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Dmitry Jakobson, Frédéric Naud & Louis Soares

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LARGE DEGREE COVERS AND SHARP RESONANCES OF HYPERBOLIC SURFACES

by Dmitry JAKOBSON, Frédéric NAUD & Louis SOARES

Abstract. — Let Γ be a convex co-compact discrete group of isometries of the hyperbolic plane $\mathbb{H}^2$, and $X = \Gamma\backslash\mathbb{H}^2$ the associated surface. In this paper we investigate the behaviour of resonances of the Laplacian $\Delta_{\widetilde{X}}$ for large degree covers of X given by $\widetilde{X} = \widetilde{\Gamma}\backslash\mathbb{H}^2$ where $\widetilde{\Gamma} \lhd \Gamma$ is a finite index normal subgroup of Γ. Using techniques of thermodynamical formalism and representation theory, we prove two new existence results of sharp non-trivial resonances close to $\{\mathrm{Re}(s) = \delta\}$, in the large degree limit, for *abelian* covers and infinite index *congruence subgroups* of $\mathrm{SL}_2(\mathbb{Z})$.

Résumé. — On considère ici des quotients $X = \Gamma\backslash\mathbb{H}^2$ du plan hyperbolique $\mathbb{H}^2$ par des groupes d'isométries convexes co-compacts Γ. On s'intéresse au comportement des résonances du Laplacien $\Delta_{\widetilde{X}}$ où $\widetilde{X} = \widetilde{\Gamma}\backslash\mathbb{H}^2$ est un revêtement Galoisien de haut degré de X. En combinant des techniques de formalisme thermodynamique et de théorie des représentations, on prouve, dans le régime de haut degré, de nouveaux théorèmes d'existence de résonances non-triviales près de l'axe $\{\mathrm{Re}(s) = \delta\}$ pour deux familles de revêtements, les cas abéliens et le cas des congruences.

1. Introduction and results

In mathematical physics, resonances generalize the L^2-eigenvalues in situations where the underlying geometry is non-compact. Indeed, when the geometry has infinite volume, the L^2-spectrum of the Laplacian is mostly continuous and the natural replacement data for the missing eigenvalues are provided by *resonances* which arise from a meromorphic continuation of the resolvent of the Laplacian.

To be more specific, in this paper we will work with the positive Laplacian Δ_X on hyperbolic surfaces $X = \Gamma\backslash\mathbb{H}^2$, where Γ is a geometrically finite,

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discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. A good reference on the subject is the book of Borthwick [7]. Here $\mathbb{H}^2$ is the hyperbolic plane endowed with its metric of constant curvature -1. Let Γ be a geometrically finite Fuchsian group of isometries acting on $\mathbb{H}^2$. This means that Γ admits a finite sided polygonal fundamental domain in $\mathbb{H}^2$. We will require that Γ has no *elliptic* elements different from the identity and that the quotient $\Gamma\backslash\mathbb{H}^2$ is of *infinite hyperbolic area*. If Γ has no parabolic elements (no cusps), then the group is called convex co-compact. We will be working with *non-elementary* groups Γ so that X is never a hyperbolic cylinder, a "trivial" case for which resonances can actually be computed. Under these assumptions, the quotient space $X = \Gamma\backslash\mathbb{H}^2$ is a Riemann surface (called convex co-compact) whose *ends geometry* is well known. The surface X can be decomposed into a compact surface N with boundary, called the Nielsen region, on which two types of ends are glued: funnels and cusps. We refer the reader to the first chapters of Borthwick [7] for a description of the metric in the ends. The limit set $\Lambda(\Gamma)$ is defined as

$$\Lambda(\Gamma) := \overline{\Gamma.z} \cap \partial\mathbb{H}^2,$$

where $z \in \mathbb{H}^2$ is a given point and $\Gamma.z$ is the orbit under the action of Γ which accumulates on the boundary $\partial\mathbb{H}^2$. The limit set Λ does not depend on the choice of z and its Hausdorff dimension $\delta(\Gamma)$ is the critical exponent of Poincaré series [56].

The spectrum of Δ_X on $L^2(X)$ has been fully described by Lax and Phillips and Patterson in [39, 56] as follows:

- The half line $[1/4, +\infty)$ is the continuous spectrum.
- There are no embedded eigenvalues inside $[1/4, +\infty)$.
- The pure point spectrum is empty if $\delta \leqslant \frac{1}{2}$, and finite and starting at $\delta(1-\delta)$ if $\delta > \frac{1}{2}$.
- Moreover, if Γ has some non-trivial parabolic elements (i.e. X has at least one cusp), then $\delta > \frac{1}{2}$.

Using the above notations, the resolvent

$$R(s) := (\Delta_X - s(1-s))^{-1} : L^2(X) \to L^2(X)$$

is a *holomorphic family of operators* for $\mathrm{Re}(s) > \frac{1}{2}$, except at a finite number of possible poles related to the eigenvalues. From the work of Mazzeo–Melrose and Guillopé–Zworski [30, 31, 46], it can be meromorphically continued (to all $\mathbb{C}$) from $C_0^\infty(X) \to C^\infty(X)$, and poles are called *resonances*. We denote in the sequel by $\mathcal{R}_X$ the set of resonances, written with multiplicities.

To each resonance $s \in \mathbb{C}$ (depending on multiplicity) are associated generalized eigenfunctions (so-called purely outgoing states) $\psi_s \in C^\infty(X)$ which provide stationary solutions of the automorphic *wave equation* given by

$$\phi(t,x) = e^{(s-\frac{1}{2})t}\psi_s(x),$$
$$\left(D_t^2 + \Delta_X - \frac{1}{4}\right)\phi = 0.$$

From a physical point of view, $\mathrm{Re}(s) - \frac{1}{2}$ is therefore a *rate of decay* while $\mathrm{Im}(s)$ is a *frequency of oscillation*. Resonances that live the longest are called *sharp resonances* and are those for which $\mathrm{Re}(s)$ is the closest to the unitary axis $\mathrm{Re}(s) = \frac{1}{2}$. In general, $s = \delta$ is the only explicitly known resonance (or eigenvalue if $\delta > \frac{1}{2}$). This resonance is called “leading resonance” since we also know from [48] that there exists a *spectral gap* i. e. one can find $\epsilon(\Gamma) > 0$ such that

$$\mathcal{R}_X \cap \{\mathrm{Re}(s) > \delta - \epsilon(\Gamma)\} = \{\delta\}.$$

A *non-trivial sharp resonance*, is a sharp resonance other than δ. There are very few effective results on the existence of non-trivial sharp resonances, and to our knowledge the best statement so far is due to the first two authors [33], where it is proved that for all $\epsilon > 0$, there are infinitely many resonances in the strip

$$\left\{\mathrm{Re}(s) > \frac{\delta(1-2\delta)}{2} - \epsilon\right\}.$$

It is conjectured in the same paper [33] that for all $\epsilon > 0$, there are infinitely many resonances in the strip $\{\mathrm{Re}(s) > \delta/2 - \epsilon\}$. However, the above result, while proving existence of non-trivial resonances, is typically a high frequency statement and does not provide estimates on the imaginary parts (the frequencies), and it is a notoriously hard problem to locate precisely non-trivial resonances. The goal of the present work is to obtain a different type of existence result by looking at families of covers of a given surface, in the large degree regime. Let us be more specific. Given a *finite index normal subgroup* $\widetilde{\Gamma} \lhd \Gamma$, we denote by

$$\mathbf{G} := \Gamma/\widetilde{\Gamma}$$

the (finite) Galois group (or covering group) of the cover $\pi_{\mathbf{G}}$

$$\pi_{\mathbf{G}} : \widetilde{X} = \widetilde{\Gamma}\backslash\mathbb{H}^2 \to X = \Gamma\backslash\mathbb{H}^2.$$

We have an associated natural projection $P_{\mathbf{G}} : \Gamma \to \mathbf{G}$ such that $\mathrm{Ker}(P_{\mathbf{G}}) = \widetilde{\Gamma}$. We will denote by $|\mathbf{G}|$ the cardinality of $\mathbf{G}$, and our purpose is to investigate the presence of non-trivial resonances, as $|\mathbf{G}|$ becomes large. We mention that since $\mathbf{G}$ is a finite group, we have $\Lambda(\Gamma) = \Lambda(\widetilde{\Gamma})$, hence the leading resonance δ remains the same for all finite covers. The end-game of this paper is to produce new resonances close to δ as $|\mathbf{G}|$ becomes large and see how the algebraic nature of $\mathbf{G}$ affects their location.

A way to attack any problem on resonances of hyperbolic surfaces is through the Selberg zeta function defined for $\mathrm{Re}(s) > \delta$ by

$$Z_\Gamma(s) := \prod_{\mathcal{C}\in\mathcal{P}} \prod_{k\in\mathbb{N}} \left(1 - e^{-(s+k)l(\mathcal{C})}\right),$$

where $\mathcal{P}$ is the set of primitive closed geodesics on $\Gamma\backslash\mathbb{H}^2$ and $l(\mathcal{C})$ is the length. This zeta function extends analytically to $\mathbb{C}$ and it is known from the work of Patterson–Perry [57] that non-trivial zeros of $Z_\Gamma(s)$ are resonances with multiplicities. This zeta function method will be our main tool in the analysis of resonances.

Let $\{\varrho\}$ denote the set of irreducible complex unitary representations of $\mathbf{G}$, and given ϱ we denote by $\chi_\varrho = \mathrm{Tr}(\varrho)$ its character, V_ϱ its linear representation space and we set

$$d_\varrho := \dim_{\mathbb{C}}(V_\varrho).$$

Our first result is the following, it will serve as a general tool to address the problem of resonances in Galois covers.

Theorem 1.1. — *Assume that Γ is convex co-compact. For $\mathrm{Re}(s) > \delta$, consider the L-function defined by*

$$L_\Gamma(s,\varrho) := \prod_{\mathcal{C}\in\mathcal{P}} \prod_{k\in\mathbb{N}} \det\left(\mathrm{Id}_{V_\varrho} - \varrho(\mathcal{C}^{\|}) e^{-(s+k)l(\mathcal{C})}\right),$$

where $\varrho(\mathcal{C})$ is understood as $\varrho(P_{\mathbf{G}}(\gamma_{\mathcal{C}}))$ where $\gamma_{\mathcal{C}} \in \Gamma$ is any representative of the conjugacy class defined by $\mathcal{C}$. Then we have the following facts.

(1) *For all ϱ irreducible, $L_\Gamma(s,\varrho)$ extends as an analytic function to $\mathbb{C}$.*
(2) *There exist $C_1, C_2 > 0$ such that for all p large, all ϱ irreducible representation of $\mathbf{G}$, and all $s \in \mathbb{C}$, we have*

$$|L_\Gamma(s,\varrho)| \leqslant C_1 \exp\left(C_2 d_\varrho \log(1+d_\varrho)(1+|s|^2)\right).$$

(3) *We have the formula valid for all $s \in \mathbb{C}$,*

$$Z_{\tilde{\Gamma}}(s) = \prod_{\varrho \text{ irreducible}} \left(L_\Gamma(s,\varrho)\right)^{d_\varrho}.$$

Notice that the L-function for the *trivial representation* is just $Z_\Gamma(s)$ and thus $Z_\Gamma(s)$ is always a factor of $Z_{\tilde{\Gamma}}(s)$. There is a long story of L-functions associated with compact extensions of geodesic flows in negative curvature, see for example [35, 54, 66]. In the case of pairs of hyperbolic pants with symmetries, a similar type of factorization has been considered for numerical purposes by Borthwick and Weich [8]. The above factorization is very similar to the factorization of Dedekind zeta functions as a product of Artin L-functions in the case of number fields. In the finite area case, after Atle Selberg, this type of factorization was known to Venkov–Zograf [74]. We also point our the related recent work of Pohl–Fedosova [22]. In the context of hyperbolic surfaces with infinite volume, although not surprising, the above statement is new and interesting in itself for various applications, especially (2) which provides necessary a priori bounds on the growth of these L-functions with respect to d_ϱ. We now describe our two main results which deal with two opposite cases, the first one when the Galois group $\mathbf{G}$ is abelian, the other when $\mathbf{G} = \mathrm{SL}_2(\mathbb{F}_p)$, which is as far from abelian as possible.

1.1. Abelian covers

An efficient way to manufacture abelian covers is to use the first homology group with integral coefficients,

$$H^1(X, \mathbb{Z}) \simeq \Gamma/[\Gamma, \Gamma],$$

where $[\Gamma, \Gamma]$ is the commutator subgroup of Γ. Since Γ is actually a free group[(1)] on r symbols (since Γ is always assumed to be non-elementary, we have $r \geqslant 2$, see Section 2 for the Schottky representation in the convex co-compact case), then

$$H^1(X, \mathbb{Z}) \simeq \mathbb{Z}^r.$$

Let us fix a surjective homomorphism $P : \Gamma \to \mathbb{Z}^r$. Fix an integer k with $1 \leqslant k \leqslant r$. Given a sequence of positive integers $(N_1^{(j)}, N_2^{(j)}, \ldots, N_k^{(j)})$ we obtain a surjective map $\widetilde{\pi}_j$ by considering

$$\widetilde{\pi}_j : \begin{cases} \mathbb{Z}^r \to \mathbb{Z}/N_1^{(j)}\mathbb{Z} \times \mathbb{Z}/N_2^{(j)}\mathbb{Z} \times \cdots \times \mathbb{Z}/N_k^{(j)}\mathbb{Z} \\ x = (x_1, \ldots, x_r) \mapsto (x_1 \bmod N_1^{(j)}, \ldots, x_k \bmod N_k^{(j)}) \end{cases}$$

One can then check that

$$\Gamma_j := \mathrm{Ker}(\widetilde{\pi}_j \circ P)$$

(1) It's a pure fact of algebraic topology that the fundamental group of a non-compact surface with finite geometry is free, see for example [71].

is indeed a normal subgroup with Galois group

$$\mathbf{G}_j = \mathbb{Z}/N_1^{(j)}\mathbb{Z} \times \mathbb{Z}/N_2^{(j)}\mathbb{Z} \times \cdots \times \mathbb{Z}/N_k^{(j)}\mathbb{Z}.$$

To avoid artificial sequence extractions, we will use the following hypothesis:

$$\lim_{j\to\infty} \min_{1\leqslant \ell\leqslant k} N_\ell^{(j)} = +\infty. \tag{H}$$

The case $k = 1$ corresponds to cyclic covers, while $k = r$ are full rank abelian covers. We will first prove the following fact.

Theorem 1.2. — *Assume that $X = \Gamma\backslash\mathbb{H}^2$ has at least one cusp, and consider a sequence of abelian covers as above with Galois group $\mathbf{G}_j$, and assume that* (H) *is satisfied. Then for all $\epsilon > 0$, one can find j such that $X_j = \Gamma_j\backslash\mathbb{H}^2$ has at least one non-trivial resonance s with $|s-\delta| \leqslant \epsilon$.*

In the case of compact hyperbolic surfaces, this is a known result proved in 1974 by Burton Randol[(2)] [64]. Note that in the compact case, it follows also from min-max techniques and the Buser inequality, see for example in the book of Bergeron [6, Chapter 3]. In the case of abelian covers of the *modular* surface, this fact was definitely first observed by Selberg, see Selberg's Collected papers [67, paper 33, "On the estimation of Fourier coefficients of modular forms", p. 12]. For more general compact manifolds, we mention the work of R. Brooks [16] (based on Cheeger's constant) which gives sufficient conditions on the fundamental group that guarantees existence of coverings with arbitrarily small spectral gaps.

The outline of the proof is (not surprisingly) as follows: since there is a cusp, we have $\delta > \frac{1}{2}$ and resonances close to δ are actually L^2-eigenvalues. One can then use the fact that Cayley graphs of abelian groups are never expanders combined with some L^2 techniques and Fell's continuity of induction to prove the result, following earlier ideas of Gamburd [27]. The proof of Theorem 1.2 is rather different than the rest of the paper and is found in the last section.

In the convex co-compact case, we can actually prove a much stronger result which goes as follows.

Theorem 1.3. — *Assume that $X = \Gamma\backslash\mathbb{H}^2$ is convex co-compact, and consider a sequence of abelian covers with Galois group $\mathbf{G}_j$ as above, with* (H) *satisfied.*

(2) although there is no interpretation in terms of abelian covers in this early work.

(1) *Then there exists* $\epsilon_0(\Gamma) > 0$ *such that for all* $j \in \mathbb{N}$,

$$\mathcal{R}_{X_j} \cap \{\mathrm{Re}(s) \geqslant \delta - \epsilon_0\}$$

consists of finitely many real *resonances included in the segment* $[\delta - \epsilon_0, \delta]$.

(2) *Moreover, up to a sequence extraction, we have weak convergence in* $C^0([\delta - \epsilon_0, \delta])^*$ *of the spectral measures:*

$$\lim_{j\to+\infty} \frac{1}{|\mathbf{G}_j|} \sum_{\lambda \in \mathcal{R}_j \cap [\delta-\epsilon_0,\delta]} \mathcal{D}_\lambda = \mu,$$

where μ *is an absolutely continuous finite measure fully supported on* $[\delta - \epsilon_0, \delta]$, *and* $\mathcal{D}_\lambda$ *is the Dirac measure at* λ.

(3) *In addition, if* $\lambda \in \mathcal{R}_X$, *then for all* $\varepsilon_0 > 0$ *small enough, one can find* $C_0 > 0$ *such that as* $j \to +\infty$,

$$C_0^{-1}|\mathbf{G}_j| \leqslant \#\mathcal{R}_{X_j} \cap D(\lambda, \varepsilon_0) \leqslant C_0|\mathbf{G}_j|.$$

- The absolutely continuous measure μ depends dramatically on the sequence of covers: a more detailed description of the density is provided in Section 3.
- Since δ belongs to the support of μ, a simple approximation argument shows that for all $\varepsilon > 0$ small enough, we have as $j \to +\infty$,

$$\#\{\lambda \in \mathcal{R}_{X_j} : |\lambda - \delta| < \varepsilon\} \sim C_\varepsilon |\mathbf{G}_j|,$$

  for some constant $C_\varepsilon > 0$.
- Another obvious corollary is that for all $\epsilon > 0$ one can find a finite abelian cover X_j of X such that X_j has a non-trivial resonance ϵ-close to δ. Both Theorems 1.2 and 1.3 *fully cover the case of all geometrically finite surfaces*. We have existence of surfaces with arbitrarily small spectral gap, which was *not known so far*.
- Note that the non-trivial resonances obtained here are *real*: for $\delta > \frac{1}{2}$, this is clear because when close enough to δ they are actually L^2-eigenvalues. However when $\delta \leqslant \frac{1}{2}$, this is not an obvious fact.
- In the general context of scattering theory on spaces with negative curvature, it is to our knowledge the first *exact* asymptotic result on the distribution of resonances, apart from the "trivial" cases of elementary groups or cylindrical manifolds where resonances can be explicitly computed. For a review of the current knowledge on counting results for resonances in various settings, we refer to the recent surveys [51, 75].
- The above theorem fully describes sharp resonances in a small vertical strip close to $\{\mathrm{Re}(s) = \delta\}$. It is the abelian analog of [53].

The proof mostly uses thermodynamical formalism and L-functions to analyse carefully the contribution of L-factors related to characters which are close to the identity. In particular we use in a fundamental way dynamical L-functions related to characters of $\mathbb{Z}^r$ and their representation as Fredholm determinants of suitable transfer operators. A key part of the proof is to prove that for large $\mathrm{Im}(s)$, there are no new resonances in covers: this fact follows from a twisted version of the analysis of [48], however we follow an alternative and much shorter route here by using Fourier decay of Patterson–Sullivan measures as obtained recently by Bourgain–Dyatlov [9].

1.2. Congruence subgroups

Let Γ be an infinite index, finitely generated, free subgroup of $\mathrm{SL}_2(\mathbb{Z})$, without parabolic elements. Because Γ is free, the projection map $\pi : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})$ is injective when restricted to Γ and we will thus identify Γ with $\pi(\Gamma)$, i.e. with its realization as a Fuchsian group. Under the above hypotheses, Γ is a convex co-compact group of isometries. For all $p > 2$ a prime number, we define the congruence subgroup $\Gamma(p)$ by

$$\Gamma(p) := \{\gamma \in \Gamma : \gamma \equiv \mathrm{Id} \mod p\},$$

and we set $\Gamma(0) = \Gamma$. Recently, these "infinite index congruence subgroups" have attracted a lot of attention because of the key role they play in number theory and graph theory. We mention the early work of Gamburd [27] and the more recent by Bourgain–Gamburd–Sarnak [10], Bourgain–Kontorovich [11, 12] and Oh–Winter [53]. In all of the previously mentioned works, the spectral theory of surfaces

$$X_p := \Gamma(p)\backslash\mathbb{H}^2,$$

plays a critical role and knowledge on resonances is mandatory. It should be stressed at this point that unlike in the case of abelian covers treated above, there is a *uniform spectral gap* as $p \to +\infty$, see [10, 27, 53], so it is a completely different situation where the non-commutative nature of $\mathbf{G}$ makes it much more difficult to exhibit new non-trivial resonances in the large p limit.

In [34], the authors have started investigating the behaviour of resonances in *the large p limit* and the present paper goes in the same direction with different techniques involving sharper tools of representation theory.

Note that it is known from Gamburd [27], that the map

$$\pi_p : \begin{cases} \Gamma \to \mathrm{SL}_2(\mathbb{F}_p) \\ \gamma \mapsto \gamma \bmod p \end{cases}$$

is onto for all p large, and we thus have a family of Galois covers $X_p \to X$ with Galois group $\mathbf{G} = \mathrm{SL}_2(\mathbb{F}_p)$. In [34], by combining trace formulae techniques with some a priori upper bounds for $Z_{\Gamma(p)}(s)$ obtained via transfer operator techniques, we proved the following fact. For all $\epsilon > 0$, there exists $C_\epsilon > 0$ such that for all p large enough,

$$C_\epsilon^{-1} p^3 \leqslant \#\mathcal{R}_{X_p} \cap \{|s| \leqslant (\log(p))^\epsilon\} \leqslant C_\epsilon p^3 (\log(p))^{1+2\epsilon}.$$

We point out that $p^3 \asymp \mathrm{Vol}(N_p)$, where $\mathrm{Vol}(N_p)$ is the volume of the convex core of X_p, therefore these bounds can be thought as a *Weyl law* in the large p regime.

In the case of covers of compact or finite volume manifolds, after the pioneering work of Heinz Huber [32], precise results for the Laplace spectrum in the "large degree" limit have been obtained in the past in [21, 29]. We also mention the recent work [40] where a precise asymptotic is proved for sequences of compact hyperbolic surfaces. In the case of infinite volume hyperbolic manifolds, we also mention the density bound obtained by Oh [52].

While this result has near optimal upper and lower bounds, it does not provide a lot of information on the precise location of non-trivial resonances. The second main result of this paper is as follows.

Theorem 1.4. — *Using the above notations, assume that $\delta > \frac{3}{4}$. Then for all $\epsilon, \beta > 0$, and for all p large,*

$$\#\mathcal{R}_{X_p} \cap \left\{\delta - \tfrac{3}{4} - \epsilon \leqslant \mathrm{Re}(s) \leqslant \delta \text{ and } |\mathrm{Im}(s)| \leqslant (\log(\log(p)))^{1+\beta}\right\} \geqslant \frac{p-1}{2}.$$

- Existence of convex co-compact subgroups Γ of $\mathrm{SL}_2(\mathbb{Z})$ with δ_Γ arbitrarily close to 1 is guaranteed by a theorem of Lewis Bowen [13]. See also [27] for some hand-made examples.
- The point of Theorem 1.4 is that we manage to produce non-trivial resonances without having to affect δ, just by moving to a finite cover, and despite the uniform spectral gap. In that sense, our result is somehow complementary to the spectral gap obtained by Gamburd [27].
- It would be interesting to know if the log log bound can be improved to a constant, but this should require different techniques (see the remarks at end of the main proof).

- It is rather clear to us that the methods of proof are robust enough to allow extensions to more general subgroups of arithmetic groups, in the spirit of the recent work of Magee [44], as long as some knowledge of the group structure of the Galois group $\mathbf{G}$ is available (see Section 5).

The outline of the proof is as follows. Having established the factorization formula, we first notice that since the dimension of any non-trivial representation of $\mathbf{G}$ is at least $\frac{p-1}{2}$, it is enough to show that at least one of the L-functions $L_\Gamma(s,\varrho)$ vanishes in the described region as $p \to \infty$. We achieve this goal through an averaging technique (over irreducible ϱ) which takes in account the "explicit" knowledge of the conjugacy classes of $\mathbf{G}$, together with the high multiplicities in the length spectrum of X. Unlike in finite volume cases where one can take advantage of a precise location of the spectrum (for example by assuming GRH), none of this strategy applies here which makes it much harder to mimic existing techniques from analytic number theory.

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2. Factorization formula and a priori bounds

2.1. Bowen-Series coding and transfer operator

The goal of this section is to prove Theorem 1.1. The technique follows closely previous works [34, 50] with the notable addition that we have to deal with vector valued transfer operators. We start by recalling Bowen-Series coding and holomorphic function spaces needed for our analysis. Let $\mathbb{H}^2$ denote the Poincaré upper half-plane

$$\mathbb{H}^2 = \{x + iy \in \mathbb{C} : y > 0\}$$

endowed with its standard metric of constant curvature -1

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}.$$

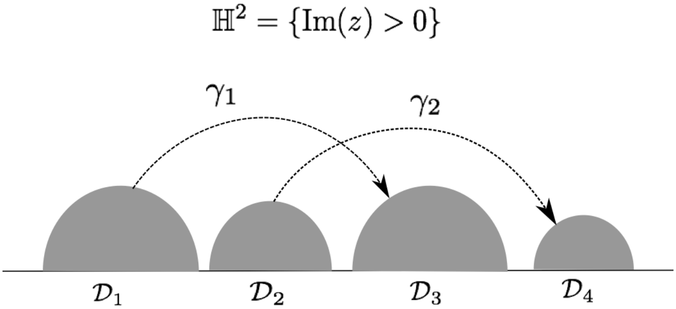

Figure 2.1. Schottky pairing

The group of isometries of $\mathbb{H}^2$ is $\mathrm{PSL}_2(\mathbb{R})$ through the action of 2×2 matrices viewed as Möbius transforms

$$z \mapsto \frac{az+b}{cz+d}, \quad ad - bc = 1.$$

Below we recall the definition of Fuchsian Schottky groups which will be used to define transfer operators. A Fuchsian Schottky group is a free subgroup of $\mathrm{PSL}_2(\mathbb{R})$ built as follows. Let $\mathcal{D}_1, \ldots, \mathcal{D}_r, \mathcal{D}_{r+1}, \ldots, \mathcal{D}_{2r}$, $r \geqslant 2$, be $2r$ Euclidean *open* discs in $\mathbb{C}$ orthogonal to the line $\mathbb{R} \simeq \partial\mathbb{H}^2$. We assume that for all $i \neq j$, $\overline{\mathcal{D}_i} \cap \overline{\mathcal{D}_j} = \emptyset$. Let $\gamma_1, \ldots, \gamma_r \in \mathrm{PSL}_2(\mathbb{R})$ be r isometries such that for all $i = 1, \ldots, r$, we have

$$\gamma_i(\mathcal{D}_i) = \widehat{\mathbb{C}} \setminus \overline{\mathcal{D}_{r+i}},$$

where $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ stands for the Riemann sphere. For notational purposes, we also set $\gamma_i^{-1} =: \gamma_{r+i}$.

Let Γ be the free group generated by γ_i, γ_i^{-1} for $i = 1, \ldots, r$, then Γ is a convex co-compact group, i.e. it is finitely generated and has no non-trivial parabolic element. The *converse is true*: up to isometry, convex co-compact hyperbolic surfaces can be obtained as a quotient by a group as above, see [17].

For all $j = 1, \ldots, 2r$, set $I_j := \mathcal{D}_j \cap \mathbb{R}$. One can define a map

$$T : I := \bigcup_{j=1}^{2r} I_j \to \mathbb{R} \cup \{\infty\}$$

by setting

$$T(x) = \gamma_j(x) \quad \text{if } x \in I_j.$$

This map encodes the dynamics of the full group Γ, and is called the Bowen-Series map, see [15] for the genesis of these type of coding. The key properties are orbit equivalence and uniform expansion of T on the maximal invariant subset $\cap_{n\geqslant 1}T^{-n}(I)$ which coincides with the limit set $\Lambda(\Gamma)$, see for example [7].

We now define the function space and the associated transfer operators. Set

$$\Omega := \bigcup_{j=1}^{2r} \mathcal{D}_j.$$

Each complex representation space V_ϱ is endowed with an inner product $\langle\,\cdot\,,\cdot\,\rangle_\varrho$ which makes each representation

$$\varrho : \mathbf{G} \to \mathrm{End}(V_\varrho)$$

unitary, where we use the notations of Section 1 i.e. $\mathbf{G}$ is the Galois group of the cover $\pi_{\mathbf{G}} : \widetilde{X} \to X$, and we have the associated natural projection $P_{\mathbf{G}} : \Gamma \to \mathbf{G}$ such that $\mathrm{Ker}(P_{\mathbf{G}}) = \widetilde{\Gamma}$.

Consider now the Hilbert space $H^2_\varrho(\Omega)$ which is defined as the set of *vector valued holomorphic functions* $F : \Omega \to V_\varrho$ such that

$$\|F\|^2_{H^2_\varrho} := \int_\Omega \|F(z)\|^2_\varrho \mathrm{d}m(z) < +\infty,$$

where $\mathrm{d}m$ is Lebesgue measure on $\mathbb{C}$. On the space $H^2_\varrho(\Omega)$, we define a “twisted” by ϱ transfer operator $\mathcal{L}_{\varrho,s}$ by

$$\mathcal{L}_{\varrho,s}(F)(z) := \sum_j ((T')(T_j^{-1}))^{-s} F(y)\varrho(T_j^{-1}) \\ = \sum_{j\neq i} (\gamma_j')^s F(\gamma_j z)\varrho(\gamma_j), \quad \text{if } z \in \mathcal{D}_i,$$

where $s \in \mathbb{C}$ is the spectral parameter. Here $\varrho(\gamma_j)$ is understood as

$$\varrho(P_{\mathbf{G}}(\gamma_j)), \gamma_j \in \mathrm{SL}_2(\mathbb{Z}).$$

We also point out that the linear map $\varrho(g)$ acts “on the right” on vectors $U \in V_\varrho$ simply by fixing an orthonormal basis $\mathcal{B} = (e_1, \dots, e_{d_\varrho})$ of V_ϱ and setting

$$U\varrho(g) := (U_1, \dots, U_{d_\varrho})\mathcal{M}at_{\mathcal{B}}(\rho(g)).$$

Notice that for all $j \neq i$, $\gamma_j : \mathcal{D}_i \to \mathcal{D}_{r+j}$ is a holomorphic contraction since $\overline{\gamma_j(\mathcal{D}_i)} \subset \mathcal{D}_{r+j}$. Therefore, $\mathcal{L}_{\varrho,s}$ is a compact *trace class* operator and thus has a *Fredholm determinant*. We start by recalling a few facts.

We need to introduce some more notations. Considering a finite sequence α with

$$\alpha = (\alpha_1, \dots, \alpha_n) \in \{1, \dots, 2r\}^n,$$

we set

$$\gamma_\alpha := \gamma_{\alpha_1} \circ \cdots \circ \gamma_{\alpha_n}.$$

We then denote by $\mathscr{W}_n$ the set of admissible sequences of length n by

$$\mathscr{W}_n := \{\alpha \in \{1, \dots, 2r\}^n : \forall\, i = 1, \dots, n-1, \alpha_{i+1} \neq \alpha_i + r \bmod 2r\}.$$

The set $\mathscr{W}_n$ is simply the set of reduced words of length n. For all $j = 1, \dots, 2r$, we define $\mathscr{W}_n^j$ by

$$\mathscr{W}_n^j := \{\alpha \in \mathscr{W}_n : \alpha_n \neq j\}.$$

If $\alpha \in \mathscr{W}_n^j$, then γ_α maps $\overline{\mathcal{D}_j}$ into $\mathcal{D}_{\alpha_1+r}$. Using this set of notations, we have the formula for all $z \in \mathcal{D}_j$, $j = 1, \dots, 2r$,

$$\mathcal{L}^N_{\varrho,s}(F)(z) = \sum_{\alpha \in \mathscr{W}_N^j} (\gamma'_\alpha(z))^s F(\gamma_\alpha z) \varrho(\gamma_\alpha).$$

A key property of the contraction maps γ_α is that they are *eventually uniformly contracting*, see [7, Proposition 15.4]: there exist $C > 0$ and $0 < \rho_2 < \rho_1 < 1$ such that for all $z \in \mathcal{D}_j$, for all $\alpha \in \mathscr{W}_n^j$ we have for all $n \geqslant 1$,

$$C^{-1}\rho_2^n \leqslant \sup_{z \in \mathcal{D}_j} |\gamma'_\alpha(z)| \leqslant C\rho_1^n \tag{2.1}$$

In addition, they have the *bounded distortion property* (see [50] for proofs): There exists $M_1 > 0$ such that for all n, j and all $\alpha \in \mathscr{W}_n^j$, we have for all $z \in \mathcal{D}_j$,

$$\left| \frac{\gamma''_\alpha(z)}{\gamma'_\alpha(z)} \right| \leqslant M_1. \tag{2.2}$$

We will also need to use the *topological pressure* as a way to estimate certain weighted sums over words. We will rely on the following fact [50]. Fix $\sigma_0 \in \mathbb{R}$, then there exists $C(\sigma_0)$ such that for all n and $\sigma \geqslant \sigma_0$, we have

$$\sum_{j=1}^{2r} \left(\sum_{\alpha \in \mathscr{W}_n^j} \sup_{\mathcal{D}_j} |\gamma'_\alpha|^\sigma \right) \leqslant C(\sigma_0) e^{nP(\sigma_0)}. \tag{2.3}$$

Here $\sigma \mapsto P(\sigma)$ is the *topological pressure*, which is a strictly convex decreasing function which vanishes at $\sigma = \delta$, see [14]. In particular, whenever $\sigma > \delta$, we have $P(\sigma) < 0$. A definition of $P(\sigma)$ is by a variational formula:

$$P(\sigma) = \sup_\mu \left(h_\mu(T) - \sigma \int_\Lambda \log |T'| \mathrm{d}\mu \right),$$

where μ ranges over the set of T-invariant probability measures, and $h_\mu(T)$ is the measure theoretic entropy. For general facts on topological pressure and thermodynamical formalism we refer to [55]. We will also use it in Section 4.

2.2. Norm estimates and determinant identity

We start with an a priori norm estimate that will be used later on, see also [34] where a similar bound (on a different function space) is proved in appendix.

Proposition 2.1. — *Fix $\sigma = \mathrm{Re}(s) \in \mathbb{R}$, then there exists $C_\sigma > 0$, independent of $\mathbf{G}, \varrho$ such that for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) = \sigma$ and all N we have*

$$\|\mathcal{L}_{\varrho,s}^N\|_{H^2_\varrho} \leqslant C_\sigma e^{C_\sigma |\mathrm{Im}(s)|} e^{2NP(\sigma)}.$$

Proof. — First we need to be more specific about the complex powers involved here. First we point out that given $z \in \mathcal{D}_i$ then for all $j \neq i$, $\gamma_j'(z)$ belongs to $\mathbb{C} \setminus (-\infty, 0]$, simply because each γ_j is in $\mathrm{PSL}_2(\mathbb{R})$. This make it possible to define $\gamma_j'(z)^s$ by

$$\gamma_j'(z)^s := e^{s\mathbb{L}(\gamma_j'(z))},$$

where $\mathbb{L}(z)$ is the complex logarithm defined on $\mathbb{C} \setminus (-\infty, 0]$ by the contour integral

$$\mathbb{L}(z) := \int_1^z \frac{\mathrm{d}\zeta}{\zeta}.$$

By analytic continuation, the same identity holds for iterates. In particular, because of bound (2.1) and also bound (2.2) one can easily show that there exists $C_1 > 0$ such that for all N, j and all $\alpha \in \mathscr{W}_N^j$, we have

$$\sup_{z \in \mathcal{D}_j} |\gamma_\alpha'(z)^s| \leqslant e^{C_1 |\mathrm{Im}(s)|} \sup_{\mathcal{D}_j} |\gamma_\alpha'|^\sigma, \tag{2.4}$$

where $\sigma = \mathrm{Re}(s)$. We can now compute, given $F \in H^2_\varrho(\Omega)$,

$$\|\mathcal{L}_{\varrho,s}^N(F)\|^2_{H^2_\varrho} := \sum_{j=1}^{2m} \sum_{\alpha,\beta \in \mathscr{W}_N^j} \int_{\mathcal{D}_j} \gamma_\alpha'(z)^s \overline{\gamma_\beta'(z)^s} \langle F(\gamma_\alpha z)\varrho(\gamma_\alpha), F(\gamma_\beta z)\varrho(\gamma_\beta)\rangle_\varrho \mathrm{d}m(z).$$

By unitarity of ϱ and Schwarz inequality we obtain

$$\|\mathcal{L}^N_{\varrho,s}(F)\|^2_{H^2_\varrho} \leqslant e^{2C_1|\mathrm{Im}(s)|}\sum_j\sum_{\alpha,\beta}\sup_{\mathcal{D}_j}|\gamma_\alpha'|^\sigma \sup_{\mathcal{D}_j}|\gamma_\beta'|^\sigma \int_{D_j}\|F(\gamma_\alpha z)\|_\varrho\|F(\gamma_\beta z)\|_\varrho \mathrm{d}m(z).$$

We now remark that $z\mapsto F(z)$ has components in $H^2(\Omega)$, the Bergman space of L^2 holomorphic functions on $\Omega=\bigcup_j \mathcal{D}_j$, so we can use the scalar reproducing kernel $B_\Omega(z,w)$ to write (in a vector valued way)

$$F(\gamma_\alpha z)=\int_\Omega F(w)B_\Omega(\gamma_\alpha z,w)\mathrm{d}m(w).$$

Therefore we get

$$\|F(\gamma_\alpha z)\|_\varrho \leqslant \int_\Omega \|F(w)\|_\varrho |B_\Omega(\gamma_\alpha z,w)|\mathrm{d}m(w),$$

and by Schwarz inequality we obtain

$$\sup_{z\in\mathcal{D}_j}\|F(\gamma_\alpha z)\|_\varrho \leqslant \|F\|_{H^2_\varrho}\left(\int_\Omega |B_\Omega(\gamma_\alpha z,w)|^2\mathrm{d}m(w)\right)^{1/2}.$$

Observe now that by uniform contraction of branches $\gamma_\alpha:\mathcal{D}_j\to\Omega$, there exists a compact subset $K\subset\Omega$ such that for all N,j and $\alpha\in\mathscr{W}_N^j$,

$$\gamma_\alpha(\mathcal{D}_j)\subset K.$$

We can therefore bound

$$\int_\Omega |B_\Omega(\gamma_\alpha z,w)|^2\mathrm{d}m(w)\leqslant C$$

uniformly in z,α. We have now reached

$$\|\mathcal{L}^N_{\varrho,s}(F)\|^2_{H^2_\varrho}\leqslant \|F\|^2_{H^2_\varrho}C_2e^{2C_1|\mathrm{Im}(s)|}\sum_j\sum_{\alpha,\beta}\sup_{\mathcal{D}_j}|\gamma_\alpha'|^\sigma\sup_{\mathcal{D}_j}|\gamma_\beta'|^\sigma,$$

and the proof is now done using the topological pressure estimate (2.3). □

The main point of the above estimate is to obtain a bound which is independent of d_ϱ. In particular the spectral radius $\rho_{sp}(\mathcal{L}_{\varrho,s})$ of $\mathcal{L}_{\varrho,s}: H^2_\varrho(\Omega)\to H^2_\varrho(\Omega)$ is bounded by

$$\rho_{sp}(\mathcal{L}_{\varrho,s})\leqslant e^{P(\mathrm{Re}(s))}, \tag{2.5}$$

which is uniform with respect to the representation ϱ, and also shows that it is a contraction whenever $\sigma=\mathrm{Re}(s)>\delta$. Notice also that using the variational principle for the topological pressure, it is possible to show that there exist $a_0,b_0>0$ such that for all $\sigma\in\mathbb{R}$,

$$|P(\sigma)|\leqslant a_0+|\sigma|b_0. \tag{2.6}$$

We continue with a key determinantal identity. We point out that representations of Selberg zeta functions as Fredholm determinants of transfer operators have a long history going back to Fried [25], Pollicott [62] and also Mayer [18, 45] for the Modular surface. For more recent works involving transfer operators and unitary representations we also mention [59, 60].

Proposition 2.2. — *For all* $\mathrm{Re}(s)$ *large, we have the identity:*

$$\det(I-\mathcal{L}_{\varrho,s}) = L_\Gamma(s,\varrho), \tag{2.7}$$

Proof. — Remark that the above statement implies analytic continuation to $\mathbb{C}$ of each L-function $L_\Gamma(s,\varrho)$, since each $s\mapsto \det(I-\mathcal{L}_{\varrho,s})$ is readily an entire function of s. For all integer $N\geqslant 1$, let us compute the trace of $\mathcal{L}_{\varrho,s}^N$. Our basic reference for the theory of Fredholm determinants on Hilbert spaces is [70]. Let $(e_1,\dots,e_{d_\varrho})$ be an orthonormal basis of V_ϱ. For each disc $\mathcal{D}_j$ let $(\varphi_\ell^j)_{\ell\in\mathbb{N}}$ be a Hilbert basis of the Bergmann space $H^2(\mathcal{D}_j)$, that is the space of square integrable holomorphic functions on $\mathcal{D}_j$. Then the family defined by

$$\Psi_{j,\ell,k}(z) := \begin{cases} \varphi_\ell^j(z)e_k & \text{if } z\in\mathcal{D}_j \\ 0 & \text{otherwise,}\end{cases}$$

is a Hilbert basis of $H^2_\varrho(\Omega)$. Writing

$$\begin{aligned}&\langle \mathcal{L}_{\varrho,s}^N(\Psi_{j,\ell,k}),\Psi_{j,\ell,k}\rangle_{H^2_\varrho(\Omega)}\\ &\qquad = \sum_{\alpha\in\mathscr{W}_N^j}\int_{\mathcal{D}_j}(\gamma'_\alpha(z))^s\varphi_\ell^j(\gamma_\alpha z)\overline{\varphi_\ell^j(z)}\langle e_k\varrho(\gamma_\alpha),e_k\rangle_\varrho \mathrm{d}m(z),\end{aligned}$$

we deduce that

$$\begin{aligned}\mathrm{Tr}(\mathcal{L}_{\varrho,s}^N) &= \sum_{j,\ell,k}\langle \mathcal{L}_{\varrho,s}^N(\Psi_{j,\ell,k}),\Psi_{j,\ell,k}\rangle_{H^2_\varrho(\Omega)}\\ &= \sum_j \sum_{\substack{\alpha\in\mathscr{W}_N^j\\ \alpha_1=r+j}} \chi_\varrho(\gamma_\alpha)\int_{\mathcal{D}_j}(\gamma'_\alpha(z))^s B_{\mathcal{D}_j}(\gamma_\alpha z,z)\mathrm{d}m(z),\end{aligned}$$

where χ_ϱ is the character of ϱ and $B_{\mathcal{D}_j}(w,z)$ is the *Bergmann reproducing kernel* of $H^2(\mathcal{D}_j)$. There is an explicit formula for the Bergmann kernel of a disc $\mathcal{D}_j = D(c_j,r_j)$:

$$B_{\mathcal{D}_\ell}(w,z) = \frac{r_j^2}{\pi\left[r_j^2-(w-c_j)(\overline{z}-c_j)\right]^2}.$$

It is now an exercise involving Stoke's and Cauchy formula (for details we refer to Borthwick [7, p. 306]) to obtain the Lefschetz identity

$$\int_{\mathcal{D}_j} (\gamma_\alpha'(z))^s B_{\mathcal{D}_j}(\gamma_\alpha z, z)\mathrm{d}m(z) = \frac{(\gamma_\alpha'(x_\alpha))^s}{1-\gamma_\alpha'(x_\alpha)},$$

where x_α is the unique fixed point of $\gamma_\alpha : \mathcal{D}_j \to \mathcal{D}_j$. Moreover,

$$\gamma_\alpha'(x_\alpha) = e^{-l(\mathcal{C}_\alpha)},$$

where $\mathcal{C}_\alpha$ is the closed geodesic represented by the conjugacy class of $\gamma_\alpha \in \Gamma$, and $l(\mathcal{C}_\alpha)$ is the length. There is a one-to-one correspondence between prime reduced words (up to circular permutations) in

$$\bigcup_{N\geqslant 1}\bigcup_{j=1}^{2r}\{\alpha \in \mathscr{W}_N^j \text{ such that } \alpha_1 = r+j\},$$

and prime conjugacy classes in Γ (see Borthwick [7, p. 303]), therefore each prime conjugacy class in Γ and its iterates appear in the above sum, when N ranges from 1 to $+\infty$.

We have therefore reached formally (absolute convergence is valid for $\mathrm{Re}(s)$ large, see later on)

$$\begin{aligned}\sum_{N\geqslant 1}\frac{1}{N}\operatorname{Tr}(\mathcal{L}_{\varrho,s}^N) &= \sum_{N\geqslant 1}\frac{1}{N}\sum_j \sum_{\substack{\alpha\in\mathscr{W}_N^j\\ \alpha_1=r+j}} \chi_\varrho(\gamma_\alpha)\frac{(\gamma_\alpha'(x_\alpha))^s}{1-\gamma_\alpha'(x_\alpha)}\\ &= \sum_{\mathcal{C}\in\mathcal{P}}\sum_{k\geqslant 1}\frac{\chi_\varrho(\mathcal{C}^k)}{k}\frac{e^{-skl(\mathcal{C})}}{1-e^{-kl(\mathcal{C})}}.\end{aligned}$$

The prime orbit theorem for convex co-compact groups says that as $T \to +\infty$, (see for example [38, 49]),

$$\#\{(k,\mathcal{C}) \in \mathbb{N}_0 \times \mathcal{P} : kl(\mathcal{C}) \leqslant T\} = \frac{e^{\delta T}}{\delta T}\left(1+o(1)\right).$$

On the other hand, since χ_ϱ takes obviously finitely many values on $\mathbf{G}$ we get absolute convergence of the above series for $\mathrm{Re}(s) > \delta$. For all $\mathrm{Re}(s)$

large, we get again formally

$$\begin{aligned}\det(I-\mathcal{L}_{\varrho,s}) &= \exp\left(-\sum_{N\geqslant 1}\frac{1}{N}\operatorname{Tr}(\mathcal{L}_{\varrho,s}^N)\right)\\ &= \exp\left(-\sum_{\mathcal{C},k,n}\frac{\chi_\varrho(\mathcal{C}^k)}{k}e^{-(s+n)kl(\mathcal{C})}\right)\\ &= \prod_{\mathcal{C}\in\mathcal{P}}\prod_{n\in\mathbb{N}}\exp\left(-\sum_{k\geqslant 1}\frac{\chi_\varrho(\mathcal{C}^k)}{k}e^{-(s+n)kl(\mathcal{C})}\right)\\ &= \prod_{\mathcal{C}\in\mathcal{P}}\prod_{k\in\mathbb{N}}\det\left(\mathrm{Id}_{V_\varrho}-\varrho(\mathcal{C}^{\|})e^{-(s+k)l(\mathcal{C})}\right).\end{aligned}$$

This formal manipulations are justified for $\mathrm{Re}(s)>\delta$ by using the spectral radius estimate (2.5) and the fact that if A is a trace class operator on a Hilbert space $\mathcal{H}$ with $\|A\|_{\mathcal{H}}<1$ then we have

$$\det(I-A)=\exp\left(-\sum_{N\geqslant 1}\frac{1}{N}\operatorname{Tr}(A^N)\right),$$

(this is a direct consequence of Lidskii's theorem, see [70, Chapter 3]). The proof is finished and we have claim (1) of Theorem 1.1. □

Claim (3) follows from the formula (valid for $\mathrm{Re}(s)>\delta$)

$$\det(I-\mathcal{L}_{\varrho,s})=\exp\left(-\sum_{\mathcal{C},k,n}\frac{\chi_\varrho(\mathcal{C}^k)}{k}e^{-(s+n)kl(\mathcal{C})}\right),$$

and the identity for the character of the regular representation (see [68, Chapter 2])

$$\sum_{\varrho\text{ irreducible}} d_\varrho\chi_\varrho(g)=|\mathbf{G}|\mathcal{D}_e(g), \tag{2.8}$$

where $\mathcal{D}_e$ is the dirac mass at the neutral element e. Indeed, using (2.8), we get

$$\prod_{\varrho\text{ irreducible}}(\det(I-\mathcal{L}_{\varrho,s}))^{d_\varrho}=\exp\left(-|\mathbf{G}|\sum_{k,n}\sum_{\substack{\mathcal{C}\in\mathcal{P}\\ r_{\mathbf{G}}(\mathcal{C})=e}}\frac{1}{k}e^{-(s+n)kl(\mathcal{C})}\right). \tag{2.9}$$

The end of the proof rests on an algebraic fact related to the splitting of conjugacy classes in $\widetilde{\Gamma}$. For the benefit of the reader, we give the outline. It is easy to check that any prime conjugacy class $\widetilde{\mathcal{C}}$ in $\widetilde{\Gamma}$ has a representative given by (representative of) a power of a prime conjugacy class (in Γ), i.e.

$$\widetilde{\mathcal{C}} = \mathcal{C}^\ell,$$

for some $1 \leqslant \ell \leqslant |G|$. It is then a fact of group theory that the conjugacy class of $\mathcal{C}^\ell$ in Γ will split in $\widetilde{\Gamma}$ in one-to-one correspondence with the cosets of

$$\Gamma/\widetilde{\Gamma}C_\Gamma(\mathcal{C}^\ell),$$

where $C_\Gamma(\mathcal{C}^\ell)$ is the centralizer in Γ of $\mathcal{C}^\ell$. Because we are in a free group, this centralizer is the elementary group generated by $\mathcal{C}$, which shows that the number of conjugacy classes in $\widetilde{\Gamma}$ is $|G|/\ell$. This factor ℓ is exactly what's needed to recognize in (2.9) the length $\ell l(\mathcal{C}) = l(\mathcal{C}^\ell) = l(\widetilde{\mathcal{C}})$.

We refer the reader to [22] for more details, including a complete proof of the factorization formula (3) for geometrically finite groups. We point out that this type of analog of the Artin factorization had already been proved by Venkov–Zograf in [74] for cofinite groups.

2.3. Singular value estimates

The proof of claim (2) will require more work and will use singular values estimates for vector-valued operators. We now recall a few facts on singular values of trace class operators. Our reference for that matter is for example the book [70]. If $T : \mathcal{H} \to \mathcal{H}$ is a compact operator acting on a Hilbert space $\mathcal{H}$, the *singular value sequence* is by definition the sequence $\mu_1(T) = \|T\| \geqslant \mu_2(T) \geqslant \cdots \geqslant \mu_n(T)$ of the eigenvalues of the positive self-adjoint operator $\sqrt{T^*T}$. To estimate singular values in a vector valued setting, we will rely on the following fact.

Lemma 2.3. — *Assume that $(e_j)_{j\in J}$ is a Hilbert basis of $\mathcal{H}$, indexed by a countable set J. Let T be a compact operator on $\mathcal{H}$. Then for any subset $I \subset J$ with $\#I = n$ we have*

$$\mu_{n+1}(T) \leqslant \sum_{j\in J\setminus I} \|Te_j\|_{\mathcal{H}}.$$

Proof. — By the min-max principle for bounded self-adjoint operators, we have

$$\mu_{n+1}(T) = \min_{\dim(F)=n} \max_{w\in F^\perp, \|w\|=1} \langle \sqrt{T^*T}w, w\rangle.$$

Set $F = \mathrm{Span}\{e_j, j \in I\}$. Given $w = \sum_{j\notin I} c_j e_j$ with $\sum_j |c_j|^2 = 1$, we obtain via Cauchy–Schwarz inequality

$$|\langle\sqrt{T^*T}w, w\rangle| \leqslant \|\sqrt{T^*T}(w)\| = \|T(w)\| \leqslant \sum_{j\notin I}\|T(e_j)\|,$$

which conclude the proof. □

Our aim is now to prove the following bound.

Proposition 2.4. — *Let $(\lambda_k(\mathcal{L}_{\varrho,s}))_{k\geqslant 1}$ denote the eigenvalue sequence of the compact operators $\mathcal{L}_{\varrho,s}$. There exists $C>0$ and $0<\eta$ such that for all $s\in\mathbb{C}$ and all representation ϱ, we have for all k,*

$$|\lambda_k(\mathcal{L}_{\varrho,s})| \leqslant C d_\varrho e^{C|s|} e^{-\frac{\eta}{d_\varrho}k}.$$

Before we prove this bound, let us show quickly how the combination of the above bound with (2.5) gives the estimate (2) of Theorem 1.1. By definition of Fredholm determinants, we have

$$\begin{aligned}\log|L_\Gamma(s,\varrho)| &\leqslant \sum_{k=1}^{\infty}\log(1+|\lambda_k(\mathcal{L}_{\varrho,s})|)\\ &= \sum_{k=1}^{N}\log(1+|\lambda_k(\mathcal{L}_{\varrho,s})|) + \sum_{k=N+1}^{\infty}\log(1+|\lambda_k(\mathcal{L}_{\varrho,s})|),\end{aligned}$$

where N will be adjusted later on. The first term is estimated via (2.6) as

$$\sum_{k=1}^{N}\log(1+|\lambda_k(\mathcal{L}_{\varrho,s})|) \leqslant \widetilde{C}(|s|+1)N,$$

for some large constant $\widetilde{C}>0$. On the other hand we have by the eigenvalue bound from Proposition 2.4

$$\begin{aligned}\sum_{k=N+1}^{\infty}\log(1+|\lambda_k(\mathcal{L}_{\varrho,s})|) &\leqslant \sum_{k=N+1}^{\infty}|\lambda_k(\mathcal{L}_{\varrho,s})|\\ &\leqslant C d_\varrho e^{C|s|}\sum_{k\geqslant N+1} e^{-\frac{\eta}{d_\varrho}k} = C d_\varrho e^{C|s|}\frac{e^{-(N+1)\eta/d_\varrho}}{1-e^{-\eta/d_\varrho}}\\ &\leqslant C'\frac{d_\varrho^2}{\eta}e^{C|s|}e^{-N\frac{\eta}{d_\varrho}}.\end{aligned}$$

Choosing $N = B[|s|d_\varrho] + B[d\varrho\log(d_\varrho+1)]$ for some large $B>0$ leads to

$$\sum_{k=N+1}^{\infty}\log(1+|\lambda_k(\mathcal{L}_{\varrho,s})|) \leqslant \widetilde{B}$$

for some constant $\widetilde{B} > 0$ uniform in $|s|$ and d_ϱ. Therefore we get

$$\log |L_\Gamma(s,\varrho)| \leqslant O\left(d_\varrho \log(d_\varrho+1)(|s|^2+1)\right),$$

which is the bound claimed in statement (2).

Proof of Proposition 2.4. — We first recall that if $\mathcal{D}_j = D(c_j, R_j)$, an explicit Hilbert basis of the Bergmann space $H^2(\mathcal{D}_j)$ is given by the functions ($\ell = 0, \ldots, +\infty$, $j = 1, \ldots, 2r$)

$$\varphi_\ell^{(j)}(z) = \sqrt{\frac{\ell+1}{\pi}}\frac{1}{R_j}\left(\frac{z-c_j}{R_j}\right)^\ell.$$

By the Schottky property, one can find $\eta_0 > 0$ such for all $z \in \mathcal{D}_j$, for all $i \neq j$ we have $\gamma_i(z) \in \mathcal{D}_{i+r}$ and

$$\frac{|\gamma_i(z) - c_{r+i}|}{R_{r+i}} \leqslant e^{-\eta_0},$$

so that we have uniformly in i, z,

$$|\varphi_\ell^{(i+r)}(\gamma_i z)| \leqslant Ce^{-\eta_1 \ell}, \tag{2.10}$$

for some $0 < \eta_1 < \eta_0$. Going back to the basis $\Psi_{j,\ell,k}(z)$ of $H^2_\varrho(\Omega)$, we can write

$$\|\mathcal{L}_{\varrho,s}(\Psi_{j,\ell,k})\|^2_{H^2_\varrho}$$
$$= \sum_{n=1}^{2r}\sum_{i,i'\neq n}\int_{\mathcal{D}_n}(\gamma_i(z))^s\overline{(\gamma_{i'}(z))^s}\langle\Psi_{j,\ell,k}(\gamma_i z)\varrho(\gamma_i), \Psi_{j,\ell,k}(\gamma_{i'}z)\varrho(\gamma_{i'})\rangle_\varrho \mathrm{d}m(z).$$

Using Schwarz inequality and unitarity of the representation ϱ for the inner product $\langle\,\cdot\,,\cdot\,\rangle_\varrho$, we get by (2.10) and also (2.4),

$$\|\mathcal{L}_{\varrho,s}(\Psi_{j,\ell,k})\|^2_{H^2_\varrho} \leqslant \widetilde{C}e^{\tilde{C}|s|}e^{-2\eta_1\ell},$$

for some large constant $\widetilde{C} > 0$. We can now use Lemma 2.3 to write

$$\mu_{2rd_\rho n+1}(\mathcal{L}_{\varrho,s}) \leqslant \sum_{j=1}^{2r}\sum_{\ell=n}^{+\infty}\sum_{k=1}^{d_\varrho}\|\mathcal{L}_{\varrho,s}(\Psi_{j,\ell,k})\|_{H^2_\varrho}$$

$$\leqslant Cd_\rho e^{\tilde{C}|s|}e^{-\eta_1 n},$$

for some $C > 0$. Given $N \in \mathbb{N}$, we write $N = 2rd_\varrho k + r$ where $0 \leqslant r < 2rd_\varrho$ and $k = [\frac{N}{2rd_\rho}]$. We end up with

$$\mu_{N+1}(\mathcal{L}_{\rho,s}) \leqslant \mu_{2rd_\varrho k+1}(\mathcal{L}_{\varrho,s}) \leqslant C' d_\varrho e^{\tilde{C}|s|}e^{-\eta_2 N/d_\varrho},$$

for some $\eta_2 > 0$. To produce a bound on the eigenvalues, we use then a variant of Weyl inequalities (see [70, Theorem 1.14]) to get

$$|\lambda_N(\mathcal{L}_{\varrho,s})| \leqslant \prod_{k=1}^{N} |\lambda_k(\mathcal{L}_{\varrho,s})| \leqslant \prod_{k=1}^{N} \mu_k(\mathcal{L}_{\varrho,s}),$$

which yields

$$|\lambda_N(\mathcal{L}_{\varrho,s})| \leqslant C_1 d_\varrho e^{C_2|s|} e^{-\frac{\eta_2}{N d_\varrho}\sum_{k=1}^N k}.$$

Using the well known identity $\sum_{k=1}^N k = \frac{N(N+1)}{2}$ we finally recover

$$|\lambda_N(\mathcal{L}_{\varrho,s})| \leqslant C_1 d_\varrho e^{C_2|s|} e^{-\frac{\eta N}{d_\varrho}},$$

for some $\eta > 0$ and the proof is done. □

3. Sharp resonances in abelian covers

In this section we prove Theorem 1.3. We use the same notations as in Section 1.

3.1. Structure of abelian covers

Let us recall the basic notations used below and previously defined in the introduction.

Let Γ be a convex co-compact group, then Γ is isomorphic to the free group of rank r, with $r \geqslant 2$ when it is non-elementary, see for example in [17] for a Schottky realization. Assume now that Γ_j is a normal subgroup of Γ such that $\mathbf{G}_j := \Gamma/\Gamma_j$ is a finite abelian group. Let $\pi_j : \Gamma \to \mathbf{G}_j$ be the associated onto homomorphism so that

$$\Gamma_j = \ker(\pi_j).$$

By universal property of the abelianized group

$$\Gamma^{ab} := \Gamma/[\Gamma,\Gamma] = H^1(X,\mathbb{Z}) \simeq \mathbb{Z}^r,$$

the homomorphism π_j can be factorized as $\pi_j = \widetilde{\pi_j} \circ P$ where $P : \Gamma \to \mathbb{Z}^r$ is a now fixed surjective homomorphism, and $\widetilde{\pi_j} : \mathbb{Z}^r \to \mathbf{G}_j$ is another (j-dependent) onto homomorphism. By the usual structure theorem for finite abelian groups, $\mathbf{G}_j$ can be written as a product of cyclic groups which we will write as

$$\mathbf{G}_j = \mathbb{Z}/N_1^{(j)}\mathbb{Z} \times \cdots \times \mathbb{Z}/N_k^{(j)}\mathbb{Z},$$

where $N_1(j), N_2(j), \ldots N_k(j)$ are integers.

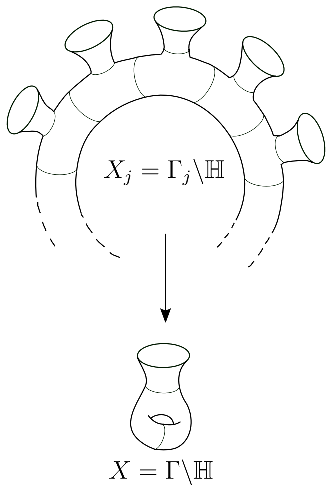

Figure 3.1. A cyclic cover of a funneled torus

In what follows, as explained in the introduction, we will be assuming that $1 \leqslant k \leqslant r$ is fixed and that the following hypothesis (H) is satisfied:

$$\text{(H)} \qquad \lim_{j\to+\infty} \inf_{\ell=1,\dots,k} N_\ell^{(j)} = +\infty.$$

Furthermore, $\widetilde{\pi}_j$ will be given by

$$\widetilde{\pi}_j(n) = (n_1 \bmod N_1^{(j)}, \dots, n_k \bmod N_k^{(j)}),$$

which is an obvious family of surjective homomorphism from $\mathbb{Z}^r$ to $\mathbf{G}_j$.

In the simplest case $k = 1$, the Galois group is the cyclic group $\mathbb{Z}/N_1^{(j)}\mathbb{Z}$, see the figure for an example, where the cover is obtained by cutting X along a simple closed geodesic and gluing cyclically several copies of the result.

3.2. Selberg's zeta function and characters

According to the result of Patterson–Perry [57], resonances on $X = \Gamma\backslash\mathbb{H}^2$ coincide with multiplicity with the non-trivial zeros of the Selberg zeta function, see also [7] for the case of surfaces. Let $\mathcal{P} = \mathcal{P}(\Gamma)$ denote the set of primitive closed geodesics on X, and if $\mathcal{C} \in \mathcal{P}$, $l(\mathcal{C})$ will be the length. Selberg zeta function is usually defined by the infinite product

$$Z_\Gamma(s) := \prod_{\mathcal{C}\in\mathcal{P}} \prod_{k\in\mathbb{N}_0} \left(1 - e^{-(s+k)l(\mathcal{C})}\right), \quad \mathrm{Re}(s) > \delta(\Gamma).$$

This infinite product has a holomorphic extension to $\mathbb{C}$. The characters of the abelian group

$$H^1(X,\mathbb{Z}) \simeq \mathbb{Z}^r$$

are given by

$$\chi_\theta(x) = e^{2i\pi\langle\theta,x\rangle}, x \in \mathbb{Z},$$

where $\langle\theta,x\rangle = \sum_{\ell=1}^r \theta_\ell x_\ell$, and $\theta = (\theta_1,\dots,\theta_r)$ belongs to the torus $\mathbb{R}^r/\mathbb{Z}^r$. Associated to each character χ_θ is a corresponding "twisted" Selberg zeta $\mathcal{Z}_\Gamma(s,\theta)$ function (or rather L-function) defined by

$$\mathcal{Z}_\Gamma(s,\theta) := \prod_{\mathcal{C}\in\mathcal{P}} \prod_{k\in\mathbb{N}_0} \left(1 - \chi_\theta(\mathcal{C}) e^{-(s+k)l(\mathcal{C})}\right), \quad \operatorname{Re}(s) > \delta(\Gamma),$$

where $\chi_\theta(\mathcal{C})$ is a shorthand for $\chi_\theta(P(\mathcal{C}))$. On the other hand, the characters of $\mathbf{G}_j$ are given by $\chi_\theta((m_1,\dots,m_k,0,\dots,0)), m \in \mathbf{G}_j$, where

$$\theta \in \mathcal{S}_j := \left\{0, \frac{1}{N_1^{(j)}} \dots, \frac{N_1^{(j)}-1}{N_1^{(j)}}\right\} \times \dots \times \left\{0, \frac{1}{N_k^{(j)}} \dots, \frac{N_k^{(j)}-1}{N_k^{(j)}}\right\} \times \underbrace{\{0\} \times \dots \times \{0\}}_{r-k \text{ times}}.$$

Notice that if $\gamma \in \Gamma$, then for all $\theta \in \mathcal{S}_j$, we have indeed

$$\chi_\theta(\widetilde{\pi}_j \circ P(\gamma)) = \chi_\theta(P(\gamma)).$$

From the results of Theorem 1.1, we know that for all $\theta \in \mathcal{S}_j$, each zeta function $s \mapsto \mathcal{Z}_\Gamma(s,\theta)$ has an analytic continuation to $\mathbb{C}$, and we have the following fundamental factorization formula, valid for all $s \in \mathbb{C}$:

$$Z_{\Gamma_j}(s) = \prod_{\theta\in\mathcal{S}_j} \mathcal{Z}_\Gamma(s,\theta). \tag{3.1}$$

Theorem 1.3 then follows from the next theorem.

Theorem 3.1. — *Assume that Γ is non-elementary. We have the following facts.*

(1) *For all $\varepsilon > 0$, one can find $\eta(\varepsilon) > 0$ such that if $\theta \in \mathbb{R}^r$ is such that $\operatorname{dist}(\theta,\mathbb{Z}^r) > \varepsilon$, then $s \mapsto \mathcal{Z}_\Gamma(s,\theta)$ does not vanish inside the strip*

$$\{\delta - \eta \leqslant \operatorname{Re}(s) \leqslant \delta\}.$$

(2) *There exists $\epsilon_0 > 0$ and $\eta_0 > 0$ such that for all θ with $\operatorname{dist}(\theta,\mathbb{Z}^r) \leqslant \epsilon_0$, the analytic function $s \mapsto \mathcal{Z}_\Gamma(s,\theta)$ has exactly one zero $\varphi(\theta)$ (which is real) inside the strip*

$$\{\delta - \eta_0 \leqslant \operatorname{Re}(s) \leqslant \delta\},$$

and the map $\theta \mapsto \varphi(\theta)$ is smooth, real valued with a non-degenerate critical point at $\theta = 0$. Moreover, the hessian $\nabla_0^2\varphi$ is negative definite.

The proof of Theorem 3.1 will occupy several sections. Let us show how one can recover Theorem 1.3 from that. We first start by picking ϵ_0 from statement (2), and then a corresponding $\eta(\epsilon_0)$ from statement (1). Set $\eta^* = \min\{\eta_0; \eta(\epsilon_0)\}$. Inside the strip

$$\Omega := \{\delta - \eta^* \leqslant \mathrm{Re}(s) \leqslant \delta\},$$

we observe that either $\mathrm{dist}(\theta, \mathbb{Z}^r) \leqslant \epsilon_0$ and $s \mapsto \mathcal{Z}_\Gamma(s,\theta)$ vanishes at most once on the real line, or $\mathrm{dist}(\theta, \mathbb{Z}^r) > \epsilon_0$ and $s \mapsto \mathcal{Z}_\Gamma(s,\theta)$ does not vanish. Going back to the factorization formula (3.1), we deduce that inside $\{\delta - \eta^* \leqslant \mathrm{Re}(s) \leqslant \delta\}$, the set of zeros of $Z_{X_j}(s)$ is given by

$$\{\varphi(\theta) : \theta \in \mathcal{S}_j \text{ and } \mathrm{dist}(\theta, \mathbb{Z}^r) \leqslant \epsilon_0\} \cap \{\delta - \eta^* \leqslant \mathrm{Re}(s) \leqslant \delta\}.$$

To complete the proof, we use Poisson summation formula. Let $f \in C_0^\infty([\delta - \epsilon_1, 1])$, where $0 < \epsilon_1 < \eta^*$ is small enough such that $\mathrm{Supp}(f \circ \varphi) \subset \{\mathrm{dist}(\theta, \mathbb{Z}^r) \leqslant \epsilon_0\}$. We therefore have

$$\frac{1}{|\mathbf{G}_j|} \sum_{\lambda \in \mathcal{R}_{X_j} \cap \Omega} f(\lambda) = \frac{1}{N_1^{(j)} \dots N_k^{(j)}} \sum_{\beta \in \mathbb{Z}^k} f \circ \varphi\left(\frac{\beta_1}{N_1^{(j)}}, \dots, \frac{\beta_k}{N_k^{(j)}}, 0, \dots, 0\right).$$

Applying Poisson summation formula,

$$\begin{aligned}\frac{1}{N_1^{(j)} \dots N_k^{(j)}} &\sum_{\beta \in \mathbb{Z}^k} f \circ \varphi\left(\frac{\beta_1}{N_1^{(j)}}, \dots, \frac{\beta_k}{N_k^{(j)}}, 0, \dots, 0\right) \\ &= \sum_{m \in \mathbb{Z}^k, m \neq 0} \widehat{\psi}(2\pi N_1^{(j)} m_1, \dots, 2\pi N_k^{(j)} m_k) + \int_{\mathbb{R}^k} \psi(x) \mathrm{d}x,\end{aligned}$$

where we have set $\psi(x) := f \circ \varphi(x, 0, \dots, 0)$ and $\widehat{\psi}$ is as usual the Fourier transform defined by

$$\widehat{\psi}(\xi) = \int_{\mathbb{R}^k} \psi(x) e^{-i\xi . x} \mathrm{d}x.$$

Since $\widehat{\psi}$ has rapid decay (Schwartz class), a simple summation argument gives

$$\frac{1}{N_1^{(j)}\dots N_k^{(j)}}\sum_\beta \psi\left(\frac{\beta_1}{N_1^{(j)}},\dots,\frac{\beta_k}{N_k^{(j)}}\right) \\ = \int \psi(x,0,\dots,0)\mathrm{d}x + O_\alpha\left(\frac{1}{(\min\{N_1^{(j)},\dots N_k^{(j)}\})^\alpha}\right),$$

for all integers α. We then set

$$\int_{\mathbb{R}^k} f\circ\varphi(x,0,\dots,0)\mathrm{d}x =: \int f\mathrm{d}\mu.$$

The fact that the push-forward measure μ is absolutely continuous follows from Radon–Nykodym's theorem and the non-degeneracy of the critical point of φ at 0, see for example [63]. We digress slightly to explain how one can describe the shape of the Radon-Nikodym derivative $\frac{\mathrm{d}\mu}{\mathrm{d}m}(u)$ in the vicinity of δ, where m is Lebesgue measure on $\mathbb{R}$. Indeed, we know from the above that locally,

$$\varphi(x,0,\dots,0) = \delta - Q(x) + O(\|x\|^3),$$

where $Q(x)$ is a positive definite quadratic form.

The Morse lemma implies that for all $\varepsilon > 0$ small enough, there is an open neighbourhood $\widetilde{U}\subset\mathbb{R}^k$ of 0 and a diffeomorphism

$$\Psi : B_\infty(0,\varepsilon)\to\widetilde{U},\quad (x_1,\dots,x_r)\mapsto(y_1,\dots,y_k)$$

such that $\Psi(0)=0$ and $\varphi\circ\Psi^{-1}(y) = \delta - y_1^2 - \dots - y_k^2$. Therefore, for any $f\in C_0^\infty([\delta-\epsilon_1,1])$, where again $\epsilon_1>0$ is taken small enough, we have

$$\begin{aligned}\int f\mathrm{d}\mu &= \int_{\mathbb{R}^k} f\circ\varphi(x,0,\dots,0)\mathrm{d}x \\ &= \int_{\tilde{U}} f(\delta - y_1^2-\dots-y_k^2)\cdot|D\Psi^{-1}(y)|\mathrm{d}y \\ &\asymp \int_{\tilde{U}} f(\delta-y_1^2-\dots-y_k^2)\mathrm{d}y,\end{aligned}$$

where $|D\Psi^{-1}(y)|$ is the Jacobian determinant. Choosing polar coordinates yields

$$\int f\mathrm{d}\mu \asymp \int_{\mathbb{R}^+}\varphi(\delta-R^2)R^{k-1}\mathrm{d}R.$$

With one last change of variables $R\mapsto\xi = R^2$ we obtain

$$\int f\mathrm{d}\mu \asymp \int_{\mathbb{R}^+}\varphi(\delta-\xi)\xi^{\frac{k-2}{2}}\mathrm{d}\xi.$$

We conclude that there exists a constant $C > 0$ such that for all u close enough to delta ($u < \delta$)

$$C^{-1}(\delta - u)^{\frac{k-2}{2}} \leqslant \frac{\mathrm{d}\mu}{\mathrm{d}m}(u) \leqslant C(\delta - u)^{\frac{k-2}{2}},$$

In particular we observe a drastic difference in the density shape when $k = 1, 2$ and $k > 2$.

By further shrinking the strip (i.e. taking a smaller η^*), and a standard approximation argument, the proof of the first two claims is complete. We now prove the last point. First we observe that using Theorem 1.1 (2), we have the existence of a constant $C_\Gamma > 0$ such that for all j and $s \in \mathbb{C}$, we have

$$|Z_{\Gamma_j}(s)| \leqslant C_\Gamma \exp\left(C_\Gamma |\mathbf{G}_j||s|^2\right). \tag{3.2}$$

On the other hand, for all $\mathrm{Re}(s) > \delta$ and $\theta \in \mathcal{S}_j$, we have

$$\mathcal{Z}_\Gamma(s, \theta) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\mathcal{C} \in \mathcal{P}(X)} \chi_\theta(\mathcal{C}^n) \frac{e^{-snl(\mathcal{C})}}{1 - e^{-nl(\mathcal{C})}}\right),$$

which combined with the factorization formula (3.1) shows that for $\mathrm{Re}(s) > \delta$,

$$|Z_{\Gamma_j}(s)| \geqslant \exp\left(-C_1 |\mathbf{G}_j| \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\mathcal{C} \in \mathcal{P}(X)} e^{-\mathrm{Re}(s) nl(\mathcal{C})}\right). \tag{3.3}$$

We now fix $\lambda \in \mathcal{R}_X$ and $\varepsilon_0 > 0$. To get the upper bound we fix $x_0 \in \mathbb{R}$ with $x_0 > \delta$ and choose $R_0 > 0$ large enough such that the disc $D(x_0, R_0)$ contains $D(\lambda, \varepsilon_0)$ in its interior. We will use Jensen's formula (or rather a consequence of it) in the following form.

Proposition 3.2. — *Let f be a holomorphic function on the open disc $D(w, R)$, and assume that $f(w) \neq 0$. let $N_f(r)$ denote the number of zeros of f in the closed disc $\overline{D}(w, r)$. For all $\widetilde{r} < r < R$, we have*

$$N_f(\widetilde{r}) \leqslant \frac{1}{\log(r/\widetilde{r})} \left(\frac{1}{2\pi} \int_0^{2\pi} \log|f(w + re^{i\theta})| \mathrm{d}\theta - \log|f(w)|\right).$$

It is now clear that by applying the above Proposition on the disc $D(x_0, R_0)$ where both x_0, R_0 are fixed we can use the bounds (3.2), and (3.3) to obtain that for all j,

$$\#\mathcal{R}_{X_j} \cap D(\lambda, \varepsilon_0) \leqslant C_\Gamma |\mathbf{G}_j|.$$

To prove the lower bound, provided ε_0 is taken small enough, we can write for all $s \in D(\lambda, \varepsilon_0)$,

$$Z_\Gamma(s) = (s-\lambda)^m \psi(s),$$

where $m \geqslant 1$ is the order of vanishing of $Z_\Gamma(s)$ at $s = \lambda$ and $s \mapsto \psi(s)$ is a holomorphic function non-vanishing on a neighborhood of $\overline{D(\lambda, \varepsilon_0)}$. On $\partial D(\lambda, \varepsilon_0)$ we have

$$|Z_\Gamma(s)| \geqslant \epsilon_0^m \inf_{s \in D(\lambda, \varepsilon_0)} |\psi(s)| > 0.$$

On the other hand, since $(s, \theta) \mapsto \mathcal{Z}(s, \theta)$ is smooth and $\mathcal{Z}(s, 0) = Z_\Gamma(s)$, there exist $\epsilon > 0$ such that for all $\|\theta\| \leqslant \epsilon$ we have

$$\sup_{s \in \partial D(\lambda, \varepsilon_0)} |\mathcal{Z}(s, \theta) - Z_\Gamma(s)| < \inf_{s \in \partial D(\lambda, \varepsilon_0)} |Z_\Gamma(s)|.$$

Applying the classical Rouché's theorem for holomorphic functions, we deduce that for each $\theta \in \mathcal{S}_j$ such that $\|\theta\| \leqslant \epsilon$, $s \mapsto \mathcal{Z}(s, \theta)$ has exactly m zeros inside $D(\lambda, \varepsilon_0)$. Using the factorization formula, we deduce that the number of zeros of $Z_{\Gamma_j}(s)$ inside $D(\lambda, \epsilon_0)$ is at least

$$m \# \{\theta \in \mathcal{S}_j : \|\theta\| \leqslant \epsilon\},$$

which is bigger than $C|\mathbf{G}_j|$ for some small constant $C > 0$, independent of j. The proof is complete.

3.3. A digression on closed geodesics in homology classes

Let $P : \Gamma \to \mathbb{Z}^r \simeq H^1(X, \mathbb{Z})$ be a fixed isomorphism as above. Let $\alpha \in \mathbb{Z}^r$ be a fixed "holomogy class", and consider the counting function

$$N(\alpha, T) = \#\{\mathcal{C} \in \mathcal{P}(X) : P(\mathcal{C}) = \alpha \text{ and } l(\mathcal{C}) \leqslant T\}.$$

In the case of infinite volume hyperbolic surfaces, the leading term is known, and follows for example from [55, Chapter 12]. (For Kleinian groups, we also mention the work of Babillot–Peigné [3]). It goes as follows: as $T \to +\infty$ we have

$$N(\alpha, T) \sim c_0 \frac{e^{\delta T}}{T^{r/2+1}}, \tag{3.4}$$

where c_0 is independent of α. Counting asymptotics for closed geodesics in homology classes has a long history of results: for compact hyperbolic manifolds it was proved independently by Phillips and Sarnak [58] and Katsuda and Sunada [35]. On compact surfaces with variable negative curvature, we mention Lalley and Pollicott [37, 61]. The most general version of the leading asymptotic (3.4) for Anosov flows is due to Sharp [69].

As a consequence of Theorem 3.1 on the non-vanishing of $\mathcal{Z}_\Gamma(s,\theta)$ and combining it with a priori estimates on zeta functions from Theorem 1.1, we obtain the following improved counting result.

Theorem 3.3. — *Assume that* Γ *is convex co-compact and non-elementary, then for all* $\alpha \in \mathbb{Z}^r$*, for all* $n \geqslant 0$*, there exists a sequence* $c_0, c_1(\alpha), \ldots, c_n(\alpha) \in \mathbb{R}$ *such that as* $T \to +\infty$*,*

$$N(\alpha,T) = \frac{e^{\delta T}}{T^{r/2+1}}\left(c_0 + c_1 T^{-1} + \cdots + c_n T^{-n} + O(T^{-n-1})\right).$$

In particular, this extends the asymptotics obtained by McGowan and Perry [47] to the case $\delta \leqslant \frac{1}{2}$, which was not known so far. This type of asymptotic expansion was first obtained for compact hyperbolic surfaces by Phillips and Sarnak [58], and for more general Anosov flows by Anantharaman [2].

The proof, knowing Theorem 3.1, is standard and goes exactly as in [47]. We recall briefly the main ideas for the benefit of the reader. One starts by picking $\phi_T \in C_0^\infty(\mathbb{R}^+)$, $\phi_T \geqslant 0$ such that $\phi_T \equiv 1$ on the interval $[\epsilon_0, T]$ and is supported in $[\epsilon_0/2, T+\beta]$, where $\epsilon_0 > 0$ is taken small and $\beta = e^{-\nu T}$ for some large $\nu > 0$. We then set

$$\psi_T(s) := \int_0^\infty e^{xs}\phi_T(x)\mathrm{d}x,$$

so that for all $A > \delta$ we have the contour integral identity

$$\frac{1}{2i\pi}\int_{A-i\infty}^{A+i\infty} \frac{\mathcal{Z}_\Gamma'(s,\theta)}{\mathcal{Z}_\Gamma(s,\theta)}\psi_T(s)\mathrm{d}s = \sum_{k,\mathcal{C}} l(\mathcal{C})\frac{\chi_\theta(\mathcal{C}^k)}{1-e^{-kl(\mathcal{C})}}\phi_T(kl(\mathcal{C})).$$

Notice that if ν is large enough and ϵ_0 small, we have for $\sigma \leqslant \delta$,

$$\phi_T(\sigma) = \frac{e^{\sigma T}}{\sigma} + O\left(e^{T\delta/2}\right).$$

Thanks to the a priori upper bound from Theorem 1.1 and Caratheodory estimates, we know that if $\mathcal{Z}_\Gamma(s,\theta) \neq 0$ for all s with $\mathrm{Re}(s) > \delta - \eta$, then we will get a polynomial upper bound for the log derivative

$$\left|\frac{\mathcal{Z}_\Gamma'(s,\theta)}{\mathcal{Z}_\Gamma(s,\theta)}\right| \leqslant M|\mathrm{Im}(s)|^2,$$

for all $|\mathrm{Im}(s)|$ large and $\mathrm{Re}(s) > \delta - \eta/2$. Integrating with respect to θ on $\mathbb{R}^r/\mathbb{Z}^r$ gives the formula

$$\frac{1}{2i\pi}\int_{A-i\infty}^{A+i\infty}\int_{\mathbb{R}^r/\mathbb{Z}^r} e^{-2i\pi\langle\alpha,\theta\rangle}\frac{\mathcal{Z}_\Gamma'(s,\theta)}{\mathcal{Z}_\Gamma(s,\theta)}\mathrm{d}\theta\psi_T(s)\mathrm{d}s = \sum_{k,\mathcal{C}:P(\mathcal{C}^k)=\alpha}\frac{l(\mathcal{C})}{1-e^{-kl(\mathcal{C})}}\phi_T(kl(\mathcal{C})).$$

Thanks to Theorem 3.1, for all $\epsilon > 0$, we can therefore deform the contour (by taking $A < \delta$) for all θ such that $\mathrm{dist}(\theta, 0) > \epsilon$ to obtain a contribution of order

$$O(e^{(\delta-\eta(\epsilon)/2)T}).$$

We are essentially left with estimating integrals over θ in a neighborhood of 0. Using the *residue formula*, the fact that $s \mapsto \mathcal{Z}_\Gamma(s,\theta)$ has a simple leading zero $\varphi(\theta)$, and neglecting error terms which are exponentially smaller than $e^{\delta T}$, we are then led to estimate integrals of the form

$$\mathcal{I}(T) = \int_{\mathbb{R}^r/\mathbb{Z}^r} e^{\varphi(\theta)T}\kappa(\theta)\mathrm{d}\theta,$$

where $\kappa(\theta)$ is a smooth function supported in an arbitrarily small neighborhood of 0. Using Morse Lemma (we know that $\theta \mapsto \varphi(\theta)$ has a non-degenerate critical point at $\theta = 0$ with negative definite Hessian) and Laplace method to deal with the *stationnary phase* at $\theta = 0$ (see [58, Lemma 2.3]) leads to expansions as $T \to +\infty$ of the form

$$\mathcal{I}(T) = \frac{e^{\delta T}}{T^{r/2}}\left(a_0 + a_1 T^{-1} + \cdots + a_n T^{-n} + O(T^{-n-1})\right).$$

Notice that there are no odd powers of $T^{-\frac{1}{2}}$ here because all the odd moments on $\mathbb{R}^r$ of $e^{-|x|^2}$ vanish. We have essentially obtained that

$$\sum_{P(\mathcal{C})=\alpha \text{ and } l(\mathcal{C})\leqslant T} l(\mathcal{C}) = \frac{e^{\delta T}}{T^{r/2}}\left(c_0 + c_1 T^{-1} + \cdots + c_n T^{-n} + O(T^{-n-1})\right).$$

To obtain the desired asymptotics for $N(\alpha, T)$ is now a simple exercise using Stieltjes integration by parts and the bound coming from the known leading term (3.4). We point out that using more delicate arguments involving the saddle point method, it is possible to derive similar asymptotics for counting functions of the type

$$N(\alpha + [T\xi], T),$$

where $\xi \in \mathbb{Z}^r \setminus \{0\}$, see Anantharaman [2].

4. Twisted zeta functions and transfer operators

We recall the function space used and the associated twisted transfer operators related to characters of the homology. Set

$$\Omega := \bigcup_{j=1}^{2r} \mathcal{D}_j.$$

Consider now the Hilbert space $H^2(\Omega)$ which is defined as the set of *holomorphic functions* $F : \Omega \to \mathbb{C}$ such that

$$\|F\|_{H^2}^2 := \int_\Omega |F(z)|^2 \mathrm{d}m(z) < +\infty,$$

where $\mathrm{d}m$ is Lebesgue measure on $\mathbb{C}$. Let $\theta \in \mathbb{R}^r/\mathbb{Z}^r$, the "character torus". On the space $H^2(\Omega)$, we define a "twisted" by θ transfer operator $\mathcal{L}_{s,\theta}$ by

$$\mathcal{L}_{s,\theta}(F)(z) := \sum_{j \neq i} (\gamma_j')^s(z) \chi_\theta(P\gamma_j) F(\gamma_j z), \text{ if } z \in \mathcal{D}_i,$$

where $s \in \mathbb{C}$ is the spectral parameter, and χ_θ is the character of $H^1(X, \mathbb{Z}) \simeq \mathbb{Z}^r$ associated to θ and $P : \Gamma \to H^1(X, \mathbb{Z})$ is the projection homomorphism. Notice that for all $j \neq i$, $\gamma_j : \mathcal{D}_i \to \mathcal{D}_{r+j}$ is a holomorphic contraction since $\overline{\gamma_j(\mathcal{D}_i)} \subset \mathcal{D}_{r+j}$. Therefore, $\mathcal{L}_{s,\theta}$ is a compact *trace class* operator and thus has a *Fredholm determinant*. We define the twisted zeta function $\mathcal{Z}_\Gamma(s, \theta)$ by

$$\mathcal{Z}_\Gamma(s, \theta) := \det(I - \mathcal{L}_{s,\theta}).$$

It follows from Theorem 1.1, but also [22] that for all $\mathrm{Re}(s) > \delta$ we have the identity

$$\det(I - \mathcal{L}_{s,\theta}) = \prod_{\mathcal{C} \in \mathcal{P}} \prod_{k \in \mathbb{N}_0} \left(1 - \chi_\theta(\mathcal{C}) e^{-(s+k)l(\mathcal{C})}\right),$$

which shows that the infinite product has actually an analytic continuation to $\mathbb{C}$.

4.1. The high and low frequency results

The proof of Theorem 3.1 will follow from two facts which will require two different types of asymptotic analysis. We state these results below.

Proposition 4.1 (The high frequency regime). — *Assume that* Γ *is non-elementary, then there exist* $\varepsilon_0 > 0$ *and* $T_0 \gg 1$ *such that for all* $\theta \in \mathbb{R}^r$ *and*

$$s \in \{\delta - \varepsilon_0 \leqslant \mathrm{Re}(s) \leqslant \delta \textit{ and } |\mathrm{Im}(s)| \geqslant T_0\},$$

we have $\mathcal{Z}_\Gamma(s, \theta) \neq 0$.

A very important feature is that $\varepsilon_0 > 0$ and T_0 can be taken *uniform* with respect to θ. This uniform high frequency (aka large $\mathrm{Im}(s)$) fact will follow from certain Dolgopyat estimates for twisted transfer operators as in [48]. In particular, this result implies that at high frequencies, there is a uniform resonance gap for all abelian covers of a given non-elementary Schottky surface, a fact that is similar to the result proved in [53] for congruence subgroups. To describe the behaviour of resonances with small $\mathrm{Im}(s)$, we will prove the following result.

Proposition 4.2 (The low frequency regime). — *Assume that* Γ *is non-elementary, then for all* $t \in \mathbb{R}$ *and* $\theta \in \mathbb{R}^r/\mathbb{Z}^r$ *we have*

$$\mathcal{Z}_\Gamma(\delta + it, \theta) = 0 \Longleftrightarrow (t, \theta) = (0, 0),$$

where 0 *in the second factor is understood mod* $\mathbb{Z}^r$.

In other words, on the vertical line $\{\mathrm{Re}(s) = \delta\}$, the zeta function $\mathcal{Z}_\Gamma(s, \theta)$ vanishes only at $s = \delta$ when $\theta \in \mathbb{Z}^r$. The proof will follow from convexity arguments in the analysis of transfer operators, as in previous works of Parry and Pollicott [55].

To conclude this section, let us show how the combination of Proposition 4.1 and Proposition 4.2 does imply Theorem 3.1. First we fix $\epsilon > 0$. We know from Proposition 4.1 that no zeta function $\mathcal{Z}_\Gamma(s, \theta)$ will vanish for $\delta - \varepsilon_0 \leqslant \mathrm{Re}(s) \leqslant \delta$ and $|\mathrm{Im}(s)| \leqslant T_0$ regardless of the value of θ. Assume that for all $\eta > 0$, there exists $\theta \in \mathbb{R}^r$ with $\mathrm{dist}(\theta, \mathbb{Z}^r) > \epsilon$ and there exists $s \in \mathbb{C}$ with $\delta - \eta \leqslant \mathrm{Re}(s) \leqslant \delta$ and $|\mathrm{Im}(s)| \leqslant T_0$ such that $\mathcal{Z}_\Gamma(s, \theta) = 0$. Then by compactness one construct a converging sequence (s_ℓ, θ_ℓ) such that

$$s_\infty := \lim_{\ell \to +\infty} s_\ell \in \delta + i[-T_0, +T_0]$$

and $\theta_\infty := \lim_{\ell \to +\infty} \theta_\ell$ satisfies $\theta_\infty \notin \mathbb{Z}^r$. By continuity, we have

$$\mathcal{Z}_\Gamma(s_\infty, \theta_\infty) = 0$$

which clearly contradicts Proposition 4.2. Therefore one can find $\widetilde{\eta}(\varepsilon) > 0$ such that if $\theta \in \mathbb{R}^r$ is such that $\mathrm{dist}(\theta, \mathbb{Z}^r) > \varepsilon$, then $s \mapsto \mathcal{Z}_\Gamma(s, \theta)$ does not vanish inside the rectangle

$$\{\delta - \widetilde{\eta} \leqslant \mathrm{Re}(s) \leqslant \delta \text{ and } |\mathrm{Im}(s)| \leqslant T_0\}.$$

By taking $\eta = \min\{\varepsilon_0, \widetilde{\eta}\}$ we have proved part (1) of Theorem 3.1.

Let us consider the family of rectangles

$$R_{T_0,\eta} := [\delta - \eta, \delta + \eta] + i[-T_0, +T_0].$$

Because we have $\mathcal{Z}_\Gamma(s, 0) = Z_\Gamma(s)$ and $(s, \theta) \mapsto \mathcal{Z}_\Gamma(s, \theta)$ is smooth, there exists a constant $C_{T_0,\eta} > 0$ such that for all $\theta \in \mathbb{R}^r$ with $\|\theta\| \leqslant \epsilon_0$ we have for all $s \in R_{T_0,\eta}$,

$$|\mathcal{Z}_\Gamma(s, \theta) - Z_\Gamma(s)| \leqslant C_{T_0,\eta}\epsilon_0.$$

On the other hand, since on the line $\{\mathrm{Re}(s) = \delta\}$, $Z_\Gamma(s)$ vanishes only at $s = \delta$, with a simple zero, one can find $\eta_0 > 0$ small enough such that for all $s \in R_{T_0,\eta_0}$ one can write

$$Z_\Gamma(s) = (s - \delta)\psi(s),$$

where $\psi(s)$ is holomorphic in a neighbourhood of R_{T_0,η_0} and does not vanish on R_{T_0,η_0}. For all $s \in \partial R_{T_0,\eta_0}$, we have

$$|Z_\Gamma(s)| \geqslant \eta_0 \inf_{R_{T_0,\eta_0}} |\psi(s)| =: M_{T_0,\eta_0}.$$

By choosing $\epsilon_0 > 0$ small enough we can make sure that $M_{T_0,\eta_0} > \epsilon_0 C_{T,\eta_0}$ so that we can apply *Rouché's theorem* to conclude that $\mathcal{Z}_\Gamma(s, \theta)$ has exactly one simple zero in R_{T_0,η_0}. By combining it with Proposition 4.1, we now know that provided $\|\theta\|$ is small enough, $s \mapsto \mathcal{Z}_\Gamma(s, \theta)$ has exactly one zero in a thin strip $\{\delta - \eta_0 \leqslant \mathrm{Re}(s) \leqslant \delta\}$. The fact that this zero is real follows from "time reversal" invariance of the length spectrum: in other words, we have the identity

$$\overline{\mathcal{Z}_\Gamma(s, \theta)} = \mathcal{Z}_\Gamma(\overline{s}, \theta).$$

To see that, first we know that for $\mathrm{Re}(s) > \delta$, we have

$$\mathcal{Z}_\Gamma(s, \theta) = \exp\left(-\sum_{\mathcal{C},k,n} \frac{\chi_\theta(\mathcal{C}^k)}{k} e^{-(s+n)kl(\mathcal{C})}\right),$$

where the sum runs over prime conjugacy classes. By complex conjugation and uniqueness of analytic continuation, we have first the identity valid for all $s \in \mathbb{C}$ and $\theta \in \mathbb{Z}^m$,

$$\overline{\mathcal{Z}_\Gamma(s, \theta)} = \mathcal{Z}_\Gamma(\overline{s}, -\theta).$$

On the other hand, if $\mathcal{C} \in \mathcal{P}$, then $\mathcal{C}^{-1} \in \mathcal{P}$ and $l(\mathcal{C}^{-1}) = l(\mathcal{C})$, while $\chi_\theta(\mathcal{C}^{-1}) = \chi_{-\theta}(\mathcal{C})$. Therefore "time reversal" invariance of $\mathcal{P}$ yields another identity (again use unique continuation) valid for all $s \in \mathbb{C}$ and $\theta \in \mathbb{Z}^m$,

$$\mathcal{Z}_\Gamma(s, \theta) = \mathcal{Z}_\Gamma(s, -\theta),$$

hence the claimed identity. Since non-real zeros must come in conjugate pairs, this forces this unique zero to be real. The fact that this unique zero can be smoothly parametrized as a (real valued) function $\varphi(\theta)$ for all $\|\theta\|$ small is just an application of the holomorphic implicit function theorem, legitimate since

$$\partial_s \mathcal{Z}_\Gamma(\delta, 0) = Z'_\Gamma(\delta) \neq 0.$$

Because we have $\varphi(0) = \delta$, and $\varphi(\theta) \leqslant \delta$ for all θ close to 0 (indeed, all zeta functions $\mathcal{Z}_\Gamma(s,\theta)$ do not vanish inside $\{\mathrm{Re}(s) > \delta\}$), the map $\theta \mapsto \varphi(\theta)$ must have a critical point at $s = \delta$. One can then show that we have

$$\det\left(\nabla^2\varphi(0)\right) \neq 0,$$

i.e. that the associated quadratic form is definite negative. We point out that the non-degeneracy of this critical point has historically played an important role on works related to prime orbit counting in homology classes, see [2, 35, 37, 58, 61]. We provide some details on that fact at the end of Section 5.

The remaining goal of this section is to prove Proposition 4.1 which is concerned with zeros of $\mathcal{Z}_\Gamma(s,\theta)$ for $\mathrm{Re}(s)$ close to δ and large $|\mathrm{Im}(s)|$. When $\theta = 0 \bmod \mathbb{Z}^r$, then this was done in [48]. The game here is to show that one can do the same *uniformly* in θ. As pointed out in [53], the fact that the extra character term $\chi_\theta(\gamma)$ is *locally constant* on $I = \bigcup_j I_j$ makes it possible to apply almost verbatim the analysis of [48], where one has essentially to check that the extra oscillating term does not interfere with the "large $\mathrm{Im}(s)$" cancellation mechanism.

In this section we will choose an *alternative route* based on the recent result of [9] which will allow us to bypass the *most technical part* of the argument in [48], allowing an easier proof of the *uniform spectral gap*. We believe this alternative proof might be interesting for future generalizations of [53] to more general families of possibly non-Galois covers, this will be pursued elsewhere.

4.2. High frequency L^2 estimates

Let $C^1(I)$ denote the Banach space of complex valued functions, C^1 on $\overline{I}$, endowed with the norm ($t \neq 0$)

$$\|f\|_{(t)} := \|f\|_\infty + \frac{1}{|t|}\|f'\|_\infty,$$

where as usual

$$\|f\|_\infty = \sup_{x\in I} |f(x)|.$$

We recall that the action of the transfer operator $\mathcal{L}_{s,\theta}$, now on $C^1(I)$, is given by

$$\mathcal{L}_{s,\theta}(F)(x) := \sum_{j\neq i} (\gamma_j')^s(x)\chi_\theta(P\gamma_j)F(\gamma_j x), \text{ if } x \in I_i.$$

We need to recall a few basic estimates that we will use throughout the rest of the paper.

We first recall some notations. We recall that $\gamma_1, \dots, \gamma_r$ are generators of the Schottky group Γ, as defined in the previous section. Considering a finite sequence α with

$$\alpha = (\alpha_1, \dots, \alpha_n) \in \{1, \dots, 2r\}^n,$$

we set

$$\gamma_\alpha := \gamma_{\alpha_1} \circ \cdots \circ \gamma_{\alpha_n}.$$

We then denote by $\mathscr{W}_n$ the set of admissible sequences of length n by

$$\mathscr{W}_n := \{\alpha \in \{1, \dots, 2r\}^n : \forall\, i = 1, \dots, n-1, \alpha_{i+1} \neq \alpha_i + r \bmod 2r\}.$$

We point out that $\alpha \in \mathscr{W}_n$ if and only if γ_α is a *reduced word* in the free group Γ. For all $j = 1, \dots, 2r$, we define $\mathscr{W}_n^j$ by

$$\mathscr{W}_n^j := \{\alpha \in \mathscr{W}_n : \alpha_n \neq j\}.$$

If $\alpha \in \mathscr{W}_n^j$, then γ_α maps $\overline{\mathcal{D}_j}$ into $\mathcal{D}_{\alpha_1+r}$. Given the above notations and $f \in C^1(I)$, we have for all $x \in I_j$ and $n \in \mathbb{N}$,

$$\mathcal{L}_{s,\theta}^n(f)(x) = \sum_{\alpha\in\mathscr{W}_n^j} (\gamma_\alpha'(x))^s \chi_\theta(P\gamma_\alpha) f(\gamma_\alpha(x)).$$

We will need in this section some distortion estimates (similar to the ones used on discs $\mathcal{D}_j$ in Section 2) for these "inverse branches" of T^n that can be found in [48]. More precisely we have:

- *(Uniform hyperbolicity).* One can find $C > 0$ and $0 < \overline{\rho} < \rho < 1$ such that for all n and all j such that $\alpha \in \mathscr{W}_n^j$, then for all $x \in I_j$ we have
$$C^{-1}\overline{\rho}^n \leqslant |\gamma_\alpha'(x)| \leqslant C\rho^n.$$
- *(Bounded distortion).* There exists $M_1 > 0$ such that for all n, j and all $\alpha \in \mathscr{W}_n^j$,
$$\sup_{I_j} \left|\frac{\gamma_\alpha''}{\gamma_\alpha'}\right| \leqslant M_1.$$

- *(Bounded distortion for third derivatives)*. There exists $Q > 0$ such that for all n, j and all $\alpha \in \mathscr{W}_n^j$,
$$\sup_{I_j} \left| \frac{\gamma_\alpha'''}{\gamma_\alpha'} \right| \leqslant Q.$$

The bounded distortion estimate has the following important consequence: there exists a uniform constant $M_2 > 0$ such that for all $x, y \in I_j$,
$$\left| \frac{\gamma_\alpha'(x)}{\gamma_\alpha'(y)} \right| \leqslant M_2.$$
We point out that the same conclusion is still valid (up to a bigger constant M_3) if x and y belong to different I_j and $I_{j'}$ such that $\alpha \in \mathscr{W}_n^j \cap \mathscr{W}_n^{j'}$. Indeed if $\alpha = \alpha_1 \dots \alpha_n$ then both $\gamma_{\alpha_n}(x), \gamma_{\alpha_n}(y) \in I_{\alpha_n + r}$ and we can apply the above estimate.

We will also need to recall facts on the *topological pressure*, already introduced in Section 2, and *Bowen's formula*. Recall that the Bowen-Series map
$$T : \bigcup_{i=1}^{2p} I_i \to \mathbb{R} \cup \{\infty\}$$
is defined by $T(x) = \gamma_i(x)$ if $x \in I_i$. The non-wandering set of this map T is exactly the limit set $\Lambda(\Gamma)$ of the group:
$$\Lambda(\Gamma) = \bigcap_{n=1}^{+\infty} T^{-n} \left(\bigcup_{i=1}^{2p} I_i \right).$$
A celebrated result of Bowen [14] says that the map
$$\sigma \mapsto P(-\sigma \log |T'|)$$
is convex, strictly decreasing and vanishes exactly at $\sigma = \delta(\Gamma)$, the Hausdorff dimension of the limit set. An alternative way to compute the topological pressure is to look at weighted sums on periodic orbits i.e. we have
$$e^{P(\varphi)} = \lim_{n \to +\infty} \left(\sum_{T^n x = x} e^{\varphi^{(n)}(x)} \right)^{1/n}, \tag{4.1}$$
with the notation $\varphi^{(n)}(x) = \varphi(x) + \varphi(Tx) + \cdots + \varphi(T^{n-1}x)$. We will use the following fact (already stated in Section 2).

Lemma 4.3. — *For all σ_0, M in $\mathbb{R}$ with $0 \leqslant \sigma_0 < M$, one can find $C_0 > 0$ such that for all n large enough and $M \geqslant \sigma \geqslant \sigma_0$, we have*
$$\sum_{j=1}^{2p} \left(\sum_{\alpha \in \mathscr{W}_n^j} \sup_{I_j} (\gamma_\alpha')^\sigma \right) \leqslant C_0 e^{nP(\sigma_0)}, \tag{4.2}$$

where $P(\sigma)$ is used as a shorthand for $P(-\sigma \log |T'|)$.

The proof of this Lemma follows rather straightforwardly from the Ruelle–Perron–Frobenius Theorem, which we state below ([54, Thm. 2.2]), and will be used several times.

Proposition 4.4 (Ruelle–Perron–Frobenius). — *Set $\mathcal{L}_\sigma = \mathcal{L}_{\sigma,0}$ where σ is real.*

- *The spectral radius of $\mathcal{L}_\sigma$ on $C^1(I)$ is $e^{P(\sigma)}$ which is a simple eigenvalue associated to a strictly positive eigenfunction $h_\sigma > 0$ in $C^1(I)$.*
- *The operator $\mathcal{L}_\sigma$ on $C^1(I)$ is quasi-compact with essential spectral radius smaller than $\kappa(\sigma)e^{P(\sigma)}$ for some $\kappa(\sigma) < 1$.*
- *There are no other eigenvalues on $|z| = e^{P(\sigma)}$.*
- *Moreover, the spectral projector $\mathbb{P}_\sigma$ on $\{e^{P(\sigma)}\}$ is given by*
$$\mathbb{P}_\sigma(f) = h_\sigma \int_{\Lambda(\Gamma)} f \mathrm{d}\mu_\sigma,$$
where μ_σ is the unique T-invariant probability measure on Λ that satisfies
$$\mathcal{L}_\sigma^*(\mu_\sigma) = e^{P(\sigma)}\mu_\sigma.$$

We continue with a basic a priori estimate.

Lemma 4.5 (Lasota–Yorke estimate). — *Fix some $\sigma_0 < \delta$, then there exists $C_0 > 0, \rho < 1$ such that for all n, θ and all $s = \sigma + it$ with $\sigma \geqslant \sigma_0$, we have*
$$\| \left(\mathcal{L}_{s,\theta}^n(f)\right)' \|_\infty \leqslant C_0 e^{nP(\sigma_0)} \left\{(1 + |t|)\|f\|_\infty + \rho^n \|f'\|_\infty\right\}.$$

Proof. — Differentiate the formula for $\mathcal{L}_{s,\theta}^n(f)$ and then use the bounded distortion property plus the uniform contraction, combined with the pressure estimate (4.2). Uniformity with respect to θ follows from the fact that $|\chi_\theta| \equiv 1$. □

The main result of this section is the following.

Proposition 4.6 (Uniform Dolgopyat estimate). — *There exist $\epsilon > 0$, $T_0 > 0$ and $C, \beta > 0$ such that for all θ and $n = [C \log |t|]$ with $s = \sigma + it$ satisfying $|\sigma - \delta| \leqslant \epsilon$ and $|t| \geqslant T_0$, we have*
$$\int_{\Lambda(\Gamma)} |\mathcal{L}_{s,\theta}^n(f)|^2 \mathrm{d}\mu_\delta \leqslant \frac{\|f\|_{(t)}^2}{|t|^\beta}.$$

This type of estimate is very similar in spirit to the ones encountered in the seminal work of Dolgopyat [20] on Anosov flows, hence the terminology.

We claim that Proposition 4.6 implies Proposition 4.1. Assume that $\sigma \leqslant \delta$. First we observe that if $g \in C^1(I)$ is *positive*, then we write ($x \in I_j$)

$$\mathcal{L}_\sigma^n(g)(x) = \sum_{\alpha \in \mathscr{W}_n^j} (\gamma'_\alpha(x))^\sigma g(\gamma_\alpha(x)),$$

and using Cauchy–Schwarz inequality and the pressure estimate (4.2), we have

$$(\mathcal{L}_\sigma^n(g)(x))^2 \leqslant A^2(\sigma, n)\mathcal{L}_\delta^n(g^2),$$

where $A(\sigma, n) \leqslant Ce^{nP(2\sigma-\delta)}$. Now write $n = n_1 + n_2$ where both n_1, n_2 will be specified later on. Given $f \in C^1(I)$, we write

$$\|\mathcal{L}_{s,\theta}^n(f)\|_\infty^2 \leqslant A^2(\sigma, n_1)\|\mathcal{L}_\delta^{n_1}(|\mathcal{L}_{s,\theta}^{n_2}(f)|^2)\|_\infty.$$

Using the Ruelle-Perron-Forbenius theorem at $\sigma = \delta$ gives (using Cauchy–Schwarz and the fact that μ_δ is a probability measure)

$$\|\mathcal{L}_{s,\theta}^n(f)\|_\infty^2 \leqslant CA^2(\sigma, n_1)\left(\int_{\Lambda(\Gamma)} |\mathcal{L}_{s,\theta}^{n_2}(f)|^2 \mathrm{d}\mu_\delta + \kappa^{n_1}\||\mathcal{L}_{s,\theta}^{n_2}(f)|^2\|_{C^1}\right),$$

for some $0 < \kappa < 1$. Using the Lasota–Yorke estimate, we know that for $\sigma_0 \leqslant \sigma \leqslant \delta$ we have (assume $|t| \geqslant 1$)

$$\|\mathcal{L}_{s,\theta}^{n_2}(f)\|_{C^1} \leqslant C_0 e^{n_2 P(\sigma_0)}|t|\|f\|_{(t)}.$$

Using Proposition 4.6 with $n_2 = [C_2 \log |t|]$, we get for $|t| \geqslant T_0$ and $\sigma_0 \leqslant \sigma \leqslant \delta$ with $|\sigma_0 - \delta| \leqslant \epsilon$,

$$\|\mathcal{L}_{s,\theta}^n(f)\|_\infty \leqslant CA(\sigma_0, n_1)\left(\frac{1}{|t|^{\beta/2}} + \kappa^{n_1}|t|^{1+C_2P(\sigma_0)}\right)\|f\|_{(t)}.$$

We know choose $n_1 = [C_1 \log |t|]$ with C_1 large enough so that for $|t| \geqslant T_0$, we have

$$\|\mathcal{L}_{s,\theta}^n(f)\|_\infty \leqslant CA(\sigma_0, n_1)\frac{\|f\|_{(t)}}{|t|^{\beta/2}},$$

and since we have

$$A(\sigma_0, n_1) \leqslant C|t|^{\tilde{C}_1(\delta-\sigma_0)},$$

with $\widetilde{C_1} = C_1|\log \overline{\rho}|$, we can make sure that σ_0 is taken close enough to δ so that

$$\|\mathcal{L}_{s,\theta}^n(f)\|_\infty \leqslant \frac{\|f\|_{(t)}}{|t|^{\beta/4}},$$

for all $|t| \geqslant T_0$. Using the Lasota–Yorke estimate, a similar computation leads to the conclusion that for all θ and $n(t) = [C_3 \log |t|]$ for some $C_3 > 0$, we get

$$\|\mathcal{L}_{s,\theta}^{n(t)}(f)\|_{(t)} \leqslant \frac{\|f\|_{(t)}}{|t|^{\overline{\beta}}}, \tag{4.3}$$

for some $\overline{\beta} > 0$ and $|t| \geqslant T_0 \gg 1$, $|\sigma - \sigma_0| \leqslant \epsilon$ with $\epsilon > 0$. Assume now that $\mathcal{Z}_\Gamma(s,\theta)$ has a zero inside the region

$$\{s \in \mathbb{C} : |\mathrm{Re}(s) - \delta| \leqslant \epsilon \text{ and } |\mathrm{Im}(s)| \geqslant T_0\}.$$

Then we get the existence of $f_{s,\theta} \in C^1(I)$ with $\|f_{s,\theta}\|_{|\mathrm{Im}(s)|} = 1$ such that

$$\mathcal{L}_{s,\theta}(f_{s,\theta}) = f_{s,\theta}.$$

Using (4.3) this leads to

$$1 \leqslant \frac{1}{|\mathrm{Im}(s)|^{\overline{\beta}}},$$

which is clearly a contradiction since $|\mathrm{Im}(s)| \gg 1$.

The remaining subsections will focus on proving Proposition 4.6.

4.3. The measure μ_δ versus Patterson–Sullivan density at i

Patterson–Sullivan densities are measures on the limit set that satisfy interesting invariance properties. In the convex co-compact case, they where first introduced by Patterson in [56]. Primarily defined on the disc model $\mathbb{D}$ of the hyperbolic plane, they are constructed via Poincaré series (with $s > \delta(\Gamma)$, $x \in \mathbb{D}$)

$$P_\Gamma(s; x, x) := \sum_{\gamma \in \Gamma} e^{-s\mathrm{d}(x,\gamma x)}.$$

By taking weak limits as $s \to \delta$ of probability measures

$$\nu_{x,s} := \frac{\sum_{\gamma\in\Gamma} e^{-s\mathrm{d}(x,\gamma x)}\mathcal{D}_{\gamma x}}{P_\Gamma(s;x,x)},$$

where $\mathcal{D}_z$ is the Dirac mass at $z \in \mathbb{D}$, one obtains a Γ-invariant measure ν_x supported on the limit set. For the upper half-plane model $\mathbb{H}^2$, one can use the push forward of ν_x by the inverse of the Cayley map given by $A(z) = i\left(\frac{1+z}{1-z}\right)$.

The Patterson–Sullivan density $\nu := \nu_i$ (centered at i) is then a probability measure supported on the limit set $\Lambda(\Gamma) \subset \mathbb{R}$ that satisfies the equivariant formula (for any integrable f on $\Lambda(\Gamma)$)

$$\forall\, \gamma \in \Gamma, \int_{\Lambda(\Gamma)} f \mathrm{d}\nu = \int_{\Lambda(\Gamma)} (f \circ \gamma)|\gamma'|_{\mathbb{D}}^{\delta} \mathrm{d}\nu,$$

where $|\gamma'(x)|_{\mathbb{D}}$ comes from the unit disc model of $\mathbb{H}^2$, given explicitly by

$$|\gamma'(x)|_{\mathbb{D}} := \gamma'(x)\left(\frac{1+x^2}{1+\gamma(x)^2}\right).$$

See for example Borthwick [7, Lemma 14.2]. This Patterson–Sullivan density ν is actually absolutely continuous with respect to μ_δ, more precisely we have the following.

Lemma 4.7. — *There exists $C_\Gamma > 0$ such that the measure μ_δ from the Ruelle–Perron–Frobenius theorem is*

$$\mu_\delta = C_\Gamma (1+x^2)^\delta \nu.$$

Proof. — From the equivariant formula, we know that for all integrable f and all bounded interval J we have for all $\gamma \in \Gamma$,

$$\int_J f \mathrm{d}\nu = \int_{\gamma^{-1}(J)} (f\circ\gamma)|\gamma'|_{\mathbb{D}}^\delta \mathrm{d}\nu.$$

Remark that

$$\Lambda(\Gamma) \subset \bigcup_{j=1,\dots,2r} \bigcup_{i\neq j} \gamma_i(I_j),$$

so that we write

$$\int_{\Lambda(\Gamma)} f \mathrm{d}\nu = \sum_j \sum_{i\neq j} \int_{\gamma_i(I_j)} f \mathrm{d}\nu.$$

By using the equivariant formula as above we get

$$\int_{\Lambda(\Gamma)} f \mathrm{d}\nu = \sum_j \int_{I_j} \sum_{i\neq j} (f\circ\gamma_i)|\gamma_i'|_{\mathbb{D}}^\delta \mathrm{d}\nu,$$

which we recognize as

$$\int_{\Lambda(\Gamma)} f \mathrm{d}\nu = \int_{\Lambda(\Gamma)} H^{-1}(x)\mathcal{L}_\delta(Hf)(x)\mathrm{d}\nu(x),$$

where $H(x) = \frac{1}{(1+x^2)^\delta}$. It is now clear that acting on measures, we have

$$\mathcal{L}_\delta^*(H^{-1}\nu) = H^{-1}\nu,$$

which by uniqueness of μ_δ (normalized as a probability measure) implies the statement. □

Since the density H^{-1} is smooth and uniformly bounded from above and below on $\Lambda(\Gamma)$, the measure μ_δ inherits straightforwardly some of the properties of Patterson–Sullivan densities. In particular we will need to use the following bound.

Proposition 4.8 (Ahlfors–David upper regularity). — *There exists $B_\Gamma > 0$ such that for all bounded interval J,*

$$\mu_\delta(J) \leqslant B_\Gamma |J|^\delta.$$

For a proof (for ν) of that fact see for example [7, Lemma 14.13]. In [9], Bourgain–Dyatlov established the following *remarkable* Theorem.

Theorem 4.9 (Decay of oscillatory integrals). — *There exist constants $\beta_1, \beta_2 > 0$ such that the following holds. Given $g \in C^1(I)$ and $\Phi \in C^2(I)$, consider the integral*

$$\mathcal{I}(\xi) := \int_{\Lambda(\Gamma)} e^{-i\xi\Phi(x)} g(x) \mathrm{d}\nu(x).$$

If we have

$$\epsilon := \inf_{\Lambda(\Gamma)} |\Phi'| > 0,$$

and $\|\Phi\|_{C^2} \leqslant M$, then for all $|\xi| \geqslant 1$, we have

$$|\mathcal{I}(\xi)| \leqslant C_M |\xi|^{-\beta_1} \epsilon^{-\beta_2} \|g\|_{C^1},$$

where $C_M > 0$ does not depend on ξ, ϵ, g.

Remark. — This result is stated as Theorem 2 in [9]. However the dependence on g and ϵ is not explicit in their statement. The fact that it can be bounded using $\|g\|_{C^1}$ is obvious: it follows from linearity in g and Banach–Steinhaus theorem. The dependence on ϵ appears only in Lemma 3.5 of [9], where one can check that the loss is polynomial in ϵ^{-1}. All we need is to allow $\epsilon \geqslant |\xi|^{-\kappa}$ for some $\kappa > 0$ without ruining the decay in $|\xi|$, see Section 4.4. We mention the recent related work of Jialun Li [41], where similar bounds on Oscillatory integrals are proved and where one has also to allow $\epsilon \to 0$ in the applications.

By Lemma 4.7, it is clear that the exact same statement holds for μ_δ. The proof of Proposition 4.6 will follow rather directly from this decay result and some additional facts that we will prove below.

4.4. A uniform non-integrability (UNI) result

Given two words $\alpha, \beta \in \mathscr{W}_n^j$, consider the quantity

$$\mathscr{D}(\alpha, \beta) := \inf_{x \in I_j} \left| \frac{\gamma_\alpha''(x)}{\gamma_\alpha'(x)} - \frac{\gamma_\beta''(x)}{\gamma_\beta'(x)} \right|.$$

We prove the following estimate, which will be used when estimating the "near-diagonal" sums (see the next section below). This type of estimate is a generalization to Schottky groups of the work done by Baladi and Vallée for the Gauss map [4].

Proposition 4.10 (UNI). — *There exist constants $M > 0$ and $\eta_0 > 0$ such that for all n and all $\epsilon = e^{-\eta n}$ with $0 < \eta < \eta_0$, we have for all $\alpha \in \mathscr{W}_n^j$,*

$$\sum_{\beta \in \mathscr{W}_n^j, \mathscr{D}(\alpha,\beta)<\epsilon} \|\gamma'_\beta\|^\delta_{I_j,\infty} \leqslant M\epsilon^\delta.$$

Proof. — First we set some notations. If α is an admissible word in say $\mathscr{W}_n^j$, we will write

$$\gamma_\alpha(x) = \frac{a_\alpha x + b_\alpha}{c_\alpha x + d_\alpha}, \quad a_\alpha d_\alpha - b_\alpha c_\alpha = 1.$$

Each γ_α is a hyperbolic isometry of $\mathbb{H}^2$ whose attracting fixed point will be denoted by x_α and repelling by x_α^*. The isometric circle of γ_α is the circle centered at

$$z_\alpha = -\frac{d_\alpha}{c_\alpha} = \gamma_\alpha^{-1}(\infty),$$

with radius $\frac{1}{c_\alpha}$. We point out that by our definition of Schottky groups, we must have

$$|\gamma_\alpha^{-1}(\infty)| \leqslant M, \tag{4.4}$$

for some uniform $M > 0$. Since x_α^* is in the disc centered at $-\frac{d_\alpha}{c_\alpha}$ and of radius $1/|c_\alpha|$, we have obviously

$$\left| x_\alpha^* + \frac{d_\alpha}{c_\alpha} \right| \leqslant \frac{1}{|c_\alpha|}.$$

On the other hand, since we have

$$\mathrm{Im}(\gamma_\alpha(i)) = \frac{1}{c_\alpha^2 + d_\alpha^2},$$

we can use (4.4) to deduce that

$$\frac{1}{|c_\alpha|} \leqslant \widetilde{M}\sqrt{\mathrm{Im}(\gamma_\alpha(i))}.$$

By the hyperbolicity estimate, it is now easy to see that one can find constants $M', \eta_0 > 0$ such that for all n we have

$$\frac{1}{|c_\alpha|} \leqslant M' e^{-\eta_0 n},$$

which in turn implies

$$\left| x_\alpha^* + \frac{d_\alpha}{c_\alpha} \right| \leqslant M' e^{-\eta_0 n}. \tag{4.5}$$

This estimate just says that repelling fixed point and center of isometric circle are exponentially close when the word length goes to infinity, a quantitative version of the well known fact that centers of isometric circles accumulate on the limit set.

Given γ_α, then $\gamma_\alpha^{-1} = \gamma_{\overline{\alpha}}$, where $\overline{\alpha} = (\alpha_n + r)\dots(\alpha_1 + r)$, understood mod $2r$. We will use below the fact that $x^*_\alpha = x_{\overline{\alpha}}$, and that $\gamma'_\alpha(x_\alpha) = \gamma'_{\overline{\alpha}}(x^*_\alpha)$. We now go back to the quantity

$$\sum_{\beta \in \mathscr{W}_n^j, \mathscr{D}(\alpha,\beta)<\epsilon} \|\gamma'_\beta\|^\delta_{I_j,\infty}.$$

For each β as above, write

$$\frac{\|\gamma'_\beta\|_{I_j,\infty}}{\gamma'_{\overline{\beta}}(x_{\overline{\beta}})} = \frac{\|\gamma'_\beta\|_{I_j,\infty}}{\gamma'_\beta(x_\beta)},$$

which by the bounded distortion estimate gives

$$\sum_{\beta \in \mathscr{W}_n^j, \mathscr{D}(\alpha,\beta)<\epsilon} \|\gamma'_\beta\|^\delta_{I_j,\infty} \leqslant M'' \sum_{\beta \in \mathscr{W}_n^j, \mathscr{D}(\alpha,\beta)<\epsilon} (\gamma'_{\overline{\beta}}(x_{\overline{\beta}}))^\delta.$$

Using the Gibbs property for the μ_δ measure, see [55, Corollary 3.2.1], we obtain that

$$\sum_{\beta \in \mathscr{W}_n^j, \mathscr{D}(\alpha,\beta)<\epsilon} \|\gamma'_\beta\|^\delta_{I_j,\infty} \leqslant C' \sum_{\beta \in \mathscr{W}_n^j, \mathscr{D}(\alpha,\beta)<\epsilon} \mu_\delta(I_{\overline{\beta}}),$$

where $I_{\overline{\beta}} = \gamma_{\overline{\beta}}(I_{j(\beta)})$, where $I_{j(\beta)}$ is chosen such that $x^*_\beta \in I_{j(\beta)}$. Because the "cylinder sets" $I_{j(\beta)}$ are pairwise disjoints, we get

$$\sum_{\beta \in \mathscr{W}_n^j, \mathscr{D}(\alpha,\beta)<\epsilon} \|\gamma'_\beta\|^\delta_{I_j,\infty} \leqslant C' \mu_\delta \left(\bigcup_{\beta \in \mathscr{W}_n^j, \mathscr{D}(\alpha,\beta)<\epsilon} I_{\overline{\beta}} \right).$$

We now conclude the proof by contemplating the implications of having $\mathscr{D}(\alpha,\beta) < \epsilon$. Roughly speaking, it implies that the *repelling fixed points* of the maps γ_α and γ_β are ϵ-close. Indeed, since we have

$$\mathscr{D}(\alpha,\beta) = 2 \inf_{x \in I_j} \frac{|c_\alpha d_\beta - c_\beta d_\alpha|}{|c_\alpha x + d_\alpha||c_\beta x + d_\beta|},$$

and

$$\gamma'_\alpha(x) = \frac{1}{(c_\alpha x + d_\alpha)^2},$$

we can use the bounded distortion property combined with (4.4) to observe that

$$\mathscr{D}(\alpha,\beta) \geqslant \frac{1}{L} \left| \frac{d_\alpha}{c_\alpha} - \frac{d_\beta}{c_\beta} \right|,$$

for some large constant $L > 0$. Using (4.5) we deduce that

$$|x^*_\alpha - x^*_\beta| \leqslant L' \left(\epsilon + e^{-\eta_0 n}\right).$$

Using the uniform contraction estimate, we get that the union of cylinder sets

$$\bigcup_{\beta \in \mathscr{W}_n^j, \mathscr{D}(\alpha,\beta)<\epsilon} I_{\overline{\beta}}$$

is included in an interval of length at most $\widetilde{L}(\rho^n + \epsilon + e^{-\eta_0 n})$. Choosing ϵ of size $e^{-\eta_1 n}$ with $\eta_1 \leqslant \min\{\eta_0, |\log \rho|\}$ and using the estimate from Proposition 4.8 we conclude the proof. □

4.5. Proof of Proposition 4.6, uniform Dolgopyat estimate

We set $s = \sigma + it$, where $\sigma_0 \leqslant \sigma \leqslant \delta$. We then write for $f \in C^1(I)$,

$$\begin{aligned}
&\int_{\Lambda(\Gamma)} |\mathcal{L}_{s,\theta}^n(f)|^2 \mathrm{d}\mu_\delta \\
&\quad = \sum_{j=1}^{2r} \int_{I_j} \sum_{\alpha,\beta \in \mathscr{W}_n^j} (\gamma_\alpha')^{\sigma+it} \left(\gamma_\beta'\right)^{\sigma-it} \chi_\theta(P\gamma_\alpha)\overline{\chi_\theta(P\gamma_\beta)} f \circ \gamma_\alpha \overline{f \circ \gamma_\beta} \mathrm{d}\mu_\delta \\
&\quad = \sum_j \sum_{\alpha,\beta \in \mathscr{W}_n^j} \chi_\theta(P\gamma_\alpha)\overline{\chi_\theta(P\gamma_\beta)} \int_{\Lambda(\Gamma)} e^{it\Phi_{\alpha,\beta}(x)} g_{\alpha,\beta}^{(j)}(x) \mathrm{d}\mu_\delta(x),
\end{aligned}$$

where we have set

$$\Phi_{\alpha,\beta}(x) := \log \gamma_\alpha'(x) - \log \gamma_\beta'(x),$$

and

$$g_{\alpha,\beta}^{(j)}(x) = \varphi_j(x) \left(\gamma_\alpha'(x)\right)^\sigma \left(\gamma_\beta'(x)\right)^\sigma f \circ \gamma_\alpha(x) \overline{f \circ \gamma_\beta(x)},$$

with φ_j being a $C^1(I)$ function which is $\equiv 1$ on I_j and $\equiv 0$ on I_i for $i \neq j$.

We point out that because they *do not depend* on the x variable, but only on the word α, the oscillating terms $\chi_\theta(P\gamma_\alpha)$ do not interfere with the oscillatory integrals, which is the *crucial reason* why we will get estimates *uniform* with respect to θ.

Using the bounded distortion estimate and the hyperbolicity estimate, we have

$$\|g_{\alpha,\beta}^{(j)}\|_\infty \leqslant C_1 \|\gamma_\alpha'\|_{\infty,j}^\sigma \|\gamma_\beta'\|_{\infty,j}^\sigma \|f\|_{(t)}^2,$$

while

$$\left\| \frac{\mathrm{d}}{\mathrm{d}x} \left(g_{\alpha,\beta}^{(j)} \right) \right\|_\infty \leqslant C_2 \|\gamma_\alpha'\|_{\infty,j}^\sigma \|\gamma_\beta'\|_{\infty,j}^\sigma \|f\|_{(t)}^2 (1 + |t|\rho^n).$$

On the other hand we have precisely

$$\inf_{x \in I_j} |\Phi_{\alpha,\beta}'(x)| = \mathscr{D}(\alpha,\beta).$$

The bounded distortion estimates for the second (and third) derivatives show that

$$\|\Phi_{\alpha,\beta}\|_{C^2(I_j)} \leqslant M$$

for some uniform $M > 0$. We now pick $\epsilon = e^{-\eta n}$, with $0 < \eta < \eta_0$ so that (UNI) holds and write

$$\int_{\Lambda(\Gamma)} |\mathcal{L}^n_{s,\theta}(f)|^2 \mathrm{d}\mu_\delta \leqslant \underbrace{C_1 \sum_j \sum_{\mathscr{D}(\alpha,\beta)<\epsilon} \|\gamma'_\alpha\|^\sigma_{\infty,j} \|\gamma'_\beta\|^\sigma_{\infty,j} \|f\|^2_{(t)}}_{\text{near diagonal sum}} + \underbrace{\sum_j \sum_{\mathscr{D}(\alpha,\beta)\geqslant\epsilon} \left| \int_{\Lambda(\Gamma)} e^{it\Phi_{\alpha,\beta}} g^{(j)}_{\alpha,\beta} \mathrm{d}\mu_\delta \right|}_{\text{off diagonal sum}}.$$

Using the pressure estimate and the (UNI) property, the "near diagonal sum" is estimated from above by

$$C_3 \|f\|^2_{(t)} A(\sigma_0, n) e^{nP(\sigma_0)} \epsilon^\delta.$$

Using the polynomial decay result on oscillatory integrals, the "off diagonal sum" is estimated from above (again using the pressure estimate) by

$$C_4 \|f\|^2_{(t)} \frac{|t|^{-\beta_1}(1 + |t|\rho^n)}{\epsilon^{\beta_2}} e^{2nP(\sigma_0)},$$

so that

$$\int_{\Lambda(\Gamma)} |\mathcal{L}^n_{s,\theta}(f)|^2 \mathrm{d}\mu_\delta \leqslant C_5 \|f\|^2_{(t)} \left(A(\sigma_0, n) e^{nP(\sigma_0)} \epsilon^\delta + \frac{|t|^{-\beta_1}(1 + |t|\rho^n)}{\epsilon^{\beta_2}} e^{2nP(\sigma_0)} \right).$$

We recall that $A(\sigma_0, n) \leqslant C\overline{\rho}^{(\sigma-\delta)n}$ and $\epsilon = e^{-\eta n}$. In the latter, n is now taken as $n = [C_0 \log |t|]$. We know <u>fix</u> $C_0 \gg 1$ so that $|t|\rho^n$ stays bounded as $|t| \to +\infty$ and choose $\eta > 0$ small enough so that we get

$$\int_{\Lambda(\Gamma)} |\mathcal{L}^n_{s,\theta}(f)|^2 \mathrm{d}\mu_\delta \leqslant C_6 \|f\|^2_{(t)} \left(A(\sigma_0, n) e^{nP(\sigma_0)} |t|^{-\beta_3} + |t|^{-\beta_1/2} e^{2nP(\sigma_0)} \right).$$

It is now clear that by taking σ_0 close enough to δ we obtain for all $|t|$ large,

$$\int_{\Lambda(\Gamma)} |\mathcal{L}^n_{s,\theta}(f)|^2 \mathrm{d}\mu_\delta \leqslant C_7 \|f\|^2_{(t)} |t|^{-\beta_4},$$

for some $\beta_4 > 0$ and the proof is complete. □

5. Zeros of $\mathcal{Z}_\Gamma(s,\theta)$ on the line $\{\mathrm{Re}(s)=\delta\}$ and hessian matrix $\nabla^2\varphi(0)$.

In this final section we prove Proposition 4.2, by combining standard ideas from [48] and [55]. Notice that we already know from [48], that $s \mapsto \mathcal{Z}_\Gamma(s,0)$ only vanishes at $s=\delta$ on the line $\{\mathrm{Re}(s)=\delta\}$, with a simple zero, a consequence of the fact that the length spectrum of $\Gamma\backslash\mathbb{H}^2$ is non-lattice. Therefore we need to show that on the line $\{\mathrm{Re}(s)=\delta\}$, if $\theta \neq 0 \bmod \mathbb{Z}^r$, then $s \mapsto \mathcal{Z}_\Gamma(s,\theta)$ does not vanish. Assume that $\theta \neq 0 \bmod \mathbb{Z}^r$ and suppose that $\mathcal{Z}_\Gamma(\delta+it_0,\theta)=0$. Then by the Fredholm determinant identity, we know that there exists $g=g_{t_0,\theta}\in C^1(I)$ with $\|g\|_\infty \neq 0$ such that

$$\mathcal{L}_{\delta+it_0,\theta}(g)=g.$$

Using the Ruelle–Perron–Frobenius theorem, we can conjugate $\mathcal{L}_{\delta+it_0,\theta}$ by the positive non-vanishing eigenfunction h_δ so that we have for all $x\in I_i$

$$\sum_{j\neq i} h_j(x)=1, \qquad \sum_{j\neq i} h_j(x)(\gamma_j'(x))^{it}\chi_\theta(P\gamma_j)\widetilde{g}\circ\gamma_j(x)=\widetilde{g}(x),$$

where $h_j(x):=h_\delta^{-1}(\gamma_j')^\delta h_\delta\circ\gamma_j$ and $\widetilde{g}=h_\delta^{-1}g$. Choosing i and $x_0\in I_i$ such that

$$|\widetilde{g}(x_0)|=\sup_{x\in\Lambda(\Gamma)}|\widetilde{g}(x)|:=\|\widetilde{g}\|_{\infty,\Lambda(\Gamma)},$$

we observe that

$$|\widetilde{g}(x_0)|\leqslant\sum_{j\neq i}h_j(x_0)|\widetilde{g}\circ\gamma_j(x_0)|\leqslant\|\widetilde{g}\|_{\infty,\Lambda(\Gamma)},$$

which implies that for all $j\neq i$,

$$|\widetilde{g}\circ\gamma_j(x_0)|=\|\widetilde{g}\|_{\infty,\Lambda(\Gamma)}.$$

Iterating this argument and using the fact that the orbit of x_0 under the semigroup generated by $\gamma_1,\dots,\gamma_{2r}$ is dense in $\Lambda(\Gamma)$, we conclude that $|\widetilde{g}|$ is actually constant on $\Lambda(\Gamma)$. Taking this constant equal to one, we can write

$$\widetilde{g}(x)=e^{i\phi(x)},$$

where ϕ is in, say, $C^0(\Lambda(\Gamma))$. We obtain that for all $x\in I_i\cap\Lambda(\Gamma)$,

$$\sum_{j\neq i}h_j(x)e^{it_0\log\gamma_j'(x)+2i\pi\langle\theta,P\gamma_j\rangle+\phi\circ\gamma_j(x)}=e^{i\phi(x)},$$

which by strict convexity of the unit circle implies that for all j,

$$t_0\log\gamma_j'(x)+\phi\circ\gamma_j(x)-\phi(x)\in 2\pi\mathbb{Z}-2\pi\langle\theta,P\gamma_j\rangle.$$

If $t_0 = 0$ then this implies (by evaluating at attracting fixed points of each γ_j) that for all $j = 1, \dots, r$,

$$\langle \theta, P\gamma_j \rangle \in \mathbb{Z}.$$

Since $\{P\gamma_1, \dots, P\gamma_r\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^r$, it implies that $\theta \in \mathbb{Z}^r$, a contradiction. Therefore $t_0 \neq 0$. Iterating the above formula, we get that for all $\gamma_\alpha \in \mathscr{W}_n^j$,

$$t_0 \log \gamma_\alpha'(x) + \phi \circ \gamma_\alpha(x) - \phi(x) \in 2\pi\mathbb{Z} - 2\pi\langle \theta, P\gamma_\alpha \rangle.$$

By evaluating at the attracting fixed point x_α of γ_α, we obtain that the translation length l_α of γ_α, given by the formula

$$e^{-l_\alpha} = \gamma_\alpha'(x_\alpha),$$

satisfies

$$l_\alpha \in \frac{2\pi}{t_0}\mathbb{Z} + \frac{2\pi}{t_0}\langle \theta, P\gamma_\alpha \rangle.$$

In term of lengths of closed geodesics, it shows in particular that the set of closed geodesics which belong to the homology class of 0 (i.e. $P\gamma_\alpha = 0$) is a subset of $\frac{2\pi}{t_0}\mathbb{Z}$. But this would imply that the length spectrum of $(\operatorname{Ker} P)\backslash\mathbb{H}^2$ is *lattice*, which is impossible since $\operatorname{Ker} P = [\Gamma, \Gamma]$ is the commutator subgroup of Γ and hence non-elementary, see for example [19].

We now explain why the Hessian matrix at 0 of $\theta \mapsto \varphi(\theta)$ is non degenerate. Let us fix $\theta \neq 0 \mod \mathbb{Z}^r$. Let $t \in \mathbb{R}$ and set

$$\chi_{t\theta}(\gamma) := e^{itg_\theta(\gamma)},$$

where $g_\theta(\gamma) := 2\pi\langle \theta, P\gamma \rangle$. We use the same argument as in [61]. For all t close to 0, we now by definition of φ that there exists $g_t \in C^1(I)$ such that for all n,

$$\mathcal{L}^n_{\varphi(t\theta), t\theta}(g_t) = g_t.$$

Perturbation theory (for simple eigenvalues) shows that the t-dependence is smooth. Differentiating two times with respect to t, and performing the same calculation(3) as in [55, p. 60–61], we obtain that

$$\langle \nabla^2 \varphi(0)\theta, \theta \rangle = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\varphi(t\theta)\right)|_{t=0} = -C \lim_{n \to +\infty} \frac{1}{n} \int_{\Lambda(\Gamma)} \left(g_\theta^{(n)}\right)^2 \mathrm{d}\mu_\delta,$$

where $C > 0$ is some normalizing constant and $g_\theta^{(n)} = g_\theta + g_\theta \circ T + \dots + g_\theta \circ T^{n-1}$. By Proposition 4.12 from [55, p. 62], we deduce that $\langle \nabla^2 \varphi(0)\theta, \theta \rangle = 0$

(3) This is the calculation of the second derivative of the pressure or the variance in the CLT for expanding maps.

implies that g_θ is cohomologous to a constant. Using the same ideas as above, we get that there exists $a \in \mathbb{R}$ such that for all $j = 1, \dots, 2r$,

$$\langle \theta, P\gamma_j \rangle = a.$$

But $P(\gamma_{j+r}) = P(\gamma_j^{-1}) = -P(\gamma_j)$, therefore $a = 0$. We are left with the fact that for all $j = 1, \dots, r$, $\langle \theta, P\gamma_j \rangle = 0$, which implies $\theta = 0$, a contradiction.

6. Zero-free regions for L-functions and explicit formulae

The goal of this section is to prove the following result which will allow us to convert zero-free regions into upper bounds on sums over closed geodesics. The results are completely general, but will be used in the last section on congruence subgroups.

Proposition 6.1. — *Fix $\overline{\alpha} > 0$, $0 \leqslant \sigma < \delta$ and $\varepsilon > 0$. Then there exists a C_0^∞ test function φ_0, with $\varphi_0 \geqslant 0$, $\mathrm{Supp}(\varphi_0) = [-1, +1]$ and such that for ϱ non-trivial, if $L_\Gamma(s, \varrho)$ has no zeros in the rectangle*

$$\{\sigma \leqslant \mathrm{Re}(s) \leqslant 1 \text{ and } |\mathrm{Im}(s)| \leqslant (\log T)^{1+\overline{\alpha}}\},$$

for some T large enough, then we have

$$\sum_{\mathcal{C},k} \chi_\varrho(\mathcal{C}^k) \frac{l(\mathcal{C})}{1 - e^{kl(\mathcal{C})}} \varphi_0\left(\frac{kl(\mathcal{C})}{T}\right) = O\left(d_\varrho \log(d_\varrho + 1) e^{(\sigma+\varepsilon)T}\right),$$

where the implied constant is uniform in T, d_ϱ.

The proof will occupy the full section and will be broken into several elementary steps.

6.1. Preliminary lemmas

We start this section by the following fact from harmonic analysis.

Lemma 6.2. — *For all $\alpha > 0$, there exists $C_1, C_2 > 0$ and a positive test function $\varphi_0 \in C_0^\infty(\mathbb{R})$ with $\mathrm{Supp}(\varphi) = [-1, +1]$ such that for all $|\xi| \geqslant 2$, we have*

$$|\widehat{\varphi_0}(\xi)| \leqslant C_1 e^{|\mathrm{Im}(\xi)|} \exp\left(-C_2 \frac{|\mathrm{Re}(\xi)|}{(\log|\mathrm{Re}(\xi)|)^{1+\alpha}}\right),$$

where $\widehat{\varphi_0}(\xi)$ is the Fourier transform, defined as usual by

$$\widehat{\varphi_0}(\xi) = \int_{-\infty}^{+\infty} \varphi_0(x) e^{-ix\xi} \mathrm{d}x.$$

Proof. — It is known from the Beurling–Malliavin multiplier Theorem, or the Denjoy–Carleman Theorem, that for compactly supported test functions ψ, one cannot beat the Fourier decay rate ($\xi \in \mathbb{R}$, large)

$$|\widehat{\psi}(\xi)| = O\left(\exp\left(-C\frac{|\xi|}{\log|\xi|}\right)\right),$$

because this rate of Fourier decay implies quasi-analyticity (hence no compactly supported test functions). We refer the reader to [36, Chapter 5] for more details. The above statement is definitely a *folklore* result. However since we need a precise control for complex valued ξ and couldn't find the exact reference for it, we provide an outline of the proof which follows closely the construction that one can find in [36, Chapter 5, Lemma 2.7].

Let $(\mu_j)_{j\geqslant 1}$ be a sequence of positive numbers such that $\sum_{j=1}^{\infty}\mu_j = 1$. For all $k \in \mathbb{Z}$, set

$$\varphi_N(k) = \prod_{j=1}^{N}\frac{\sin(\mu_j k)}{\mu_j k}, \varphi(k) = \prod_{j=1}^{\infty}\frac{\sin(\mu_j k)}{\mu_j k}.$$

Consider the Fourier series given by

$$f(x) := \sum_{k\in\mathbb{Z}}\varphi(k)e^{ikx}, f_N(x) := \sum_{k\in\mathbb{Z}}\varphi_N(k)e^{ikx},$$

then one can observe that by rapid decay of $\varphi(k)$, $f(x)$ defines a C^∞ function on $[-2\pi, 2\pi]$. On the other hand, one can check that $f_N(x)$ converges uniformly to f as N goes to ∞ and that

$$f_N(x) = (g_1 \star g_2 \star \cdots \star g_N)(x),$$

where $\star$ is the convolution product and each g_j is given by

$$g_j(x) := \begin{cases} \frac{2\pi}{\mu_j} & \text{if } |x| \leqslant \mu_j \\ 0 & \text{elsewhere.} \end{cases}$$

From this observation one deduces that f is positive and supported on $[-1, +1]$ since we assume

$$\sum_{j=1}^{\infty}\mu_j = 1.$$

We now extend f outside $[-1, +1]$ by zero and write by integration by parts and Schwarz inequality,

$$|\widehat{f}(\xi)| \leqslant \frac{e^{|\mathrm{Im}(\xi)|}}{|\mathrm{Re}(\xi)|^N}\|f^{(N)}\|_{L^2(-1,+1)}.$$

By Plancherel formula, we get

$$\|f^{(N)}\|^2_{L^2(-1,+1)} = \sum_{k\in\mathbb{Z}} k^{2N}(\varphi(k))^2 \leqslant C \prod_{j=1}^{N+1} \mu_j^{-2},$$

where $C > 0$ is some universal constant. Fixing $\epsilon > 0$, we now choose

$$\mu_j = \frac{\widetilde{C}}{j(\log(1+j))^{1+\epsilon}},$$

where $\widetilde{C}$ is adjusted so that $\sum_{j=1}^{\infty} \mu_j = 1$, and we get

$$|\widehat{f}(\xi)| \leqslant \frac{e^{|\mathrm{Im}(\xi)|}}{|\mathrm{Re}(\xi)|^N}(C_1)^N N! e^{N(1+\epsilon)\log\log(N)}.$$

Using Stirling's formula and choosing N of size

$$N = \left[\frac{|\mathrm{Re}(\xi)|}{(\log(|\mathrm{Re}(\xi)|)^{1+2\epsilon}}\right]$$

yields (after some calculations) to

$$|\widehat{f}(\xi)| \leqslant O\left(e^{|\mathrm{Im}(\xi)|} e^{-C_2 \frac{|\mathrm{Re}(\xi)|}{(\log(|\mathrm{Re}(\xi)|)^{1+2\epsilon}}}\right),$$

and the proof is finished. □

One can obviously push the above construction further below the threshold $\frac{|\xi|}{\log|\xi|}$ by obtaining decay rates of the type

$$\exp\left(-\frac{|\xi|}{\log|\xi| \log(\log|\xi|) \dots (\log_{(n)}|\xi|)^{1+\alpha}}\right),$$

where $\log_{(n)}(x) = \log\log\dots\log(x)$, iterated n times. However this would only yield a very mild improvement to the main statement, so we will content ourselves with the above lemma.

We continue with another result which will allow us to estimate the size of the log-derivative of $L_\Gamma(s,\varrho)$ in a narrow rectangular zero-free region. More precisely, we have the following:

Proposition 6.3. — *Fix $\sigma < \delta$. For all $\epsilon > 0$, there exist $C(\epsilon), R(\epsilon) > 0$ such that for all $R \geqslant R(\epsilon)$, if $L_\Gamma(s,\varrho)$ (ϱ is non-trivial) has no zeros in the rectangle*

$$\{\sigma \leqslant \mathrm{Re}(s) \leqslant 1 \text{ and } |\mathrm{Im}(s)| \leqslant R\},$$

then we have for all s in the smaller rectangle

$$\{\sigma + \epsilon \leqslant \mathrm{Re}(s) \leqslant 1 \text{ and } |\mathrm{Im}(s)| \leqslant C(\epsilon)R\},$$

$$\left|\frac{L'_\Gamma(s,\varrho)}{L_\Gamma(s,\varrho)}\right| \leqslant B(\epsilon) d_\varrho \log(d_\varrho + 1) R^6.$$

Proof. — We will use Caratheodory's Lemma and take advantage of the a priori bound from Theorem 1.1. More precisely, our goal is to rely on this estimate (see Titchmarsh [73, 5.51]).

Lemma 6.4. — *Assume that f is a holomorphic function on a neighborhood of the closed disc $\overline{D}(0,r)$, then for all $r'<r$, we have*

$$\max_{|z|\leqslant r'}|f'(z)|\leqslant \frac{8r}{(r-r')^2}\left(\max_{|z|\leqslant r}|\mathrm{Re}(f(z))|+|f(0)|\right).$$

First we recall that for all $\mathrm{Re}(s)>\delta$, $L_\Gamma(s,\varrho)$ does not vanish and has a representation as

$$L_\Gamma(s,\varrho)=\exp\left(-\sum_{\mathcal{C},k}\frac{\chi_\varrho(\mathcal{C}^k)}{k}\frac{e^{-skl(\mathcal{C})}}{1-e^{kl(\mathcal{C})}}\right),$$

so that we get for all $\mathrm{Re}(s)\geqslant A>\delta$,

$$|\log|L_\Gamma(s,\varrho)||\leqslant C_A d_\varrho,\qquad \left|\frac{L_\Gamma'(s,\varrho)}{L_\Gamma(s,\varrho)}\right|\leqslant C_A' d_\varrho \tag{6.1}$$

where $C_A, C_A'>0$ are uniform constants on all half-planes

$$\{\mathrm{Re}(s)\geqslant A>\delta\}.$$

We have simply used the prime orbit theorem and the trivial bound on characters of unitary representations:

$$|\chi_\varrho(g)|\leqslant d_\varrho,\ \text{for all } g\in\mathbf{G}.$$

Let us now assume that $L_\Gamma(s,\varrho)$ does not vanish on the rectangle

$$\{\sigma\leqslant\mathrm{Re}(s)\leqslant 1 \text{ and } |\mathrm{Im}(s)|\leqslant R\}.$$

Consider the disc $D(M,r)$ centered at M and with radius r where $M(\sigma,R)$ and $r(\sigma,R)$ are given by

$$M(\sigma,R)=\frac{R^2}{2(1-\sigma)}+\frac{\sigma+1}{2};\quad r(\sigma,R)=M(\sigma,R)-\sigma,$$

see the figure below.

Since by assumption $s\mapsto L_\Gamma(s,\varrho)$ does not vanish on the closed disc $\overline{D}(M,r)$, we can choose a determination of the complex logarithm of $L_\Gamma(s,\varrho)$ on this disc to which we can apply Lemma 6.4 on the smaller disc $D(M,r-\varepsilon)$, which yields (using the a priori bound from Theorem 1.1 and estimate (6.1))

$$\begin{aligned}\left|\frac{L_\Gamma'(s,\varrho)}{L_\Gamma(s,\varrho)}\right|&\leqslant C\frac{r}{\varepsilon}\left(d_\varrho\log(d_\varrho+1)r^2+A_1d_\varrho\right)\\&=O\left(R^6d_\varrho\log(d_\varrho+1)\right),\end{aligned}$$

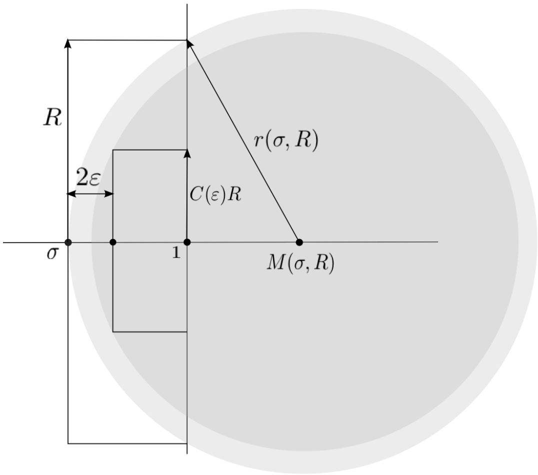

Figure 6.1. Covering rectangles by discs.

where the implied constant is uniform with respect to R and d_ϱ. Looking at the picture, the smaller disc $D(M, r-\varepsilon)$ contains a rectangle

$$\{\sigma + 2\varepsilon \leqslant \mathrm{Re}(s) \leqslant 1 \text{ and } |\mathrm{Im}(s)| \leqslant L(\varepsilon)\},$$

where $L(\varepsilon)$ satisfies the identity (Pythagoras Theorem!)

$$L^2(\varepsilon) = \varepsilon(2M - 2\sigma - 3\varepsilon),$$

which shows that

$$L(\epsilon) \geqslant C(\varepsilon)R,$$

with $C(\varepsilon) > 0$, as long as $R \geqslant R_0(\epsilon)$, for some $R_0 > 0$. The proof is done. $\square$

6.2. Proof of Proposition 6.1

We are now ready to prove the main result of this section, by combining the above facts with a standard contour deformation argument. We fix a small $\varepsilon > 0$ and $0 < \alpha < \overline{\alpha}$. We use Lemma 6.2 to pick a test function φ_0 with Fourier decay as described, with same exponent α. We set for all

$T > 0$, and $s \in \mathbb{C}$,

$$\begin{aligned}\psi_T(s) &= \int_{-\infty}^{+\infty} e^{sx} \varphi_0\left(\frac{x}{T}\right) \mathrm{d}x \\ &= T\widehat{\varphi_0}(isT).\end{aligned}$$

By the estimate from Lemma 6.2, we have

$$|\psi_T(s)| \leqslant C_1 T e^{T|\mathrm{Re}(s)|} \exp\left(-C_2 \frac{|\mathrm{Im}(s)|T}{(\log(T|\mathrm{Im}(s)|))^{1+\alpha}}\right). \tag{6.2}$$

We fix now $A > \delta$ and consider the contour integral

$$I(\varrho, T) = \frac{1}{2i\pi} \int_{A-i\infty}^{A+i\infty} \frac{L_\Gamma'(s,\varrho)}{L_\Gamma(s,\varrho)} \psi_T(s) \mathrm{d}s.$$

Convergence is guaranteed by estimate (6.1) and rapid decay of $|\psi_T(s)|$ on vertical lines. Because we choose $A > \delta$, we have absolute convergence of the series

$$\frac{L_\Gamma'(s,\varrho)}{L_\Gamma(s,\varrho)} = \sum_{\mathcal{C},k} \chi_\varrho(\mathcal{C}^k) \frac{l(\mathcal{C}) e^{-skl(\mathcal{C})}}{1 - e^{kl(\mathcal{C})}}$$

on the vertical line $\{\mathrm{Re}(s) = A\}$, and we can use Fubini to write

$$I(\varrho, T) = \sum_{\mathcal{C},k} \chi_\varrho(\mathcal{C}^k) \frac{l(\mathcal{C})}{1 - e^{kl(\mathcal{C})}} e^{-Akl(\mathcal{C})} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itkl(\mathcal{C})} \widehat{\psi_T}(iA - t) \mathrm{d}t,$$

and Fourier inversion formula gives

$$I(\varrho, T) = \sum_{\mathcal{C},k} \chi_\varrho(\mathcal{C}^k) \frac{l(\mathcal{C})}{1 - e^{kl(\mathcal{C})}} \varphi_0\left(\frac{kl(\mathcal{C})}{T}\right).$$

Assuming that $L_\Gamma(s, \varrho)$ has no zeros in

$$\{\sigma \leqslant \mathrm{Re}(s) \leqslant 1 \text{ and } |\mathrm{Im}(s)| \leqslant R\},$$

where R will be adjusted later on, our aim is to use Proposition 6.3 to deform the contour integral $I(\varrho, T)$ as depicted in the figure below.

Writing $I(\varrho, T) = \sum_{j=1}^{5} I_j$ (see the above figure), we need to estimate carefully each contribution. In the course of the proof, we will use the following basic fact.

Lemma 6.5. — *Let $\phi : [M_0, +\infty) \to \mathbb{R}^+$ be a C^2 map with $\phi'(x) > 0$ on $[M_0, +\infty)$ and satisfying*

$$\sup_{x \geqslant M_0} \left| \frac{\phi''(x)}{(\phi'(x))^3} \right| \leqslant C, \tag{$*$}$$

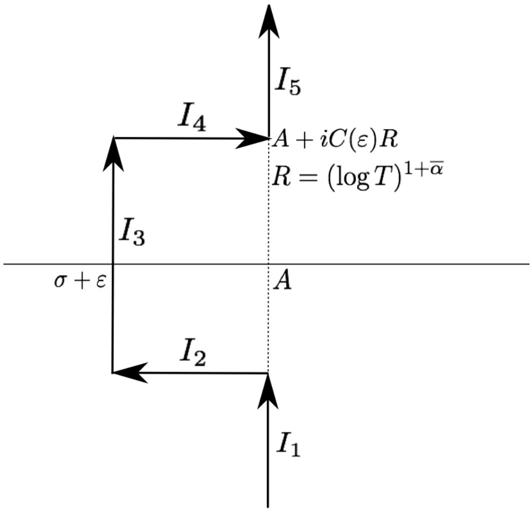

Figure 6.2. The contour deformation

then we have for all $M \geqslant M_0$,

$$\int_M^{+\infty} e^{-\phi(t)} \mathrm{d}t \leqslant \frac{e^{-\phi(M)}}{\phi'(M)} + Ce^{-\phi(M)}.$$

Proof. — First observe that condition $(*)$ implies that

$$x \mapsto \frac{1}{(\phi'(x))^2}$$

has a uniformly bounded derivative, which is enough to guarantee that

$$\lim_{x\to+\infty} \frac{e^{-\phi(x)}}{\phi'(x)} = 0.$$

In particular $\lim_{x\to+\infty} \phi(x) = +\infty$ and for all $M \geqslant M_0$, $\phi : [M, +\infty) \to [\phi(M), +\infty)$ is a C^2-diffeomorphism. A change of variable gives

$$\int_M^{+\infty} e^{-\phi(t)} \mathrm{d}t = \int_{\phi(M)}^{+\infty} e^{-u} \frac{\mathrm{d}u}{\phi'(\phi^{-1}(u))},$$

and integrating by parts yields the result. □

- First we start with I_1 and I_5. Using estimate (6.1) combined with (6.2), we have

$$|I_5| \leqslant Cd_\varrho Te^{TA} \int_{C(\varepsilon)R}^{+\infty} e^{-C_2 \frac{tT}{(\log(tT))^{1+\alpha}}} \mathrm{d}t,$$

which by a change of variable leaves us with

$$|I_5| \leqslant C d_\varrho e^{TA} \int_{C(\varepsilon)RT}^{+\infty} e^{-C_2 \frac{u}{(\log(u))^{1+\alpha}}} \mathrm{d}u.$$

This where we use Lemma 6.5 with

$$\phi(x) = C_2 \frac{x}{(\log(x))^{1+\alpha}}.$$

Computing the first two derivatives, we can check that condition $(*)$ is fulfilled and therefore

$$\int_M^{+\infty} e^{-C_2 \frac{u}{(\log(u))^{1+\alpha}}} \leqslant C(\log(M))^{1+\alpha} e^{-C_2 \frac{M}{(\log(M))^{1+\alpha}}},$$

for some universal constant $C > 0$. We have finally obtained

$$|I_5| \leqslant C d_\varrho e^{TA} (\log(RT))^{1+\alpha} e^{-C_2 \frac{RT}{(\log(RT))^{1+\alpha}}}.$$

Choosing $R = (\log(T))^{1+\overline{\alpha}}$, with $\overline{\alpha} > \alpha$ gives

$$\begin{aligned} |I_5| &= O\left(d_\varrho e^{TA} (\log(T))^{1+\alpha} e^{-C_2 T (\log(T))^{\overline{\alpha}-\alpha}}\right) \\ &= O\left(d_\rho e^{-BT}\right), \end{aligned}$$

where $B > 0$ can be taken as large as we want. The exact same estimate is valid for I_1.

- The case of I_4 and I_2. Here we use the bound from Proposition 6.3 and again (6.2) to get

$$|I_4| + |I_2| = O\left(d_\varrho \log(d_\varrho + 1) e^{-BT}\right),$$

where B can be taken again as large as we want.

- We are left with I_3 where

$$I_3 = \frac{1}{2\pi} \int_{-C(\epsilon)R}^{+C(\epsilon)R} \frac{L'_\Gamma(\sigma + \varepsilon + it, \varrho)}{L_\Gamma(\sigma + \varepsilon + it, \varrho)} \psi_T(\sigma + \varepsilon + it) \mathrm{d}t.$$

Using Proposition 6.3 and (6.2) we get

$$|I_3| = O\left(d_\varrho \log(d_\varrho + 1)(\log(T))^{7(1+\overline{\alpha})} e^{(\sigma+\epsilon)T}\right).$$

Clearly the leading term in the contour integral is provided by I_3, and the proof of Proposition 6.1 is now complete.

We conclude this section by a final observation. If $\varrho = \mathrm{id}$ is the *trivial representation*, then $L_\Gamma(s, \mathrm{id}) = Z_\Gamma(s)$ has a zero at $s = \delta$, thus the best estimate for the contour integral $I(\mathrm{id}, T)$ is given by (6.1) and (6.2) which yields (by a change of variable)

$$\begin{aligned}|I(\mathrm{id}, T)| &\leqslant C_A d_\varrho \int_{-\infty}^{+\infty} |\psi_T(A+it)| \mathrm{d}t \\ &\leqslant \widetilde{C}_A d_\varrho T e^{TA} \int_{-\infty}^{+\infty} \exp\left(-C_2 \frac{|t|T}{(\log(T|t|))^{1+\alpha}}\right) \mathrm{d}t = O\left(d_\varrho e^{TA}\right).\end{aligned}$$

Since $d_\varrho = 1$ and A can be taken as close to δ as we want, the contribution from the trivial representation is of size

$$I(\mathrm{id}, T) = O\left(e^{(\delta+\epsilon)T}\right). \tag{6.3}$$

7. Congruence subgroups and existence of "low lying" zeros for $L_\Gamma(s, \varrho)$

7.1. Conjugacy classes in G

In this section, we will use more precise knowledge on the group structure of

$$\mathbf{G} = \mathrm{SL}_2(\mathbb{F}_p).$$

Our basic reference is the book [72], see Section 6 of Chapter 3 for much more general statements over finite fields. We start by describing the conjugacy classes in $\mathbf{G}$. Since we are only interested in the large p behaviour, we will assume that p is an odd prime strictly bigger than 3. Conjugacy classes of elements $g \in \mathbf{G}$ are essentially determined by the roots of the characteristic polynomial

$$\det(xI_2 - g) = x^2 - \mathrm{tr}(g)x + 1,$$

which are denoted by λ, λ^{-1}, where $\lambda \in \mathbb{F}_{p}^{\times}$. There are three different possibilities.

- $\lambda \neq \lambda^{-1} \in \mathbb{F}_p^\times$. In that case g is diagonalizable over $\mathbb{F}_p$ and g is conjugate to the matrix

$$D(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

  The centralizer $\mathcal{Z}(D(\lambda)) = \{h \in \mathbf{G} : hD(\lambda)h^{-1} = D(\lambda)\}$ is then equal to the "maximal torus"

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{F}_p^\times \right\},$$

and we have $|A| = p - 1$, the conjugacy class of g has $p(p+1)$ elements.

- $\lambda \neq \lambda^{-1} \notin \mathbb{F}_p^\times$. In that case λ belongs to $\mathcal{F} \simeq \mathbb{F}_{p^2}$ the unique quadratic extension of $\mathbb{F}_p$. The root λ can be written as
$$\lambda = a + b\sqrt{\epsilon}, \lambda^{-1} = a - b\sqrt{\epsilon},$$
where $\{1, \sqrt{\epsilon}\}$ is a fixed $\mathbb{F}_p$-basis of $\mathcal{F}$. Therefore g is conjugate to
$$\begin{pmatrix} a & \epsilon b \\ b & a \end{pmatrix},$$
and $|\mathcal{Z}(g)| = p + 1$, its conjugacy class has $p(p-1)$ elements.
- $\lambda = \lambda^{-1} \in \{\pm 1\}$. In that case g is non-diagonalizable unless $g \in \mathcal{Z}(\mathbf{G}) = \{\pm I_2\}$, and is conjugate to $\pm u$ or $\pm u'$ where
$$u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad u' = \begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}.$$
The centralizer $\mathcal{Z}(g)$ has cardinality $2p$ and the four conjugacy classes have $p(p+1)$ elements.

Using this knowledge on conjugacy classes, one can construct all irreducible representations and write a character table for $\mathbf{G}$, but we won't need it. There are two facts that we highlight and will use in the sequel:

(1) For all $g \in \mathbf{G}$, $|\mathcal{Z}(g)| \geqslant p - 1$.
(2) For all ϱ non-trivial we have $d_\rho \geqslant \frac{p-1}{2}$.

We will also rely on the very important observation below.

Proposition 7.1. — *Let Γ be a convex co-compact subgroup of $\mathrm{SL}_2(\mathbb{Z})$ as above. Fix $0 < \beta < 2$, and consider the set $\mathcal{E}_T$ of conjugacy classes $\overline{\gamma} \subset \Gamma \setminus \{\mathrm{Id}\}$ such that for all $\overline{\gamma} \in \mathcal{E}_T$, we have $\mathcal{L}(\gamma) \leqslant T := \beta \log(p)$. Then for all p large and all $\overline{\gamma_1}, \overline{\gamma_2} \in \mathcal{E}_T$, the following are equivalent:*

(1) $\mathrm{tr}(\gamma_1) = \mathrm{tr}(\gamma_2)$.
(2) *γ_1 and γ_2 are conjugate in $\mathbf{G}$.*

Proof. — Clearly (1) implies that γ_1 and γ_2 have the same trace modulo p. Unless we are in the cases $\mathrm{tr}(\gamma_1) = \mathrm{tr}(\gamma_2) = \pm 2 \mod p$, we know from the above description of conjugacy classes that they are determined by the knowledge of the trace. To eliminate these "parabolic mod p" cases, we observe that if $\overline{\gamma} \in \mathcal{E}_T$ satisfies $\mathrm{tr}(\gamma) = \pm 2 + kp$ with $k \neq 0$, then
$$2\cosh(l(\gamma)/2) = |\mathrm{tr}(\gamma)| \geqslant p - 2,$$
and we get
$$p - 2 \leqslant 1 + p^{\frac{\beta}{2}},$$

which leads to an obvious contradiction if p is large, therefore $k=0$. Then it means that $|\mathrm{tr}(\gamma)|=2$ which is impossible since Γ has no non-trivial parabolic element (convex co-compact hypothesis). Conversely, if γ_1 and γ_2 are conjugate in $\mathbf{G}$, then we have

$$\mathrm{tr}(\gamma_1)=\mathrm{tr}(\gamma_2) \bmod p.$$

If $\mathrm{tr}(\gamma_1)\neq \mathrm{tr}(\gamma_2)$ then this gives

$$p\leqslant |\mathrm{tr}(\gamma_1)-\mathrm{tr}(\gamma_2)|\leqslant 4\cosh(T/2)\leqslant 2(p^{\frac{\beta}{2}}+1),$$

again a contradiction for p large. □

7.2. Proof of Theorem 1.4

Before we can rigorously prove Theorem 1.4, we need one last fact from representation theory which is a handy folklore formula.

Lemma 7.2. — *Let $\mathbf{G}$ be a finite group and let $\varrho:\mathbf{G}\to\mathrm{End}(V_\varrho)$ be an irreducible representation. Then for all $x,y\in\mathbf{G}$, we have*

$$\chi_\varrho(x)\overline{\chi_\varrho(y)}=\frac{d_\varrho}{|G|}\sum_{g\in\mathbf{G}}\chi_\varrho(xgy^{-1}g^{-1}).$$

Proof. — Writing

$$\sum_{g\in\mathbf{G}}\chi_\varrho(xgy^{-1}g^{-1})=\mathrm{Tr}\left(\varrho(x)\sum_g\varrho(gy^{-1}g^{-1})\right),$$

we observe that

$$U_y:=\sum_g\varrho(gy^{-1}g^{-1})$$

commutes with the irreducible representation ϱ, therefore by Schur's Lemma [68, Chapter 2], it has to be of the form

$$U_y=\lambda(y)I_{V_\varrho},$$

with $\lambda(y)\in\mathbb{C}$, which shows that

$$\sum_{g\in\mathbf{G}}\chi_\varrho(xgy^{-1}g^{-1})=\chi_\varrho(x)\lambda(y).$$

Similarly we obtain

$$\sum_{g\in\mathbf{G}}\chi_\varrho(xgy^{-1}g^{-1})=\overline{\chi_\varrho(y)}\lambda(x),$$

and evaluating at the neutral element $x=e_{\mathbf{G}}$ ends the proof since we have

$$U_{e_{\mathbf{G}}}=|\mathbf{G}|I_{V_\varrho}.$$ □

We fix some $0 \leqslant \sigma < \delta$. We take $\varepsilon > 0$ and $\alpha > 0$. We assume that for all non-trivial representation ϱ, the corresponding L-function $L_\Gamma(s,\varrho)$ does not vanish on the rectangle

$$\{\sigma \leqslant \mathrm{Re}(s) \leqslant 1 \text{ and } |\mathrm{Im}(s)| \leqslant (\log T)^{1+\alpha}\},$$

where $T = \beta \log(p)$ with $0 < \beta < 2$. The idea is to look at the average

$$S(p) := \sum_{\varrho \text{ irreducible}} |I(\varrho, T)|^2,$$

where $I(\varrho, T)$ is the sum given by

$$I(\varrho, T) = \sum_{\mathcal{C},k} \chi_\varrho(\mathcal{C}^k) \frac{l(\mathcal{C})}{1 - e^{kl(\mathcal{C})}} \varphi_0\left(\frac{kl(\mathcal{C})}{T}\right).$$

While each term $I(\varrho, T)$ is hard to estimate from below because of the oscillating behaviour of characters, the mean square is tractable thanks to Lemma 7.2. Let us compute $S(p)$.

$$S(p) = \sum_{\varrho \text{ irreducible}} \sum_{\mathcal{C},k} \sum_{\mathcal{C}',k'} \frac{l(\mathcal{C})l(\mathcal{C}')}{(1 - e^{kl(\mathcal{C})})(1 - e^{k'l(\mathcal{C}')})} \varphi_0\left(\frac{kl(\mathcal{C})}{T}\right) \times \varphi_0\left(\frac{k'l(\mathcal{C}')}{T}\right) \chi_\varrho(\mathcal{C}^k)\overline{\chi_\varrho(\mathcal{C}'^{k'})}.$$

Using Lemma 7.2, we have

$$\chi_\varrho(\mathcal{C}^k)\overline{\chi_\varrho(\mathcal{C}'^{k'})} = \frac{d_\varrho}{|G|} \sum_{g \in \mathbf{G}} \chi_\varrho(\mathcal{C}^k g (\mathcal{C}')^{-k'} g^{-1}),$$

and Fubini plus the identity

$$\sum_{\varrho \text{ irreducible}} d_\varrho \chi_\varrho(g) = |\mathbf{G}| \mathcal{D}_e(g)$$

allow us to obtain

$$S(p) = \sum_{\mathcal{C},k} \sum_{\mathcal{C}',k'} \frac{l(\mathcal{C})l(\mathcal{C}')}{(1 - e^{kl(\mathcal{C})})(1 - e^{k'l(\mathcal{C}')})} \varphi_0\left(\frac{kl(\mathcal{C})}{T}\right) \times \varphi_0\left(\frac{k'l(\mathcal{C}')}{T}\right) \Phi_{\mathbf{G}}(\mathcal{C}^k, \mathcal{C}'^{k'}),$$

where

$$\Phi_{\mathbf{G}}(\mathcal{C}^k, \mathcal{C}'^{k'}) := \sum_{g \in \mathbf{G}} \mathcal{D}_e(\mathcal{C}^k g (\mathcal{C}')^{-k'} g^{-1}).$$

Since all terms in this sum are now positive and $\mathrm{Supp}(\varphi_0) = [-1, +1]$, we can fix a small $\varepsilon > 0$ and find a constant $C_\varepsilon > 0$ such that

$$S(p) \geqslant C_\varepsilon \sum_{\substack{kl(\mathcal{C}) \leqslant T(1-\varepsilon) \\ k'l(\mathcal{C}') \leqslant T(1-\varepsilon)}} \Phi_{\mathbf{G}}(\mathcal{C}^k, \mathcal{C}'^{k'}).$$

Observe now that

$$\Phi_{\mathbf{G}}(\mathcal{C}^k, \mathcal{C}'^{k'}) = \sum_{g \in \mathbf{G}} \mathcal{D}_e(\mathcal{C}^k g (\mathcal{C}')^{-k'} g^{-1}) \neq 0$$

if and only if $\mathcal{C}^k$ and $\mathcal{C}'^{-k'}$ *are in the same conjugacy class mod* p, and in that case,

$$\Phi_{\mathbf{G}}(\mathcal{C}^k, \mathcal{C}'^{k'}) = |\mathcal{Z}(\mathcal{C}^k)| = |\mathcal{Z}(\mathcal{C}'^{k'})|.$$

Using the lower bound for the cardinality of centralizers, we end up with

$$S(p) \geqslant C_\varepsilon (p-1) \sum_{\substack{\overline{\mathcal{C}^k} = \overline{\mathcal{C}'^{k'}} \bmod p \\ kl(\mathcal{C}), k'l(\mathcal{C}') \leqslant T(1-\varepsilon)}} 1.$$

Notice that since we have taken $T = \beta \log(p)$ with $\beta < 2$, we can use Proposition 7.1 which says that $\mathcal{C}^k$ and $\mathcal{C}'^{-k'}$ are in the same conjugacy class mod p iff they have the same traces (in $\mathrm{SL}_2(\mathbb{Z})$). It is therefore natural to rewrite the lower bound for $S(p)$ in terms of traces. We need to introduce a bit more notations. Let $\mathscr{L}_\Gamma$ be set of traces i.e.

$$\mathscr{L}_\Gamma = \{\mathrm{tr}(\gamma) : \gamma \in \Gamma\} \subset \mathbb{Z}.$$

Given $t \in \mathscr{L}_\Gamma$, we denote by $m(t)$ the multiplicity of t in the trace set by

$$m(t) = \#\{\text{conj class } \overline{\gamma} \subset \Gamma : \mathrm{tr}(\gamma) = t\}.$$

We have therefore (notice that multiplicities are squared in the double sum)

$$S(p) \geqslant C_\varepsilon (p-1) \sum_{\substack{t \in \mathscr{L}_\Gamma \\ |t| \leqslant 2\cosh(T(1-\varepsilon)/2)}} m^2(t).$$

To estimate from below this sum, we use a trick that goes back to Selberg. By the prime orbit theorem [38, 49, 65] applied to the surface $\Gamma\backslash\mathbb{H}^2$, we know that for all T large, we have

$$Ce^{(\delta - 2\varepsilon)T} \leqslant \sum_{\substack{t \in \mathscr{L}_\Gamma \\ |t| \leqslant 2\cosh(T(1-\varepsilon)/2)}} m(t),$$

and by Schwarz inequality we get for T large

$$Ce^{(\delta-2\varepsilon)T} \leqslant C_0 \left(\sum_{\substack{t\in\mathscr{L}_\Gamma \\ |t|\leqslant 2\cosh(T(1-\varepsilon)/2)}} m^2(t) \right)^{1/2} e^{T/4},$$

where we have used the obvious bound

$$\#\{n \in \mathbb{Z} : |n| \leqslant 2\cosh(T(1-\varepsilon)/2)\} = O(e^{T/2}).$$

This yields the lower bound

$$\sum_{\substack{t\in\mathscr{L}_\Gamma \\ |t|\leqslant 2\cosh(T(1-\varepsilon)/2)}} m^2(t) \geqslant C'_\varepsilon e^{(2\delta-1/2-\varepsilon)T},$$

which shows that one can take advantage of exponential multiplicities in the length spectrum when $\delta > \frac{1}{2}$, thus beating the simple bound coming from the prime orbit theorem. In a nutshell, we have reached the lower bound (for all $\varepsilon > 0$),

$$S(p) \geqslant C_\varepsilon (p-1)e^{(2\delta-1/2-\varepsilon)T}.$$

Keeping that lower bound in mind, we now turn to upper bounds using Proposition 6.1. Writing

$$S(p) = |I(\mathrm{id}, T)|^2 + \sum_{\varrho\neq\mathrm{id}} |I(\varrho, T)|^2,$$

and using the bound (6.3) combined with the conclusion of Proposition 6.1, we get

$$S(p) = O(e^{(2\delta+\varepsilon)T}) + O\left(\sum_{\varrho\neq\mathrm{id}} d_\varrho^2 (\log(d_\varrho+1))^2 e^{2(\sigma+\varepsilon)T} \right).$$

Using the formula

$$|\mathbf{G}| = \sum_\varrho d_\varrho^2,$$

combined with the fact that $|\mathbf{G}| = p(p^2-1) = O(p^3)$, we end up with

$$S(p) = O(e^{(2\delta-\varepsilon)T}) + O\left(p^3 \log(p) e^{2(\sigma+\varepsilon)T} \right).$$

Since $T = \beta\log(p)$, we have obtained for all p large[4]

$$Cp^{(2\delta-1/2-\varepsilon)\beta} \leqslant p^{(2\delta+\varepsilon)\beta-1} + p^{2+2(\sigma+\varepsilon)\beta+\varepsilon}.$$

[4] Note that the $\log(p)$ term has been absorbed in p^ε.

Remark that since $\beta < 2$, then if ε is small enough we always have

$$(2\delta+\varepsilon)\beta - 1 < (2\delta - 1/2 - \varepsilon)\beta,$$

so up to a change of constant C, we actually have for all large p

$$Cp^{(2\delta-1/2-\varepsilon)\beta} \leqslant p^{2+2(\sigma+\varepsilon)\beta+\varepsilon}.$$

We have contradiction for p large provided

$$\sigma < \left(\delta - \frac{1}{4} - \frac{1}{\beta}\right) - \varepsilon - \frac{\varepsilon}{2\beta}.$$

Since β can be taken arbitrarily close to 2 and ε arbitrarily close to 0, we have a contradiction whenever $\delta > \frac{3}{4}$ and $\sigma < \delta - \frac{3}{4}$. Therefore for all p large, at least one of the L-function $L_\Gamma(s,\varrho)$ for non-trivial ϱ has to vanish inside the rectangle

$$\left\{\delta - \tfrac{3}{4} - \epsilon \leqslant \mathrm{Re}(s) \leqslant \delta \ \text{ and } \ |\mathrm{Im}(s)| \leqslant (\log(\log(p)))^{1+\alpha}\right\},$$

but then by the product formula we know that this zero appears as a zero of $Z_{\Gamma(p)}(s)$ with multiplicity d_ρ which is greater or equal to $\frac{p-1}{2}$ by Frobenius. The main theorem is proved. □

We end by a few comments. It would be interesting to know if the $\log^{1+\epsilon}(\log(p))$ bound can be improved to a uniform constant. However, it would likely require a completely different approach since $\log(\log(p))$ is the very limit one can achieve with compactly supported test functions. Indeed, to achieve a uniform bound with our approach would require the use of test functions $\varphi \not\equiv 0$ with Fourier bounds

$$|\widehat{\varphi}(\xi)| \leqslant C_1 e^{|\mathrm{Im}(\xi)|} e^{-C_2|\mathrm{Re}(\xi)|},$$

but an application of the Paley–Wiener theorem shows that these test functions do not exist (they would be both compactly supported and analytic on the real line).

8. Fell's continuity and Cayley graphs of abelian groups

In this section we prove Theorem 1.2. The arguments follow closely those of Gamburd in [27]. Roughly speaking, since Cayley graphs of finite abelian groups can never form a family of expanders, one should expect strongly that there is no uniform spectral gap in the family of covers $X_j = \Gamma_j \backslash \mathbb{H}^2$. We give a rigourous proof of that fact using Fell's continuity.

Let $\mathcal{G}$ be a finite graph with set of vertices $\mathcal{V}$ and of degree k. That is, for every vertex $x \in \mathcal{V}$ there are k edges adjacent to x. For a subset of vertices

$A \subset \mathcal{V}$ we define its boundary ∂A as the set of edges with one extremity in A and the other in $\mathcal{G} - A$. The Cheeger isoperimetric constant $h(\mathcal{G})$ is defined as

$$h(\mathcal{G}) = \min\left\{\frac{|\partial A|}{|A|} : A \subset \mathcal{V} \text{ and } 1 \leqslant |A| \leqslant \frac{|\mathcal{V}|}{2}\right\}.$$

Let $L^2(\mathcal{V})$ be the Hilbert space of complex-valued functions on $\mathcal{V}$ with inner product

$$\langle F, G\rangle_{L^2(\mathcal{V})} = \sum_{x\in\mathcal{V}} F(x)\overline{G(x)}.$$

Let Δ be the discrete Laplace operator acting on $L^2(\mathcal{V})$ by

$$\Delta F(x) = F(x) - \frac{1}{k}\sum_{y\sim x} F(y),$$

where $F \in L^2(\mathcal{V})$, $x \in \mathcal{V}$ is a vertex of $\mathcal{G}$, and $y \sim x$ means that y and x are connected by an edge. The operator Δ is self-adjoint and positive. Let $\lambda_1(\mathcal{G})$ denote the first non-zero eigenvalue of Δ.

The following result due to Alon and Milman [1] relates the spectral gap $\lambda_1(\mathcal{G})$ and Cheeger's isoperimetric constant.

Proposition 8.1. — *For finite graphs $\mathcal{G}$ of degree k we have*

$$\frac{1}{2}k \cdot \lambda_1(\mathcal{G}) \leqslant h(\mathcal{G}) \leqslant k\sqrt{\lambda_1(\mathcal{G})(2-\lambda_1(\mathcal{G}))}.$$

We note that large first non-zero eigenvalue $\lambda_1(\mathcal{G})$ implies fast convergence of random walks on $\mathcal{G}$, that is, high connectivity (see Lubotzky [42]).

Definition 8.2. — *A family of finite graphs $\{\mathcal{G}_j\}$ of bounded degree is called a family of expanders if there exists a constant $c > 0$ such that $h(\mathcal{G}_j) \geqslant c$.*

The family of graphs we are interested in is built as follows. Let $\Gamma = \langle S\rangle$ be a Fuchsian group generated by a finite set $S \subset \mathrm{PSL}_2(\mathbb{R})$. We will assume that S is symmetric, i.e. $S^{-1} = S$. Given a sequence Γ_j of finite index normal subgroups of Γ, let S_j be the image of S under the natural projection $r_{\mathbf{G}_j} : \Gamma \to \mathbf{G}_j = \Gamma/\Gamma_j$. Notice that S_j is a symmetric generating set for the group $\mathbf{G}_j$. Let $\mathcal{G}_j = \mathrm{Cay}(\mathbf{G}_j, S_j)$ denote the Cayley graph of $\mathbf{G}_j$ with respect to the generating set S_j. That is, the vertices of $\mathbf{G}_j$ are the elements of $\mathbf{G}_j$ and two vertices x and y are connected by an edge if and only if $xy^{-1} \in S_j$.

The connection of uniform spectral gap with the graphs constructed above comes from the following result.

Proposition 8.3. — *Assume that $\delta = \delta(\Gamma) > \frac{1}{2}$ and assume that there exists $\epsilon > 0$ such that for all j all non-trivial resonances s of $X_j = \Gamma_j \setminus \mathbb{H}^2$ satisfy $|s - \delta| > \epsilon$. Then the Cayley graphs $\mathcal{G}_j$ form a family of expanders.*

Let us see how Proposition 8.3 implies Theorem 1.2.

Proof of Theorem 1.2. — Since $X = \Gamma \setminus \mathbb{H}^2$ has at least one cusp by assumption, we have $\delta > \frac{1}{2}$ so that we can apply Proposition 8.3. Suppose by contradiction that there exists $\epsilon > 0$ such that for all j we have $|s-\delta| > \epsilon$ for all non-trivial resonances s of X_j. Then Proposition 8.3 implies that the Cayley graphs $\mathcal{G}_j = \mathrm{Cay}(\mathbf{G}_j, S_j)$ form a family of expanders. We will show that this is never true for the sequence of abelian groups $\mathbf{G}_j$ defined in Section 1.1, thus showing Theorem 1.2. Using the same notations as in Section 1, write

$$\mathbf{G}_j = \mathbb{Z}/N_1^{(j)}\mathbb{Z} \times \mathbb{Z}/N_2^{(j)}\mathbb{Z} \times \cdots \times \mathbb{Z}/N_k^{(j)}\mathbb{Z},$$

where $1 \leqslant k \leqslant r$ is fixed. The space $L^2(\mathbf{G}_j)$ is spanned by the characters χ_α given by

$$\chi_\alpha(x) = \exp\left(2\pi i \sum_{\ell=1}^{k} \frac{\alpha_\ell}{N_\ell^{(j)}} x_\ell\right)$$

where $x = (x_1, \dots, x_k)$ and $\alpha = (\alpha_1, \dots, \alpha_k)$ with $\alpha_\ell \in \{0, \dots, N_\ell^{(j)} - 1\}$. Note that the trivial character $\chi_\alpha \equiv 1$ corresponds to $\alpha = 0$. Applying the discrete Laplace operator Δ on $\mathcal{G}_j$ to χ_α yields

$$\begin{aligned}
\Delta\chi_\alpha(x) &= \chi_\alpha(x) - \frac{1}{|S_j|} \sum_{s \in S_j} \chi_\alpha(x+s) \\
&= \chi_\alpha(x) - \frac{1}{|S_j|} \sum_{s \in S_j} \exp\left(2\pi i \sum_{\ell=1}^{k} \frac{\alpha_\ell}{N_\ell^{(j)}} s_\ell\right) \chi_\alpha(x) \\
&= \chi_\alpha(x) - \frac{1}{|S_j|} \sum_{s \in S_j} \cos\left(2\pi i \sum_{\ell=1}^{k} \frac{\alpha_\ell}{N_\ell^{(j)}} s_\ell\right) \chi_\alpha(x) \\
&= \left(1 - \frac{1}{|S_j|} \sum_{s \in S_j} \cos\left(2\pi i \sum_{\ell=1}^{m} \frac{\alpha_\ell}{N_\ell^{(j)}} s_\ell\right)\right) \chi_\alpha(x),
\end{aligned}$$

where we exploited the symmetry of the set S_j in the third line. Thus every character χ_α is an eigenfunction of Δ with eigenvalue

$$\lambda_\alpha^{(j)} := \frac{1}{|S_j|} \sum_{s \in S_j} \left(1 - \cos\left(2\pi i \sum_{\ell=1}^{k} \frac{\alpha_\ell}{N_\ell^{(j)}} s_\ell\right)\right).$$

Note that we can view S_j as a subset of $\{0,\dots,N_1^{(j)}-1\}\times\cdots\times\{0,\dots,N_k^{(j)}-1\}\subset\mathbb{Z}^k$. Since S is a finite subset of $\mathrm{PSL}_2(\mathbb{R})$, there exists a constant $M>0$ independent of j such that $\max_{s\in S_j}\|s\|_\infty\leqslant M$, where $\|s\|_\infty=\max_{1\leqslant\ell\leqslant k}|s_\ell|$ is the supremum norm. Since we assume that

$$\lim_{j\to+\infty}\min_{\ell} N_\ell^{(j)}=+\infty,$$

we know that $N_1^{(j)}\to+\infty$. Set $\alpha=(1,0,\dots,0)$. Then we have

$$0\leqslant\eta^{(j)}:=\max_{s\in S_j}\sum_{\ell=1}^{k}\frac{\alpha_\ell}{N_\ell^{(j)}}s_\ell=\max_{s\in S_j}\frac{1}{N_1^{(j)}}s_1\leqslant\frac{M}{N_1^{(j)}}\to 0$$

as $j\to+\infty$. Using $1-\cos x\ll x^2$ for $|x|$ sufficiently small we obtain

$$\lambda_\alpha^{(j)}\ll(\eta^{(j)})^2\to 0$$

as $j\to+\infty$. We need to exclude the possibility that $\lambda_\alpha^{(j)}$ is zero. Note that $\mathcal{G}_j$ is a connected graph because S_j is a generating set for $\mathbf{G}_j$. Hence the zero eigenvalue of the discrete Laplacian is simple and therefore

$$\lambda_\alpha^{(j)}=0\Leftrightarrow\alpha=0.$$

In particular, for $\alpha=(1,0,\dots,0)$ we have $\lambda_\alpha^{(j)}>0$. We have thus shown that the spectral gap $\lambda_1(\mathcal{G}_j)$ of $\mathcal{G}_j$ tends to zero as $j\to+\infty$. By Proposition 8.1 this implies that the $\mathcal{G}_j$ do not form a family of expanders. The proof of Theorem 1.2 is therefore complete. □

8.1. Proof of Proposition 8.3

A very similar statement to that of Proposition 8.3 was given by Gamburd [27, Section 7]. The key ingredient in Gamburd's proof is Fell's continuity of induction and we will follow this line of thought.

For the remainder of this section set $G=\mathrm{SL}_2(\mathbb{R})$ and let $\hat{G}$ be its unitary dual, that is, the set of equivalence classes of (continuous) irreducible unitary representations of G. We endow the set $\hat{G}$ with the *Fell topology*. We refer the reader to [23] and [5, Chapter F] for more background on the Fell topology. A representation of G is called *spherical* if it has a non-zero K-invariant vector, where $K=\mathrm{SO}(2)$. Let us consider the subset $\hat{G}^1\subset\hat{G}$ of irreducible spherical unitary representations.

According to Lubotzky [43, Chapter 5], the set $\hat{G}^1$ can be parametrized as

$$\hat{G}^1=i\mathbb{R}^+\cup\left[0,\frac{1}{2}\right],$$

where $s \in i\mathbb{R}^+$ corresponds to the spherical unitary principal series representations, $s \in (0, \frac{1}{2})$ corresponds to the complementary series representation, and $s = \frac{1}{2}$ corresponds to the trivial representation. See also Gelfand–Graev–Pyatetskii-Shapiro [28, Chapter 1, §3] for a classification of the irreducible (spherical and non-spherical) unitary representations with a different parametrization. Moreover the Fell topology on $\hat{G}^1$ is the same as that induced by viewing the set of parameters s as a subset of $\mathbb{C}$, see [43, Chapter 5]. In particular, the spherical unitary principal series representations are bounded away from the identity.

Let us now recall the connection between the exceptional eigenvalues $\lambda \in (0, \frac{1}{4})$ and the complementary series representation. Consider the (left) quasiregular representation $(\lambda_{G/\Gamma}, L^2(G/\Gamma))$ of G defined by

$$\lambda_{G/\Gamma}(g)f(h\Gamma) = f(hg^{-1}\Gamma).$$

(We will denote this representation simply by $L^2(G/\Gamma)$.) Define the function $s(\lambda) = \sqrt{1/4 - \lambda}$ for $\lambda \in (0, \frac{1}{4})$. Then, $\lambda \in (0, \frac{1}{4})$ is an exceptional eigenvalue of $\Delta_{\Gamma\backslash\mathbb{H}}$ if and only if the complementary series $\pi_{s(\lambda)}$ occurs as a subrepresentation of $L^2(G/\Gamma)$. This is the so-called Duality Theorem [28, Chapter 1, §4].

Let us return to the proof of Proposition 8.3. Let Γ and Γ_j be as in Proposition 8.3. Let $\Omega(\Gamma)$ denote eigenvalues of the Laplacian Δ_X on $X = \Gamma \setminus \mathbb{H}$. Let $\lambda_0(\Gamma) = \delta(1-\delta) = \inf \Omega(\Gamma)$ denote the bottom of the spectrum. Since Γ_j is by assumption a finite-index subgroup of Γ, we have $\delta(\Gamma_j) = \delta$ and consequently

$$\lambda_0(\Gamma_j) = \lambda_0(\Gamma) =: \lambda_0$$

for all j. Let V_{s_0} be the invariant subspace corresponding to the representation π_{s_0} and let $L^2_0(G/\Gamma_j)$ be its orthogonal complement in $L^2(G/\Gamma_j)$. For each j we can decompose the quasiregular representation of G into direct sum of subrepresentations

$$L^2(G/\Gamma_j) = L^2_0(G/\Gamma_j) \oplus V_{s_0}.$$

Recall that λ_0 is a simple eigenvalue by the result of Patterson [56]. By the Duality Theorem it follows that V_{s_0} is one-dimensional. The following lemma provides us with a link between uniform spectral gap and representation theory.

Lemma 8.4. — *Let $\mathcal{R} \subset \hat{G}^1$ be the following set:*

$$\mathcal{R} = \bigcup_j \left\{ (\pi, \mathcal{H}) : \begin{array}{c} \pi \text{ is spherical irreducible} \\ \text{unitary subrep. of } L^2_0(G/\Gamma_j) \end{array} \right\} / \sim,$$

where $\sim$ denotes the equivalence of representations. Then the following are equivalent.

(1) *There exists $\varepsilon_0 > 0$ such that $|s-\delta| > \epsilon_0$ for all j and all non-trivial resonances s of X_j.*
(2) *The representation π_{s_0} is isolated in the set $\mathcal{R} \cup \{\pi_{s_0}\}$ with respect to the Fell topology.*

Proof. — Since the resonances s of $X_j = \Gamma_j \setminus \mathbb{H}$ with $\mathrm{Re}(s) > \frac{1}{2}$ correspond to the eigenvalues $\lambda = s(1-s) \in [\lambda_0, \frac{1}{4})$, the uniform spectral gap condition (1) can be stated as follows. There exists $\epsilon_1 > 0$ such that for all j we have

$$\Omega(\Gamma_j) \cap [0, \lambda_0 + \varepsilon_1) = \{\lambda_0\}. \tag{8.1}$$

Now we can reformulate (8.1) in representation-theoretic language. Set $s_0 = s(\lambda_0)$. Then by the Duality Theorem, there exists $\epsilon > 0$ such that for all j and all $s \in (s_0 - \varepsilon, \frac{1}{2}]$, the complementary series representation π_s does not occur as a subrepresentation of $L^2(G/\Gamma_j)$. Since V_{s_0} is one-dimensional (and each representation π_s with $s \neq \frac{1}{2}$ is infinite-dimensional), (1) is equivalent to

$$\mathcal{R} \cap \left(s_0 - \varepsilon, \frac{1}{2}\right] = \{s_0\}. \tag{8.2}$$

Since the Fell topology on $\hat{G}^1$ is equivalent to the one induced by viewing $\hat{G}^1$ as the subset $i\mathbb{R}^+ \cup \left[0, \frac{1}{2}\right]$ of the complex plane, the equivalence of (1) and (2) is now evident. $\square$

Let 1_{Γ_j} denote the trivial representation of Γ_j on $\mathbb{C}$. Then the induced representation $\mathrm{Ind}_{\Gamma_j}^{\Gamma} 1_{\Gamma_j}$ is equivalent to the (left) quasiregular representation $(\lambda_{\mathbf{G}_j}, L^2(\mathbf{G}_j))$ of Γ defined by

$$(\lambda_{\mathbf{G}_j}(\gamma)F)(h\Gamma_j) = (\gamma.F)(h\Gamma_j) = F(h\gamma^{-1}\Gamma_j).$$

The action of Γ on $L^2(\mathbf{G}_j)$ given by $\gamma.F = \lambda_{\mathbf{G}_j}(\gamma)F$ is transitive. Hence the only Γ-fixed vectors are the constants. Thus we can decompose the representation of Γ on $L^2(\mathbf{G}_j)$ into a direct of subrepresentations

$$L^2(\mathbf{G}_j) = L^2_0(\mathbf{G}_j) \oplus \mathbb{C},$$

where $L^2_0(\mathbf{G}_j)$ is the subspace of functions orthogonal to the constant function, and $(1_\Gamma, \mathbb{C})$ does not occur as a subrepresentation of $L^2_0(\mathbf{G}_j)$.

Consider the following subset of $\hat{\Gamma}$:

$$\mathcal{T} = \bigcup_{j\in\mathbb{N}} \{(\rho, V) : \rho \text{ is irreducible unitary subrepresentation of } L^2_0(\mathbf{G}_j)\}/\sim,$$

We claim the following.

Lemma 8.5. — *Assume that one of the equivalent statements in Lemma 8.4 holds true. Then the trivial representation 1_Γ is isolated in $\mathcal{T}\cup\{1_\Gamma\}$ with respect to the Fell topology.*

Proof. — Let K be a closed subgroup of a locally compact group H. Given a unitary representation (π, V) of K, the induced representation $\mathrm{Ind}_K^H \pi$ of H is defined as follows. Let μ be a quasi-invariant regular Borel measure on H/K and set

$$\mathrm{Ind}_K^H \pi := \left\{ f : H \to V : \begin{array}{c} f(hk) = \pi(k^{-1}) f(h) \\ \text{for all } k \in K \text{ and } f \in L^2_\mu(H/K) \end{array} \right\}. \tag{8.3}$$

Note that the requirement $f \in L^2_\mu(H/K)$ makes sense, since the norm of $f(g)$ is constant on each left coset of H. The action of G on $\mathrm{Ind}_H^G \pi$ is defined by

$$g.f(x) = f(g^{-1}x)$$

for all $x, g \in G$, $f \in \mathrm{Ind}_H^G \pi$. We also note that the equivalence class of the induced representation $\mathrm{Ind}_K^H \pi$ is independent of the choice of μ. We refer the reader to [5, Chapter E] for a more thorough discussion on properties of induced representations.

If two representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ are equivalent, we write $\mathcal{H}_1 = \mathcal{H}_2$ by abuse of notation. Using induction by stages (see [24] or [26] for a proof) we have

$$\begin{aligned} V_{s_0} \oplus L^2_0(G/\Gamma_j) &= L(G/\Gamma_j) \\ &= \mathrm{Ind}_{\Gamma_j}^G 1_{\Gamma_j} \\ &= \mathrm{Ind}_\Gamma^G \mathrm{Ind}_{\Gamma_j}^\Gamma 1_{\Gamma_j} \\ &= \mathrm{Ind}_\Gamma^G L^2(\mathbf{G}_j) \\ &= \mathrm{Ind}_\Gamma^G 1_\Gamma \oplus \mathrm{Ind}_\Gamma^G L^2_0(\mathbf{G}_j) \\ &= V_{s_0} \oplus L^2_0(G/\Gamma) \oplus \mathrm{Ind}_\Gamma^G L^2_0(\mathbf{G}_j). \end{aligned}$$

Choose an index j and an irreducible unitary subrepresentation (τ, V) of $L^2_0(\mathbf{G}_j)$. The above calculation implies that $\mathrm{Ind}_\Gamma^G \tau$ is a unitary subrepresentation of $L^2_0(G/\Gamma_j)$. Since τ is unitary and irreducible, so is $\mathrm{Ind}_\Gamma^G \tau$. Moreover $\mathrm{Ind}_\Gamma^G \tau$ is a spherical representation of G, since any non-zero function $f \in L^2(\mathbb{H}/\Gamma)$ and non-zero vector $v \in V$ gives rise to a non-zero K-invariant function $F \in \mathrm{Ind}_\Gamma^G \tau$. Indeed, we have $\mathbb{H} \cong K \setminus G$, so that we my view f as function $f : G \to \mathbb{C}$ satisfying $f(kg\gamma) = f(g)$ for all

$g \in G, k \in K, \gamma \in \Gamma$. Now one easily verifies that $F = fv : G \to V$ belongs to $\operatorname{Ind}_\Gamma^G \tau$ and is invariant under K. In other words, $\operatorname{Ind}_\Gamma^G \tau$ belongs to $\mathcal{R}$.

Now suppose the lemma is false. Then there exists a sequence $(\tau_n)_{n\in\mathbb{N}} \subset \mathcal{T}$ that converges to 1_Γ as $n \to \infty$. On the other hand, π_{s_0} is weakly contained in $\operatorname{Ind}_\Gamma^G 1_\Gamma$. By Fell's continuity of induction [23] we have

$$\pi_{s_0} \prec \operatorname{Ind}_\Gamma^G 1_\Gamma = \lim_{n\to\infty} \operatorname{Ind}_\Gamma^G \tau_n \in \overline{\mathcal{R}},$$

which contradicts Lemma 8.4. ☐

We can now prove Proposition 8.3.

Proof of Proposition 8.3. — Let us recall the definition of the Fell topology on $\hat{\Gamma}$ (for further reading consult [5, Chapter F]). For an irreducible unitary representation (π, V) of Γ, for a unit vector $\xi \in V$, for a finite set $Q \subset \Gamma$, and for $\varepsilon > 0$ let us define the set $W(\pi, \xi, Q, \varepsilon)$ that consists of all irreducible unitary representations (π', V') of Γ with the following property. There exists a unit vector $\xi' \in V'$ such that

$$\sup_{\gamma\in Q} |\langle \pi(\gamma)\xi, \xi\rangle_V - \langle \pi'(\gamma)\xi', \xi'\rangle_{V'}| < \varepsilon.$$

The Fell topology is generated by the sets $W(\pi, \xi, Q, \varepsilon)$. By Lemma 8.5 and the definition of the Fell topology, there exists $c_0 = c_0(\Gamma, S) > 0$ only depending on Γ and the generating set S of Γ, but not on j, such that for all $F \in L_0^2(\mathbf{G}_j)$

$$\sup_{\gamma\in S} |\langle \gamma.F - F, F\rangle_{L^2(\mathbf{G}_j)}| \geqslant c_0 \|F\|^2. \tag{8.4}$$

By the Cauchy–Schwarz inequality we have

$$\sup_{\gamma\in S} \|\gamma.F - F\| \geqslant c_0 \|F\|.$$

Fix a non-empty subset A of $\mathbf{G}_j$ with $|A| \leqslant \frac{1}{2}|\mathbf{G}_j|$ and define the function

$$F(x) = \begin{cases} |\mathbf{G}_j| - |A| & \text{if } x \in A \\ -|A| & \text{if } x \notin A. \end{cases}$$

One can verify that $F \in L_0^2(\mathbf{G}_j)$ and $\|F\|^2 = |A||\mathbf{G}_j|(|\mathbf{G}_j| - |A|)$. On the other hand,

$$\|\gamma.F - F\|^2 = |\mathbf{G}_j|^2 E_\gamma(A, \mathbf{G}_j \setminus A),$$

where

$$E_\gamma(A, B) := |\{x \in \mathbf{G}_j : x \in A \text{ and } x\gamma \in B \text{ or } x \in B \text{ and } x\gamma \in A\}|.$$

Therefore there exists $\gamma \in S$ such that

$$E_\gamma(A, \mathbf{G}_j \setminus A) = \frac{\|\gamma.F - F\|^2}{|\mathbf{G}_j|^2} \geqslant \frac{c_0^2\|F\|^2}{|\mathbf{G}_j|^2} = c_0^2\left(1 - \frac{|A|}{|\mathbf{G}_j|}\right)|A|.$$

Thus we obtain a lower bound for the size of the boundary of A in the graph $\mathcal{G}_j = \mathrm{Cay}(\mathbf{G}_j, S_j)$:

$$|\partial A| \geqslant \frac{1}{2}\sup_{\gamma\in S} E_\gamma(A, \mathbf{G}_j \setminus A) \geqslant \frac{c_0^2}{2}\left(1 - \frac{|A|}{|\mathbf{G}_j|}\right)|A| \geqslant \frac{c_0^2}{4}|A|.$$

Consequently, $h(\mathcal{G}_j) \geqslant c_0^2/4$ for all j and thus, the graphs $\mathcal{G}_j$ form a family of expanders. The proof of Proposition 8.3 is complete. □

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Dmitry JAKOBSON
McGill University
Department of Mathematics and Statistics
805 Sherbrooke Street West
Montreal, Quebec, H3A0B9 (Canada)
jakobson@math.mcgill.ca

Frédéric NAUD
Laboratoire de Mathématiques d’Avignon
Avignon Université, Campus Jean-Henri Fabre, 301
rue Baruch de Spinoza
84916 Avignon Cedex 9 (France)
frederic.naud@univ-avignon.fr

Louis SOARES
Friedrich-Schiller-Universität Jena
Institut für Mathematik
Ernst-Abbe-Platz 2, 07743 Jena (Germany)
louis.soares@uni-jena.de