Y(X, \omega, [M_1, M_2])$ with respect to the lattice $\Lambda \times (M_2 - M_1)\mathbb{Z}$. This allows us to construct inductively an $(n-1)$-complex $\mathfrak{X}$ and a function $f : \mathfrak{X} \times \mathbb{R} \to \mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2}$ whose image gives the cones in the signed fundamental domain. The second part of the proof (see Section 4) is mainly a calculation of certain global and local topological degrees associated to f. As in [4, 6], degrees enter because Theorem 2.1 can be interpreted as an instance of the local-global principle on suitable manifolds.

3. Lattice-adapted ordered simplicial complexes

3.1. Affine preliminaries

Let V be a real vector space and $\{v_0, \ldots, v_p\} \subset V$ a finite subset. It is called affinely independent if for a fixed j the set $\{v_i - v_j\}_{\substack{0 \leqslant i \leqslant p \\ i \neq j}}$ is linearly independent. This notion does not depend on the choice of j. If $p = 0$, any $v_0 \in V$ is affinely independent.

The convex hull S of $\{v_0, \ldots, v_p\}$ is

$$(3.1) \qquad S = [v_0, v_1, \ldots, v_p] := \left\{ \omega = \sum_{j=0}^p t_j v_j \,\middle|\, t_j \geqslant 0, \ \sum_{j=0}^p t_j = 1 \right\}.$$

We call S a simplex if the v_j are affinely independent. Then $\mathrm{Vt}(S) := \{v_0, v_1, \ldots, v_p\}$, its set of vertices, is uniquely determined by the point set S. If we wish to note the dimension of S, we call it a p-simplex. An ℓ-face, or simply a face, of S is an ℓ-simplex K such that $\mathrm{Vt}(K) \subset \mathrm{Vt}(S)$.

The t_j in (3.1) are called the barycentric coordinates of ω and $\sum_{j=0}^p t_j v_j$ is called its barycentric expansion (with respect to S). If S is a simplex we write $\mathrm{SpVt}(\omega) = \mathrm{SpVt}(\omega; S)$ for the set of spanning vertices of ω, i.e. those $v_j \in \mathrm{Vt}(S)$ with $t_j > 0$ in (3.1). Note that if $\omega \in K \subset S$, where K is a face of the simplex S, then

$$(3.2) \qquad\qquad \mathrm{SpVt}(\omega; S) = \mathrm{SpVt}(\omega; K).$$

If U is a real vector space, a map $A : V \to U$ is affine if $A(v) = L(v) + C$, where $L : V \to U$ is $\mathbb{R}$-linear and $C \in U$ is fixed. If $S = [v_0, v_1, \ldots, v_p] \subset V$

is a simplex, a map $H : S \to U$ is called affine if it is the restriction to S of an affine map $A : \mathbb{R}S \to U$. Here $\mathbb{R}S$ is the vector subspace of V spanned by the v_i. In terms of the barycentric expansion such a map satisfies

$$(3.3) \quad H(\omega) = \sum_{j=0}^{p} t_j H(v_j) \quad \left(\omega = \sum_{j=0}^{p} t_j v_j, \ \sum_{j=0}^{p} t_j = 1, \ t_j \geqslant 0 \text{ for } 0 \leqslant j \leqslant p \right),$$

and so is determined by the $H(v_j)$. In fact, $H(\omega)$ is determined by the values of H on the spanning vertices of ω. Conversely, if for each vertex v_j of S we choose some $H(v_j) \in U$, then (3.3) defines a unique affine map $H : S \to U$.

If $S' = [u_0, u_1, \ldots, u_\ell] \subset U$ is a simplex in U, a map $T : S \to S'$ is called simplicial if it is an affine map from S to U that takes vertices of S to vertices of S'. An injective simplicial map T preserves barycentric coordinates, i.e. if t_j is the barycentric coordinate of $\omega \in S$ corresponding to a vertex v_j, then t_j is also the barycentric coordinate of $T(\omega) \in S'$ corresponding to the vertex $T(v_j)$.

3.2. Λ-complexes

Recall that in Section 2.1 we defined an ordered complex $X = \bigcup_{\alpha \in \mathfrak{I}} X_\alpha$. A map of complexes $X \to X'$, where $X' = \bigcup_{\beta \in \mathfrak{I}'} X'_\beta$, is a map of index sets $\widetilde{T} : \mathfrak{I} \to \mathfrak{I}'$ together with a map of point sets $T : X \to X'$ such that T restricted to each X_α is a simplicial map to $X'_{\widetilde{T}(\alpha)}$. If $\widetilde{T}$ and T are bijections, the set-theoretic inverse of T is also a map of complexes, and so the complexes are isomorphic.

DEFINITION 3.1. — *Suppose V is a p-dimensional real vector space, $\Lambda \subset V$ is a full lattice (i.e. a discrete subgroup of V whose $\mathbb{R}$-span is V), and $X = \bigcup_{\alpha \in \mathfrak{I}} X_{\alpha, \prec_\alpha} \subset V$ is an ordered p-complex in V. We shall say that X is a Λ-complex if it satisfies the following five properties.*

(i) *X is a simplicial complex, i.e. for $\alpha, \beta \in \mathfrak{I}$, $X_\alpha \cap X_\beta$ is empty or*

$$(3.4) \quad X_\alpha \cap X_\beta = [v_1, \ldots, v_\ell], \quad \text{where} \quad \{v_1, \ldots, v_\ell\} := \mathrm{Vt}(X_\alpha) \cap \mathrm{Vt}(X_\beta).$$

   In other words, simplices intersect along common faces.
(ii) *The orders are compatible, i.e.*

$$(3.5) \quad v \prec_\alpha w \quad \Longleftrightarrow \quad v \prec_\beta w \qquad (\alpha, \beta \in \mathfrak{I}, \ v, w \in \mathrm{Vt}(X_\alpha) \cap \mathrm{Vt}(X_\beta)).$$

(iii) *The orders are Λ-invariant, i.e. for $\alpha, \alpha' \in \mathfrak{I}$, if $v, w \in \mathrm{Vt}(X_\alpha)$ and $v + \lambda, w + \lambda \in \mathrm{Vt}(X_{\alpha'})$ for some $\lambda \in \Lambda$, then*

$$(3.6) \qquad\qquad v \prec_\alpha w \quad\Longleftrightarrow\quad v + \lambda \prec_{\alpha'} w + \lambda.$$

(iv) *X is nearly a fundamental domain for Λ, i.e. the restriction to X of the natural quotient map from V to V/Λ is surjective, and it is injective when restricted to the union $\bigcup_{\alpha \in \mathfrak{I}} \overset{\circ}{X}_\alpha$ of the interiors of the X_α.*

(v) *The spanning vertices are Λ-equivariant,[4] i.e.*

$$\bigl[x, x' \in X, \ \lambda \in \Lambda, \ x' = x + \lambda\bigr] \quad\Longrightarrow\quad \mathrm{SpVt}(x') = \mathrm{SpVt}(x) + \lambda.$$

Note that if X is simplicial (in the sense of (i) above) and X' is an isomorphic complex, then X' is also simplicial.

Next we state the main result of this section.

PROPOSITION 3.2. — *Let the p-complex $X = \bigcup_{\alpha \in \mathfrak{I}} X_{\alpha, \prec_\alpha}$ be a Λ-complex in $V = \mathbb{R}\Lambda$, let $M_1 < M_2$ be integers, let $\widehat{\Lambda} := \Lambda \times (M_2 - M_1)\mathbb{Z}$, and let $\omega : V \to \mathbb{R}$ be a linear function. Then the ordered $(p+1)$-complex defined in Section 2.1,*

$$Y(X, \omega, [M_1, M_2]) = \bigcup_{\gamma \in J} Y_{\gamma, \prec_\gamma} \subset V \times \mathbb{R},$$

is a $\widehat{\Lambda}$-complex. Furthermore, if $\gamma = (\alpha, v, \ell) \in J$ and we define $\Omega : V \times \mathbb{R} \to \mathbb{R}$ by $\Omega(s \times t) := \omega(s) + t$, then for all $\kappa \in Y_\gamma$ we have

$$(3.7) \qquad\qquad A(v) + \ell - 1 \leqslant \Omega(\kappa) \leqslant A(v) + \ell,$$

where $A(v)$ is the upper fractional part of $\omega(v)$ defined in (2.5).

In the next three subsections we prove a series of lemmas leading to a proof in Section 3.6 of the above proposition.

3.3. Simplicial decomposition of $X \times [M_1, M_2]$

LEMMA 3.3. — *Let $I = [0, 1] \subset \mathbb{R}$ be the unit interval, let $S = S_\prec$ be an ordered p-simplex in some real p-dimensional vector space V, and write*

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[4] As the complex $X = \bigcup_{\alpha \in \mathfrak{I}} X_\alpha$ is assumed simplicial, (3.2) shows that the set $\mathrm{SpVt}(x) = \mathrm{SpVt}(x; X_\alpha)$ of spanning vertices of $x \in X$ is independent of the simplex X_α containing x used to calculate the barycentric expansion of x. We will write $\mathrm{SpVt}(x; X)$ when we wish to specify the simplicial complex involved.

$S = [v_0, v_1, \ldots, v_p]$ as the convex hull of its $p + 1$ vertices, ordered so that $v_0 \prec v_1 \prec \cdots \prec v_p$. For any vertex $v = v_j \in \mathrm{Vt}(S)$, let

$$(3.8) \quad S^v = S_\prec^v := [v_0 \times 1, v_1 \times 1, \ldots, v_j \times 1, v_j \times 0, v_{j+1} \times 0, \ldots, v_p \times 0]$$
$$\subset S \times I \subset V \times \mathbb{R}.$$

Then S^v is an ordered $(p + 1)$-simplex if we define, for $w, w' \in \mathrm{Vt}(S^v)$,

$$(3.9) \quad w \prec^v w' \Longleftrightarrow$$
$$\big[[\pi_V(w) \prec \pi_V(w')] \text{ or } [\pi_V(w) = \pi_V(w') \text{ and } \pi_\mathbb{R}(w') < \pi_\mathbb{R}(w)] \big],$$

where $\pi_V : V \times \mathbb{R} \to V$ and $\pi_\mathbb{R} : V \times \mathbb{R} \to \mathbb{R}$ are the natural projections. Furthermore, $S \times I = \bigcup_{v \in \mathrm{Vt}(S)} S^v$, and this is an ordered simplicial $(p+1)$-complex, i.e. it satisfies (3.4) and (3.5).

Proof. — It is immediate that S^v is a $(p+1)$-simplex and that (3.9) makes $\bigcup_v S^v$ into an ordered complex satisfying (3.5). To see that it is simplicial, i.e. satisfies (3.4), we will show that it is isomorphic to the standard simplicial decomposition of $\Delta_p \times I$, where Δ_p is the standard p-simplex. Indeed, let $e_1, \ldots, e_p$ be the standard basis of $\mathbb{R}^p$ and set $E_0 := 0_{\mathbb{R}^p} \in \mathbb{R}^p$, $E_i := e_i + E_{i-1}$ $(1 \leqslant i \leqslant p)$, and let $A : V \to \mathbb{R}^p$ be the unique affine isomorphism satisfying $A(v_i) := E_{p-i}$ $(0 \leqslant i \leqslant p)$, so that

$$A(S) = [E_p, E_{p-1}, \ldots, E_0] =: \Delta_p = \{ x \in \mathbb{R}^p \mid 1 \geqslant x_1 \geqslant x_2 \geqslant \cdots \geqslant x_p \geqslant 0 \}.$$

Let $\widehat{A} : V \times \mathbb{R} \to \mathbb{R}^p \times \mathbb{R}$ be the affine isomorphism given by $\widehat{A}(v \times y) := A(v) \times y \in \mathbb{R}^p \times \mathbb{R} = \mathbb{R}^{p+1}$. Then $\widehat{A}(S \times I) = \Delta_p \times I$, and

$$\widehat{A}(S^{v_j}) = [E_p \times 1, \ldots, E_{p-j} \times 1, E_{p-j} \times 0, E_{p-j-1} \times 0, \ldots, E_0 \times 0]$$
$$= \{ x \in \mathbb{R}^{p+1} \mid 1 \geqslant x_1 \geqslant \cdots \geqslant x_{p-j} \geqslant x_{p+1} \geqslant x_{p-j+1} \geqslant \cdots \geqslant x_p \geqslant 0 \}$$

$(0 \leqslant j \leqslant p)$. This is the standard decomposition of the product $\Delta_p \times I$, easily seen to be simplicial. $\square$

Next we record some simple properties of the above decomposition of $S \times I$.

(i) The projection $\pi_V : V \times \mathbb{R} \to V$ maps S^v onto S, and maps the vertices of S^v bijectively onto the vertices of S, except for the vertices $v \times 0$ and $v \times 1$, both of which map to v.

(ii) Let S^{v-} and S^{v+} be the p-faces of S^v defined by

$$(3.10)$$
$$S_\prec^{v-} = S^{v-} := [v_0 \times 1, v_1 \times 1, \ldots, v_{j-1} \times 1, v_j \times 0, v_{j+1} \times 0, \ldots, v_p \times 0],$$
$$S_\prec^{v+} = S^{v+} := [v_0 \times 1, v_1 \times 1, \ldots, v_{j-1} \times 1, v_j \times 1, v_{j+1} \times 0, \ldots, v_p \times 0],$$

If $w \in S$, then $w \times 0$ is contained in a proper face of S^{v_0}, namely in $S^{v_0 -} := [v_0 \times 0, v_1 \times 0, \dots, v_p \times 0] \subset S^{v_0}$. Similarly, $w \times 1$ is contained in a proper face of S^{v_p},

$$(3.11) \qquad S^{v_p +} := [v_0 \times 1, v_1 \times 1, \dots, v_p \times 1] = S^{v_0 -} + (0_V \times 1) \subset S^{v_p},$$

where $0_V \in V$ is the origin in V.

LEMMA 3.4. — Suppose $X = \bigcup_{\alpha \in \mathfrak{I}} X_{\alpha, \prec_\alpha}$ is an ordered simplicial p-complex. Then

$$(3.12) \qquad X \times I = \bigcup_{\alpha \in \mathfrak{I}} \bigcup_{v \in \mathrm{Vt}(X_\alpha)} X^v_{\alpha, \prec^v_\alpha}$$

is an ordered simplicial $(p+1)$-complex. Here I is the closed interval $[0,1]$ and $X^v_{\alpha, \prec^v_\alpha}$ was defined in Lemma 3.3, taking $S_\prec := X_{\alpha, \prec_\alpha}$.

Proof. — The equality of sets in (3.12) is clear from Lemma 3.3. By the same lemma, it is easy to see that (3.12) is an ordered $(p+1)$-complex. We now show that the decomposition is simplicial. Suppose $\delta = \omega \times y \in X^v_\alpha \cap X^w_\beta$. We will show that the spanning verticies of δ satisfy $\mathrm{SpVt}(\delta; X^v_\alpha) = \mathrm{SpVt}(\delta; X^w_\beta)$. This suffices as it shows that δ lies in the convex hull of $\mathrm{Vt}(X^v_\alpha) \cap \mathrm{Vt}(X^w_\beta)$. Let $L := X_\alpha \cap X_\beta$, a non-empty simplex as $\omega \in L$, and a common face of the simplices X_α and X_β. Then $\delta \in L \times I$, so by Lemma 3.3, $\delta \in L^\mu$ for some vertex $\mu \in \mathrm{Vt}(L) = \mathrm{Vt}(X_\alpha) \cap \mathrm{Vt}(X_\beta)$. But L^μ is a face of X^μ_α and of X^μ_β, as follows immediately from the construction of Lemma 3.3. Therefore $\delta \in X^\mu_\alpha \cap X^v_\alpha$ and $\delta \in X^\mu_\beta \cap X^w_\beta$. Since $X_\alpha \times I = \bigcup_{\rho \in \mathrm{Vt}(X_\alpha)} X^\rho_\alpha$ is simplicial by Lemma 3.3, we have

$$\mathrm{SpVt}(\delta; X^v_\alpha) = \mathrm{SpVt}(\delta; X^\mu_\alpha) = \mathrm{SpVt}(\delta; L^\mu)$$
$$= \mathrm{SpVt}(\delta; X^\mu_\beta) = \mathrm{SpVt}(\delta; X^w_\beta). \qquad \square$$

Next we use translations in the last coordinate to extend the decomposition of Lemma 3.4 from $X \times I$ to $X \times [M_1, M_2]$. Given an ordered p-complex $X = \bigcup_{\alpha \in \mathfrak{I}} X_{\alpha, \prec_\alpha} \subset V$, define the ordered $(p+1)$-simplex $X^\gamma = X^\gamma_{\prec_\gamma}$ by

$$(3.13) \quad X^\gamma := X^v_\alpha + (0_V \times \ell) \quad (\gamma = (\alpha, v, \ell),\ \alpha \in \mathfrak{I},\ v \in \mathrm{Vt}(X_\alpha),\ \ell \in \mathbb{Z}),$$

where $\prec_\gamma$ is defined by (3.9). Thus, for $w, w' \in \mathrm{Vt}(X^\gamma)$,

$$(3.14) \quad w \prec_\gamma w' \Longleftrightarrow$$
$$\big[[\pi_V(w) \prec_\alpha \pi_V(w')] \text{ or } [\pi_V(w) = \pi_V(w') \text{ and } \pi_{\mathbb{R}}(w') < \pi_{\mathbb{R}}(w)] \big].$$

Note that when $\gamma \in J$ in (2.3), then $M_1 \leqslant \ell < M_2$.

LEMMA 3.5. — *Let $M_1 < M_2$ be integers, suppose $X = \bigcup_{\alpha \in \mathfrak{I}} X_{\alpha, \prec_\alpha}$ is an ordered simplicial p-complex, and let $X^\gamma = X^\gamma_{\prec_\gamma}$ be as in (3.13) and (3.14). Then*

$$(3.15) \qquad X \times [M_1, M_2] = \bigcup_{\gamma \in J} X^\gamma = \bigcup_{\gamma \in J} X^\gamma_{\prec_\gamma}$$

is an ordered simplicial $(p+1)$-complex.

Proof. — As in the proof of Lemma 3.3, applying an affine isomorphism it is easy to see that

$$X_\alpha \times [M_1, M_2] = \bigcup_{\substack{M_1 \leqslant \ell < M_1 \\ v \in \mathrm{Vt}(X_\alpha)}} (X^v_\alpha + (0_V \times \ell))$$

is an ordered simplicial $(p+1)$-complex. The rest follows the proof of Lemma 3.4.   □

We can take $M_1 \to -\infty$ and $M_2 \to +\infty$ in (3.15), as we record next.

COROLLARY 3.6. — *Let X be as in Lemma 3.5 and let*

$$J_\infty := \big\{ (\alpha, v, \ell) \mid \alpha \in \mathfrak{I}, \ v \in \mathrm{Vt}(X_\alpha), \ \ell \in \mathbb{Z} \big\}.$$

Then $X \times \mathbb{R} = \bigcup_{\gamma \in J_\infty} X^\gamma$ is an ordered simplicial $(p+1)$-complex.

Lemma 3.5 shows that any $\rho \in X \times [M_1, M_2]$ belongs to some simplex X^γ with $\gamma = (\alpha, v, \ell)$. Our next result shows that v can be chosen to be a spanning vertex of the projection $\pi_V(\rho) \in X$.

LEMMA 3.7. — *Let X, M_1 and M_2 be as in Lemma 3.5. Suppose $\rho \in X \times [M_1, M_2]$ and $\pi_V(\rho) \in X_\alpha$. Then there is some $v \in \mathrm{SpVt}(\pi_V(\rho)) \subset \mathrm{Vt}(X_\alpha)$ and an integer ℓ satisfying $M_1 \leqslant \ell < M_2$ such that $\rho \in X^\gamma$, where $\gamma = (\alpha, v, \ell)$.*

Proof. — Lemma 3.5 shows that $\rho \in X^\Delta$ for some $\Delta = (\alpha, w, \ell) \in J$. If $w \times \ell$ or $w \times (\ell + 1)$ is a spanning vertex of ρ (with respect to X^Δ), then $w = \pi_V(w \times \ell) = \pi_V(w \times (\ell + 1))$ is a spanning vertex of $\pi_V(\rho)$, and we may pick $v := w$. Otherwise,

$$(3.16) \quad \mathrm{SpVt}(\rho) = \big\{ v_1 \times (\ell + 1), \ldots, v_t \times (\ell + 1), w_1 \times \ell, \ldots, w_r \times \ell \big\},$$

where, by definition (3.15) of X^γ and definition (3.8) of X^v_α, the v_i and w_j are vertices of X_α satisfying

$$v_1 \prec_\alpha v_2 \prec_\alpha \cdots \prec_\alpha v_t \prec_\alpha w \prec_\alpha w_1 \prec_\alpha w_2 \prec_\alpha \cdots \prec_\alpha w_r.$$

Applying π_V to (3.16) we obtain

$$\mathrm{SpVt}(\pi_V(\rho)) = \big\{ v_1, \ldots, v_t, w_1, \ldots, w_r \big\}.$$

If $t > 0$, let $v := v_t$. Otherwise, let $v := w_1$. Then, from (3.16), (3.15) and (3.8), $\mathrm{SpVt}(\rho) \subset \mathrm{Vt}(X^\gamma)$. Hence $\rho \in X^\gamma$ and $v \in \mathrm{SpVt}\left(\pi_V(\rho)\right)$. $\qquad\square$

3.4. The complex Y

Throughout this subsection we assume, as in Proposition 3.2, that $M_1 < M_2$ are integers and that $X = \bigcup_{\alpha \in \mathfrak{I}} X_{\alpha, \prec_\alpha} \subset V$ is a simplicial p-complex contained in some vector space V, with compatible orders $\prec_\alpha$ in the sense of (3.5). So far in this section we have made no use of the linear function $\omega : V \to \mathbb{R}$ in Proposition 3.2. Now we will use ω to make two changes in the construction of the simplices X^γ in Lemma 3.5. First we will replace the given order $\prec_\alpha$ on the vertices of X_α by a new order $\prec_\alpha^A$ which depends on ω. Then, using the piecewise affine map T defined below, we will make a change in the last coordinate of elements of $X \times \mathbb{R}$ to obtain the simplices Y_γ in Proposition 3.2.

Recall that for $u, u' \in \mathrm{Vt}(X_\alpha)$ we defined in (2.6)

$$(3.17) \quad u \prec_\alpha^A u' \iff \left[\left[A(u) < A(u') \right] \text{ or } \left[A(u) = A(u') \text{ and } u \prec_\alpha u' \right] \right],$$

where A is the upper fractional part of ω. A trivial verification shows that if the orders $\prec_\alpha$ are compatible, then so are the orders $\prec_\alpha^A$.

Applying Lemma 3.5 to the ordered simplicial complex $X := \bigcup_{\alpha \in \mathfrak{I}} X_{\alpha, \prec_\alpha^A}$, we obtain the simplicial complex

$$(3.18) \qquad X \times [M_1, M_2] = \bigcup_{\gamma \in J} X^{\gamma, A} =: X^A,$$

where we have called the new simplices $X^{\gamma, A}$ (instead of X^γ) to clarify that the original order $\prec_\alpha$ on the vertices of X_α has been replaced by $\prec_\alpha^A$.[5] We have also denoted by X^A the corresponding simplicial decomposition of $X \times [M_1, M_2]$. By Corollary 3.6, we also get a simplicial complex

$$X \times \mathbb{R} = \bigcup_{\gamma \in J_\infty} X^{\gamma, A} =: X_\infty^A.$$

For $\gamma = (\alpha, v, \ell) \in J_\infty$, i.e. $\alpha \in \mathfrak{I}$, $v \in \mathrm{Vt}(X_\alpha)$ and $\ell \in \mathbb{Z}$, let

$$T_\gamma := V \times \mathbb{R} \to V \times \mathbb{R}$$

be the unique affine function which on any vertex $\sigma \in \mathrm{Vt}\left(X^{\gamma, A}\right)$ satisfies

$$(3.19) \quad T_\gamma(\sigma) := \sigma + (0_V \times (A(\pi_V(\sigma)) - \omega(\pi_V(\sigma)) - 1)) \quad \left(\sigma \in \mathrm{Vt}(X^{\gamma, A})\right),$$

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[5] We do not consider X^A as an ordered complex since this is not the complex Y appearing in Proposition 3.2. To get Y we will still need to apply the map T studied in Lemma 3.8 below.

where $\pi_V : V \times \mathbb{R} \to V$ is the projection to the first component and 0_V is the origin in V. Actually, $T_\gamma(\sigma) \in X \times \mathbb{Z}$ since $\sigma \in X \times \mathbb{Z}$ by (3.8) and (3.13), while $A(v) - \omega(v) \in \mathbb{Z}$ by (2.5). Since $T_\gamma(\sigma)$ depends only on σ (and not on the simplex $X^{\gamma,A}$ to which it belongs) and X_∞^A is a simplicial complex, there is a unique piecewise affine function

(3.20) $T : X \times \mathbb{R} \to X \times \mathbb{R}$,

$$T(\rho) = T_\gamma(\rho) := \sum_{\sigma \in \mathrm{Vt}(X^{\gamma,A})} c_\sigma T_\gamma(\sigma) \quad (\rho \in X^{\gamma,A}),$$

where

(3.21) $$\rho = \sum_{\sigma \in \mathrm{Vt}(X^{\gamma,A})} c_\sigma \sigma \qquad \left(\sum_{\sigma \in \mathrm{Vt}(X^{\gamma,A})} c_\sigma = 1, \ c_\sigma \geqslant 0 \right).$$

Note that $T(\rho) = \rho$ if ω vanishes identically.

LEMMA 3.8. — *Suppose* $X = \bigcup_{\alpha \in \mathfrak{I}} X_{\alpha, \prec_\alpha} \subset V$ *is a simplicial complex in some vector space* V, *with compatible orders* $\prec_\alpha$. *Then the piecewise affine function* $T : X \times \mathbb{R} \to X \times \mathbb{R}$ *defined in* (3.20) *is a bijection. It satisfies the identities*

(3.22) $\pi_V \circ T = \pi_V$,

$$T\left((s \times t) + (0_V \times t')\right) = T(s \times t) + (0_V \times t') \quad (s \in X, \ t, t' \in \mathbb{R}),$$

and

(3.23) $T(s \times t) = s \times (r_s + t) \quad (s \in X, \ t \in \mathbb{R}, \ s \times r_s := T(s \times 0))$.

Proof. — The last identity implies the first two, and hence that T is a bijection. To prove (3.23), let $\rho = s \times t \in X^{\gamma,A}$, where $\gamma = (\alpha, v, \ell)$. From (3.19), (3.20) and (3.21), we get

(3.24)
$$T(\rho) = \sum_{\sigma \in \mathrm{Vt}(X^{\gamma,A})} c_\sigma \left(\sigma + (0_V \times (A(\pi_V(\sigma)) - \omega(\pi_V(\sigma)) - 1))\right)$$
$$= \rho - (0_V \times \omega(\pi_V(\rho))) + (0_V \times q),$$

where $q = q(s \times t) := -1 + \sum_\sigma c_\sigma A\left(\pi_V(\sigma)\right)$.

Now, π_V restricted to $\mathrm{Vt}(X^{\gamma,A})$ is a bijection onto $\mathrm{Vt}(X_\alpha)$, except for the two vertices $v \times \ell$ and $v \times (\ell + 1)$, both of which map to v (see (3.13) and (i) after the proof of Lemma 3.3). From (3.21) we have

$$s = \pi_V(s \times t) = \sum_{\sigma \in \mathrm{Vt}(X^{\gamma,A})} c_\sigma \pi_V(\sigma) = \sum_{\delta \in \mathrm{Vt}(X_\alpha)} d_\delta \delta,$$

where in the last equation we have simply written the barycentric expansion of s with respect to X_α (see the remarks after (3.1)). Since barycentric coordinates with respect to a simplex are unique, we have $c_{v \times \ell} + c_{v \times (\ell+1)} = d_v$ and $c_\sigma = d_{\pi_V(\sigma)}$ for $\pi_V(\sigma) \neq v$. Hence

$$q(s \times t) := -1 + \sum_\sigma c_\sigma A\left(\pi_V(\sigma)\right) = -1 + \sum_{\delta \in \mathrm{Vt}(X_\alpha)} d_\delta A(\delta).$$

But the d_δ are uniquely determined by s, so $q(s \times t) = q(s \times 0)$. Now (3.23) follows from (3.24). $\qquad\square$

By the lemma just proved, the affine function $T_\gamma : V \times \mathbb{R} \to V \times \mathbb{R}$ is injective when restricted to a non-empty open subset of $V \times \mathbb{R}$, namely on the interior of any $X^{\gamma,A}$. Hence T_γ is an affine bijection. Thus

$$T(X^{\gamma,A}) = T_\gamma(X^{\gamma,A}) \subset X \times \mathbb{R}$$

is a $(p+1)$-simplex and

$$T(X \times [M_1, M_2]) = \bigcup_{\gamma \in J} T(X^{\gamma,A})$$

is a $(p+1)$-complex, the isomorphic image by T of the $(p+1)$-complex in (3.18)

(3.25) $$X^A = X \times [M_1, M_2] = \bigcup_{\gamma \in J} X^{\gamma,A}.$$

Since, as remarked before (3.18), X^A is a simplicial complex, so is $T(X^A)$.

We can now prove part of Proposition 3.2.

LEMMA 3.9. — *With the hypotheses and notation of Proposition 3.2, the following hold.*

(i) $Y_\gamma = T(X^{\gamma,A})$, $\mathrm{Vt}(Y_\gamma) = T\left(\mathrm{Vt}(X^{\gamma,A})\right)$ $(\gamma \in J)$, and so by (3.25), $Y = T(X \times [M_1, M_2])$.

(ii) $Y = Y(X, \omega, [M_1, M_2]) = \bigcup_{\gamma \in J} Y_{\gamma, \prec_\gamma}$ is an ordered simplicial $(p+1)$-complex and the orders $\prec_\gamma$ are compatible.

(iii) *Define* $\Omega : V \times \mathbb{R} \to \mathbb{R}$ *by* $\Omega(s \times t) := \omega(s) + t$. *Then for all* $\kappa \in Y_\gamma$,

$$A(v) + \ell - 1 \leqslant \Omega(\kappa) \leqslant A(v) + \ell \qquad (\gamma = (\alpha, v, \ell) \in J).$$

Proof. — The second and third equations in (i) follow directly from the first one and Lemma 3.8. To prove the first equation in (i), we start with the case $\gamma = (\alpha, v, 0)$ with $\alpha \in \mathfrak{I}$ and $v \in \mathrm{Vt}(X_\alpha)$. Unraveling definition (3.18), $X^{\gamma,A} = S^v$, where we have applied Lemma 3.3 to $S_\prec := X_{\alpha, \prec_\alpha^A}$. If we order

the vertices of X_α as $v_0 \prec_\alpha^A v_1 \prec_\alpha^A \cdots \prec_\alpha^A v_p$, then by (3.8) the vertices of $X^{\gamma,A}$ are

$$\left\{ v_0 \times 1, v_1 \times 1, \ldots, v_j \times 1, v_j \times 0, v_{j+1} \times 0, \ldots, v_p \times 0 \right\} \qquad (v_j = v).$$

Applying definition (3.19) of T_γ, the vertices of $T(X^{\gamma,A}) = T_\gamma(X^{\gamma,A})$ are

$$\left\{ v_0 \times (A(v_0) - \omega(v_0)), \ldots, v_j \times (A(v_j) - \omega(v_j)), v_j \times (A(v_j) - \omega(v_j) - 1), \right.$$
$$\left. v_{j+1} \times (A(v_{j+1}) - \omega(v_{j+1}) - 1), \ldots, v_p \times (A(v_p) - \omega(v_p) - 1) \right\}.$$

Comparing this with the definition (2.7) of the vertices of Y_γ, we find that we have proved (i) for $\gamma = (\alpha, v, 0)$. This implies (i) for any $\gamma = (\alpha, v, \ell)$ since $Y_{(\alpha,v,\ell)} = Y_{(\alpha,v,0)} + (0_V \times \ell)$ by (2.7), while by (3.13) and (3.22)

$$X^{(\alpha,v,\ell),A} = X^{(\alpha,v,0),A} + (0_V \times \ell),$$
$$T\left((s \times t) + (0_V \times \ell) \right) = T\left((s \times t) \right) + (0_V \times \ell).$$

As we remarked just before stating Lemma 3.9, the $(p + 1)$-complex $\bigcup_{\gamma \in J} T(X^{\gamma,A})$ is simplicial. By (i), $T([X \times [M_1, M_2]]) = \bigcup_{\gamma \in J} Y_\gamma =: Y$ is simplicial. This proves (ii) since the assumed compatibility of the $\prec_\alpha$ immediately implies that of the $\prec_\gamma$ defined in (2.8).

We now prove (iii). Just as in the proof of (i), it suffices to take $\gamma = (\alpha, v, 0)$. Since Ω is linear, it suffices to prove (iii) for all vertices $\kappa \in \text{Vt}(Y_\gamma)$. Since $\ell = 0$, by (i) we have $\kappa = T(\delta \times t)$, where $\delta \in \text{Vt}(X_\alpha)$, $t = 1$ if $\delta \prec_\alpha^A v$ and $t = 0$ if $v \prec_\alpha^A \delta$. By (3.19),

$$(3.26) \qquad \Omega(\kappa) = \Omega(\delta \times t) + \Omega\left(0_V \times (A(\delta) - \omega(\delta) - 1)\right) = t + A(\delta) - 1.$$

There are two vertices with $\delta = v$, namely $T(v \times 0)$ and $T(v \times 1)$. From (3.26) we have $\Omega\left(T(v \times 0)\right) = A(v) - 1$ and $\Omega\left(T(v \times 1)\right) = A(v)$, proving (iii) for the vertices $T(v \times 0)$ and $T(v \times 1)$.

We may therefore restrict to $\kappa = T(\delta \times t) \in \text{Vt}(Y_\gamma)$ with $\delta \neq v$. We consider first the case $A(\delta) = A(v)$. Then (3.26) gives $\Omega(\kappa) = A(v) - 1$ if $t = 0$, while if $t = 1$ we have $\Omega(\kappa) = A(v)$. Hence (iii) holds if $A(\delta) = A(v)$. If $A(\delta) \neq A(v)$, then by definition (3.17) of the order $\prec_\alpha^A$, we have $t = 1$ if and only if $A(\delta) < A(v)$. If $t = 1$, then (iii) follows from

$$A(v) - 1 \leqslant 0 < A(\delta) = \Omega(\kappa) < A(v),$$

where we again used (3.26) for the equality. If $t = 0$, so $A(\delta) > A(v)$, then

$$A(v) > 0 \geqslant A(\delta) - 1 = \Omega(\kappa) > A(v) - 1,$$

which proves (iii) in the last remaining case. $\qquad\square$

3.5. Λ-invariance

Although we have assumed in Proposition 3.2 that the orders $\prec_\alpha$ on $\mathrm{Vt}(X_\alpha)$ are invariant with respect to a lattice Λ, the new orders $\prec_\alpha^A$ on $\mathrm{Vt}(X_\alpha)$ in general are not Λ-invariant. We shall prove now that this deviation is determined by

$$(3.27) \qquad \Delta_{v,\lambda} := A(v) + A(\lambda) - A(v+\lambda) = \begin{cases} 0 & \text{if } A(v) + A(\lambda) \leqslant 1, \\ 1 & \text{if } A(v) + A(\lambda) > 1, \end{cases}$$

where the last equality used (2.5) and the linearity of ω.

LEMMA 3.10. — *Let $\Lambda \subset V$ be a lattice for which $X = \bigcup_{\alpha \in J} X_{\alpha,\prec_\alpha}$ has Λ-invariant orders $\prec_\alpha$ (see (3.6)), let $\omega : V \to \mathbb{R}$ be a linear function, define $\prec_\alpha^A$ and Δ as in (3.17) and (3.27). Suppose v and w are vertices of X_α and, for some $\lambda \in \Lambda$, $v + \lambda$ and $w + \lambda$ are vertices of $X_{\alpha'}$ ($\alpha, \alpha' \in J$).*

Then, translation preserves order if $\Delta_{v,\lambda} = \Delta_{w,\lambda}$, i.e.

$$(3.28) \qquad v \prec_\alpha^A w \iff v + \lambda \prec_{\alpha'}^A w + \lambda \qquad (\Delta_{v,\lambda} = \Delta_{w,\lambda}),$$

while it reverses order if $\Delta_{v,\lambda} \neq \Delta_{w,\lambda}$, i.e.

$$(3.29) \qquad v \prec_\alpha^A w \iff w + \lambda \prec_{\alpha'}^A v + \lambda \qquad (\Delta_{v,\lambda} \neq \Delta_{w,\lambda}).$$

Moreover,

$$(3.30) \qquad\qquad A(v) = A(w) \iff A(v+\lambda) = A(w+\lambda),$$

and

$$(3.31) \qquad\qquad A(v) = A(w) \implies \Delta_{v,\lambda} = \Delta_{w,\lambda}.$$

Proof. — From (3.27) we obtain

$$(3.32)\ \ \big(A(v+\lambda) - A(w+\lambda)\big) - \big(A(v) - A(w)\big) = \Delta_{w,\lambda} - \Delta_{v,\lambda} = -1, 0,\ \text{or } 1.$$

As A takes values in $(0,1]$, (3.30) and (3.31) follow.

We now turn to the first two claims in the lemma. If $A(v) = A(w)$, then $\Delta_{w,\lambda} = \Delta_{v,\lambda}$ and the orders $\prec_\alpha^A$ and $\prec_{\alpha'}^A$ in (3.28) reduce to $\prec_\alpha$ and $\prec_{\alpha'}$, respectively. In this case (3.28) follows from the assumed Λ-invariance of the order (3.6). We may therefore suppose $A(v) \neq A(w)$. Then $A(v+\lambda) \neq A(w+\lambda)$ and the orders $\prec_\alpha^A$ and $\prec_{\alpha'}^A$ in (3.28) and (3.29) reduce to the usual order of A-values (see (3.17)). If $\Delta_{w,\lambda} = \Delta_{v,\lambda}$, then (3.32) shows that $A(v) - A(w) = A(v+\lambda) - A(w+\lambda)$, so (3.28) follows. If $\Delta_{w,\lambda} \neq \Delta_{v,\lambda}$, then

$$\big|\big(A(v+\lambda) - A(w+\lambda)\big) - \big(A(v) - A(w)\big)\big| = 1.$$

But this is the distance between two real numbers in the open interval $(-1, 1)$, which must therefore have opposite signs. This proves (3.29). $\qquad\square$

LEMMA 3.11. — *Let $\Lambda \subset V$ be a full lattice in a vector space V, let $X = \bigcup_{\alpha \in \mathfrak{I}} X_\alpha \subset V$ be a Λ-complex (see Definition 3.1), $\omega : V \to \mathbb{R}$ a linear function, $Y = \bigcup_{\gamma \in J} Y_{\gamma, \prec_\gamma} \subset V \times \mathbb{R}$ as in (2.2) and $\pi_V : V \times \mathbb{R} \to V$ the projection. Assume that for some $\alpha \in \mathfrak{I}$, $v \in \mathrm{Vt}(X_\alpha)$ and $\lambda \in \Lambda$ we have $(v + \lambda) \in \mathrm{Vt}(X_{\alpha'})$ for some $\alpha' \in \mathfrak{I}$. Suppose also that $\kappa \in \mathrm{Vt}(Y_{(\alpha, v, \ell)})$ and $(\pi_V(\kappa) + \lambda) \in \mathrm{Vt}(X_{\alpha'})$. Then $(\kappa + \widehat{\lambda}) \in \mathrm{Vt}(Y_{(\alpha', v+\lambda, \ell)})$, where*

$$(3.33) \qquad \widehat{\lambda} = \lambda \times j \in \Lambda \times \mathbb{Z}, \quad j := A(\lambda) - \omega(\lambda) - \Delta(v, \lambda),$$

A being as in (2.5) and Δ as in (3.27).

Note that $\widehat{\lambda}$ and $\gamma' := (\alpha', v + \lambda, \ell)$ depend on λ and on $\gamma := (\alpha, v, \ell)$, but not on $\kappa \in \mathrm{Vt}(Y_\gamma)$.

Proof. — That $j \in \mathbb{Z}$ is immediate from (2.5) and (3.27). Now, $\kappa \in \mathrm{Vt}(Y_\gamma) = T\left(\mathrm{Vt}(X^{\gamma, A})\right)$, by Lemma 3.9 (i). Thus,

$$\kappa = T(w \times t), \quad w \times t \in \mathrm{Vt}(X^{\gamma, A}), \quad t \in \{\ell, \ell+1\}, \quad w = \pi_V(\kappa) \in \mathrm{Vt}(X_\alpha).$$

By (3.19),

$$(3.34) \qquad \kappa = T(w \times t) = w \times t + 0_V \times (A(w) - \omega(w) - 1).$$

By assumption, both $v + \lambda$ and $w + \lambda$ are vertices of $X_{\alpha'}$. Thus there exists a vertex

$$(3.35) \quad \rho \in \mathrm{Vt}(X^{\gamma', A}), \ \rho = (w + \lambda) \times t', \ t' \in \{\ell+1, \ell\}, \ \gamma' := (\alpha', v + \lambda, \ell).$$

Notice that for $w \neq v$, the value of t' in (3.35) is determined by $w + \lambda$, but for $w = v$ there are vertices $\rho = (w + \lambda) \times t' \in \mathrm{Vt}(X^{\gamma', A})$ for both values of t'. We now calculate

$$
\begin{aligned}
T(\rho) &= \rho + 0_V \times (A(w + \lambda) - \omega(w + \lambda) - 1) && \text{[by (3.19) and (3.35)]} \\
&= (w + \lambda) \times t' + 0_V \times (A(w) + A(\lambda) - \Delta(w, \lambda) - \omega(w) - \omega(\lambda) - 1) \\
&&& \text{[by (3.27)]} \\
&= w \times t + 0_V \times (A(w) - \omega(w) - 1) + \lambda \times (A(\lambda) - \omega(\lambda) - \Delta(v, \lambda)) \\
&\quad + 0_V \times (\Delta(v, \lambda) - \Delta(w, \lambda) + t' - t) && \text{[t as in (3.34)]} \\
&= \kappa + \widehat{\lambda} + 0_V \times (\Delta(v, \lambda) - \Delta(w, \lambda) + t' - t) && \text{[by (3.33) and (3.34)].}
\end{aligned}
$$

Since $T(\rho) \in \mathrm{Vt}(Y_{\gamma'})$, Lemma 3.11 will be proved once we show

$$(3.36) \qquad\qquad \Delta(v, \lambda) - \Delta(w, \lambda) + t' - t = 0.$$

We will do this by checking various cases, bearing in mind that $\Delta = 0$ or 1 and $t, t' = \ell$ or $\ell + 1$.

If $w = v$, both $(w + \lambda) \times \ell$ and $(w + \lambda) \times (\ell + 1)$ are vertices of $X^{\gamma', A}$. Hence we can pick the vertex ρ so that $t' = t$, making (3.36) hold trivially.

We may therefore assume $w \neq v$, which makes t and t' uniquely determined by $\prec_\alpha^A$ and $\prec_{\alpha'}^A$. Namely, $t = \ell + 1$ is equivalent to $w \prec_\alpha^A v$ (see Lemma 3.3). Otherwise, $t = \ell$. Similarly, $t' = \ell + 1$ if and only if $w + \lambda \prec_{\alpha'}^A v + \lambda$.

If $\Delta(v, \lambda) = \Delta(w, \lambda)$, Lemma 3.10 shows $t = t'$, proving (3.36) in this case. We may therefore assume $\Delta(v, \lambda) \neq \Delta(w, \lambda)$. Thus, again by Lemma 3.10, $t \neq t'$, $A(v) \neq A(w)$ and $A(v + \lambda) \neq A(w + \lambda)$.

We consider first the case $t = \ell$. Then $v \prec_\alpha^A w$, so $A(w) > A(v)$ by definition (3.17) of $\prec_\alpha^A$. Also, $t' \neq t$, so $t' = \ell + 1$. We claim $\Delta(w, \lambda) = 1$. Indeed, otherwise $\Delta(w, \lambda) = 0$ and so $\Delta(v, \lambda) = 1$. Therefore, by (3.27), $A(v) + A(\lambda) > 1$, whence $A(w) + A(\lambda) > 1$. Then, again by (3.27), $\Delta(w, \lambda) = 1$, a contradiction. Hence, $\Delta(w, \lambda) = 1$ and $\Delta(v, \lambda) = 0$, from which (3.36) follows.

Lastly, if $t = \ell + 1$, then $A(w) < A(v)$ and $t' = \ell$. We claim $\Delta(v, \lambda) = 1$. Otherwise $\Delta(v, \lambda) = 0$ and $\Delta(w, \lambda) = 1$. By (3.27), $A(v) + A(\lambda) \leqslant 1$. Hence $A(w) + A(\lambda) < 1$, proving $\Delta(w, \lambda) = 0$, a contradiction. Thus, $\Delta(v, \lambda) = 1$, $\Delta(w, \lambda) = 0$, and (3.36) follows. $\square$

3.6. Proof of Proposition 3.2

We need to show that $Y = \bigcup_{\gamma \in J} Y_{\gamma, \prec_\gamma} \subset V \times \mathbb{R}$ is an ordered $(p + 1)$-complex satisfying inequality (3.7), and properties (i) to (v) in Definition 3.1 with respect to the lattice

$$\widehat{\Lambda} := \Lambda \times (M_2 - M_1)\mathbb{Z}.$$

Parts (ii) and (iii) of Lemma 3.9 show that Y is an ordered simplicial $(p + 1)$-complex with compatible orders and satisfies inequality (3.7). We must still prove that Y has properties (iii) to (v).

We now prove (iii), i.e. if $\kappa, \delta \in \mathrm{Vt}(Y_\gamma)$ and $\kappa + \widehat{\lambda}, \delta + \widehat{\lambda} \in \mathrm{Vt}(Y_{\gamma'})$ for some $\widehat{\lambda} \in \widehat{\Lambda}$, $\gamma = (\alpha, v, \ell)$, $\gamma' = (\alpha', v', \ell') \in J$, then we must show

$$(3.37) \qquad \kappa \prec_\gamma \delta \qquad \Longleftrightarrow \qquad \kappa + \widehat{\lambda} \prec_{\gamma'} \delta + \widehat{\lambda}.$$

By (3.22) $\pi_V \circ T = \pi_V$, so π_V maps $\mathrm{Vt}(Y_\gamma) = T\left(\mathrm{Vt}(X^{\gamma, A})\right)$ to $\mathrm{Vt}(X_\alpha)$ (see (i) in Lemma 3.9 and remark (i) following the proof of Lemma 3.3). Of course, π_V maps $\widehat{\Lambda}$ to Λ. Since X is by assumption a Λ-complex,

$$(3.38) \qquad \pi_V(\kappa) \prec_\alpha \pi_V(\delta) \qquad \Longleftrightarrow \qquad \pi_V(\kappa + \widehat{\lambda}) \prec_{\alpha'} \pi_V(\delta + \widehat{\lambda}).$$

If $\pi_V(\kappa) \neq \pi_V(\delta)$, then $\pi_V(\kappa + \widehat{\lambda}) \neq \pi_V(\delta + \widehat{\lambda})$, so (3.37) is clear from (2.8) and (3.38). When $\pi_V(\kappa) = \pi_V(\delta)$, (2.8) defines

$$\kappa \prec_\gamma \delta \Longleftrightarrow t' < t \qquad (\kappa = u \times t, \ \delta = u \times t').$$

Hence (3.37) is again clear and we have proved (iii).

We now turn to the proof of (iv) in Definition 3.1, i.e. that Y is nearly a fundamental domain for $\widehat{\Lambda} \subset V \times \mathbb{R}$. By assumption, X surjects onto V/Λ. Recall from (3.23) that $T(x \times s) = x \times (r_x + s)$, where $r_x \in \mathbb{R}$ is independent of $s \in \mathbb{R}$. Thus, $Y = T(X \times [M_1, M_2])$ surjects onto $(V \times \mathbb{R})/\widehat{\Lambda}$.

To complete the proof of (iv), we show that the union of interiors $\bigcup_{\gamma \in J} \overset{\circ}{Y_\gamma}$ injects into $(V \times \mathbb{R})/\widehat{\Lambda}$. Suppose $\rho = T(s \times t) \in \overset{\circ}{Y_\gamma}$ and $\rho' = T(s' \times t') \in \overset{\circ}{Y_{\gamma'}}$ map to the same point in $(V \times \mathbb{R})/\widehat{\Lambda}$, for some $\gamma = (\alpha, v, \ell)$, $\gamma' = (\alpha', v', \ell') \in J$. Since ρ is in the interior of $Y_\gamma = T(X^{\gamma, A})$ and $\pi_V(Y_\gamma) = X_\alpha$, it follows that $s = \pi_V(\rho) \in \overset{\circ}{X_\alpha}$. Similarly, $s' \in \overset{\circ}{X_{\alpha'}}$. But we are assuming the congruence

$$(s, r_s + t) = T(s \times t) \equiv T(s' \times t') = s' \times (r_{s'} + t') \pmod{\Lambda \times (M_2 - M_1)\mathbb{Z}},$$

so $s \equiv s' \pmod{\Lambda}$. By assumption, $X = \bigcup_\alpha X_\alpha$ injects into V/Λ. Thus, $s = s'$. Then

$$s \times (r_s + t) = T(s \times t) \equiv T(s \times t') = s \times (r_s + t') \pmod{\widehat{\Lambda}},$$

showing $t - t' \in (M_2 - M_1)\mathbb{Z}$. It $t \neq t'$, then $t = M_1$ or M_2 as $t, t' \in [M_1, M_2]$. However, $t \in \mathbb{Z}$ means that $s \times t$ lies in a proper face of some $X^{\gamma'', A}$ (see remark (ii) after Lemma 3.3). Hence $\rho = T(s \times t)$ lies in a proper face of $Y_{\gamma''}$. Since Y is a simplicial complex, this contradicts $\rho \in \overset{\circ}{Y_\gamma}$, proving (iv).

We now turn to the proof of (v), i.e. the $\widehat{\Lambda}$-equivariance of the spanning vertices. To this end, suppose

$$(3.39) \qquad y, y' \in Y = T(X \times [M_1, M_2]), \quad y' = y + \widetilde{\lambda}, \quad \widetilde{\lambda} = \lambda \times j \in \widehat{\Lambda}.$$

We must show that their spanning vertices satisfy

$$(3.40) \qquad \qquad \mathrm{SpVt}(y'; Y) = \widetilde{\lambda} + \mathrm{SpVt}(y; Y),$$

where for clarity we have written $\mathrm{SpVt}(y; Y)$ to indicate that the spanning vertices are calculated with respect to Y, i.e. with respect to any simplex $Y_\gamma \subset Y$ containing y. We claim that it suffices to prove

$$(3.41) \qquad \qquad \kappa \in \mathrm{SpVt}(y; Y) \quad \Longrightarrow \quad \kappa + \widetilde{\lambda} \in \mathrm{Vt}(Y_{\widetilde{\gamma}})$$

for some $\widetilde{\gamma} \in J$ independent of κ. Indeed, the barycentric expansion

$$(3.42) \qquad y = \sum_{\kappa} c_{\kappa} \kappa \qquad \left(\kappa \in \mathrm{SpVt}(y; Y), \quad \sum_{\kappa} c_{\kappa} = 1, \quad c_{\kappa} > 0 \right)$$

implies $y' = y + \widetilde{\lambda} = \sum_{\kappa} c_{\kappa}(\kappa + \widetilde{\lambda})$, proving (3.40) since $\kappa + \widetilde{\lambda} \in \mathrm{Vt}(Y_{\widetilde{\gamma}})$ by (3.41).

We now prove (3.41). By (3.22) and (3.39),

$$x' = x + \lambda \in X, \quad x := \pi_V(y) \in X, \quad x' := \pi_V(y') \in X, \quad \lambda := \pi_V(\widetilde{\lambda}) \in \Lambda.$$

Since $X = \bigcup_{\beta \in J} X_{\beta}$ is by assumption a Λ-complex,

$$(3.43) \qquad\qquad \mathrm{SpVt}(x'; X) = \lambda + \mathrm{SpVt}(x; X).$$

As $y \in Y = T(X \times [M_1, M_2])$, we have $y = T(\rho) \in Y_{\gamma}$ for some $\gamma = (\alpha, v, \ell) \in J$ and $\rho \in X^{\gamma, A}$. Therefore $\mathrm{SpVt}(y) \subset \mathrm{Vt}(Y_{\gamma})$. Similarly, $y' \in Y_{\gamma'}$ for some $\gamma' = (\alpha', v', \ell') \in J$. By Lemma 3.7 and (3.22), we can assume $v \in \mathrm{SpVt}(x) \subset \mathrm{Vt}(X_{\alpha})$.

Let $\kappa \in \mathrm{SpVt}(y; Y) \subset \mathrm{Vt}(Y_{\gamma})$. Then $\pi_V(\kappa) \in \mathrm{SpVt}(x) \subset \mathrm{Vt}(X_{\alpha})$. By (3.43), $v + \lambda \in \mathrm{Vt}(X_{\alpha'})$ and $\pi_V(\kappa) + \lambda \in \mathrm{Vt}(X_{\alpha'})$. By Lemma 3.11, there is a $\widehat{\lambda} = \lambda \times \widehat{j} \in \Lambda \times \mathbb{Z}$, independent of κ, such that $\kappa + \widehat{\lambda} \in \mathrm{Vt}(Y_{(\alpha', v + \lambda, \ell)})$. By adding $0_V \times q$ to $\widehat{\lambda}$ for some $q \in \mathbb{Z}$, we may ensure $\widehat{\lambda} \in \widehat{\Lambda}$ and

$$(3.44) \qquad\qquad \kappa + \widehat{\lambda} \in \mathrm{Vt}(Y_{(\alpha', v + \lambda, \ell')}),$$

where $\ell' := \ell + q$. By again adding $0_V \times q'(M_2 - M_1)$ to $\widehat{\lambda}$ for some $q' \in \mathbb{Z}$, we may assume $\ell' \in [M_1, M_2)$. Note that $\gamma' = (\alpha', v + \lambda, \ell')$ is independent of κ. Using the barycentric expansion (3.42) of y, let

$$(3.45) \qquad \widehat{y} := \sum_{\kappa} c_{\kappa}(\kappa + \widehat{\lambda}) = y + \widehat{\lambda} \in Y_{\gamma'} \subset Y = T(X \times [M_1, M_2]).$$

As $y', \widehat{y} \in Y$, we can write

$$y' = T(x' \times t'), \qquad \widehat{y} = T(\widehat{x} \times \widehat{t}) \qquad (x', \widehat{x} \in X, \; t', \widehat{t} \in [M_1, M_2]).$$

Applying the projection π_V and (3.22), we find

$$x' = \pi_V(y') = \pi_V(y + \widetilde{\lambda}) = x + \lambda = \pi_V(y + \widehat{\lambda}) = \pi_V(\widehat{y}) = \widehat{x}.$$

Therefore, using (3.23),

$$\begin{aligned} \widetilde{\lambda} - \widehat{\lambda} = y' - \widehat{y} &= T(x' \times t') - T(x' \times \widehat{t}) \\ &= x' \times (r_{x'} + t') - x' \times (r_{x'} + \widehat{t}) = 0_V \times (t' - \widehat{t}). \end{aligned}$$

Since $\widetilde{\lambda} - \widehat{\lambda} \in \widehat{\Lambda} := \Lambda \times (M_2 - M_1)\mathbb{Z}$ and $t', \widehat{t} \in [M_1, M_2]$, there are only three possibilities:

(i) $t' = \widehat{t}$,

(ii) $t' = M_1$, $\widehat{t} = M_2$,

(iii) $t' = M_2$, $\widehat{t} = M_1$.

Note that t' and $\widehat{t}$ are independent of $\kappa \in \mathrm{SpVt}(y)$ since $\widetilde{\lambda}$ and $\widehat{\lambda}$ are independent of κ. Hence the same case (i), (ii) or (iii) above occurs, independently of κ.

In case (i), $\widetilde{\lambda} = \widehat{\lambda}$, so (3.44) implies (3.41) with $\widetilde{\gamma} := \gamma'$. In this case we are done. Consider now case (ii), so $\widehat{y} = T(\widehat{x} \times M_2)$ and

$$(3.46) \qquad \widetilde{\lambda} = \widehat{\lambda} + (0_V \times (M_1 - M_2)) .$$

Order the vertices v_j of $X_{\alpha'}$ so that $v_0 \prec_{\alpha'}^A v_1 \prec_{\alpha'}^A \cdots \prec_{\alpha'}^A v_p$. Using the notation of (3.10), let

$$\left(X^{(\alpha,v,\ell),A} \right)^{\pm} := \left(X_{\alpha,\prec_\alpha^A}^v \right)^{\pm} + (0_V \times \ell) \subset X^{(\alpha,v,\ell),A} \qquad (\ell \in \mathbb{Z}).$$

From (3.11) we have

$$(\widehat{x} \times M_2) \in \left(X^{(\alpha',v_p,M_2-1),A} \right)^{+} \subset X^{(\alpha',v_p,M_2-1),A}.$$

Hence

$$\mathrm{SpVt}(\widehat{x} \times M_2; X^A) \subset \left(X^{(\alpha',v_p,M_2-1),A} \right)^{+},$$

where X^A is the simplicial complex $X^A := \bigcup_\gamma X^{\gamma,A} = X \times [M_1, M_2]$ in (3.18). Thus,

$$\widehat{y} = T(\widehat{x} \times M_2) \in T\left((X^{(\alpha',v_p,M_2-1),A})^+ \right) \subset T(X^{(\alpha',v_p,M_2-1),A})$$
$$= Y_{(\alpha',v_p,M_2-1)}.$$

As $\widehat{y} := \sum_\kappa c_\kappa (\kappa + \widehat{\lambda})$ by (3.45), we see from (3.44) that

$$(3.47) \qquad \kappa + \widehat{\lambda} \in \mathrm{SpVt}(\widehat{y}) \subset \mathrm{Vt}\left(T((X^{(\alpha',v_p,M_2-1),A})^+) \right).$$

By (3.11) again,

$$(X^{(\alpha',v_p,M_2-1),A})^+ = (X^{(\alpha',v_0,M_1),A})^- + (0_V \times (M_2 - M_1)) .$$

Applying T and (3.22) we find

$$(3.48) \quad T\left((X^{(\alpha',v_p,M_2-1),A})^+ \right)$$
$$= T\left((X^{(\alpha',v_0,M_1),A})^- \right) + (0_V \times (M_2 - M_1)) .$$

Using (3.46), (3.47) and (3.48) we obtain

$$\kappa + \widetilde{\lambda} = \kappa + \widehat{\lambda} + (0_V \times (M_1 - M_2)) \subset \mathrm{Vt}\left((Y_{(\alpha',v_0,M_1)})^- \right) \subset \mathrm{Vt}\left(Y_{(\alpha',v_0,M_1)} \right),$$

proving (3.41) with $\widetilde{\gamma} := (\alpha', v_0, M_1)$.

Case (iii) is analogous, for then $\widehat{y} = T(\widehat{x} \times M_1) \in (Y_{(\alpha', v_0, M_1)})^-$ and $\widetilde{\lambda} = \widehat{\lambda} + (0_V \times (M_2 - M_1))$. Reasoning as above, we find $\kappa + \widetilde{\lambda} \in \mathrm{Vt}(Y_{(\alpha', v_p, M_2 - 1)})$, concluding the proof of Proposition 3.2. $\qquad \blacksquare$

4. Degree theory and the signed fundamental domain

In this section we prove Theorem 2.1. Thus we have fixed independent totally positive units $\varepsilon_\ell \in k$, principal logarithms $\log \varepsilon_\ell^{(r_1+j)}$ of these units at the complex embeddings, and integers $N_j \geqslant 3$ ($1 \leqslant \ell \leqslant r := r_1 + r_2 - 1$, $1 \leqslant j \leqslant r_2$). Lastly, we have also fixed a twister function $\beta : \mathbb{Z}^{n-1} \to k^*$, as defined in Section 2.3.

4.1. The function f on $\mathfrak{X} \times \mathbb{R}$

PROPOSITION 4.1. — The $(n-1)$-complex $\mathfrak{X} = \bigcup_{\alpha \in \mathfrak{I}} \mathfrak{X}_\alpha \subset \mathbb{R}^{n-1}$ defined in Section 2.2 is a Λ-complex (see Definition 3.1), where $\Lambda := \mathbb{Z}^r \times N_1 \mathbb{Z} \times N_2 \mathbb{Z} \times \cdots \times N_{r_2} \mathbb{Z} \subset \mathbb{Z}^{n-1}$. It has the following additional properties.

(i) The vertices have integral coordinates, i.e. $\mathrm{Vt}(\mathfrak{X}_\alpha) \subset \mathbb{Z}^{n-1}$ for all $\alpha \in \mathfrak{I}$.

(ii) There are $|\mathfrak{I}| = (n-1)! \prod_{j=1}^{r_2} N_j$ simplices $\mathfrak{X}_\alpha$.

(iii) For any integer $1 \leqslant j \leqslant r_2$ and any index $\alpha \in \mathfrak{I}$, there is an $\alpha_j \in \mathbb{R}$ such that any $\kappa \in \mathfrak{X}_\alpha$ satisfies $\alpha_j \leqslant \Omega_j(\kappa) \leqslant \alpha_j + 1$, where

$$(4.1) \quad \Omega_j(\kappa) := \kappa[r+j] + \frac{N_j}{2\pi} \sum_{\ell=1}^r \kappa[\ell] \arg\left(\varepsilon_\ell^{(r_1+j)}\right)$$

$$(\kappa = (\kappa[1], \ldots, \kappa[n-1]) \in \mathbb{R}^{n-1})$$

and $-\pi < \arg\left(\varepsilon_\ell^{(r_1+j)}\right) \leqslant \pi$.

Proof. — Recall from Section 2.2 that $\mathfrak{X} := X_{n-1}$ was constructed inductively starting from the trivial one-point complex X_0, letting

$$X_j := Y(X_{j-1}, \omega_{j-1}, [0, Q_j])$$

where $Q_j := 1$, $\omega_{j-1}(x) := 0$ if $1 \leqslant j \leqslant r$, and

$$Q_{r+j} := N_j, \quad \omega_{r+j-1}(x) := \frac{N_j}{2\pi} \sum_{\ell=1}^r x[\ell] \arg\left(\varepsilon_\ell^{(r_1+j)}\right) \quad (1 \leqslant j < r_2).$$

Since X_0 is obviously a Λ_0-complex for the trivial lattice $\Lambda_0 := \{0\}$, Proposition 3.2 and induction on j shows that $X_j \subset \mathbb{R}^j$ is a $Q_1\mathbb{Z} \times \cdots \times Q_j\mathbb{Z}$-complex for $1 \leqslant j \leqslant n-1$.[6] Taking $j = n-1$ shows that $\mathfrak{X}$ is a Λ-complex.

Property (i) in Proposition 4.1 was already proved in (2.12), while (ii) follows from (2.4). To prove (iii), let $\kappa \in \mathfrak{X}_\alpha$ and let $\pi_{r+j} : \mathbb{R}^{n-1} \to \mathbb{R}^{r+j}$ be the projection to the first $r + j$ coordinates $(1 \leqslant j \leqslant r_2)$. Then $\pi_{r+j}(\kappa)$ is contained in some $(r + j)$-simplex $Y_\gamma := \pi_{r+j}(\mathfrak{X}_\alpha) \subset X_{r+j}$ for some index $\gamma = (\beta, v, \ell)$ depending only on j and α. Here the complex X_{r+j} was defined in (2.10), v is a vertex in a simplex in X_{j+r-1} indexed by β, and ℓ is an integer in the range $0 \leqslant \ell < N_j$. Inequality (3.7) in Proposition 3.2, applied to $\pi_{r+j}(\kappa) \in Y_\gamma$, to $X := X_{r+j-1}$, $V = \mathbb{R}^{r+j-1}$, $\omega := \omega_{r+j-1}$ and to $[M_1, M_2] := [0, N_j]$, yields (iii) with $\alpha_j := A(v) + \ell - 1$ and $A(v)$ as in (3.7). $\qquad\square$

We now define the function f in (1.1). Henceforth we keep the notation of Proposition 4.1, i.e. $\mathfrak{X}, \mathfrak{I}, \Lambda$ and Ω_j. For $\alpha \in \mathfrak{I}$, let $A_\alpha : \mathbb{R}^{n-1} \to \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ be the unique affine function which on vertices $v = (v[1], \ldots, v[n-1]) \in \mathrm{Vt}(\mathfrak{X}_\alpha) \subset \mathbb{Z}^{n-1}$ is given by

$$(4.2) \qquad A_\alpha(v) := \widetilde{f}(v) := \beta(v) \prod_{\ell=1}^{r} \varepsilon_\ell^{v[\ell]} \in k^* \subset \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}.$$

Here we used (i) of Proposition 4.1 and the twister function β recalled at the beginning of this section. Note that since by definition β is Λ-periodic,

$$(4.3) \qquad \widetilde{f}(v + \lambda) = \widetilde{f}(v) \prod_{\ell=1}^{r} \varepsilon_\ell^{\lambda[\ell]} \qquad (v \in \mathbb{Z}^{n-1}, \ \lambda \in \Lambda).$$

Let $A : \mathfrak{X} \to \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ be the piecewise affine function which equals A_α on $\mathfrak{X}_\alpha$. Since $\mathfrak{X} = \bigcup_{\alpha \in \mathfrak{I}} \mathfrak{X}_\alpha$ is a simplicial complex by Proposition 4.1, and $\widetilde{f}(v)$ in (4.2) does not depend on $\alpha \in \mathfrak{I}$, the A_α define a unique (and continuous) piecewise affine function $A : \mathfrak{X} \to \mathbb{R}$ which restricts to A_α on $\mathfrak{X}_\alpha$. Let

$$(4.4) \qquad f(x \times y) := e^y A(x), \qquad f : \mathfrak{X} \times \mathbb{R} \to \mathbb{R}^{r_1} \times \mathbb{C}^{r_2},$$

where e^y is regarded as a real scalar multiplying each one of the $r_1 + r_2$ coordinates of $A(x)$. Writing $x = \sum_{i=0}^{n-1} c_i v_i$, where $\sum_{i=0}^{n-1} c_i = 1$, $c_i \geqslant 0$ $(0 \leqslant i \leqslant n - 1)$, (3.3) gives

$$(4.5) \ \ f(x \times y) = e^y \sum_{i=1}^{n} c_i w_i \ \left(x \in \mathfrak{X}_\alpha = [v_0, \ldots, v_{n-1}], \ y \in \mathbb{R}, \ w_i := \widetilde{f}(v_i) \right).$$

---

[6] We identify $0 \times \mathbb{R}^j$ with $\mathbb{R}^j$, of course.

We now prove some basic properties of f.

LEMMA 4.2. — *The function f defined in (4.4) above has the following properties.*

(i)

$$(4.6) \quad f(\mathfrak{X}_\alpha \times \mathbb{R}) \subset \mathcal{H}_\alpha := \mathbb{R}_+^{r_1} \times H\left(\frac{\alpha_1 + \frac{1}{2}}{N_1}\right) \times \cdots \times H\left(\frac{\alpha_{r_2} + \frac{1}{2}}{N_{r_2}}\right),$$

where $\alpha_j \in \mathbb{R}$ was defined in Proposition 4.1 (iii) and where $H(t) \subset \mathbb{C}^*$ is the open half-plane

$$H(t) := \left\{ r\, e^{2\pi\, i(t+\theta)} \in \mathbb{C}^* \mid r > 0,\ \theta \in (-1/4, 1/4) \right\}.$$

Thus,

$$(4.7) \qquad\qquad f(\mathfrak{X} \times \mathbb{R}) \subset \mathbb{R}_+^{r_1} \times \mathbb{C}^{*\, r_2}.$$

(ii) *For $\alpha \in J$, let $v_0, \ldots, v_{n-1}$ be the vertices of $\mathfrak{X}_\alpha$ (in any order) and let $w_i := \widetilde{f}(v_i) = f(v_i \times 0) \in k \subset \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, and let $\overline{C}_\alpha$ be as defined in (2.18). Then*

$$(4.8) \quad \overline{C}_\alpha = f(\mathfrak{X}_\alpha \times \mathbb{R})$$
$$= \left\{ t_0 w_0 + t_1 w_1 + \cdots + t_{n-1} w_{n-1} \,\middle|\, t_i \geqslant 0 \text{ for } 0 \leqslant i \leqslant n-1 \right\} - \{0\},$$

the closed k-rational polyhedral cone generated by the w_i, minus the origin.

(iii) *If $y \in \mathbb{R}$ and $x' = x + \lambda$ with $x, x' \in \mathfrak{X}$ and $\lambda \in \Lambda$ (see Proposition 4.1), then*

$$(4.9)\ \ f(x' \times y) = \varepsilon(\lambda) f(x \times y), \quad \varepsilon(\lambda) := \prod_{\ell=1}^{r} \varepsilon_\ell^{\lambda[\ell]} \ \ (\lambda = (\lambda[1], \ldots, \lambda[n-1])).$$

Note that $\varepsilon(\lambda)$ is a unit belonging to the subgroup generated by $\varepsilon_1, \ldots, \varepsilon_r$. The product $\varepsilon(\lambda) f(x \times y)$ is taken in the ring $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. Recall that we regard $k \subset \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$.

Proof. — We begin with the proof of (4.6), i.e. $f(\mathfrak{X}_\alpha \times \mathbb{R}) \subset \mathcal{H}_\alpha$. The set $\mathcal{H}_\alpha$ is a convex cone, so from (4.5) it is clear that it suffices to prove $f(\kappa \times 0) \in \mathcal{H}_\alpha$ for $\kappa \in \mathrm{Vt}(\mathfrak{X}_\alpha)$. From (4.3) and (4.5),

$$(4.10) \qquad f(\kappa \times 0)^{(j)} = \beta(\kappa)^{(j)} \prod_{\ell=1}^{r} \left(\varepsilon_\ell^{(j)}\right)^{\kappa[\ell]} \qquad (1 \leqslant j \leqslant r_1 + r_2).$$

For $1 \leqslant j \leqslant r_1$, $\beta(\kappa)^{(j)} > 0$ and $\varepsilon_\ell^{(j)} > 0$, so $f(\kappa \times 0)^{(j)} > 0$. For $1 \leqslant j \leqslant r_2$, recall from (2.13) in the definition of twisters,

$$(4.11) \quad \beta(\kappa)^{(r_1+j)} = |\beta(\kappa)^{(r_1+j)}| \exp\left(2\pi \,\mathrm{i}(\kappa[r+j] + t_j(\kappa))/N_j\right)$$
$$\left(\frac{|t_j(\kappa)|}{N_j} < \frac{1}{4} - \frac{1}{2N_j}\right).$$

Using Proposition 4.1 (iii) and (4.1), we can write

$$(4.12) \quad \Omega_j(\kappa) := \kappa[r+j] + \frac{N_j}{2\pi} \sum_{\ell=1}^{r} \kappa[\ell] \arg\left(\varepsilon_\ell^{(r_1+j)}\right) = \alpha_j + \frac{1}{2} + \theta_j(\kappa)$$

for some $\theta_j(\kappa) \in [-1/2, 1/2]$. Then, letting $r_j(\kappa) := \left|\widetilde{f}(\kappa)^{(r_1+j)}\right| > 0$, from (4.10) we have

$$\widetilde{f}(\kappa)^{(r_1+j)}$$

$$= r_j(\kappa) \exp\left(2\pi \,\mathrm{i}(\kappa[r+j] + t_j(\kappa))/N_j\right) \exp\left(i \sum_{\ell=1}^{r} \kappa[\ell] \arg\left(\varepsilon_\ell^{(r_1+j)}\right)\right)$$

$$= r_j(\kappa) \exp\left(\frac{2\pi \,\mathrm{i}}{N_j}\left(\kappa[r+j] + \frac{N_j}{2\pi} \sum_{\ell=1}^{r} \kappa[\ell] \arg\left(\varepsilon_\ell^{(r_1+j)}\right)\right)\right) \exp(2\pi \,\mathrm{i}\, t_j(\kappa)/N_j)$$

$$= r_j(\kappa) \exp(2\pi \,\mathrm{i}\, \Omega_j(\kappa)/N_j) \exp(2\pi \,\mathrm{i}\, t_j(\kappa)/N_j)$$

$$= r_j(\kappa) \exp\left(2\pi \,\mathrm{i}(\alpha_j + \tfrac{1}{2}/N_j\right) \exp\left(2\pi \,\mathrm{i}(t_j(\kappa) + \theta_j(\kappa))/N_j\right).$$

By (4.11) and (4.12), $|t_j(\kappa) + \theta_j(\kappa)|/N_j < \frac{1}{4}$, showing that $\widetilde{f}(\kappa)^{(r_1+j)} \in H\left((\alpha_j + \tfrac{1}{2})/N_j\right)$ and proving (4.6).

To prove (4.9), write the barycentric expansion of $x = \sum_i c_i v_i \in \mathfrak{X}_\alpha$ as in (4.5), but only include vertices v_i with $c_i > 0$, i.e. $v_i \in \mathrm{SpVt}(x)$. Since $\mathfrak{X}$ is a Λ-complex by Proposition 4.1, every $v_i + \lambda$ is a spanning vertex of x', and therefore a vertex of some $\mathfrak{X}_{\alpha'}$ where α' is independent of v_i (see property (v) in Definition 3.1). Hence we obtain the barycentric expansion of x' as

$$x' = x + \lambda = \lambda + \sum_i c_i v_i = \sum_i c_i (v_i + \lambda).$$

By (4.5) and (4.3),

$$f(x' \times y) = \mathrm{e}^y \sum_i c_i \widetilde{f}(v_i + \lambda) = \mathrm{e}^y \sum_i c_i \widetilde{f}(v_i)\varepsilon(\lambda) = \varepsilon(\lambda) f(x \times y),$$

proving (4.9).

We now prove (4.8). Since $f(x \times 0) = A_\alpha(x)$ for $x \in \mathfrak{X}_\alpha$, and A_α is an affine function, it follows that $f(\mathfrak{X}_\alpha \times 0) = [w_0, \ldots, w_{n-1}]$, the convex hull of the $w_i := A_\alpha(v_i)$. From (4.4), $f(x \times y) = \mathrm{e}^y f(x \times 0)$. It follows that

$f(\mathfrak{X}_\alpha \times \mathbb{R}) \subset \mathcal{H}_\alpha$ is the closed cone generated by the w_i, minus the origin. It does not contain the origin since $0 \notin \mathcal{H}_\alpha$. $\qquad\square$

4.2. Global and local degree computations

As in Theorem 2.1, let $\varepsilon_1, \ldots, \varepsilon_r$ be fixed independent totally positive units of k, where $r = r_1 + r_2 - 1$. We use them to define a homomorphism g from the additive group $\mathbb{R}^n$ to the multiplicative group $\mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2}$. Namely, let the j-th coordinate of g $(1 \leqslant j \leqslant r_1 + r_2)$ be

(4.13)
$$g(x)^{(j)} := \begin{cases} \exp\left(x[n] + \sum_{\ell=1}^{r} x[\ell] \log \varepsilon_\ell^{(j)}\right) & \text{if } j \leqslant r_1, \\ \exp\left(x[n] + \frac{2\pi\,\mathrm{i}\,x[j+r_2-1]}{N_{j-r_1}} + \sum_{\ell=1}^{r} x[\ell] \log \varepsilon_\ell^{(j)}\right) & \text{if } r_1 < j. \end{cases}$$

Here N_j is as in Theorem 2.1 and $\log z$ denotes the branch with $-\pi < \operatorname{Im}(\log z) \leqslant \pi$.

LEMMA 4.3. — *The homomorphism g in (4.13) is onto, infinitely differentiable and*

(4.14) $g^{-1}\left(\langle \varepsilon_1, \ldots, \varepsilon_r \rangle\right) = \widehat{\Lambda} := \Lambda \times 0 = \mathbb{Z}^r \times N_1\mathbb{Z} \times \cdots \times N_{r_2}\mathbb{Z} \times 0 \subset \mathbb{R}^n$,

where $\langle \varepsilon_1, \ldots, \varepsilon_r \rangle$ is the subgroup of the units of k generated by the ε_ℓ.

Proof. — From (4.13) it is clear that g is infinitely differentiable. Let us show that g is onto. Let $\{E_i\}_{i=1}^n$ be the standard basis for $\mathbb{R}^n$. Then $g(E_i) = \varepsilon_i$ for $1 \leqslant i \leqslant r$, and $g(E_n) = (\mathrm{e}, \ldots, \mathrm{e})$. Define $\mathrm{L} : \mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2} \to \mathbb{R}^{r_1+r_2}$ at $z = \left(z^{(1)}, \ldots, z^{(r_1+r_2)}\right)$ by

(4.15) $\quad (\mathrm{L}(z))_j := e_j \log |z^{(j)}|$

$$(e_j := 1 \text{ if } 1 \leqslant j \leqslant r_1, \ e_j := 2 \text{ if } r_1 < j \leqslant r_1 + r_2).$$

Then $\mathrm{L}(g(x)) = x[n]F + \sum_{\ell=1}^{r} x[\ell]\,\mathrm{L}(\varepsilon_\ell)$, where $F_j := e_j$. As the ε_i are assumed independent and F does not lie in the span of the $\mathrm{L}(\varepsilon_\ell)$, given $z \in \mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2}$, there exists $x \in \mathbb{R}^n$ such that $\mathrm{L}(g(x)) = \mathrm{L}(z)$. But then all coordinates of $zg(-x) \in \mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2}$ have absolute value 1, so $z^{(j)} = g(x)^{(j)} \exp(2\pi\,\mathrm{i}\,\theta_j)$ with $\theta_j \in \mathbb{R}$ $(r_1 < j \leqslant r_1 + r_2)$. On letting $x'[\ell] := x[\ell] + N_{\ell-r}\theta_{\ell-r_2+1}$ for $r < \ell \leqslant n - 1$ and $x'[\ell] := x[\ell]$ for all other ℓ, we have $g(x') = z$.

A similar use of L shows that $x \in \ker(g)$ if and only if $x[j] = 0$ for $1 \leqslant j \leqslant r$ and $j = n$, and $x[r+j] \in N_j\mathbb{Z}$ for $1 \leqslant j \leqslant r_2$. Hence (4.14) follows. $\qquad\square$

LEMMA 4.4. — *With g and $\mathcal{H}_\alpha$ as in (4.13) and (4.6), $g(\mathfrak{X}_\alpha \times \mathbb{R}) \subset \mathcal{H}_\alpha$.*

Proof. — For $1 \leqslant j \leqslant r_1$ it is clear that $g(x \times y)^{(j)} > 0$ for $x \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}$. For $1 \leqslant j \leqslant r_2$, observe that $g(x \times y)^{(r_1+j)} = e^y\, g(x \times 0)^{(r_1+j)}$. Hence it suffices to show that $g(x \times 0) \in \mathcal{H}_\alpha$ for $x \in \mathfrak{X}_\alpha$. Now $g(x \times 0)^{(r_1+j)} = r_{x,j} \exp\left(2\pi\, i\, \Omega_j(x)/N_j\right)$, where $r_{x,j} > 0$ and $\Omega_j(x)$ was defined in (4.1). Property (iii) in Proposition 4.1 concludes the proof since $N_j \geqslant 3$ by assumption. $\qquad\square$

Let $\widehat{\Lambda} \subset \mathbb{R}^n$ be as in (4.14), and define the n-manifolds

$$(4.16) \qquad \widehat{T} := \mathbb{R}^n/\widehat{\Lambda}, \qquad T := (\mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2})/\langle \varepsilon_1, \dots, \varepsilon_r \rangle.$$

It follows from Lemma 4.3 that g induces a bijection $G : \widehat{T} \to T$,

$$(4.17) \qquad \begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\ g\ } & \mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2} \\ {\scriptstyle \widehat{\pi}}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi} \\ \widehat{T} & \xrightarrow{\ G\ } & T \end{array}$$

making the diagram commute. Here $\widehat{\pi}$ and π are the natural quotient maps.

LEMMA 4.5. — *The map $G : \widehat{T} \to T$ defined above is a homeomorphism.*

Proof. — Since $\widehat{T}$ is not compact, it is not quite obvious that G^{-1} is continuous. But, as G is a continuous and bijective map between locally compact metric spaces, it suffices to prove that G is proper. Thus, let $K \subset T$ be compact, and let us show that $G^{-1}(K) \subset \widehat{T}$ is compact. Define the "norm" map $\mathcal{N} : \mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2} \to \mathbb{R}_+$ by

$$(4.18) \qquad \mathcal{N}(z) := \prod_{j=1}^{r_1+r_2} \left| z^{(j)} \right|^{e_j} \qquad \left(z = (z^{(1)}, \dots, z^{(r_1+r_2)}) \right),$$

with $e_j = 1$ or 2 as in (4.15). Then $\mathcal{N}(\varepsilon_i) = 1$ $(1 \leqslant i \leqslant r)$, so $\mathcal{N}$ induces a continuous map $\widetilde{\mathcal{N}} : T \to \mathbb{R}_+$. Since K is compact, $\widetilde{\mathcal{N}}(K) \subset [a,b]$ for some positive real numbers a and b. As $\mathcal{N}\left(g(x[1], \dots, x[n])\right) = e^{nx[n]}$, we have by Lemma 4.3 and (4.16),

$$G^{-1}(K) \subset \widehat{\pi}\left([0,1]^r \times [0,N_1] \times \cdots \times [0,N_{r_2}] \times [(\log a)/n, (\log b)/n]\right),$$

a compact subset of T. $\qquad\square$

From Lemma 4.5 and $\widehat{T} := \mathbb{R}^n/\widehat{\Lambda}$, it is clear that $\widehat{T}$ and T are homeomorphic to an $(n-1)$-torus $\times\, \mathbb{R}$. Thus, G is a homeomorphism between

connected, orientable n-manifolds. We fix any orientation of $\mathbb{R}^n$ and orient $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ by declaring the homeomorphism $\mathcal{I} : \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \to \mathbb{R}^n$,

$$(4.19)\quad \mathcal{I}(z_1, \ldots, z_{r_1+r_2})$$
$$:= (z_1, \ldots, z_{r_1}, \operatorname{Re}(z_{r_1+1}), \operatorname{Im}(z_{r_1+1}), \ldots, \operatorname{Re}(z_{r_1+r_2}), \operatorname{Im}(z_{r_1+r_2})),$$

to be orientation-preserving. The orientation on $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ induces an orientation on its open subset $\mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2} = \operatorname{domain}(\pi)$. Finally, we orient $\widehat{T}$ and T by declaring π and $\widehat{\pi}$ in (4.17) to be orientation-preserving.

In practice, this means for the homeomorphism G that $\deg(G) = \pm 1$ is the sign of the determinant of the Jacobian matrix $\operatorname{Jac}_x(\mathcal{I} \circ g)$ of $\mathcal{I} \circ g : \mathbb{R}^n \to \mathbb{R}^n$ at any point x of its domain, a number which we now compute.[7]

LEMMA 4.6. — *With the orientation on $\widehat{T}$ and T defined above,*

$$\deg(G) = (-1)^{n-1}(-1)^{r_2(r_2-1)/2} \operatorname{sign}(\det(R)) \neq 0,$$

where $R = (r_{ij})$ is the matrix defined in (2.16).

Proof. — We have $\det(R) \neq 0$ since the rows with $i > 1$ span the hyperplane

$$\left\{ x = (x_j)_j \in \mathbb{R}^{r_1+r_2} \,\middle|\, x_1 + \cdots + x_{r_1} + 2x_{r_1+1} + \cdots + 2x_{r_1+r_2} = 0 \right\},$$

while the first row is off the hyperplane. Since G is a homeomorphism, $\deg(G) = \operatorname{locdeg}_p(\widehat{T})$, the local degree at any $p \in \widehat{T}$ [4, (39)]. As $\widehat{\pi}$ and π are orientation-preserving local homeomorphisms, $\deg(G) = \operatorname{locdeg}_x(g)$ for any $x \in \mathbb{R}^n$. By definition of the orientation on $\mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2}$, $\operatorname{locdeg}_x(g) = \operatorname{locdeg}_x(\mathcal{I} \circ g)$, with $\mathcal{I}$ as in (4.19). If the differential $\operatorname{d}(\mathcal{I} \circ g)_x : \mathbb{R}^n \to \mathbb{R}^n$ is invertible at some x, then [4, Prop. 22]

$$\deg(G) = \operatorname{locdeg}_x(\mathcal{I} \circ g) = \operatorname{sign}(\det(\operatorname{d}(\mathcal{I} \circ g)_x)) = \pm 1.$$

To compute the sign of $\det(\operatorname{Jac}_x(\mathcal{I} \circ g))$, note that (4.13) and (4.19) show that the coordinates corresponding to $1 \leqslant j \leqslant r_1$ are

$$((\mathcal{I} \circ g)(x))_j = \mathrm{e}^{x[n]+\sum_{\ell=1}^r x[\ell]\log(\varepsilon_\ell^{(j)})} = \mathrm{e}^{x[n]+\sum_{\ell=1}^r x[\ell]\log|\varepsilon_\ell^{(j)}|}$$

since the ε_ℓ are assumed positive at all real embeddings. For the complex embeddings we have two real coordinates, corresponding to the real and

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[7] The degree of any (proper, continuous) function from an oriented manifold L to itself is independent of the orientation on L. Thus, $\deg(\mathcal{I} \circ g)$ is independent of the orientation of $\mathbb{R}^n$ fixed above. A summary of degree theory in five pages, based on Dold's textbook [5] and sufficient for our purposes, can be found in [4, §7].

imaginary parts of $g(x)^{(j)}$. Thus, setting $\Theta_\ell^{(j)} := \arg(\varepsilon_\ell^{(j+r_1)})$ for $1 \leqslant j \leqslant r_2$,

$$((\mathcal{I} \circ g)(x))_{2j-1+r_1} = e^{x[n]+\sum_{\ell=1}^r x[\ell] \log |\varepsilon_\ell^{(j)}|} \cos\left(\frac{2\pi x[j+r]}{N_j} + \sum_{\ell=1}^r x[\ell] \Theta_\ell^{(j)} \right),$$

$$((\mathcal{I} \circ g)(x))_{2j+r_1} = e^{x[n]+\sum_{\ell=1}^r x[\ell] \log |\varepsilon_\ell^{(j)}|} \sin\left(\frac{2\pi x[j+r]}{N_j} + \sum_{\ell=1}^r x[\ell] \Theta_\ell^{(j)} \right).$$

It is now easy to calculate the matrix $J := \operatorname{Jac}_{x=0}(\mathcal{I} \circ g)$. Namely, abbreviating $L_\ell^{(j)} := \log |\varepsilon_\ell^{(j)}|$ $(1 \leqslant j \leqslant r_1 + r_2,\ 1 \leqslant \ell \leqslant r)$,

$$J = \begin{pmatrix}
L_1^{(1)} & \cdots & L_1^{(r_1)} & L_1^{(r_1+1)} & \Theta_1^{(1)} & \cdots & L_1^{(r_1+r_2)} & \Theta_1^{(r_2)} \\
L_2^{(1)} & \cdots & L_2^{(r_1)} & L_2^{(r_1+1)} & \Theta_2^{(1)} & \cdots & L_2^{(r_1+r_2)} & \Theta_2^{(r_2)} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
L_r^{(1)} & \cdots & L_r^{(r_1)} & L_r^{(r_1+1)} & \Theta_r^{(1)} & \cdots & L_r^{(r_1+r_2)} & \Theta_r^{(r_2)} \\
0 & \cdots & 0 & 0 & 2\pi/N_1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 2\pi/N_{r_2} \\
1 & \cdots & 1 & 1 & 0 & \cdots & 1 & 0
\end{pmatrix}.$$

To compute $\det(J)$, permute the rows so that the bottom row becomes the first row, and row i becomes row $i+1$ $(1 \leqslant i \leqslant n-1)$. This introduces a factor of $(-1)^{n-1}$. Now permute columns so that all columns coming from imaginary parts at complex places are placed to the right of all those coming from the real parts (with the places in the same relative order as before). This gives a factor of $(-1)^{\sum_{j=1}^{r_2-1} j} = (-1)^{r_2(r_2-1)/2}$, and results in a lower-right $r_2 \times r_2$ diagonal block and a lower-left block of 0's of size $r_2 \times (r_1 + r_2)$. Hence,

$$\det(J) = (-1)^{n-1}(-1)^{r_2(r_2-1)/2} \det(R) \prod_{j=1}^{r_2}(2\pi/N_j) \neq 0,$$

from which the lemma follows. $\square$

We now incorporate the map f of Lemma 4.2 into our maps between manifolds. By (4.7) and (4.9), f induces a (continuous) map $F : \widehat{T} \to T$ making

(4.20)

$$\begin{CD}
\mathfrak{X} \times \mathbb{R} @>f>> \mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2} \\
@V\hat{\pi}VV @VV\pi V \\
\widehat{T} @>>F> T
\end{CD}$$

a commutative diagram (4.20),where $\widehat{T}$, $\widehat{\pi}$, T and π are as in (4.17).

LEMMA 4.7. — *Fix the orientation on $\widehat{T}$ and T as in Lemma 4.6. Then F in (4.20) is a proper map and*

$$\deg(F) = \deg(G) = (-1)^{n-1}(-1)^{r_2(r_2-1)/2} \operatorname{sign}(\det(R)) = \pm 1.$$

Proof. — By Lemma 4.6, it suffices to prove that F is proper and that $\deg(F) = \deg(G)$. This will follow once we construct a proper homotopy between G and F. Suppose $x \times y \in \mathfrak{X} \times \mathbb{R}$. Then $x \in \mathfrak{X}_\alpha$ for some $\alpha \in \mathfrak{I}$. By Lemmas 4.2 and 4.4, both $f(x \times y)$ and $g(x \times y)$ lie in the same convex set $\mathcal{H}_\alpha \subset \mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2}$. Hence we can define $\vartheta : \mathfrak{X} \times \mathbb{R} \times [0,1] \to \mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2}$,

$$\vartheta(x \times y \times t) := tf(x \times y) + (1-t)g(x \times y) \quad (x \in \mathfrak{X}, \; y \in \mathbb{R}, \; t \in [0,1]).$$

Note that

$$\vartheta(x \times y \times t) = e^y \, \vartheta(x \times 0 \times t),$$

as $f(x \times y) = e^y f(x \times 0)$ and $g(x \times y) = e^y g(x \times 0)$. For $x \in \mathfrak{X}$, $y \in \mathbb{R}$ and $\lambda \in \Lambda := \mathbb{Z}^r \times N_1 \mathbb{Z} \times \cdots \times N_{r_2} \mathbb{Z}$, we have by (4.9) and (4.13),

$$(4.21) \quad f\left((x+\lambda) \times y\right) = \varepsilon(\lambda) f(x \times y), \quad g\left((x+\lambda) \times y\right) = \varepsilon(\lambda) g(x \times y).$$

As $\varepsilon(\lambda) \in E := \langle \varepsilon_1, \ldots, \varepsilon_r \rangle$, the homotopy ϑ descends to the quotient manifolds,

$$\Theta : \widehat{T} \times [0,1] \to T, \qquad \Theta\left(\widehat{\pi}(x \times y) \times t\right) = \pi\left(\vartheta(x \times y \times t)\right).$$

Since Θ provides a homotopy between F and G, our only remaining task is to show that Θ is a proper map [4, Prop. 21 (4)]. To this end, let $K \subset T$ be compact. We must prove that $\widehat{K} := \Theta^{-1}(K) \subset \widehat{T} \times [0,1]$ is compact. From this it will follow that F is proper, since $F^{-1}(K) = \{\widehat{x} \in \widehat{T} \mid \widehat{x} \times 1 \in \Theta^{-1}(K)\}$.

As the Λ-complex $\mathfrak{X} \subset \mathbb{R}^{n-1}$ is nearly a fundamental domain for Λ (see (iv) in Definition 3.1),

$$(4.22) \qquad\qquad\qquad \widehat{\pi}(\mathfrak{X} \times \mathbb{R}) = \widehat{T},$$

so it suffices to prove the compactness of

$$\mathcal{K} := \widehat{\pi}^{-1}(\widehat{K}) = \vartheta^{-1}\left(\pi^{-1}(K)\right) \subset \mathfrak{X} \times \mathbb{R} \times [0,1].$$

As in the proof of Lemma 4.5, $\widetilde{\mathcal{N}}(K) \subset [a,b]$ for some $a, b > 0$. Since $L := \pi\left(\vartheta(\mathfrak{X} \times 0 \times [0,1])\right)$ is a compact subset of T, $\widetilde{\mathcal{N}}(L) \subset [c,d]$ for some $c, d > 0$. If $x \times y \times t \in \mathcal{K}$, then

$$[a,b] \ni \widetilde{\mathcal{N}}\left(\pi(\vartheta(x \times y \times t))\right) = e^{ny} \widetilde{\mathcal{N}}\left(\pi(\vartheta(x \times 0 \times t))\right) \in e^{ny}[c,d].$$

It follows that $\frac{1}{n} \log(a/d) \leqslant y \leqslant \frac{1}{n} \log(b/c)$. Thus, y is bounded and $\mathcal{K}$ is compact. $\qquad\square$

We now calculate another Jacobian determinant.

LEMMA 4.8. — *Given n affinely independent elements $v_0, v_1, \ldots, v_{n-1} \in \mathbb{R}^{n-1}$, and any n elements $w_0, w_1, \ldots, w_{n-1} \in \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, let $B : \mathbb{R}^{n-1} \to \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ be the unique affine function such that $B(v_i) = w_i$ $(0 \leqslant i \leqslant n-1)$. Define $K : \mathbb{R}^n \to \mathbb{R}^n$ by*

$$K(x[1], \ldots, x[n]) := \mathcal{I}\left(\mathrm{e}^{x[n]} B(x[1], \ldots, x[n-1])\right),$$

where the linear isomorphism $\mathcal{I} : \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \to \mathbb{R}^n$ was defined in (4.19). Then

(4.23) $$\det(\mathrm{Jac}_x(K)) = (-1)^{n-1} \, \mathrm{e}^{nx[n]} \det(W)/\det(V)$$

where $x = (x[1], \ldots, x[n]) \in \mathbb{R}^n$, W is the $n \times n$ matrix whose i-th column is $\mathcal{I}(w_{i-1})$ $(1 \leqslant i \leqslant n)$, and V is the invertible $(n-1) \times (n-1)$ matrix whose i-th column is $v_i - v_0$ $(1 \leqslant i \leqslant n-1)$.

Proof. — The matrix V is invertible since, by definition of affine independence, the $v_i - v_0$ $(1 \leqslant i \leqslant n-1)$ are linearly independent. By (3.3), B satisfies $B\left(\sum_{i=1}^{n} y_i v_i\right) = \sum_{i=1}^{n} y_i w_i$ provided $\sum_{i=1}^{n} y_i = 1$, $y_i \in \mathbb{R}$. Let $E_n := (0, 0, \ldots, 0, 1) \in \mathbb{R}^n$ and $\widetilde{v}_i := v_i \times 0 \in \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$, and define $U : \mathbb{R}^n \to \mathbb{R}^n$ by

$$U(x[1], \ldots, x[n]) := x[n]E_n + \widetilde{v}_0 + \sum_{i=1}^{n-1} x[i](\widetilde{v}_i - \widetilde{v}_0).$$

Then, for $1 \leqslant j \leqslant r_1$,

$$((K \circ U)(x))\,[j] = \mathrm{e}^{x[n]}\left(w_0^{(j)} + \sum_{i=1}^{n-1} x[i](w_i - w_0)^{(j)}\right),$$

while for $1 \leqslant j \leqslant r_2$,

$$((K \circ U)(x))\,[2j-1+r_1]$$
$$= \mathrm{e}^{x[n]}\left(\mathrm{Re}\left(w_0^{(r_1+j)}\right) + \sum_{i=1}^{n-1} x[i]\,\mathrm{Re}\left((w_i - w_0)^{(r_1+j)}\right)\right),$$

$$((K \circ U)(x))\,[2j+r_1]$$
$$= \mathrm{e}^{x[n]}\left(\mathrm{Im}\left(w_0^{(r_1+j)}\right) + \sum_{i=1}^{n-1} x[i]\,\mathrm{Im}\left((w_i - w_0)^{(r_1+j)}\right)\right).$$

The above formulas and elementary row operations yield

$$\det(\mathrm{Jac}_x(K \circ U)) = \mathrm{e}^{nx[n]}(-1)^{n-1}\det(\mathcal{I}(w_0), \ldots, \mathcal{I}(w_n))$$
$$= \mathrm{e}^{nx[n]}(-1)^{n-1}\det(W).$$

Now,

$$\det(\mathrm{Jac}_x(K \circ U)) = \det\big(\mathrm{Jac}_{U(x)}(K)\big) \det(\mathrm{Jac}_x(U))$$
$$= \det\big(\mathrm{Jac}_{U(x)}(K)\big) \det(V),$$

so

$$\det\big(\mathrm{Jac}_{U(x)}(K)\big) = \mathrm{e}^{nx[n]}(-1)^{n-1}\frac{\det(W)}{\det(V)} = \mathrm{e}^{nU(x)[n]}(-1)^{n-1}\frac{\det(W)}{\det(V)}.$$

Since U is surjective, the proof is done. $\qquad\square$

We now use (4.20) to calculate the local degree of F.

LEMMA 4.9. — Let $\widehat{x} = \widehat{\pi}(x) \in \widehat{T}$, where $x = \kappa \times y \in \mathfrak{X}_\alpha \times \mathbb{R}$, let $v_0, \dots, v_{n-1}$ be the vertices of $\mathfrak{X}_\alpha$ (in any order) and let $w_i := f(v_i \times 0)$. Assume that the w_i are $\mathbb{R}$-linearly independent and that κ is an interior point of $\mathfrak{X}_\alpha$. Then the local degree of F at $\widehat{x}$ is defined and

$$(4.24) \qquad \mathrm{locdeg}_{\widehat{x}}(F) = (-1)^{n-1}\, \mathrm{sign}\,(\det(V))\, \mathrm{sign}\,(\det(W)) =: d_\alpha,$$

with V and W as in Lemma 4.8.

Proof. — By Lemma 4.8 and the inverse function theorem, f is a local homeomorphism in a neighborhood of x, so $\mathrm{locdeg}_x(f) = \pm 1$ is defined. Note that $\deg(\mathcal{I}) = 1$ as $\mathcal{I}$ in (4.19) is an orientation-preserving homeomorphism, by definition of the orientation of $\mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2}$. Hence (4.23) shows that $\mathrm{locdeg}_x(f)$ is given by the right-hand side of (4.24). Here we used the fact that the local degree is invariant under composition with orientation-preserving local homeomorphisms [4, Prop. 21 (7)].

We now consider the commutative diagram (4.20). Note that $\widehat{\pi}$ is a local orientation-preserving homeomorphism in a neighborhood of x since $\mathfrak{X}$ is nearly a fundamental domain (see property (iv) in Definition 3.1). The same holds for the covering map π at any point of its domain, and in particular at $f(x)$. Thus, $\mathrm{locdeg}_{\widehat{x}}(F) = \mathrm{locdeg}_x(f)$. $\qquad\square$

Next we prove that the number of points in any orbit $E \cdot x$ inside a cone $\overline{C}_\alpha$ is bounded independently of x, as claimed at the end of Theorem 2.1.

LEMMA 4.10. — Let $\overline{C}_\alpha := f(\mathfrak{X}_\alpha \times \mathbb{R})$ be the cone in (4.8) and let $E := \langle \varepsilon_1, \dots, \varepsilon_r \rangle$ be the subgroup of the units of k generated by the ε_ℓ. Then there exists $c_\alpha \in \mathbb{N}$ such that for any $z \in \mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2}$, the orbit $E \cdot z$ has at most c_α elements in $\overline{C}_\alpha$. Moreover, given any compact subset $K \subset \mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2}$, there are at most finitely many $\varepsilon \in E$ such that $\varepsilon K \cap \overline{C}_\alpha \neq \varnothing$.

Proof. — We begin by proving the first claim. Let $\mathcal{N}$ be the "norm" map defined in (4.18). As $\overline{C}_\alpha$ is a cone, the map $z \to z^\dagger := \mathcal{N}(z)^{-1/n}z$ gives a

bijection between $(E \cdot z) \cap \overline{C}_\alpha$ and $(E \cdot z^\dagger) \cap \overline{C}_\alpha$. Hence we may assume $\mathcal{N}(z) = 1$.

Let m and M be respectively the minimum and maximum values of $\mathcal{N}$ on the convex hull $[w_0, \ldots, w_{n-1}]$ of the generators w_i of $\overline{C}_\alpha$. Then $m, M > 0$, and by homogeneity we have

$$m \left(\sum_{i=0}^{n-1} t_i \right)^n \leqslant \mathcal{N} \left(\sum_{i=0}^{n-1} t_i w_i \right) \leqslant M \left(\sum_{i=0}^{n-1} t_i \right)^n \qquad (t_i \geqslant 0,\ 0 \leqslant i \leqslant n-1).$$

Hence $\Gamma := \ker(\mathcal{N}) \cap \overline{C}_\alpha$ is compact. Applying the logarithmic map L in (4.15), we are reduced to bounding the number of $\varepsilon \in E$ such that $L(z) + L(\varepsilon) \in L(\Gamma)$. This is bounded independently of z since the lattice $L(E)$ is discrete, proving the first claim.

The second claim is proved similarly. Namely, if $\varepsilon K \cap \overline{C}_\alpha \neq \varnothing$ with K compact, then $\varepsilon K^\dagger \cap \overline{C}_\alpha \neq \varnothing$. As $\varepsilon K^\dagger \cap \overline{C}_\alpha \subset \Gamma := \ker(\mathcal{N}) \cap \overline{C}_\alpha$, we need to bound the number of $\varepsilon \in E$ such that $L(z) + L(\varepsilon) \in L(\Gamma)$ for $z \in K^\dagger$. But this is again bounded since $K^\dagger$ is compact. $\qquad\square$

We can now prove that generic points $z \in \mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2}$ satisfy the basic count formula (2.20) in Theorem 2.1. More precisely, let $\partial \mathfrak{X}_\alpha := \mathfrak{X}_\alpha - \overset{\circ}{\mathfrak{X}}_\alpha$ be the boundary of $\mathfrak{X}_\alpha$ and let

$$(4.25) \qquad B := \bigcup_{\substack{\alpha \in \mathfrak{I} \\ \mu_\alpha = 0}} (\mathfrak{X}_\alpha \times \mathbb{R}) \ \cup \bigcup_{\substack{\alpha \in \mathfrak{I} \\ \mu_\alpha \neq 0}} (\partial \mathfrak{X}_\alpha \times \mathbb{R}), \qquad \mathcal{B} := \bigcup_{\varepsilon \in E} \varepsilon f(B),$$

where $E := \langle \varepsilon_1, \ldots, \varepsilon_r \rangle$ and μ_α was defined in (2.15). Note that $\mathcal{B} \subset \mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2}$ is a subset of Lebesgue measure 0 since the cone $\overline{C}_\alpha$ is degenerate if $\mu_\alpha = 0$.

We now prove the following claims.

  (a) If $\mu_\alpha \neq 0$, then f maps $\mathfrak{X}_\alpha \times \mathbb{R}$ bijectively onto the cone $\overline{C}_\alpha$ in (4.8).
  (b) The restriction of $\widehat{\pi}$ to $\mathfrak{X} \times \mathbb{R} - B$ is a bijection onto $\widehat{T} - \widehat{\pi}(B)$.
  (c) F is surjective.

Because of (4.8), to prove (a) it suffices to show that f is injective on $\mathfrak{X}_\alpha \times \mathbb{R}$. Let $x = \sum_{i=0}^{n-1} c_i v_i \in \mathfrak{X}_\alpha$ with $\sum_i c_i = 1$ be the barycentric expansion of $x \in \mathfrak{X}_\alpha$. Similarly, let $x' = \sum_i c_i' v_i \in \mathfrak{X}_\alpha$ and suppose $f(x \times y) = f(x' \times y')$. From (4.5) we find $\mathrm{e}^y \sum_i c_i w_i = \mathrm{e}^{y'} \sum_i c_i' w_i$. As we are assuming $\mu_\alpha \neq 0$, the w_i are linearly independent. Hence $\mathrm{e}^y\, c_i = \mathrm{e}^{y'}\, c_i'$ $(0 \leqslant i \leqslant n-1)$. Summing over i and using $\sum_i c_i = 1 = \sum_i c_i'$ gives $y = y'$ and $c_i = c_i'$, proving claim (a).

Claim (b) follows since the quotient map $\mathbb{R}^{n-1} \to \mathbb{R}^{n-1}/\Lambda$ is surjective when restricted to $\mathfrak{X}$ and injective when restricted to $\bigcup_{\alpha \in \mathfrak{I}} \overset{\circ}{\mathfrak{X}}_\alpha$ (see Proposition 4.1 and (iv) in Definition 3.1). Claim (c) follows from Lemma 4.7, as $\deg(F) \neq 0$ implies the surjectivity of F [4, Prop. 21 (3)].

LEMMA 4.11. — *Let $z \in \mathbb{R}^{r_1}_+ \times \mathbb{C}^{*r_2} - \mathcal{B}$, where $\mathcal{B}$ was defined in (4.25). Then*

$$\sum_{\substack{\alpha \in \mathfrak{I} \\ \mu_\alpha \neq 0}} \mu_\alpha \sum_{\substack{\varepsilon \in E \\ \varepsilon \cdot z \in \overline{C}_\alpha}} 1 = 1,$$

where $\mu_\alpha = d_\alpha/\deg(F)$ was given in (2.15) using (4.24) and Lemma 4.7.

Proof. — With the notation of (4.20), set $\gamma := \pi(z) \in T$ and suppose $\widehat{x} \in F^{-1}(\gamma) \subset \widehat{T}$. Such $\widehat{x}$ exists since F is surjective by (c) above. By (4.22), $\widehat{x} = \widehat{\pi}(x)$ for some $x \in \mathfrak{X} \times \mathbb{R}$. But

$$\pi\left(f(x)\right) = F\left(\widehat{\pi}(x)\right) = F(\widehat{x}) = \gamma = \pi(z),$$

shows that $f(x) = \varepsilon z$ for some $\varepsilon \in E$. As we assumed $z \notin \mathcal{B}$, we have $x \notin B$. By (b) above, $\widehat{x}$ uniquely determines x. Conversely, given $x \in \mathfrak{X} \times \mathbb{R}$ such that $f(x) = \varepsilon z$ for some $\varepsilon \in E$, then $\widehat{x} := \widehat{\pi}(x) \in F^{-1}(\gamma)$. As $f(x) \in \overline{C}_\alpha$ for some $\alpha \in \mathfrak{I}$ with $\mu_\alpha \neq 0$, Lemma 4.10 and (a) above show that the set of such $\widehat{x}$ is finite and that F is a local homeomorphism in a neighborhood of $\widehat{x}$. The local-global principle [4, Prop. 21 (9)], Lemma 4.9 and (a) above give

$$\deg(F) = \deg_\gamma(F) = \sum_{\substack{x \in \mathfrak{X} \times \mathbb{R} \\ \widehat{\pi}(x) \in F^{-1}(\gamma)}} \text{locdeg}_{\widehat{x}}(F) = \sum_{\substack{x \in \mathfrak{X} \times \mathbb{R} \\ f(x) \in E \cdot z}} \text{locdeg}_{\widehat{\pi}(x)}(F)$$

$$= \sum_{\substack{\alpha \in \mathfrak{I} \\ \mu_\alpha \neq 0}} d_\alpha \sum_{\substack{x \in \mathfrak{X}_\alpha \times \mathbb{R} \\ f(x) \in E \cdot z}} 1 = \sum_{\substack{\alpha \in \mathfrak{I} \\ \mu_\alpha \neq 0}} d_\alpha \sum_{\substack{\varepsilon \in E \\ \varepsilon \cdot z \in \overline{C}_\alpha}} 1. \qquad \square$$

4.3. End of proof of Theorem 2.1

Having proved the basic count (2.20) for generic z, we extend it to all $z \in \mathbb{R}^{r_1}_+ \times \mathbb{C}^{*r_2}$ following Colmez's unpublished idea for selecting boundary parts of the $\overline{C}_\alpha$'s. It is only here that we must finally assume that k has at least one real embedding.

LEMMA 4.12. — *Let $v_1, \ldots, v_\ell \in k \subset V := \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ be elements of k with $\ell < n := [k : \mathbb{Q}]$, assume $r_1 \geqslant 1$, and define $e_1 \in V$ by $e_1^{(1)} = 1$ and $e_1^{(j)} = 0$ if $2 \leqslant j \leqslant r_1 + r_2$. Then e_1 is not contained in the real span of $v_1, \ldots, v_\ell$.*

Proof. — Following the proof in [4, Lemma 9], define the $\mathbb{R}$-bilinear form on V,

$$(4.26) \quad \langle z, w \rangle := \sum_{j=1}^{r_1} z^{(j)} w^{(j)} + 2 \sum_{j=r_1+1}^{r_1+r_2} \mathrm{Re}\left(z^{(j)} w^{(j)} \right) \qquad (z, w \in V).$$

Note $\langle z, w \rangle = \mathrm{Trace}_{k/\mathbb{Q}}(zw)$ for $z, w \in k$. As $\ell < n$, there exists $x \in k^*$ such that $\mathrm{Trace}(x v_i) = 0$ for $1 \leqslant i \leqslant \ell$. Let $\psi : V \to \mathbb{R}$ be the $\mathbb{R}$-linear function $\psi(w) := \langle x, w \rangle$. Thus, $\psi(v_i) = 0$, and so $\psi(e_1) = 0$ if $e_1 \in \mathbb{R}v_1 + \cdots + \mathbb{R}v_\ell$. But (4.26) and $r_1 \neq 0$ give $x^{(1)} = \langle x, e_1 \rangle = \psi(e_1) = 0$, contradicting $x \in k^*$. □

For any subset $C \subset V$ and $x, y \in V$, we shall say that $\overrightarrow{x, y}$ pierces C if $y \in C$ and the closed line segment $\overline{x, y}$ connecting x and y intersects the interior of C. We now characterize piercing of a cone in terms of coordinates [4, Lemma 14].

LEMMA 4.13. — *Let $w_0, \ldots, w_{n-1} \in V$ be a basis of a real vector space V, let*

$$\overline{C} = \overline{C}(w_0, \ldots, w_{n-1}) := \left\{ t_0 w_0 + \cdots + t_{n-1} w_{n-1} \,\middle|\, t_i \geqslant 0 \text{ for } 0 \leqslant i \leqslant n-1 \right\}$$

be the corresponding closed polyhedral cone. In the basis $w_0, \ldots, w_{n-1}$, write $x = \sum_{j=0}^{n-1} x_j w_j$ and $y = \sum_{j=0}^{n-1} y_j w_j$. Then

$$\overrightarrow{x, y} \text{ pierces } \overline{C} \quad \Longleftrightarrow \quad \left[y_j \geqslant 0 \ (0 \leqslant j \leqslant n-1) \text{ and } \left[y_j = 0 \Rightarrow x_j > 0 \right] \right].$$

Furthermore, if $\overrightarrow{x, y}$ pierces $\overline{C}$ and $s \in \overrightarrow{x, y}$ is an interior point of $\overline{C}$, then every point of $\overrightarrow{s, y}$ is an interior point of $\overline{C}$, except possibly for y.

Proof. — Suppose $\overrightarrow{x, y}$ pierces $\overline{C}$. Then $y_j \geqslant 0$ since $y \in \overline{C}$. Let $s \in \overrightarrow{x, y}$ be in the interior of $\overline{C}$. Then $s = (1 - t)x + ty$ for some $t \in [0, 1]$ and $s = \sum_j s_j w_j$ with $s_j > 0$ $(0 \leqslant j \leqslant n - 1)$. But $s_j = (1 - t)x_j + ty_j > 0$ implies $x_j > 0$ whenever $y_j = 0$. Conversely, if $y_j \geqslant 0$, and $y_j = 0 \Rightarrow x_j > 0$, then for some sufficiently small positive ϵ and all $t \in [1 - \epsilon, 1)$, the point $s := (1 - t)x + ty$ lies in the interior of $\overline{C}$. To prove the last claim in the lemma, assume $s_j > 0$ $(1 \leqslant j \leqslant n)$. Then $((1 - t)s + ty)_j \geqslant (1 - t)s_j > 0$ if $t \in [0, 1)$. □

Lemma 4.13 shows that an equivalent definition of the cone C_α in (2.19) is

$$C_\alpha = \left\{ y \in \overline{C}_\alpha \mid \overrightarrow{e_1, y} \text{ pierces } \overline{C}_\alpha \right\}.$$

This is useful because of the following "piercing invariance" of e_1.

LEMMA 4.14. — *Assume $r_1 > 0$, let $y \in V := \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ and let $\varepsilon \in \mathbb{R}^{r_1}_+ \times \mathbb{C}^{*r_2}$. Then $\overrightarrow{e_1, y}$ pierces a closed polyhedral cone $\overline{C} = \overline{C}(w_0, \ldots, w_{n-1}) \subset V$ if and only if $\overrightarrow{\varepsilon e_1, y}$ pierces $\overline{C}$.*

Proof. — We may assume that the n generators of $\overline{C}$ are $\mathbb{R}$-linearly independent, for otherwise there is no piercing at all and the lemma is trivial. As $\varepsilon e_1 = \varepsilon^{(1)} e_1$, where the real scalar $\varepsilon^{(1)} > 0$, the lemma follows from Lemma 4.13. □

To complete the proof of Theorem 2.1 we will prove the basic count (2.20) in the form

$$(4.27) \quad 1 = \sum_{\substack{\alpha \in \mathfrak{J} \\ \mu_\alpha \neq 0}} \mu_\alpha \sum_{\varepsilon \in I_\alpha(y)} 1 = \sum_{\substack{\alpha \in \mathfrak{J} \\ \mu_\alpha \neq 0}} \mu_\alpha \operatorname{Card}(I_\alpha(y)) \quad (y \in \mathbb{R}^{r_1}_+ \times \mathbb{C}^{*r_2}),$$

where we have set

$$I_\alpha(y) := \left\{ \varepsilon \in E \,\middle|\, \varepsilon y \in C_\alpha \right\} = \left\{ \varepsilon \in E \,\middle|\, \overrightarrow{e_1, \varepsilon y} \text{ pierces } \overline{C}_\alpha \right\}.$$

Note that Lemma 4.11 established (4.27) only for $y \in \mathbb{R}^{r_1}_+ \times \mathbb{C}^{*r_2} - \mathcal{B}$. If $\mu_\alpha = 0$ we have $I_\alpha(y) = \varnothing$, for in this case $\overline{C}_\alpha$ has an empty interior. As in [4, Lemma 25], we will prove that the I_α stabilize along the path from e_1 to y.

LEMMA 4.15. — *For $t \in [0, 1]$ and $y \in \mathbb{R}^{r_1}_+ \times \mathbb{C}^{*r_2}$, parametrize the line-segment $\overrightarrow{e_1, y}$ by $P_y(t) := (1-t)e_1 + ty$. Then there exists $T_0 = T_0(y) \in (0, 1)$ such that $I_\alpha(y) = I_\alpha(P_y(t))$ for all $\alpha \in \mathfrak{J}$ and all $t \in [T_0, 1]$. Moreover, T_0 can be chosen so that $P_y(t) \notin \mathcal{B}$ for all $t \in [T_0, 1)$.*

Proof. — Suppose $\varepsilon \in I_\alpha(y)$ for some $\alpha \in \mathfrak{J}$, i.e. $\overrightarrow{e_1, \varepsilon y}$ pierces $\overline{C}_\alpha$. By Lemma 4.14, $\overrightarrow{\varepsilon e_1, \varepsilon y}$ pierces $\overline{C}_\alpha$. Thus, for some $s = s(\alpha, \varepsilon) \in (0, 1)$, the point $(1-s)\varepsilon e_1 + s\varepsilon y$ is an interior point of $\overline{C}_\alpha$. By Lemma 4.13, $(1-t)\varepsilon e_1 + t\varepsilon y$ is also an interior point of $\overline{C}_\alpha$ for $s \leqslant t < 1$. But $\varepsilon P_y(t) = (1-t)\varepsilon e_1 + t\varepsilon y$. Hence $\varepsilon \in I_\alpha(P_y(t))$ for $s \leqslant t \leqslant 1$. Thus $I_\alpha(y) \subset I_\alpha(P_y(t))$ for $T_0 \leqslant t \leqslant 1$, where $T_0 := \sup_{\alpha \in \mathfrak{J}, \varepsilon \in I_\alpha(y)} \{s(\alpha, \varepsilon)\}$. Since $\mathfrak{J}$ and $I_\alpha(y)$ are finite sets, we have $T_0 \in (0, 1)$.

We now prove the reverse inclusion, i.e. $I_\alpha(P_y(t)) \subset I_\alpha(y)$ for $t \in [1-\epsilon, 1]$ for some $\epsilon > 0$. Note that $P_y(t) \in \mathbb{R}^{r_1}_+ \times \mathbb{C}^{*r_2}$ for t near 1, as $P_y(1) = y \in \mathbb{R}^{r_1}_+ \times \mathbb{C}^{*r_2}$. If the inclusion claimed is false, then there is a sequence $t_j \in (0, 1)$ converging to 1, and corresponding $\varepsilon_j \in I_\alpha(P_y(t_j))$ with $\varepsilon_j \notin I_\alpha(y)$. Thus $\varepsilon_j P_y(t_j) \in C_\alpha$ but $\varepsilon_j y \notin C_\alpha$. By Lemma 4.10, if $\Gamma \subset \mathbb{R}^{r_1}_+ \times \mathbb{C}^{*r_2}$ is a small neighborhood of y, there are finitely many $\varepsilon \in E$ such that $\varepsilon \Gamma \cap C_\alpha \neq \varnothing$. Hence the set of ε_j is finite. Passing to a subsequence, we may therefore assume $\varepsilon_j = \varepsilon$ is fixed, $\varepsilon \in I_\alpha(P_y(t_j))$, $\varepsilon \notin I_\alpha(y)$. Thus $\overrightarrow{e_1, \varepsilon P_y(t_j)}$ pierces $\overline{C}_\alpha$. In particular, $\varepsilon P_y(t_j) \in \overline{C}_\alpha$. As $P_y(t_j)$ converges to

y and $\overline{C}_\alpha \cup \{0\}$ is closed in $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, it follows that $\varepsilon y \in \overline{C}_\alpha \cup \{0\}$. Since $\varepsilon y \neq 0$, it follows that $\varepsilon y \in \overline{C}_\alpha$. As $\overrightarrow{e_1, \varepsilon P_y(t_j)}$ pierces $\overline{C}_\alpha$, Lemma 4.14 shows that $\overrightarrow{\varepsilon e_1, \varepsilon P_y(t_j)}$ pierces $\overline{C}_\alpha$, i.e. contains an interior point of $\overline{C}_\alpha$. But

$$\overrightarrow{\varepsilon e_1, \varepsilon P_y(t_j)} = \varepsilon \cdot \left(\overrightarrow{e_1, P_y(t_j)} \right) \subset \varepsilon \cdot \left(\overrightarrow{e_1, P_y(1)} \right) = \varepsilon \cdot (\overrightarrow{e_1, y}) = \overrightarrow{\varepsilon e_1, \varepsilon y}.$$

Hence $\overrightarrow{\varepsilon e_1, \varepsilon y}$ contains an interior point of $\overline{C}_\alpha$. As we have already shown that $\varepsilon y \in \overline{C}_\alpha$, we have proved that $\overrightarrow{\varepsilon e_1, \varepsilon y}$ pierces $\overline{C}_\alpha$. Hence, again by Lemma 4.14, $\overrightarrow{e_1, \varepsilon y}$ pierces $\overline{C}_\alpha$. Thus $\varepsilon \in I_\alpha(y)$, contradicting our choice of ε.

To prove the final claim in the lemma, suppose it is false. Then $P_y(t_j) \in \mathcal{B}$ for a sequence $t_j \in (0,1)$ converging to 1. From the definition (4.25) of $\mathcal{B}$ we see that there are $\varepsilon_j \in E$ and $\alpha_j \in \mathfrak{I}$ such that $P_y(t_j) \in \varepsilon_j f(\partial X_{\alpha_j} \times \mathbb{R})$ if $\mu_{\alpha_j} \neq 0$ or $P_y(t_j) \in \varepsilon_j f(X_{\alpha_j} \times \mathbb{R})$ if $\mu_{\alpha_j} = 0$. Thus $\varepsilon_j^{-1} P_y(t_j) \in \overline{C}_{\alpha_j} := f(X_{\alpha_j} \times \mathbb{R})$. Hence the ε_j belong to a finite set. Passing to a subsequence we can assume that the $\varepsilon_j = \varepsilon$ and $\alpha_j = \alpha$ are fixed. If $\mu_\alpha = 0$, the cone generators $w_0, \ldots, w_{n-1}$ in (4.8) are $\mathbb{R}$-linearly dependent. Hence $f(\mathfrak{X}_\alpha \times \mathbb{R}) = \overline{C}_\alpha$ is contained in the $\mathbb{R}$-span H_α of ℓ elements of the number field k, with $\ell < n$. Thus, for two distinct values t_j we have $P_y(t_j) \in \varepsilon H_\alpha$. In particular, the straight line connecting both of these points lies in εH_α. As the $P_y(t_j) \in \overrightarrow{e_1, y}$, this line includes e_1. Thus $e_1 \in \varepsilon H_\alpha$, contradicting Lemma 4.12. If $\mu_\alpha \neq 0$,

$$f(\partial X_\alpha \times \mathbb{R}) \subset \bigcup_{i=0}^{n-1} H_{\alpha,i}, \qquad H_{\alpha,i} := \sum_{\substack{0 \leqslant m \leqslant n-1 \\ m \neq i}} \mathbb{R} w_m.$$

Passing again to a subsequence of the t_j, we may assume $P_y(t_j) \in \varepsilon H_{\alpha,i}$ for a fixed i. This again contradicts Lemma 4.12. $\qquad\square$

We can now finish the proof of Theorem 2.1, i.e. the basic count (2.20) for all $y \in \mathbb{R}_+^{r_1} \times \mathbb{C}^{*r_2}$, as reformulated in (4.27). Using Lemma 4.15, we have $I_\alpha(P_y(t)) = I_\alpha(y)$ and $P_y(t) \notin \mathcal{B}$ for some $t \in (0,1)$. Hence, by Lemma 4.11,

$$\sum_{\substack{\alpha \in \mathfrak{I} \\ \mu_\alpha \neq 0}} \mu_\alpha \sum_{\varepsilon \in I_\alpha(y)} 1 \;=\; \sum_{\substack{\alpha \in \mathfrak{I} \\ \mu_\alpha \neq 0}} \mu_\alpha \sum_{\varepsilon \in I_\alpha(P_y(t))} 1 \;=\; 1.$$

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Milton ESPINOZA
Instituto de Matemáticas
Facultad de Ciencias
Universidad de Valparaíso
Gran Bretaña 1091, 3er piso, Valparaíso (Chile)
milton.espinoza@uv.cl

Eduardo FRIEDMAN
Departamento de Matemática
Facultad de Ciencias
Universidad de Chile
Casilla 653, Santiago (Chile)
friedman@uchile.cl