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FINITE QUOTIENTS OF THREE-DIMENSIONAL COMPLEX TORI

by Patrick GRAF & Tim KIRSCHNER

ABSTRACT. — We provide a characterization of quotients of three-dimensional complex tori by finite groups that act freely in codimension one via a vanishing condition on the first and second orbifold Chern class. We also treat the case of free action in codimension two, using instead the “birational” second Chern class, as we call it.

Both notions of Chern classes are introduced here in the setting of compact complex spaces with klt singularities. In such generality, this topic has not been treated in the literature up to now. We also discuss the relation of our definitions to the classical Schwartz–MacPherson Chern classes.

RÉSUMÉ. — Nous fournissons une caractérisation des quotients des tores complexes de dimension trois par l’action libre en codimension un d’un groupe fini, par une condition d’annulation de la première et deuxième classe de Chern orbifold. Nous traitons aussi le cas des actions libres en codimension deux, utilisant la deuxième classe de Chern « birationnelle », comme nous l’appelons, au lieu de la classe de Chern orbifold.

Toutes les deux notions des classes de Chern sont introduites ici dans le cadre des espaces complexes compacts avec des singularités klt. Dans cette généralité, le sujet n’a pas été traité dans la littérature jusqu’à maintenant. Nous discutons aussi le rapport de notre définition aux classes de Chern classiques de Schwartz–MacPherson.

1. Introduction

Consider an n -dimensional compact Kähler manifold (X, ω) with $c_1(X) = 0$ and $\int_X c_2(X) \wedge \omega^{n-2} = 0$. The first condition implies, via Yau’s solution to the Calabi conjecture, that X can be equipped with a Ricci-flat Kähler metric [29]. As a consequence of the second condition, X is then uniformized by \mathbb{C}^n , i.e. the universal cover of X is affine space. Equivalently, X is the quotient of a complex torus T by a finite group G acting freely on T .

Keywords: Complex tori, torus quotients, vanishing Chern classes, second orbifold Chern class, Minimal Model Program, klt singularities.

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The philosophy of the Minimal Model Program (MMP) is that the most natural bimeromorphic models of a given Kähler manifold will in general have mild singularities [12]. From this point of view, it is certainly important to extend the above result to singular complex spaces. That is, one would like to have a criterion for a singular space X to be the quotient of a complex torus by a finite group acting freely *in codimension one*.

This problem has attracted considerable interest in the past, but results are available only in the projective case, i.e. for quotients of abelian varieties: see the article [25] by Shepherd-Barron and Wilson for the three-dimensional case, and the more recent ones [11] by Greb, Kebekus, and Peternell and [26] by Lu and Taji in higher dimensions. In this paper, we make a step towards settling the problem in general by proving the following uniformization results for Kähler threefolds with canonical singularities.

THEOREM 1.1 (Characterization of three-dimensional torus quotients, I). *Let X be a compact complex threefold with canonical singularities. The following are equivalent:*

- (1) *We have $c_1(X) = 0 \in H^2(X, \mathbb{R})$, and there exists a Kähler class $\omega \in H^2(X, \mathbb{R})$ such that $\tilde{c}_2(X) \cdot \omega = 0$.*
- (2) *There exists a 3-dimensional complex torus T and a holomorphic action of a finite group $G \subset T$, free in codimension one, such that $X \cong T/G$.*

Here $\tilde{c}_2(X)$ denotes the *second orbifold Chern class* of X , see Definition 5.2.

COROLLARY 1.2 (Characterization of three-dimensional torus quotients, II). — *Let X be a compact complex threefold with canonical singularities. The following are equivalent:*

- (1) *We have $c_1(X) = 0 \in H^2(X, \mathbb{R})$, and there exists a Kähler class ω on X as well as a resolution of singularities $f: Y \rightarrow X$, minimal in codimension two, such that*

$$\int_Y c_2(Y) \wedge f^*(\omega) = 0.$$

- (2) *There exists a 3-dimensional complex torus T and a holomorphic action of a finite group $G \subset T$, free in codimension two, such that $X \cong T/G$.*

Remark. — The second Chern class condition in (1) is a way of saying “ $c_2(X) \cdot \omega = 0$ ” that does not involve showing independence of the choice of resolution $f: Y \rightarrow X$. In Section 5, we discuss in detail both notions of

second Chern class appearing above. We also relate them to the classical Schwartz–MacPherson Chern classes (Remark 5.5).

Remark. — Assume that X is projective. In [25] and the other references cited above, $\tilde{c}_2(X)$ needs to intersect an ample Cartier divisor trivially, while for us it is sufficient to have a Kähler form with this property. In this sense, our result is new even in the projective case. Of course, a posteriori both conditions are equivalent, but this is precisely what we need to prove.

Further problems

Theorem 1.1 does not yield a full characterization of torus quotients because quotient singularities are in general not canonical, but only klt. Therefore it would be most natural to drop the a priori assumption on canonicity. Also the restriction to dimension three should obviously not be necessary. That said, we propose the following conjecture.

CONJECTURE 1.3 (Characterization of torus quotients). — *Let X be a compact complex space of dimension $n \geq 2$. The following are equivalent:*

- (1) *X has klt singularities, $c_1(X) = 0 \in H^2(X, \mathbb{R})$, and there exists a Kähler class $\omega \in H^2(X, \mathbb{R})$ such that $\tilde{c}_2(X) \cdot \omega^{n-2} = 0$.*
- (2) *There exists a complex torus T and a holomorphic action of a finite group $G \subset T$, free in codimension one, such that $X \cong T/G$.*

In dimension $n = 2$, this is well-known, see Proposition 7.2. In dimension 3, the conjecture would follow from Theorem 1.1 and the following special case of the Abundance Conjecture (see Subsection 8.3): *Let X be a compact Kähler threefold with klt singularities and $c_1(X) = 0$. Then the canonical sheaf of X is torsion, that is, the m -th reflexive tensor power $\omega_X^{[m]} \cong \mathcal{O}_X$ for some $m > 0$.* This is already known in important special cases, namely

- if X has canonical singularities (due to Campana–Höring–Peternell, see [6, Proposition 8.2]), and
- if X is projective (in any dimension, due to Nakayama [21, Corollary 4.9]).

In dimensions $n \geq 4$ our methods do not seem to apply. Cf. Remark 6.11, and note that also the Serre duality argument in Subsection 8.1 breaks down in higher dimensions.

Remark 1.4. — If in Conjecture 1.3, we replace $\tilde{c}_2(X)$ with $c_2(X)$ (the birational second Chern class, cf. Proposition 5.3) and “free in codimension

one” by “free in codimension two”, we obtain another conjecture. Let us call it Conjecture 1.3’. Currently, we do not know whether Conjecture 1.3 implies Conjecture 1.3’. The most natural approach to attack this problem would be to prove a generalization of Miyaoka semipositivity (Proposition 6.9) to the Kähler case.

On a side note, we also do not know whether Corollary 1.2 and Abundance imply Conjecture 1.3’ in dimension three. The problem is that we may not assume a priori that X is smooth in codimension two and hence taking an index one cover does not preserve the second Chern class vanishing condition (Remark 5.7). Again, this issue could be bypassed by the above-mentioned generalization of Proposition 6.9.

Outline of proof of Theorem 1.1

The non-trivial direction of our result is of course “(1) \Rightarrow (2)”. The first step is to observe that by abundance, the canonical sheaf of X is torsion. Taking an index one cover, we may assume that X has trivial canonical sheaf. We then distinguish two cases, according to whether X is projective or not. If X is projective, we decompose $H^2(X, \mathbb{R})$ into an algebraic and a transcendental part (Proposition 4.2) in order to replace the Kähler class ω intersecting $\tilde{c}_2(X)$ trivially by the first Chern class of an ample \mathbb{R} -Cartier divisor. Using Miyaoka’s famous semipositivity theorem, that divisor can even be chosen to be Cartier, i.e. with integral coefficients. By the result of Shepherd-Barron and Wilson mentioned above, X is then a quotient of an abelian threefold.

If X is not projective, its Albanese map is a fibre bundle over a positive-dimensional complex torus. The fibre F has trivial canonical sheaf and at worst canonical singularities. We calculate that its second orbifold Chern class vanishes, $\tilde{c}_2(F) = 0$. Since Yau’s result extends to spaces with quotient singularities, F can be equipped with a Ricci-flat Kähler metric and we obtain that the tangent bundle of the smooth locus of F is flat. Combined with a new result about étale fundamental groups of complex klt surfaces (Proposition 7.3), this implies that F is a torus quotient and then so is X .

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2. Basic conventions and definitions

All complex spaces are assumed to be separated, connected and reduced, unless otherwise stated.

DEFINITION 2.1 (Resolutions). — *A resolution of singularities of a complex space X is a proper bimeromorphic morphism $f: Y \rightarrow X$, where Y is smooth.*

- (1) *We say that the resolution is projective if f is a projective morphism. That is, f factors as $Y \hookrightarrow X \times \mathbb{P}^n \rightarrow X$, where the first map is a closed embedding and the second one is the projection. In this case, if X is compact Kähler then so is Y . Any compact complex space X has a projective resolution by [15, Theorem 3.45].*
- (2) *A resolution is said to be strong if it is an isomorphism over the smooth locus of X .*
- (3) *The resolution f is said to be minimal if it is projective and the canonical sheaf ω_Y is f -nef. This means that $\deg(\omega_Y|_C) \geq 0$ for every compact curve $C \subset Y$ mapped to a point by f .*
- (4) *The resolution f is said to be minimal in codimension two if there exists an analytic subset $S \subset X$ with $\text{codim}_X(S) \geq 3$ such that for $U := X \setminus S$, the restriction $f^{-1}(U) \rightarrow U$ is minimal.*

For the definition of canonical and klt singularities we refer to [17, Definition 2.34].

Notation 2.2. — *Sheaf cohomology is denoted by $H^k(X, \mathcal{F})$ as usual. By $H_c^k(X, \mathcal{F})$ we mean cohomology with compact support, that is, the right derived functors of taking global sections with compact support. We will also use homology $H_k(X, \mathbb{R})$ and Borel–Moore homology with integer coefficients $H_k^{\text{BM}}(X)$. The dual of a (not necessarily finite-dimensional) real vector space V is denoted $V^\vee := \text{Hom}_{\mathbb{R}}(V, \mathbb{R})$.*

DEFINITION 2.3 (Quasi-étaleté). — *A finite surjective map $f: Y \rightarrow X$ of normal complex spaces is said to be étale in codimension k if for some open subset $X^\circ \subset X$ with $\text{codim}_X(X \setminus X^\circ) \geq k + 1$, the restriction $f^{-1}(X^\circ) \rightarrow X^\circ$ is étale. The map f is called quasi-étaleté if it is étale in codimension one.*

3. Kähler metrics on singular spaces

In this section we collect several technical results about Kähler metrics and their cohomology classes on singular complex spaces. The statements in this section are probably well-known to experts. Unfortunately, we have been unable to find published proofs of these results, at least not in the exact form we need. Since our arguments in the rest of the paper depend crucially on these facts, and also for the reader's convenience, we have chosen to include full proofs here. As far as notation is concerned, we mostly follow [2] and [28].

3.1. Singular Kähler spaces

First, we set up some notation and we define what a Kähler metric on a complex space is.

Notation 3.1 (Pluriharmonic functions, [28, pp. 17 and 23]). — Let X be a reduced complex space. We denote by $\mathcal{C}_{X,\mathbb{R}}^\infty$ the sheaf of smooth real-valued functions on X . Moreover, we denote by $\text{PH}_{X,\mathbb{R}}$ the image of the real part map $\text{Re}: \mathcal{O}_X \rightarrow \mathcal{C}_{X,\mathbb{R}}^\infty$, which is called the sheaf of real-valued pluriharmonic functions on X , and we set $\mathcal{K}_{X,\mathbb{R}}^1 := \mathcal{C}_{X,\mathbb{R}}^\infty / \text{PH}_{X,\mathbb{R}}$.

The sheaf $\text{PH}_{X,\mathbb{R}}$ appears in two different short exact sequences:

$$(3.1) \quad 0 \longrightarrow \text{PH}_{X,\mathbb{R}} \longrightarrow \mathcal{C}_{X,\mathbb{R}}^\infty \longrightarrow \mathcal{K}_{X,\mathbb{R}}^1 \longrightarrow 0$$

and

$$(3.2) \quad 0 \longrightarrow \underline{\mathbb{R}}_X \xrightarrow{i} \mathcal{O}_X \xrightarrow{\text{Re}} \text{PH}_{X,\mathbb{R}} \longrightarrow 0.$$

We denote by

$$\delta^0: \mathcal{K}_{X,\mathbb{R}}^1(X) \longrightarrow \text{H}^1(X, \text{PH}_{X,\mathbb{R}})$$

the connecting homomorphism in degree 0 associated to the sequence (3.1), and by

$$\delta^1: \text{H}^1(X, \text{PH}_{X,\mathbb{R}}) \longrightarrow \text{H}^2(X, \mathbb{R})$$

the connecting homomorphism in degree 1 associated to the sequence (3.2).

Remark 3.2.

- (1) Since partitions of unity exist for the sheaf $\mathcal{C}_{X,\mathbb{R}}^\infty$, it is acyclic. Hence the map δ^0 is always surjective.

- (2) If X is compact and normal and the natural map $H^1(X, \mathbb{C}) \rightarrow H^1(X, \mathcal{O}_X)$ is surjective (e.g. if X is normal projective with Du Bois singularities), then also $H^1(X, i\mathbb{R}) \rightarrow H^1(X, \mathcal{O}_X)$ is surjective by Proposition 3.5. In this case, δ^1 is injective.
- (3) If X is a compact Kähler manifold, (2) can be made more precise: the map δ^1 induces an isomorphism $H^1(X, \text{PH}_{X, \mathbb{R}}) \xrightarrow{\sim} H^{1,1}(X) \cap H^2(X, \mathbb{R})$.

Notation 3.3 (Period class, [2, p. 525]). — We write

$$P = \delta^1 \circ \delta^0: \mathcal{K}_{X, \mathbb{R}}^1(X) \rightarrow H^2(X, \mathbb{R}).$$

For an element $\kappa \in \mathcal{K}_{X, \mathbb{R}}^1(X)$ we call $P(\kappa)$ the *period class* of κ on X .

DEFINITION 3.4 (Kähler metrics, [28, pp. 23 and 18]). — *Let X be a reduced complex space.*

- (1) A Kähler metric on X is an element κ of $\mathcal{K}_{X, \mathbb{R}}^1(X)$ which can be represented by a family $(U_i, \varphi_i)_{i \in I}$ such that φ_i is a smooth strictly plurisubharmonic function on U_i for all $i \in I$. That is, locally φ_i is induced by a smooth strictly plurisubharmonic function on an open subset of \mathbb{C}^{N_i} under a local embedding $U_i \hookrightarrow \mathbb{C}^{N_i}$.
- (2) We say that $c \in H^2(X, \mathbb{R})$ is a Kähler class on X if there exists a Kähler metric κ on X such that $c = P(\kappa)$.
- (3) We say that X is Kähler if there exists a Kähler metric on X .

PROPOSITION 3.5. — *Let X be a normal compact Kähler space such that the natural map $H^1(X, \mathbb{C}) \rightarrow H^1(X, \mathcal{O}_X)$ is surjective. Then also $H^1(X, \mathbb{R}) \rightarrow H^1(X, \mathcal{O}_X)$ and $H^1(X, i\mathbb{R}) \rightarrow H^1(X, \mathcal{O}_X)$ are surjective.*

Proof. — Let $X' \rightarrow X$ be a resolution, and note that $H^1(X, \mathcal{O}_X) \rightarrow H^1(X', \mathcal{O}_{X'})$ is injective by the Leray spectral sequence. Let F^\bullet and W_\bullet be the Hodge and weight filtrations on any given mixed Hodge structure. If no Hodge structure is given, it will be understood that by default we are considering $H^1(X, \mathbb{C})$. Since $W_0 H^1(X', \mathbb{C}) = 0$, it follows that $H^1(X, \mathbb{C}) \rightarrow H^1(X, \mathcal{O}_X)$ factorizes via a map

$$\alpha: H^1(X, \mathbb{C}) / W_0 H^1(X, \mathbb{C}) \rightarrow H^1(X, \mathcal{O}_X),$$

which clearly remains surjective. We have that $F^1 / W_0 \cap F^1 \subset \ker \alpha$ because $F^1 H^1(X', \mathbb{C})$ maps to zero in $H^1(X', \mathcal{O}_{X'})$. Now, the source of α is nothing but W_1 / W_0 and so F^\bullet induces on it a pure Hodge structure

$$(3.3) \quad W_1 / W_0 = \left(F^1 / W_0 \cap F^1 \right) \oplus \left(\overline{F^1} / W_0 \cap \overline{F^1} \right).$$

Let $d \in H^1(X, \mathcal{O}_X)$ be arbitrary and pick $c \in W_1/W_0$ with $\alpha(c) = d$. Write $c = c_1 + c'$ according to the decomposition (3.3). We have seen that $\alpha(c_1) = \alpha(\overline{c'}) = 0$ and hence $\alpha(c') = d$. Let $c'' \in H^1(X, \mathbb{C})$ be a preimage of c' . Then the class $c'' + \overline{c''}$ is contained in $H^1(X, \mathbb{R})$, as it is invariant under conjugation, and it maps to d . Likewise, $c'' - \overline{c''}$ is contained in $H^1(X, i\mathbb{R})$ and also maps to d . This proves the desired surjectivity of $H^1(X, \mathbb{R}) \rightarrow H^1(X, \mathcal{O}_X)$ and $H^1(X, i\mathbb{R}) \rightarrow H^1(X, \mathcal{O}_X)$. \square

3.2. Properties of Kähler metrics

Our first proposition is the Kähler analog of a well-known property of ample line bundles.

PROPOSITION 3.6 (Finite pullbacks). — *Let $f: Y \rightarrow X$ be a finite morphism of complex spaces, $c \in H^2(X, \mathbb{R})$ a Kähler class on X . Then $f^*(c)$ is a Kähler class on Y .*

Proof. — Since c is Kähler class on X , there exists a Kähler metric κ on X such that $P(\kappa) = c$. By [27, p. 253, Claim A in the proof of Theorem 1], there exists a smooth function $\varphi: Y \rightarrow \mathbb{R}$ such that $f^*(\kappa) + \varphi$ is a Kähler metric on Y . Here, $f^*: \mathcal{K}_{X, \mathbb{R}}^1 \rightarrow f_*(\mathcal{K}_{Y, \mathbb{R}}^1)$ is the sheaf map induced by the pullback of smooth functions $\mathcal{C}_{X, \mathbb{R}}^\infty \rightarrow f_*(\mathcal{C}_{Y, \mathbb{R}}^\infty)$. Explicitly this means there exists a family $(U_i, \kappa_i)_{i \in I}$ of smooth strictly plurisubharmonic functions on X representing κ as well as an open cover $(V_j)_{j \in J}$ of Y and a map $\lambda: J \rightarrow I$ such that for all $j \in J$ we have $V_j \subset f^{-1}(U_{\lambda(j)})$ and $\kappa_{\lambda(j)} \circ f|_{V_j} + \varphi|_{V_j}$ is a strictly plurisubharmonic function on V_j .

Since $\varphi \in \mathcal{C}_{Y, \mathbb{R}}^\infty(Y)$, we see that $\delta^0(\varphi) = 0$. Hence, given that the period class map P commutes with pullback along f ,

$$P(f^*(\kappa) + \varphi) = P(f^*(\kappa)) = f^*(P(\kappa)) = f^*(c),$$

so $f^*(c)$ is a Kähler class. \square

The next two results concern openness properties of Kähler metrics.

PROPOSITION 3.7 (Being Kähler is an open property, I). — *Let X be a compact complex space, κ a Kähler metric on X , and $\varphi \in \mathcal{K}_{X, \mathbb{R}}^1(X)$ an arbitrary element. Then there exists a number $\varepsilon > 0$ such that for all $t \in \mathbb{R}$ with $|t| < \varepsilon$, we have that $\kappa + t\varphi$ is a Kähler metric on X .*

Proof. — Since κ is a Kähler metric, there exists a family $(U_i, \kappa_i)_{i \in I}$ representing κ in the quotient sheaf $\mathcal{K}_{X, \mathbb{R}}^1$ such that $(U_i)_{i \in I}$ is an open cover of X and κ_i is strictly plurisubharmonic on U_i for all $i \in I$. Likewise,

φ is represented by a family $(V_j, \varphi_j)_{j \in J}$ where $\varphi_j \in \mathcal{C}_{X, \mathbb{R}}^\infty(V_j)$ for all $j \in J$. Passing to a common refinement and using the definitions of $\mathcal{C}_{X, \mathbb{R}}^\infty$ and strict plurisubharmonicity on X , we may assume that

- $(U_i)_{i \in I} = (V_j)_{j \in J}$,
- there exist $W_i \subset \mathbb{C}^{N_i}$ open, $N_i \in \mathbb{N}$, and closed embeddings $g_i: U_i \hookrightarrow W_i$,
- there exist $\tilde{\kappa}_i, \tilde{\varphi}_i \in \mathcal{C}^\infty(W_i, \mathbb{R})$ such that the $\tilde{\kappa}_i$ are strictly plurisubharmonic functions, $[\tilde{\kappa}_i] = \kappa_i$, and $[\tilde{\varphi}_i] = \varphi_i$,
- there exist relatively compact, open subsets $W'_i \Subset W_i$ such that, setting $U'_i := g_i^{-1}(W'_i)$, we have $\bigcup_{i \in I} U'_i = X$.

Furthermore, since X is compact, we may assume that I is finite.

Let $i \in I$. Then, since strict plurisubharmonicity for smooth functions on \mathbb{C}^{N_i} is equivalent to their Levi form being positive definite at each point, there exists $\varepsilon_i > 0$ such that $(\tilde{\kappa}_i + t\tilde{\varphi}_i)|_{W'_i}$ is strictly plurisubharmonic on W'_i for all $t \in \mathbb{R}$ with $|t| < \varepsilon_i$. Define $\varepsilon := \min\{\varepsilon_i : i \in I\} > 0$. When t is a real number such that $|t| < \varepsilon$, then $\kappa + t\varphi$ is a Kähler metric on X , for it is represented by the family $(U'_i, (\kappa_i + t\varphi_i)|_{U'_i})$ and $(\kappa_i + t\varphi_i)|_{U'_i}$ is induced by $(\tilde{\kappa}_i + t\tilde{\varphi}_i)|_{W'_i}$ for all $i \in I$. \square

PROPOSITION 3.8 (Being Kähler is an open property, II). — *Let X be a compact complex space. Then*

- (1) *the real vector space $H^1(X, \text{PH}_{X, \mathbb{R}})$ is finite-dimensional, and*
- (2) *the set $\mathcal{K}_X := \{\delta^0(\kappa) : \kappa \text{ is a Kähler metric on } X\}$ is an open convex cone in $H^1(X, \text{PH}_{X, \mathbb{R}})$, called the Kähler cone of X .*

Proof. — (1) is clear by looking at the long exact sequence in cohomology associated to the short exact sequence of sheaves (3.2).

If κ and λ are Kähler metrics on X , then $s\kappa + t\lambda$ is a Kähler metric on X for all $s, t \in \mathbb{R}_{\geq 0}$ with $s + t > 0$, since the analogous statement holds for strictly plurisubharmonic functions on X . Therefore the set of Kähler metrics on X is a convex cone in the real vector space $\mathcal{K}_{X, \mathbb{R}}^1(X)$. Since δ^0 is a linear map, \mathcal{K}_X is a convex cone in $H^1(X, \text{PH}_{X, \mathbb{R}})$.

For openness, consider an arbitrary element $c = \delta^0(\kappa) \in \mathcal{K}_X$, for κ a Kähler metric on X . By (1), there exists a finite basis (b_1, \dots, b_ρ) for $H^1(X, \text{PH}_{X, \mathbb{R}})$. As we noted in Remark 3.2, the map $\delta^0: \mathcal{K}_{X, \mathbb{R}}^1(X) \rightarrow H^1(X, \text{PH}_{X, \mathbb{R}})$ is surjective. Hence, there exist $\varphi_1, \dots, \varphi_\rho \in \mathcal{K}_{X, \mathbb{R}}^1(X)$ such that $\delta^0(\varphi_i) = b_i$ for all $1 \leq i \leq \rho$. By Proposition 3.7, there exists a number $\varepsilon > 0$ such that for all $1 \leq i \leq \rho$ and all $t \in \mathbb{R}$ with $|t| < \varepsilon$ we have that $\kappa + t\varphi_i$ is a Kähler metric on X . Consequently $c + tb_i \in \mathcal{K}_X$ for all i and t

as before. Since \mathcal{K}_X is a convex cone, we deduce that $c + \sum_{i=1}^{\rho} t_i b_i \in \mathcal{K}_X$ for all $t = (t_i) \in \mathbb{R}^{\rho}$ with $|t_i| < \varepsilon/\rho$ for all i , and we obtain (2). \square

4. A decomposition of the second cohomology group

The purpose of this section is to associate to any Kähler class on a mildly singular projective variety X an \mathbb{R} -ample divisor class having the same intersection numbers with all curves in X (Proposition 4.5). Before we can give the statement, we need to introduce some notation.

Notation 4.1.

Let X be a non-empty complex space, of pure dimension n .

- (1) We denote by $[X] \in H_{2n}^{BM}(X)$ the *fundamental class* of X in Borel–Moore homology.
- (2) If $i: A \hookrightarrow X$ is a nonempty purely k -dimensional closed analytic subset, abusing notation we write $[A]$ too for $i_*[A] \in H_{2k}^{BM}(X)$.

Assume furthermore that X is compact. In this case, $H_*(X, \mathbb{Z}) = H_*^{BM}(X)$.

- (3) For any integer $k \geq 0$, we define $B_{2k}(X, \mathbb{R}) \subset H_{2k}(X, \mathbb{R})$ to be the real linear subspace spanned by the set of all $[A] \in H_{2k}(X, \mathbb{R})$, where $A \subset X$ is a k -dimensional irreducible closed analytic subset.
- (4) $N^1(X)_{\mathbb{R}} \subset H^2(X, \mathbb{R})$ is the real linear subspace spanned by the image of the first Chern class map $H^1(X, \mathcal{O}_X^*) \rightarrow H^2(X, \mathbb{R})$.
- (5) $T(X) \subset H^2(X, \mathbb{R})$ is the subspace orthogonal to $B_2(X, \mathbb{R})$ with respect to the canonical pairing $\langle \cdot, \cdot \rangle: H^2(X, \mathbb{R}) \times H_2(X, \mathbb{R}) \rightarrow \mathbb{R}$. That is,

$$T(X) = \{a \in H^2(X, \mathbb{R}) \mid \forall b \in B_2(X, \mathbb{R}): \langle a, b \rangle = 0\}.$$

- (6) $N_1(X)_{\mathbb{R}}$ is the quotient of $B_2(X, \mathbb{R})$ by *numerical equivalence*. That is, $N_1(X)_{\mathbb{R}} = B_2(X, \mathbb{R}) / B_2(X, \mathbb{R}) \cap S$ where $S \subset H_2(X, \mathbb{R})$ is the subspace orthogonal to $N^1(X)_{\mathbb{R}}$ with respect to $\langle \cdot, \cdot \rangle$.

PROPOSITION 4.2 (Decomposition of singular cohomology). — *Let X be a projective variety with rational (e.g., canonical) singularities only. Then*

$$H^2(X, \mathbb{R}) = N^1(X)_{\mathbb{R}} \oplus T(X).$$

Proof. — First we show that $N^1(X)_{\mathbb{R}} \cap T(X) = \{0\}$. To this end, let $a \in N^1(X)_{\mathbb{R}}$ be an element such that $\langle a, b \rangle = 0$ for all $b \in B_2(X, \mathbb{R})$. By definition, we may write $a = \sum_{i=1}^k a_i c_1(L_i)$ with $a_1, \dots, a_k \in \mathbb{R}$, $L_1, \dots, L_k \in$

$\text{Pic}(X)$, and $k \in \mathbb{N}$. By a linear algebra argument⁽¹⁾, there are line bundles $M_1, \dots, M_\ell \in \text{Pic}(X)$ such that $c_1(M_j) \in T(X)$ for all $1 \leq j \leq \ell$ and $a = \sum_{j=1}^{\ell} b_j c_1(M_j)$ for some $b_1, \dots, b_\ell \in \mathbb{R}$. Now [18, Corollary 1.4.38] implies that there exist integers $N_j > 0$ such that $M_j^{\otimes N_j} \in \text{Pic}^0(X)$, i.e. $M_j^{\otimes N_j}$ is a deformation of \mathcal{O}_X . In particular, $c_1(M_j) = 0$ for each j and then clearly $a = 0$.

To conclude, it suffices to show that

$$\dim N^1(X)_{\mathbb{R}} + \dim T(X) \geq \dim H^2(X, \mathbb{R}).$$

By [16, Corollary 12.1.5.2], an element $b \in B_2(X, \mathbb{R})$ is zero if $\langle a, b \rangle = 0$ for all $a \in N^1(X)_{\mathbb{R}}$. In other words, $N_1(X)_{\mathbb{R}} = B_2(X, \mathbb{R})$ and hence the map $B_2(X, \mathbb{R}) \rightarrow N^1(X)_{\mathbb{R}}^{\vee}$ induced by $\langle \cdot, \cdot \rangle$ is injective. Thus

$$\begin{aligned} \dim N^1(X)_{\mathbb{R}} + \dim T(X) &\geq \dim B_2(X, \mathbb{R}) + \dim T(X) \\ &= \dim H^2(X, \mathbb{R}), \end{aligned}$$

the last equality being due to the orthogonality of $B_2(X, \mathbb{R})$ and $T(X)$ with respect to the perfect pairing $\langle \cdot, \cdot \rangle$. \square

Remark 4.3. — The hypothesis “rational singularities” in Proposition 4.2 can be weakened to “1-rational”, that is, for some/any resolution $f: Y \rightarrow X$, the higher direct image sheaf $R^1 f_* \mathcal{O}_Y$ vanishes. Cf. the remark after the proof of [16, Corollary 12.1.5.2].

LEMMA 4.4 (Pullback of transcendental classes). — *Let $f: Y \rightarrow X$ be a morphism between compact complex spaces. Then $f^*(T(X)) \subset T(Y)$.*

Proof. — Let $a \in T(X)$ be arbitrary and $D \subset Y$ an irreducible reduced closed complex subspace of dimension 1. Then, by Remmert’s mapping theorem, $f(D) \subset X$ too is an irreducible reduced closed complex subspace. Moreover, either $\dim f(D) = 0$ or $\dim f(D) = 1$. If $f(D)$ is 0-dimensional, then clearly $\langle f^*a, [D] \rangle = 0$. If $f(D)$ is 1-dimensional, then there exists a number $d > 0$ such that $D \rightarrow f(D)$ is a d -sheeted analytic covering. Therefore $\langle f^*a, [D] \rangle = d \cdot \langle a, [f(D)] \rangle = 0$. We conclude that $f^*a \in T(Y)$. \square

PROPOSITION 4.5 (Algebraic part of a Kähler class). — *Let X be a projective variety with rational singularities, $c \in H^2(X, \mathbb{R})$ a Kähler class on X . Write $c = h + t$ according to the direct sum decomposition of Proposition 4.2. Then $h \in N^1(X)_{\mathbb{R}}$ is an \mathbb{R} -ample divisor class.*

⁽¹⁾ Write the condition “ $a \in T(X)$ ” as a finite system of linear equations in the a_i , defined over \mathbb{Q} , and note that any real solution to such a system is a real linear combination of rational solutions. See also the proof of [18, Proposition 1.3.13].

Proof. — Let $\overline{NE}(X) \subset N_1(X)_{\mathbb{R}}$ be the cone of curves. By Kleiman’s ampleness criterion [18, Theorem 1.4.29], it suffices to show that $h \cdot a > 0$ for all $a \in \overline{NE}(X) \setminus \{0\}$. But $h \cdot a = c \cdot a - t \cdot a = c \cdot a$, so we only need to show that $c \cdot a > 0$.

To begin with, we remark that $c \cdot a \geq 0$, since clearly $c \cdot [C] > 0$ for any irreducible and reduced curve $C \subset X$. Now since $a \neq 0$ in $N_1(X)_{\mathbb{R}}$, there exists a line bundle $L \in \text{Pic}(X)$ such that $c_1(L) \cdot a \neq 0$. We may assume that $c_1(L) \cdot a > 0$. We know [2, (4.15)] that there exists a group homomorphism $\ell: H^1(X, \mathcal{O}_X^*) \rightarrow H^1(X, \text{PH}_{X, \mathbb{R}})$ such that the following diagram commutes:

$$\begin{CD} H^1(X, \mathcal{O}_X^*) @>c_1>> H^2(X, \mathbb{Z}) \\ @V\ell VV @VVV \\ H^1(X, \text{PH}_{X, \mathbb{R}}) @>\delta^1>> H^2(X, \mathbb{R}). \end{CD}$$

Since c is a Kähler class on X , there exists a Kähler metric κ on X with the property $P(\kappa) = c$. In particular, $\delta^0(\kappa) \in \mathcal{K}_X$. By openness of the Kähler cone, Proposition 3.8, there is a number $t < 0$ such that $\delta^0(\kappa) + t\ell(L) \in \mathcal{K}_X$. As a consequence,

$$0 \leq \delta^1(\delta^0(\kappa) + t\ell(L)) \cdot a = (P(\kappa) + t\delta^1(\ell(L))) \cdot a = (c + tc_1(L)) \cdot a.$$

Thus

$$0 < -tc_1(L) \cdot a \leq c \cdot a,$$

which was to be demonstrated. □

5. Chern classes on singular spaces

In order to prove our main result, we need to discuss the Chern classes of the tangent sheaf of a singular Kähler space X and their intersection numbers with a given Kähler class. On complex manifolds there is a unique notion of Chern classes, but in the singular case there are at least two competing approaches: Firstly, if X has quotient singularities in sufficiently high codimension, one can define the “orbifold” Chern classes of X . Secondly, one may pull back everything to an appropriate resolution of singularities $\tilde{X} \rightarrow X$ to define a “birational” notion of Chern classes. For our purposes, both approaches will be useful.

References for these matters include [4, 11, 25, 26], but they all either make assumptions on smoothness in high codimension that are not satisfied in our setting, or they define intersection numbers with ample divisors only

and not with arbitrary Kähler classes. Therefore we have chosen to include here a self-contained presentation of the material. We restrict ourselves to considering the first and second Chern class of a space with klt singularities, which is more than sufficient for this paper. These notions should be of independent interest.

DEFINITION 5.1 (First Chern class). — *Let X be a normal complex space which is \mathbb{Q} -Gorenstein, i.e. for some $m > 0$ the reflexive tensor power $\omega_X^{[m]} := (\omega_X^{\otimes m})^{**}$ is invertible, where ω_X is the dualizing sheaf. The first Chern class of X is the cohomology class*

$$c_1(X) := -\frac{1}{m}c_1(\omega_X^{[m]}) \in H^2(X, \mathbb{Q}).$$

This is independent of the choice of m .

Spaces with klt singularities are \mathbb{Q} -Gorenstein by definition (see [17, Definition 2.34]), so Definition 5.1 applies to them.

Both our notions of second Chern class will be elements of $H^{2n-4}(X, \mathbb{Q})^\vee = \text{Hom}_{\mathbb{Q}}(H^{2n-4}(X, \mathbb{Q}), \mathbb{Q})$, that is, linear forms on the appropriate cohomology group. Of course, this is the same as giving a homology class, but this is not how we usually think about Chern classes.

DEFINITION 5.2 (“Orbifold” second Chern class). — *Let X be a compact complex space with klt singularities, of pure dimension n . Let $X^\circ \subset X$ be the (open) locus of quotient singularities of X . The second orbifold Chern class of X is the unique element $\tilde{c}_2(X) \in H^{2n-4}(X, \mathbb{Q})^\vee$ whose restriction to $H_c^{2n-4}(X^\circ, \mathbb{Q})^\vee$ is the Poincaré dual of the second orbifold Chern class $\tilde{c}_2(X^\circ) \in H^4(X^\circ, \mathbb{Q})$.*

In Subsection 5.2 below, we will discuss more carefully why this definition makes sense. Using de Rham cohomology, we will also interpret it in terms of integrating differential forms.

PROPOSITION 5.3 (“Birational” second Chern class). — *Let X be a compact complex space with klt singularities, of pure dimension n . Then there exists a resolution of singularities $f: Y \rightarrow X$ which is minimal in codimension two. The birational second Chern class of X is the element $c_2(X) \in H^{2n-4}(X, \mathbb{Q})^\vee$ defined by*

$$c_2(X) \cdot a := \int_Y c_2(Y) \cup f^*(a) \quad \text{for any } a \in H^{2n-4}(X, \mathbb{Q}),$$

where $c_2(Y) \in H^4(Y, \mathbb{Q})$ is the usual second Chern class of the complex manifold Y . This definition is independent of the resolution f chosen (provided it is minimal in codimension two).

In particular, for classes $a_1, \dots, a_{n-2} \in H^2(X, \mathbb{R})$ we may set

$$\tilde{c}_2(X) \cdot a_1 \cdots a_{n-2} := \tilde{c}_2(X) \cdot (a_1 \cup \cdots \cup a_{n-2})$$

and likewise for $c_2(X)$. In this way, $\tilde{c}_2(X)$ and $c_2(X)$ yield $(n-2)$ -multilinear forms on $H^2(X, \mathbb{R})$.

Remark 5.4 (Comparison of \tilde{c}_2 and c_2). — Let X_1 be a complex 2-torus, and consider the quotient map $g: X_1 \rightarrow X = X_1/\pm 1$. Then $\tilde{c}_2(X) = 0$, as follows from (5.1) below. But $c_2(X) = 24$ under the natural identification $H^0(X, \mathbb{R})^\vee = \mathbb{R}$, since the minimal resolution of X is a K3 surface. This shows that the two notions of Chern classes do not agree even on spaces with only canonical quotient singularities.

On the other hand, $\tilde{c}_2(X) = c_2(X)$ whenever X is smooth in codimension two. This can be seen as follows. Let $i: U \hookrightarrow X$ be the smooth locus and let $f: Y \rightarrow X$ be a strong resolution, i.e. $V := f^{-1}(U) \xrightarrow{f_U} U$ is an isomorphism. Any $a \in H^{2n-4}(X, \mathbb{R})$ can be written uniquely as $a = i_*(b)$ for some $b \in H_c^{2n-4}(U, \mathbb{R})$ (see the long exact sequence in Subsection 5.2). Then, with $j: V \hookrightarrow Y$ the inclusion,

$$\begin{aligned} \tilde{c}_2(X) \cdot a &= \int_U c_2(U) \cup b && \text{almost by definition} \\ &= \int_V c_2(V) \cup f_U^*(b) && \text{since } f_U \text{ is an isomorphism} \\ &= \int_V j^* c_2(Y) \cup j^*(f^*a) && \text{since } c_2(V) = j^* c_2(Y) \text{ and } b = i^*(a) \\ &= \int_Y c_2(Y) \cup f^*(a) && \text{since } Y \setminus V \subset Y \text{ is analytic} \\ &= c_2(X) \cdot a && \text{by definition.} \end{aligned}$$

Remark 5.5 (Schwartz–MacPherson Chern classes). — The earliest treatment of Chern classes on singular varieties is due to Schwartz and MacPherson [24, 19]. It is natural to ask how our Chern classes relate to theirs. Note that in [19], the construction is carried out only for compact complex algebraic varieties X , and that the k -th Schwartz–MacPherson Chern class $c_k^{\text{SM}}(X)$ lives in $H_{2n-2k}(X, \mathbb{Z})$, where $n = \dim X$. This is however not a problem, since we may use the isomorphism $H_{2n-4}(X, \mathbb{R}) \cong H^{2n-4}(X, \mathbb{R})^\vee$ from the universal coefficient theorem to compare $c_2^{\text{SM}}(X)$ (with real coefficients) to $c_2(X)$ and $\tilde{c}_2(X)$.

We give an example where both $\tilde{c}_2(X) \neq c_2^{\text{SM}}(X)$ and $c_2(X) \neq c_2^{\text{SM}}(X)$. Let $X = X_1/\pm 1$ be as in Remark 5.4, but assume that X_1 is algebraic, i.e. an abelian surface. Let $f: \tilde{X} \rightarrow X$ be the minimal resolution, where \tilde{X} is a K3 surface. Denote $p_1, \dots, p_{16} \in X$ the sixteen singular points of X , corresponding to the 2-torsion points of X_1 . If $i_k: \{p_k\} \hookrightarrow X$ is the

inclusion, then using notation from [19, Proposition 1] we have

$$\mathbf{1}_X = f_* (\mathbf{1}_{\tilde{X}}) - \sum_{k=1}^{16} i_{k*} (\mathbf{1}_{\{p_k\}}).$$

By [19, proof of Proposition 2] it follows that the total Chern class of X in homology is

$$c^{\text{SM}}(X) = f_* (\text{PD}(c(\tilde{X}))) - \sum_{k=1}^{16} i_{k*} (\text{PD}(c(\{p_k\}))) \in H_*(X, \mathbb{Z}),$$

where $\text{PD}: H^i(\tilde{X}, \mathbb{Z}) \rightarrow H_{4-i}(\tilde{X}, \mathbb{Z})$ is the Poincaré duality map. We obtain the second Chern class by looking at the degree 0 part:

$$c_2^{\text{SM}}(X) = f_*(24) - \sum_{k=1}^{16} i_{k*}(1) = 24 - 16 = 8 \in H_0(X, \mathbb{Z}).$$

Hence also $c_2^{\text{SM}}(X) = 8$ as an element of $H^0(X, \mathbb{R})^\vee = \mathbb{R}$. Comparing this to Remark 5.4, we see that even on algebraic varieties with only canonical quotient singularities, $c_2^{\text{SM}}(X)$, $\tilde{c}_2(X)$ and $c_2(X)$ are three pairwise distinct notions.

On the other hand, $c_2^{\text{SM}}(X) = c_2(X)$ if X is algebraic and smooth in codimension two, and so all three versions coincide in this case: Let $f: \tilde{X} \rightarrow X$ be a strong resolution. Then we can write

$$\mathbf{1}_X = f_* (\mathbf{1}_{\tilde{X}}) - \sum_k g_{k*} (\gamma_k),$$

where $g_k: Y_k \rightarrow X$ are maps from smooth varieties Y_k of dimension $\dim_{\mathbb{C}} Y_k \leq n - 3$, $n = \dim_{\mathbb{C}} X$, and the γ_k are constructible functions on Y_k . Arguing as before and taking the degree $2n - 4$ part, we get

$$c_2^{\text{SM}}(X) = f_* (\text{PD}(c_2(\tilde{X})))$$

since $H_{2n-4}(Y_k, \mathbb{Z}) = 0$ for all k . Rewriting this statement in cohomology yields the claim, since f is in particular minimal in codimension two.

PROPOSITION 5.6 (Behavior of \tilde{c}_2 and c_2 under quasi-étale maps).

Let $g: X_1 \rightarrow X$ be a finite surjective map between normal compact complex spaces of pure dimension n , where X has klt singularities. Assume that g is étale in codimension one. Then also X_1 has klt singularities and for all $a \in H^{2n-4}(X, \mathbb{R})$ we have

$$(5.1) \quad \tilde{c}_2(X_1) \cdot g^*(a) = \text{deg}(g) \cdot \tilde{c}_2(X) \cdot a.$$

If g is étale in codimension two, then we also have

$$(5.2) \quad c_2(X_1) \cdot g^*(a) = \text{deg}(g) \cdot c_2(X) \cdot a$$

for all $a \in H^{2n-4}(X, \mathbb{R})$.

Remark 5.7. — For the map $g: X_1 \rightarrow X$ from Remark 5.4, the left-hand side of (5.2) is zero, while the right-hand side evaluates to 48. This shows that (5.2) fails if g is only étale in codimension one.

5.1. Auxiliary results

Before we prove the above propositions, we collect some preliminary lemmas.

LEMMA 5.8 (Local structure of klt singularities). — *Let X be a complex space with klt singularities. Then there exists an analytic subset Z in X such that $\text{codim}_X(Z) \geq 3$ and such that for all $x \in X \setminus Z$, either X is smooth at x or there exist a klt surface singularity (S, o) and an integer $n \geq 0$ for which we have $(X, x) \cong (S, o) \times (\mathbb{C}^n, 0)$ as germs of complex spaces. In particular, $X \setminus Z$ has quotient singularities. Furthermore, if X has canonical singularities then $X \setminus Z$ is Gorenstein.*

Proof. — Assuming that X is quasi-projective, [9, Proposition 9.3] shows the existence of a closed analytic set $Z \subset X$ of codimension greater than or equal to 3 such that, for all $x \in X \setminus Z$, the germ (X, x) is a quotient singularity. The proof of the cited result, however, shows more precisely that either (X, x) is smooth or $(X, x) \cong (S, o) \times (\mathbb{C}^n, 0)$ for a klt surface singularity (S, o) and an integer $n \geq 0$. If X has canonical singularities, (S, o) will be a canonical singularity too. By [17, Theorem 4.20], every such (S, o) is a hypersurface singularity, whence (X, x) is Gorenstein.

In the general case, the arguments of [9] go through with minor modifications. For example, in order to obtain [9, Proposition 2.26] (“projection to a subvariety”) we need to employ the Open Projection Lemma [8, p. 71] instead of Noether normalization. \square

LEMMA 5.9 (Pullback lemma). — *Let X, Y, Y_1 be complex spaces, $h: Y_1 \rightarrow Y$ and $f: Y \rightarrow X$ proper morphisms, and U an open subspace of X with analytic complement such that the restriction of h to $V_1 := h^{-1}(f^{-1}(U))$ yields a d -fold étale covering $h_V: V_1 \rightarrow V := f^{-1}(U)$. Assume that Y and Y_1 are pure n -dimensional complex manifolds. Then, for all natural numbers $k \leq \text{codim}_X(X \setminus U) - 1$ and all $a \in H_c^{2n-2k}(X, \mathbb{R})$, we have*

$$\int_{Y_1} c_k(Y_1) \cup h^*(f^*(a)) = d \cdot \int_Y c_k(Y) \cup f^*(a).$$

Proof. — Let $i: U \hookrightarrow X$, $j: V \hookrightarrow Y$ and $j_1: V_1 \hookrightarrow Y_1$ be the inclusions and consider the following commutative diagram, where we write h_* for $(h^*)^\vee$ and analogously for the other maps.

$$\begin{array}{ccccc}
 H^{2k}(Y_1, \mathbb{R}) & \xrightarrow{\text{PD}} & H_c^{2n-2k}(Y_1, \mathbb{R})^\vee & \xrightarrow{(j_1^*)^\vee} & H_c^{2n-2k}(V_1, \mathbb{R})^\vee \\
 \downarrow & & \downarrow h_* & & \downarrow h_{V_*} \\
 H^{2k}(Y, \mathbb{R}) & \xrightarrow{\text{PD}} & H_c^{2n-2k}(Y, \mathbb{R})^\vee & \xrightarrow{(j^*)^\vee} & H_c^{2n-2k}(V, \mathbb{R})^\vee \\
 & & \downarrow f_* & & \downarrow f_{U_*} \\
 & & H_c^{2n-2k}(X, \mathbb{R})^\vee & \xrightarrow{(i^*)^\vee} & H_c^{2n-2k}(U, \mathbb{R})^\vee.
 \end{array}$$

Here PD denotes the Poincaré duality isomorphism on the complex manifolds Y and Y_1 , respectively. Since h_V is étale, we have $h_V^* \mathcal{S}_V \cong \mathcal{S}_{V_1}$ and hence $h_V^* c_k(V) = h_V^* c_k(\mathcal{S}_V) = c_k(h_V^* \mathcal{S}_V) = c_k(\mathcal{S}_{V_1}) = c_k(V_1) \in H^{2k}(V_1, \mathbb{R})$. Furthermore, $\int_{V_1} h_V^* \sigma = d \cdot \int_V \sigma$ for all $\sigma \in H_c^{2n}(V, \mathbb{R})$. Thus we see that

$$h_{V_*}(\text{PD}(c_k(V_1))) = d \cdot \text{PD}(c_k(V)).$$

By a similar argument, $(j^*)^\vee(\text{PD}(c_k(Y))) = \text{PD}(c_k(V))$ and analogously for j_1 . This shows that the classes $c_k(Y_1) \in H^{2k}(Y_1, \mathbb{R})$ and $d \cdot c_k(Y) \in H^{2k}(Y, \mathbb{R})$ in the left-hand side column are mapped to the same element in $H_c^{2n-2k}(V, \mathbb{R})^\vee$. In particular, they are mapped to the same element in $H_c^{2n-2k}(U, \mathbb{R})^\vee$. But then their images in $H_c^{2n-2k}(X, \mathbb{R})^\vee$ are also the same, because $(i^*)^\vee$ is injective. The latter claim follows from the long exact sequence in compactly supported cohomology associated to the inclusions $i: U \hookrightarrow X$ and $\iota: X \setminus U \hookrightarrow X$ [13, III.7.6],

$$\dots \longrightarrow H_c^{2n-2k}(U, \mathbb{R}) \xrightarrow{i_*} H_c^{2n-2k}(X, \mathbb{R}) \xrightarrow{\iota^*} \underbrace{H_c^{2n-2k}(X \setminus U, \mathbb{R})}_{=0} \longrightarrow \dots,$$

the last vanishing being due to the fact that $\dim_{\mathbb{R}}(X \setminus U) \leq 2n - 2k - 2$. We have thus shown that $f_* h_* c_k(Y_1) = d \cdot f_* c_k(Y)$. This is exactly the claim of the lemma. □

5.2. Explanation of Definition 5.2

As far as complex spaces with quotient singularities (“V-manifolds”, “orbifolds”) are concerned, we use the terminology of [23]. In particular, by a *local uniformization* (“orbifold chart”) of an open subset U of such a

space X we mean a triple (\tilde{U}, G, φ) , where

- $\tilde{U} \subset \mathbb{C}^n$ is a contractible open set,
- G is a finite group acting on \tilde{U} linearly and freely in codimension one,
- $\varphi: \tilde{U} \rightarrow U$ is a G -invariant map exhibiting U as the quotient \tilde{U}/G .

We will use Poincaré duality in the following guise.

PROPOSITION 5.10 (Poincaré duality for orbifolds). — *Let U be an n -dimensional connected complex space with quotient singularities. Then for any $0 \leq k \leq 2n$, the bilinear pairing*

$$H^k(U, \mathbb{Q}) \times H_c^{2n-k}(U, \mathbb{Q}) \longrightarrow H_c^{2n}(U, \mathbb{Q}) \cong \mathbb{Q}$$

gives rise to an isomorphism $PD: H^k(U, \mathbb{Q}) \xrightarrow{\sim} H_c^{2n-k}(U, \mathbb{Q})^\vee$. In terms of de Rham cohomology, the pairing is given by

$$([\alpha], [\beta]) \longmapsto \int_U \alpha \wedge \beta.$$

Proof. — This is a special case of Verdier duality [13, V.2.1]. If U is a topological manifold, it is explained in [13, proof of V.3.2] how to deduce our statement from Verdier duality. A closer look at the proof reveals that the only property of U being used there is that every point $x \in U$ has a neighborhood basis consisting of open sets V satisfying

$$H_c^k(V, \mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & k = 2n, \\ 0, & \text{otherwise.} \end{cases}$$

This continues to hold if U has at worst quotient singularities, since using local uniformizations (\tilde{V}, G, φ) of V we have $H_c^k(V, \mathbb{Q}) = H_c^k(\tilde{V}, \mathbb{Q})^G$.

For the statement about de Rham cohomology, see [23, Theorem 3]. Note that [23] assumes for simplicity that U is compact, however this is not essential for the argument because β has compact support. □

Now consider the setting of Definition 5.2. We start with the second orbifold Chern class $\tilde{c}_2(X^\circ) \in H^4(X^\circ, \mathbb{R})$, defined differential-geometrically as explained for example in [26, Section 2.2.1]. Its Poincaré dual $PD(\tilde{c}_2(X^\circ))$ is an element of $H_c^{2n-4}(X^\circ, \mathbb{R})^\vee$. Consider the inclusions

$$i: X^\circ \hookrightarrow X \quad \text{and} \quad \iota: Z = X \setminus X^\circ \hookrightarrow X$$

and the following excerpt from the associated long exact sequence:

$$H^{2n-5}(Z, \mathbb{R}) \longrightarrow H_c^{2n-4}(X^\circ, \mathbb{R}) \xrightarrow{i_*} H^{2n-4}(X, \mathbb{R}) \xrightarrow{\iota^*} H^{2n-4}(Z, \mathbb{R}).$$

By Lemma 5.8, $\dim_{\mathbb{R}} Z \leq 2n - 6$, so the outer terms vanish and i_* is an isomorphism. We now define the second orbifold Chern class $\tilde{c}_2(X) \in H^{2n-4}(X, \mathbb{R})^\vee$ to be $(i_*^{-1})^\vee(\text{PD}(\tilde{c}_2(X^\circ)))$.

Remark 5.11. — The intersection of $\tilde{c}_2(X)$ with a class in $H^{2n-4}(X, \mathbb{R})$ can be described more explicitly in terms of differential forms. For simplicity, assume that X is a compact klt threefold with just one single isolated singularity $p \in X$. We may assume $p \in X$ to be non-quotient, so that the orbifold locus $X^\circ = X \setminus \{p\}$ is smooth. Let $c = P(\kappa) \in H^2(X, \mathbb{R})$ be a cohomology class, where $\kappa \in \mathcal{K}_{X, \mathbb{R}}^1(X)$ is represented by a family $(U_i, \varphi_i)_{i \in I}$ of smooth functions whose differences $\varphi_i - \varphi_j$ are pluriharmonic. Pick an index $\ell \in I$ with $p \in U_\ell$. Let $\lambda: X \rightarrow [0, 1]$ be a cutoff function near p , that is, $\lambda \equiv 1$ in a neighborhood of p and $\text{supp}(\lambda) \subset U_\ell$. Then $\lambda \cdot \varphi_\ell$, extended by zero outside of U_ℓ , is a smooth function on X .

The element $\tilde{\kappa} \in \mathcal{K}_{X, \mathbb{R}}^1(X)$ represented by $(U_i, \tilde{\varphi}_i)_{i \in I}$, where $\tilde{\varphi}_i = \varphi_i - \lambda \varphi_\ell$, satisfies $P(\tilde{\kappa}') = c$. For each index i , we may consider the real $(1, 1)$ -form $i\partial\bar{\partial}\tilde{\varphi}_i$ on the complex manifold $U_i^\circ := U_i \setminus \{p\}$. Since $\tilde{\varphi}_i - \tilde{\varphi}_j = \varphi_i - \varphi_j$ is pluriharmonic, these forms glue to a real $(1, 1)$ -form ω on X° . Furthermore, since $\tilde{\varphi}_\ell$ is zero in a neighborhood of p , the form ω has compact support. Picking a Kähler metric h on X° , we obtain the second Chern form $c_2(X^\circ, h)$, which is a real $(2, 2)$ -form on X° . The 6-form $c_2(X^\circ, h) \wedge \omega$ has compact support, hence integrates to a finite value. We then have

$$\tilde{c}_2(X) \cdot c = \int_{X^\circ} c_2(X^\circ, h) \wedge \omega \in \mathbb{R}.$$

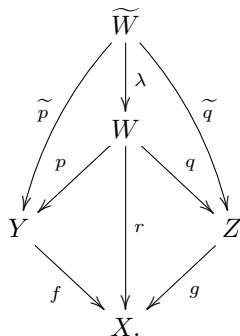
Remark 5.12. — Definition 5.2 can obviously be extended to define the k -th orbifold Chern class $\tilde{c}_k(X) \in H^{2n-2k}(X, \mathbb{R})^\vee$ of a compact complex space X whose non-orbifold locus has codimension $\geq k + 1$. However, we do not know any non-trivial natural condition guaranteeing this property for $k \geq 3$.

5.3. Proof of Proposition 5.3

First we show the existence of a resolution which is minimal in codimension two. Consider the functorial resolution $f: Y \rightarrow X$, which is projective (see [15, Theorems. 3.35 and 3.45]). Let $Z \subset X$ be the subset from Lemma 5.8. Locally at any point $x \in X \setminus Z$, either X is smooth and f is an isomorphism, or $X \cong S \times \mathbb{C}^n$ for a surface S with klt singularities. In the latter case, we have $Y \cong \tilde{S} \times \mathbb{C}^n$ with $\tilde{S} \rightarrow S$ the functorial resolution, since taking the functorial resolution commutes with smooth morphisms in

the sense of [15, 3.34.1]. But $\widetilde{S} \rightarrow S$ is the minimal resolution, and then also f is minimal at x .

It remains to show well-definedness, i.e. independence of $c_2(X)$ of the resolution chosen. To this end, suppose that $f: Y \rightarrow X$ and $g: Z \rightarrow X$ are two resolutions minimal in codimension two. Let $S \subset X$ be an analytic subset of codimension ≥ 3 such that $Y \setminus f^{-1}(S) \rightarrow X \setminus S$ and $Z \setminus g^{-1}(S) \rightarrow X \setminus S$ are minimal resolutions. Consider W the normalization of the main component of the fibre product $Y \times_X Z$ and pick a projective strong resolution $\lambda: \widetilde{W} \rightarrow W$ of W . We then have the following commutative diagram:



Furthermore we set $\tilde{r} := r \circ \lambda: \widetilde{W} \rightarrow X$.

CLAIM 5.13. — *Let $E_0 \subset \widetilde{W}$ be a prime divisor with $\tilde{r}(E_0) \not\subset S$. Then E_0 is \tilde{p} -exceptional if and only if it is \tilde{q} -exceptional.*

Proof of Claim 5.13. — This is well-known, but we recall the proof. Disregarding S and its respective preimages, we may assume that f and g are minimal. Write as usual

$$(5.3) \quad K_{\widetilde{W}} = \tilde{p}^* K_Y + E = \tilde{q}^* K_Z + F,$$

where E is effective and \tilde{p} -exceptional with support equal to $\text{Exc}(\tilde{p})$, and likewise for F . Arguing by contradiction, assume that there is a \tilde{p} -exceptional prime divisor $E_0 \subset \widetilde{W}$ that is not \tilde{q} -exceptional. Set

$$E' := E - \min\{E, F\}, \quad F' := F - \min\{E, F\},$$

where the minimum is taken coefficient-wise. Then E' and F' are effective with no common components. Furthermore $E' \neq 0$ since $E_0 \subset \text{supp}(E')$. Since \tilde{p} is a projective morphism, the Negativity Lemma implies that some component of E' is covered by \tilde{p} -exceptional curves C satisfying $E' \cdot C < 0$. For a general such curve, $F' \cdot C \geq 0$ since C is not contained in F' . Now

by (5.3),

$$\underbrace{(\tilde{p}^* K_Y + E')}_{=E' \cdot C < 0} \cdot C = (\tilde{q}^* K_Z + F') \cdot C = \underbrace{K_Z \cdot \tilde{q}_* C}_{\geq 0} + \underbrace{F' \cdot C}_{\geq 0}.$$

Here the first summand on the right-hand side is non-negative since K_Z is g -nef and $\tilde{q}_* C$ is g -exceptional (or zero). This is the desired contradiction. If instead there is a \tilde{q} -exceptional prime divisor that is not \tilde{p} -exceptional, the argument is similar. □

CLAIM 5.14. — We have $\text{codim}_X(\tilde{r}(\text{Exc}(\tilde{p}) \cup \text{Exc}(\tilde{q}))) \geq 3$.

Proof of Claim 5.14. — We will only show $\text{codim}_X(\tilde{r}(\text{Exc}(\tilde{p}))) \geq 3$, since the argument for \tilde{q} is similar. Since Y is smooth and λ is a strong resolution, we have $\lambda(\text{Exc}(\lambda)) \subset W_{\text{sg}} \subset \text{Exc}(p)$ and hence

$$\tilde{r}(\text{Exc}(\tilde{p})) \subset r(\lambda(\text{Exc}(\lambda) \cup \lambda^{-1}(\text{Exc}(p)))) \subset r(\text{Exc}(p)).$$

Thus it suffices to show $\text{codim}_X(r(B)) \geq 3$ for any irreducible component $B \subset \text{Exc}(p)$. We may assume that B is a divisor and that $r(B) \not\subset S$, since otherwise the claim is clear. The divisor $\lambda_*^{-1}(B)$ is \tilde{p} -exceptional and hence also \tilde{q} -exceptional by Claim 5.13. Thus B is p - and q -exceptional. So we have maps $p: B \rightarrow Y$ and $q: B \rightarrow Z$ with $\dim p(B), \dim q(B) \leq \dim B - 1$ and the further property that $(p, q): B \rightarrow Y \times Z$ is finite (by the construction of W). Since r factors through both p and q , this easily implies $\dim r(B) \leq \dim B - 2 = \dim X - 3$. □

By Claim 5.14, we may apply Lemma 5.9 with $Y_1 = \widetilde{W}$ and $U = X \setminus \tilde{r}(\text{Exc}(\tilde{p}))$ to conclude that for any $a \in H^{2n-4}(X, \mathbb{R})$ we have

$$\int_{\widetilde{W}} c_2(\widetilde{W}) \cup \tilde{r}^*(a) = \int_Y c_2(Y) \cup f^*(a).$$

By the same reasoning applied to Z instead of Y ,

$$\int_{\widetilde{W}} c_2(\widetilde{W}) \cup \tilde{r}^*(a) = \int_Z c_2(Z) \cup g^*(a).$$

This shows that $c_2(X) \cdot a = \int_Y c_2(Y) \cup f^*(a) = \int_Z c_2(Z) \cup g^*(a)$ is well-defined, as desired. □

5.4. Proof of Proposition 5.6

That X_1 again has klt singularities follows from [17, Proposition 5.20]. eq. (5.1) holds since in local uniformizations, the map g becomes étale. For more details, see the proof of [26, Lemma 2.7]. For (5.2), let $f: Y \rightarrow X$

be a resolution minimal in codimension two and consider the commutative diagram

$$\begin{array}{ccc}
 Y_1 & \xrightarrow{h} & Y \\
 f_1 \downarrow & & \downarrow f \\
 X_1 & \xrightarrow{g} & X,
 \end{array}$$

where Y_1 is a strong resolution of the main component of the fibre product $Y \times_X X_1$. Then $f_1 : Y_1 \rightarrow X_1$ is minimal in codimension two and we have

$$\begin{aligned}
 c_2(X_1) \cdot g^*(a) &= \int_{Y_1} c_2(Y_1) \cup f_1^*(g^*a) && \text{by definition} \\
 &= \int_{Y_1} c_2(Y_1) \cup h^*(f^*a) && \text{since } g \circ f_1 = f \circ h \\
 &= \deg(h) \cdot \int_Y c_2(Y) \cup f^*(a) && \text{by Lemma 5.9} \\
 &= \deg(g) \cdot c_2(X) \cdot a && \text{since } \deg g = \deg h.
 \end{aligned}$$

This ends the proof. □

6. The projective case

In this section we prove Theorem 1.1 and Corollary 1.2 in the case where X is assumed to be projective. In this case, we can even weaken the assumption of canonicity to being klt.

THEOREM 6.1. — *Let X be a projective threefold with klt singularities and $c_1(X) = 0$. Assume that $\tilde{c}_2(X) \cdot \omega = 0$ for some Kähler class $\omega \in H^2(X, \mathbb{R})$. Then there exists an abelian threefold T and a finite group G acting on T holomorphically and freely in codimension one such that $X \cong T/G$.*

COROLLARY 6.2. — *Let X be as above, but assume that $c_2(X) \cdot \omega = 0$ for some Kähler class ω on X . Then $X \cong T/G$ as above, where G acts freely in codimension two.*

The following proposition is crucial to the proof of Theorem 6.1.

PROPOSITION 6.3 (Algebraicity of Chern classes). — *Let X be an n -dimensional normal projective variety with klt singularities. Then under the isomorphism $H^{2n-4}(X, \mathbb{R})^\vee \cong H_{2n-4}(X, \mathbb{R})$ from the universal coefficient theorem, $\tilde{c}_2(X)$ is mapped to an element of $B_{2n-4}(X, \mathbb{R})$ (see notation 4.1).*

Unfortunately, the proof of Proposition 6.3 is slightly involved. Essentially, it consists in comparing our definition of $\tilde{c}_2(X)$ for complex spaces to the algebraic definition for quasi-projective \mathbb{Q} -varieties given by Mumford [20, Part I], denoted here by $\hat{c}_2(X)$. We will freely use notation from [20] and from [10, Section 3], to which we refer the reader.

DEFINITION 6.4 (Cycle class map for orbifolds). — *Let U be a quasi-projective variety with quotient singularities, of pure dimension n . For any integer $k \geq 0$, we define the cycle class map*

$$[\cdot]: A_k(U) \rightarrow H^{2(n-k)}(U, \mathbb{R})$$

from the Chow group of U to cohomology by sending an algebraic k -cycle Z on U first to its fundamental class $[Z] \in H_{2k}^{\text{BM}}(U)$ (see notation 4.1) and then using the composition

$$H_{2k}^{\text{BM}}(U) \longrightarrow \text{Hom}_{\mathbb{Z}}(H_c^{2k}(U, \mathbb{Z}), \mathbb{Z}) \xrightarrow{-\otimes \mathbb{R}} H_c^{2k}(U, \mathbb{R})^\vee \xrightarrow{\text{PD}^{-1}} H^{2(n-k)}(U, \mathbb{R}),$$

where the first map is [13, IX.1.7]. The de Rham interpretation of this map after tensorizing by \mathbb{R} is that of integrating compactly supported $2k$ -forms on U over Z .

Proof of Proposition 6.3. — By [10, Lemma 3.19], there exists a closed subset $Z \subset X$ with $\text{codim}_X(Z) \geq 3$ such that $X^\circ := X \setminus Z$ can be equipped with the structure of a quasi-étale \mathbb{Q} -variety admitting a global Cohen–Macaulay cover. That is, there is a finite set A and for each $\alpha \in A$ a quasi-étale Galois map $p_\alpha: X_\alpha^\circ \rightarrow X^\circ$ from a smooth quasi-projective variety X_α° , say with Galois group G_α , such that $X^\circ = \bigcup p_\alpha(X_\alpha^\circ)$. Furthermore, there is a finite Galois map $p: \tilde{X}^\circ \rightarrow X^\circ$ from a normal Cohen–Macaulay variety \tilde{X}° , say with Galois group G , such that for each $\alpha \in A$ there is a commutative diagram

$$\begin{array}{ccc} \tilde{X}_\alpha^\circ & \xrightarrow{\text{incl. of open}} & \tilde{X}^\circ & \xleftarrow{\pi} & Z \\ q_\alpha \downarrow & & \downarrow p & \swarrow \alpha & \\ X_\alpha^\circ & \xrightarrow{p_\alpha} & X^\circ & & \end{array}$$

where $\pi: Z \rightarrow \tilde{X}^\circ$ is a resolution of singularities. The tangent sheaf \mathcal{T}_{X° of X° gives rise to a \mathbb{Q} -sheaf $\{\mathcal{F}_\alpha\}$ defined by setting $\mathcal{F}_\alpha := p_\alpha^{[*]} \mathcal{T}_{X^\circ}$. Since p_α is quasi-étale and X_α° is smooth, $p_\alpha^{[*]} \mathcal{T}_{X^\circ} = \mathcal{T}_{X_\alpha^\circ}$ is locally free, so $\{\mathcal{F}_\alpha\}$ even is a \mathbb{Q} -bundle. As described in [20, p. 277], the pulled-back sheaves $q_\alpha^* \mathcal{F}_\alpha$ on \tilde{X}_α° glue together to a locally free G -sheaf $\tilde{\mathcal{F}}$ on \tilde{X}° . Note that in fact we have $\tilde{\mathcal{F}} = p^{[*]} \mathcal{T}_{X^\circ}$, cf. [10, Remark 3.9].

Mumford now considers the Chern classes $c_k(\widetilde{\mathcal{F}}) \in \text{opA}^k(\widetilde{X}^\circ)^G$. The algebraic orbifold Chern class $\hat{c}_k(X^\circ) \in A_{n-k}(X^\circ)$ then is, by definition, the image of $\frac{1}{\text{deg } p} \cdot c_k(\widetilde{\mathcal{F}})$ under the isomorphism [20, Theorem 3.1]

$$(6.1) \quad A_{n-k}(X^\circ) \cong \text{opA}^k(\widetilde{X}^\circ)^G.$$

CLAIM 6.5. — For each $1 \leq k \leq n$, we have $\tilde{c}_k(X^\circ) = [\hat{c}_k(X^\circ)] \in H^{2k}(X^\circ, \mathbb{R})$, where $[-]$ denotes the cycle class map as in Definition 6.4.

Proof of Claim 6.5. — Consider the vector bundle $\mathcal{F}_Z := \pi^* \widetilde{\mathcal{F}}$. Since Z is smooth, for any bundle \mathcal{E} it is well-known that $c_k(\mathcal{E}) = [c_k^{\text{CH}}(\mathcal{E})]$, where $c_k^{\text{CH}}(-)$ denotes the usual Chern class mapping to the Chow group of Z . Again as Z is smooth, Mumford’s Chern class $\hat{c}_k(-)$ coincides with $c_k^{\text{CH}}(-)$. Taking $\mathcal{E} = \mathcal{F}_Z$, we get

$$(6.2) \quad c_k(\mathcal{F}_Z) = [\hat{c}_k(\mathcal{F}_Z)].$$

Translating the definition of orbifold Chern classes from [26, Section 2.B.1] to our setting, we see that $\tilde{c}_k(X^\circ)$ is calculated by choosing a collection $\{h_\alpha\}$ of G_α -invariant hermitian metrics on $\mathcal{T}_{X^\circ_\alpha}$. Their pullbacks to $\widetilde{X}^\circ_\alpha$ glue to a metric \tilde{h} on $\widetilde{\mathcal{F}}$, which in turn yields a metric h_Z on \mathcal{F}_Z . Using h_Z to calculate $c_k(\mathcal{F}_Z)$, we see

$$(6.3) \quad c_k(\mathcal{F}_Z) = \alpha^*(\tilde{c}_k(X^\circ)).$$

On the other hand, as we have already remarked above, $\mathcal{F}_Z = \pi^* p^{[*]} \widetilde{\mathcal{F}} = \pi^* p^{[*]} \mathcal{T}_{X^\circ}$. It thus follows directly from Mumford’s construction of the isomorphism (6.1) that

$$(6.4) \quad [\hat{c}_k(\mathcal{F}_Z)] = \alpha^*[\hat{c}_k(X^\circ)].$$

Claim 6.5 is now a consequence of (6.2), (6.3), (6.4) and the injectivity of α^* (see Lemma 6.6 below). □

Pick an algebraic $(n - 2)$ -cycle $\gamma = \sum n_i \gamma_i$ on X° (with rational coefficients) which represents the algebraic second Chern class $\hat{c}_2(X^\circ) \in A_{n-2}(X^\circ)$, and let $\bar{\gamma} := \sum n_i \bar{\gamma}_i \in A_{n-2}(X)$ be its Zariski closure. By definition, its fundamental class $[\bar{\gamma}] \in H_{2n-4}(X, \mathbb{R})$ is contained in $B_{2n-4}(X, \mathbb{R})$. We claim that the image of $\tilde{c}_2(X)$ in $H_{2n-4}(X, \mathbb{R})$ is exactly $[\bar{\gamma}]$, which will finish the proof of Proposition 6.3.

Equivalently, the image of $[\bar{\gamma}]$ in $H^{2n-4}(X, \mathbb{R})^\vee$ is $\tilde{c}_2(X)$. Consider the chain of isomorphisms

$$H^{2n-4}(X, \mathbb{R})^\vee \xrightarrow{(i_*)^\vee} H_c^{2n-4}(X^\circ, \mathbb{R})^\vee \xrightarrow{\text{PD}^{-1}} H^4(X^\circ, \mathbb{R}),$$

where $i: X^\circ \hookrightarrow X$ is the inclusion and PD denotes the Poincaré duality map (cf. Subsection 5.2 and Proposition 5.10). By definition, the image of $\tilde{c}_2(X)$ on the right-hand side is $\tilde{c}_2(X^\circ)$. Since $\bar{\gamma} \cap X^\circ = \gamma$, it is also clear that the image of $[\bar{\gamma}]$ on the right-hand side is the cycle class of γ , which in turn equals $[\hat{c}_2(X^\circ)]$. These two elements agree by Claim 6.5. \square

LEMMA 6.6. — *Let U and \tilde{U} be (not necessarily compact) complex spaces with quotient singularities and $f: \tilde{U} \rightarrow U$ a proper, surjective and generically finite map. Then for any $k \geq 0$, the map $f^*: H^k(U, \mathbb{R}) \rightarrow H^k(\tilde{U}, \mathbb{R})$ is injective.*

Proof. — Let $n := \dim U = \dim \tilde{U}$. We define the Gysin map $f_*: H^k(\tilde{U}, \mathbb{R}) \rightarrow H^k(U, \mathbb{R})$ as the composition

$$H^k(\tilde{U}, \mathbb{R}) \xrightarrow{\text{PD}} H_c^{2n-k}(\tilde{U}, \mathbb{R})^\vee \xrightarrow{(f^*)^\vee} H_c^{2n-k}(U, \mathbb{R})^\vee \xrightarrow{\text{PD}^{-1}} H^k(U, \mathbb{R}).$$

It is easy to see that

$$f_* \circ f^*: H_c^{2n}(U, \mathbb{R}) \cong H_0(U, \mathbb{R}) \longrightarrow H_0(U, \mathbb{R}) \cong H_c^{2n}(U, \mathbb{R})$$

is multiplication by $\deg(f) \neq 0$, the cardinality of a general fibre of f . Assume now that $\alpha \in H^k(U, \mathbb{R})$ is nonzero, but $f^*\alpha = 0$. Let $\beta \in H_c^{2n-k}(U, \mathbb{R})$ be the Poincaré dual of α . Then

$$0 \neq \deg(f) = (\deg f) \cdot (\alpha \cup \beta) = f_* f^*(\alpha \cup \beta) = f_*(f^* \alpha \cup f^* \beta) = 0,$$

which is a contradiction. \square

Remark 6.7. — Lemma 6.6 continues to hold even if f is not generically finite, as long as U is compact and \tilde{U} is Kähler. However, we will not need this.

Notation 6.8. — Let X be as in Theorem 6.1 and let H be an ample line bundle on X . Then we will write $\tilde{c}_2(X) \cdot H$ as a shorthand for $\tilde{c}_2(X) \cdot c_1(\mathcal{O}_X(H))$. This is compatible with the notation introduced in [25, Introduction], cf. [10, Theorem 3.13.2].

PROPOSITION 6.9 (Miyaoka semipositivity). — *Let X be a projective threefold with klt singularities. Assume that K_X is numerically trivial. Then for all ample line bundles H on X we have*

$$0 \leq \tilde{c}_2(X) \cdot H \stackrel{(*)}{\leq} c_2(X) \cdot H,$$

with equality in $(*)$ if and only if X is smooth in codimension two.

Proof. — By [26, Proposition 2.9], the sheaf $\Omega_X^{[1]}$ is generically semipositive, hence semistable with respect to any polarization (as $K_X \equiv 0$). The first inequality then is [26, (2.3.2)]. If X is smooth in codimension two, then (*) is an equality by Remark 5.4. If X does have singularities in codimension two, then $\tilde{c}_2(X) \cdot H < c_2(X) \cdot H$ by [25, Proposition 1.1]. \square

PROPOSITION 6.10 (Criterion for vanishing of \tilde{c}_2 and c_2). — *Let X be a projective threefold with klt singularities and $K_X \equiv 0$. Then the following are equivalent:*

- (1) *There exists a Kähler class $\omega \in H^2(X, \mathbb{R})$ such that $\tilde{c}_2(X) \cdot \omega = 0$.*
- (2) *There exists an \mathbb{R} -ample class $h \in N^1(X)_{\mathbb{R}}$ such that $\tilde{c}_2(X) \cdot h = 0$.*
- (3) *We have $\tilde{c}_2(X) = 0 \in H^2(X, \mathbb{R})^\vee$.*

Furthermore, the three statements remain equivalent if $\tilde{c}_2(X)$ is replaced by $c_2(X)$.

Proof. — (3) trivially implies (1). So assume (1). By Proposition 4.2, there exist elements $h \in N^1(X)_{\mathbb{R}}$ and $t \in T(X)$ such that $\omega = h + t$ in $H^2(X, \mathbb{R})$. In view of Proposition 6.3, we have $\tilde{c}_2(X) \cdot t = 0$ and hence $\tilde{c}_2(X) \cdot h = \tilde{c}_2(X) \cdot \omega = 0$. By Proposition 4.5 we know that h is \mathbb{R} -ample. This yields (2).

Assume (2) now. Let $b \in H^2(X, \mathbb{R})$ be arbitrary, and let $b = d + s$ be the decomposition according to Proposition 4.2. We aim to show that $\tilde{c}_2(X) \cdot b = 0$, which by Proposition 6.3 again is equivalent to $\tilde{c}_2(X) \cdot d = 0$. Arguing by contradiction, assume first that $\tilde{c}_2(X) \cdot d > 0$. Since h is \mathbb{R} -ample, there exists a number $\varepsilon > 0$ such that $h - \varepsilon d$ is still \mathbb{R} -ample. Moreover $\tilde{c}_2(X) \cdot (h - \varepsilon d) < 0$. Perturbing $h - \varepsilon d$ slightly and clearing denominators, we find an ample Cartier divisor class h' such that $\tilde{c}_2(X) \cdot h' < 0$. This however contradicts Proposition 6.9. In case $\tilde{c}_2(X) \cdot d < 0$ we argue similarly. (3) follows.

The argument for the $c_2(X)$ version is exactly the same, except that the reference to Proposition 6.3 needs to be replaced by Lemma 4.4. \square

Proof of Theorem 6.1. — By means of Proposition 6.10, we deduce that $\tilde{c}_2(X) = 0$. The claim now follows from [26, Theorem 1.2]. \square

Proof of Corollary 6.2. — By Proposition 6.10, we deduce that $c_2(X) = 0$. Then also $\tilde{c}_2(X) = 0$ by Proposition 6.9 and Proposition 6.10 again. Hence by Theorem 6.1 we can write $X \cong T/G$, where T is an abelian threefold, $G = \text{Gal}(T/X)$ is a finite group and the quotient map $T \rightarrow X$ is quasi-étale. Now X is smooth in codimension two by the second part of Proposition 6.9, so $T \rightarrow X$ is étale in codimension two by purity of

branch locus. This implies that the action $G \curvearrowright T$ is free in codimension two. \square

Remark 6.11. — The above approach for proving Theorem 6.1 does not work in higher dimensions. To be more precise, suppose we are in the setting of Proposition 6.10, but X has dimension 4. Given a Kähler class ω on X such that $\tilde{c}_2(X) \cdot \omega^2 = 0$ and writing $\omega = h + t$ according to Proposition 4.2, we would like to have $\tilde{c}_2(X) \cdot h^2 = 0$. Writing $\omega^2 = h^2 + 2h \cup t + t^2$ and observing that the middle term integrates to zero because $\tilde{c}_2(X) \cup h \in B_2(X, \mathbb{R})$, we are led to the following question: *Let Y be a complex projective variety with canonical singularities and $\sigma \in T(Y) \subset H^2(Y, \mathbb{R})$. Do we have $\int_S \sigma \cup \sigma = 0$ for any algebraic surface $S \subset Y$?* The (easy) example below shows that this fails even if Y is a manifold.

Example 6.12. — Let E be an elliptic curve without complex multiplication, and set $Y = E \times E$ with projections $p, q: Y \rightarrow E$. Pick two linearly independent classes $\alpha, \beta \in H^1(E, \mathbb{R})$, and consider $\sigma := p^* \alpha \cup q^* \alpha + p^* \beta \cup q^* \beta$. Then $\sigma \cup \sigma = -2 \cdot p^*(\alpha \cup \beta) \cup q^*(\alpha \cup \beta) \neq 0 \in H^4(Y, \mathbb{R})$, but σ is zero on the fibres of p and q as well as on the diagonal of Y . It follows that $\sigma \in T(Y)$ since these classes generate $B_2(Y, \mathbb{R})$.

The following additional example, which is even simpler, was kindly communicated to us by the referee.

Example 6.13. — Let S be a projective K3 surface, and let $\sigma \in H^{2,0}(S)$ be a nonzero holomorphic 2-form. Then $\sigma + \bar{\sigma} \in T(S)$, but $(\sigma + \bar{\sigma})^2$ is proportional to the volume form $\sigma \wedge \bar{\sigma} \in H^4(S, \mathbb{R})$.

7. The case of nonzero irregularity

In this section we prove Theorem 1.1 and Corollary 1.2 in case X has trivial canonical bundle and non-trivial Albanese torus.

THEOREM 7.1. — *Let X be a compact complex threefold with canonical singularities. Assume that $\omega_X \cong \mathcal{O}_X$ and $q(X) := \dim_{\mathbb{C}} H^1(X, \mathcal{O}_X) > 0$.*

- (1) *If $\tilde{c}_2(X) \cdot \omega = 0$ for some Kähler class $\omega \in H^2(X, \mathbb{R})$, then there exists a 3-dimensional complex torus T and a finite group G acting on T holomorphically and freely in codimension one such that $X \cong T/G$.*
- (2) *If $c_2(X) \cdot \omega = 0$ for some Kähler class ω on X , then $X \cong T/G$ as before, where G acts freely. In particular, X is smooth.*

7.1. Uniformization in dimension two

The idea of the proof is, in a sense, to use the Albanese map of X as a replacement for cutting down by hyperplane sections. The following proposition is then applied to the fibres of alb_X .

PROPOSITION 7.2 (Torus quotients in dimension two). — *Let S be a compact Kähler surface with klt singularities satisfying $c_1(S) = 0$ and $\tilde{c}_2(S) = 0$. Then there exists a 2-dimensional complex torus T and a finite group G acting on T holomorphically and freely in codimension one such that $S \cong T/G$.*

The proof of Proposition 7.2 relies on the following statement about étale fundamental groups of klt surfaces. Here, we *define* the étale fundamental group $\pi_1^{\text{ét}}(X)$ of a complex space X to be the profinite completion of its topological fundamental group $\pi_1(X)$. This is compatible with the standard usage in algebraic geometry [11, Fact 1.6].

PROPOSITION 7.3 (Étale fundamental groups of surfaces). — *Let S be a compact complex surface with klt singularities. Then there exists a finite quasi-étale Galois cover $\gamma: T \rightarrow S$, with T normal (hence klt), such that the map $\pi_1^{\text{ét}}(T_{\text{sm}}) \rightarrow \pi_1^{\text{ét}}(T)$ induced by the inclusion of the smooth locus is an isomorphism.*

This result has been proven in [11, Theorem 1.5] for quasi-projective klt varieties of any dimension. In [1], the first author together with his coauthors generalized it to positive characteristic, namely to F -finite Noetherian integral strongly F -regular schemes. It would be equally interesting to consider the case of arbitrary (compact) complex spaces with klt singularities. However, this is not needed for our present purposes.

Proof of Proposition 7.3. — We follow the proof of [11, Theorem 1.5]. Assume that the desired cover does not exist. Then for every finite quasi-étale Galois cover $\tilde{S} \rightarrow S$ there exists a further cover $\hat{S} \rightarrow \tilde{S}$ which is quasi-étale but not étale. Iterating this argument and taking Galois closure, one obtains a sequence of covers

$$\dots \xrightarrow{\gamma_3} S_2 \xrightarrow{\gamma_2} S_1 \xrightarrow{\gamma_1} S_0 = S$$

such that each γ_i is quasi-étale but not étale, and each $\delta_i := \gamma_1 \circ \dots \circ \gamma_i$ is Galois. For every index i , there exists a (necessarily singular) point $p_i \in S$ such that γ_i is not étale over some point of $\delta_{i-1}^{-1}(p_i)$. Since the singular set S_{sg} is finite, we have $p_i = p_0$ (say) for infinitely many i .

By Lemma 5.8, we may choose a sufficiently small neighborhood $p_0 \in U_0 \subset S$ admitting a local uniformization (V_0, G, φ) , where G acts on $0 \in V_0 \subset \mathbb{C}^2$ freely outside of the origin. Set $U_0^\times := U_0 \setminus \{p_0\}$. Shrinking U_0 if necessary, we may assume that each connected component of each $\delta_i^{-1}(U_0)$ contains exactly one point mapping to p_0 . Choose a sequence of connected components $W_i \subset \delta_i^{-1}(U_0)$ such that $W_i \subset \gamma_i^{-1}(W_{i-1})$. Let $t_i \in W_i$ be the unique point mapping to p_0 , and set $W_i^\times := W_i \setminus \{t_i\}$. Then

$$G_i := \delta_{i*}(\pi_1(W_i^\times)) \subset \pi_1(U_0^\times) \cong G$$

defines a decreasing sequence of (normal) subgroups of G . Whenever $W_i \xrightarrow{\gamma_i} W_{i-1}$ is not étale, $W_i^\times \xrightarrow{\gamma_i} W_{i-1}^\times$ has degree ≥ 2 and consequently $G_i \subsetneq G_{i-1}$. As this happens for infinitely many indices i , the sequence (G_i) does not stabilize. This is impossible because G is finite. \square

Proof of Proposition 7.2. — Since S has only quotient singularities (Lemma 5.8), it is an orbifold pure in the sense of [3, Définition 3.1]. By [3, Théorème 4.1], S carries a Ricci-flat orbifold Kähler metric g . This means that the tangent \mathbb{Q} -bundle (\mathcal{T}_S, g) is Hermite–Einstein over (S, g) . By [14, Theorem 4.4.11], $\tilde{c}_2(S) = 0$ implies that the Chern connection on (\mathcal{T}_S, g) is flat⁽²⁾. In particular, the tangent bundle of the smooth locus $\mathcal{T}_{S_{\text{sm}}}$ is given by a linear representation $\rho: \pi_1(S_{\text{sm}}) \rightarrow \text{GL}(2, \mathbb{C})$. The finitely generated group $G := \rho(\pi_1(S_{\text{sm}})) \subset \text{GL}(2, \mathbb{C})$ is residually finite, meaning that the natural map to the profinite completion $G \rightarrow \widehat{G}$ is injective [22]. Let $\gamma: T \rightarrow S$ be the cover given by Proposition 7.3, and set $T^\circ := \gamma^{-1}(S_{\text{sm}})$. Then the tangent bundle $\mathcal{T}_{T^\circ} = \gamma^* \mathcal{T}_{S_{\text{sm}}}$ is given by $\rho^\circ: \pi_1(T^\circ) \rightarrow \text{GL}(2, \mathbb{C})$, the pullback of ρ . We have a commutative diagram

$$\begin{array}{ccccc}
 \widehat{G} & \xleftarrow{\widehat{\rho}^\circ} & \pi_1^{\text{ét}}(T^\circ) & \xrightarrow{\sim} & \pi_1^{\text{ét}}(T_{\text{sm}}) & \xrightarrow{\sim} & \pi_1^{\text{ét}}(T) \\
 \uparrow & & \uparrow & & & & \uparrow \\
 G & \xleftarrow{\rho^\circ} & \pi_1(T^\circ) & \longrightarrow & & \longrightarrow & \pi_1(T)
 \end{array}$$

Here, the natural map $\pi_1^{\text{ét}}(T^\circ) \rightarrow \pi_1^{\text{ét}}(T_{\text{sm}})$ is an isomorphism because $T_{\text{sm}} \setminus T^\circ$ is a finite set. To be more precise, since the link of a smooth surface point is a 3-sphere and hence in particular simply connected, $\pi_1(T^\circ) = \pi_1(T_{\text{sm}})$ by Seifert–van Kampen and this remains valid after profinite completion.

(2) The cited reference only treats vector bundles over Kähler manifolds. But note that the proof consists of purely local calculations, which in our situation can still be done in local uniformizations $(\tilde{U}_\alpha, G_\alpha, \varphi_\alpha)$ of open sets $U_\alpha \subset S$ covering S . Therefore the result continues to hold for complex spaces S with at worst quotient singularities.

It now follows from an easy diagram chase that ρ° factorizes via a representation $\bar{\rho}: \pi_1(T) \rightarrow G$. By construction, on T° the associated locally free sheaf $\mathcal{F}_{\bar{\rho}}$ agrees with the tangent sheaf \mathcal{T}_T . As both sheaves are reflexive and $\text{codim}_T(T \setminus T^\circ) \geq 2$, they are in fact isomorphic. In particular, \mathcal{T}_T is locally free. By the known cases of the Lipman–Zariski conjecture⁽³⁾, T is smooth [5, Theorem 1.1],[7, Corollary 1.3]. Now classical differential geometry implies that T is the quotient of a complex torus by a finite group acting freely [14, Corollary 4.4.15]. We conclude by Lemma 7.4 below. \square

The following well-known argument will be used several times in the sequel.

LEMMA 7.4 (Galois closure trick). — *Let X, Y be normal compact complex spaces, and let $\gamma: Y \rightarrow X$ be a finite surjective map étale in codimension $k \geq 1$. If Y is the quotient of a complex torus T by a finite group acting freely in codimension k , then the same is true of X (for a complex torus T' isogenous to T).*

Proof. — Let $T \rightarrow Y$ be the quotient map and consider the Galois closure of the composition $q: T \rightarrow Y \xrightarrow{\gamma} X$,

$$T' \begin{array}{c} \xrightarrow{\text{Galois}} \\ \longrightarrow T \xrightarrow{q} \end{array} X.$$

Since q is étale in codimension $k \geq 1$, so is $T' \rightarrow T$. By purity of branch locus, $T' \rightarrow T$ is étale and hence T' is a complex torus, isogenous to T . The Galois group $G' := \text{Gal}(T'/X)$ acts on T' freely in codimension k . Thus $X = T'/G'$ is a torus quotient as desired. See [11, proof of Corollary 1.16] for more details. \square

7.2. Proof of Theorem 7.1

By [6, Theorem 1.10], the Albanese map $\alpha: X \rightarrow A = \text{Alb}(X)$ is “étale locally trivial” in the following sense: there exists a finite étale cover $A_1 \rightarrow A$ such that $X \times_A A_1 \cong F \times A_1$ over A_1 , where F is connected. Here, necessarily F is compact, of dimension ≤ 2 , with canonical (hence quotient) singularities and having trivial canonical sheaf $\omega_F \cong \mathcal{O}_F$. Write $p: X_1 := X \times_A A_1 \rightarrow F$ for the projection onto the first factor and $g: X_1 \rightarrow X$ for the natural map.

⁽³⁾The cited references only consider algebraic varieties. However, T has at worst quotient singularities and these are automatically algebraic.

CLAIM 7.5. — In case (1), we have $\tilde{c}_2(F) = 0$. In case (2), we have $c_2(F) = 0$.

Proof. — For the first statement, note that $X_1 \cong F \times A_1$ has at worst quotient singularities, hence we may use Poincaré duality (Proposition 5.10) to define the Gysin map $p_*: H^*(X_1, \mathbb{R}) \rightarrow H^{*-2}(F, \mathbb{R})$ as the composition

$$H^*(X_1, \mathbb{R}) \xrightarrow{\text{PD}} H^{6-*}(X_1, \mathbb{R})^\vee \xrightarrow{(p^*)^\vee} H^{6-*}(F, \mathbb{R})^\vee \xrightarrow{\text{PD}^{-1}} H^{*-2}(F, \mathbb{R}).$$

By the projection formula we have

$$(7.1) \quad p_*(p^*\tilde{c}_2(F) \cup g^*\omega) = \tilde{c}_2(F) \cup p_*(g^*\omega) \in H^4(F, \mathbb{R}) \cong \mathbb{R}.$$

According to (5.1), it holds that $\tilde{c}_2(X_1) \cdot g^*(\omega) = (\deg g) \cdot \tilde{c}_2(X) \cdot \omega = 0$. Since $\tilde{c}_2(X_1) = p^*\tilde{c}_2(F)$, the left-hand side of (7.1) is zero. On the other hand, $p_*(g^*\omega) \neq 0$ since $g^*\omega$ restricted to the fibres of p is a Kähler class by Proposition 3.6. Thus (7.1) shows that $\tilde{c}_2(F) = 0$, as desired.

The proof of the second statement is similar, arguing on $\tilde{F} \times A_1$ instead of X_1 , where $\tilde{F} \rightarrow F$ is the minimal resolution. The details are omitted. \square

Proof of (1). — If $\dim F \leq 1$, clearly F is a torus. If $\dim F = 2$, by Claim 7.5 $\tilde{c}_2(F) = 0$ and then by Proposition 7.2, $F = T/G$ is the quotient of a complex 2-torus T by a finite group G acting freely in codimension one. Letting G act trivially on A_1 , the same is true of $X_1 = T \times A_1/G$. Now item 1 follows from Lemma 7.4.

Proof of (2). — By Claim 7.5, $c_2(F) = 0$. But $c_2(F) = c_2(\tilde{F})$, where \tilde{F} is the minimal resolution. Hence \tilde{F} is a complex torus. Since a complex torus does not contain any exceptional curves, $\tilde{F} \rightarrow F$ is an isomorphism. This shows that $X_1 = F \times A_1$ is a complex torus. Again, item 2 follows from Lemma 7.4.

This ends the proof of Theorem 7.1. \square

8. Proof of main results

In this section, we prove Theorem 1.1 and Corollary 1.2.

8.1. Proof of Theorem 1.1

For “(2) \Rightarrow (1)”, let $q: T \rightarrow X = T/G$ be the quotient map. Then the reflexive tensor power $\omega_X^{[n]} \cong \mathcal{O}_X$, where $n = |G|$, and in particular

$c_1(X) = 0$ in $H^2(X, \mathbb{R})$. Since T is Kähler, so is X , see [28, Chapter IV, Corollary 1.2]. Let $\omega \in H^2(X, \mathbb{R})$ be a Kähler class on X . By (5.1) we have

$$\tilde{c}_2(X) \cdot \omega = \frac{1}{n} \cdot \tilde{c}_2(T) \cdot q^*(\omega) = 0,$$

as $\tilde{c}_2(T) = c_2(T) = 0$.

For “(1) \Rightarrow (2)”, note that ω_X is torsion by abundance (cf. e.g. [6, Proposition 8.2]). Let $g: X_1 \rightarrow X$ be the index one cover of X , where $\omega_{X_1} \cong \mathcal{O}_{X_1}$ and X_1 likewise has canonical singularities [17, Definition 5.19 and Proposition 5.20]. If X_1 is the quotient of a complex torus T by a finite group acting freely in codimension one, then the same is true of X by Lemma 7.4. Note that $c_1(X_1) = g^*(c_1(X)) = 0$ and that (5.1) implies

$$\tilde{c}_2(X_1) \cdot g^*(\omega) = \deg(g) \cdot \tilde{c}_2(X) \cdot \omega = 0.$$

Since $g^*(\omega)$ is a Kähler class by Proposition 3.6, replacing X by X_1 we may assume from now on that $\omega_X \cong \mathcal{O}_X$. We make a case distinction:

- If X is projective, then (2) follows from Theorem 6.1.
- If X is not projective, then $q(X) = h^1(X, \mathcal{O}_X) \neq 0$ by the Kodaira embedding theorem and Serre duality (cf. the argument in the proof of [6, Theorem 4.1]). Now item 2 follows from Theorem 7.1, (1).

This ends the proof of Theorem 1.1. □

8.2. Proof of Corollary 1.2

This proof is completely analogous to the previous one, and thus it is omitted. All one needs to do is use (5.2) instead of (5.1), Corollary 6.2 instead of Theorem 6.1, and (2) of Theorem 7.1 instead of (1).

8.3. On Conjecture 1.3 in dimension three

Here we explain how to prove Conjecture 1.3 in dimension three, assuming the special case of the Abundance Conjecture mentioned in the introduction. The direction “(2) \Rightarrow (1)” is proved exactly the same way as “(2) \Rightarrow (1)” of Theorem 1.1. For the other direction, ω_X is torsion by abundance. Let $g: X_1 \rightarrow X$ be the index one cover of X . Then X_1 is also klt and since ω_{X_1} is invertible, the singularities of X_1 are in fact canonical. We may therefore apply Theorem 1.1 to X_1 and conclude that X_1 is a torus quotient. By Lemma 7.4, also X is a torus quotient.

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