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HOMOTOPY THEORY OF HOMOTOPY ALGEBRAS

by Bruno VALLETTE (*)

In memoriam JLL

ABSTRACT. — This paper studies the homotopy theory of algebras and homotopy algebras over an operad. It provides an exhaustive description of their higher homotopical properties using the more general notion of morphism called infinity-morphism. The method consists in using the operadic calculus to endow the category of coalgebras over the Koszul dual cooperad or the bar construction with a new type of model category structure, Quillen equivalent to that of algebras. We provide an explicit homotopy equivalence for infinity-morphisms, which gives a simple description of the homotopy category, and we endow the category of homotopy algebras with an infinity-category structure.

RÉSUMÉ. — Cet article porte sur la théorie homotopique des algèbres et des algèbres à homotopie près sur une opérade. Il fournit une description exhaustive de leurs propriétés homotopiques supérieures en utilisant la notion générale de morphisme appelé infini-morphisme. La méthode consiste à utiliser le calcul opéradique pour munir la catégorie des cogèbres sur la coopérade duale de Koszul ou sur la construction bar d'un nouveau type de structure de modèles, équivalente au sens de Quillen de celle des algèbres. Nous introduisons une notion d'équivalence homotopique explicite pour les infinis-morphismes, qui induit une description simple de la catégorie homotopique, et nous munissons la catégorie des algèbres à homotopie près d'une structure d'infinie-catégorie.

Introduction

To define derived functors in non-necessarily additive setting, D. Quillen generalized the ideas of A. Grothendieck [16] and introduced the notion of model category [27]. A derived functor, being defined up to quasi-isomorphisms, finds its source in the homotopy category, which is the original category localized with respect to quasi-isomorphisms. (This process is the

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categorical analogue of the construction of the field of rational numbers, where one starts from the ring of integers and formally introduce inverses for the non-zero numbers). For instance, Quillen homology theory for algebras of "any" type is defined by deriving the functor of indecomposables, see [24, Chapter 12].

So it becomes crucial to be able to describe the homotopy category of algebras, and more generally the homotopy theory of algebras. Using the free algebra functor, Quillen explained how to transfer the cofibrantly generated model category of chain complexes to the category of differential graded algebras. His main theorem, then asserts that the homotopy category is equivalent to the full subcategory of fibrant-cofibrant objets, with morphisms up to a certain homotopy equivalence relation. In this model category structure [18], all the algebras are fibrant but the cofibrant objects are not so easily described: they are actually the retracts of quasi-free algebras on generators endowed with a suitable filtration.

In his seminal paper on Rational Homotopy Theory [28], Quillen proved that several algebraic and topological homotopy categories are equivalent (differential graded Lie algebras, differential graded cocommutative coalgebras, topological spaces, simplicial spaces, etc.). For instance, one can find there a way to describe the homotopy category of differential graded Lie algebras as the homotopy category of differential graded cocommutative coalgebras. However, one *problem* and one *question* arise there. The aforementioned Quillen equivalences hold only under a strong connectivity assumption; and why does the category of Lie algebras admits the "dual" category of cocommutative coalgebras?

The *problem* was solved by Hinich [19], see also Lefevre-Hasegawa [23], who showed how to bypass the connectivity assumption by considering on cocommutative coalgebras a new class of weak equivalences, which is strictly included in the class of quasi-isomorphisms. The underlying idea is quite natural: the cobar functor going from differential graded cocommutative coalgebras to differential graded Lie algebras does not preserve quasi-isomorphisms. So if one wants this functor to form a Quillen equivalence, one has to find a class of morphisms of coalgebras which are sent to quasi-isomorphisms of Lie algebras. This forces the definition of weak equivalences of coalgebras.

The *question* is answered by the Koszul duality for operads [14, 15]. One encodes the category of Lie algebras with an operad and its Koszul dual cooperad is the one which encodes the category of cocommutative coalgebras.

In the present paper, we settle the homotopy theory of algebras over an operad as follows. First, we consider the category of coalgebras over the Koszul dual cooperad, when the original operad is Koszul, or over the bar construction of the operad, in general. Then, we endow it with a new type of model category structure, Quillen equivalent to the one on algebras and where the class of weak equivalences is strictly included in the class of quasi-isomorphisms. In this model category, all the coalgebras are cofibrant and the fibrant ones are the quasi-free ones, that is the ones for which the underlying coalgebra, forgetting the differential, is cofree. This already provides us with a simpler subcategory of fibrant-cofibrant objects.

Theorem 2.2. — *The category of conilpotent dg $\mathcal{P}^{¡}$-coalgebras admits a model category structure, Quillen equivalent to that of dg $\mathcal{P}$-algebras, in which every object is cofibrant and in which fibrant objects are the quasi-free ones.*

The method consists in extending Lefevre-Hasegawa's strategy from associative algebras to any algebra over an operad $\mathcal{P}$, using heavily the operadic calculus developed in [24, Chapters 10–11]. (Notice that we try to provide complete proofs for all the results, like the form of fibrant objects, for instance, which was not given on the level of associative algebras).

Notice how the present theory parallels that of Joyal–Lurie of ∞-categories. To get a suitable notion of higher category, one can endow the category of simplicial sets with a new model category made up of less weak equivalences than the Quillen–Kan classical ones. In this case, all the objects are cofibrant and the fibrant ones provide us with the notion of quasi-category, which is one model of ∞-category. In our case, we consider a new model category on $\mathcal{P}^{¡}$-coalgebras with less weak equivalences. All the objects are cofibrant and the fibrant ones give us the suitable notion of $\mathcal{P}_\infty$-algebra together with a good notion of morphisms that we call ∞-morphisms.

	Classical notion	Homotopy generalization	Context
Joyal–Lurie	Category	∞-category	Simplicial sets
Present paper	$\mathcal{P}$-algebras	$\mathcal{P}_\infty$-algebras with ∞-morphisms	$\mathcal{P}^{¡}$-coalgebras

So this new model category structure induces a new homotopy theory for homotopy algebras with their ∞-morphisms. To complete the picture, we solve the problem of a functorial cylinder object inducing a universal homotopy relation for ∞-morphisms. (Notice that this question is not trivial, for instance, the author was not able to fix a crucial gap in [3] related to this issue.) The cylinder object that we present here is proved to be equivalent to "all" the equivalence relations that have been considered so far in the literature. With it, we prove that any ∞-quasi-isomorphism admits a homotopy inverse, a property which does not hold for strict quasi-isomorphisms. Moreover, the initial category of algebras sits inside the category of homotopy algebras with their ∞-morphisms. And it can be proved that the second one retracts onto the first one. We are done: the homotopy category of algebras is equivalent the following simple one.

Theorem 3.11. — *The following categories are equivalent*

$$\mathsf{Ho}(\mathsf{dg}\ \mathcal{P}\text{-}\mathsf{alg}) \simeq \infty\text{-}\mathcal{P}\text{-}\mathsf{alg}/\sim_h .$$

This gives a simple description of the first homotopical level of information about algebras over an operad. However, using the simplicial localization methods of Dwyer–Kan [10], one can use the full power of this new model category structure on coalgebras to endow the category of homotopy algebras together with ∞-morphism with an ∞-category structure, thereby encoding all their higher homotopical information.

Theorem 3.12. — *The category $\infty\text{-}\mathcal{P}_\infty\text{-}\mathsf{alg}$ of $\mathcal{P}_\infty$-algebra with ∞-morphisms extends to a simplicial category giving the same underlying homotopy category.*

Another direct corollary of the model category structure on coalgebras endows the category of homotopy algebras with a fibrant objects category structure [2]. We reinforce this statement by proving that it actually carries a model category structure, except for the existence of some limits and colimits. In this case, the description of the three classes of structure maps is simple: weak equivalences (respectively cofibrations, respectively fibrations) are given by ∞-quasi-isomorphisms (respectively ∞-monomorphisms, respectively ∞-epimorphisms). For instance, this provides a neat description of fibrations between quasi-free coalgebras.

There is already an extensive literature about model category structures on coalgebras over cooperads, see [1, 13, 17, 19, 23, 27, 28, 30] for instance. The present result is more general for the following three reasons. First, no assumption is needed here like bounded below chain complexes or finite dimensional space. Second, it treats the general case of any operad. Third, in most of the cases, the model category structures on coalgebras considers quasi-isomorphisms for weak equivalences. We show that such model category structures can be obtained from the present one by means of Bousfield localization.

In the end, applied to the three graces, the present general treatise recovers many of the results of Hinich [19] for cocommutative algebras (operad $\mathcal{P} = \text{Lie}$), of Lefevre-Hasegawa [23] for coassociative algebras (operad $\mathcal{P} = \text{Ass}$), and of Lazarev–Markl [22] for inverse limits of finite-dimensional nilpotent Lie algebras (operad $\mathcal{P} = \text{Com}$). This latter case is obtained by considering the linear duals of conilpotent Lie coalgebras. (Notice that this latter article was written independently during the preparation of the present one.)

Last but not least, let us mention the large range of applications of the present general homotopy theory. They include all the examples of algebraic structures treated in the compendium of Chapter 13 of [24]. Therefore this applies to all the fields where homotopy algebras and ∞-morphisms play a role. So far, applications have been found, at least, in the following fields.

∞-*morphisms of mixed complexes (D_∞-morphisms):*
: cyclic homology, spectral sequences.

∞-*morphisms of assocative algebras (A_∞-morphisms):*
: algebraic topology (loop spaces, Massey products), symplectic topology (Fukaya categories), probability theory (free probability).

∞-*morphisms of Lie algebras (L_∞-morphisms):*
: deformation theory, differential geometry (deformation quantization of Poisson manifolds), rational homotopy theory.

∞-*morphisms of commutative algebras (C_∞-morphisms):*
: rational homotopy theory (Kähler manifolds).

∞-*morphisms of Gerstenhaber algebras (G_∞-morphisms):*
: Deligne conjecture, Drinfeld associators.

∞-*morphisms of Batalin–Vilkovisky algebras (BV_∞-morphisms):*
: Quantum cohomology (Frobenius manifolds), mirror symmetry.

In most of these cases, the homotopy theory of ∞-morphisms was possible to achieve "by hands" because the associated operad is rather small. This is not necessarily the case in the new appearing algebraic structures, like the homotopy Batalin–Vilkovisky algebras for instance. This point was the starting motivation for the development of the present general theory.

Layout

The paper is organized as follows. We begin with some recollections on operadic homological algebra and on the model category for algebras. In the second section, we endow the category of coalgebras over a Koszul dual cooperad with a new model category structure. The induced homotopy theory for ∞-morphisms is developed in Section 3. The last section proves that there is almost a model category structure on homotopy algebras. Appendix A deals with the obstruction theory of ∞-morphisms and Appendix B contains the proof of a technical lemma.

Prerequisites

The reader is supposed to be familiar with the notion of an operad and operadic homological algebra, for which we refer to the book [24]. In the present paper, we use the same notations as in loc. cit..

Framework

Throughout this paper, we work over a field $\mathbb{K}$ of characteristic 0. Every chain complex is $\mathbb{Z}$-graded with homological degree convention, i.e. with degree -1 differential. All the $\mathbb{S}$-modules $M = \{M(n)\}_{n\in\mathbb{N}}$ are reduced, that is $M(0) = 0$.

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1. Recollections

In this section, we recall the main results about algebras over an operad [24, Chapters 10–11], the (inhomogeneous) Koszul duality theory of operads [11, Appendix A] and a (cofibrantly generated) model category structure for algebras [18].

1.1. Operad and cooperad

Recall that a *differential graded operad* (respectively a *differential graded cooperad*) is a monoid (resp. a comonoid) in the monoidal category $(\mathsf{dg}\ \mathbb{S}\text{-}\mathsf{Mod}, \circ, \mathrm{I})$ of differential graded $\mathbb{S}$-modules. A dg operad $\mathcal{P}$ (resp. a dg cooperad $\mathcal{C}$) is called *augmented* (resp. *coaugmented*) when it is equipped with an augmentation morphism $\epsilon : \mathcal{P} \to \mathrm{I}$ of dg operads (resp. with a coaugmentation morphism $\eta : \mathrm{I} \to \mathcal{C}$ of dg cooperads).

1.2. Algebra and coalgebra

Let $\mathcal{P}$ be a dg operad. A *dg $\mathcal{P}$-algebra* (A, γ_A) is a left $\mathcal{P}$-module concentrated in arity 0:

$$\gamma_A : \mathcal{P}(A) := \mathcal{P} \circ A = \bigoplus_{n\geqslant 1} \mathcal{P}(n) \otimes_{\mathbb{S}_n} A^{\otimes n} \to A.$$

A morphism of dg $\mathcal{P}$-algebras is a morphism of dg modules $f : A \to B$ which commutes with the structure maps, i.e. $f\gamma_A = \gamma_B \mathcal{P}(f)$. This category is denoted by $\mathsf{dg}\ \mathcal{P}\text{-}\mathsf{alg}$.

Dually, let $\mathcal{C}$ be a dg cooperad. A *conilpotent dg $\mathcal{C}$-coalgebra* (C, Δ_C) is a left $\mathcal{C}$-comodule concentrated in arity 0:

$$\Delta_C : C \to \mathcal{C}(C) := \mathcal{C} \circ C = \bigoplus_{n\geqslant 1} \mathcal{C}(n) \otimes_{\mathbb{S}_n} C^{\otimes n}.$$

This category is denoted by $\mathsf{conil}\ \mathsf{dg}\ \mathcal{C}\text{-}\mathsf{coalg}$.

In general, one defines the notion of a $\mathcal{C}$-coalgebra with the product and invariant elements: $\Delta_C(C) \subset \prod_{n\geqslant 1} (\mathcal{C}(n) \otimes C^{\otimes n})^{\mathbb{S}_n}$. Since we work over a field of characteristic 0, we can identify invariant and coinvariant elements. We restrict here to the case where the image of the coproduct lies in the sum. We refer to [24, Chapter 5] for more details.

A dg $\mathcal{P}$-algebra A (resp. a dg $\mathcal{C}$-coalgebra C) is called *quasi-free* when its underlying graded module, i.e. after forgetting the differential map, is isomorphic to a free $\mathcal{P}$-algebra: $A \cong \mathcal{P}(V)$ (resp. to a cofree $\mathcal{C}$-coalgebra: $C \cong \mathcal{C}(V)$).

1.3. Operadic homological algebra

Let $\mathcal{P}$ be a dg operad and let $\mathcal{C}$ be a dg cooperad. The graded vector space of $\mathbb{S}$-equivariant maps $\mathrm{Hom}_{\mathbb{S}}(\mathcal{C}, \mathcal{P}) := \prod_{n\geqslant 1} \mathrm{Hom}_{\mathbb{S}_n}(\mathcal{C}(n), \mathcal{P}(n))$ from

$\mathcal{C}$ to $\mathcal{P}$ carries a natural dg Lie algebra structure, called the *convolution Lie algebra*. An *operadic twisting morphism* $\alpha \in \mathrm{Tw}(\mathcal{C}, \mathcal{P})$ is an element $\alpha : \mathcal{C} \to \mathcal{P}$ of the convolution Lie algebra which satisfies the Maurer–Cartan equation:

$$\partial\alpha + \frac{1}{2}[\alpha, \alpha] = 0.$$

This operadic twisting morphism bifunctor is represented by the bar construction on the right-hand side and by the cobar construction on the left-hand side

$$\mathrm{Hom}_{\mathsf{dg\ Op}}\left(\Omega\mathcal{C},\, \mathcal{P}\right) \cong \mathrm{Tw}(\mathcal{C},\, \mathcal{P}) \cong \mathrm{Hom}_{\mathsf{conil\ dg\ coOp}}\left(\mathcal{C},\, \mathrm{B}\,\mathcal{P}\right).$$

So these latter ones form a pair of adjoint functors.

$$\mathrm{B}\ :\ \textsf{augmented dg operads}\ \rightleftharpoons\ \textsf{conilpotent dg cooperads}\ :\ \Omega.$$

Any twisting morphism $\alpha \in \mathrm{Tw}(\mathcal{C}, \mathcal{P})$ gives rise to twisted differentials d_α on the composite products $\mathcal{C}\circ\mathcal{P}$ and $\mathcal{P}\circ\mathcal{C}$. The resulting chain complexes are called *left* and *right twisting composite product* and are denoted $\mathcal{C} \circ_\alpha \mathcal{P}$ and $\mathcal{P} \circ_\alpha \mathcal{C}$ respectively, see [24, Chapter 6].

1.4. Bar and cobar constructions on the algebra level

Let $\alpha \in \mathrm{Tw}(\mathcal{C}, \mathcal{P})$ be an operadic twisting morphism. Let (A, γ_A) be a dg $\mathcal{P}$-algebra and let (C, Δ_C) be a dg $\mathcal{C}$-coalgebra. We consider the following unary operator $\star_\alpha$ of degree -1 on $\mathrm{Hom}(C, A)$:

$$\star_\alpha(\varphi)\ :\ C \xrightarrow{\Delta_C} \mathcal{C} \circ C \xrightarrow{\alpha\circ\varphi} \mathcal{P} \circ A \xrightarrow{\gamma_A} A,\ \text{ for } \varphi \in \mathrm{Hom}(C, A).$$

A *twisting morphism with respect to* α is a linear map $\varphi : C \to A$ of degree 0 which is a solution to the *Maurer-Cartan* equation

$$\partial(\varphi) + \star_\alpha(\varphi) = 0.$$

The space of twisting morphisms with respect to α is denoted by $\mathrm{Tw}_\alpha(C, A)$.

This twisting morphism bifunctor is represented by the bar construction on the right-hand side and by the cobar construction on the left-hand side

$$\mathrm{B}_\alpha\ :\ \textsf{dg}\ \mathcal{P}\textsf{-alg}\ \rightleftharpoons\ \textsf{conil dg}\ \mathcal{C}\textsf{-coalg}\ :\ \Omega_\alpha.$$

So they form a pair of adjoint functors. The underlying spaces are given by a cofree $\mathcal{C}$-coalgebra, $\mathrm{B}_\alpha\, A = \mathcal{C}(A)$, and by a free $\mathcal{P}$-algebra, $\Omega_\alpha C = \mathcal{P}(C)$, respectively. For more details, see [24, Chapter 11].

1.5. Koszul duality theory

Let (E, R) be a quadratic-linear data, that is $R \subset E \oplus \mathcal{T}(E)^{(2)}$. It gives rise to a quadratic operad $\mathcal{P} := \mathcal{P}(E, R) = \mathcal{T}(E)/(R)$, where $\mathcal{T}(E)$ stands for the free operad on E and where (R) stands for the ideal generated by R.

Let $q : \mathcal{T}(E) \twoheadrightarrow \mathcal{T}(E)^{(2)}$ be the projection onto the quadratic part of the free operad. The image of R under q, denoted qR, is homogeneous quadratic. So the associated quotient operad $q\mathcal{P} := \mathcal{P}(E, qR)$ is homogeneous quadratic. We consider the homogeneous quadratic cooperad $q\mathcal{P}^{\text{i}} := \mathcal{C}(sE, s^2 qR)$, where s denotes the suspension map. We assume that the space of relations R satisfies the condition

$$(ql_1) \ : \ R \cap E = \{0\},$$

which means that the space of generators is minimal. Under this assumption, there exists a map $\varphi : qR \to E$ such that R is the graph of φ. If R satisfies the condition

$$(ql_2) \ : \ \{R \circ_{(1)} E + E \circ_{(1)} R\} \cap \mathcal{T}(E)^{(2)} \subset R \cap \mathcal{T}(E)^{(2)},$$

which amounts to the maximality of the space of relations R, then the map φ induces a square-zero coderivation d_φ on the cooperad $q\mathcal{P}^{\text{i}}$. For more details, we refer the reader to [11, Appendix A] and to [24, Chapter 7]. From now on, we will always assume the two conditions (ql_1) and (ql_2).

The dg cooperad $\mathcal{P}^{\text{i}} := (q\mathcal{P}^{\text{i}}, d_\varphi)$ is called the *Koszul dual cooperad* of $\mathcal{P}$. Notice that when the data (E, R) is homogeneous quadratic, $qR = R$ and the differential map d_φ vanishes. In this case, one recovers the homogeneous Koszul duality theory of Ginzburg–Kapranov and Getzler–Jones [14, 15].

There is a canonical operadic twisting morphism $\kappa \in \mathrm{Tw}(\mathcal{P}^{\text{i}}, \mathcal{P})$ defined by the following composite

$$\kappa \ : \ \mathcal{P}^{\text{i}} = \mathcal{C}(sE, s^2 qR) \twoheadrightarrow sE \xrightarrow{s^{-1}} E \rightarrowtail \mathcal{P}.$$

The associated twisted composite product $\mathcal{P} \circ_\kappa \mathcal{P}^{\text{i}}$ (resp. $\mathcal{P}^{\text{i}} \circ_\kappa \mathcal{P}$) is called the *Koszul complex*.

Definition 1.1.

(1) *A homogeneous quadratic operad is called a* homogeneous Koszul operad *when its Koszul complex is acyclic.*
(2) *An quadratic-linear operad $\mathcal{P}$ is called a* Koszul operad *when its presentation $\mathcal{P}(E, R)$ satisfies conditions (ql_1) and (ql_2) and when the associated homogeneous quadratic operad $q\mathcal{P}$ is homogeneous Koszul.*

When an inhomogeneous operad $\mathcal{P}$ is Koszul, then its Koszul complexes $\mathcal{P} \circ_\kappa \mathcal{P}^{\text{!`}}$ and $\mathcal{P}^{\text{!`}} \circ_\kappa \mathcal{P}$ are acyclic.

1.6. The general case

For simplicity and to avoid discrepancy, the rest of the paper is written in the case of a Koszul presentation of the operad $\mathcal{P}$ using coalgebras over the Koszul dual cooperad $\mathcal{P}^{\text{!`}}$. In general, one can always consider the trivial presentation made up of all the elements of $\mathcal{P}$ as generators. This presentation is quadratic-linear and always Koszul. In this case, the Koszul dual cooperad is nothing but the bar construction $\mathrm{B}\,\mathcal{P}$ of the operad $\mathcal{P}$. So all the results of the present paper are always true if one considers this presentation and $\mathrm{B}\,\mathcal{P}$-coalgebras.

1.7. Weight grading

Throughout the present paper, we will require an extra grading, other than the homological degree, to make all the proofs work. We work over the ground category of *weight graded* dg $\mathbb{S}$-modules. This means that every dg $\mathbb{S}$-module is a direct sum of sub-dg $\mathbb{S}$-modules indexed by this weight $M_d = \bigoplus_{\omega \in \mathbb{N}} M_d^{(\omega)}$, where d stands for the homological degree and where ω stands for the weight grading. In this context, the free operad is a weight graded dg operad, where the weight is given by the number of generators, or equivalently by the number of vertices under the tree representation. This induces a filtration on any quadratic operad $\mathcal{P} = \mathcal{T}(E)/(R)$, where

$$F_0\mathcal{P} = \mathrm{I},\ F_1\mathcal{P} = \mathrm{I} \oplus E,\ \text{and}\ F_2\mathcal{P} = \mathrm{I} \oplus \frac{E \oplus \mathcal{T}(E)^{(2)}}{R}.$$

The underling cooperad of any Koszul dual cooperad is *connected weight graded*:

$$q\mathcal{P}^{\text{!`}} = \mathbb{K}\,\mathrm{I} \oplus q\mathcal{P}^{\text{!`}\,(1)} \oplus \cdots \oplus q\mathcal{P}^{\text{!`}\,(\omega)} \oplus \cdots = \mathbb{K}\,\mathrm{I} \oplus sE \oplus s^2R \oplus \cdots .$$

The coderivation d_φ of the Koszul dual cooperad does not preserve this weight grading but lowers it by 1:

$$d_\varphi : q\mathcal{P}^{\text{!`}\,(\omega)} \to q\mathcal{P}^{\text{!`}\,(\omega-1)}.$$

1.8. Homotopy algebra

We denote by $\mathcal{P}_\infty := \Omega\mathcal{P}^{\text{i}}$ the cobar construction of the Koszul dual cooperad of $\mathcal{P}$. When the operad $\mathcal{P}$ is a Koszul operad, this is a resolution $\mathcal{P}_\infty \xrightarrow{\sim} \mathcal{P}$ of $\mathcal{P}$. Algebras over this dg operads are called *$\mathcal{P}_\infty$-algebras* or *homotopy $\mathcal{P}$-algebras*.

A $\mathcal{P}_\infty$-algebra structure $\mu : \mathcal{P}^{\text{i}} \to \mathrm{End}_A$ on a dg module A is equivalently given by a square-zero coderivation d_μ extending the underlying differential on A, called a codifferential, on the cofree $\mathcal{P}^{\text{i}}$-coalgebra $\mathcal{P}^{\text{i}}(A)$:

$$\mathrm{Hom}_{\mathsf{dg\ Op}}(\Omega\mathcal{P}^{\text{i}}, \mathrm{End}_A) \cong \mathrm{Codiff}(\mathcal{P}^{\text{i}}(A)).$$

By definition, an *∞-morphism of $\mathcal{P}_\infty$-algebras* is a morphism

$$F : \left(\mathcal{P}^{\text{i}}(A), d_\mu\right) \to \left(\mathcal{P}^{\text{i}}(B), d_\nu\right)$$

of dg $\mathcal{P}^{\text{i}}$-coalgebras. The composite of two ∞-morphisms is defined as the composite of the associated morphisms of dg $\mathcal{P}^{\text{i}}$-coalgebras:

$$F \circ G := \mathcal{P}^{\text{i}}(A) \to \mathcal{P}^{\text{i}}(B) \to \mathcal{P}^{\text{i}}(C).$$

The category of $\mathcal{P}_\infty$-algebras with their ∞-morphisms is denoted by ∞-$\mathcal{P}_\infty$-alg. An ∞-morphism between $\mathcal{P}_\infty$-algebras is denoted by $A \rightsquigarrow B$ to avoid confusion with the classical notion of morphism.

Since an ∞-morphism $A \rightsquigarrow B$ is a morphism of $\mathcal{P}^{\text{i}}$-coalgebras $\mathcal{P}^{\text{i}}(A) \to \mathcal{P}^{\text{i}}(B)$, it is characterized by its projection $\mathcal{P}^{\text{i}}(A) \to B$ onto the space of generators B. The first component $A \cong \mathrm{I}(A) \subset \mathcal{P}^{\text{i}}(A) \to B$ of this map is a morphism of chain complexes. When this is a quasi-isomorphism (resp. an isomorphism), we say that the map F is an *∞-quasi-isomorphism* (resp. an *∞-isomorphism*). One of the main property of ∞-quasi-isomorphisms, which does not hold for quasi-isomorphisms, lies in the following result.

Theorem 1.2 ([24, Theorem 10.4.4]). — *Let $\mathcal{P}$ be a Koszul operad and let A and B be two $\mathcal{P}_\infty$-algebras. If there exists an ∞-quasi-isomorphism $A \overset{\sim}{\rightsquigarrow} B$, then there exists an ∞-quasi-isomorphism in the opposite direction $B \overset{\sim}{\rightsquigarrow} A$, which is the inverse of $H(A) \xrightarrow{\cong} H(B)$ on the level on holomogy.*

So being ∞-quasi-isomorphic is an equivalence relation, which we call the *homotopy equivalence*. A complete treatment of the notion of ∞-morphism is given in [24, Chapter 10].

1.9. The various categories

We apply the arguments of Section 1.4 to the universal twisting morphism $\iota : \mathcal{P}^{\text{¡}} \to \Omega\mathcal{P}^{\text{¡}} = \mathcal{P}_\infty$ and to the canonical twisting morphism $\kappa : \mathcal{P}^{\text{¡}} \to \mathcal{P}$. This provides us with two bar-cobar adjunctions respectively.

By definition of ∞-morphisms, the bar construction

$$\mathrm{B}_\iota : \mathcal{P}_\infty\textsf{-alg} \to \textsf{conil dg } \mathcal{P}^{\text{¡}}\textsf{-coalg}$$

extends to a functor

$$\widetilde{\mathrm{B}}_\iota : \infty\text{-}\mathcal{P}_\infty\textsf{-alg} \to \textsf{conil dg } \mathcal{P}^{\text{¡}}\textsf{-coalg}.$$

These two functors actually lands in quasi-free $\mathcal{P}^{\text{¡}}$-coalgebras, yielding an isomorphism of categories. These various functors form the following diagram.

$$\begin{tikzcd}
\textsf{dg } \mathcal{P}\textsf{-alg} \arrow[d, hook, "i"] \arrow[r, shift right, "\mathrm{B}_\kappa"'] & \textsf{conil dg } \mathcal{P}^{\text{¡}}\textsf{-coalg} \arrow[l, shift right, "\Omega_\kappa"'] \\
\infty\text{-}\mathcal{P}_\infty\textsf{-alg} \arrow[u, bend left, "\Omega_\kappa\widetilde{\mathrm{B}}_\iota"] \arrow[r, "\cong", "\widetilde{\mathrm{B}}_\iota"'] & \textsf{quasi-free } \mathcal{P}^{\text{¡}}\textsf{-coalg} \arrow[u, hook]
\end{tikzcd}$$

Theorem 1.3 (Rectification [11]). — *Let $\mathcal{P}$ be a Koszul operad.*

(1) *The functors*

$$\Omega_\kappa\widetilde{\mathrm{B}}_\iota \ : \ \infty\text{-}\mathcal{P}_\infty\textsf{-alg} \ \rightleftharpoons \ \textsf{dg } \mathcal{P}\textsf{-alg} \ : \ i$$

form a pair of adjoint functors, where i is right adjoint to $\Omega_\kappa\widetilde{\mathrm{B}}_\iota$.

(2) *Any homotopy $\mathcal{P}$-algebra A is naturally ∞-quasi-isomorphic to the dg $\mathcal{P}$-algebra $\Omega_\kappa\widetilde{\mathrm{B}}_\iota A$:*

$$A \overset{\sim}{\rightsquigarrow} \Omega_\kappa\widetilde{\mathrm{B}}_\iota A.$$

The dg $\mathcal{P}$-algebra $\Omega_\kappa\widetilde{\mathrm{B}}_\iota A$, homotopically equivalent to the $\mathcal{P}_\infty$-algebra A is called the *rectified* $\mathcal{P}$-algebra. We refer the reader to [24, Chapter 11] for more details.

1.10. Homotopy categories

Recall that the *homotopy category* $\textsf{Ho}(\textsf{dg } \mathcal{P}\textsf{-alg})$ (resp. $\textsf{Ho}(\infty\text{-}\mathcal{P}_\infty\textsf{-alg})$) is the localization of the category of dg $\mathcal{P}$-algebras with respect to the class of quasi-isomorphisms (resp. $\mathcal{P}_\infty$-algebras with respect to the class of ∞-quasi-isomorphisms). The rectification adjunction of Theorem 1.3 induces an equivalence of categories between these two homotopy categories.

THEOREM 1.4 ([24, Theorem 11.4.8]). — *Let $\mathcal{P}$ be a Koszul operad. The homotopy category of dg $\mathcal{P}$-algebras and the homotopy category of $\mathcal{P}_\infty$-algebras with the ∞-morphisms are equivalent*

$$\mathsf{Ho}(\mathsf{dg}\ \mathcal{P}\text{-}\mathsf{alg}) \cong \mathsf{Ho}(\infty\text{-}\mathcal{P}_\infty\text{-}\mathsf{alg}).$$

1.11. Model category for algebras

A *model category structure* consists of the data of three distinguished classes of maps: *weak equivalences*, *fibrations* and *cofibration*, subject to five axioms. This extra data provided by fibrations and cofibrations gives a way to describe the homotopy category, defined by localization with respect to the weak equivalences. This notion is due to D. Quillen [27]; we refer the reader to the reference book of M. Hovey [21] for a comprehensive presentation.

THEOREM 1.5 ([18]). — *The following classes of morphisms endow the category of dg $\mathcal{P}$-algebras with a model category structure.*

- *The class $\mathfrak{W}$ of* weak equivalences *is given by the quasi-isomorphisms;*
- *the class $\mathfrak{F}$ of* fibrations *is given by degreewise epimorphisms, $f_n : A_n \twoheadrightarrow B_n$;*
- *the class $\mathfrak{C}$ of* cofibrations *is given by the maps which satisfy the left lifting property with respect to acyclic fibrations $\mathfrak{F} \cap \mathfrak{W}$.*

Notice that this model category structure is cofibrantly generated since it is obtained by transferring the cofibrantly generated model category structure on dg modules thought the free $\mathcal{P}$-algebra functor, which is left adjoint to the forgetful functor, see [27, Section II.4].

Following D. Sullivan [31], we call *triangulated dg $\mathcal{P}$-algebra* any quasi-free dg $\mathcal{P}$-algebra $(\mathcal{P}(V), d)$ equipped with an exhaustive filtration

$$V_0 = \{0\} \subset V_1 \subset V_2 \subset \cdots \subset \mathrm{Colim}_i\, V_i = V,$$

satisfying $d(V_i) \subset \mathcal{P}(V_{i-1})$.

PROPOSITION 1.6 ([18]). — *With respect to the aforementioned model category structure, every dg $\mathcal{P}$-algebra is fibrant and a dg $\mathcal{P}$-algebra is cofibrant if and only if its a retract of a triangulated dg $\mathcal{P}$-algebra.*

So this model category structure is right proper.

2. Model category structure for coalgebras

For a Koszul operad $\mathcal{P}$, we endow the category of conilpotent dg $\mathcal{P}^{¡}$-coalgebras with a model category structure which makes the bar-cobar adjunction a Quillen equivalence with dg $\mathcal{P}$-algebras.

2.1. Main theorem

Definition 2.1. — *In the category of conilpotent dg $\mathcal{P}^{¡}$-coalgebras, we consider the following three classes of morphisms.*

- *The class $\mathfrak{W}$ of* weak equivalences *is given by the morphisms of dg $\mathcal{P}^{¡}$-coalgebras $f : C \to D$ whose image $\Omega_\kappa f : \Omega_\kappa C \xrightarrow{\sim} \Omega_\kappa D$ under the cobar construction is a quasi-isomorphism of dg $\mathcal{P}$-algebras;*
- *the class $\mathfrak{C}$ of* cofibrations *is given by degreewise monomorphisms, $f_n : C_n \rightarrowtail D_n$;*
- *the class $\mathfrak{F}$ of* fibrations *is given by the maps which satisfy the right lifting property with respect to acyclic cofibrations $\mathfrak{C} \cap \mathfrak{W}$.*

Theorem 2.2.

(1) *Let $\mathcal{P}$ be a Koszul operad. The aforementioned three classes of morphisms form a model category structure on conilpotent dg $\mathcal{P}^{¡}$-coalgebras.*
(2) *With this model category structure, every conilpotent dg $\mathcal{P}^{¡}$-coalgebra is cofibrant; so this model category is left proper. A conilpotent dg $\mathcal{P}^{¡}$-coalgebra is fibrant if and only if it is isomorphic to a quasi-free dg $\mathcal{P}^{¡}$-coalgebra.*
(3) *The bar-cobar adjunction*

$$\mathrm{B}_\kappa \ : \ \mathsf{dg}\ \mathcal{P}\text{-}\mathsf{alg} \ \rightleftharpoons \ \mathsf{conil\ dg}\ \mathcal{P}^{¡}\text{-}\mathsf{coalg} \ : \ \Omega_\kappa .$$

is a Quillen equivalence.

2.2. Weight filtration

Definition 2.3. — *Any $\mathcal{P}^{¡}$-coalgebra (C, Δ_C) admits the following* weight filtration*:*

$$F_n C := \left\{ c \in C \;\middle|\; \Delta_C(c) \in \bigoplus_{\omega=0}^{n} \mathcal{P}^{¡\,(\omega)}(C) \right\}.$$

For instance, the first terms are

$$F_{-1}C = \{0\} \subset F_0C := \{c \in C \,|\, \Delta_C(c) = c\}$$
$$\subset F_1C := \left\{ c \in C \,\middle|\, \Delta_C(c) \in C \oplus \mathcal{P}^{¡\,(1)}(C) \right\} \subset \cdots .$$

Examples.

- For any cofree coalgebra $\mathcal{P}^{¡}(V)$, the weight filtration is equal to
$$F_n\mathcal{P}^{¡}(V) := \bigoplus_{\omega=0}^{n} \mathcal{P}^{¡\,(\omega)}(V).$$
- When the operad $\mathcal{P}$ is the operad As, which encodes associative algebras, the Koszul dual cooperad $As^{¡} = As^c$ encodes coassociative coalgebras. In this case, the weight filtration is equal, up to a shift of indices, to the coradical filtration of coassociative coalgebras, cf. [28, Appendix B] and [24, Section 1.2.4].

We consider the reduced coproduct $\bar{\Delta}_C(c) := \Delta_C(c) - c$. Its kernel is equal to $\operatorname{Prim} C := F_0C$, which is called the *space of primitive elements*.

PROPOSITION 2.4. — *Let C be a conilpotent dg $\mathcal{P}^{¡}$-coalgebra (C, d_c, Δ_C).*

(1) *Its weight filtration is exhaustive: $\bigcup_{n\in\mathbb{N}} F_nC = C$.*
(2) *Its weight filtration satisfies*
$$\bar{\Delta}_C(F_nC) \subset \bigoplus_{\substack{1\leqslant\omega\leqslant n,\ k\geqslant 1,\\ n_1+\cdots+n_k=n-\omega}} \mathcal{P}^{¡\,(\omega)}(k) \otimes_{\mathbb{S}_k} (F_{n_1}C \otimes \cdots \otimes F_{n_k}C).$$
(3) *The differential preserves the weight filtration: $d_c(F_nC) \subset F_nC$.*

Proof. — The first point follows from the definition of a conilpotent coalgebra and from the fact that $\mathcal{P}^{¡}$ is a connected weight graded cooperad. The second point is a direct corollary of the relation $\mathcal{P}^{¡}(\Delta_C)\Delta_C = \Delta_{\mathcal{P}^{¡}}(C)\Delta_C$ in the definition of a coalgebra over a cooperad. The last point is a consequence of the commutativity of the differential d_C and the structure map Δ_C. □

This proposition shows that the weight filtration is made up of dg $\mathcal{P}^{¡}$-subcoalgebras. Notice that any morphism $f : C \to D$ of $\mathcal{P}^{¡}$-coalgebras preserves the weight filtration: $f(F_nC) \subset F_nD$.

2.3. Filtered quasi-isomorphisms

In this section, we refine the results of [24, Chapter 11] and of [23, Section 1.3] about the behavior of the bar and cobar constructions with respect to quasi-isomorphisms.

Definition 2.5. — *A* filtered quasi-isomorphism *of conilpotent dg $\mathcal{P}^{¡}$-coalgebras is a morphism $f : C \to D$ of dg $\mathcal{P}^{¡}$-coalgebras such that the induced morphisms of chain complexes*

$$\mathrm{gr}_n\, f : F_nC/F_{n-1}C \xrightarrow{\sim} F_nD/F_{n-1}D$$

are quasi-isomorphisms, for any $n \geqslant 0$.

Proposition 2.6. — *The class of filtered quasi-isomorphisms of conilpotent dg $\mathcal{P}^{¡}$-coalgebras is included in the class of weak equivalences.*

Proof. — Let $f : C \to D$ be a filtered quasi-isomorphism of conilpotent dg $\mathcal{P}^{¡}$-coalgebras. We consider the following filtration on the cobar construction $\Omega_\kappa C = (\mathcal{P}(C), d_1 + d_2)$ induced by the weight filtration:

$$\mathcal{F}_n\, \Omega_\kappa C := \sum_{\substack{k\geqslant 1,\\ n_1+\cdots+n_k\leqslant n}} \mathcal{P}(k) \otimes_{\mathbb{S}_k} (F_{n_1}C \otimes \cdots \otimes F_{n_k}C).$$

Recall from [24, Section 11.2] that the differential of the cobar construction is made up of two terms $d_1 + d_2$, where $d_1 = \mathcal{P} \circ' d_C$ and where d_2 is the unique derivation which extends

$$C \xrightarrow{\Delta_C} \mathcal{P}^{¡} \circ C \xrightarrow{\kappa\circ \mathrm{Id}_C} \mathcal{P} \circ C.$$

It is explicitly given by

$$\mathcal{P} \circ C \xrightarrow{\mathrm{Id}_\mathcal{P}\, \circ' \Delta_C} \mathcal{P} \circ (C; \mathcal{P}^{¡} \circ C) \xrightarrow{\mathrm{Id}_\mathcal{P}\, \circ (\mathrm{Id}_C; \kappa\circ \mathrm{Id}_C)} \mathcal{P} \circ (C; \mathcal{P} \circ C)$$
$$\cong (\mathcal{P} \circ_{(1)} \mathcal{P})(C) \xrightarrow{\gamma_{(1)} \circ \mathrm{Id}_C} \mathcal{P} \circ C.$$

So, Proposition 2.4 implies

$$d_1(\mathcal{F}_n) \subset \mathcal{F}_n \quad \text{and} \quad d_2(\mathcal{F}_n) \subset \mathcal{F}_{n-1}.$$

The first page of the associated spectral sequence is equal to

$$\mathcal{P}(\mathrm{gr}\ f)\ :\ (\mathcal{P}(\mathrm{gr}\ C), \mathcal{P} \circ' d_{\mathrm{gr}\ C}) \xrightarrow{\sim} (\mathcal{P}(\mathrm{gr}\ D), \mathcal{P} \circ' d_{\mathrm{gr}\ D}),$$

which is a quasi-isomorphism by assumption. Since the weight filtration is exhaustive, this filtration is exhaustive. It is also bounded below, so we conclude by the classical convergence theorem of spectral sequences [25, Chapter 11]:

$$\Omega_\kappa f\ :\ \Omega_\kappa C \xrightarrow{\sim} \Omega_\kappa D. \qquad \square$$

Proposition 2.7. — *If $f : A \xrightarrow{\sim} A'$ is a quasi-isomorphism of dg $\mathcal{P}$-algebras, then $B_\kappa f : \mathrm{B}_\kappa A \to \mathrm{B}_\kappa A'$ is a filtered quasi-isomorphism of conilpotent dg $\mathcal{P}^{¡}$-coalgebras.*

Proof. — Since the bar construction $\mathrm{B}_\kappa A$ is a quasi-free coalgebra, its weight filtration is equal to $F_n \mathrm{B}_\kappa A = \bigoplus_{\omega=0}^n \mathcal{P}^{¡(\omega)}(A)$. Recall that its differential is the sum of three terms $d_\varphi \circ \mathrm{Id}_A + \mathrm{Id}_{\mathcal{P}^{¡}} \circ' d_A + d_2$, where d_2 is the unique coderivation which extends

$$\mathcal{P}^{¡} \circ A \xrightarrow{\kappa \circ \mathrm{Id}_A} \mathcal{P} \circ A \xrightarrow{\gamma_A} A.$$

So, the coderivation d_2 is equal to the composite

$$\begin{aligned}\mathcal{P}^{¡} \circ A \xrightarrow{\Delta_{(1)} \circ \mathrm{Id}_A} (\mathcal{P}^{¡} \circ_{(1)} \mathcal{P}^{¡}) \circ A &\xrightarrow{(\mathrm{Id}_{\mathcal{P}^{¡}} \circ_{(1)} \kappa) \circ \mathrm{Id}_A} (\mathcal{P}^{¡} \circ_{(1)} \mathcal{P}) \circ A \\ &\cong \mathcal{P}^{¡} \circ (A; \mathcal{P} \circ A) \xrightarrow{\mathrm{Id}_{\mathcal{P}^{¡}} \circ (\mathrm{Id}_A; \gamma_A)} \mathcal{P}^{¡} \circ A.\end{aligned}$$

Since the maps κ and d_φ lowers the weight grading by 1, we get

$$\mathrm{Id}_{\mathcal{P}^{¡}} \circ' d_A(F_n) \subset F_n, \quad d_\varphi \circ \mathrm{Id}_A(F_n) \subset F_{n-1}, \quad \text{and} \quad d_2(F_n) \subset F_{n-1}.$$

Hence, the graded analogue of $\mathrm{B}_\kappa f$ is equal to

$$\mathrm{gr}_n \mathrm{B}_\kappa f = \mathcal{P}^{¡(n)}(f) : \left(\mathcal{P}^{¡(n)}(A), \mathrm{Id}_{\mathcal{P}^{¡}} \circ' d_A\right) \xrightarrow{\sim} \left(\mathcal{P}^{¡(n)}(A'), \mathrm{Id}_{\mathcal{P}^{¡}} \circ' d_{A'}\right),$$

which is a quasi-isomorphism for any $n \in \mathbb{N}$. $\square$

Proposition 2.8. — *The class of weak equivalences of conilpotent dg $\mathcal{P}^{¡}$-coalgebras is included in the class of quasi-isomorphisms.*

Proof. — Let $f : C \to D$ be a weak equivalence of conilpotent dg $\mathcal{P}^{¡}$-coalgebras. By definition, its image under the bar construction $\Omega_\kappa f : \Omega_\kappa C \xrightarrow{\sim} \Omega_\kappa D$ is a quasi-isomorphism of dg $\mathcal{P}$-algebras. Since the bar construction B_κ preserves quasi-isomorphisms, by [24, Proposition 11.2.3], and since the counit of the bar-cobar adjunction $\upsilon_\kappa C : C \xrightarrow{\sim} \mathrm{B}_\kappa \Omega_\kappa C$ is a quasi-isomorphism, we conclude with the following commutative diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\upsilon_\kappa C"', "\sim"] & D \arrow[d, "\upsilon_\kappa D"', "\sim"] \\ \mathrm{B}_\kappa \Omega_\kappa C \arrow[r, "\sim"] & \mathrm{B}_\kappa \Omega_\kappa D. \end{tikzcd}$$

$\square$

Without any assumption on the connectivity of the underlying chain complexes, the aforementioned inclusion can be strict. Examples of quasi-isomorphisms, which is not weak equivalences, are given in [19, Section 9.1.2] of dg cocommutative coalgebras and in [23, Section 1.3.5] of dg coassociative coalgebras.

The following diagram sums up the aforementioned propositions.

$$\begin{tikzcd}
 & \boxed{\text{filtered quasi-isomorphisms}} \arrow[d, hook] \\
\boxed{\text{quasi-isomorphisms}} \arrow[ur, bend left, "\mathrm{B}_\kappa"] & \\
 & \boxed{\text{weak equivalences}} \arrow[ul, bend left, "\Omega_\kappa"] \arrow[d, hook, "\neq"] \\
 & \boxed{\text{quasi-isomorphisms}} \\
\underline{\text{dg } \mathcal{P}\text{-algebras}} & \underline{\text{dg } \mathcal{P}^{¡}\text{-coalgebras}}
\end{tikzcd}$$

Theorem 2.9. — *Let $\mathcal{P}$ be a Koszul operad.*

(1) *The counit of the bar-cobar adjunction $\varepsilon_\kappa : \Omega_\kappa \mathrm{B}_\kappa A \xrightarrow{\sim} A$ is a quasi-isomorphism of dg $\mathcal{P}$-algebras, for every dg $\mathcal{P}$-algebra A.*

(2) *The unit of the bar-cobar adjunction $\upsilon_\kappa : C \xrightarrow{\sim} \mathrm{B}_\kappa \Omega_\kappa C$ is a weak equivalence of conilpotent dg $\mathcal{P}^{¡}$-coalgebras, for every conilpotent dg $\mathcal{P}^{¡}$-coalgebra C.*

Proof. — The first point follows from [24, Theorem 11.3.3] and [24, Corollary 11.3.5]. For the second point, we consider the following filtration induced by the weight filtration of C:

$$\mathcal{F}_n \, \mathrm{B}_\kappa \Omega_\kappa C := \sum_{\substack{k \geqslant 1,\\ n_1+\cdots+n_k \leqslant n}} (\mathcal{P}^{¡} \circ \mathcal{P})(k) \otimes_{\mathbb{S}_k} (F_{n_1} C \otimes \cdots \otimes F_{n_k} C).$$

Since the unit of adjunction υ_κ is equal to the composite

$$C \xrightarrow{\Delta_C} \mathcal{P}^{¡}(C) \cong \mathcal{P}^{¡} \circ \mathrm{I} \circ C \rightarrowtail \mathcal{P}^{¡} \circ \mathcal{P} \circ C,$$

it preserves the respective filtrations by Proposition 2.4 (2). The associated graded morphism is equal to

$$\mathrm{gr}_n \, \upsilon_\kappa : F_n C / F_{n-1} C \to \mathcal{F}_n \, \mathrm{B}_\kappa \Omega_\kappa C / \mathcal{F}_{n-1} \, \mathrm{B}_\kappa \Omega_\kappa C,$$

where the right-hand side is isomorphic to $\mathrm{B}_\kappa \Omega_\kappa \, \mathrm{gr}\, C \cong (\mathcal{P}^{¡} \circ_\kappa \mathcal{P}) \circ (\mathrm{gr}\, C, d_{\mathrm{gr}\, C})$. Since the operad $\mathcal{P}$ is Koszul, its Koszul complex is acyclic, $\mathcal{P}^{¡} \circ_\kappa \mathcal{P} \xrightarrow{\sim} \mathrm{I}$, which proves that the unit υ_κ is a filtered quasi-isomorphism.

Finally, we conclude that the unit υ_κ is a weak-equivalence by Proposition 2.6. $\square$

2.4. Fibrations and cofibrations

Let us first recall that the coproduct $A \vee B$ of two $\mathcal{P}$-algebras (A, γ_A) and (B, γ_B) is given by the following coequalizer

$$\mathcal{P} \circ (A \oplus B; \mathcal{P}(A) \oplus \mathcal{P}(B)) \rightrightarrows \mathcal{P}(A \oplus B) \twoheadrightarrow A \vee B,$$

where one map is induced by the partial composition product $\mathcal{P} \circ_{(1)} \mathcal{P} \to \mathcal{P}$ of the operad $\mathcal{P}$ and where the other one is equal to $\mathcal{P} \circ (\mathrm{Id}_{A \oplus B}; \gamma_A + \gamma_B)$. When $B = \mathcal{P}(V)$ is a free $\mathcal{P}$-algebra, the coproduct $A \vee \mathcal{P}(V)$ is simply equal to the following coequalizer

$$\mathcal{P} \circ (A \oplus V; \mathcal{P}(A)) \rightrightarrows \mathcal{P}(A \oplus V) \twoheadrightarrow A \vee \mathcal{P}(V).$$

As usual, see [21], we denote by D^n the acyclic chain complex

$$\cdots \to 0 \to \mathbb{K} \xrightarrow{\sim} \mathbb{K} \to 0 \to \cdots$$

concentrated in degrees n and $n-1$. We denote by S^n the chain complex

$$\cdots \to 0 \to \mathbb{K} \to 0 \to \cdots$$

concentrated in degrees n. The generating cofibrations of the model category of dg modules are the embeddings $I^n : S^{n-1} \rightarrowtail D^n$ and the generating acyclic cofibrations are the quasi-isomorphisms $J^n : 0 \overset{\sim}{\rightarrowtail} D^n$. So, in the cofibrantly generated model category of dg $\mathcal{P}$-algebras, the relative $\mathcal{P}(I)$-cell complexes, also known as *standard cofibrations*, are the sequential colimits of pushouts of coproducts of $\mathcal{P}(I)$-maps. Since we are working over a field $\mathbb{K}$, such a pushout is equivalent to

$$\begin{tikzcd}
\mathcal{P}(s^{-1}V) \arrow[r, "\gamma_A \mathcal{P}(s\alpha)"] \arrow[d, tail] & A \arrow[d, tail] \\
\mathcal{P}(V \oplus s^{-1}V) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & A \vee_\alpha \mathcal{P}(V),
\end{tikzcd}$$

where V is a graded module, $\alpha : V \to A$ is a degree -1 map, with image in the cycles of A. The dg $\mathcal{P}$-algebra $A \vee_\alpha \mathcal{P}(V)$ is equal to the coproduct of $\mathcal{P}$-algebras $A \vee \mathcal{P}(V)$ endowed with the differential given by d_A and by the unique derivation which extends the map

$$V \xrightarrow{\alpha} A \rightarrowtail A \vee \mathcal{P}(V).$$

Hence a standard cofibration of dg $\mathcal{P}$-algebras is a morphism of dg $\mathcal{P}$-algebras $A \rightarrowtail (A \vee \mathcal{P}(S), d)$, where the graded module S admits an exhaustive filtration

$$S_0 = \{0\} \subset S_1 \subset S_2 \subset \cdots \subset \mathrm{Colim}_i\ S_i = S,$$

satisfying $d(S_i) \subset A \vee \mathcal{P}(S_{i-1})$.

In the same way, a *standard acyclic cofibration*, or relative $\mathcal{P}(J)$-cell complex, is a morphism of dg $\mathcal{P}$-algebras $A \rightarrowtail A \vee \mathcal{P}(M)$, where the chain complex M is a direct sum $M = \bigoplus_{i=1}^{\infty} M^i$ of acyclic chain complexes. Finally, any cofibration (resp. acyclic cofibration) is a retract of a standard cofibration (resp. standard acyclic cofibration) with isomorphisms on domains.

Let (A, d_A) be a dg $\mathcal{P}$-algebra and let (V, d_V) be a chain complex. Let $\alpha : V \to A$ be a degree -1 map such that the unique derivation on the coproduct $A \vee \mathcal{P}(V)$, defined by d_A, d_V and α, squares to 0. In this case, the dg $\mathcal{P}$-algebra produced is still denoted by $A \vee_\alpha \mathcal{P}(V)$.

Lemma 2.10. — *The embedding $A \rightarrowtail A \vee_\alpha \mathcal{P}(V)$ is a standard cofibration of dg $\mathcal{P}$-algebras.*

Proof. — Since we are working over a field $\mathbb{K}$, any chain complex V decomposes into

$$V \cong B \oplus H \oplus sB,$$

where $d_V(B) = d_V(H) = 0$ and where $d_V : sB \xrightarrow{s^{-1}} B$. It is enough to consider the following filtration to conclude

$$S_0 = \{0\} \subset S_1 := B \oplus H \subset S_2 := V. \qquad \square$$

Proposition 2.11. — *Let (C, Δ_C) be a dg $\mathcal{P}^{\text{¡}}$-coalgebra and let $C' \subset C$ be a dg sub-$\mathcal{P}^{\text{¡}}$-coalgebra such that $\bar{\Delta}_C(C) \subset \mathcal{P}^{\text{¡}}(C')$. The image of the inclusion $C' \rightarrowtail C$ under the cobar construction Ω_κ is a standard cofibration of dg $\mathcal{P}$-algebras.*

Proof. — Since we are working over a field $\mathbb{K}$, there exists a graded submodule E of C such that $C \cong C' \oplus E$ in the category of graded $\mathbb{K}$-modules. Forgetting the differentials, the underlying $\mathcal{P}$-algebra of the cobar construction of C is isomorphic to

$$\Omega_\kappa C \cong \mathcal{P}(C' \oplus E) \cong \mathcal{P}(C') \vee \mathcal{P}(E).$$

Under the decomposition $C \cong C' \oplus E$, the differential d_C of C is the sum of the following three terms:

$$d_{C'} : C' \to C', \quad d_E : E \to E, \quad \text{and} \quad \alpha : E \to C'.$$

By assumption, the degree -1 map

$$\beta : E \rightarrowtail C \xrightarrow{\Delta_C} \mathcal{P}^{\text{¡}}(C) \xrightarrow{\kappa(C)} \mathcal{P}(C)$$

actually lands in $\mathcal{P}(C')$. So the morphism of dg $\mathcal{P}$-algebras $\Omega_\kappa C' \to \Omega_\kappa C$ is equal to the embedding $A \rightarrowtail A \vee_{\alpha+\beta} \mathcal{P}(E)$, where A stands for the dg $\mathcal{P}$-algebra $\Omega_\kappa C'$. We conclude the present proof with Lemma 2.10. □

Theorem 2.12.

(1) *The cobar construction Ω_κ preserves cofibrations and weak equivalences.*
(2) *The bar construction B_κ preserves fibrations and weak equivalences.*

Proof.

(1). — Let $f : C \rightarrowtail D$ be a cofibration of conilpotent dg $\mathcal{P}^{\text{¡}}$-coalgebras. For any $n \in \mathbb{N}$, we consider the dg sub-$\mathcal{P}^{\text{¡}}$-coalgebra of D defined by

$$D^{[n]} := f(C) + F_{n-1}D,$$

where $F_{n-1}D$ stands for the weight filtration of the $\mathcal{P}^{\text{¡}}$-coalgebra D. By convention, we set $D^{[0]} := C$. Proposition 2.4 (2) implies

$$\bar{\Delta}_{D^{[n+1]}}(D^{[n+1]}) \subset \mathcal{P}^{\text{¡}}(D^{[n]}).$$

So, we can apply Proposition 2.11 to show that the maps $\Omega_\kappa D^{[n]} \to \Omega_\kappa D^{[n+1]}$ are standard cofibrations of dg $\mathcal{P}$-algebras. Finally, the map $\Omega_\kappa f$ is a cofibration as a sequential colimit of standard cofibrations.

The cobar construction Ω_κ preserves weak equivalences by definition.

(2). — Let $g : A \twoheadrightarrow A'$ be a fibration of dg $\mathcal{P}$-algebras. Its image $\mathrm{B}_\kappa\, g$ is a fibration if and only if it satisfies the right lifting property with respect to any acyclic cofibration $f : C \overset{\sim}{\rightarrowtail} D$. Under the bar-cobar adjunction 1.4, this property is equivalent to the left lifting property of $\Omega_\kappa f$ with respect to g, which holds true by the above point (1).

The bar construction B_κ sends quasi-isomorphisms of dg $\mathcal{P}$-algebras to weak equivalences of dg $\mathcal{P}^{\text{¡}}$-coalgebras by Propositions 2.6 and 2.7. □

2.5. Proof of Theorem 2.2 (1)

(MC 1) *(Limits and Colimits).* — Since we are working over a field of characteristic 0, Proposition 1.20 of [14] applies and shows that the category of conilpotent dg $\mathcal{P}^{\text{¡}}$-coalgebras admits finite limits and finite colimits.

(MC 2) *(Two out of three).* — Let $f : C \to D$ and $g : D \to E$ be two morphisms of conilpotent dg $\mathcal{P}^{\text{!`}}$-coalgebras. If any two of f, g and gf are weak equivalences, then so is the third. This is a direct consequence of the definition of weak equivalences and the axiom (MC 2) for dg $\mathcal{P}$-algebras.

(MC 3) *(Retracts).* — Since the cofibrations $\mathfrak{C}$ are the degreewise monomorphisms, they are stable under retracts.

Since the image of a retract under the cobar construction Ω_κ is again a retract, weak equivalences $\mathfrak{W}$ of conilpotent dg $\mathcal{P}^{\text{!`}}$-coalgebras are stable under retract by the axiom (MC 3) for dg $\mathcal{P}$-algebras.

Let $f : C \twoheadrightarrow D$ be a fibration $\mathfrak{F}$ of conilpotent dg $\mathcal{P}^{\text{!`}}$-coalgebras and let $g : E \to F$ be a retract of f. Let $c : G \overset{\sim}{\rightarrowtail} H$ be an acyclic cofibration fitting into the following commutative diagram.

$$\begin{tikzcd}
G \arrow[r] \arrow[d, tail, "c"', "\sim"] & E \arrow[r] \arrow[d, "g"'] & C \arrow[r, "p"] \arrow[d, two heads, "f"'] & E \arrow[d, "g"'] \\
H \arrow[r] \arrow[urr, dotted, "\alpha"] & F \arrow[r] & D \arrow[r] & F
\end{tikzcd}$$

By the lifting property, there exists a map $\alpha : H \to C$ making the first rectangle into a commutative diagram. Finally, the composite $p\alpha$ makes the first square into a commutative diagram, which proves that the map g is a fibration.

(MC 4) *(Factorization).* — Let $f : C \to D$ be a morphism of conilpotent dg $\mathcal{P}^{\text{!`}}$-coalgebras. The factorization axiom (MC 5) for dg $\mathcal{P}$-algebras allows us to factor $\Omega_\kappa f$ into

$$\begin{tikzcd}
\Omega_\kappa C \arrow[rr, "\Omega_\kappa f"] \arrow[dr, tail, "i"'] & & \Omega_\kappa D \\
& A, \arrow[ur, two heads, "p"'] &
\end{tikzcd}$$

where i is a cofibration and p a fibration and where one of these two is a quasi-isomorphism. So, the morphism $\mathrm{B}_\kappa\, \Omega_\kappa f$ factors into $\mathrm{B}_\kappa\ p \circ \mathrm{B}_\kappa\ i$. We consider the following commutative diagram in the category of conilpotent dg $\mathcal{P}^{¡}$-coalgebras.

$$
\begin{tikzcd}
\mathrm{B}_\kappa \Omega_\kappa C \arrow[rr, "\mathrm{B}_\kappa \Omega_\kappa f"] \arrow[dr, "\mathrm{B}_\kappa\ i"'] & & \mathrm{B}_\kappa \Omega_\kappa D \\
& \mathrm{B}_\kappa A \arrow[ur, "\mathrm{B}_\kappa\ p"'] & \\
C \arrow[uu, "\upsilon_\kappa C"] \arrow[rr, "f"] \arrow[dr, dotted, "\tilde\imath"'] & & D \arrow[uu, "\upsilon_\kappa D"'] \\
& \mathrm{B}_\kappa A \times_{\mathrm{B}_\kappa \Omega_\kappa D} D \arrow[uu] \arrow[ur, "\tilde p"'] \arrow[ur, phantom, very near start, "\urcorner"] &
\end{tikzcd}
$$

By definition of the pullback, there exists a morphism

$$\tilde\imath \colon C \to \mathrm{B}_\kappa A \times_{\mathrm{B}_\kappa \Omega_\kappa D} D,$$

such that $f = \tilde p\, \tilde\imath$. We shall now prove that the two maps $\tilde p$ and $\tilde\imath$ are respectively a fibration and a cofibration.

First, the map $\mathrm{B}_\kappa\ p$ is a fibration by Theorem 2.12 (2). Since fibrations are stable under base change, the morphism $\tilde p$ is also a fibration.

As a cofibration of dg $\mathcal{P}$-algebras, the map $i : \Omega_\kappa C \rightarrowtail A$ is a retract of a standard cofibration, with isomorphisms on domains, and so is a monomorphism. The composite $(\mathrm{B}_\kappa\, i)(\upsilon_\kappa C)$ is actually equal to the following composite

$$C \xrightarrow{\Delta_C} \mathcal{P}^{¡}(C) \xrightarrow{\mathcal{P}^{¡}(i_C)} \mathcal{P}^{¡}(A).$$

Since its first component on $A \cong \mathrm{I}(A) \subset \mathcal{P}^{¡}(A)$ is equal to the restriction i_C on C, it is a monomorphism. We conclude that the morphism $\tilde\imath$ is a monomorphism by Lemma B.1, proved in the Appendix B.

If the map i (resp. p) is a quasi-isomorphim, then the map $\mathrm{B}_\kappa\ i$ (resp. $\mathrm{B}_\kappa\ p$) is a weak equivalence by Theorem 2.12 (2). Recall that the unit of adjunction υ_κ is a weak equivalence by Theorem 2.9 (2). Assuming Lemma B.1, that is $j : \mathrm{B}_\kappa A \times_{\mathrm{B}_\kappa \Omega_\kappa D} D \xrightarrow{\sim} \mathrm{B}_\kappa A$ being a weak equivalence, we conclude that the map $\tilde\imath$ (resp. $\tilde p$) is a weak equivalence by the above axiom (MC 2).

(MC 5) *(Lifting property).* — We consider the following commutative diagram in the category of conilpotent dg $\mathcal{P}^{\text{i}}$-colagebras

$$
\begin{tikzcd}
E \arrow[r] \arrow[d, tail, "c"'] & C \arrow[d, two heads, "f"] \\
F \arrow[r] \arrow[ur, dotted, "\alpha"] & D,
\end{tikzcd}
$$

where c is a cofibration and where f is a fibration. If moreover the map c is a weak equivalence, then there exists a morphism α such that the two triangles commute, by the definition of the class $\mathfrak{F}$ of fibrations.

Let us now prove the same lifting property when the map f is a weak equivalence. Using the aforementioned axiom (MC 5), we factor the map f into $f = \tilde{p}\,\tilde{\imath}$, where $\tilde{\imath}$ is a cofibration and $\tilde{p}$ a fibration. By the axiom (MC 2), both maps $\tilde{p}$ and $\tilde{\imath}$ are weak equivalences. By the definition of fibrations, there exists a lifting $r : \mathrm{B}_\kappa\, A \times_{\mathrm{B}_\kappa\, \Omega_\kappa D} D \to C$ in the diagram

$$
\begin{tikzcd}
C \arrow[r, "\mathrm{id}_C"] \arrow[d, tail, "\tilde{\imath}"', "\sim"] & C \arrow[d, two heads, "f"] \\
\mathrm{B}_\kappa\, A \times_{\mathrm{B}_\kappa\, \Omega_\kappa D} D \arrow[r, two heads, "\tilde{p}"'] \arrow[ur, dotted, "r"] & D.
\end{tikzcd}
$$

It remains to find a lifting in the diagram

$$
\begin{tikzcd}
E \arrow[r] \arrow[d, tail, "c"'] & \mathrm{B}_\kappa\, A \times_{\mathrm{B}_\kappa\, \Omega_\kappa D} D \arrow[d, two heads, "\tilde{p}"] \\
F \arrow[r] \arrow[ur, dotted] & D,
\end{tikzcd}
$$

which, by the pullback property, is equivalent to finding a lifting in

$$
\begin{tikzcd}
E \arrow[r] \arrow[d, tail, "c"'] & \mathrm{B}_\kappa\, A \times_{\mathrm{B}_\kappa\, \Omega_\kappa D} D \arrow[r] \arrow[d, two heads, "\tilde{p}"] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{B}_\kappa\, A \arrow[d, two heads, "\mathrm{B}_\kappa\ p"] \\
F \arrow[r] \arrow[urr, dotted] & D \arrow[r, "\upsilon_\kappa D"'] & \mathrm{B}_\kappa\, \Omega_\kappa D.
\end{tikzcd}
$$

To prove that such a lifting exists, it is enough to consider the following dual diagram under the bar-cobar adjunction 1.4.

$$
\begin{tikzcd}
\Omega_\kappa E \arrow[r] \arrow[d, tail, "\Omega_\kappa c"'] & A \arrow[d, two heads, "\sim"', "p"] \\
\Omega_\kappa F \arrow[r] \arrow[ur, dotted] & \Omega_\kappa D
\end{tikzcd}
$$

Since the cobar construction preserves cofibrations, by Theorem 2.12 (1), and since the map p is an acyclic cofibration, we conclude by the lifting axiom (MC 4) in the model category of dg $\mathcal{P}$-algebras. $\square$

2.6. Fibrant and cofibrant objects

Since cofibrations are monomorphisms, every conilpotent dg $\mathcal{P}^{\text{!`}}$-coalgebra is cofibrant. Let us now prove that a conilpotent dg $\mathcal{P}^{\text{!`}}$-coalgebra is fibrant if and only if it is isomorphic to a quasi-free dg $\mathcal{P}^{\text{!`}}$-coalgebra.

Proof of Theorem 2.2 (2). — Let $(C, d_C) \cong (\mathcal{P}^{\text{!`}}(A), d_\mu)$ be a conilpotent dg $\mathcal{P}^{\text{!`}}$-coalgebra isomorphic to a quasi-free dg $\mathcal{P}^{\text{!`}}$-coalgebra. The codifferential d_μ endows A with a $\mathcal{P}_\infty$-algebra structure, so $C \cong \widetilde{\mathrm{B}}_\iota A$. We consider the unit $\upsilon \;:\; A \rightsquigarrow \Omega_\kappa \widetilde{\mathrm{B}}_\iota A$ of the $(\Omega_\kappa \widetilde{\mathrm{B}}_\iota, i)$-adjunction of Theorem 1.3. Its first component $\upsilon_{(0)} : A \cong \mathrm{I} \circ \mathrm{I} \circ A \rightarrowtail \mathcal{P} \circ \mathcal{P}^{\text{!`}} \circ A$ is a monomorphism. We denote by $\rho_{(0)} : \mathcal{P} \circ \mathcal{P}^{\text{!`}} \circ A \twoheadrightarrow \mathrm{I} \circ \mathrm{I} \circ A \cong A$ its right inverse. We define a map $\rho : \mathcal{P}^{\text{!`}} \to \mathrm{End}_A^{\Omega_\kappa \widetilde{\mathrm{B}}_\iota A}$ by the formula of [24, Theorem 10.4.1]. The proof given in loc. cit. shows that the map ρ is an ∞-morphism, which is right inverse to υ, i.e. $\rho\upsilon = \mathrm{id}_A$. This allows us to write the conilpotent dg $\mathcal{P}^{\text{!`}}$-coalgebra C as a retract of its bar-cobar construction $\mathrm{B}_\kappa\, \Omega_\kappa C$:

$$
\begin{tikzcd}
C \arrow[r, "\upsilon_\kappa C"] \arrow[rr, bend right, "\mathrm{id}"'] & \mathrm{B}_\kappa\, \Omega_\kappa C \arrow[r, "\widetilde{\mathrm{B}}_\iota \rho"] & C.
\end{tikzcd}
$$

Since the dg $\mathcal{P}$-algebra $\Omega_\kappa C$ is fibrant and since the bar construction B_κ preserves fibrations by Theorem 2.12 (2), then the bar-cobar construction $\mathrm{B}_\kappa\, \Omega_\kappa C$ is a fibrant conilpotent dg $\mathcal{P}^{\text{!`}}$-coalgebra. We conclude with the general property that fibrant objects are stable under retract.

In the other way round, let C be a fibrant conilpotent dg $\mathcal{P}^{\text{!`}}$-coalgebra. By definition of the fibrations of conilpotent dg $\mathcal{P}^{\text{!`}}$-coalgebras, there exists a lifting r in the following commutative diagram

$$\begin{tikzcd}
C \arrow[r, equal] \arrow[d, tail, "\upsilon_\kappa C"', "\sim"] & C \arrow[d, two heads] \\
\mathrm{B}_\kappa\, \Omega_\kappa C \arrow[r] \arrow[ur, dotted, "r"] & 0,
\end{tikzcd}$$

which makes C into a retract of its bar-cobar construction. Since the map r preserves the respective weight filtrations, its first component $\mathcal{P}(C) \to \operatorname{Prim} C$ on F_0 induces the following projection onto $\operatorname{Prim} C$

$$\xi : C \rightarrowtail \mathcal{P}(C) \to \operatorname{Prim} C.$$

Let us now prove that the induced morphism of conilpotent $\mathcal{P}^{\textrm{!`}}$-coalgebras $\Xi : C \to \mathcal{P}^{\textrm{!`}}(\operatorname{Prim} C)$ is an isomorphism. To that extend, we show, by induction on $n \in \mathbb{N}$, that the graded morphism

$$\operatorname{gr}_n \Xi : \operatorname{gr}_n C \xrightarrow{\cong} \mathcal{P}^{\textrm{!`}\,(n)}(\operatorname{Prim} C),$$

associated to the weight filtration, is an isomorphism. The case $n = 0$ is trivially satisfied, since $\operatorname{gr}_0 C = \operatorname{Prim} C$. Suppose now the result true up to n and let us prove it for $n + 1$.

We consider the cobar construction $\Omega_\kappa C$ of the $\mathcal{P}^{\textrm{!`}}$-coalgebra (C, Δ_C) *without* its internal differential, equipped with the filtration induced by the weight filtration of C :

$$\mathscr{F}_n \Omega_\kappa C := \sum_{\substack{k \geqslant 1, \\ l+n_1+\cdots+n_k \leqslant n}} F_l \mathcal{P}(k) \otimes_{\mathbb{S}_k} F_{n_1} C \otimes \cdots \otimes F_{n_k} C.$$

This filtration is stable under the boundary map $d_2 : \mathscr{F}_n \Omega_\kappa C \to \mathscr{F}_n \Omega_\kappa C$. The associated chain complex

$$\operatorname{gr}_n \Omega_\kappa C \cong \sum_{\substack{k \geqslant 1, \\ l+n_1+\cdots+n_k = n}} \operatorname{gr}_l \mathcal{P}(k) \otimes_{\mathbb{S}_k} \operatorname{gr}_{n_1} C \otimes \cdots \otimes \operatorname{gr}_{n_k} C$$

is cohomologically graded by the weight l of the operad $\mathcal{P}$:

$$\operatorname{gr}_n C \xrightarrow{\mathrm{d}} \sum_{\substack{k\geqslant 1,\\ n_1+\cdots+n_k=n-1}} E(k)\otimes_{\mathbb{S}_k} \operatorname{gr}_{n_1} C\otimes\cdots\otimes \operatorname{gr}_{n_k} C \xrightarrow{\mathrm{d}}$$
$$\sum_{\substack{k\geqslant 1,\\ n_1+\cdots+n_k=n-2}} \operatorname{gr}_2 \mathcal{P}(k)\otimes_{\mathbb{S}_k} \operatorname{gr}_{n_1} C\otimes\cdots\otimes \operatorname{gr}_{n_k} C \xrightarrow{\mathrm{d}}$$
$$\cdots \xrightarrow{\mathrm{d}} (\operatorname{gr}_n \mathcal{P})(\operatorname{Prim} C) \xrightarrow{\mathrm{d}} 0.$$

Notice that if one considers the same construction $\operatorname{gr}\,\Omega_\kappa \mathcal{P}^{\mathrm{i}}(V)$ for any the cofree $\mathcal{P}^{\mathrm{i}}$-coalgebra $\mathcal{P}^{\mathrm{i}}(V)$, one gets a (co)chain complex isomorphic to the Koszul complex $q\mathcal{P}\circ_\kappa q\mathcal{P}^{\mathrm{i}}(V)$ of the quadratic analogous operad $q\mathcal{P}$ “with coefficients” in V. Since the Koszul property for the operad $\mathcal{P}$ includes the Koszul property of the quadratic analgue operad $q\mathcal{P}$, this later chain complex is acyclic. Recall that it decomposes as a direct sum of sub-chain complexes labelled by the global weight, so that it is isomorphic and thus quasi-isomorphic to V in weight 0 and its homology groups are all trivial in higher weights.

The morphism of $\mathcal{P}^{\mathrm{i}}$-coalgebras $\Xi : C \to \mathcal{P}^{\mathrm{i}}(\operatorname{Prim} C)$ induces morphisms of (co)chain complexes, which is $\operatorname{gr}_{n+1}\Omega_\kappa\Xi$ on weight $n+1$:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\cong"] & \operatorname{gr}_{n+1} C \quad \displaystyle\sum_{\substack{k\geqslant 1,\\ n_1+\cdots+n_k=n}} E(k)\otimes_{\mathbb{S}_k} \operatorname{gr}_{n_1} C\otimes\cdots\otimes \operatorname{gr}_{n_k} C \arrow[r] \arrow[d, "\operatorname{gr}_{n+1}\Xi"] & \cdots \arrow[d, "\cong"] \\
0 \arrow[r] & \mathcal{P}^{\mathrm{i}\,(n+1)}(\operatorname{Prim} C) D \arrow[r] & \cdots
\end{tikzcd}$$

where

$$D=\sum_{\substack{k\geqslant 1,\\ n_1+\cdots+n_k=n}} E(k)\otimes_{\mathbb{S}_k} \mathcal{P}^{\mathrm{i}\,(n_1)}(\operatorname{Prim} C)\otimes\cdots\otimes \mathcal{P}^{\mathrm{i}\,(n_k)}(\operatorname{Prim} C).$$

The top (co)chain complex is acyclic since it can be written as retract of a similar one for a cofree $\mathcal{P}^{\mathrm{i}}$-coalgebra, the bar-cobar resolution $\mathrm{B}_\kappa\,\Omega_\kappa C$. The bottom one is also acyclic. This allows to conclude that the map $\operatorname{gr}_{n+1}\Xi$ is an isomorphism. □

2.7. Quillen equivalence

Now that Theorem 2.2 (1) is proved, Theorem 2.12 states that the bar-cobar adjunction

$$\mathrm{B}_\kappa \ : \ \mathsf{dg}\ \mathcal{P}\text{-}\mathsf{alg} \ \rightleftharpoons \ \mathsf{conil}\ \mathsf{dg}\ \mathcal{P}^{\text{¡}}\text{-}\mathsf{coalg} \ : \ \Omega_\kappa.$$

forms a Quillen functor. Let us now prove Point (3) of Theorem 2.2: the bar-cobar adjunction is a Quillen equivalence.

Proof of Theorem 2.2 (3). — Recall that any dg $\mathcal{P}$-algebra is fibrant and that any conilpotent dg $\mathcal{P}^{\text{¡}}$-coalgebra is cofibrant, in the respective model category structures considered here. Let A be a dg $\mathcal{P}$-algebra and let C be a conilpotent dg $\mathcal{P}^{\text{¡}}$-coalgebra. We consider two maps

$$f \ : \ \Omega_\kappa C \to A \quad \text{and} \quad g \ : \ C \to \mathrm{B}_\kappa A,$$

which are sent to one another under the bar-cobar adjunction.

If the map f is a quasi-isomorphism of dg $\mathcal{P}$-algebras, then the map $\mathrm{B}_\kappa f$ is a filtered quasi-isomorphism of conilpotent dg $\mathcal{P}^{\text{¡}}$-coalgebras by Proposition 2.7 and so a weak equivalence by Proposition 2.6. Since the map g is equal to the following composite with the unit of adjuction

$$g \ : \ C \xrightarrow{\upsilon_\kappa C} \mathrm{B}_\kappa\, \Omega_\kappa C \xrightarrow{\mathrm{B}_\kappa f} \mathrm{B}_\kappa A,$$

then it is a weak equivalence by Theorem 2.9 (2).

In the other way round, if the map g is a weak equivalence of conilpotent dg $\mathcal{P}^{\text{¡}}$-coalgebras, then the map $\Omega_\kappa g$ is a quasi-isomorphism of dg $\mathcal{P}$-algebras by definition. Since the map f is equal to the following composite with the counit of adjuction

$$f \ : \ \Omega_\kappa C \xrightarrow{\Omega_\kappa g} \Omega_\kappa\, \mathrm{B}_\kappa A \xrightarrow{\varepsilon_\kappa A} A,$$

then it is a quasi-isomorphism by Theorem 2.9 (1). $\square$

Corollary 2.13. — *The induced adjunction*

$$\mathbb{R}\,\mathrm{B}_\kappa \ : \ \mathsf{Ho}(\mathsf{dg}\ \mathcal{P}\text{-}\mathsf{alg}) \ \rightleftharpoons \ \mathsf{Ho}(\mathsf{conil}\ \mathsf{dg}\ \mathcal{P}^{\text{¡}}\text{-}\mathsf{coalg}) \ : \ \mathbb{L}\Omega_\kappa$$

is an equivalence between the homotopy categories.

2.8. Comparison between model category structures on coalgebras

In order to understand the homotopy theory of conilpotent dg $\mathcal{P}^{\text{¡}}$-coalgebras with respect to quasi-isomorphisms, one can endow them with a model category structure.

Theorem 2.14 ([17]). — *The category of bounded below dg $\mathcal{P}^{\text{i}}$-coalgebras with respectively quasi-isomorphisms, degree wise monomorphisms and the induced fibrations forms a model category.*

Remark. — There is a rich literature on model category structures for coalgebras with respect to quasi-isomorphisms, starting from the original work of Quillen [27, 28]. Getzler–Goerss treated the case of non-negatively graded but not necessarily conilpotent coassociative coalgebras in [13]. Aubry–Chataur covered the case of conilpotent coalgebras over a quasi-cofree cooperad in [1]. J.R. Smith worked out in [30] the case of coalgebras over operads satisfying a certain condition (Condition 4.3 in loc. cit.) and with chain homotopy equivalences.

The model category structure with quasi-isomorphisms can be obtained from the present model category structure by means of Bousfield localization.

Proposition 2.15. — *The model category structure on conilpotent dg $\mathcal{P}^{\text{i}}$-coalgebras with quasi-isomorphisms is the left Bousfield localization of the model category structure of Theorem 2.2 with respect to the class of quasi-isomorphisms.*

Proof. — In this proof, we will denote by $\mathsf{coalg}_{\mathsf{we}}$ the category of conilpotent dg $\mathcal{P}^{\text{i}}$-coalgebras equipped with the model category structure of Theorem 2.2 and we will denote by $\mathsf{coalg}_{\mathsf{qi}}$ the same underlying category but equipped with the model category structure with quasi-isomorphisms.

We first prove that the model category $\mathsf{coalg}_{\mathsf{qi}}$ is the localization of the model category $\mathsf{coalg}_{\mathsf{we}}$ with respect to the class of quasi-isomorphisms. Since the class of weak equivalences sits inside the class of quasi-isomorphisms (Proposition 2.8) and since the cofibrations are the same in both model categories, the identity functor $\mathrm{id} : \mathsf{coalg}_{\mathsf{we}} \to \mathsf{coalg}_{\mathsf{qi}}$ is a left Quillen functor. (Its right adjoint is the identity too.) We now show that the identity satisfies the universal property of begin a unital object, see [20, Definition 3.1.1]. Let $F : \mathsf{coalg}_{\mathsf{we}} \leftrightarrows \mathsf{C} : G$ be a Quillen adjunction such that the total left derived functor $\mathbb{L}F$ sends quasi-isomorphisms into isomorphisms in the homotopy category $\mathrm{Ho}(\mathsf{C})$. Obviously, there is a unique way to factor this adjunction by the identity adjunction: $\widetilde{F} : \mathsf{coalg}_{\mathsf{qi}} \leftrightarrows \mathsf{C} : \widetilde{G}$. Since every object in $\mathsf{coalg}_{\mathsf{we}}$ is cofibrant, Theorem 3.1.6 (1)(b) of [20] shows that the functor F sends quasi-isomorphisms of coalgebras into weak equivalences in C. Therefore, the functor $\widetilde{F}$ is a left Quillen functor.

We now prove that this localisation of model categories is a Bousfield localization. For this, it is enough to prove that the class of quasi-isomorphisms of coalgebras is equal to the class of local equivalences with respect to quasi-isomorphisms. By definition, the former is included into the latter. The inclusion in the other way round is provided by Theorem 3.1.6 (1)(d) of [20] applied to the identity Quillen adjunction id : $\mathsf{coalg}_{\mathsf{we}} \to \mathsf{coalg}_{\mathsf{qi}}$. □

From the present study and the Bousfield localization, we can obtain a more precise description of the model category with quasi-isomorphisms of [17].

Corollary 2.16. — *In the model category of conilpotent dg $\mathcal{P}^{\text{i}}$-coalgebras with quasi-isomorphisms, the class of acyclic fibrations is the same as the class of acyclic fibrations in the present model category. Its fibrant objects are the dg $\mathcal{P}^{\text{i}}$-coalgebras isomorphic to quasi-free ones which are local with respect to quasi-isomorphisms.*

Proof. — The first point is a direct corollary of Proposition 2.15 and Proposition 3.3.3 (1)(b) of [20]. The second point follows from the left properness of the present model category (Theorem 2.2 (2)), Proposition 2.15 and Proposition 3.4.1 (1) of [20]. □

We refer the reader to Proposition 4.5 for a complete description of acyclic fibration between quasi-free $\mathcal{P}^{\text{i}}$-coalgebras. For more elaborate results and a full comparison between the possible model category structures on conilpotent dg $\mathcal{P}^{\text{i}}$-coalgebras, we refer the reader to the recent preprint of Drummond-Cole–Hirsh [8].

3. Homotopy theory of infinity-morphisms

The purpose of this section is to apply the previous model category structure on conilpotent dg $\mathcal{P}^{\text{i}}$-coalgebras to get general results about ∞-morphisms. For instance, the model category structure provides us automatically with a good notion of homotopy equivalence between morphisms of fibrant-cofibrant objects, that is a homotopy equivalence between ∞-morphisms of $\mathcal{P}_\infty$-algebras. In this section, we realize this homotopy equivalence with a functorial cylinder object. We also show that this new simple homotopy equivalence is equivalent to "all" the equivalence relations that have been considered so far on ∞-morphisms in the literature. Then, we state and prove one of the main results of this paper: the homotopy category of $\mathcal{P}$-algebras is equivalent to the category of $\mathcal{P}$-algebras with

∞-morphisms up to homotopy equivalence. Finally, we explain how the present model category does not only encaptures this first level homotopical data, but all the higher homotopical properties of $\mathcal{P}_\infty$-algebras. This is achieved by upgrading the category of $\mathcal{P}_\infty$-algebras with ∞-morphism into an ∞-category.

3.1. Functorial cylinder objects

To define functorial cylinder objets in the category of dg $\mathcal{P}^{\text{!`}}$-coalgebras, we consider two algebraic models for the interval, the first one in the category of dg coassociative coalgebras and the second one in the category of dg commutative algebras.

Definition 3.1 (Coassociative model for the interval). — *We consider the cellular chain complex of the interval:*

$I := \mathbb{K}\mathfrak{0} \oplus \mathbb{K}\mathfrak{1} \oplus \mathbb{K}\mathfrak{i}$, *with* $|\mathfrak{0}| = |\mathfrak{1}| = 0,\ |\mathfrak{i}| = 1,\ d(\mathfrak{0}) = d(\mathfrak{1}) = 0,\ d(\mathfrak{i}) = \mathfrak{1} - \mathfrak{0}$.

It is equipped with a dg coassociative coalgebra structure by

$$\Delta(\mathfrak{0}) = \mathfrak{0} \otimes \mathfrak{0},\ \Delta(\mathfrak{1}) = \mathfrak{1} \otimes \mathfrak{1},\ \textit{and}\ \Delta(\mathfrak{i}) = \mathfrak{0} \otimes \mathfrak{i} + \mathfrak{i} \otimes \mathfrak{1}.$$

When $\mathcal{P}$ is a nonsymmetric operad, the tensor product of any conilpotent dg $\mathcal{P}^{\text{!`}}$-coalgebra $(C, d_C, \Delta_C)$ with $I$ provides us with a functorial conilpotent dg $\mathcal{P}^{\text{!`}}$-coalgebra. The arity n component of its structure map is given by

$$C \otimes I \xrightarrow{\Delta_C(n) \otimes \Delta^{n-1}} \mathcal{P}^{\text{!`}}(n) \otimes C^{\otimes n} \otimes I^{\otimes n} \xrightarrow{\cong} \mathcal{P}^{\text{!`}}(n) \otimes (C \otimes I)^{\otimes n}.$$

Proposition 3.2. — *Let $\mathcal{P}$ be a nonsymmetric Koszul operad and let C be a conilpotent dg $\mathcal{P}^{\text{!`}}$-coalgebra. The $\mathcal{P}^{\text{!`}}$-coalgebra $C \otimes I$ provides us with a functorial good cylinder object*

$$C \oplus C \rightarrowtail C \otimes I \xrightarrow{\sim} C$$

in the model category of conilpotent dg $\mathcal{P}^{\text{!`}}$-coalgebras of Theorem 2.2.

Proof. — One can notice that the weight filtration satisfies $F_n(C \otimes I) \cong F_n C \otimes I$. Since the $\mathcal{P}^{\text{!`}}$-coalgebra C is conilpotent, then so is $C \otimes I$. The left-hand map is the embedding $c + c' \mapsto c \otimes \mathfrak{0} + c' \otimes \mathfrak{1}$, hence it is a cofibration. The right-hand map is equal to $c \otimes \mathfrak{0} \mapsto c$, $c \otimes \mathfrak{1} \mapsto c$, and $c \otimes \mathfrak{i} \mapsto 0$. To prove that it is a weak-equivalence, we show that it is a filtered quasi-isomorphism. The graded part of the above morphism of dg $\mathcal{P}^{\text{!`}}$-coalgebras is equal to

$$F_n C / F_{n-1} C \otimes I \to F_n C / F_{n-1} C,$$

with the same kind of formula. So, this is a quasi-isomorphism and we conclude with Proposition 2.6. □

In the symmetric case, the situation is more involved since it is difficult to find a suitable model for the interval in the category of dg cocommutative coalgebras equipped with two different group-like elements. Instead, we proceed as follows.

Definition 3.3 (Commutative model for the interval). — *Let* $\Lambda(t,dt) := \mathbb{K}[t] \oplus \mathbb{K}[t]dt$ *be the dg commutative algebra made up of the polynomial differential forms on the interval. The degree is defined by* $|t| = 0$, $|dt| = -1$ *and the differential is the unique derivation extending* $\mathrm{d}(t) = dt$ *and* $\mathrm{d}(dt) = 0$. *This* dg commutative algebra model for the interval $(\Lambda(t, dt), \mathrm{d})$ *is called the* Sullivan algebra.

Let $\mathcal{P}$ be a Koszul operad and let (A, μ) be a $\mathcal{P}_\infty$-algebra. The tensor product $A \otimes \Lambda(t, dt)$ inherits a natural $\mathcal{P}_\infty$-algebra, given by

$$\widetilde{\mu} \ : \ \mathcal{P}^{\mathrm{i}} \cong \mathcal{P}^{\mathrm{i}} \otimes \mathrm{Com} \xrightarrow{\mu\otimes\nu} \mathrm{End}_A \otimes \mathrm{End}_{\Lambda(t,dt)} \cong \mathrm{End}_{A\otimes\Lambda(t,dt)},$$

where Com denotes the operad of commutative algebras, whose arity-wise components are one-dimensional, and where ν denotes the commutative algebra structure on $\Lambda(t, dt)$. We consider the cellular chain complex of the interval I^*, which is isomorphic to the sub-complex of $\Lambda(t, dt)$ made up of $J := \mathbb{K}1 \oplus \mathbb{K}t \oplus \mathbb{K}dt$ under the identification $\mathfrak{0}^* + \mathfrak{1}^* \leftrightarrow 1$, $\mathfrak{1}^* \leftrightarrow t$, and $\mathfrak{i}^* \leftrightarrow dt$. This latter chain complex is a deformation retract of the polynomial differential forms on the interval; a particularly elegant contraction was given by J. Dupont in his proof of the de Rham theorem [9], see also [4, 12].

Definition 3.4 (Dupont's contraction). — *The Dupont's contraction amounts to the following deformation retract:*

$$\begin{tikzcd} (\Lambda(t, dt), \mathrm{d}) \arrow[loop left, "\mathrm{h}"] \arrow[r, shift left, "\mathrm{p}"] & (J, \mathrm{d}) \arrow[l, shift left, "\mathrm{i}"] \end{tikzcd}$$

$$\mathrm{h}(t^k dt) := \frac{t^{k+1} - t}{k+1}, \ \mathrm{h}(t^k) := 0 \quad \textit{and}$$

$$\mathrm{p}(t^k dt) := \frac{1}{k+1} dt, \ \mathrm{p}(1) := 1, \ \mathrm{p}(t^k) = t \quad \textit{for} \quad k \geqslant 1.$$

We now consider the contraction $\mathrm{id}_A \otimes h$ on $A \otimes \Lambda(t, dt)$ and then the induced $\mathcal{P}_\infty$-algebra structure $\hat{\mu}$ on $A \otimes J$ obtained by applying the Homotopy Transfer Theorem [24, Theorem 10.3.1], see also [7, Section 8] for more insight. This transferred $\mathcal{P}_\infty$-algebra structure satisfies the following properties.

LEMMA 3.5. — *The $\mathcal{P}_\infty$-algebra structure $\hat{\mu}$ on $A \otimes J$ satisfies*

$$\hat{\mu}(p; a_1 \otimes b_1, \ldots, a_n \otimes b_n) = \mu(p; a_1, \ldots, a_n) \otimes \mathrm{p}(b_1 \cdots b_n),$$

for any $p \in \mathcal{P}^{¡}(n)$, $a_1, \ldots, a_n \in A$ when $b_1, \ldots, b_n = 1$ or t and

$$\hat{\mu}(p; a_1 \otimes b_1, \ldots, a_i \otimes dt, \ldots, a_n \otimes b_n) \in A \otimes \mathbb{K}dt,$$

for any $p \in \mathcal{P}^{¡}(n)$, $a_1, \ldots, a_n \in A$, and $b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n \in J$.

Proof of Lemma 3.5. — The formula for the Homotopy Transfer Theorem given in [24, Theorem 10.3.3] is a sum of terms where one has always to apply the homotopy h to some of the b_i's, except for the term $\mu(p; a_1, \ldots, a_n) \otimes \mathrm{p}(b_1 \cdots b_n)$. Since h applied to t^k gives 0, when all the b_i's are equal to 1 or t, only remains the last term.

When one b_i is equal to dt, the last term is of the form $\mu(p; a_1, \ldots, a_n) \otimes P(t)dt$, with $P(t) \in \mathbb{K}[t]$. Each other term involves applying at least one homotopy h above the root vertex. Therefore the upshot is of the form

$$a \otimes \mathrm{p}\left(\sum_{l=1}^{L} P_l(t)(t^{k_l+1} - t)\right),$$

with $a \in A$ and $P_l(t) \in \mathbb{K}[t]$. Since $\mathrm{p}\big(t^m(t^{k+1} - t)\big) = 0$, all these terms vanish and the second formula is proved. □

Finally, this produces the required cylinder for quasi-free dg $\mathcal{P}^{¡}$-coalgebras.

PROPOSITION 3.6. — *Let $\mathcal{P}$ be a Koszul operad and let $(\mathcal{P}^{¡}(A), d_\mu)$ be a quasi-free dg $\mathcal{P}^{¡}$-coalgebras. The $\mathcal{P}^{¡}$-coalgebra $(\mathcal{P}^{¡}(A \otimes J), d_{\bar{\mu}})$ provides us with a functorial good cylinder object*

$$\mathcal{P}^{¡}(A) \oplus \mathcal{P}^{¡}(A) \rightarrowtail \mathcal{P}^{¡}(A \otimes J) \xrightarrow{\sim} \mathcal{P}^{¡}(A)$$

in the model category of conilpotent dg $\mathcal{P}^{¡}$-coalgebras of Theorem 2.2.

Proof. — For the left-hand map, we consider the embedding $i_0 + i_1 \ : a + b \mapsto a \otimes 1 + b \otimes t$, which extends to the following unique morphism of $\mathcal{P}^{¡}$-coalgebras:

$$\begin{aligned}\mathcal{P}^{¡}(i_0) + \mathcal{P}^{¡}(i_1) \ : \ & p(a_1, \ldots, a_n) + q(b_1, \ldots, b_n) \\ & \mapsto p(a_1 \otimes 1, \ldots, a_n \otimes 1) + q(b_1 \otimes t, \ldots, b_n \otimes t),\end{aligned}$$

for $p, q \in \mathcal{P}^{¡}(n)$ and for $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$. It commutes with the respective differentials since the following diagram, where the middle map is $\mathcal{P}^{¡}\big(\mathcal{P}^{¡}(i_0);i_0\big)$, and the other similar one with i_1, are commutative

$$\begin{tikzcd}
\mathcal{P}^{¡}(A) \arrow[rrr, "\mathcal{P}^{¡}(i_0)"] \arrow[dr, "\Delta_{(1)}(A)"] \arrow[dd, "d_\mu"'] & & & \mathcal{P}^{¡}(A\otimes \mathbb{K}1) \arrow[dl, "\Delta_{(1)}(A\otimes\mathbb{K}1)"'] \arrow[dd, "d_{\hat\mu}"] \\
& \mathcal{P}^{¡}\big(\mathcal{P}^{¡}(A);A\big) \arrow[r] \arrow[dl, "\mathcal{P}^{¡}(\mu_A;A)"] & \mathcal{P}^{¡}\big(\mathcal{P}^{¡}(A\otimes \mathbb{K}1);A\otimes \mathbb{K}1\big) \arrow[dr, "\mathcal{P}^{¡}(\hat\mu_{A\otimes\mathbb{K}1};A\otimes\mathbb{K}1)"'] & \\
\mathcal{P}^{¡}(A) \arrow[rrr, "\mathcal{P}^{¡}(i_0)"] & & & \mathcal{P}^{¡}(A\otimes \mathbb{K}1)
\end{tikzcd}$$

by Lemma 3.5, where μ_A (respectively $\hat\mu_{A\otimes\mathbb{K}1}$) denotes the map $\mathcal{P}^{¡}(A)\to A$ induced from $\mu : \mathcal{P}^{¡}\to \mathrm{End}_A$. It is clearly a degreewise monomorphism, and so a cofibration.

The second map $\mathcal{P}^{¡}(A\otimes J)\to\mathcal{P}^{¡}(A)$ is the unique morphism $\mathcal{P}^{¡}(j)$ of $\mathcal{P}^{¡}$-coalgebras which extends the map $j : A\otimes J\to A$, defined by $j(a\otimes 1)=j(a\otimes t)=a$ and $j(a\otimes dt)=0$. It commutes with the respective differentials since the following diagram, where the middle map is $\mathcal{P}^{¡}\big(\mathcal{P}^{¡}(j);j\big)$, is commutative

$$\begin{tikzcd}
\mathcal{P}^{¡}(A\otimes J) \arrow[rrr, "\mathcal{P}^{¡}(j)"] \arrow[dr, "\Delta_{(1)}(A\otimes J)"] \arrow[dd, "d_{\hat\mu}"'] & & & \mathcal{P}^{¡}(A) \arrow[dl, "\Delta_{(1)}(A)"'] \arrow[dd, "d_\mu"] \\
& \mathcal{P}^{¡}\big(\mathcal{P}^{¡}(A\otimes J);A\otimes J\big) \arrow[r] \arrow[dl, "\mathcal{P}^{¡}(\hat\mu_{A\otimes J};A\otimes J)"] & \mathcal{P}^{¡}\big(\mathcal{P}^{¡}(A);A\big) \arrow[dr, "\mathcal{P}^{¡}(\mu_A;A)"'] & \\
\mathcal{P}^{¡}(A\otimes J) \arrow[rrr, "\mathcal{P}^{¡}(j)"] & & & \mathcal{P}^{¡}(A)
\end{tikzcd}$$

by Lemma 3.5. Notice the map $j : A\otimes J\to A$ is a quasi-isomorphism. We consider the canonical weight filtrations on the quasi-free dg $\mathcal{P}^{¡}$-coalgebras $\mathcal{P}^{¡}(A\otimes J)$ and $\mathcal{P}^{¡}(A)$. They induce a filtered quasi-isomorphism and thus a weak equivalence by Proposition 2.6.

The data of a morphism $F : (\mathcal{P}^{¡}(A), d_\mu)\to(\mathcal{P}^{¡}(B), d_\omega)$ between two quasi-free dg $\mathcal{P}^{¡}$-coalgebras is equivalent to the data of an ∞-morphism $f : (A,\mu)\rightsquigarrow(B,\omega)$ between the two associated $\mathcal{P}_\infty$-algebras. The Homotopy

Transfer Theorem [24, Theorem 10.3.3] produces the following natural ∞-morphisms

$$\begin{tikzcd}
(A \otimes \Lambda(t,dt), \widetilde{\mu}) \arrow[d, squiggly, "f\otimes\nu"] & (A \otimes J, \hat{\mu}) \arrow[l, squiggly, "(A\otimes \mathrm{i})_\infty"'] \arrow[d, squiggly] \\
(B \otimes \Lambda(t,dt), \widetilde{\omega}) \arrow[r, squiggly, "(B\otimes \mathrm{p})_\infty"] & (B \otimes J, \hat{\omega})
\end{tikzcd} ,$$

whose composition gives the functoriality of the present cylinder object. $\square$

3.2. Homotopy equivalence of ∞-morphisms

The ultimate goal of these two sections, is to provide the present theory with a suitable explicit notion of homotopy equivalence of ∞-morphisms, which allows us to obtain a simple description of the homotopy category of $\mathcal{P}$-algebras, for instance.

Definition 3.7 (Homotopy relation). — *Two morphisms $f, g : C \to D$ of conilpotent dg $\mathcal{P}^{¡}$-coalgebras are* homotopic *if there exists a morphism h of conilpotent dg $\mathcal{P}^{¡}$-coalgebras fulling the commutative diagram*

$$\begin{tikzcd}
C \arrow[r, "j_0"] \arrow[dr, "f"'] & \mathrm{Cyl(C)} \arrow[d, dotted, "h"] & C \arrow[l, "j_1"'] \arrow[dl, "g"] \\
 & D, &
\end{tikzcd}$$

where $\mathrm{Cyl}(C)$ *is a cylinder for* C.

Proposition 3.8. — *This homotopy relation of morphisms of conilpotent dg $\mathcal{P}^{¡}$-coalgebras is an equivalence relation.*

Proof. — This a direct consequence of the general theory of model categories [21, Proposition 1.2.5] since every object of the present category is cofibrant by Theorem 2.2 (2). $\square$

This homotopy relation restricts naturally to maps between fibrant-cofibrant objects, that is to ∞-morphisms of $\mathcal{P}_\infty$-algebras. In this case, we can use the small cylinder given in Proposition 3.6 for instance. In the recent paper [6], V. Dotsenko and N. Poncin study several equivalence relations for ∞-morphisms and they prove that they are all equivalent. The next proposition shows that these equivalence relations are actually homotopy equivalences in the sense of the present model category of conilpotent dg $\mathcal{P}^{¡}$-coalgebras.

Proposition 3.9. — *All the equivalence relations for ∞-morphisms of* [6] *are equivalent to the above homotopy relation.*

Proof. — It is enough to prove that

$$\mathcal{P}^{\text{¡}}(A) \xrightarrow{\sim} \mathcal{P}^{\text{¡}}\big(A \otimes \Lambda(t, dt)\big) \twoheadrightarrow \mathcal{P}^{\text{¡}}(A \oplus A)$$

is a good path object in the model category of conilpotent dg $\mathcal{P}^{\text{¡}}$-coalgebras of Theorem 2.2. Then, by the general theory of model categories, the associated right homotopy equivalence will be an equivalence relation, equivalent to the above left homotopy relation defined by the above good cylinder object. Finally, one can notice that this new equivalence is nothing but the one called concordance in [6, Definition 3], which is proved in loc. cit. to be equivalent to the other ones.

This statement is proved in the same way as Proposition 3.6, so we will only give the various arguments and constructions. We first notice that the right-hand term is the categorical product of the dg $\mathcal{P}^{\text{¡}}$-coalgebra $(\mathcal{P}^{\text{¡}}(A), d_\mu)$ with itself; the details can be found in the proof of Axiom (MC 1$'$) of Theorem 4.2 (1) in the sequel. The first map is defined by the unique morphism of $\mathcal{P}^{\text{¡}}$-coalgebras extending $a \mapsto a\otimes 1$. It is straightforward to check that it commutes with the differentials. Since it is the extension of a quasi-isomorphism $A \xrightarrow{\sim} A \otimes \Lambda(t, dt)$, it is a filtered quasi-isomorphism and therefore a weak equivalence of conilpotent dg $\mathcal{P}^{\text{¡}}$-coalgebras. The second map is defined by the unique morphism of $\mathcal{P}^{\text{¡}}$-coalgebras extending $a \otimes (P(t) + Q(t)dt) \mapsto a \otimes P(0) + a \otimes P(1)$. It is again straightforward to check that it commutes with the differentials. To prove that it forms a fibration, we use the characterization of fibrations between quasi-free dg $\mathcal{P}^{\text{¡}}$-coalgebras given in Proposition 4.5 (and whose proof does not depend on the present result). Since the map $A \otimes \Lambda(t, dt) \twoheadrightarrow A \oplus A$ is a degreewise epimorphism, the map $\mathcal{P}^{\text{¡}}\big(A \otimes \Lambda(t, dt)\big) \twoheadrightarrow \mathcal{P}^{\text{¡}}(A \oplus A)$ is a fibration. Finally, we check that the composite of these two maps is equal to the product of the identity with itself, which concludes the proof. □

The following result refines Theorem 1.2: it gives a finer control of the “inverse” ∞-quasi-isomorphism.

Theorem 3.10. — *Any ∞-quasi-isomorphism admits a homotopy inverse.*

Proof. — By Proposition 11.4.7 of [24], any ∞-quasi-isomorphism $f : A \overset{\sim}{\rightsquigarrow} A'$ induces a weak equivalence

$$\widetilde{\mathrm{B}}_\iota f \,:\, \widetilde{\mathrm{B}}_\iota A \xrightarrow{\sim} \widetilde{\mathrm{B}}_\iota A'$$

between fibrant-cofibrant conilpotent dg $\mathcal{P}^{\text{i}}$-coalgebras. By the general model category arguments, this latter one admits a homotopy inverse, which translated back on the level of $\mathcal{P}_\infty$-algebras gives the result. □

The other general consequence of the above mentioned theory is the following description of the homotopy category of dg $\mathcal{P}$-algebras, defined as a localized category, as the category of $\mathcal{P}$-algebras with ∞-morphism up to homotopy equivalence.

Theorem 3.11. — *The following categories are equivalent*

$$\mathsf{Ho}(\mathsf{dg}\ \mathcal{P}\text{-}\mathsf{alg}) \ \simeq\ \mathsf{Ho}(\infty\text{-}\mathcal{P}_\infty\text{-}\mathsf{alg}) \ \simeq\ \infty\text{-}\mathcal{P}_\infty\text{-}\mathsf{alg}/\!\sim_h\ \simeq\ \infty\text{-}\mathcal{P}\text{-}\mathsf{alg}/\!\sim_h.$$

Proof. — By Corollary 2.13 of Theorem 2.2 (3), the total derived functors

$$\mathbb{R}\,\mathrm{B}_\kappa\ :\ \mathsf{Ho}(\mathsf{dg}\ \mathcal{P}\text{-}\mathsf{alg}) \ \overset{\cong}{\rightleftharpoons}\ \mathsf{Ho}(\mathsf{conil\ dg}\ \mathcal{P}^{\text{i}}\text{-}\mathsf{coalg})\ :\ \mathbb{L}\Omega_\kappa$$

form an equivalence of categories. By the general arguments of model categories, the right-hand category is equivalent to the category of fibrant-cofibrant conilpotent dg $\mathcal{P}^{\text{i}}$-coalgebras modulo the homotopy relation. So by Theorem 2.2 (2), we get the following equivalence of categories

$$\mathsf{Ho}(\mathsf{conil\ dg}\ \mathcal{P}^{\text{i}}\text{-}\mathsf{coalg}) \overset{\cong}{\rightleftharpoons} \mathsf{quasi\text{-}free}\ \mathcal{P}^{\text{i}}\text{-}\mathsf{coalg}/\!\sim_h.$$

We then use the equivalence between the category of quasi-free $\mathcal{P}^{\text{i}}$-coalgebras and the category of $\mathcal{P}_\infty$-algebras with their ∞-morphisms. Finally, we conclude with Theorem 1.3, which shows that any $\mathcal{P}_\infty$-algebra can be rectified into a dg $\mathcal{P}$-algebra:

$$\infty\text{-}\mathcal{P}_\infty\text{-}\mathsf{alg}/\!\sim_h\ \overset{\cong}{\rightleftharpoons}\ \infty\text{-}\mathcal{P}\text{-}\mathsf{alg}/\!\sim_h. \qquad □$$

Using the explicit homotopy relation defined by the cylinder object associated to I, this recovers the classical homotopy relation of A_∞-morphisms and the description of the homotopy category for *unbounded* dg associative algebras, see [23, 26].

3.3. An ∞-category enrichment of homotopy algebras

The previous result deals with the homotopy category of $\mathcal{P}_\infty$-algebras, which is only the first homotopical level of information. At the present stage of the theory, we have objects (the $\mathcal{P}_\infty$-algebras), 1-morphisms (the ∞-morphisms) and 2-morphisms (the homotopy relation). However, one can go further, thanks to the model category structure established in the previous section, and prove that the category of $\mathcal{P}_\infty$-algebras actually extends to an ∞-category.

THEOREM 3.12. — *The category* ∞-$\mathcal{P}_\infty$-alg *of* $\mathcal{P}_\infty$*-algebras with* ∞*-morphisms extends to a simplicial category giving the same underlying homotopy category.*

Proof. — This is a direct application of the simplicial localization methods of Dwyer–Kan [10]. □

During the preparation of this paper, Dolgushev–Hoffnung–Rogers [5] used the integration theory of L_∞-algebras [12, 18] to endow the category of $\mathcal{P}_\infty$-algebras with another model of ∞-category, that we present now using the langage of the present paper. Let us denote by $\Omega_\bullet$ the simplicial dg commutative algebra made up of the polynomial differential forms on the geometric simplicies, for instance $\Omega_1 = \Lambda(t, dt)$. Given a $\mathcal{P}_\infty$-algebra (A, μ), the associated quasi-free dg $\mathcal{P}^{\text{!`}}$-coalgebra $(\mathcal{P}^{\text{!`}}(A), d_\mu)$ admits the following simplicial resolution $(\mathcal{P}^{\text{!`}}(A\otimes\Omega_\bullet), d_{\widetilde{\mu}})$, where $(A\otimes\Omega_\bullet, \widetilde{\mu})$ is the simplicial $\mathcal{P}_\infty$-algebra obtained by tensoring with the various commutative algebras. Then, the simplicial enrichment

$$\mathrm{Hom}_{\mathsf{conil\,dg}\,\mathcal{P}^{\text{!`}}\text{-}\mathsf{coalg}}\Big((\mathcal{P}^{\text{!`}}(A), d_\mu), (\mathcal{P}^{\text{!`}}(B\otimes\Omega_\bullet), d_{\tilde{\omega}})\Big)$$

coincides with Dolgushev–Hoffnung–Rogers simplicial enrichment; this can be proved using the methods recently developed by Robert-Nicoud in [29]. Since the simplicial category of Theorem 3.12 is produced out of a cylinder whose definition comes from a homotopy equivalent form of Ω_1, one expects that these two simplicial categories of $\mathcal{P}_\infty$-algebras are weakly equivalent.

Remark. — One can also notice that Hinich introduces another type of simplicial enrichment in the case of dg cocommutative colagebras by extending the ground ring to Ω_n. One can perform the same kind of simplicial enrichment here, which heuristically should produce again a weakly equivalent simplicial category, since the objects involved are the same. However the details of such a non-trivial result goes beyond the scope of the present paper.

4. Homotopy algebras

The model category structure for algebras over an operad of Theorem 1.5 applies as well to the category $\mathcal{P}_\infty$-alg of $\mathcal{P}_\infty$-algebras with their *strict* morphisms. But if we consider the category ∞-$\mathcal{P}_\infty$-alg of $\mathcal{P}_\infty$-algebras with their ∞-morphisms, then it cannot admit a model category structure strictly speaking since it lacks some colimits like coproducts.

With the abovementioned isomorphism between the category of quasi-free $\mathcal{P}^{\mathsf{i}}$-coalgebras and the category of $\mathcal{P}_\infty$-algebras with their ∞-morphisms, Theorem 2.2 shows that the category ∞-$\mathcal{P}_\infty$-alg is endowed with a *fibrant objects category structure*, see [2] for the definition. Such a notion is defined by two classes of maps: the weak equivalences and the fibrations. Notice that the fibrations of $\mathcal{P}^{\mathsf{i}}$-coalgebras has not been made explicit so far.

In this section, we refine this result: we provide the category ∞-$\mathcal{P}_\infty$-alg of homotopy $\mathcal{P}$-algebras and their ∞-morphisms with almost a model category structure, only the first axiom on limits and colimits is not completely fulfilled. As a consequence, this will allow us to describe the fibrations between quasi-free $\mathcal{P}^{\mathsf{i}}$-coalgebras.

4.1. Almost a model category

Definition 4.1. — *In the category of homotopy $\mathcal{P}$-algebras with their ∞-morphisms, we consider the following three classes of morphisms.*

- *The class $\mathfrak{W}$ of* weak equivalences *is given by the ∞-quasi-isomorphisms $f : A \overset{\sim}{\rightsquigarrow} A'$, i.e. the ∞-morphisms whose first component $f_{(0)} : A \xrightarrow{\sim} A'$ is a quasi-isomorphism;*
- *the class $\mathfrak{C}$ of* cofibrations *is given by the ∞-monomorphisms $f : A \rightsquigarrow A'$, i.e. the ∞-morphisms whose first component $f_{(0)} : A \rightarrowtail A'$ is a monomorphism;*
- *the class $\mathfrak{F}$ of* fibrations *is given by ∞-epimorphisms $f : A \rightsquigarrow A'$, i.e. the ∞-morphisms whose first component $f_{(0)} : A \twoheadrightarrow A'$ is a epimorphism;*

Theorem 4.2.

(1) *The category ∞-$\mathcal{P}_\infty$-alg of $\mathcal{P}_\infty$-algebras with their ∞-morphisms, endowed with the three classes of maps $\mathfrak{W}$, $\mathfrak{C}$, and $\mathfrak{F}$, satisfies the axioms* (MC 2)–(MC 5) *of model categories and the following axiom.*

(MC 1′). — This category admits finite products and pullbacks of fibrations.

(2) *Every $\mathcal{P}_\infty$-algebra is fibrant and cofibrant.*

Recall that a category admits finite colimits if and only if it admits finite coproducts and coequalizers. The present category lacks coproducts. It is enough to consider the two dimension 1 trivial A_∞-algebras $\bar{T}^c(sx)$ and $\bar{T}^c(sy)$, viewed as quasi-free coassocaitive coalgebras, and to see that they do not admit coproducts in that category. The situation about equalizers and coequalizers is more subtle and requires further studies.

4.2. Properties of ∞-morphisms

The proof of Theorem 4.2 relies on the algebraic properties of ∞-morphism given in this section and on the obstruction theory developped in Appendix A.

Recall from [24] that the dg module $\mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{¡}, \mathrm{End}_A)$ is endowed with a preLie product defined by

$$f \star g := \gamma_{\mathcal{P}} \circ \big(f \otimes g\big) \circ \Delta_{(1)},$$

where $\Delta_{(1)} : \mathcal{P}^{¡} \to \mathcal{T}(\mathcal{P}^{¡})^{(2)}$ is the partial decomposition map of the cooperad $\mathcal{P}^{¡}$. This preLie product induces the Lie bracket of Section 1.3 by antisymetrization $[f, g] := f \star g - (-1)^{|f||g|} g \star f$. So the Maurer–Cartan equation encoding $\mathcal{P}_\infty$-algebra structures is equivalently written

$$\partial(\alpha) + \alpha \star \alpha = 0.$$

We consider the dg $\mathbb{S}$-module End_B^A defined by

$$\mathrm{End}_B^A := \big(\{\mathrm{Hom}(A^{\otimes n}, B)\}_{n\in\mathbb{N}}, \partial_B^A\big).$$

Let $\mu \in \mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{¡}, \mathrm{End}_A)$, $\nu \in \mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{¡}, \mathrm{End}_B)$ and $f \in \mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{¡}, \mathrm{End}_B^A)$. We consider the following two operations:

$$f * \mu := \mathcal{P}^{¡} \xrightarrow{\Delta_{(1)}} \mathcal{P}^{¡} \circ_{(1)} \mathcal{P}^{¡} \xrightarrow{f \circ_{(1)} \mu} \mathrm{End}_B^A \circ_{(1)} \mathrm{End}_A \to \mathrm{End}_B^A,$$

$$\nu \circledast f := \mathcal{P}^{¡} \xrightarrow{\Delta} \mathcal{P}^{¡} \circ \mathcal{P}^{¡} \xrightarrow{\nu \circ f} \mathrm{End}_B \circ \mathrm{End}_B^A \to \mathrm{End}_B^A,$$

where the right-hand maps is the usual composite of functions.

Theorem 4.3 ([24, Theorem 10.2.3]). — *Let (A, d_A, μ) and (B, d_B, ν) be two $\mathcal{P}_\infty$-algebras. An ∞-morphism $F : A \rightsquigarrow B$ of $\mathcal{P}_\infty$-algebras is equivalent to a morphism of dg $\mathbb{S}$-modules $f : \mathcal{P}^{¡} \to \mathrm{End}_B^A$ satisfying*

$$\partial(f) = f * \mu - \nu \circledast f \tag{4.1}$$

in $\mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{¡}, \mathrm{End}_B^A)$.

Using this equivalent definition of ∞-morphisms, the composite of $f : A \rightsquigarrow B$ with $g : B \rightsquigarrow C$ is given by

$$g \circledcirc f := \mathcal{P}^{¡} \xrightarrow{\Delta} \mathcal{P}^{¡} \circ \mathcal{P}^{¡} \xrightarrow{g \circ f} \mathrm{End}_C^B \circ \mathrm{End}_B^A \to \mathrm{End}_C^A.$$

Notice that the product $\circledcirc$ is associative and left linear.

For any element $\mu \in \mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{¡}, \mathrm{End}_A)$ and any element $f \in \mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{¡}, \mathrm{End}^A_B)$, we denote by

$$\mu_{(n)} \in \mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{¡\,(n)}, \mathrm{End}_A) \text{ and by } f_{(n)} \in \mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{¡\,(n)}, \mathrm{End}^A_B)$$

the respective restrictions to the weight n part $\mathcal{P}^{¡\,(n)}$ of the cooperad $\mathcal{P}^{¡}$. Recall that an ∞-morphism $f : A \rightsquigarrow B$ of $\mathcal{P}_\infty$-algebras is a (strict) morphism of $\mathcal{P}_\infty$-algebras if and only if its higher components $f_{(n)} = 0$ vanish for $n \geqslant 1$.

Proposition 4.4.

(1) *Let $f : (A, d_A, \mu) \rightsquigarrow (B, d_B, \nu)$ be an ∞-monomorphism. There exists a $\mathcal{P}_\infty$-algebra structure (B, d_B, ξ) and an ∞-isomorphism $g : (B, d_B, \nu) \rightsquigarrow (B, d_B, \xi)$ with first component $g_{(0)} = \mathrm{id}_B$ such that the composite gf is a (strict) morphism of $\mathcal{P}_\infty$-algebras equal to $f_{(0)}$.*

(2) *Let $g : (A, d_A, \mu) \rightsquigarrow (B, d_B, \nu)$ be an ∞-epimorphism. There exists a $\mathcal{P}_\infty$-algebra structure (A, d_A, ω) and an ∞-isomorphism $f : (A, d_A, \omega) \rightsquigarrow (A, d_A, \mu)$ with first component $f_{(0)} = \mathrm{id}_A$ such that the composite gf is a (strict) morphism of $\mathcal{P}_\infty$-algebras equal to $g_{(0)}$.*

Proof.

(1). — Since the map $f_{(0)} : A \rightarrowtail B$ is a monomorphism of graded modules, it admits a retraction $r : B \twoheadrightarrow A$, such that $rf_{(0)} = \mathrm{id}_A$. We define a series of linear maps $g_{(n)} : \mathcal{P}^{¡\,(n)} \to \mathrm{End}_B$ by induction as follows. Let $g_{(0)}$ be equal to $\mathcal{P}^{¡\,(0)} = \mathrm{I} \mapsto \mathrm{id}_B$. Suppose the maps $g_{(k)}$ constructed up to $k = n-1$, we define the map $g_{(n)}$ by the formula

$$g_{(n)} := -\sum_{k=0}^{n-1} g_{(k)} \circledcirc (r^* f),$$

where the map $r^* f$ is equal to the composite

$$r^* f : \mathcal{P}^{¡}(n) \xrightarrow{f} \mathrm{Hom}(A^{\otimes n}, B) \xrightarrow{(r^{\otimes n})^*} \mathrm{Hom}(B^{\otimes n}, B).$$

So for $n \geqslant 1$, the weight n part of the composite gf is equal to

$$\begin{aligned}(gf)_{(n)} = \sum_{k=0}^{n} g_{(k)} \circledcirc f &= \sum_{k=0}^{n-1} g_{(k)} \circledcirc f + g_{(n)} \circledcirc f_{(0)} \\ &= \sum_{k=0}^{n-1} \big(g_{(k)} \circledcirc f - g_{(k)} \circledcirc (\underbrace{r^* f \circledcirc f_{(0)}}_{= f \circledcirc (r f_{(0)}) = f}) \big) = 0.\end{aligned}$$

Since the image of $g_{(0)}$ is an invertible map, the full map $g \in \mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{¡}, \mathrm{End}_B)$ induces an isomorphism $G : \mathcal{P}^{¡}(B) \to \mathcal{P}^{¡}(B)$ of $\mathcal{P}^{¡}$-coalgebras, with inverse G^{-1} given by the formulae of [24, Theorem 10.4.1]. Let d_ν denote the codifferential of $\mathcal{P}^{¡}(B)$ corresponding to the $\mathcal{P}_\infty$-algebra structure $\nu \in \mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{¡}, \mathrm{End}_B)$ on B. We consider the square-zero degree -1 map on $\mathcal{P}^{¡}(B)$ given by $d_\xi := G d_\nu G^{-1}$. The following commutative diagram shows that d_ξ is a coderivation.

$$\begin{tikzcd}
\mathcal{P}^{¡}(B) \arrow[r, "G^{-1}"] \arrow[d, "\Delta(B)"] \arrow[rrr, bend left, "d_\xi"] & \mathcal{P}^{¡}(B) \arrow[r, "d_\nu"] \arrow[d, "\Delta(B)"] & \mathcal{P}^{¡}(B) \arrow[r, "G"] \arrow[d, "\Delta(B)"] & \mathcal{P}^{¡}(B) \arrow[d, "\Delta(B)"] \\
\mathcal{P}^{¡} \circ \mathcal{P}^{¡}(B) \arrow[r, "\mathcal{P}^{¡} \circ G^{-1}"] \arrow[rrr, bend right, "\mathcal{P}^{¡} \circ' d_\xi"'] & \mathcal{P}^{¡} \circ \mathcal{P}^{¡}(B) \arrow[r, "\mathcal{P}^{¡} \circ' d_\nu"] & \mathcal{P}^{¡} \circ \mathcal{P}^{¡}(B) \arrow[r, "\mathcal{P}^{¡} \circ G"] & \mathcal{P}^{¡} \circ \mathcal{P}^{¡}(B).
\end{tikzcd}$$

So it defines another $\mathcal{P}_\infty$-algebra structure (B, d_B, ξ) on the underlying chain complex B, such that the map $g : (B, d_B, \nu) \rightsquigarrow (B, d_B, \xi)$ becomes the required ∞-isomorphism. This concludes the proof.

(2). — The second point is shown by the same kind of arguments, where one has to use a splitting of the epimorphism $g_{(0)} : A \twoheadrightarrow B$ this time. $\square$

4.3. Proof of Theorem 4.2

Proof of Theorem 4.2(1).

(MC 1′) *(Finite products and pullbacks of fibrations).* — This is a direct corollary of the fibrant objects category structure [2]. Let us first make the product construction explicit. Let (A, d_A, μ) and (B, d_B, ν) be two $\mathcal{P}_\infty$-algebras. Their product is given by $A \oplus B$ with $\mathcal{P}_\infty$-algebra structure:

$$\mathcal{P}^{¡} \xrightarrow{\mu + \nu} \mathrm{End}_A \oplus \mathrm{End}_B \to \mathrm{End}_{A \oplus B}.$$

The structures maps are the classical projections $A \oplus B \twoheadrightarrow A$ and $A \oplus B \twoheadrightarrow B$. Any pair $C \overset{f}{\rightsquigarrow} A$ and $C \overset{f}{\rightsquigarrow} B$ of ∞-morphisms extend to the following unique ∞-morphism:

$$\mathcal{P}^{¡} \xrightarrow{f+g} \mathrm{End}^C_A \oplus \mathrm{End}^C_B \to \mathrm{End}^C_{A \oplus B}.$$

The pullbacks of fibrations is given as follows. Let us now consider a third $\mathcal{P}_\infty$-algebra (C, d_C, ω) together with two ∞-morphisms $f : A \rightsquigarrow C$ and $g : B \rightsquigarrow C$. Requiring that this latter ∞-morphism is a fibration amounts to requiring that its first component $g_{(0)} : B \twoheadrightarrow C$ is an epimorphism of chain complexes. Notice that using Proposition 4.4, it is enough to do the case where g is a strict morphism, that is when $g_{(0)}$ is its unique non-trivial component. We denote by $s : C \to B$ a section of $g_{(0)}$ as a morphism of graded vector spaces, that is $g_{(0)}s = \mathrm{id}_C$. In the underlying category of graded vector spaces, the pullback is given by the following diagram

$$(*)\qquad \begin{tikzcd} A \oplus \ker g_{(0)} \arrow[r, "sf_{(0)}+i"] \arrow[d, "p"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, two heads, "g_{(0)}"] \\ A \arrow[r, "f_{(0)}"] & C, \end{tikzcd}$$

where $p : A \oplus \ker g_{(0)} \twoheadrightarrow A$ denotes the canonical projection and where $i : \ker g_{(0)} \rightarrowtail B$ denotes the canonical inclusion. We consider the following two elements φ, ψ in $\mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{¡}, \mathrm{End}_{A\oplus B})$ defined respectively by

$$\varphi := \mathrm{id}_A + \mathrm{id}_B + s_* f \qquad \text{and} \qquad \psi := \mathrm{id}_A + \mathrm{id}_B - s_* f.$$

They are inverse to each other under the composite $\circledcirc$. We first consider on $A \oplus B$ the $\mathcal{P}_\infty$-algebra product structure $\mu + \nu$ described above, that we transport on $A \oplus B$ under φ and ψ, as in the proof of Proposition 4.4, that is $d_\xi := \psi d_{\mu+\nu}\varphi$. The weight 0 part of this formula produces the following twisted underlying differential on $A \oplus B$: $\mathrm{d} = d_A + d_B + d_B s f_{(0)} - s f_{(0)} d_A$; the transported $\mathcal{P}_\infty$-algebra structure ξ on $(A \oplus B, \mathrm{d})$ should be understood with this differential. One can then check that this twisted differential d restricts to $A \oplus \ker g_{(0)}$; this chain complex becomes the pullback $(*)$ but the category of chain complexes now.

It is then straightforward to check that $A \oplus \ker g_{(0)}$ is stable under the $\mathcal{P}_\infty$-algebra structure ξ. Under the previous definitions, the collection φ becomes an ∞-morphism from $(A \oplus \ker g_{(0)}, \mathrm{d}, \xi)$ to $(A \oplus B, d_A + d_B, \mu + \nu)$. The following commutative diagram provides us with the desired pullback in the category of $\mathcal{P}_\infty$-algebras with ∞-morphisms:

$$\begin{tikzcd} A \oplus \ker g_{(0)} \arrow[r, squiggly, "\varphi"] \arrow[d, squiggly, "\varphi"] \arrow[dr, phantom, "\lrcorner", very near start] & A \oplus B \arrow[r, two heads] & B \arrow[dd, squiggly, two heads, "g"] \\ A \oplus B \arrow[d] & & \\ A \arrow[rr, squiggly, "f"] & & C. \end{tikzcd}$$

The universal property of the pullback is then an easy consequence of the universal property of the product and the definition of the ∞-isomorphism ψ.

(MC 2) *(Two out of three).* — Straightforward.

(MC 3) *(Retracts).* — Straightforward.

(MC 4) *(Lifting property).* — We consider the following commutative diagram in the category $\infty\text{-}\mathcal{P}_\infty\text{-}\mathsf{alg}$

$$\begin{tikzcd} A \arrow[r, rightsquigarrow, "h"] \arrow[d, rightarrowtail, rightsquigarrow, "f"] & C \arrow[d, twoheadrightarrow, rightsquigarrow, "g"] \\ B \arrow[r, rightsquigarrow, "k"] & D, \end{tikzcd}$$

where f is a cofibration and where g is a fibration. Using Proposition 4.4, we can equivalently suppose that the morphisms f and g are strict. Let us prove by induction on the weight n the existence of a lifting $l : B \rightsquigarrow C$ of the diagram

$$\begin{tikzcd} A \arrow[r, rightsquigarrow, "h"] \arrow[d, rightarrowtail, "f"] & C \arrow[d, twoheadrightarrow, "g"] \\ B \arrow[r, rightsquigarrow, "k"] \arrow[ur, dotted, "l"] & D, \end{tikzcd} \tag{4.2}$$

when either f or g is a quasi-isomorphism. The lifting property (MC 4) of the model category structure on unbounded chain complexes [21] provides us with a chain map $l_{(0)} : (B, d_B) \to (C, d_C)$ such that the following diagram commutes

$$\begin{tikzcd} (A, d_A) \arrow[r, "h_{(0)}"] \arrow[d, rightarrowtail, "f"] & (C, d_C) \arrow[d, twoheadrightarrow, "g"] \\ (B, d_B) \arrow[r, "k_{(0)}"] \arrow[ur, dotted, "l_{(0)}"] & (D, d_D). \end{tikzcd}$$

Suppose constructed the components $l_{(0)}, l_{(1)}, \ldots, l_{(n-1)}$ of the map l such that Diagram (4.2) commutes up to weight $n-1$. Let us look for a map $l_{(n)} \in \mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{\text{i}\,(n)}, \mathrm{End}^B_C)$ such that the diagram (4.2) commutes up in weight n and such that Equation (4.1) is satisfied in weight n, i.e.:

$$f^* l_{(n)} = h_{(n)}, \tag{4.3a}$$

$$g_* l_{(n)} = k_{(n)}, \tag{4.3b}$$

$$\partial^A_B\, l_{(n)} = \sum_{k=1}^{n} l_{(n-k)} * \mu_{(k)} - \sum_{k=1}^{n} \nu_{(k)} \circledast l_{(n-k)} + l_{(n-1)} d_\varphi, \tag{4.3c}$$

where μ and ν stand respectively for the $\mathcal{P}_\infty$-algebra structures on B and C. We consider a retraction $r : B \twoheadrightarrow A$ of f, $rf = \mathrm{id}_A$, and a section $s : D \rightarrowtail C$ of g, $gs = \mathrm{id}_D$, in the category of graded modules. The map ℓ defined by

$$\ell := r^* h_{(n)} + s_* k_{(n)} - (sg)_* r^* h_{(n)}$$

is a solution to (4.3a) and (4.3b). Let us denote by $\widetilde{l}_{(n)}$ the right-hand side of (4.3c), as in Appendix A. Since f is a morphism of $\mathcal{P}_\infty$-algebras, then one can see, by a direct computation from the definition, that the obstruction $\widetilde{(f^* l)}_{(n)}$ to lift $f^* l = l \circledcirc f$ is equal to $f^* \widetilde{l}_{(n)}$. This implies

$$f^* \big(\partial^B_C\, \ell - \widetilde{l}_{(n)}\big) = \partial^A_C \big(f^* \ell\big) - \widetilde{(f^* l)}_{(n)} = \partial^A_C \big(h_{(n)}\big) - \widetilde{h}_{(n)} = 0.$$

In the same way, the relation $\widetilde{(g_* l)}_{(n)} = g_* \widetilde{l}_{(n)}$ gives

$$g_* \big(\partial^B_C\, \ell - \widetilde{l}_{(n)}\big) = \partial^B_D \big(g_* \ell\big) - \widetilde{(g_* l)}_{(n)} = \partial^A_C \big(k_{(n)}\big) - \widetilde{k}_{(n)} = 0.$$

Let $\lambda : \mathcal{P}^{\text{!`}(n)}(B) \to C$ be the image of $\partial^B_C\, \ell - \widetilde{l}_{(n)}$ under the isomorphism $\mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{\text{!`}(n)}, \mathrm{End}^B_C) \cong \mathrm{Hom}(\mathcal{P}^{\text{!`}(n)}(B), C)$. Since $\lambda \circ \mathcal{P}^{\text{!`}(n)}(f) = 0$ and since $g \circ \lambda = 0$, then the map λ factors through a map $\bar{\lambda} : \operatorname{coker} \mathcal{P}^{\text{!`}(n)}(f) \to \ker g$, that is $\lambda = i\bar{\lambda}p$, where $i$ and $p$ are the respective canonical injection and projection. If $f$ is a quasi-isomorphism, then so is the map $\mathcal{P}^{\text{!`}(n)}(f)$, by the operadic Künneth formula [24, Theorem 6.2.3] and hence the chain complex $\operatorname{coker} \mathcal{P}^{\text{!`}(n)}(f)$ is acyclic. Respectively, if g is a quasi-isomorphism, then the chain complex $\ker g$ is acyclic. Theorem A.1 shows that λ is a cycle for the differential $(\partial^B_C)_*$, then so is $\bar{\lambda}$. Hence, in either of the two aforementioned cases, the cycle $\bar{\lambda}$ is boundary element : $\bar{\lambda} = \partial(\theta)$. By a slight abuse of notation, we denote by $i\theta p$ the corresponding element in $\mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{\text{!`}(n)}, \mathrm{End}^B_C)$. Finally, we consider the element

$$l_{(n)} := \ell - i\theta p,$$

which satisfies (4.3a), (4.3b) and (4.3c).

(MC 5) *(Factorization).* — Let $f : A \rightsquigarrow B$ be an ∞-morphism of $\mathcal{P}_\infty$-algebras.

(a) We consider first the factorization of the chain map $f_{(0)}$ into an acyclic monomorphism followed by an epimorphism:

$$\begin{tikzcd} A \arrow[r, tail, "\sim", "i"'] \arrow[rr, bend left, "f_{(0)}"] & C \arrow[r, two heads, "p_{(0)}"'] & B, \end{tikzcd}$$

where C stands for the mapping cocylinder of $f_{(0)}$. Let us recall that this latter chain complex is given by $C := A \oplus s^{-1}B \oplus B$ with differential

$$(a, s^{-1}b, b') \mapsto \left(d_A(a), -s^{-1}f(a) - s^{-1}d_B(b) + s^{-1}b', d_B(b')\right).$$

The inclusion i is equal to $i(a) := (a, 0, f(a))$ and the projection $p_{(0)}$ is equal to $p_{(0)}(a, s^{-1}b, b') := b'$. The right-hand side of C is the cocone of the identity id_B, that is the cone of $-\mathrm{id}_{s^{-1}B}$; so this is an acyclic chain complex. The canonical projection $A \oplus s^{-1}B \oplus B \twoheadrightarrow A$ induced a chain map $q : C \twoheadrightarrow A$ which admits i for section, i.e. $qi = \mathrm{id}_C$. As in the classical case, the short exact sequence

$$0 \longrightarrow \mathrm{Cone}\,(-\mathrm{id}_{s^{-1}B}) \rightarrowtail C \twoheadrightarrow A \longrightarrow 0,$$

induces a long exact sequence in homology which proves that i is a quasi-isomorphism. The maps q and i show that C is a homotopy retract of A with respect to the trivial homotopy. We can thus apply the homotopy transfer theorem [24, Theorem 10.3.1]. If we denote by (A, d_A, μ) the $\mathcal{P}_\infty$-algebra structure on A, the formulae [24, Theorem 10.3.3] for the homotopy transfer theorem endow C with the $\mathcal{P}_\infty$-algebra structure $(C, d_C, q^* i_* \mu)$ in this particular case. It is then straightforward to check that i is a strict ∞-morphism between these two $\mathcal{P}_\infty$-algebras. All of this provides us with the following commutative diagram in the category of chain complexes

$$\begin{tikzcd} A \arrow[r, "f_{(0)}"] \arrow[d, tail, "i"', "\sim"] & B \arrow[d, two heads] \\ C \arrow[r] \arrow[ur, dotted, two heads, "p_{(0)}"] & 0, \end{tikzcd}$$

where i actually forms an injective (strict) ∞-quasi-isomorphim. Using the same arguments as in the aforementioned proof of Axiom (MC 4), there exists an ∞-morphism $p : C \rightsquigarrow B$, extending $p_{(0)}$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, rightsquigarrow, "f"] \arrow[d, tail, "i"', "\sim"] & B \arrow[d] \\ C \arrow[r] \arrow[ur, rightsquigarrow, two heads, "p"] & 0. \end{tikzcd}$$

In the end, the factorization $f = pi$ concludes the proof.

(b) Let $C := s^{-1}A \oplus A$ be the mapping cone of the identity id_A of A. By Corollary A.2, the canonical inclusion $A \rightarrowtail C$ extends to an ∞-monomorphism denoted by $j : A \rightsquigarrow C$. We consider the product i of the two ∞-morphisms f and j:

$$\begin{tikzcd} & A \arrow[dl, rightsquigarrow, "f"'] \arrow[dr, rightsquigarrow, tail, "j"] \arrow[dd, rightsquigarrow, tail, "i"] & \\ B & & C \\ & B \prod C. \arrow[ul, rightsquigarrow, two heads, "p", "\sim"'] \arrow[ur, rightsquigarrow, two heads] & \end{tikzcd}$$

Since the underlying chain complex of the product $B \prod C$ is equal to $B \oplus C$, the ∞-morphism i is an cofibration and the projection p is an acyclic fibration. So the factorization $f = pi$ concludes the proof. $\square$

4.4. Relationship with the model category structure on conilpotent dg $\mathcal{P}^{\text{!`}}$-coalgebras

Since the bar construction $\widetilde{\mathrm{B}}_\iota$ provides us with an isomorphism of categories

$$\widetilde{\mathrm{B}}_\iota \;:\; \infty\text{-}\mathcal{P}_\infty\text{-}\mathsf{alg} \cong \mathsf{quasi\text{-}free}\ \mathcal{P}^{\text{!`}}\text{-}\mathsf{coalg},$$

we can compare the model category structure without equalizers on $\mathcal{P}_\infty$-algebras (Theorem 4.2) with the model category structure on conilpotent

$\mathcal{P}^{\text{!`}}$-coalgebras (Theorem 2.2). The following proposition shows that the various notions of weak equivalences, cofibrations and fibrations agree.

PROPOSITION 4.5.

(1) *An ∞-morphism f is a weak-equivalence of $\mathcal{P}_\infty$-algebras if and only if its image under the bar construction $\widetilde{\mathrm{B}}_\iota\, f$ is a weak equivalence of conilpotent dg $\mathcal{P}^{\text{!`}}$-coalgebras.*
(2) *An ∞-morphism f is a cofibration of $\mathcal{P}_\infty$-algebras if and only if its image under the bar construction $\widetilde{\mathrm{B}}_\iota\, f$ is a cofibration of conilpotent dg $\mathcal{P}^{\text{!`}}$-coalgebras.*
(3) *An ∞-morphism f is a fibration of $\mathcal{P}_\infty$-algebras if and only if its image under the bar construction $\widetilde{\mathrm{B}}_\iota\, f$ is a fibration of conilpotent dg $\mathcal{P}^{\text{!`}}$-coalgebras.*

Proof.

(1). — This is Proposition 11.4.7 of [24].

(2). — Let $f : A \rightsquigarrow A'$ be an ∞-morphism of $\mathcal{P}_\infty$-algebras. If $\widetilde{\mathrm{B}}_\iota\, f : \widetilde{\mathrm{B}}_\iota A \rightarrowtail \widetilde{\mathrm{B}}_\iota A'$ is a cofibration of conilpotent dg $\mathcal{P}^{\text{!`}}$-coalgebras, then it is a monomorphism by definition. So its restriction to A is again a monomorphism. Since this restriction is equal to the composite

$$A \xrightarrow{f_{(0)}} A' \rightarrowtail \mathcal{P}^{\text{!`}}(A'),$$

this implies that the first component $f_{(0)}$ is a monomorphism.

In the other way round, suppose that the ∞-morphism $f : A \rightsquigarrow A'$ is an ∞-monomorphism, i.e. $f_{(0)} : A \rightarrowtail A'$ is injective. Let $r_{(0)} : A' \twoheadrightarrow A$ be a retraction of $f_{(0)}$. The formula of [24, Theorem 10.4.1] produces an ∞-morphism r, which is right inverse to f. Therefore, the image $\widetilde{\mathrm{B}}_\iota\, r$ is a right inverse to $\widetilde{\mathrm{B}}_\iota\, f$, which proves that this latter one is a monomorphism.

(3). — Let us first recall that axioms (MC 3) and (MC 5) imply that fibrations are characterized by the right lifting property with respect to acyclic cofibrations. So this characterization also holds for ∞-epimorphisms in the model category without equalizers of Theorem 4.2.

Let $f : A \rightsquigarrow A'$ be an ∞-morphism of $\mathcal{P}_\infty$-algebras. Suppose that $\widetilde{\mathrm{B}}_\iota\, f : \widetilde{\mathrm{B}}_\iota A \twoheadrightarrow \widetilde{\mathrm{B}}_\iota A'$ is a fibration of conilpotent dg $\mathcal{P}^{\text{!`}}$-coalgebras. We consider a commutative diagram in $\infty\text{-}\mathcal{P}_\infty\text{-}\mathsf{alg}$

$$\begin{tikzcd} B \arrow[r, squiggly, "h"] \arrow[d, tail, squiggly, "g"', "\sim"] & A \arrow[d, two heads, squiggly, "f"] \\ B' \arrow[r, squiggly, "k"] & A', \end{tikzcd} \tag{4.4}$$

where g is an acyclic cofibration. Its image $\widetilde{\mathrm{B}}_\iota\, g : \widetilde{\mathrm{B}}_\iota B \to \widetilde{\mathrm{B}}_\iota B'$ under the bar construction functor is an acyclic cofibration of conilpotent dg $\mathcal{P}^{\text{!`}}$-coalgebras by the two previous points (1) and (2). So, by the axiom (MC 4) of Theorem 2.2, there exists a lifting map $\widetilde{\mathrm{B}}_\iota\, l$ in following diagram

$$\begin{tikzcd} \widetilde{\mathrm{B}}_\iota B \arrow[r] \arrow[d, tail, "\widetilde{\mathrm{B}}_\iota\, g"', "\sim"] & \widetilde{\mathrm{B}}_\iota A \arrow[d, two heads, "\widetilde{\mathrm{B}}_\iota\, f"] \\ \widetilde{\mathrm{B}}_\iota B' \arrow[r] \arrow[ur, dotted, "\widetilde{\mathrm{B}}_\iota\, l"] & \widetilde{\mathrm{B}}_\iota A', \end{tikzcd}$$

which proves that l is a lifting map in Diagram (4.4). So the ∞-morphism $f : A \rightsquigarrow A'$ is a fibration.

In the other way round, suppose that the ∞-morphism $f : A \rightsquigarrow A'$ is a fibration of $\mathcal{P}_\infty$-algebras and let

$$\begin{tikzcd} C \arrow[r, "H"] \arrow[d, tail, "G"', "\sim"] & \widetilde{\mathrm{B}}_\iota A \arrow[d, "\widetilde{\mathrm{B}}_\iota\, f"] \\ C' \arrow[r, "K"] & \widetilde{\mathrm{B}}_\iota A' \end{tikzcd}$$

be a commutative diagram in the category of conilpotent dg $\mathcal{P}^{\text{i}}$-coalgebras, where G is an acyclic cofibration. We consider the diagram

$$\begin{tikzcd}
C \arrow[rr, "H"] \arrow[dd, tail, "G"', "\sim"] \arrow[dr, tail, "\upsilon_\kappa C", "\sim"'] & & \widetilde{\mathrm{B}}_\iota A \arrow[dd, "\widetilde{\mathrm{B}}_\iota f"] \\
& \mathrm{B}_\kappa \Omega_\kappa C \arrow[dd, tail, "\sim"', "\mathrm{B}_\kappa \Omega_\kappa G", near start] & \\
C' \arrow[rr, "K", near end] \arrow[dr] & & \widetilde{\mathrm{B}}_\iota A' \\
& \mathrm{B}_\kappa \Omega_\kappa C'. &
\end{tikzcd}$$

Since the unit $\upsilon_\kappa C$ of the bar-cobar adjunction is an acyclic cofibration by Theorem 2.9 (2) and since $\widetilde{\mathrm{B}}_\iota A$ is a fibrant dg $\mathcal{P}^{\text{i}}$-coalgebra, then there exists a morphism $H' : \mathrm{B}_\kappa \Omega_\kappa C \to \widetilde{\mathrm{B}}_\iota A$ which factors H, i.e. $H = H'\upsilon_\kappa C$. Since G is a weak-equivalence, then $\Omega_\kappa G$ is a quasi-isomorphism, by definition, and so $\mathrm{B}_\kappa \Omega_\kappa G$ is a weak-equivalence by point (1). In the same way, since G is a cofibration, then it is a monomorphism, by definition, and so is $\Omega_\kappa G = \mathrm{id}_{\mathcal{P}}(G) : \mathcal{P}(C) \rightarrowtail \mathcal{P}(C')$; point (2) shows that $\mathrm{B}_\kappa \Omega_\kappa G$ is a cofibration. All together, this proves that the map $\mathrm{B}_\kappa \Omega_\kappa G$ is an acyclic cofibration of conilpotent dg $\mathcal{P}^{\text{i}}$-coalgebras, and, since $\widetilde{\mathrm{B}}_\iota A'$ is fibrant, then there exists a morphism $K' : \mathrm{B}_\kappa \Omega_\kappa C' \to \widetilde{\mathrm{B}}_\iota A'$ which factors $(\widetilde{\mathrm{B}}_\iota f)H'$, i.e. $(\widetilde{\mathrm{B}}_\iota f)H' = K'(\mathrm{B}_\kappa \Omega_\kappa G)$. Finally, we apply the lifting property axiom (MC 4) of Theorem 4.2 in the diagram

$$\begin{tikzcd}
\Omega_\kappa C \arrow[r, rightsquigarrow, "h'"] \arrow[d, tail, "\Omega_\kappa G"', "\sim"] & A \arrow[d, rightsquigarrow, two heads, "f"] \\
\Omega_\kappa C' \arrow[r, rightsquigarrow, "k'"] \arrow[ur, dotted] & A'
\end{tikzcd}$$

to conclude that $\widetilde{\mathrm{B}}_\iota f$ is a fibration. □

Appendix A. Obstruction theory for infinity-morphisms

In this appendix, we settle the obstruction theory for ∞-morphisms of homotopy $\mathcal{P}$-algebras.

Recall from Theorem 4.3, that an ∞-morphism $f : A \rightsquigarrow B$ between two $\mathcal{P}_\infty$-algebras (A, d_A, μ) and (B, d_B, ν) is a map $f : \mathcal{P}^{¡} \to \mathrm{End}^A_B$ satisfying Equation (4.1)

$$\partial(f) = f * \mu - \nu \circledast f$$

in $\mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{¡}, \mathrm{End}^A_B)$.

For any $n \geqslant 0$, we denote by $\mu_{(n)}$, $\nu_{(n)}$, and $f_{(n)}$ the respective restrictions of the maps μ, ν, and f to the weight n part ${\mathcal{P}^{¡}}^{(n)}$ of the cooperad $\mathcal{P}^{¡}$. Using these notations, Equation (4.1) becomes

$$\partial^A_B f_{(n)} - f_{(n-1)} d_\varphi = \sum_{k=1}^{n} f_{(n-k)} * \mu_{(k)} - \sum_{k=1}^{n} \nu_{(k)} \circledast f_{(n-k)} \tag{A.1}$$

on ${\mathcal{P}^{¡}}^{(n)}$, for any $n \geqslant 0$, where $\nu_{(k)} \circledast f_{(n-k)}$ means, by a slight abuse of notations, that the total weight of the various maps f involved on the right-hand side is equal to $n - k$.

Theorem A.1. — *Let (A, d_A, μ) and (B, d_B, ν) be two $\mathcal{P}_\infty$-algebras. Let $n \geqslant 0$ and suppose given $f_{(0)}, \dots, f_{(n-1)}$ satisfying Equation* (A.1) *up to $n-1$. The element*

$$\widetilde{f}_{(n)} := \sum_{k=1}^{n} f_{(n-k)} * \mu_{(k)} - \sum_{k=1}^{n} \nu_{(k)} \circledast f_{(n-k)} + f_{(n-1)} d_\varphi$$

is a cycle in the chain complex $(\mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{i}, \mathrm{End}^A_B), (\partial^A_B)_*)$. Therefore, there exists an element $f_{(n)}$ satisfying Equation (4.1) *at weight n if and only if the cycle $\widetilde{f}_{(n)}$ is a boundary element.*

Proof. — Let us prove that $\partial^A_B \widetilde{f}_{(n)} = 0$; the second statement is then straightforward. We have

$$\begin{aligned}\partial^A_B \widetilde{f}_{(n)} = \sum_{k=1}^{n} \big(&(\partial^A_B f_{(n-k)}) * \mu_{(k)} + f_{(n-k)} * (\partial_A \mu_{(k)}) \\ &- (\partial_B \nu_{(k)}) \circledast f_{(n-k)} + \nu_{(k)} \circledast (f; \partial^A_B f)_{(n-k)} \big) + \partial^A_B f_{(n-1)} d_\varphi,\end{aligned}$$

where the notation $(f; \partial^A_B f)$ means that we apply the definition of the product $\circledast$ with many f but one $\partial^A_B f$. (With the notations of [24], this coincides to composing by $\nu\circ(f;\partial^A_B f)$.) Applying Equation (A.1) at weight strictly less than n and the Maurer–Cartan equation

$$\partial_A\mu_{(n)} = -\sum_{k=1}^{n}\mu_{(k)}\star\mu_{(n-k)} - \mu_{(n-1)}d_\varphi$$

for μ and ν respectively, we get

$$\begin{aligned}\partial^A_B\widetilde{f}_{(n)} = \sum_{k+l+m=n} &\Big((f_{(k)} * \mu_{(l)}) * \mu_{(m)} - f_{(k)} * (\mu_{(l)} \star \mu_{(m)})\\ &\quad + (\nu_{(k)}\star\nu_{(l)}) \circledast f_{(m)} - \nu_{(k)} \circledast (f;\nu\circledast f)_{(l+m)}\\ &\quad + \nu_{(k)} \circledast (f; f*\mu)_{(l+m)} - (\nu_{(k)}\circledast f_{(l)}) * \mu_{(m)}\Big)\\ + \sum_{k=1}^{n-1} &\Big((f_{(n-k-1)}d_\varphi) * \mu_{(k)} - f_{(n-k-1)} * (\mu_{(k)}d_\varphi)\\ &\quad + (f_{(n-k-1)} * \mu_{(k)})d_\varphi + (\nu_{(k)}d_\varphi)\circledast f_{n-k-1})\\ &\quad - (\nu_{(k)}\circledast f_{n-k-1})d_\varphi + \nu_{(k)}\circledast (f; f_{(\bullet-1)}d_\varphi)_{(n-k-1)}\Big)\\ + f_{(n-2)}&(d_\varphi)^2.\end{aligned}$$

Since μ has degree -1, the preLie relations of the operations $\star$ and $*$ imply $(f*\mu)*\mu = f*(\mu\star\mu)$. The coassociativity of the decomposition coproduct Δ of the cooperad $\mathcal{P}^{¡}$ implies $(\nu\star\nu)\circledast f = \nu(f;\nu\circledast f)$ and $(\nu\circledast f)*\mu = \nu\circledast(f; f\circledast\mu)$. Since d_φ is a coderivation of the cooperad $\mathcal{P}^{¡}$, it implies $(f*\mu)d_\varphi = f*(\mu d_\varphi) - (fd_\varphi)*\mu$ and $(\nu\circledast f)d_\varphi = (\nu d_\varphi)\circledast f + \nu\circledast(f; f_{(\bullet-1)}d_\varphi)$. Finally, the coderivation d_φ squares to zero, which concludes the proof. □

Corollary A.2. — *Let (A, d_A, μ) be a $\mathcal{P}_\infty$-algebra and let (B, d_B) be an acyclic chain complex, viewed as a trivial $\mathcal{P}_\infty$-algebra. Any chain map $(A, d_A)\to(B, d_B)$ extends to an ∞-morphism $(A, d_A, \mu)\rightsquigarrow(B, d_B, 0)$.*

Proof. — We prove the existence of a series of maps $f_{(n)}$, for $n\geqslant 0$ satisfying Equation (A.1) by induction on n using Theorem A.1. Let us denote the map $A\to B$ by $f_{(0)}$. Since this is a chain map, it satisfies Equation (A.1) for $n=0$. Since the chain complex (B, d_B) is acyclic, then so is the chain complex $(\mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{¡}, \mathrm{End}^A_B), (\partial^A_B)_*)$. Therefore, all the obstructions vanish and Theorem A.1 applies. □

Appendix B. A technical lemma

Lemma B.1. — *Let A be a dg $\mathcal{P}$-algebra and let D be a conilpotent dg $\mathcal{P}^{¡}$-coalgebra. Let $p : A \twoheadrightarrow \Omega_\kappa D$ be a fibration of dg $\mathcal{P}$-algebras. The* *morphism* $j : \mathrm{B}_\kappa\, A \times_{\mathrm{B}_\kappa\, \Omega_\kappa D} D \overset{\sim}{\rightarrowtail} \mathrm{B}_\kappa\, A$*, produced by the pullback diagram*

$$\begin{tikzcd}
\mathrm{B}_\kappa\, A \times_{\mathrm{B}_\kappa\, \Omega_\kappa D} D \arrow[r] \arrow[d, tail, "j"', "\sim"] \arrow[dr, phantom, "\lrcorner"] & D \arrow[d, "\upsilon_\kappa D"] \\
\mathrm{B}_\kappa\, A \arrow[r, "\mathrm{B}_\kappa\, p"] & \mathrm{B}_\kappa\, \Omega_\kappa D,
\end{tikzcd}$$

is an acyclic cofibration of conilpotent dg $\mathcal{P}^{\text{i}}$*-coalgebras.*

Proof. — We consider the kernel $K := \mathrm{Ker}(p)$ of the map $p : A \twoheadrightarrow \Omega_\kappa D$, which is a sub-dg $\mathcal{P}$-algebra of A. The short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, tail] & A \arrow[r, two heads] & \Omega_\kappa D = \mathcal{P}(D) \arrow[r] \arrow[l, dotted, bend right] & 0
\end{tikzcd}$$

of dg $\mathcal{P}$-algebras splits in the category of graded $\mathcal{P}$-algebras since the underlying $\mathcal{P}$-algebra of $\Omega_\kappa D = \mathcal{P}(D)$ is free. The induced isomorphism of graded $\mathcal{P}$-algebras $A \cong K \oplus \mathcal{P}(D)$ becomes an isomorphism of dg $\mathcal{P}$-algebras when the right-hand side is equipped with the transferred differential, which the sum of the following three terms

$$d_K \ : \ K \to K, \quad d_{\Omega_\kappa D} \ : \ \mathcal{P}(D) \to \mathcal{P}(D), \text{ and } \quad d' \ : \ \mathcal{P}(D) \to K.$$

Notice that $K \oplus \mathcal{P}(D)$ endowed with the $\mathcal{P}$-algebra structure given by

$$\mathcal{P}(K \oplus \mathcal{P}(D)) \twoheadrightarrow \mathcal{P}(K) \oplus \mathcal{P}(\mathcal{P}(D)) \xrightarrow{\gamma_K \oplus \gamma_{\mathcal{P}(D)}} K \oplus \mathcal{P}(D)$$

is the product of K and $\mathcal{P}(D)$ in the category of graded $\mathcal{P}$-algebras. Since the bar construction is right adjoint, it preserves the limits and thus the products. This induces the following two isomorphisms of conilpotent dg $\mathcal{P}^{\text{i}}$-coalgebras

$$\mathrm{B}_\kappa\, A \cong \mathrm{B}_\kappa\, K \times \mathrm{B}_\kappa\, \Omega_\kappa D, \quad \text{and} \quad \mathrm{B}_\kappa\, A \times_{\mathrm{B}_\kappa\, \Omega_\kappa D} D \cong \mathrm{B}_\kappa\, K \times D,$$

with both right-hand sides equipped with an additional differential coming from d'. Under these identifications, the initial pullback becomes

$$\begin{tikzcd}
\mathrm{B}_\kappa\, K \times D \arrow[r, "\mathrm{proj}"] \arrow[d, "\mathrm{id} \times \upsilon_\kappa D"'] \arrow[dr, phantom, "\lrcorner"] & D \arrow[d, "\upsilon_\kappa D"] \\
\mathrm{B}_\kappa\, K \times \mathrm{B}_\kappa\, \Omega_\kappa D \arrow[r, "\mathrm{proj}"] & \mathrm{B}_\kappa\, \Omega_\kappa D.
\end{tikzcd}$$

Since the unit of adjunction $\upsilon_\kappa D$ is monomorphism, then so is the map j, which is therefore a cofibration.

It remains to prove that $\mathrm{id}\times \upsilon_\kappa D$ is again a weak-equivalence when the twisted differentials, coming from d', are taken into account. We will prove that this is a filtered quasi-isomorphism with the same kind of filtrations as in the proof of Theorem 2.9.

We notice first that the product of two $\mathcal{P}^{¡}$-coalgebras is given by an equalizer dual to the coequalizer introduced at the beginning of Section 2.4 to describe coproducts of $\mathcal{P}$-algebras. So the product $\mathcal{P}^{¡}(K)\times D$ is a conilpotent sub-$\mathcal{P}^{¡}$-coalgebra of the conilpotent cofree $\mathcal{P}^{¡}$-coalgebra $\mathcal{P}^{¡}(K\oplus D)$. We filter the latter one by

$$\mathcal{F}_n\left(\mathcal{P}^{¡}(K\oplus D)\right) := \sum_{\substack{k\geqslant 1,\\ n_1+\cdots+n_k\leqslant n}} \mathcal{P}^{¡}(k)\otimes_{\mathbb{S}_k}\big((K\oplus F_{n_1}D)\otimes\cdots\otimes(K\oplus F_{n_k}D)\big)$$

and we denote by $\mathcal{F}_n\left(\mathrm{B}_\kappa\, K\times D\right)$ the induced filtration on the product $\mathrm{B}_\kappa\, K\times D$. In the same way, we filter the product $\mathrm{B}_\kappa\, K\times \mathrm{B}_\kappa\,\Omega_\kappa D$, whose underlying conilpotent $\mathcal{P}^{¡}$-coalgebra is $\mathcal{P}^{¡}(K\oplus\mathcal{P}(D))$, by

$$\mathcal{F}_n\left(\mathrm{B}_\kappa\, K\times \mathrm{B}_\kappa\,\Omega_\kappa D\right) := \sum_{\substack{k\geqslant 1,\\ n_1+\cdots+n_k\leqslant n}} \mathcal{P}^{¡}(k)\otimes_{\mathbb{S}_k}\big((K\oplus F_{n_1}\mathcal{P}(D))\otimes\cdots\otimes(K\oplus F_{n_k}\mathcal{P}(D))\big),$$

where

$$F_n\mathcal{P}(D) := \sum_{\substack{k\geqslant 1,\\ n_1+\cdots+n_k\leqslant n}} \mathcal{P}(k)\otimes_{\mathbb{S}_k}(F_{n_1}D\otimes\cdots\otimes F_{n_k}D).$$

The respective differentials and the map $\mathrm{id}\times\upsilon_\kappa D$ preserve these two filtrations. Let us now prove that the associated map $\mathrm{gr}(\mathrm{id}\times\upsilon_\kappa D)=\mathrm{id}\times\mathrm{gr}(\upsilon_\kappa D)$ is a quasi-isomorphism with the twisted differentials coming for d'. We now consider the filtration $\mathscr{F}_n\left(\mathrm{B}_\kappa\, K\times \mathrm{gr}\, D\right)$ induced by

$$\mathscr{F}_n\left(\mathcal{P}^{¡}(K\oplus \mathrm{gr}\, D)\right) := \sum_{\substack{k\geqslant 1,\\ n_1+\cdots+n_k+k\leqslant n}} \mathcal{P}^{¡}(k)\otimes_{\mathbb{S}_k}\big((K\oplus \mathrm{gr}_{n_1} D)\otimes\cdots\otimes(K\oplus \mathrm{gr}_{n_k} D)\big).$$

We introduce the filtration $\mathscr{F}_n(\mathrm{B}_\kappa\, K\times \mathrm{B}_\kappa\,\Omega_\kappa\;\mathrm{gr}\, D)$ given by

$$\mathscr{F}_n\left(\mathcal{P}^{¡}(K\oplus \mathcal{P}(\mathrm{gr}\, D))\right) := \sum_{\substack{k\geqslant 1,\\ n_1+\cdots+n_k\leqslant n}} \mathcal{P}^{¡}(k)\otimes_{\mathbb{S}_k}\big((K\oplus \mathscr{F}_{n_1}\mathcal{P}(\mathrm{gr}\, D))\otimes\cdots\otimes(K\oplus \mathscr{F}_{n_k}\mathcal{P}(\mathrm{gr}\, D))\big),$$

where

$$\mathscr{F}_n\mathcal{P}(\operatorname{gr} D) := \sum_{\substack{k\geqslant 1,\\ n_1+\cdots+n_k+k\leqslant n}} \mathcal{P}(k) \otimes_{\mathbb{S}_k} (\operatorname{gr}_{n_1} D \otimes \cdots \otimes \operatorname{gr}_{n_k} D).$$

The respective differentials and the map $\mathrm{id} \times \operatorname{gr}(\upsilon_\kappa D)$ preserve these two filtrations. By their definitions, the part of the differential coming from d', and only this part, is killed on the first page of the respective spectral sequences. By the same arguments as in the proof of Theorem 2.9, we get a quasi-isomorphism between these first pages. Since the two aforementioned filtrations are bounded below and exhaustive, we conclude by the classical convergence theorem of spectral sequences [25, Chapter 11]. ☐

BIBLIOGRAPHY

[1] M. Aubry & D. Chataur, “Cooperads and coalgebras as closed model categories”, *J. Pure Appl. Algebra* **180** (2003), no. 1-2, p. 1-23.

[2] K. S. Brown, “Abstract homotopy theory and generalized sheaf cohomology”, *Trans. Am. Math. Soc.* **186** (1973), p. 419-458.

[3] U. Buijs & A. Murillo, “Algebraic models of non-connected spaces and homotopy theory of L_∞-algebras”, *Adv. Math.* **236** (2013), p. 60-91.

[4] X. Z. Cheng & E. Getzler, “Transferring homotopy commutative algebraic structures”, *J. Pure Appl. Algebra* **212** (2008), no. 11, p. 2535-2542.

[5] V. A. Dolgushev, A. E. Hoffnung & C. L. Rogers, “What do homotopy algebras form?”, *Adv. Math.* **274** (2015), p. 562-605.

[6] V. Dotsenko & N. Poncin, “A tale of three homotopies”, *Appl. Categ. Struct.* **24** (2016), no. 6, p. 845-873.

[7] V. Dotsenko, S. Shadrin & B. Vallette, “Pre-Lie deformation theory”, *Mosc. Math. J.* **16** (2016), no. 3, p. 505-543.

[8] G. C. Drummond-Cole & J. Hirsh, “Model structures for coalgebras”, *Proc. Am. Math. Soc.* **144** (2016), no. 4, p. 1467-1481.

[9] J. L. Dupont, “Simplicial de Rham cohomology and characteristic classes of flat bundles”, *Topology* **15** (1976), no. 3, p. 233-245.

[10] W. G. Dwyer & D. M. Kan, “Function complexes in homotopical algebra”, *Topology* **19** (1980), no. 4, p. 427-440.

[11] I. Gálvez-Carrillo, A. Tonks & B. Vallette, “Homotopy Batalin–Vilkovisky algebras”, *J. Noncommut. Geom.* **6** (2012), no. 3, p. 539-602.

[12] E. Getzler, “Lie theory for nilpotent L_∞-algebras”, *Ann. Math.* **170** (2009), no. 1, p. 271-301.

[13] E. Getzler & P. Goerss, “A model category structure for differential graded coalgebras”, 1999, `http://www.math.northwestern.edu/~pgoerss/papers/model.ps`.

[14] E. Getzler & J. D. S. Jones, “Operads, homotopy algebra and iterated integrals for double loop spaces”, `https://arxiv.org/abs/hep-th/9403055`, 1994.

[15] V. Ginzburg & M. Kapranov, “Koszul duality for operads”, *Duke Math. J.* **76** (1994), no. 1, p. 203-272.

[16] A. Grothendieck, “Sur quelques points d’algèbre homologique”, *Tôhoku Math. J.* **9** (1957), p. 119-221.

[17] K. Hess & B. Shipley, "The homotopy theory of coalgebras over a comonad", *Proc. Lond. Math. Soc.* **108** (2014), no. 2, p. 484-516.

[18] V. Hinich, "Descent of Deligne groupoids", *Int. Math. Res. Not.* (1997), no. 5, p. 223-239.

[19] ———, "DG coalgebras as formal stacks", *J. Pure Appl. Algebra* **162** (2001), no. 2-3, p. 209-250.

[20] P. S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, 2003, xvi+457 pages.

[21] M. Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, 1999, xii+209 pages.

[22] A. Lazarev & M. Markl, "Disconnected rational homotopy theory", *Adv. Math.* **283** (2015), p. 303-361.

[23] K. Lefevre-Hasegawa, "Sur les A-infini catégories", `https://arxiv.org/abs/math/0310337`, 2003.

[24] J.-L. Loday & B. Vallette, *Algebraic operads*, Grundlehren der Mathematischen Wissenschaften, vol. 346, Springer, 2012, xviii+512 pages.

[25] S. Mac Lane, *Homology*, Classics in Mathematics, Springer, 1995, Reprint of the 1975 edition, x+422 pages.

[26] H. J. Munkholm, "DGA algebras as a Quillen model category. Relations to shm maps", *J. Pure Appl. Algebra* **13** (1978), no. 3, p. 221-232.

[27] D. G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, vol. 43, Springer, 1967.

[28] ———, "Rational homotopy theory", *Ann. Math.* **90** (1969), p. 205-295.

[29] D. Robert-Nicoud, "Deformation theory with homotopy algebra structures on tensor products", *Doc. Math.* **23** (2018), p. 189-240.

[30] J. R. Smith, "Model-categories of coalgebras over operads", *Theory Appl. Categ.* **25** (2011), p. 189-246.

[31] D. Sullivan, "Infinitesimal computations in topology", *Publ. Math., Inst. Hautes Étud. Sci.* (1977), no. 47, p. 269-331.

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