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The algebraic functional equation of Riemann's theta function

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THE ALGEBRAIC FUNCTIONAL EQUATION OF RIEMANN'S THETA FUNCTION

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ABSTRACT. — We give an algebraic analog of the functional equation of Riemann's theta function. More precisely, we define a "theta multiplier" line bundle over the moduli stack of principally polarized abelian schemes with theta characteristic and prove that its dual is isomorphic to the determinant bundle over the moduli stack. We do so by explicit computations involving the Picard group of the moduli stack. This is all done over the ring $R = \mathbb{Z}[1/2, i]$: passing to the complex numbers, we recover the classical functional equation.

RÉSUMÉ. — Nous donnons un analogue algébrique de l'équation fonctionnelle de la fonction thêta de Riemann. Plus précisément, nous définissons un fibré en droites « multiplicateur thêta » sur le champ de modules de schémas abéliens principalement polarisés avec une caractéristique thêta et prouvons que son dual est isomorphe au fibré déterminant sur le champ de modules. Nous le faisons par des calculs explicites impliquant le groupe de Picard du champ de modules. Tout cela se fait sur l'anneau $R = \mathbb{Z}[1/2, i]$: en passant aux nombres complexes, on retrouve l'équation fonctionnelle classique.

1. Introduction

In 1829 C.G.J Jacobi introduced the theta function $\vartheta(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau}$ and proved the remarkable transformation formula

$$(1.1) \qquad \vartheta\left(\frac{-1}{\tau}\right) = \sqrt{\frac{\tau}{i}} \, \vartheta(\tau), \quad \tau \in \mathfrak{h} := \{z \in \mathbb{C} : \text{Im}[z] > 0\},$$

where $\sqrt{\cdot}$ is defined using the principal branch of the logarithm. Along with the trivial identity $\vartheta(\tau + 2) = \vartheta(\tau)$, equation (1.1) expresses the fact that ϑ is a *modular form* of weight $1/2$ on the *theta group* $\Gamma_1^+(1,2) \subseteq \text{SL}_2(\mathbb{Z})$, a level 2 congruence subgroup. Much effort has been devoted in recent years to study the algebro-geometric nature of modular forms, which has led to

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great progress in understanding their Fourier coefficients and their relation to Galois representations. Yet despite all this success, statements such as (1.1) have remained rather mysterious from an algebro-geometric point of view, their algebraic nature obscured by the use of analytic techniques such as Poisson summation. Given the importance of theta functions and of modular forms of half-integral weight in general, this has to be viewed as a serious gap in our understanding of the theory. It is the aim of this paper to fill this gap.

The natural geometric framework to study this problem is the moduli space of principally polarized abelian varieties of dimension g with theta characteristic. Over the complex numbers this moduli space is given by the analytic quotient stack $\mathscr{A}_{g,\mathrm{an}}^{\pm} := \left[\Gamma_g^{\pm}(1,2)\backslash\mathfrak{h}_g\right]$, where $\mathfrak{h}_g$ is the g-dimensional Siegel upper half-space and, according to whether the characteristic is even $(+)$ or odd $(-)$, the subgroups $\Gamma_g^{\pm}(1,2) \subseteq \mathrm{Sp}_{2g}(\mathbb{Z})$ are the *theta groups*, generalizing $\Gamma_1^+(1,2)$ above (e.g. [7, VIII, §3.4.1, §3.4.2]). For example, in the even case we have

$$(1.2)\ \ \Gamma_g^+(1,2) := \left\{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_{2g}(\mathbb{Z}) : AB^t \text{ and } CD^t \text{ have even diagonal}\right\},$$

which is the group of transformations of the *Riemann theta function* $\vartheta_g(\tau)$. This is the analytic function of $\tau \in \mathfrak{h}_g$ given by

$$\vartheta_g(\tau) := \sum_{n \in \mathbb{Z}^g} e^{\pi i\, n^t \tau n},$$

a higher-dimensional generalization of $\vartheta(\tau)$. In this geometric setting, the theta function $\vartheta_g(\tau)$ is the value at zero of a section of a certain relatively ample, normalized, symmetric line bundle Θ of degree 1 over the universal abelian variety $\pi : \mathscr{A} \to \Gamma_g^+(1,2)\backslash\mathfrak{h}_g$, whose isomorphism class is determined by the theta characteristic. In particular, ϑ_g is a section of the dual $(\pi_*\Theta)^{-1}$ of the line bundle $\pi_*\Theta$. A similar theta function and corresponding geometric analog can be obtained in the case of odd theta characteristics.

Theta functions such as $\vartheta_g(\tau)$ have well-known functional equations analogous to the $g = 1$ case (1.1). In this article, we would like to explain the relationship between these transformation laws and the moduli problem at hand. Moreover, we want to do this *algebraically*: namely, we work over the algebraic moduli stack $\mathscr{A}_g^{\pm}$ of principally polarized abelian schemes with theta characteristic. This is an algebraic stack which is smooth over $\mathbb{Z}[1/2]$ ([7, §VIII.3.2.4]), and whose analytification is the analytic stack $\mathscr{A}_{g,\mathrm{an}}^{\pm}$ defined above. The formation of ϑ_g can be carried out in this algebraic setting as well. However, any analytic statement such as (1.1) seems now meaningless, since ϑ_g is no longer an analytic function on $\mathfrak{h}_g$, but rather a section

of an algebraic line bundle over an algebraic stack. Our goal then is to translate the functional equation into a statement which makes sense over any commutative ring, and not just over the complex numbers.

The first results in this direction have been obtained in [7, App. 1 (the "canonical key formula")], where the Grothendieck–Riemann–Roch Theorem is employed to show that the *determinant bundle*

$$\Delta(\Theta) := (\pi_*\Theta)^{\otimes 2} \otimes \underline{\omega},$$

where $\underline{\omega}$ is the Hodge bundle of π, is a canonical torsion element in $\mathrm{Pic}(\mathscr{A}_g^{\pm})$. This fact alone shows that some high power r of the algebraic Riemann theta function ϑ_g is an algebraic Siegel modular form of weight $r/2$, i.e. a section of $\underline{\omega}^{r/2}$. This result was later improved by [3, Thm. I.5.1], who show that the smallest such r is 8 and essentially that $\Delta(\Theta)$ has the structure of a μ_4-torsor over $\mathscr{A}_g^{\pm}$. Moreover, it was shown in [8] that the trivialization of $\Delta(\Theta)^{\otimes 4}$ can be computed analytically using the functional equation of Riemann's theta function. In fact, the functional equation (1.1) and its higher-dimensional analogs, together with the above-mentioned results, suggest that there must exist a natural "theta multiplier bundle" $\mathcal{M}(\Theta)$, a μ_4-torsor over $\mathscr{A}_g^{\pm}$, and an isomorphism

$$(1.3) \qquad\qquad \Delta(\Theta) \xrightarrow{\simeq} \mathcal{M}(\Theta)^{-1}$$

of the underlying line bundles over $\mathscr{A}_g^{\pm}$. Over $\mathbb{C}$, $\mathcal{M}(\Theta)$ must correspond to the character

$$\lambda : \Gamma(1,2)_g^{\pm} \to \mu_4 \subseteq \mathbb{C}^{\times}$$

giving the 4-th roots of unity appearing in the transformation laws of $\vartheta_g^2(\tau)$. We construct the theta multiplier bundle $\mathcal{M}(\Theta)$ in Section 3 and prove the isomorphism (1.3) in Section 5. Moreover, we show in Section 6 that (1.3) is given analytically by the functional equation. The current article can thus be viewed as a refinement of [3, Thm. I.5.1], who give a canonical trivialization of $\Delta(\Theta)^{\otimes 4}$, and [8], where such trivialization is computed in terms of the functional equation.

Our construction of theta multiplier bundles is based on a geometric interpretation of λ, which could be of independent interest. This interpretation is due to Pierre Deligne (private communication), and is given in Section 2. In the even case, a different geometric interpretation of λ had been already given in the beautiful paper [5]: that construction was also one of the original inspirations for this article and the relation with our λ is given at the end of Section 2.

There are many other interesting modular forms on levels other than $\Gamma_g^{\pm}(1,2)$, whose algebraic construction can be similarly worked out. Indeed,

the isomorphism (1.3) can be generalized to the case when Θ is replaced by a relatively ample line bundle of higher degree, in which case (1.3) is replaced by an isomorphism of vector bundles. This level of generality requires the machinery of Heisenberg groups ([7], [11, §5], and [15]) and is contained in the sequel of the present paper [1]. This sequel gives algebraic definitions of modular form (of integral and half-integral weight) taking values in Weil representations: in particular, it provides algebraic counterparts to modular forms of half-integral weight in the classical sense, that of Shimura. It should also be possible to employ similar ideas to obtain analogous results for non-degenerate line bundles of higher index, although this direction has not yet been pursued.

2. Symplectic 4-groups with theta characteristic

In this section we study free $\mathbb{Z}/4\mathbb{Z}$-modules V equipped with a symplectic form, together with a quadratic form on the reduction of V modulo 2 whose associated bilinear form is the reduction of the symplectic form modulo 2. For these objects we construct a type of "determinant" λ (Theorem 2.4), which lies at the heart of the construction of the theta multiplier bundles appearing in the isomorphism 1.3 in the Introduction (Theorem 5.1 below). The construction of λ was inspired by [5], and is due to Pierre Deligne (private communication).

DEFINITION 2.1. — *A symplectic 4-group is a free $\mathbb{Z}/4\mathbb{Z}$-module V of rank $2g$ together with a non-degenerate alternating bilinear form*

$$\psi : V \times V \longrightarrow \mathbb{Z}/4\mathbb{Z}.$$

Let $\overline{V} := V/2V$ be the free $\mathbb{Z}/2\mathbb{Z}$-module of rank $2g$ obtained from V by reducing the coordinates of V modulo 2. The bilinear form 2ψ descends to the quotient $\overline{V}$ to give a non-degenerate alternating (thus symmetric) bilinear form $\overline{\psi}$ on $\overline{V}$. Suppose then we are given a function

$$q : \overline{V} \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

which is a quadratic form for $\overline{\psi}$, i.e.

$$q(v_1 + v_2) - q(v_1) - q(v_2) = \overline{\psi}(v_1, v_2), \quad \text{for all } v_1, v_2 \in \overline{V}.$$

DEFINITION 2.2. — *The triple* (V, ψ, q) *is called a symplectic 4-group with theta characteristic. The group* $\Gamma := \mathrm{Aut}(V, \psi, q)$, *of* $\mathbb{Z}/4\mathbb{Z}$*-linear automorphisms* $\varphi : V \to V$ *such that* φ *preserves* ψ *and* $\overline{\varphi}$ *preserves* q, *is called the* theta group *of* (V, ψ, q) *(cf. [5]).*

Up to isomorphism, there are only two quadratic forms q for each g. We say that q is *even* if there exists a subspace $L \subseteq \overline{V}$ such that $q(L) = 0$ and $\dim L = g$ (i.e. L is *maximal isotropic*), and *odd* otherwise. We say that the symplectic 4-group (V, ψ) with theta characteristic q is *even* or *odd*, according to the parity of q. In terms of the Arf invariant, q is even if $\mathrm{Arf}(q) = 0$ and odd otherwise. Accordingly, the isomorphism class of the theta group of (V, ψ, q) is entirely determined by g and by the parity of q. There are special elements of this group which are essential to understand its structure.

DEFINITION 2.3 ([5, §1]). — *A* anisotropic transvection *is any linear map* $t_v \in \Gamma$ *of the form*

$$t_v(x) = x + \psi(v, x)\, v,$$

where v *is a vector such that* $q(\overline{v}) \neq 0$.

Note that any such anisotropic transvection t_v satisfies
  (i) $t_v^2(x) = x + 2\,\psi(v, x)\, v$,
  (ii) $\gamma t_v \gamma^{-1} = t_{\gamma v}, \;\; \gamma \in \Gamma$.
In particular, (i) implies that t_v^2 reduces to the identity modulo 2 and (ii) implies that all anisotropic transvections are conjugates of each other.

2.1.

Given a symplectic 4-group with theta characteristic (V, ψ, q), denote by $\Gamma(2)$ the kernel of the "reduction modulo 2" homomorphism $\mathrm{GL}(V) \to \mathrm{GL}(\overline{V})$, and let $\mathrm{O}(\overline{V}, q)$ be the orthogonal group of the quadratic module $(\overline{V}, q)$ modulo 2. The theta group Γ is then a group extension

(2.1) $$0 \to \Gamma(2) \to \Gamma \xrightarrow{\gamma \mapsto \overline{\gamma}} \mathrm{O}(\overline{V}, q) \to 0.$$

Each flanking term in this exact sequence is endowed with a natural nontrivial homomorphism to a cyclic group of order 2, which we now describe.

First, we have the *Dickson invariant* (e.g. [2, §3])

$$D_q : \mathrm{O}(\overline{V}, q) \longrightarrow \mathbb{Z}/2\mathbb{Z},$$

defined by the action of $\mathrm{O}(\overline{V}, q)$ on $Z(\mathrm{Cliff}^+(\overline{V}, q))$. Given an isomorphism $\mathrm{O}(\overline{V}, q) \simeq \mathrm{O}^\pm(2g, 2)$, there are explicit formulas for $D_q(t)$ which are quadratic in the entries of the matrix t. Alternatively, D_q is given by the formula ([16, §3.8.1])

$$(2.2) \qquad\qquad D_q(t) = \mathrm{rk}(\mathrm{id} + t) \mod 2.$$

Second, there is a homomorphism

$$q : \Gamma(2) \longrightarrow \mathbb{Z}/2\mathbb{Z},$$

canonically induced by q, defined as follows. There is an isomorphism

$$\Gamma(2) \xrightarrow{\simeq} \mathfrak{sp}(\overline{V}, \overline{\psi})$$
$$\mathrm{id} + 2\beta \longmapsto \beta,$$

as $\mathbb{Z}/2\mathbb{Z}$-vector spaces of rank $g(2g+1)$, and the symmetric bilinear form $\overline{\psi}$ induces a canonical isomorphism

$$\mathfrak{gl}(\overline{V}, \overline{\psi}) \simeq \mathrm{End}(\overline{V}) \simeq \overline{V}^* \otimes \overline{V} \xrightarrow{\overline{\psi}} \overline{V} \otimes \overline{V},$$

under which $\mathfrak{sp}(\overline{V}, \overline{\psi})$ corresponds to the subspace of symmetric 2-tensors $\Sigma^2(\overline{V})$. Therefore $\Gamma(2)$ and $\Sigma^2(\overline{V})$ are canonically isomorphic as $\mathbb{Z}/2\mathbb{Z}$-vector spaces. The quadratic form q now induces a linear form $q : \Sigma^2(\overline{V}) \longrightarrow \mathbb{Z}/2\mathbb{Z}$, by the universal property of $\Sigma^2(\overline{V})$ with respect to degree 2 maps.

It turns out that the two homomorphisms D_q and q can be combined to construct a remarkable $\mathbb{Z}/4\mathbb{Z}$-valued character on $\Gamma := \mathrm{Aut}(V, \psi, q)$, whose existence implies the existence of the theta multiplier bundle appearing in our main Theorem 5.1.

THEOREM 2.4. — *Let (V, ψ, q) be a symplectic 4-group with theta characteristic, and let $\Gamma = \mathrm{Aut}(V, \psi, q)$ be its theta group. Then there is a unique group homomorphism*

$$\lambda : \Gamma \longrightarrow \mathbb{Z}/4\mathbb{Z},$$

such that

    (1) *$\lambda|_{\Gamma(2)} = 2 \cdot q$, where $\mathbb{Z}/2\mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z}/4\mathbb{Z}$ is the canonical injection,*
    (2) *$\lambda(\gamma) \equiv D_q(\overline{\gamma}) \mod 2$, for all $\gamma \in \Gamma$,*
    (3) *$\lambda(t_v) = 1$, for any anistropic transvection t_v.*

Proof. — The orthogonal group $\mathrm{O}(\overline{V}, q)$, by definition, preserves the quadratic form q and therefore it preserves $\ker(q : \Gamma(2) \to \mathbb{Z}/2\mathbb{Z})$ under the outer action given by (2.1). The group $\ker(q)$ is thus normal in Γ, and the quotient is a central extension

$$(2.3) \qquad 0 \to \Gamma(2)/\ker(q) \to \Gamma/\ker(q) \xrightarrow{p} \mathrm{O}(\overline{V}, q) \to 0.$$

Indeed, $\Gamma(2)/\ker(q) = \{\pm \mathrm{id} + \ker(q)\}$ since $q(-\mathrm{id}) = \det(\overline{\psi})$ and $\overline{\psi}$ is non-degenerate.

Next, consider $\Omega(\overline{V}, q) := \ker(D_q)$, the "special orthogonal group" in characteristic 2. The central extension

$$0 \to \Gamma(2)/\ker(q) \to p^{-1}(\Omega(\overline{V}, q)) \xrightarrow{p} \Omega(\overline{V}, q) \to 0,$$

deduced from (2.3), has a *unique* splitting σ for all $g \geqslant 5$. In fact, for such g the group $\Omega(\overline{V}, q)$ is simple with trivial Schur multiplier ([16, §3.8.2]). We may thus form the quotient $G := (\Gamma/\ker(q))/\sigma(\Omega(\overline{V}, q))$, a central extension of $\Gamma(2)/\ker(q) \overset{q}{\simeq} \mathbb{Z}/2\mathbb{Z}$ by $\mathrm{O}(\overline{V}, q)/\Omega(\overline{V}, q) \overset{D_q}{\simeq} \mathbb{Z}/2\mathbb{Z}$.

We now claim that there is an isomorphism of G with $\mathbb{Z}/4\mathbb{Z}$. To find this isomorphism, let t_v be any anisotropic transvection. Clearly $D_q(\bar{t}_v) = 1$ from (2.2), so $t_v \neq 0 \in G$. Moreover, $t_v^2 \in \Gamma(2)$ so that

$$\lambda(t_v^2) = q(t_v^2) = q(\overline{v} \otimes \overline{v}) = q(\overline{v}) \neq 0,$$

thus t_v gives an element of exact order 4 in G.

To summarize, for $g \geqslant 5$ we have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(2) & \longrightarrow & \Gamma & \longrightarrow & \mathrm{O}(\overline{V}, q) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \to & \Gamma(2)/\ker(q) & \to & \Gamma/\ker(q) & \longrightarrow & \mathrm{O}(\overline{V}, q) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \\
0 & \to & \Gamma(2)/\ker(q) & \longrightarrow & G & \longrightarrow & \mathrm{O}(\overline{V}, q)/\Omega(\overline{V}, q) & \to & 0 \\
& & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \simeq\, t_v} & & \downarrow{\scriptstyle D_q} & & \\
0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\;2\cdot\;} & \mathbb{Z}/4\mathbb{Z} & \xrightarrow{\;\mathrm{mod}\ 2\;} & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0.
\end{array}
$$

We define $\lambda : \Gamma \to \mathbb{Z}/4\mathbb{Z}$ to be the homomorphism given by composing the arrows in the middle vertical column. Properties (1) and (2) are then clear from the definition. Property (3), and uniqueness, follow by requiring one (all) anisotropic transvection t_v to map to 1 under the isomorphism $G \simeq \mathbb{Z}/4\mathbb{Z}$. If $g < 5$, we may choose an embedding

$(V, \psi, q) \hookrightarrow (V \oplus V', \psi \oplus \psi', q \oplus q')$ into a symplectic 4-group $V \oplus V'$ with large enough g, in which case there is a canonical injection

$$\mathrm{Aut}(V, \psi, q) \hookrightarrow \mathrm{Aut}(V \oplus V', \psi \oplus \psi', q \oplus q'),$$

and we may define λ on $\Gamma = \mathrm{Aut}(V, \psi, q)$ by restriction. $\qquad\square$

By construction, λ satisfies the following important compatibility, whose proof we omit since it follows from an elementary computation.

PROPOSITION 2.5. — *For any two symplectic 4-groups with theta characteristic* $(V, \psi, q), (V', \psi', q')$, *the diagram*

$$
\begin{array}{ccc}
\Gamma(V) \times \Gamma(V') & \longrightarrow & \Gamma(V \oplus V') \\
\downarrow{\scriptstyle \lambda(V)+\lambda(V')} & & \downarrow{\scriptstyle \lambda(V \oplus V')} \\
\mathbb{Z}/4\mathbb{Z} & \xrightarrow{\ =\ } & \mathbb{Z}/4\mathbb{Z}
\end{array}
$$

is commutative.

We now compute the character λ explicitly in the $g = 1$ case. These computations will later be used in the proof of our main Theorem 5.1.

2.2. Case $g = 1$, q even

By choosing a basis for V we may assume that $V \simeq \mathbb{Z}/4\mathbb{Z}^{\oplus 2}$, ψ is the standard symplectic form

$$\psi(x_1, y_1, x_2, y_2) = x_1 y_2 - y_1 x_2,$$

and $q(x, y) = xy$. In this case $\mathrm{O}(\overline{V}, q) \simeq \mathrm{O}^+(2, 2)$,

$$\mathrm{O}^+(2, 2) = \left\{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \right\} \simeq \mathbb{Z}/2\mathbb{Z},$$

and $\Gamma(2) \simeq \mathbb{Z}/2\mathbb{Z}^{\oplus 3}$. The theta group Γ is generated by $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T^2 = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$. Now $T^2 \in \Gamma(2)$ and it corresponds to the symmetric 2-tensor $\overline{v_1} \otimes \overline{v_1}$, where $v_1 = (1, 0)$. Thus $\lambda(T^2) = q(\overline{v_1} \otimes \overline{v_1}) = q(1, 0) = 0$. On the other hand $D_q(\overline{S}) = 1$, thus $\lambda(S) = \pm 1$. The sign can be fixed by choosing an anisotropic transvection. For example, the vector $(1, 1) \in V$ reduces mod 2 to the unique anisotropic vector of $\overline{V}$, and the corresponding transvection t_v is given by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 2 \end{smallmatrix}\right)$. Setting $\lambda(t_v) = 1$ then forces $\lambda(S) = -1$, since $S\, t_v = T^2$.

2.3. Case $g = 1$, q odd

Again let $V = \mathbb{Z}/4\mathbb{Z}^{\oplus 2}$, ψ as above and q now given by $q(x, y) = x^2 + y^2 + xy$. We have $\mathrm{O}(\overline{V}, q) \simeq \mathrm{O}^-(2, 2) \simeq S_3$ and $\Gamma(2) \simeq \mathbb{Z}/2\mathbb{Z}^{\oplus 3}$. The theta group Γ is isomorphic to $\mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$ and is generated by S and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Now $T = t_{v_1}$ is the anisotropic transvection of the vector $v_1 = (1, 0)$, thus $\lambda(T) = 1$. On the other hand $\lambda(S) = \pm 1$ as before. To fix the sign, let $v = (1, 1)$ as before and apply λ to $S\,t_v = T^2$ to obtain $\lambda(S) = 1$.

2.4. Case $g \geqslant 3$, q even

By Proposition 2.5, in order to compute λ explicitly it suffices to find a formula for it in the case of q even and g large (say $g \geqslant 3$). In this case, a different construction of λ has already been given in [5], together with a remarkably simple algorithm to compute it. We now recall this construction and show how it is a special case of ours.

DEFINITION 2.6. — *Let (V, ψ, q) be a symplectic 4-group of rank $2g$ with even theta characteristic. A free $\mathbb{Z}/4\mathbb{Z}$-submodule $L \subseteq V$ is called an isotropic lagrangian submodule if it is a direct summand of V of rank g, with $\psi = 0$ on $L \times L$ and $q(\overline{L}) = 0$, where $\overline{L} \subseteq \overline{V}$ is the reduction of L modulo 2.*

Let L be any isotropic lagrangian, and let $\{v_i\}_{i=1}^g$ be a $\mathbb{Z}/4\mathbb{Z}$-basis for it. Any two such bases differ by a uniquely defined element of $\mathrm{GL}(L)$, of determinant ± 1. We may thus define an equivalence relation on the set of all bases for L by declaring $\{v_i\}_{i=1}^g \sim \{v_i'\}_{i=1}^g$ if they differ by an element of determinant 1. There are only 2 such equivalence classes, which we call *orientations* of L.

DEFINITION 2.7. — *A pair $(L, [\{v_i\}_{i=1}^g])$ of an isotropic lagrangian L and a choice of orientation $[\{v_i\}_{i=1}^g]$ is called an* oriented *isotropic lagrangian. The set of all such pairs is denoted by $\Lambda_0(V)$.*

The authors of [5] define the function $m_{JM} : \Lambda_0(V) \times \Lambda_0(V) \to \mathbb{Z}/4\mathbb{Z}$ by

$$(L_1, L_2) \longmapsto \sigma(L_1, L_2) + (g - \dim \overline{L}_1 \cap \overline{L}_2) - 1 \mod 4,$$

where $\sigma(L_1, L_2) \in \{\pm 1\}$ is a sign function, depending on the orientations, defined as follows: if $L_1 \cap L_2 = \{0\}$ (i.e. L_1 and L_2 are *transversal*), then $\psi : L_1 \to L_2^*$ is an isomorphism, and σ is the determinant of the matrix of this isomorphism with respect to the orientations given. If $\overline{L}_1 = \overline{L}_2$, then

we choose an isotropic lagrangian D which is transversal to both L_1 and L_2 and set $\sigma(L_1, L_2) = \sigma(L_1, D)\sigma(L_2, D)$. All other cases can be reduced to these two ([5, §2]). To define a character $\Gamma \to \mathbb{Z}/4\mathbb{Z}$, fix an oriented isotropic lagrangian L_0 and let

$$\lambda_{JM}(\gamma) := m_{JM}(L_0, \gamma L_0).$$

This is a homomorphism satisfying $\lambda_{JM}(t_v) = 1$ ([5, §3]) for any anisotropic transvection t_v, thus $\lambda_{JM} = \lambda$ since the abelianization of Γ is equal to $\mathbb{Z}/4\mathbb{Z}$, generated by the conjugacy classes of anisotropic transvections ([5, Thm. 1.1.(i)]).

Conversely, the function m_{JM} is entirely determined by the character λ_{JM}, since this is trivial on commutators. In particular, m_{JM} can be recovered from our definition of λ. To see this, note that the theta group $\Gamma = \mathrm{Aut}(V, \psi, q)$ acts transitively on $\Lambda_0(V)$, and we may define a function

$$m : \Lambda_0(V) \times \Lambda_0(V) \longrightarrow \mathbb{Z}/4\mathbb{Z}$$
$$(L_1, L_2) \longmapsto \lambda(\gamma_{1,2}),$$

by choosing $\gamma_{1,2} \in \Gamma$ so that $\gamma_{1,2}L_1 = L_2$.

LEMMA 2.8. — *The function m is well-defined, and $m = m_{JM}$.*

Proof. — Choose an oriented isotropic lagrangian L and a splitting $V = L \oplus M$ compatible with ψ and q. Let $\Gamma(L)$ be the stabilizer of L under the action of Γ on $\Lambda_0(V)$, i.e. the subgroup of all $\gamma \in \Gamma$ which preserve L, along with its orientation. We need to show that λ factors through $\Gamma(L)$. Now any element of $\Gamma(L)$, with respect to the chosen orientation, has the form $\left(\begin{smallmatrix} A & B \\ 0 & A^{-1,t} \end{smallmatrix}\right)$, where $A \in \mathrm{SL}(L)$. In other words, $\Gamma(L)$ is a group extension

$$0 \to T \longrightarrow \Gamma(L) \overset{\gamma \mapsto A}{\longrightarrow} \mathrm{SL}(L) \to 0,$$

where elements of T are of the form $\left(\begin{smallmatrix} I & B \\ 0 & I \end{smallmatrix}\right)$. Since $\mathrm{SL}(L)$ has no non-trivial characters, it suffices to show that λ factors through T. Indeed, T is a group extension

$$0 \to T(2) \longrightarrow T \overset{\mathrm{mod}\ 2}{\longrightarrow} \overline{T} \to 0,$$

where elements of $T(2) \subseteq \Gamma(2)$ are of the form $\left(\begin{smallmatrix} I & B \\ 0 & I \end{smallmatrix}\right) = I + 2\left(\begin{smallmatrix} 0 & B' \\ 0 & 0 \end{smallmatrix}\right)$. These elements map under $\overline{\psi}$ to symmetric 2-tensors in $\Sigma^2(L)$, where the value of $\lambda = q$ is always 0 since L is isotropic. Thus λ factors through $T(2)$. But $\lambda(\gamma) \equiv D_q(\overline{\gamma}) \mod 2$, and clearly the elements of T have even rank mod 2, thus λ factors through all of T. Now the value of m is entirely determined by $\lambda = \lambda_{JM}$, and the same is true for m_{JM}. □

Remark 2.9. — For q odd, note that $\Lambda_0(V) = \emptyset$ and the geometric construction of [5] cannot be applied directly.

3. Theta multiplier bundles

Let now S be a scheme over $\mathbb{Z}[1/2, i]$, where i denotes a choice of primitive 4th root of unity. Let $K \to S$ be a finite étale commutative group scheme with geometric fibers isomorphic to $\mathbb{Z}/4\mathbb{Z}^{\oplus 2g}$, together with a non-degenerate symplectic pairing

$$e_K : K \times K \longrightarrow \mu_4,$$

where μ_N is the finite flat S-group scheme of N-th roots of unity. We call (K, e_K) a *symplectic 4-group scheme* of rank $2g$. The group scheme $\overline{K} := K/2K$ is endowed with the alternating (thus symmetric) pairing $e_{\overline{K}} := e_K^2$. Let

$$e_*^K : \overline{K} \longrightarrow \mu_2$$

be a quadratic character for $e_{\overline{K}}$.

DEFINITION 3.1. — *The triple* (K, e_K, e_*^K) *is called a* symplectic 4-group scheme with theta characteristic. *The theta group of* (K, e_K, e_*^K) *is the finite étale group scheme* $\underline{\Gamma}_S$ *representing the functor*

$$\underline{\mathrm{Aut}}(K, e_K, e_*^K)(T \to S) := \mathrm{Aut}(K \times_S T, e_K, e_K^*).$$

The character λ of Theorem 2.4, composed with $[i] : \underline{\mathbb{Z}/4\mathbb{Z}}_S \simeq \mu_4$, gives by descent a group scheme homomorphism

$$(3.1) \qquad\qquad \lambda : \underline{\Gamma}_S \longrightarrow \mu_4,$$

which is compatible under taking direct sums, as in Proposition 2.5.

3.1.

Given a symplectic 4-group scheme with theta characteristic (K, e_K, e_*^K), consider the constant triple $(\underline{\mathbb{Z}/4\mathbb{Z}}_S^{2g}, e_4, e_*^{\pm})$ of rank $2g$, equipped with the standard symplectic form e_4 and a theta characteristic $e_*^{\pm}$ of the same type (i.e. even or odd) as that of e_*^K. The functor on S-schemes given by

$$\{T \to S\} \mapsto \mathrm{Isom}\left((K \times_S T, e_K, e_*^K), (\underline{\mathbb{Z}/4\mathbb{Z}}_S^{2g}, e_4, e_*^{\pm})\right)$$

is representable by a $\underline{\Gamma}_S$-torsor $\underline{\mathrm{Isom}}_S\left((K, e_K, e_*^K), (\underline{\mathbb{Z}/4\mathbb{Z}}_S^{2g}, e_4, e_*^\pm)\right)$. Define

$$\mathcal{M}(K, e_K, e_*) := \lambda_* \underline{\mathrm{Isom}}\left((K, e_K, e_*^K), (\underline{\mathbb{Z}/4\mathbb{Z}}_S^{2g}, e_4, e_*^\pm)\right),$$

a μ_4-torsor over S, whose formation is compatible under base-change.

DEFINITION 3.2. — *Given a symplectic 4-group scheme* $(K \to S, e_K, e_*^K)$ *with theta characteristic, the* μ_4-*torsor* $\mathcal{M}(K, e_K, e_*^K)$ *over* S *is called the theta multiplier bundle associated to* (K, e_K, e_*).

By Proposition 2.5, the formation of theta multiplier bundles is compatible with direct sums. More precisely, given any two symplectic 4-group schemes $(K, e_K, e_*^K), (K', e_{K'}, e_*^{K'})$ with theta characteristic over S, there is a μ_4-torsor isomorphism

$$(3.2) \quad \mathcal{M}(K \oplus K', e_K \oplus e_{K'}, e_*^K \oplus e_*^{K'}) \xrightarrow{\cong} \mathcal{M}(K, e_K, e_*^K) \otimes \mathcal{M}(K', e_{K'}, e_*^{K'}).$$

4. Determinant line bundles on abelian schemes

Having described in Section 3 the term on the right-hand side of the isomorphism (1.3), we now turn to the left-hand side, the so called "determinant bundle". We assume from now on that all schemes are locally noetherian and that all abelian schemes are projective, as in [13, §6.2].

4.1.

Let $\pi : A \to S$ be an abelian scheme of relative dimension g with identity section $e : S \to A$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_A$-module which has been normalized at the identity, i.e. we have chosen an $\mathcal{O}_S$-module isomorphism $e^*\mathcal{L} \simeq \mathcal{O}_S$. Any such invertible $\mathcal{O}_A$-module defines a canonical morphism $\varphi_\mathcal{L} : A \longrightarrow A^t$ to the dual abelian scheme. Let

$$K(\mathcal{L}) := \ker \phi_\mathcal{L},$$

a commutative group scheme over S. When $\mathcal{L}$ is relatively ample, $K(\mathcal{L})$ is finite flat over S and it is canonically endowed with a non-degenerate symplectic pairing

$$e_\mathcal{L} : K(\mathcal{L}) \times K(\mathcal{L}) \longrightarrow \mathbb{G}_m.$$

In this case, $\mathrm{rk}(K(\mathcal{L})) = d^2$, where d is the degree of $\mathcal{L}$, so that $K(\mathcal{L})$ is étale over $S[1/d]$ ([10, §1], [11, §6]).

Suppose next that S is a scheme where $1/2 \in \mathcal{O}_S$ and suppose that the (relatively ample, normalized) invertible sheaf $\mathcal{L}$ is *symmetric*. This means that there is a (unique) isomorphism of normalized invertible $\mathcal{O}_A$-modules $\iota_{\mathcal{L}} : [-1]^*\mathcal{L} \xrightarrow{\simeq} \mathcal{L}$, where $[-1] : A \to A$ is the inversion morphism. In this situation the isomorphism $\iota_{\mathcal{L}}$, restricted to the fixed locus $A[2]$ of $[-1]^*$, is multiplication by ± 1 and thus it defines a function

$$e_*^{\mathcal{L}} : A[2] \longrightarrow \mu_2.$$

This function is quadratic for the symmetric pairing $e_{\mathcal{L}^2}$ ([10, §2, Cor. 1]).

PROPOSITION 4.1. — *Let Θ be a normalized, relatively ample, symmetric invertible sheaf of degree $d = 1$ over an abelian scheme $A \to S$ of relative dimension g. Then $(K(\Theta^4), e_{\Theta^4}, e_*^{\Theta})$ is a symplectic 4-group of rank $2g$ with theta characteristic.*

Proof. — The invertible sheaf Θ^2 (resp. Θ^4) is also relatively ample and of degree 2^g (resp. 4^g), so that $K(\Theta^2)$ (resp. $K(\Theta^4)$) is finite étale of rank 2^{2g} (resp. 4^{2g}). In fact, since $\varphi_{\mathcal{L}}$ is additive in $\mathcal{L}$, there is a canonical isomorphism

$$K(\Theta^2) = \ker 2 \, \varphi_L \simeq A[2]$$

and similarly $K(\Theta^4) = \ker 4 \, \varphi_L \simeq A[4]$. In particular, the function e_*^{Θ} is a theta characteristic on the symplectic 4-group scheme $(K(\Theta^4), e_{\Theta^4})$. $\qquad\square$

Suppose additionally that S is over $\mathbb{Z}[1/2, i]$. Given Θ as in Proposition 4.1, we can form the corresponding theta multiplier bundle (Definition 3.2)

$$(4.1) \qquad \mathcal{M}(\Theta) := \mathcal{M}(K(\Theta^4), e_{\Theta^4}, e_*^{\Theta}),$$

which is a μ_4-torsor over S.

PROPOSITION 4.2. — *The formation of $\mathcal{M}(\Theta)$ is compatible under base-change; that is, given any two pairs (A, Θ) and (A', Θ') as in Proposition 4.1, a base-change morphism of abelian schemes*

$$
\begin{array}{ccc}
A' \overset{\phi}{\simeq} A \times_S S' & \longrightarrow & A \\
\downarrow & & \downarrow \\
S' \overset{\varphi}{\longrightarrow} & & S,
\end{array}
$$

together with an $\mathcal{O}_{S'}$-module isomorphism $\psi : \phi^*\Theta \simeq \Theta'$, then there is a canonical isomorphism $\varphi^*\mathcal{M}(\Theta) \simeq \mathcal{M}(\Theta')$ as μ_4-torsors.

Proof. — This is clear, since the formation of the symplectic 4-group $(K(\Theta^4), e_{\Theta^4})$ and the theta characteristic e_*^Θ are all compatible under such base-change. $\square$

In addition, the formation of $\mathcal{M}(\Theta)$ is compatible under taking direct products, i.e. for any two pairs $(A_1, \Theta_1), (A_2, \Theta_2)$ there is a canonical μ_4-torsor isomorphism

$$(4.2) \qquad \mathcal{M}(\Theta_1 \boxtimes \Theta_2) \simeq \mathcal{M}(\Theta_1) \otimes \mathcal{M}(\Theta_2),$$

as follows from (3.2).

4.2.

There is another μ_4-torsor that can be canonically attached to the pair (A, Θ) of Proposition 4.1. Namely, let

$$(4.3) \qquad \Delta(\Theta) := (\pi_*\Theta)^{\otimes 2} \otimes \underline{\omega}_{A/S}$$

be the *determinant bundle*, where $\underline{\omega}_{A/S} := \det(\pi_*\Omega^1_{A/S})$ is the Hodge bundle of $\pi : A \to S$. Since Θ is relatively ample of degree 1, the determinant bundle is an invertible sheaf over S ([7, VIII.1.0]), and its formation is compatible with base-change ([7, VIII.1.1.1]). Moreover, there is an $\mathcal{O}_S$-module isomorphism ([3, Thm. I.5.1])

$$(4.4) \qquad \Delta(\Theta)^{\otimes 4} \simeq \mathcal{O}_S$$

which is compatible under base-change ([14, Remark after Thm. 0.2]) giving $\Delta(\Theta)$ a canonical μ_4-torsor structure over S.

Remark 4.3. — The right level of generality in which to study determinant bundles is that of a non-degenerate symmetric line bundle $\mathcal{L}$ over A of arbitrary degree $d \geqslant 0$, in which case $\Delta(\mathcal{L}) := (\det R\pi_*\mathcal{L})^{\otimes 2} \otimes \underline{\omega}_{A/S}$ is a "true" determinant ([3], [6], [7], [14]). Note that the cohomology of non-degenerate line bundles is concentrated in degree $i(\mathcal{L})$, the *index* of $\mathcal{L}$ ([12, §16]). Therefore, the complex $R\pi_*\mathcal{L}$ is concentrated in the term $R^{i(\mathcal{L})}\pi_*\mathcal{L}$, by [12, §II.5], so that

$$\Delta(\mathcal{L}) = (\det R^{i(\mathcal{L})}\pi_*\mathcal{L})^{\otimes(-1)^{i(\mathcal{L})}\cdot 2} \otimes \underline{\omega}_{A/S}.$$

By the Künneth formula, determinant bundles are also compatible under direct products, i.e. for any two pairs $(A_1, \Theta_1), (A_2, \Theta_2)$ there is a canonical μ_4-torsor isomorphism

$$(4.5) \qquad \Delta(\Theta_1 \boxtimes \Theta_2) \simeq \Delta(\Theta_1) \otimes \Delta(\Theta_2).$$

We now "compute" determinants in the genus 1 case, following [14, §5.1]. These computations will later be used in the proof of our main Theorem 5.1.

4.3. Case $g = 1$, e_* even

Let $\pi : E \to S$ be an elliptic curve over a scheme S with $1/2 \in \mathcal{O}_S$, and let $e : S \to E$ be its identity section. Let $P : S \to E$ be a non-trivial section of order 2. Then we claim that $\Theta := \mathcal{O}_E(P)$ is a normalized, relatively ample, symmetric invertible sheaf of degree 1 over E with even e_*^{Θ}. Indeed $e^*\Theta \simeq \mathcal{O}_S$ (since $P \neq e$), $\mathcal{O}_E(P)$ is ample of degree 1 on geometric fibers and $\mathcal{O}_E(P) \simeq \mathcal{O}_E([-P])$ since P is 2-torsion. The theta characteristic is even, since e_*^{Θ} is given by the formula ([10, §2, Prop. 2])

$$e_*^{\Theta}(x) = (-1)^{\mathrm{mult}_x(P) - \mathrm{mult}_e(P)}$$

where $x \in E[2]$. We claim that there is an isomorphism

(4.6) $$\Delta(\Theta) \simeq \underline{\omega}_{E/S},$$

induced by the isomorphism $\pi_*\mathcal{O}_E(P) \simeq \mathcal{O}_S$. The latter can be deduced from the adjunction exact sequence

$$0 \longrightarrow \mathcal{O}_E \longrightarrow \mathcal{O}_E(P) \longrightarrow P_*P^*\mathcal{O}_E(P) \longrightarrow 0.$$

Indeed, taking right-derived functors of π_* we get a long exact sequence

$$0 \to \pi_*\mathcal{O}_E \to \pi_*\mathcal{O}_E(P) \to P^*\mathcal{O}_E(P) \to R^1\pi_*\mathcal{O}_E \to R^1\pi_*\mathcal{O}_E(P) \to \ldots$$

But $R^1\pi_*\mathcal{O}_E(P) = 0$, as can be checked on the geometric fibers, and all the other sheaves in the exact sequence are locally free of rank 1, thus $\pi_*\mathcal{O}_E(P) \simeq \pi_*\mathcal{O}_E \simeq \mathcal{O}_S$.

4.4. Case $g = 1$, e_* odd

In this case, let

$$\Theta := \mathcal{O}_E(e) \otimes \Omega^1_{E/S}.$$

Note that $e^*\mathcal{O}_E(e) \simeq R^1\pi_*\mathcal{O}_E \simeq \underline{\omega}_{E/S}^{-1}$, as can be deduced as above by taking right-derived functors of π_* applied to the adjunction exact sequence

$$0 \longrightarrow \mathcal{O}_E \longrightarrow \mathcal{O}_E(e) \longrightarrow e_*e^*\mathcal{O}_E(e) \longrightarrow 0.$$

Therefore Θ is normalized. It is also symmetric, relatively ample of degree 1 as before. The characteristic is now odd, and there is an isomorphism

(4.7) $$\Delta(\Theta) \simeq \underline{\omega}_{E/S}^{\otimes 3},$$

which follows from the isomorphism $\pi_*\mathcal{O}_E(e) \simeq \mathcal{O}_E$ and by the projection formula applied to $\Omega^1_{E/S} = \pi^*\underline{\omega}_{E/S}$.

5. The canonical key formula

Let (A, Θ) be a pair of an abelian scheme $\pi : A \to S$ of relative dimension g, where S is a scheme over $R = \mathbb{Z}[1/2, i]$, together with a normalized, relatively ample, symmetric invertible sheaf Θ of degree $d = 1$. We have two canonical μ_4-torsors over S attached to this pair:

   (i) the *theta multiplier bundle*
$$\mathcal{M}(\Theta) := \mathcal{M}(K(\Theta^4), e_{\Theta^4}, e_*^\Theta)$$
   of (4.1), and
   (ii) the *determinant bundle*
$$\Delta(\Theta) := \pi_*\Theta^{\otimes 2} \otimes \underline{\omega}_{A/S}$$
   of (4.3).

This section is devoted to showing the "canonical key formula" below (Theorem 5.1, cf. the weaker $\mathrm{FCC}^{\mathrm{ab}}(\mathrm{Spec}(R), g, 1)$ of [7, VIII.1.2]), comparing the image of $\mathcal{M}(\Theta)$ and $\Delta(\Theta)$ under the natural map

$$\left\{ \begin{array}{c} \text{iso. classes of} \\ \mu_4\text{-torsors over } S \end{array} \right\} = H^1_{\text{ét}}(S, \mu_4) \to H^1_{\text{ét}}(S, \mathbb{G}_m)[4] = \mathrm{Pic}(S)[4].$$

The key observation in the proof of the formula is that the Picard groups of the classifying stacks $\widetilde{\mathscr{A}_g}$ of pairs (A, Θ) as above are very small ([14, Thm. 5.6]). Therefore to compare the two invertible sheaves it suffices to consider the case $g = 1$ and then use the respective compatibilities under direct products.

THEOREM 5.1 (Canonical Key Formula). — *For each pair $(A \to S, \Theta)$ as above, there is a functorial (i.e. compatible under base-change) $\mathcal{O}_S$-module isomorphism*
$$\Delta(\Theta) \simeq \mathcal{M}(\Theta)^{-1}$$
between the underlying invertible sheaves.

Proof. — Consider the algebraic stack $\widetilde{\mathscr{A}_g}$ over R classifying all pairs (A, Θ) as above (morphisms as in Proposition 4.2), and let $\Delta_g, \mathcal{M}_g$ be the μ_4-torsors defined over $\widetilde{\mathscr{A}_g}$ by the functors $(A, \Theta) \mapsto \Delta(\Theta)$ and $(A, \Theta) \mapsto \mathcal{M}(\Theta)$, respectively. To prove Theorem 5.1, it suffices to show that there exists an isomorphism $\Delta_g \simeq \mathcal{M}_g^{-1}$ as invertible sheaves over $\widetilde{\mathscr{A}_g}$. Now the

stack $\widetilde{\mathscr{A}_g}$ is smooth ([7, VIII.3.2.4]) with two irreducible components $\mathscr{A}_g^{\pm}$, each classifying pairs (A, Θ) with e_*^{Θ} even $(+)$ or odd $(-)$ ([8, Prop. 1.1.4]). For each $g_1, g_2 \geqslant 1$, there is a map

$$m_{g_1, g_2} : \widetilde{\mathscr{A}_{g_1}} \times \widetilde{\mathscr{A}_{g_2}} \longrightarrow \widetilde{\mathscr{A}_{g_1+g_2}}$$
$$((A_1, \Theta_1), (A_2, \Theta_2)) \longmapsto (A_1 \times A_2, \Theta_1 \boxtimes \Theta_2)$$

and μ_4-torsor isomorphisms ((4.5) and (4.2))

$$m_{g_1, g_2}^* \Delta_{g_1+g_2} \simeq \Delta_{g_1} \otimes \Delta_{g_2}$$
$$m_{g_1, g_2}^* \mathcal{M}_{g_1+g_2} \simeq \mathcal{M}_{g_1} \otimes \mathcal{M}_{g_2}.$$

Moreover, the map m_{g_1, g_2} sends $\mathscr{A}_{g_1}^- \times \mathscr{A}_{g_2}^-$ to $\mathscr{A}_{g_1+g_2}^+$, $\mathscr{A}_{g_1}^+ \times \mathscr{A}_{g_2}^+$ to $\mathscr{A}_{g_1+g_2}^+$ and so on ([8, §1.3]). It suffices then to prove $\Delta_g \simeq \mathcal{M}_g^{-1}$ over $\mathscr{A}_1^-$ and over $\mathscr{A}_g^+$, $g \geqslant 3$.

First, consider $\mathscr{A}_1^-$. Following a technique of Mumford ([9, §6]), we are going to construct an explicit isomorphism

$$\alpha_1^- : \operatorname{Pic}(\mathscr{A}_1^-)[4] \overset{\simeq}{\longrightarrow} \mathbb{Z}/4\mathbb{Z},$$

and then show that the images of $\Delta_1, \mathcal{M}_1^{-1}$ under α_1^- are equal. To do so, let k be an algebraically closed field of characteristic not 2, and consider the elliptic curve E/k given by the Weierstrass equation $y^2 = x^3 - x$. Let $\Theta_1^- := \mathcal{O}_E(e) \otimes \Omega^1_{E/k}$. The pair (E, Θ_1^-) is classified by a point (Section 4.4)

$$\kappa_1^- : \operatorname{Spec}(k) \longrightarrow \mathscr{A}_1^-.$$

The curve E has a special automorphism of order 4 defined by

$$(5.1) \qquad\qquad [i](x) = -x, \quad [i](y) = i\, y,$$

where $i \in k^{\times}$ is a choice of primitive 4-th root of unity. Under a suitable choice of basis for $E[4] \simeq (\mathbb{Z}/4\mathbb{Z})^2$, we may assume that the action of $[i]$ is given by the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and that $e_*^{\Theta_1^-}(a, b) = (-1)^{a+b+ab}$. Thus $[i]$ extends to an automorphism of the pair (E, Θ_1^-), since it preserves e_*. Now given an invertible sheaf $\mathcal{L}$ on $\mathscr{A}_1^-$, we may uniquely define a 4-th root of unity $\alpha_1^-(\mathcal{L})$ by

$$\alpha_1^-(\mathcal{L}) := [i]^* \in \operatorname{Aut}(\kappa_1^{-*}\mathcal{L}) \simeq k^{\times},$$

which gives a homomorphism $\alpha_1^- : \operatorname{Pic}(\mathscr{A}_1^-) \longrightarrow \mathbb{Z}/4\mathbb{Z}$. If we let $\mathcal{L} = \Delta_1$, then $\kappa_1^{-*}\Delta_1 \simeq \underline{\omega}_{E/k}^{\otimes 3}$ by (4.7). Now $\underline{\omega}_{E/k}$ is the 1-dimensional k-vector space generated by the regular differential $\frac{dx}{y}$ and thus

$$\alpha_1^-(\Delta_1) = i^3 = i^{-1},$$

as follows from the explicit formulas (5.1) defining $[i]$. Since i is primitive, α_1^- is surjective. Moreover, since there is only one odd, normalized, symmetric, relatively ample invertible sheaf of degree 1 on an elliptic curve, we have $\mathscr{A}_1^- = \mathscr{M}_{1,1}[1/2, i]$, where $\mathscr{M}_{1,1}$ is the moduli stack of elliptic curves, for which we have (e.g. [4])

$$\mathrm{Pic}(\mathscr{M}_{1,1}[1/2, i])[4] \simeq \mathbb{Z}/4\mathbb{Z}.$$

Therefore α_1^- is an isomorphism when restricted to $\mathrm{Pic}(\mathscr{A}_1^-)[4]$. To show that $\Delta_1 \simeq \mathcal{M}_1^{-1}$, note that

$$\alpha_1^-(\mathcal{M}_1^{-1}) = \lambda^{-1}\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = i^{-1},$$

as follows from the computations of Section 2.3.

Next, we employ the same technique for the case $g \geqslant 3$ even. In particular, we are going to construct an explicit isomorphism

$$\alpha_g^+ : \mathrm{Pic}(\mathscr{A}_g^+)[4] \xrightarrow{\simeq} \mathbb{Z}/4\mathbb{Z}$$

for all $g \geqslant 3$, and then prove that $\alpha_g^+(\Delta_g) = \alpha_g^+(\mathcal{M}_g^{-1})$. To do so, consider again the elliptic curve E/k. Let $\Theta_1^+ := \mathcal{O}_E(P)$, where P is the unique nontrivial point of order 2 fixed by $\mathrm{Aut}(E/k)$. The pair (E, Θ_1^+) is classified by a point (Section 4.3) $\kappa_1^+ : \mathrm{Spec}(k) \to \mathscr{A}_1^+$. If g is odd, consider the point

$$\kappa_g^+ : \mathrm{Spec}(k) \longrightarrow \mathscr{A}_g^+$$

classifying $(E^{\times g}, (\Theta_1^+)^{\boxtimes g})$. The special automorphism $[i]$ preserves the characteristic and it extends to an automorphism $[i]^{\times g}$ of the pair $(E^{\times g}, (\Theta_1^+)^{\boxtimes g})$. Given an invertible sheaf $\mathcal{L}$ on $\mathscr{A}_g^+$, the action of $[i]^{\times g}$ on $\kappa_g^{+\,*}(\mathcal{L})$ gives a homomorphism $\alpha_g^+ : \mathrm{Pic}(\mathscr{A}_g^+) \longrightarrow \mathbb{Z}/4\mathbb{Z}$. This is surjective, since (4.6) shows that

$$\kappa_g^{+\,*}(\Delta_g) \simeq \underline{\omega}_{E/k}^g = k\left(\frac{dx}{y}\right)^{\otimes g}$$

and therefore $\alpha_g^+(\Delta_g) = i^g$, a primitive 4-th root of unity. If g is even, apply the same argument with the automorphism $[i]^{\times g-1} \times \mathrm{id}$ replacing $[i]^{\times g}$. In both cases, there is an isomorphism ([14, Thm. 5.6])

$$\mathrm{Pic}(\mathscr{A}_g^+)[4] \simeq \mathbb{Z}/4\mathbb{Z}$$

thus α_g^+ restricts to the desired isomorphism on 4-torsion.

To prove that $\Delta_g \simeq \mathcal{M}_g^{-1}$, note that for g odd

$$\alpha_g^+(\mathcal{M}_g^{-1}) = \lambda^{-g}\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = i^g,$$

as follows from the computations of Section 4.3 and by the compatibility of λ under direct sums. Similarly, for g even we have

$$\alpha_g^+(\mathcal{M}_g^{-1}) = \lambda^{-g+1}\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = i^{g-1},$$

which shows that $\Delta_g \simeq \mathcal{M}_g^{-1}$ in all cases. $\qquad\square$

Remark 5.2. — Theorem 5.1 can also be proved using the classical transformation laws of theta functions. The advantage of using the "theta-function-free" approach above is that we can now use Theorem 5.1 to give a new proof of the transformation laws of theta functions (this is done in Section 6.1 below).

Remark 5.3. — Any two choices of isomorphism $\Delta_g \simeq \mathcal{M}_g^{-1}$ must differ by an element of $\Gamma(\mathscr{A}_g^\pm, \mathbb{G}_m)$. The same arguments as in [8, §1], show that $\Gamma(\mathscr{A}_g^\pm, \mathbb{G}_m) = R^\times$, thus the isomorphism of Theorem 5.1 is unique up to multiplication by a constant in $R^\times$.

6. Algebraic and analytic functional equations

We now explain how the canonical key formula (Theorem 5.1, or rather its refinement (6.1) below) can be viewed as the algebraic analog of the functional equation of Riemann's theta function. For simplicity, we only treat the case of even theta characteristic, the odd case differing only in the explicit analytic formulas for the theta function. Thus let $(\pi : \mathscr{A} \to \mathscr{A}_g^+, \Theta)$ be the universal pair over $\mathscr{A}_g^+ \to \mathrm{Spec}(R)$. The canonical key formula gives by duality an isomorphism

$$(\pi_*\Theta)^{-2} \simeq \mathcal{M}(\Theta) \otimes \underline{\omega},$$

where $\mathcal{M}(\Theta) = \mathcal{M}_g$ and $\underline{\omega}$ is the Hodge bundle of $\pi : \mathscr{A} \to \mathscr{A}_g^+$. The *bundle of half-forms*

$$\underline{\omega}_\Theta^{1/2} := \sqrt{\mathcal{M}(\Theta) \otimes \underline{\omega}},$$

a square-root which a priori can only be defined over a μ_2-gerbe over $\mathscr{A}_g^+$, descends to a well-defined invertible sheaf over $\mathscr{A}_g^+$, together with an isomorphism

$$(6.1) \qquad\qquad \iota_{\mathrm{alg}} : (\pi_*\Theta)^{-1} \overset{\simeq}{\to} \underline{\omega}_\Theta^{1/2}.$$

The algebraic Riemann theta function is a section of $(\pi_*\Theta)^{-1}$, defined as follows (e.g. [3, §I.5]): since Θ is normalized, there is a well-defined "evaluation-at-the-identity" map $\mathrm{ev} : e^*\Theta \longrightarrow \mathcal{O}_{\mathscr{A}_g^+}$ which by adjunction gives a section ϑ_g of $(\pi_*\Theta)^{-1} = \mathrm{Hom}(\pi_*\Theta, \mathcal{O}_{\mathscr{A}_g^+})$. By (6.1), we know that

ϑ_g maps isomorphically to a section of $\underline{\omega}_\Theta^{1/2}$, an "algebraic modular form" of weight $1/2$ over $\mathscr{A}_g^+$.

6.1.

Over the category of analytic spaces, isomorphism classes of line bundles over the analytic quotient stack $\left[\Gamma_g^+(1,2)\backslash\mathfrak{h}_g\right] = \mathscr{A}_{g,\mathrm{an}}^+$ are in 1-1 correspondence with group cohomology classes in $H^1(\Gamma_g^+(1,2), \mathcal{O}_{\mathfrak{h}_g}^\times)$, since $\mathfrak{h}_g$ is a Stein manifold. In particular, let $\mathrm{pr} : \mathfrak{h}_g \to \mathscr{A}_{g,\mathrm{an}}^+$ be the projection map, classifying the universal abelian variety

$$A_g := \mathbb{C}^g \times \mathfrak{h}_g/\Lambda_g, \quad \Lambda_g = \{(z,\tau) \in \mathbb{C}^g \times \mathfrak{h}_g : z \in \mathbb{Z}^g + \tau\mathbb{Z}^g\},$$

together with the symmetric line bundle Th over $A_g \to \mathfrak{h}_g$ defined by the divisor of the (2-variable) Riemann theta function $\sum_{n\in\mathbb{Z}^g} e^{\pi i\, {}^t n \tau n + 2\pi i {}^t n z}$. The line bundle $\mathrm{pr}^*\underline{\omega}_\Theta^{1/2}$ is trivialized over $\mathfrak{h}_g$ by the section

$$s_g := \sqrt{c\, \mathrm{d}z_1 \cdots \mathrm{d}z_g},$$

where c is a constant section of $\mathcal{M}(Th)$, the theta multiplier bundle over the universal abelian variety A_g, and $z = (z_1,\ldots,z_g)$ are coordinates on $\mathbb{C}^g$. This choice of trivialization corresponds to the 1-cocycle $j_{1/2} \in Z(\Gamma_g(1,2)^+, \mathcal{O}_{\mathfrak{h}_g^\times})$ given by

$$j_{1/2} : \gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \longmapsto \sqrt{\lambda(\gamma) \cdot \det(C\tau + D)},$$

where $\lambda : \Gamma_g^+(1,2) \to \mu_4$ is the character obtained from the λ of Section 2 by factoring through the principal congruence subgroup $\Gamma_g(4)$ (technically, this is a 1-cocycle of the metaplectic cover $\widetilde{\Gamma}_g^+(1,2)$, but formula (6.1) shows that the square-roots can be chosen in a compatible way so that the resulting cocycle descends to $\Gamma_g^+(1,2)$). On the other hand, the line bundle $\mathrm{pr}^*(\pi_*\Theta)^{-1}$ is trivialized over $\mathfrak{h}_g$ by Riemann's theta function $\vartheta_g(\tau) = \sum_{n\in\mathbb{Z}^g} e^{\pi i\, {}^t n \tau n}$, giving rise to a 1-cocycle

$$\gamma \longmapsto \vartheta_g(\gamma\tau)\vartheta_g(\tau)^{-1}.$$

By the classical functional equation of Riemann's theta function, the composition of the two trivializations s_g and ϑ_g is a $\Gamma_g^+(1,2)$-invariant isomorphism

$$\iota_{\mathrm{an}} : \mathrm{pr}^*(\pi_*\Theta)^{-1} \xrightarrow{\simeq} \mathrm{pr}^*\underline{\omega}_\Theta^{1/2},$$

and so it must descend to an isomorphism over the quotient $\mathscr{A}_{g,\mathrm{an}}^+$. Another such isomorphism is given by the base-change $\iota_{\mathrm{alg},\mathbb{C}}$ of (6.1) from $R = \mathbb{Z}[1/2, i]$ to $\mathbb{C}$. The composition $\iota_{\mathrm{alg},\mathbb{C}} \circ \iota_{\mathrm{an}}^{-1}$ is then a non-vanishing

analytic function on $\mathscr{A}_{g,\mathrm{an}}^+$ (compatible under products of abelian varieties) and is thus equal to a constant $c_0 \in \mathbb{C}$ ([8, 1.4.1]). Therefore

$$\iota_{\mathrm{alg},\mathbb{C}} = c_0\, \iota_{\mathrm{an}}.$$

In this sense, the isomorphism ι_{alg} of (6.1) can be viewed as the algebraic analog to the functional equation of Riemann's theta function, just as the classical, analytic functional equation is equivalent to the existence of the $\Gamma_g^+(1,2)$-invariant isomorphism ι_{an}.

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