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GEODESIC FLOW OF NONSTRICTLY CONVEX HILBERT GEOMETRIES

by Harrison BRAY (*)

Abstract. — In this paper we describe the topological behavior of the geodesic flow for a class of closed 3-manifolds realized as quotients of nonstrictly convex Hilbert geometries. The structure of these 3-manifolds is described explicitly by Benoist; they are Finsler with isometrically embedded flats, but hyperbolic away from flats. We prove the geodesic flow of the quotient is topologically mixing and satisfies a nonuniform Anosov Closing Lemma, with applications to entropy and orbit counting. We also prove entropy-expansivity for the geodesic flow of any compact quotient of a Hilbert geometry, which implies existence of a measure of maximal entropy.

Résumé. — Dans cet article, nous décrivons le comportement topologique du flot géodésique pour une classe de 3-variétés fermées réalisées sous forme de quotients de géométries de Hilbert non strictement convexes. La structure de ces 3-variétés est explicitement décrite par Benoist; elles sont de Finsler avec des parties plates plongées de façon isométrique, mais hyperboliques loin des parties plates. Nous prouvons que le flot géodésique du quotient est topologiquement mélangeant et satisfait un lemme fermant d'Anosov non uniforme, avec applications au comptage d'entropie et d'orbites. Nous prouvons également l'expansivité de l'entropie pour le flot géodésique de tout quotient compact d'une géométrie de Hilbert, ce qui implique l'existence d'une mesure d'entropie maximale.

1. Introduction

We study topological behavior of the geodesic flow of a class of closed 3-manifolds which are only *Finsler*, meaning the tangent space admits a norm which does not come from an inner product, and for which the geodesic flow is nonuniformly hyperbolic due to the presence of isometrically embedded flats of dimension two. The 3-manifolds arise as quotients of properly

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convex domains in real projective space by discrete groups of projective transformations. Such objects are known as *Hilbert geometries* or *convex real projective structures*. The structure of the domain and the quotient is well-described thanks to Benoist ([6], see Theorem 2.2). As such, we refer to the 3-manifolds of interest as *Benoist 3-manifolds*.

We prove several recurrence properties of the geodesic flow of the Benoist 3-manifolds, such as topological transitivity and a nonuniform Anosov Closing Lemma. Though stable and unstable sets are not even defined for a dense set of points, we prove that strong unstable leaves are defined and dense for closed *hyperbolic* orbits, which are dense in the phase space. These results culminate in the following:

Theorem 5.7. — *The geodesic flow of a Benoist 3-manifold is topologically mixing*[1].

The geometric properties of the universal cover which Benoist verifies in dimension three are essential for the arguments, hence the results do not immediately generalize.

This paper also serves as a precursor to work of the author on the Bowen–Margulis measure of maximal entropy [9]. To that end, we verify conditions of Bowen [8] for easier computability of topological entropy:

Theorem 6.2. — *The geodesic flow of any closed Hilbert geometry satisfies Bowen's entropy-expansive property.*

One corollary of Theorem 6.2 is existence of a measure of maximal entropy for the flow. Another consequence is the following proposition, which implies positive topological entropy.

Proposition 7.1. — *The topological entropy of the geodesic flow of a Benoist 3-manifold is bounded below by the exponential growth rate of lengths of hyperbolic closed orbits.*

The structure of the paper is as follows: we first introduce the objects of interest and the relevant background in Section 2. In Section 3 we study automorphisms of the universal cover, and prove that the additive subgroup of $\mathbb{R}$ generated by lengths of closed hyperbolic orbits is dense (Proposition 3.9). This result, along with transitivity (Proposition 4.3) and nonuniform Anosov Closing (Theorem 4.4) from Section 4 will be crucial for the proof of topological mixing in Section 5. In the same section we also prove a

[1] A continuous dynamical system $f^t\colon X \to X$ is *topologically mixing* if for any open $U, V \subset X$ there exists a $T > 0$ such that $f^T(U) \cap V \neq \varnothing$.

nonuniform orbit gluing lemma (Lemma 5.3) which suffices for topological mixing but requires no control over exponential contraction or expansion rates. Section 6 is devoted to the proof of entropy-expansiveness and Section 7 to orbit counting, with remarks on the relationships between the topological entropy of the geodesic flow, the volume entropy of the metric space, and the critical exponent of the group.

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2. Background

A domain Ω in n-dimensional real projective space $\mathbb{RP}^n$ is *proper* if there is an affine chart in which Ω is bounded, and *properly convex* if moreover Ω is convex in this affine chart, meaning the intersection of Ω with any line is connected. Furthermore, Ω is *strictly convex* if the intersection of the topological boundary $\partial\Omega$ with any line in the complement of Ω contains at most one point. A *Hilbert geometry* on a properly convex domain Ω in $\mathbb{RP}^n$ is determined by the Hilbert metric, defined on an affine chart for Ω as follows: let $d_\Omega(x,x) = 0$. Then for any distinct points $x, y \in \Omega$, there is a unique projective line $\overline{xy}$ passing through x and y. Take a and b to be the intersection points of $\overline{xy}$ with $\partial\Omega$ such that a is closer to x than y. Then the *Hilbert distance* between x and y is

$$d_\Omega(x,y) := \frac{1}{2}\log[a,x,y,b],$$

where $[a,x,y,b] := \frac{|ay|}{|ax|}\frac{|bx|}{|by|}$ and $|\cdot|$ is a Euclidean norm for the chosen affine chart. One can verify that d_Ω satisfies the properties of a metric, is complete on Ω, and is well-defined for any affine representation of Ω by projective invariance of the cross-ratio. Projective lines are always geodesic in this metric, but not all geodesics are lines.

The Hilbert metric comes from a Finsler norm, which is Riemannian only when Ω is an ellipsoid. One can compute that for $(x,v) \in T\Omega$, the Finsler norm is given by

$$F(x,v) := \frac{|v|}{2}\left(\frac{1}{|xv^+|} + \frac{1}{|xv^-|}\right)$$

where v^- and v^+ are the intersection points with $\partial\Omega$ of the projective line determined by v in the direction of $-v$ and $+v$, respectively. A properly convex domain Ω in $\mathbb{RP}^2$ is uniquely geodesic if and only if there is at most one open line segment in $\partial\Omega$ (this can be verified using the well-definedness of the cross-ratio of four lines). The ellipsoid in $\mathbb{RP}^n$ is isometric to $\mathbb{H}^n$ when endowed with the Hilbert metric. In this metric, angles are defined, though distorted. This model for hyperbolic space is known as the Beltrami–Klein model or Cayley–Klein model.

For a properly convex open $\Omega \subset \mathbb{RP}^n$, define the *automorphism group* of Ω to be

$$\mathrm{Aut}(\Omega) := \{g \in \mathrm{PSL}(n+1,\mathbb{R}) \mid g\Omega = \Omega\}.$$

Note that $\mathrm{Aut}(\Omega)$ is a subgroup of $\mathrm{Isom}(\Omega)$, the isometry group of (Ω, d_Ω), since projective transformations preserve the cross-ratio. The full isometry group of (Ω, d_Ω) is, up to index 2, the group of collineations which preserve Ω [29]. A properly convex domain Ω in $\mathbb{RP}^n$ is *divisible* if it admits a cocompact action by a discrete subgroup Γ of $\mathrm{PSL}(n+1,\mathbb{R})$, in which case we say Γ *divides* Ω. As a first example, the ellipse is divisible by any Fuchsian group. The projective triangle, isometric to $\mathbb{R}^2$ with a hexagonal norm when endowed with the Hilbert metric [11, 19], admits a $\mathbb{Z}^2$-action with quotient a flat torus.

Suppose $\Gamma < \mathrm{PSL}(n+1,\mathbb{R})$ acts properly discontinuously without torsion on $\Omega \subset \mathbb{RP}^n$, so that the quotient $M = \Omega/\Gamma$ is a manifold. The *geodesic flow* of M is defined on SM, the Finsler unit tangent bundle to M, by flowing unit tangent vectors along projective lines at unit Hilbert speed:

$$\begin{aligned} \phi^t \colon SM &\longrightarrow SM \\ (x,v) &\longmapsto (x+tv, v). \end{aligned}$$

In other words, $(x,v) \in SM$ determines a unique oriented projective line $\ell_v \colon \mathbb{R} \to M$ parameterized at unit Hilbert speed, with $\ell_v(0) = x$ and $\phi^t(v)$ the Finsler unit tangent vector to ℓ_v based at $\ell_v(t)$. In the strictly convex case, all geodesics are projective lines and this definition coincides with the standard definition of geodesic flow. In our setting, geodesics are not unique so we require defining the geodesic flow in this very natural way. The geodesic flow on SM can be lifted to a flow $\widetilde{\phi}^t \colon S\Omega \to S\Omega$ which acts equivariantly with respect to Γ. Note from the definitions that the regularity of the boundary of Ω determines the regularity of the geodesic flow on $S\Omega$ and hence SM.

2.1. Benoist's dichotomy

The following landmark theorem of Benoist for the study of divisible Hilbert geometries is equivalence of the regularity of the boundary, convexity of the boundary, and hyperbolicity of the flow based on an abstract property of the group.

THEOREM 2.1 ([4, Theorem 1.1]). — *Suppose* Γ *is a discrete torsion-free subgroup of* $\mathrm{PSL}(n+1,\mathbb{R})$ *dividing a properly convex domain* $\Omega \subset \mathbb{RP}^n$. *Then the following are equivalent:*

(1) *The domain* Ω *is strictly convex.*
(2) *The boundary* $\partial\Omega$ *is of class* C^1.
(3) *The group* Γ *is Gromov-hyperbolic.*
(4) *The geodesic flow on the quotient manifold* $M = \Omega/\Gamma$ *is Anosov.*

A finitely presented group G is Gromov-hyperbolic if geodesic triangles in the Cayley graph of G endowed with the word metric are δ-thin, meaning the δ-neighborhood of any two sides of a geodesic triangle contains the third side. Examples include hyperbolic manifold groups, and nonexamples include $\mathbb{Z}^n$ for $n \geqslant 2$.

Essential to Benoist's Theorem 2.1 is Benzecri's thesis work on the PGL-orbits of marked properly convex sets in projective space [7]. In fact, an application of the work of Benzecri shows that in dimension two, a divisible properly convex domain Ω is either strictly convex with C^1-boundary or a projective triangle.

2.2. The Benoist 3-manifolds

One might ask whether, as in dimension two, a divisible Hilbert geometry in any dimension is either strictly convex with C^1-boundary or a simplex. Benoist proved, in the contrary, existence of Hilbert geometries in dimension three which are nonstrictly convex and indecomposable via a modification of the Kac–Vinberg Coxeter construction [6, Proposition 1.3]. Moreover, Benoist proved geometric properties of such Hilbert geometries. Before stating the theorem, we introduce some terms: let C be the cone in

$\mathbb{R}^{n+1}$ over a properly convex domain Ω in $\mathbb{RP}^n$, and define C to be properly convex if and only if Ω is properly convex. Then Ω is *decomposable* if there exist vector subspaces $V_1, V_2 \subset \mathbb{R}^{n+1}$ and properly convex cones $C_1 \subset V_1, C_2 \subset V_2$ such that $C = C_1 + C_2$. Else, Ω is *indecomposable*. Note that a simplex is always decomposable.

A *properly embedded triangle* in Ω is a projective triangle $\triangle \subset \Omega$ such that $\partial\triangle \subset \partial\Omega$. Let $\mathcal{T}$ denote the collection of triangles $\triangle$ which are properly embedded in Ω, and let $\mathrm{Stab}_\Gamma(\triangle) := \{\gamma \in \Gamma \mid \gamma\triangle = \triangle\}$ be the subgroup of Γ stabilizing $\triangle \in \mathcal{T}$.

Theorem 2.2 ([6, Theorem 1.1]). — *Let $\Gamma < \mathrm{SL}(4,\mathbb{R})$ be a discrete torsion-free subgroup which divides an open, properly convex, indecomposable $\Omega \subset \mathbb{RP}^3$, and $M = \Omega/\Gamma$. Then*

(1) *Every subgroup in Γ isomorphic to $\mathbb{Z}^2$ stabilizes a unique triangle $\triangle \in \mathcal{T}$.*
(2) *If $\triangle_1, \triangle_2 \in \mathcal{T}$ are distinct, then $\overline{\triangle_1} \cap \overline{\triangle_2} = \varnothing$.*
(3) *For every $\triangle \in \mathcal{T}$, the group $\mathrm{Stab}_\Gamma(\triangle)$ contains an index-two $\mathbb{Z}^2$ subgroup.*
(4) *The group Γ has only finitely many orbits in $\mathcal{T}$.*
(5) *The image in M of triangles in $\mathcal{T}$ is a finite collection $\mathcal{F}$ of disjoint tori and Klein bottles. If one cuts M along each $\mathbb{T} \in \mathcal{F}$, each of the resulting connected components is atoroidal.*
(6) *Every nontrivial line segment is included in the boundary of some $\triangle \in \mathcal{T}$.*
(7) *If Ω is not strictly convex, then the set of vertices of triangles in $\mathcal{T}$ is dense in $\partial\Omega$.*

We will call a compact quotient of a nonstrictly convex, indecomposable, divisible Hilbert geometry in dimension three a *Benoist 3-manifold*. The topological decomposition as in Benoist's Theorem 2.2 (5) is an example of a Jaco–Shalen–Johansson (JSJ) decomposition [20, 21]. Benoist remarks after Theorem 2.2 that as a consequence of Thurston's geometrization, the atoroidal components of the quotient are diffeomorphic to finite volume hyperbolic 3-manifolds [6, p. 4–5].

3. Automorphisms of Benoist's 3-manifolds

Let Ω be a properly convex domain in $\mathbb{RP}^n$. Then for any automorphism g of Ω, we can define the *translation length* of g by

$$\tau(g) := \inf_{x\in\Omega} d_\Omega(x, g.x).$$

An *axis* of g is a g-invariant projective line in Ω.

We will diverge slightly from the literature here in our terminology. We define $g \in \mathrm{Aut}(\Omega)$ to be *hyperbolic* if $\tau(g) > 0$ and the infimum is attained along a unique axis of g. Any other $f \in \mathrm{Aut}(\Omega)$ for which $\tau(f)$ is positive and realized, but not along a unique axis of f, will be called *flat*. Typically, both flat and hyperbolic automorphisms are called hyperbolic (see [16, Section 3]), and in the strictly convex case there would be no need for the distinction we introduce here. A projective transformation is *quasi-hyperbolic* if $\tau(g) > 0$ and the infimum is not attained, *parabolic* if $\tau(g) = 0$ and the infimum is not attained, and *elliptic* if $\tau(g) = 0$ and the infimum is attained.

There is an important property of the Benoist 3-manifolds which has dynamical implications for the group elements. If the quotient is a Benoist 3-manifold, then Ω must be indecomposable, hence Γ is irreducible [28]. A subgroup $H < \mathrm{PSL}(4, \mathbb{R})$ is *irreducible* if it does not stabilize a projective point, line, or plane in $\mathbb{RP}^3$, and H is *strongly irreducible* if every finite-index subgroup of H is irreducible.

Remark 3.1. — An element g of $\mathrm{SL}(n, \mathbb{R})$ is *proximal* if $|\lambda_1(g)| > |\lambda_2(g)|$, where $|\lambda_1(g)| \geqslant |\lambda_2(g)| \geqslant \cdots \geqslant |\lambda_n(g)|$ are the moduli of the eigenvalues of g, and *biproximal* if g^{-1} is also proximal. On the other hand, g in $\mathrm{SL}(n, \mathbb{R})$ is *positively semi-proximal* if $\lambda_1(g) = |\lambda_1(g)|$. Every element g of a group Γ which preserves a properly convex set is positively semi-proximal ([5, Lemma 3.2], see [24, Proposition 2.2] for a translation). Since Γ contains inverses, every element of Γ has an eigenline associated to each of the top and bottom eigenvalues. The projection of these eigenlines must lie in $\partial\Omega$ for Γ to preserve Ω.

Moreover, for the Benoist 3-manifolds, all elements of any $\mathbb{Z}^2$ subgroup of Γ preserving a properly embedded triangle $\triangle$ have only real positive eigenvalues [6, Corollary 2.4]. It is straightforward to verify that top and bottom eigenvalues for these elements of the stabilizer of $\triangle$ correspond to eigenlines which project to vertices of $\triangle$.

The following proposition is also proved in greater generality in [24, Proposition 2.9]. We include the simpler proof only for the case of the Benoist 3-manifolds here.

Proposition 3.2. — *Let $M = \Omega/\Gamma$ be a Benoist 3-manifold with discrete, torsion-free dividing group Γ. Then there are no quasi-hyperbolic automorphisms of Ω.*

Proof. — Suppose $\tau(g) > 0$. Let the eigenvalues of a representative of g in $\mathrm{SL}(4,\mathbb{R})$ be given by $\lambda_i(g)$ such that $\lambda_0 = |\lambda_0(g)| \geqslant |\lambda_1(g)| \geqslant |\lambda_2(g)| \geqslant |\lambda_3(g)| = \lambda_3 > 0$. Since g is hyperbolic, $\tau(g) = \log \frac{\lambda_0(g)}{\lambda_3(g)}$ and is realized along a projective line joining the projections of the eigenvectors e_0 and e_3 associated to λ_0 and λ_3, respectively. Let ℓ_g be the open projective line segment connecting the projections of the eigenvectors e_0 and e_3 of g. Since g preserves Ω and Ω is properly convex, the line segment ℓ_g is contained in either Ω or $\partial\Omega$. If $\ell_g \subset \Omega$ then $\tau(g)$ is realized along ℓ_g inside Ω. Thus, suppose $\ell_g \subset \partial\Omega$. Then by Benoist's Theorem 2.2 (6), $\ell_g \subset \partial\triangle$ for some properly embedded triangle $\triangle$ in Ω. Since g acts by projective transformations and preserves Ω, for any properly embedded triangle $\triangle$ we have that $g\triangle$ is also a properly embedded triangle. Then since g stabilizes ℓ_g we have $\ell_g \subset g\overline{\triangle} \cap \overline{\triangle} \neq \varnothing$ implying g preserves the projective triangle $\triangle$ by Benoist's Theorem 2.2 (2). It follows that $\tau(g)$ is realized in $\triangle$ by a standard cross-ratio argument. $\square$

We now introduce terminology for points in $\partial\Omega$. Following classical theory in convex geometry, we say that a boundary point ξ is *smooth* if there is a unique supporting hyperplane to Ω at ξ, where a hyperplane H in $\mathbb{RP}^n$ is a *supporting hyperplane* at ξ if H is contained in the complement of Ω and H contains ξ. We also define $\xi \in \partial\Omega$ to be *extremal* if there is no open line segment containing ξ embedded in $\partial\Omega$; in other words, $\xi \in \partial\Omega$ is extremal if and only if $\overline{\Omega} \smallsetminus \{\xi\}$ is still a convex set. We remark that smooth points need not even be C^1; for instance, Lebesgue almost every point in the boundary of the universal cover of a Benoist 3-manifold is smooth, but no points are C^1 because vertices of properly embedded triangles are dense (Benoist's Theorem 2.2 (7)). Note that by Benoist's Theorem 2.2 (6), duality of divisibility [4, Lemma 2.8], and duality of smooth points to extremal points, the set of smooth extremal points forms the complement of the boundaries of properly embedded triangles.

Proposition 3.3. — *Let $M = \Omega/\Gamma$ be a Benoist 3-manifold with discrete, torsion-free dividing group Γ. Then for all $g \in \Gamma$,*

- *g is hyperbolic if and only if g has exactly two fixed points g^- and g^+ in $\overline{\Omega}$ which are smooth extremal points in the boundary. These fixed points are respectively repelling and attracting under the dynamics of g on $\overline{\Omega}$.*
- *g is flat if and only if $g \in \mathrm{Stab}_\Gamma(\triangle)$ for some properly embedded $\triangle$.*

These are the only possible behaviors of elements of Γ.

Proof. — Since Γ is discrete and torsion-free, there are no elliptic isometries in Γ. Since M is compact without boundary, closed homotopically nontrivial loops in M cannot have arbitrarily small length, hence there are no parabolic isometries in Γ (see also [6, Lemma 2.8]). By Proposition 3.2, there are no quasi-hyperbolic elements of Γ. Thus, it suffices to characterize the dynamics of group elements with translation length realized in Ω. By Remark 3.1, the conclusion is straightforward. □

3.1. Lengths of hyperbolic orbits

The goal of this subsection is to prove that the additive subgroup of $\mathbb{R}$ generated by translation lengths of closed hyperbolic orbits is dense via Zariski density of an immersed hyperbolic surface group in Γ.

Theorem 3.4 ([2, 25]). — *The fundamental group of a complete, finite volume, noncompact hyperbolic 3-manifold contains a closed quasi-Fuchsian surface subgroup.*

Let Γ_{hyp} denote the set of hyperbolic elements of Γ, and let $\Sigma < \mathrm{PSL}(4,\mathbb{R})$ be the subgroup of Γ which is isomorphic to the hyperbolic surface subgroup given by Theorem 3.4 and Benoist's remark following Theorem 2.2. Since Σ is a quasi-Fuchsian subgroup, no element of Σ can preserve any properly embedded triangle. Then by Proposition 3.3, Σ is a subgroup in Γ_{hyp}.

Corollary 3.5. — *There exist infinitely many noncommuting hyperbolic group elements in* Γ.

Let G be any subset of $\mathrm{Aut}(\Omega)$ and $\mathcal{L}(G) := \langle \tau(g) \rangle_{g \in G}$ the additive subgroup of $\mathbb{R}$ generated by translation lengths of group elements in G. Note that if G is a subset of the group Γ which divides Ω then $\mathcal{L}(G)$ is the additive subgroup of $\mathbb{R}$ generated by lengths of closed geodesics in the quotient M associated to conjugacy classes in Γ of elements of G.

Corollary 3.6 (of [4, Fact 5.5]). — *If* Γ *is a Zariski dense subgroup of* $\mathrm{SL}(n+1,\mathbb{R})$ *preserving a properly convex domain* $\Omega \subset \mathbb{RP}^n$*, then* $\mathcal{L}(\Gamma)$ *is dense in* $\mathbb{R}$.

If Ω is not an ellipsoid, then the hypotheses of Corollary 3.6 hold whenever Γ is acting cocompactly on an indecomposable properly convex and *strictly convex* domain Ω in real projective space [3, Theorem 1.2]. In our case, Ω, the universal cover of a Benoist 3-manifold, is indecomposable but is not strictly convex, so Corollary 3.6 does not apply directly to Γ the fundamental group of a Benoist 3-manifold.

Proposition 3.7 (restatement of [17, Proposition 6.5]). — *Suppose* Γ *is a strongly irreducible subgroup of* $\mathrm{SL}(n+1,\mathbb{R})$ *which preserves a properly convex* $\Omega \subset \mathbb{RP}^n$. *Let* G *be the Zariski closure of* Γ. *Then* G *is a Zariski-connected real semi-simple Lie group.*

Let $\log(G) := \{(\log|\lambda_1(g)|, \log|\lambda_2(g)|, \dots, \log|\lambda_{n+1}(g)|) \in \mathbb{R}^{n+1} \mid g \in G\}$ with λ_i decreasing in magnitude. Taking G to be the Zariski closure of Γ, then Proposition 3.7 implies $\log(G)$ is a subspace of $\mathbb{R}^{n+1}$, and $\log(\Gamma)$ is dense in $\log(G)$. It is straightforward to verify that $\overline{\mathcal{L}(\Gamma)} \neq \mathbb{R}$ implies $\overline{\log(\Gamma)}$ cannot be a subspace of $\mathbb{R}^{n+1}$. Thus, it suffices to prove that Σ is strongly irreducible to conclude that $\mathcal{L}(\Sigma)$ is dense in $\mathbb{R}$.

For the following lemma, we will use that Γ preserves (divides) Ω if and only if the transpose Γ^t preserves (divides) the projective dual Ω^* [4, Lemma 2.8].

Lemma 3.8. — *The surface subgroup* Σ *is either strongly irreducible or* $\mathcal{L}(\Sigma)$ *is dense in* $\mathbb{R}$.

Proof. — First, since Σ is a surface group, every finite-index subgroup is also a surface subgroup. Thus, it suffices to show any surface group in $\mathrm{PSL}(4,\mathbb{R})$ preserving a properly convex domain $\Omega \subset \mathbb{RP}^3$ is irreducible. By contradiction, suppose Σ fixes a point $p \in \mathbb{RP}^3$. Clearly $p \notin \Omega$ because Γ acting discretely without torsion cannot have elliptic elements. Also, $p \notin \partial\Omega$: this is because elements of Σ do not stabilize any properly embedded triangles, hence all fixed points of elements of Σ are smooth and extremal. But noncommuting hyperbolic isometries cannot fix the same smooth extremal point since Γ acts properly discontinuously on Ω. If $p \notin \overline{\Omega}$, then we consider the dual case: when Σ^t preserves a projective plane Π which intersects Ω^*. Then Σ^t is acting cocompactly on a totally geodesic hypersurface $\Pi \cap \Omega^*$. By [3, Theorem 1.2], Σ^t is either Zariski dense and hence $\mathcal{L}(\Sigma^t)$ is dense by Corollary 3.6 or $\Pi \cap \Omega^*$ is homogeneous and $\mathcal{L}(\Sigma^t)$ is dense in $\mathbb{R}$ anyways. Then so is $\mathcal{L}(\Sigma)$ since dual groups preserving dual properly convex sets are isospectral. Thus, if Σ preserves Ω and fixes a point, then $\mathcal{L}(\Sigma)$ is dense in $\mathbb{R}$.

Now suppose Σ preserves a line l. The case where $l \subset \Omega$ is impossible because Ω is properly convex. The case where l is disjoint from $\overline{\Omega}$ is impossible because $\mathrm{Aut}(l) = \mathbb{R}$ and Σ is a surface group. If l intersects $\partial\Omega$ then either Σ does not preserve Ω (when $l \cap \partial\Omega$ contains exactly one point), or Σ preserves an open line segment inside Ω which is again impossible (when $l \cap \partial\Omega$ contains exactly two points), or $\Sigma \subset \mathrm{Stab}_\Gamma(\triangle)$ (when $l \cap \Omega$ is an open line segment) which is again impossible because Σ contains only hyperbolic isometries. Therefore, Σ cannot preserve a projective line.

If Σ stabilizes a plane, then we revisit the dual cases where Σ^t stabilizes a point, unless the plane intersects Ω. In this case, we have already seen that $\mathcal{L}(\Sigma)$ is dense in $\mathbb{R}$. $\square$

Proposition 3.9. — *Let $\Omega/\Gamma = M$ be a Benoist 3-manifold. Then $\mathcal{L}(\Gamma_{hyp})$ is dense in $\mathbb{R}$.*

Proof. — By Lemma 3.8 and Proposition 3.7, the group $\Sigma \subset \Gamma_{\mathrm{hyp}}$ is either Zariski dense or $\mathcal{L}(\Sigma)$ is dense in $\mathbb{R}$. By Corollary 3.6, density of $\mathcal{L}(\Sigma)$ holds in both cases. $\square$

4. Recurrence behavior

Recall that we define a boundary point $\xi \in \partial\Omega$ to be smooth if there exists a unique supporting hyperplane to Ω at ξ and extremal if ξ is not contained in any open line segment embedded in $\partial\Omega$. For the Benoist 3-manifolds, vertices of properly embedded triangles are the only nonsmooth points, and all nonextremal points are contained in the side of some properly embedded triangle. Thus, the smooth extremal points form the complement in $\partial\Omega$ of the boundaries of properly embedded triangles.

We will say $v \in S\Omega$ is *regular* if its endpoints at infinity v^+ and v^- are smooth extremal points. Else, v is *singular*. Let $S\Omega_{\mathrm{reg}}$ be the collection of regular vectors and the complement, $S\Omega_{\mathrm{sing}}$, the set of $v \in \Omega$ such that v^+ or v^- is in the boundary of some properly embedded triangle. The collection of vectors tangent to projective lines contained entirely in properly embedded triangles is denoted $S\Omega_{\mathrm{flat}}$. These notions descend to the quotient since Γ is acting by projective transformations, and we assign the analogous definitions to $SM_{\mathrm{reg}}, SM_{\mathrm{sing}}$, and SM_{flat}.

Lastly, an orbit $\phi \cdot v$ which is closed will be called *hyperbolic* if when v is lifted to $\widetilde{v}$ in the universal cover, $\ell_{\tilde{v}}$ is preserved by a hyperbolic group element. We call a vector v in SM *periodic* if the orbit of v is closed. Note that any periodic vector with a closed orbit which is hyperbolic must be regular (Proposition 3.3).

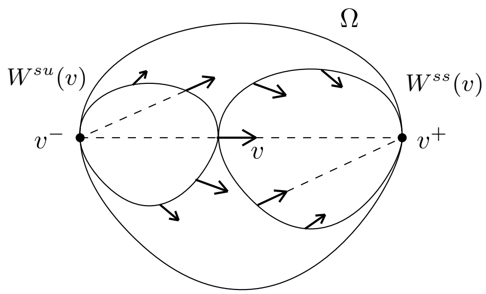

Figure 4.1. The stable and unstable sets at a point v in $S\Omega$, the universal cover of SM.

4.1. Stable and unstable sets

Recall that the stable and unstable sets at a point are defined to be

$$W^{ss}(v) := \{u \in SM \mid d(\phi^t v, \phi^t u) \to 0 \mid t \to +\infty\},$$
$$W^{su}(v) := \{u \in SM \mid d(\phi^{-t} v, \phi^{-t} u) \to 0 \mid t \to +\infty\}.$$

The weak stable and unstable sets of v (denoted $W^{os}(v)$ and $W^{ou}(v)$, respectively) are the points which stay bounded distance from v under the geodesic flow in positive and negative time, respectively. The strong stable and unstable sets are *global* if for all regular $u \neq v$, at least one of $W^{ss}(v) \cap W^{ou}(u)$ or $W^{ss}(v) \cap W^{ou}(-u)$ are nonempty.

To define the stable and unstable sets in our setting, we will need a Finsler metric on SM compatible with the topology, see [26, p. 161–206]. We will soon see that there is a geometric description of stable and unstable sets. Consequently, there exists an adapted metric on $S\Omega$ such that the distance between two points in the same weak stable set is monotone decreasing under the geodesic flow in positive time for the adapted metric [15], which can be verified by properties of the cross-ratio. Similarly, the distance between points in the same unstable set is monotone decreasing under the flow in negative time. We denote this metric by d and note that it descends to the quotient SM. We denote the d-metric ball of radius r about v by $B(v, r)$.

Proposition 4.1. — *If v in SM is regular then $W^{ss}(v)$ and $W^{su}(v)$ are defined globally, and the weak stable and unstable sets $W^{os}(v)$ and $W^{ou}(v)$ admit a flow invariant foliation by strong stable (respectively, strong unstable) leaves.*

Proof. — We will see it is enough to verify the proposition in the universal cover. For smooth extremal points, horospheres are well-defined and the geometric description of stable and unstable sets applies as for the strictly convex case (as in [4, Lemma 3.4]): that is, for regular vectors v in $S\Omega$,

$$\begin{aligned} W^{ss}(v) &= \{u \in S\Omega \mid \pi u \in \mathcal{H}_{v^+}(\pi v),\ u^+ = v^+\}, \\ W^{su}(v) &= \{u \in S\Omega \mid \pi u \in \mathcal{H}_{v^-}(\pi v),\ u^- = v^-\}, \end{aligned}$$

where $\mathcal{H}_\xi(x)$ is the horosphere based at $\xi \in \partial\Omega$ passing through $x \in \Omega$ (see [4, Figures 5 and 6]), and $\pi\colon T\Omega \to \Omega$ is the footpoint projection. See Figure 4.1. Moreover, the strong stable and unstable sets foliate the weak stable and unstable sets

$$\begin{aligned} W^{ou}(v) &= \bigcup_{t\in\mathbb{R}} W^{su}(\widetilde{\phi}^t v) \\ &= \{w \in S\Omega \mid w^- = v^-\}, \end{aligned}$$

such that the foliation is both Γ-invariant and $\widetilde{\phi}^t$-invariant. It is then clear that for u, v any two distinct regular vectors, $W^{ss}(v) \cap W^{ou}(u) \neq \varnothing$ as long as $u^- \neq v^+$. In the case where $u^- = v^+$, apply the same arguments to show that $W^{ss}(v) \cap W^{ou}(-u) \neq \varnothing$. □

Note that in fact, in the universal cover, if they do intersect then $W^{ss}(v)$ and $W^{ou}(u)$ intersect at a unique point when u and v are distinct regular vectors. Conversely, nonsmooth and nonextremal points do not have well-defined stable and unstable sets which foliate the weak stable and unstable sets. This can be verified by basic properties of the cross-ratio. By Benoist's Theorem 2.2 (7), the vertices of properly embedded triangles in Ω are dense in $\partial\Omega$, and as such the singular points are dense in $S\Omega$. Since these points do not admit stable and unstable sets, the geodesic flow cannot have local product structure in the classical sense (see for instance [12, Section 2] for the definition of local product structure), and thus we cannot employ any results depending on it.

However, the Bowen bracket for regular vectors is almost always well-defined by Proposition 4.1. In the universal cover, the *Bowen–Bracket* of u with v, denoted $\langle u, v\rangle$, is the point of intersection w of the strong stable and weak stable sets of u and v respectively, assuming $v^- \neq u^+$. Geometrically, w is uniquely determined by $w^- = v^-$, $w^+ = u^+$, and $\pi w \in \mathcal{H}_{u^+}(\pi u)$. The notion of Bowen–Bracket descends canonically to the quotient by choosing a fundamental domain for the action of Γ on $S\Omega$. See Figure 4.2.

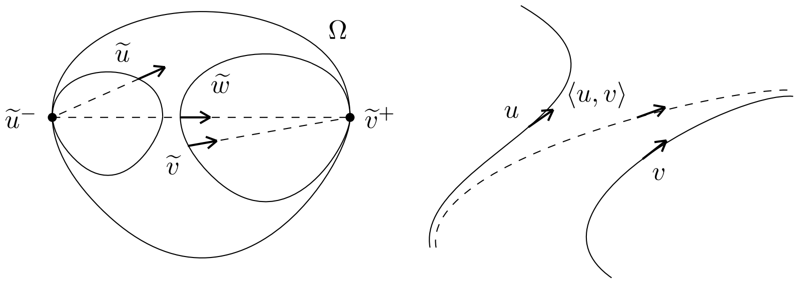

Figure 4.2. In the universal cover, pictured left, the Bowen Bracket of $\widetilde{u}$ with $\widetilde{v}$ is the unique point $\widetilde{w}$ in both the weak unstable set of $\widetilde{u}$ and the strong unstable set of $\widetilde{v}$. The notion descends to the quotient vectors u and v pictured on the right. The drawing on the right depicts the key feature of the Bowen bracket of u with v; the vector $\langle u, v\rangle$ has the same past as u up to a time change and the same future as v.

4.2. Topological transitivity

In this subsection we prove topological transitivity, which is equivalent to existence of a dense orbit when the phase space is compact [22, Lemma 1.4.2], as in the case of the Benoist 3-manifolds. A continuous dynamical system $f^t \colon X \to X$ is *topologically transitive* if for every pair of open sets $U, V \subset X$, there exists a time $0 < T \in \mathbb{R}$ such that $f^T(U) \cap V \neq \varnothing$. If X is a metric space then the system is *uniformly transitive* if for all $\delta > 0$, there exists a $T > 0$ such that for all $x, y \in X$, there is some $t \leqslant T$ such that $f^t\big(B(x,\delta)\big) \cap B(y,\delta) \neq \varnothing$. It is straightforward to check that transitivity implies uniform transitivity when X is a compact metric space.

Lemma 4.2. — *Hyperbolic closed orbits are dense for the geodesic flow of a Benoist 3-manifold.*

Proof. — We want to show any pair (ξ, η) in $\partial\Omega \times \partial\Omega \smallsetminus \Delta$, the pairs of distinct points in the boundary of Ω, can be approximated by pairs (g^-, g^+) of fixed points of hyperbolic group elements g. Take two noncommuting hyperbolic elements $g, h \in \Gamma$, which exist by Corollary 3.5. Construct the sequence $k_n = g^n h^n$. Then by Remark 3.1, there are fixed points k_n^+ and k_n^- in $\partial\Omega$ of k_n such that $k_n^+ \to g^+$ and $k_{-n}^- \to h^-$ as $n \to \infty$. Although the notation is suggestive, these k_n could a priori stabilize properly embedded triangles and have three fixed points in $\partial\Omega$. Using the sequence k_n and minimality of the action of Γ on $\partial\Omega$ [6, Proposition 3.10], we conclude that any $(\xi, \eta) \in \partial\Omega \times \partial\Omega \smallsetminus \Delta$ is approximable by such k_n. If any k_n

admits a projective line axis, then this projective line axis corresponds to a periodic orbit for the flow and we conclude that any vector tangent to the projective line $(\xi\eta)$ is approximable by closed orbits. It suffices to show there necessarily exists a subsequence k_{n_i} of only hyperbolic elements, which we know have a unique projective line axis by Proposition 3.3.

By contradiction, suppose there is no such subsequence. There exists an N such that for all $n \geqslant N$, each k_n preserves a properly embedded triangle $\triangle_n$. Consider the accumulation of the boundary of the triangles, $\partial\triangle_n$, in $\partial\Omega$, which will contain h^- and g^+. This set will be either the boundary of a properly embedded triangle, a line segment, or a point. None of the above are possible; since h and g are hyperbolic and do not commute in the discrete group Γ, h^- cannot equal g^+, and h^- and g^+ are smooth extremal points, so the line segment $[h^-g^+]$ cannot be contained in $\partial\Omega$. □

Proposition 4.3. — *The geodesic flow of a Benoist 3-manifold is topologically transitive.*

Proof. — Take two open sets U and V in SM. By Lemma 4.2, there are regular $u \in U$ and $v \in V$ with closed orbits such that $v \neq -u$. We now construct a heteroclinic orbit between v and u. Lifting to the universal cover, we have $\widetilde{u} \in \widetilde{U}, \widetilde{v} \in \widetilde{V} \subset S\Omega$ such that $\widetilde{u}^-$ and $\widetilde{v}^+$ are distinct smooth extremal points of $\partial\Omega$. Then the open projective line segment $(\widetilde{u}^-\widetilde{v}^+)$ is contained in Ω and is the footpoint projection of an orbit of the geodesic flow. Let $\widetilde{w} \in S\Omega$ denote the Bowen bracket of $\widetilde{v}$ with $\widetilde{u}$, defined as the unique intersection point of $W^{ss}(\widetilde{v})$ and $W^{su}(\phi^t\widetilde{u})$ for some $t \in \mathbb{R}$. Since u and v have closed orbits, there are hyperbolic group elements $\gamma_{\tilde{u}}$ and $\gamma_{\tilde{v}}$ preserving $\ell_{\tilde{u}}$ and $\ell_{\tilde{v}}$, respectively. Thus $D\gamma_{\tilde{u}}^n(\widetilde{U}) \cap \widetilde{\phi} \cdot \widetilde{u}$ and $D\gamma_{\tilde{v}}^n(\widetilde{V}) \cap \widetilde{\phi} \cdot \widetilde{v}$ each contain lifts of u and v respectively for all $n \in \mathbb{Z}$, where D is the differential. Since $\gamma_{\tilde{u}}, \gamma_{\tilde{v}}$ are isometries and $\widetilde{u}^- = \widetilde{w}^-, \widetilde{v}^+ = \widetilde{w}^+$ are smooth extremal points, there is an N such that for all $n \geqslant N$, $d\gamma_{\tilde{u}}^{-n}(\widetilde{U}) \cap \widetilde{\phi} \cdot \widetilde{w} \neq \varnothing$ and $d\gamma_{\tilde{v}}^n(\widetilde{V}) \cap \widetilde{\phi} \cdot \widetilde{w} \neq \varnothing$. Then choosing times t_1, t_2 so that $\phi^{t_1}\widetilde{w}$ is in $D\gamma_{\tilde{u}}^{-n}(\widetilde{U}) \cap \phi \cdot \widetilde{w}$ and $\phi^{t_2}\widetilde{w}$ is in $D\gamma_{\tilde{v}}^n(\widetilde{V}) \cap \phi \cdot \widetilde{w}$, we can project $\phi^{t_1}\widetilde{w}$ to SM and obtain $T = -t_1 + t_2$ such that $w' := D\pi_\Gamma \phi^{t_1}\widetilde{w} \in U$, where $\pi_\Gamma \colon \Omega \to M$ is the quotient map, and $\phi^T w' \in V$ as desired. □

4.3. The Anosov Closing Lemma

In this subsection, we prove Anosov closing of recurrent orbits, originally due to Anosov for geodesic flows in negative curvature [1].

Define a filtration of $SM \smallsetminus SM_{\text{flat}}$ by compact sets bounded away from flats:

$$\Lambda_\eta := \{v \in SM \mid d(v,w) \geqslant \eta \text{ for all } w \in SM_{\text{flat}}\}.$$

We say for vectors $u, v \in SM$ and $\epsilon > 0$ that u *ϵ-shadows* v for time t if $d_s(u,v) < \epsilon$ for $s \in [0,t]$.

Theorem 4.4. — *Let Ω be an indecomposable, nonstrictly convex domain in $\mathbb{RP}^3$. Suppose $\Gamma < \mathrm{PSL}(4,\mathbb{R})$ is a discrete, torsion-free group dividing Ω, with compact quotient $M = \Omega/\Gamma$. Then for all $\eta > 0$ and sufficiently small $\epsilon > 0$, there exists a $\delta > 0$ and $T > 0$ such that:*

For any $t \geqslant T$, $v \in \Lambda_\eta$ with $d(\phi^t v, v) < \delta$, there exists a regular periodic v' of period $t' \in \,]t-\epsilon, t+\epsilon[$ which ϵ-shadows v for time $\min\{t,t'\}$.

Proof. — We adapt a proof by contradiction following Eberlein [18] (see also [13, Theorem 7.1]). Assume we have particular $\eta, \epsilon > 0$ and a sequence of $v_n \in \Lambda_\eta$ paired with a sequence $t_n \to \infty$ such that $d(v_n, \phi^{t_n} v_n) \to 0$, yet any w_n which ϵ-shadows v_n for time t_n is not periodic of any period $t_n' \in \,]t_n - \epsilon, t_n + \epsilon[$.

We can assume up to extraction of subsequences that the v_n converge to some $v \in \Lambda_\eta$. Lifting SM to a compact fundamental domain SD containing $\widetilde{v}$ in $S\Omega$, we have some $\widetilde{v} \in S\Omega$ with endpoints v^+, v^- in $\partial\Omega$, and lifts $\widetilde{v_n}$ of the v_n which converge to $\widetilde{v}$ in SD. Also, since $\phi \cdot v_n$ almost closes up after time t_n, there are group elements γ_n which take $\widetilde{v_n}$ close to $\widetilde{\phi}^{t_n}\widetilde{v_n}$. Note that the γ_n need not be hyperbolic a priori.

Again, the contradiction hypothesis is that if w_n ϵ-shadows v_n for time t_n, then w_n cannot be periodic of any period $t_n' \in \,]t_n - \epsilon, t_n + \epsilon[$. Eberlein's geometric observation is that in the universal cover, if w_n ϵ-shadows v_n for time t_n, then the same γ_n which moves $\widetilde{v_n}$ close to $\widetilde{\phi}^{t_n}\widetilde{v_n}$ must also be responsible for moving $\widetilde{w_n}$ close to $\widetilde{\phi}^{t_n}\widetilde{w_n}$. Because Γ is acting on Ω properly discontinuously and cocompactly by isometries, the assumption that w_n is not periodic of period approximately t_n is realized in the universal cover as follows: if $d(\widetilde{w_n}, \widetilde{v_n}) < \epsilon$, then $\gamma_n.\widetilde{w_n} \neq \phi^{t_n'}\widetilde{w_n}$ for any $t_n' \in \,]t_n - \epsilon, t_n + \epsilon[$.

The goal of the following lemmas will be to show that nonexistence of an axis of γ_n which is ϵ-close to $\ell_{\widetilde{v_n}}[0, t_n]$ for infinitely many n is mutually exclusive with the assumption that the v_n and v are in Λ_η, producing the desired contradiction.

Lemma 4.5. — *Let $x \in \Omega$ be the footpoint of $\widetilde{v}$. Then $\gamma_n.x \to \widetilde{v}^+$ and $\gamma_n^{-1}.x \to \widetilde{v}^-$ as $n \to \infty$.*

Proof. — Take any convex open neighborhood $\mathcal{N}(\widetilde{v}^+)$ of $\widetilde{v}^+$ in $\overline{\Omega}$. Since the sequence v_n converges to v, for all suffiently large n the point v_n^+ will be in the neighborhood $\mathcal{N}(v^+)$. Then as t_n diverges to infinity, each point $\ell_{\tilde{v_n}}(t_n)$ will be in $\mathcal{N}(\widetilde{v}^+)$ by convexity of $\mathcal{N}(\widetilde{v}^+)$. Since γ_n is chosen so that $d(D\gamma_n.\widetilde{v_n}, \phi^{t_n}\widetilde{v_n})$ decreases to 0 as n grows, then $d_\Omega(\gamma_n.x_n, \ell_{\tilde{v_n}}(t_n))$ decreases to 0 with n, where x_n is the footpoint of $\widetilde{v_n}$ in Ω. Once $\gamma_n.x_n$ is sufficiently close to $\ell_{\tilde{v_n}}(t_n)$, then $\gamma_n.x_n$ will also be in $\mathcal{N}(\widetilde{v}^+)$. Finally, since x_n converges to x and γ_n is an isometry, we can ensure for large n that $\gamma_n.x$ is in $\mathcal{N}(\widetilde{v}^+)$.

Now consider $\mathcal{N}(\widetilde{v}^-)$ a convex open neighborhood of $\widetilde{v}^-$ in $\overline{\Omega}$. As $D\gamma_n.\widetilde{v_n}$ approaches $\phi^{t_n}\widetilde{v_n}$, the group element γ_n^{-1} brings the line segment $\ell_{\tilde{v_n}}[-s_n+t_n, s_n+t_n]$ back very close to the line segment $\ell_{\tilde{v_n}}[-s_n, s_n]$ for some sequence of times s_n which diverge to infinity with n. Then as s_n gets very large, $\ell_{\tilde{v_n}}(-s_n)$ will approach $\widetilde{v}_n^-$, as will $\gamma_n^{-1}.\ell_{\tilde{v_n}}(-s_n+t_n)$. But $\gamma_n^{-1}.\ell_{\tilde{v_n}}(-s_n+t_n)$ converges to $\gamma_n^{-1}.\widetilde{v}_n^-$ as n increases, so $\gamma_n^{-1}.\widetilde{v}_n^-$ must approach $\widetilde{v}_n^-$ in the boundary. Then since $\widetilde{v_n}$ approches $\widetilde{v}$, for sufficiently large n, the boundary point $\gamma_n^{-1}.\widetilde{v}^-$ will be in the neighborhood $\mathcal{N}(\widetilde{v}^-)$.

Lastly, since $\gamma_n^{-1}.x_n$ is a point on the line $\gamma_n^{-1}.\ell_{\tilde{v_n}}$, it suffices to observe that $d_\Omega(\gamma_n^{-1}.x_n, x_n)$ grows like t_n, which diverges with n, to conclude $\gamma_n^{-1}.x_n$ and hence $\gamma_n^{-1}.x$ are in the convex neighborhood $\mathcal{N}(\widetilde{v}^-)$ for all sufficiently large n. $\square$

We next define $V_k(\widetilde{v}^+)$, $V_k(\widetilde{v}^-)$ open neighborhoods in $\partial\Omega$ of $\widetilde{v}^+, \widetilde{v}^-$, respectively, such that for any boundary points $\xi \in V_k(\widetilde{v}^+)$ and $\zeta \in V_k(\widetilde{v}^-)$, the projective line $(\zeta\xi)$ is distance less than $\frac{1}{k}$ from $\ell_v(0)$ in the Hilbert metric. The existence of such V_k is immediate in a Hilbert geometry by the definition of d_Ω. The V_k are also homeomorphic to open balls in $\mathbb{R}^2$. Choose k large enough that $\frac{1}{k} < \frac{\epsilon}{2}$.

Lemma 4.6. — *For all sufficiently large n, $\gamma_n(\overline{V_k(\widetilde{v}^+)}) \subset V_k(\widetilde{v}^+)$ and $\gamma_n^{-1}(\overline{V_k(\widetilde{v}^-)}) \subset V_k(\widetilde{v}^-)$.*

Proof. — Note that as the sequence of boundary points $\gamma_n.\widetilde{v}_n^+$ converge to $\widetilde{v}_n^+$ and the sequence of vectors $\widetilde{v_n}$ converge to $\widetilde{v}$, then the boundary points $\gamma_n.\widetilde{v}^+$ approach $\widetilde{v}^+$ (and similarly, $\gamma_n^{-1}.\widetilde{v}^-$ converges to $\widetilde{v}^-$ with n). If $\gamma_n.\widetilde{v}^+$ is very close to $\widetilde{v}^+$, then by Proposition 3.3, γ_n either fixes $\widetilde{v}^+$, is contracting near $\widetilde{v}^+$, or both. The only way that $\gamma_n(\overline{V_k(\widetilde{v}^+)})$ would not be contained in $V_k(\widetilde{v}^+)$ is if γ_n stabilized a properly embedded triangle $\triangle_n$ whose boundary $\partial\triangle_n$ intersects the boundary of the neighborhood $V_k(\widetilde{v}^+)$. If this happened infinitely often, then $\widetilde{v}^+$ would necessarily be the limit of vertices of $\triangle_n$ which are attracting eigenvectors for the γ_n. Similarly, since γ_n^{-1} also stabilizes $\triangle_n$, vertices of $\triangle_n$ which are repelling eigenvectors for γ_n must accumulate on $\widetilde{v}^-$. Then in the quotient SM, the vector v must be distance less than ϵ from the flat tangent to a quotient of $\triangle_n$ for large enough n. This contradicts the assumption that $v \in \Lambda_\eta$ for small ϵ.

An analogous argument applies to show, up to extraction of subsequences, that $\gamma_n^{-1}(\overline{V_k(\widetilde{v}^-)}) \subset V_k(\widetilde{v}^-)$ for all sufficiently large n. $\square$

So we now have that for large n, $\gamma_n(\overline{V_k(\widetilde{v}^+)}) \subset V_k(\widetilde{v}^+)$ and similarly $\gamma_n^{-1}(\overline{V_k(\widetilde{v}^-)}) \subset V_k(\widetilde{v}^-)$, both of which are homeomorphic to open balls in $\mathbb{R}^2$. Applying Brouwer's fixed point theorem, it follows that γ_n fixes points in $V_k(\widetilde{v}^-)$ and $V_k(\widetilde{v}^+)$. Then γ_n has an axis distance less than $\frac{1}{k} < \frac{\epsilon}{2}$ from $\ell_{\tilde{v}}(0)$, hence ϵ-close to $\ell_{\tilde{v_n}}(0)$ for all sufficiently large n. We also assume that $\gamma_n.\widetilde{v_n}$ is arbitrarily close to $\phi^{t_n}\widetilde{v_n}$, so the axis of γ_n will eventually and thereafter be ϵ-close to $\ell_{\tilde{v_n}}[0, t_n]$ and the translation length of γ_n must be ϵ-close to t_n. And so we have a closed orbit of length $t'_n \in \,]t_n - \epsilon, t_n + \epsilon[$ which ϵ-shadows v_n for time $\max\{t_n, t'_n\}$, contradicting the assumption. If we obey our hypothesis that such a closed orbit is impossible, then we would necessarily have $v \notin \Lambda_\eta$ as proven in Lemma 4.6 – a contradiction. $\square$

5. Topological mixing

We prove the geodesic flow of a Benoist 3-manifold is topologically mixing following the strategies of Coudene [12], but without the local product structure property. Key properties will be a nonuniform orbit gluing lemma (Lemma 5.3) and density of the unstable leaves for periodic regular vectors (Proposition 5.5).

Let $W^{su}_\epsilon(v) := W^{su}(v) \cap B(v, \epsilon)$, and similarly for the strong stable sets and the weak stable and unstable sets. Recall that $\langle v, u\rangle$ denotes the Bowen bracket of regular vectors v and u; again, the Bowen bracket is the vector with the same past as v up to a time change and the same future as u. By Proposition 4.1, the Bowen bracket is defined for all ordered pairs of regular vectors (v, u), as long as $-v$ does not have the same future as u up to a time change. See also Figure 4.2.

Proposition 5.1. — *For all $\epsilon > 0$ and u regular, there is a $\delta > 0$ such that for all regular $v \in B(u, \delta)$ and for some $|t| < \epsilon$,*

$$\langle v, u\rangle \in W^{su}_\epsilon(\phi^t v) \cap W^{ss}_\epsilon(u).$$

Proof. — It suffices to make the arguments in a small neighborhood in the universal cover. Keep δ small enough that $v^- \neq u^+$, so that the Bowen bracket $\langle v, u\rangle$ is defined. For all $\epsilon > 0$, there are neighborhoods U^+ of u^+ and U^- of u^- in $\partial\Omega$ such that any regular vector v, if v is within ϵ of u

then v^+ is in U^+ and v^- is in U^-. Conversely, for these neighborhoods, if v is such that $v^+ \in U^+$ and $v^- \in U^-$ and moreover, the footpoint πv is sufficiently close to πu, then v is within ϵ of u. Make U^- small enough that $U^- \subset \{w^- \in \partial\Omega \mid w \in W^{ss}_\epsilon(u)\}$ guarantees that any v with $v^- \in U^-$ satisfies $\langle v, u\rangle \in W^{ss}_\epsilon(u)$. Taking U^+ be as small as needed, we can ensure such vectors v with $v^- \in U^-$ are arbitrarily close to $\phi^{t_v}\langle v, u\rangle$ in this small neighborhood of u. It suffices to choose $\delta > 0$ sufficiently small as to ensure $|t_v| < \epsilon$. □

5.1. Orbit gluing in Hilbert geometries

Uniform orbit gluing is also known as shadowing of pseudo-orbits in the literature. We introduce a weaker notion here. We can associate to any orbit segment $\phi^{[0,t]}v$ the pair $(v, t) \in SM \times \mathbb{R}^+_0$. An n-length δ-*pseudo-orbit* is a collection of n-many finite length orbit segments $\{(v_i, t_i)\}_{i=1}^n \subset SM \times \mathbb{R}^+_0$ such that $d(\phi^{t_i} v_i, v_{i+1}) < \delta$ for $i = 1, \dots, n-1$.

Definition 5.2. — *The dynamics satisfies* weak orbit gluing *if for all $\epsilon > 0$ and $\{v_i\}_{i=1}^n$ there exists $\delta > 0$ such that for all n-length δ-pseudo orbits $\{(v_i, t_i)\}_{i=1}^n$ there is a point w which ϵ-shadows the pseudo-orbit $\{(v_i, t_i)\}_{i=1}^n$. More explicitly: for some $|t| < \epsilon$,*

$$w \in W^{su}_\epsilon(\phi^t v_1) \quad \textit{and} \quad \phi^{\sum_{i=1}^{n-1} t_i} w \in W^{ss}_\epsilon(v_n),$$

and there are numbers $|s'_i| < \epsilon$ for $i = 1, \dots, n-1$ such that for all $k = 2, \dots, n-1$,

$$d(\phi^{t_1+\cdots+t_{k-1}+s}(w), \phi^{s'_k+s}(v_k)) < \epsilon, \qquad \textit{if } \ 0 < s < t_k,$$
$$d(\phi^{s}(w), \phi^{s'_1+s}(v_1)) < \epsilon, \qquad \textit{if } \ 0 < s < t_1.$$

Lemma 5.3 (Weak orbit gluing). — *The geodesic flow of a Benoist 3-manifold satisfies weak orbit gluing for pseudo-orbits $\{(v_i, t_i)\}_{i=1}^n$ when $v_1, \dots, v_n$ are regular.*

The proof of the lemma is not difficult but is technical. For the readers' convenience, we provide first a proof in the case where $n = 3$. Note that the case where $n = 2$ is simply a single Bowen bracket operation. Ultimately, we will only need to use Lemma 5.3 to shadow three orbit segments in the proof of topological mixing. The general statement and proof are simply of independent interest.

Proof of Lemma 5.3 in the case where $n = 3$. — The proof is effectively a finite recursive application of taking Bowen brackets (Proposition 5.1). Suppose $d(\phi^{t_i} v_i, v_{i+1}) < \delta_1$ for all $i = 1, \ldots, n-1$. We will refine this δ_1 with each successive Bowen bracket. For sufficiently small $\delta_1 > 0$, the Bowen bracket w_1 of $\phi^{t_1} v_1$ and v_2 satisfies

$$w_1 \in W^{su}_{\delta_2}(\phi^{t'_1+t_1} v_1) \cap W^{ss}_{\delta_2}(v_2)$$

for some $|t'_1| < \delta_2$, where δ_2 is not yet defined. Then by monotonicity of stable sets in the adapted metric and the triangle inequality, $\phi^{t_2} w_1$ is within $\delta_1 + \delta_2$ of v_3. Again, we take a Bowen bracket w_2 of $\phi^{t_2} w_1$ with v_3, keeping $\delta_1 + \delta_2$ small enough that

$$w_2 \in W^{su}_{\delta_3}(\phi^{t'_2+t_2} w_1) \cap W^{ss}_{\delta_3}(v_3)$$

for $|t'_2| < \delta_3$ and some small δ_3 which is not yet determined.

Let $w = \phi^{-t_1-t_2} w_2$. First, w is clearly in the same weak unstable manifold as v_1. We will see at the end of the proof that w is also sufficiently close to v_1. If $\delta_3 < \epsilon$, then $\phi^{t_1+t_2} w = w_2 \in W^{ss}_\epsilon(v_3)$ is immediate. Let $s'_1 = t'_1 + t'_2$ and $s'_2 = t'_2$. Then for $0 \leqslant s \leqslant t_2$,

$$\begin{aligned} d(\phi^{t_1+s} w, \phi^{s'_2+s} v_2) &= d(\phi^{-t_2+s} w_2, \phi^{s'_2+s} v_2) \\ &\leqslant d(\phi^{-t_2+s} w_2, \phi^{t'_2+s} w_1) + d(\phi^{t'_1+s} w_1, \phi^{t'_2+s} v_2) \\ &\leqslant d(w_2, \phi^{t'_2+t_2} w_1) + d(\phi^{t'_2} w_1, \phi^{t'_2} v_2) < \delta_3 + \delta_2 + 2|t'_2| \end{aligned}$$

since $w_2 \in W^{su}_{\delta_3}(\phi^{t_2+t'_2} w_1)$ and $w_1 \in W^{ss}_{\delta_2}(v_2)$ and d is monotone under the flow on leaves of the stable and unstable foliations.

Also, for $0 \leqslant s \leqslant t_2$,

$$\begin{aligned} &d(\phi^s w, \phi^{s'_1+s} v_1) \\ &\qquad = d(\phi^{-t_1-t_2+s} w_2, \phi^{t'_1+t'_2+s} v_1) \\ &\qquad \leqslant d(\phi^{-t_1-t_2+s} w_2, \phi^{-t_1+t'_2+s} w_1) + d(\phi^{-t_1+t'_2+s} w_1, \phi^{t'_1+t'_2+s} v_1) \\ &\qquad \leqslant d(w_2, \phi^{t_2+t'_2} w_1) + d(\phi^{t'_2} w_1, \phi^{t'_1+t'_2+t_1} v_1) < \delta_3 + \delta_2 + 2|t'_2| \end{aligned}$$

because $w_2 \in W^{su}_{\delta_3}(\phi^{t_2+t'_2} w_1)$ and $w_1 \in W^{su}_{\delta_2}(\phi^{t_1+t'_1} v_1)$. It remains to refine the δ_i. It would be enough for $\delta_i < \epsilon/4$ for $i = 1, 2, 3$, but δ_2 depends on δ_3 and δ_1 depends on both δ_2 and δ_3, so we take a minimum as needed. Then choose $\delta = \delta_1$ to conclude the proof. □

Proof of Lemma 5.3 for $n > 3$. — We resume after the construction of w_2 as in the $n = 3$ proof. Repeating the argument, we have $\phi^{t_3} w_2$ within $\delta_1 + \delta_3$ of v_4, allowing us to take the Bowen bracket of $\phi^{t_3} w_2$ with v_4 such that

$$w_3 \in W^{su}_{\delta_4}(\phi^{t'_3 + t_3} w_2) \cap W^{ss}_{\delta_4}(v_4)$$

for some $|t'_3| < \delta_4$ and some undetermined δ_4.

The process terminates with the Bowen bracket of $\phi^{t_{n-1}} w_{n-2}$ with v_n which is distance less than $\delta_1 + \delta_{n-1}$ away, providing the final vector w_{n-1}, determined by

$$w_{n-1} = \langle \phi^{t_{n-1}} w_{n-2}, v_n \rangle \in W^{su}_{\delta_n}(\phi^{t'_n + t_{n-1}} w_{n-2}) \cap W^{ss}_{\delta_n}(v_n)$$

for some $|t'_{n-1}| < \delta_n$. Let $w_0 = v_1$ and observe the following:

$$w_k \in W^{su}_{\delta_{k+1}}(\phi^{t'_k + t_k} w_{k-1}) \text{ for all } k = 1, \dots, n-1, \tag{5.1}$$

$$w_k \in W^{ss}_{\delta_{k+1}}(v_{k+1}) \text{ for all } k = 1, \dots, n-1, \tag{5.2}$$

$$\delta_k \text{ depends only on } v_{k+1} \text{ and } \delta_{k+1} \text{ for } k = 2, \dots, n-1 \tag{5.3}$$
$$\text{and } |t'_k| < \delta_{k+1} \text{ for } k = 2, \dots, n-1.$$

Note that δ_1 depends on $\delta_2, \dots, \delta_n$ and $v_2, \dots, v_n$. This δ_1 will be the δ taken to meet the conclusion of the lemma. Also,

$$w = \phi^{-\sum_{i=1}^{n-1} t_i} w_{n-1}$$

will be the w which satisfies the conclusion of the lemma. We now refine the δ_i in terms of ϵ and verify that w has the desired properties.

If $\delta_n < \epsilon$, then $\phi^{\sum_{i=1}^{n-1} t_i} w = w_{n-1}$ is in $W^{ss}_\epsilon(v_n)$ by Equation (5.2). Let $s'_k = \sum_{i=k}^{n-1} t'_i$. Then for $k = 1, \dots, n-1$ and $s \in [0, t_k]$,

$$\begin{aligned}
d&\left(\phi^{\left(\sum_{i=1}^{k-1} t_i\right)+s} w, \phi^{s'_k + s} v_k\right) \\
&= d\left(\phi^{-\left(\sum_{i=k}^{n-1} t_i\right)+s} w_{n-1}, \phi^{\left(\sum_{i=k}^{n-1} t'_i\right)+s} v_k\right) \\
&\leqslant d(\phi^{-t_k - \dots - t_{n-1} + s} w_{n-1}, \phi^{-t_k - \dots - t_{n-2} + t'_{n-1} + s} w_{n-2}) \\
&\qquad + d(\phi^{-t_k - \dots - t_{n-2} + s} w_{n-2}, \phi^{-t_k - \dots - t_{n-3} + t'_{n-1} + t'_{n-2} + s} w_{n-3}) \\
&\qquad + \dots + d(\phi^{-t_k + t'_{k+1} + \dots + t'_{n-1} + s} w_k, \phi^{t'_k + t'_{k+1} + \dots + t'_{n-1}} w_{k-1}) \\
&\qquad + d(\phi^{t'_k + \dots + t'_{n-1} + s} w_{k-1}, \phi^{t'_k + \dots + t'_{n-1} + s} v_k) \\
&< \sum_{i=k}^{n} \delta_i + 2 \sum_{i=k}^{n-1} |t'_i|
\end{aligned}$$

by Equation (5.1) for terms $k+1,\dots,n-1,n$ and Equation (5.2) for the $k-1$ term. Recalling that $w_0 = v_1$, it suffices to recognize that the δ_i and hence t'_i can be made sufficiently small to meet the definition of weak orbit gluing by the remark in Equation (5.3). □

5.2. Density of unstable sets

Using Proposition 3.9, that the additive subgroup generated by translation lengths of closed hyperbolic orbits is dense in $\mathbb{R}$, we now show that unstable sets for hyperbolic closed orbits are dense and shortly thereafter conclude the geodesic flow is topologically mixing.

Let $\mathcal{P}$ be the set of hyperbolic periodic vectors up to orbit equivalence. Let T_p denote the length of the orbit of a hyperbolic periodic vector p. Recall that every hyperbolic closed orbit is preserved by a hyperbolic isometry. Thus $T_p = \tau(\gamma_{\tilde{p}})$, the translation length of $\gamma_{\tilde{p}}$, where $\widetilde{p}$ is any lift of p to $S\Omega$ and $\gamma_{\tilde{p}} \in \Gamma$ is the hyperbolic isometry which perserves the projective line $\ell_{\tilde{p}}$ in Ω.

The following lemma uses transitivity (Proposition 4.3), Anosov Closing (Theorem 4.4), orbit gluing of 3 orbit segments (Lemma 5.3), and density of $\langle T_p\rangle_{p\in\mathcal{P}}$ in $\mathbb{R}$ (Proposition 3.9).

Lemma 5.4. — *For all open U in SM, the lengths of periodic orbits passing through U generate a dense subgroup of $\mathbb{R}$.*

Proof. — Fix a hyperbolic periodic vector p. Since SM_{flat} is closed, it suffices to assume $U \cap SM_{\text{flat}} = \varnothing$. Choose $\eta > 0$ such that U is contained in Λ_η (recall that Λ_η is the complement of the η-neighborhood of the flats in SM). By transitivity (Proposition 4.3), consider a point $v_0 \in U$ with a dense forward orbit. Let ϵ be any positive real number small enough that $B(v_0, 2\epsilon) \subset U$. Choose $0 < \delta(\epsilon,\eta) < \epsilon$ small enough to satisfy Anosov Closing (Theorem 4.4) on Λ_η with shadowing distance ϵ. Choose $\delta'(\eta, \frac{\delta}{6}, 3) < \frac{\delta}{3}$ such that any δ'-pseudo orbit of three orbit segments with starting points v_0, p, v_0 can be $\frac{\delta}{6}$-shadowed by a true orbit (Lemma 5.3).

Since v_0 has dense forward orbit, there exist times $s_0, s_1 > 0$ such that $d(\phi^{s_0} v_0, p) < \delta'$ and $d(\phi^{s_0+s_1} v_0, v_0) < \delta'$. Thus, the orbit segments $\{(v_0, s_0), (p, nT_p), (\phi^{s_0} v_0, s_1)\}$ form a δ' pseudo-orbit for all $n \in \mathbb{N}$. Applying the orbit gluing lemma for each n, we obtain some point v_n in SM

which $\frac{\delta}{6}$-shadows each pseudo-orbit. In particular,

$$\phi^s v_n \in B\left(\phi^{r_1'+s}v_0, \frac{\delta}{6}\right) \text{ for all } s \in [0, s_0], \tag{5.4}$$

$$\phi^{s_0+s} v_n \in B\left(\phi^{r_2'+s}p, \frac{\delta}{6}\right) \text{ for all } s \in [0, nT_p], \text{ and} \tag{5.5}$$

$$\phi^{s_0+nT_p+s} v_n \in B\left(\phi^{r_3'+s_0+s}v_0, \frac{\delta}{6}\right) \text{ for all } s \in [0, s_1] \tag{5.6}$$

for some $|r_i'| < \frac{\delta}{6}$. Then taking $s = 0$ in Equation (5.4) and $s = s_1$ in Equation (5.6) yields

$$\begin{aligned} d(v_n, \phi^{s_0+nT_p+s_1}(v_n)) \\ &\leqslant d(v_n, v_0) + d(v_0, \phi^{s_0+s_1}v_0) + d(\phi^{s_0+s_1}v_0, \phi^{s_0+nT_p+s_1}v_n) \\ &< 2\left(\frac{\delta}{6}\right) + \left(\frac{\delta}{3}\right) + 2\left(\frac{\delta}{6}\right) = \delta. \end{aligned}$$

Note that $v_n \in B(v_0, \delta/3) \subset U \subset \Lambda_\eta$. Since $v_n \in \Lambda_\eta$ returns within δ of itself after time $s_0 + nT_p + s_1$ we can apply Anosov Closing to find a nearby orbit which shadows v_n and has approximately this length. More precisely, there exists a regular w_n which has period length $s_0 + nT_p + s_1 + t_n'$ for some $|t_n'| < \epsilon$ and also ϵ-shadows v_n for the length of its period. Then w_n must be within 2ϵ of v, and therefore in U. We can repeat the above argument for all n with the same η, ϵ, p and v_0, hence the same δ, δ' and the same s_0, s_1. Then for all n, the hyperbolic periodic w_n and w_{n+1} are in U and

$$s_0 + (n+1)T_p + s_1 + t_{n+1}' - (s_0 + nT_p + s_1 + t_n') = T_p + t_{n+1}' - t_n'$$

implies $T_p + t_{n+1}' - t_n' \in \langle T_q \rangle_{q \in U \cap \mathcal{P}}$, the additive subgroup generated by lengths of closed hyperbolic orbits passing through U. Since $|t_{n+1}' - t_n'| < 2\epsilon$, letting ϵ go to zero we conclude $T_p \in \overline{\langle T_q \rangle}_{q \in U \cap \mathcal{P}}$ for all hyperbolic periodic p, which proves the lemma because $\overline{\langle T_p \rangle}_{p \in \mathcal{P}} = \mathbb{R}$ by Proposition 3.9. ☐

We are now prepared to prove a key proposition.

Proposition 5.5. — *If $v \in SM$ is a hyperbolic periodic vector, then $W^{su}(v)$ is dense in SM.*

Proof. — Let U be an open subset of SM. By Lemma 5.4 there exists a hyperbolic periodic vector u in U such that $\overline{\langle T_v, T_u \rangle} = \mathbb{R}$. Take $\epsilon > 0$ to be small enough that $B(u, \epsilon) \subset U$. Since v and u are regular vectors and ϵ can be as small as needed, the Bowen bracket is defined and we conclude there exists a $T \in \mathbb{R}$ such that $w \in W^{su}(v) \cap W^{ss}(\phi^T(u))$. Then $\phi^{-T}w \in W^{ss}(u)$ so choose $M \in \mathbb{N}$ large enough that $d(\phi^{mT_u - T}w, u) < \epsilon/2$

for any $m \geqslant M$. Because $\overline{\langle T_v, T_u\rangle} = \mathbb{R}$, there are large enough $m, n \in \mathbb{N}$ with $m \geqslant M$ such that

$$-\epsilon/2 < |-nT_v + mT_u - T| < \epsilon/2$$

implying that $d(\phi^{-nT_v}w, \phi^{mT_u - T}w) < \epsilon/2$. Thus, $\phi^{-nT_v}w \in B(u, \epsilon) \subset U$ by the triangle inequality. Note that $\phi^{-nT_v}w \in \phi^{-nT_v}W^{su}(v) = W^{su}(v)$ to conclude the proof. □

The following lemma, which is the final piece preceding the proof of topological mixing, is generally taken as fact. We have included the proof for completeness.

Lemma 5.6. — *Let $f^t \colon X \to X$ be any continuous flow of a compact metric space and p in X a periodic point for the flow. Then density of $W^{su}(p)$ implies that for all $\delta > 0$ and for all $x \in X$, there is a time $T(p, \delta, x) > 0$ such that*

$$f^t W^{su}_\delta(p) \cap B(x, \delta) \neq \varnothing$$

for all $t \geqslant T$.

Proof. — By density of the strong unstable set, there exists some $z \in W^{su}(p)$ within $\delta/2$ of x. Then there exists an $S > 0$ such that $s \geqslant S$ implies $d(f^{-s}p, f^{-s}z) < \delta$. In particular, for all $n \in \mathbb{N}$ such that $nT_p \geqslant S$,

$$d(p, f^{-nT_p}z) = d(f^{-nT_p}p, f^{-nT_p}z) < \delta,$$

hence $f^{-nT_p}z$ is in $f^{-nT_p}W^{su}(p) = W^{su}(p)$ and $B(p,\delta)$. Thus, $f^{nT_p}(W^{su}_\delta(p)) \cap B(x, \delta/2)$ is nonempty; the point z is in the intersection.

Take a finite $\delta/2$-cover $\{t_1, \dots, t_k\}$ of $[0, T_p]$. Repeat the above argument for each periodic point $f^{t_i}p$ of period T_p; this produces a point z_i in $W^{su}(f^{t_i}p)$ which is within $\delta/2$ of x, and some positive integer n_i such that

$$z_i \in f^{nT_p}(W^{su}_\delta(f^{t_i}p)) \subset f^{nT_p + t_i}(W^{su}_\delta(p))$$

Let $N = \max_{1 \leqslant i \leqslant k} n_i$ and $T = (N+1)T_p$. Then for all $t \geqslant T$, there is some integer $M_t \geqslant N + 1$, some $i \in \{1, \dots, k\}$, and some $0 \leqslant \epsilon \leqslant \delta/2$ such that $t = M_t T_p + t_i + \epsilon$ and thus

$$z_i \in f^{M_t T_p + t_i}(W^{su}_\delta(p)) \cap B(x, \delta/2)$$

so $f^\epsilon z_i \in f^t(W^{su}_\delta(p)) \cap B(x, \delta)$ as desired. □

We are now prepared to prove the main theorem of the section.

Theorem 5.7. — *The geodesic flow on SM is topologically mixing.*

Proof. — Let U and V be open subsets of SM and p a vector in U with closed orbit. Let $\delta > 0$ be small enough that $W^{su}_{\delta}(p) \subset U$ and $B(v,\delta) \subset V$ for some regular vector $v \in V$. By Proposition 5.5 and Lemma 5.6, there is a $T(U,V) > 0$ such that for all $t \geqslant T$,

$$\varnothing \neq \phi^t W^{su}_{\delta}(p) \cap B(v,\delta) \subset \phi^t U \cap V. \qquad \square$$

6. Entropy-expansiveness of Hilbert geometries

In this section, we prove entropy-expansiveness for the geodesic flow of any compact Hilbert geometry. First, we review some preliminary notions from entropy theory. Given any metric space admitting a flow, one can define the *Bowen distance* by

$$d_t(v,u) := \max_{0 \leqslant s \leqslant t} d(\phi^s v, \phi^s u).$$

Then d_t is a metric on SM, nondecreasing in t. Metric d_t-balls are called *Bowen balls*, denoted $B_t(v,\delta)$. A (t,δ)-*spanning set* for $K \subset SM$ is one which is δ-dense in K for the d_t metric. For any compact $K \subseteq SM$, we can choose a minimal finite (t,δ)-spanning set and denote the cardinality by $S(t,\delta,K)$. Then we define the *topological entropy* of ϕ^t on K by

$$h_{\mathrm{top}}(\phi, K) := \lim_{\delta\to 0} \lim_{t\to\infty} \frac{1}{t} \log S(t,\delta,K).$$

There are many equivalent definitions of h_{top} [22], and we include one other here. For $K \subseteq SM$ compact, we define a (t,δ)-separated set $R \subset K$ such that for all $u, v \in R$ which are distinct, $d_t(v,u) \geqslant \delta$. Let $R(t,\delta,K)$ denote the maximal cardinality for (t,δ)-separated sets, which is again finite by compactness of K. Then

$$h_{\mathrm{top}}(\phi, K) = \lim_{\delta\to 0} \lim_{t\to\infty} \frac{1}{t} \log R(t,\delta,K).$$

When $K = SM$, we abbreviate $S(t,\epsilon) := S(t,\epsilon,SM)$, $R(t,\epsilon) := R(t,\epsilon,SM)$, and $h_{\mathrm{top}}(\phi) := h_{\mathrm{top}}(\phi, SM)$.

For the purposes of applying Bowen's work, we take

$$h_{\mathrm{top}}(\phi, K, \delta) := \lim_{t\to\infty} \frac{1}{t} \log S(t,\delta,K).$$

so that $h_{\mathrm{top}}(\phi,K) = \lim_{\delta\to 0} h_{\mathrm{top}}(\phi,K,\delta)$, and for $K = SM$ we have $h_{\mathrm{top}}(\phi) := \lim_{\delta\to 0} h_{\mathrm{top}}(\phi,\delta)$. For any point v in SM, we define the *infinite Bowen balls* about v in positive or negative time:

$$\Phi_\epsilon(v) := \bigcap_{t\in\mathbb{R}^+} \phi^{-t}\overline{B(\phi^t v, \epsilon)} = \{y \in M \mid d(\phi^t y, \phi^t v) \leqslant \epsilon \text{ for all } t \in \mathbb{R}^+\}.$$

Intuitively, we should think of the $\Phi_\epsilon(v)$ as the exceptions to expansivity. An expansive map (not flow) is defined by the existence of an $\epsilon > 0$ such that $\Phi_\epsilon(v) = \{v\}$ for all v. An expansive flow would satisfy that $\Phi_\epsilon(v) = W^{os}_\epsilon(v)$ for all v. These are special cases of entropy expansive systems. Define

$$h^*(\epsilon) := \sup_{v \in SM} h_{\mathrm{top}}(\phi, \Phi_\epsilon(v)).$$

Then ϕ is *h-expansive* with expansivity constant $\epsilon > 0$ if $h^*(\epsilon) = 0$. In other words, there is an $\epsilon > 0$ such that the exceptions to ϵ-expansivity have no influence on the entropy of the system. For an h-expansive system, Bowen proved that we can bypass the cumbersome limit over $\delta \to 0$ of $h_{\mathrm{top}}(\phi, \delta)$ to compute $h_{\mathrm{top}}(\phi)$.

Theorem 6.1 ([8, Theorem 2.4]). — *If ϵ is an h-expansive constant for ϕ, then*

$$h_{\mathrm{top}}(\phi) = h_{\mathrm{top}}(\phi, \epsilon).$$

Moreover, to compute the metric entropy of a system, one can simply take a sufficiently fine measurable partition rather than an infimum over all possible partitions. An immediate consequence is existence of a measure of maximal entropy (see [30, Theorem 8.6 (2)]).

For any manifold, the *injectivity radius* of $x \in M$ is defined to be

$$\mathrm{inj}(x) := \frac{1}{2} \inf_{\ell} \{\mathrm{length}(\ell)\},$$

where ℓ varies over all homotopically nontrivial loops through x. Then define the *injectivity radius of M* to be

$$\mathrm{inj}(M) := \inf_{x \in M} \mathrm{inj}(x).$$

If M is compact then $\mathrm{inj}(M) > 0$.

Theorem 6.2. — *The geodesic flow ϕ^t on any compact Hilbert geometry is h-expansive.*

Proof. — Lift v to $\widetilde{v}$ in $S\Omega$. If $\widetilde{v}^+$ is extremal, then by properties of the Hilbert metric, $\widetilde{u}^+ \neq \widetilde{v}^+$ for any lift $\widetilde{u}$ of u implies $u \notin \Phi_\epsilon(v)$ for $0 < \epsilon < \mathrm{inj}(M)/3$. Then a $(0, \delta)$-spanning set for $\Phi_\epsilon(v)$ is a (t, δ)-spanning set for all $t > 0$ and $h_{\mathrm{top}}(\Phi_\epsilon(v)) = 0$.

Suppose now that v^+ is not extremal. Let $C \subset \partial\Omega$ be the intersection of all supporting hyperplanes to Ω at $\widetilde{v}^+$. Note that if $\widetilde{v}^+$ is not extremal then C has nonempty interior for the subspace topology in the minimal projective subspace containing C. Since C is properly convex in this projective subspace, we can extend the Hilbert metric to the interior of C, which we will denote d_C, with metric balls denoted by B_C. Now define

$$\Phi^+_C(\widetilde{v},\epsilon) := \{u \in \overline{B(\widetilde{v},\epsilon)} \mid u^+ \in \overline{B_C(\widetilde{v}^+,\epsilon)}\}.$$

Then $\Phi_\epsilon(v)$ is contained in the quotient projection of $\Phi^+_C(\widetilde{v},\epsilon)$. For all $\eta \in B_C(\widetilde{v}^+,\epsilon)$ let v_η be such that $\pi v_\eta = \pi\widetilde{v}$ and $v^+_\eta = \eta$. Then $d(v_\eta, v) \leqslant d_C(\eta,\widetilde{v}^+) \leqslant \epsilon$. Then for all $w \in \Phi^+_C(\widetilde{v},\epsilon)$, there is an $\eta = w^+$ implying $d(w, v_\eta) \leqslant d(w,\widetilde{v}) + d(\widetilde{v}, v_\eta) = \epsilon + \epsilon = 2\epsilon$, hence

$$\Phi^+_C(\widetilde{v},\epsilon) \subset \bigcup_{\eta\in\overline{B_C(\widetilde{v}^+,\epsilon)}} \Phi^+(v_\eta, 2\epsilon).$$

Choose a finite $\delta/2$-cover of $B_C(\widetilde{v}^+,\epsilon/2)$ by $\{\eta_i\}_{i=1}^k$ and $v_i := v_{\eta_i}$. Then for all $u \in \Phi^+(v_\eta, 2\epsilon)$, there is an η_i such that $d_C(\eta,\eta_i) < \delta/2$ and $d(u, v_i) \leqslant d(u, v_\eta) + d(v_\eta, v_i) < 2\epsilon + d_C(\eta,\eta_i) < \frac{5\epsilon}{2}$ for δ small. Describe all such u by

$$\Phi^+_C\,(v_i, 5\epsilon/2, \delta/2) := \{u \in \overline{B(\widetilde{v}, 5\epsilon/2)} \mid u^+ \in \overline{B_C(\widetilde{v}^+,\delta/2)}\}.$$

Then for $\epsilon < \mathrm{inj}(M)/3$,

$$\Phi_\epsilon(v) \subset \bigcup_{i=1}^k \Phi^+_C\,(D\pi_\Gamma v_i, 5\epsilon/2, \delta/2)$$

Note that for each compact $\Phi^+_C(D\pi_\Gamma v_i, 5\epsilon/2, \delta/2)$, a minimal $(0,\delta)$-spanning set E_i will be a (t,δ)-spanning set for all $t \geqslant 0$. Thus,

$$h_{\mathrm{top}}(\Phi_\epsilon(v),\delta) \leqslant \lim_{t\to\infty} \frac{1}{t}\log\left(\sum_{i=1}^k |E_i|\right) = 0. \qquad \square$$

7. Applications to counting and entropy

Let $P_t(\phi)$ denote the collection of hyperbolic ϕ-periodic orbits of period at most t, modulo orbit equivalence, and

$$\rho(\phi) := \lim_{t\to\infty} \frac{1}{t}\log \#P_t(\phi).$$

Note that the hyperbolic closed orbits for the flow are isolated and countable, since they are disjoint from the flats (Proposition 3.3).

The next proposition is a straightforward consequence of h-expansivity (Theorem 6.2), included for completeness.

Proposition 7.1. — *The geodesic flow ϕ^t of a Benoist 3-manifold satisfies*

$$\rho(\phi) \leqslant h_{\mathrm{top}}(\phi).$$

Proof. — Choose $0 < \epsilon \leqslant \operatorname{inj} M/3$ an h-expansivity constant for the geodesic flow on SM. We show that P_t is a (t,ϵ)-separated set. If $v,u \in P_T$ such that $d_T(v,u) < \epsilon$, then $d_t(v,u) < \epsilon$ for all $t \in \mathbb{R}$. Since Γ acts discretely and $\epsilon < \operatorname{inj}(M)/3$, this is only possible if $v = u$ or if v and u lift to tangent vectors to a properly embedded triangle $\triangle$ such that $\ell_{\tilde{v}}, \ell_{\tilde{u}} \subset \triangle$. Then v,u are in a flat so they are not counted in P_T.

Thus, P_t is (t,ϵ)-separated and has cardinality at most $R(t,\epsilon)$, the cardinality of a maximal (t,ϵ)-separated set. We conclude by h-expansivity and Bowen's Theorem 6.1 that

$$\rho(\phi) = \lim_{t\to\infty} \frac{1}{t} \log \#P_t(\phi) \leqslant \lim_{t\to\infty} \frac{1}{t} \log R(t,\epsilon) = h_{\text{top}}(\phi). \qquad \square$$

Proposition 7.2. — *The geodesic flow of a compact Benoist 3-manifold has positive topological entropy.*

Proof. — By Corollary 3.5, there exist noncommuting hyperbolic elements $g,h \in \Gamma$ which generate a free subgroup. There is a positive lower bound for the exponential growth rate of lengths of closed curves associated to this subgroup, which bounds below $\rho(\phi)$ and hence $h_{\text{top}}(\phi)$. □

7.1. Volume entropy

We remark in this section that A. Manning's proof that volume entropy and topological entropy agree for compact nonpositively curved Riemannian manifolds generalizes to our context immediately [23]. Let vol be a natural projectively invariant volume on Ω, such as the Busemann volume [24, Section 1]. Then

$$h_{\text{vol}}(\Omega) = \lim_{r\to\infty} \frac{1}{r} \log \operatorname{vol}(B_\Omega(x,r))$$

is the volume entropy of Ω. Let δ_Γ denote the critical exponent of the action of Γ on Ω, equivalently: $\delta_\Gamma = \limsup_{t\to\infty} \frac{1}{t} \log N_\Gamma(t)$ where $N_\Gamma(t) = \#\{\gamma \in \Gamma \mid d_\Omega(x,\gamma.x) \leqslant t\}$.

Proposition 7.3. — *If Ω is any divisible properly convex domain in $\mathbb{RP}^n$, then*

$$\delta_\Gamma = h_{\text{vol}} = h_{\text{top}}(\phi).$$

Proof. — Whenever a discrete group of isometries acts cocompactly on a metric space with finite critical exponent, one has $\delta_\Gamma = h_{\text{vol}}$ (a proof is available in [27, Lemma 4.5]). The statement in [23, Theorem 1] that

$h_{\mathrm{vol}} \leqslant h_{\mathrm{top}}(\phi)$ holds as long as M is compact and (Ω, d_Ω) is complete. The proof of the opposite inequality in Theorem 2 uses nonpositive sectional curvature to prove a technical lemma. We can bypass curvature and prove the lemma immediately in Hilbert geometries. The rest of the proof follows in the same way.

This lemma has already been proven by Crampon in the strictly convex case, but in the proof Crampon only uses strict convexity to state the lemma with a strict inequality for all *geodesics*, which can only be projective lines when the domain is strictly convex. Since our geodesic flow is defined to follow projective lines, the lemma suffices.

Lemma 7.4 ([14, Lemma 8.3]). — *For any two projective lines $\sigma, \tau\colon [0,r] \to M$, $r > 0$,*

$$d_\Omega(\sigma(t), \tau(t)) \leqslant d_\Omega(\sigma(0), \tau(0)) + d_\Omega(\sigma(r), \tau(r)).$$

In fact, Lemma 7.4 is an immediate consequence of work by Busemann, who observed that the Hilbert distance from a geodesic projective line to any convex subset of Ω is *peakless*, where Ω is any properly convex domain. The definition of peakless is equation (18.11) and the statement is (18.12), in [10, Chapter 18: Hilbert Geometry, p. 109]. We thank the referee for this reference. □

BIBLIOGRAPHY

[1] D. V. Anosov, "Geodesic flows on closed Riemann manifolds with negative curvature", *Proc. Steklov Inst. Math.* **90** (1967), p. 1-235, translated from the Russian by S. Feder.

[2] M. D. Baker & D. Cooper, "Finite-volume hyperbolic 3–manifolds contain immersed quasi-Fuchsian surfaces", *Algebr. Geom. Topol.* **15** (2015), no. 2, p. 1199-1228.

[3] Y. Benoist, "Convexes divisibles. II", *Duke Math. J.* **120** (2003), no. 1, p. 97-120.

[4] ———, "Convexes divisibles. I", in *Algebraic groups and arithmetic*, Tata Institute of Fundamental Research Studies in Mathematics, vol. 17, Tata Institute of Fundamental Research, 2004, p. 339-374.

[5] ———, "Convexes divisibles. III", *Ann. Sci. Éc. Norm. Supér.* **38** (2005), no. 5, p. 793-832.

[6] ———, "Convexes divisibles. IV. Structure du bord en dimension 3", *Invent. Math.* **164** (2006), no. 2, p. 249-278.

[7] J. P. Benzécri, "Sur les variétés localement affines et localement projectives", *Bull. Soc. Math. Fr.* **88** (1960), p. 229-332.

[8] R. Bowen, "Entropy-expansive maps", *Trans. Am. Math. Soc.* **164** (1972), p. 323-331.

[9] H. Bray, "Ergodicity of Bowen–Margulis measure for the Benoist 3-manifolds", `https://arxiv.org/abs/1705.08519`, 2019.

[10] H. Busemann, *The geometry of geodesics*, Pure and Applied Mathematics, vol. 6, Academic Press Inc., 1955, x+422 pages.

[11] ———, "Timelike spaces", *Diss. Math.* **53** (1967), p. 1-52.

[12] Y. Coudene, "Topological dynamics and local product structure", *J. Lond. Math. Soc.* **69** (2004), no. 2, p. 441-456.

[13] Y. Coudene & B. Schapira, "Generic measures for hyperbolic flows on non-compact spaces", *Isr. J. Math.* **179** (2010), p. 157-172.

[14] M. Crampon, "Entropies of strictly convex projective manifolds", *J. Mod. Dyn.* **3** (2009), no. 4, p. 511-547.

[15] ———, "Lyapunov exponents in Hilbert geometry", *Ergodic Theory Dyn. Syst.* **34** (2014), no. 2, p. 501-533.

[16] M. Crampon & L. Marquis, "Finitude géométrique en géométrie de Hilbert", *Ann. Inst. Fourier* **64** (2014), no. 6, p. 2299-2377.

[17] ———, "Le flot géodésique des quotients géométriquement finis des géométries de Hilbert", *Pac. J. Math.* **268** (2014), no. 2, p. 313-369.

[18] P. B. Eberlein, *Geometry of nonpositively curved manifolds*, Chicago Lectures in Mathematics, University of Chicago Press, 1996, vii+449 pages.

[19] P. de la Harpe, "On Hilbert's metric for simplices", in *Geometric group theory, Vol. 1 (Sussex, 1991)*, London Mathematical Society Lecture Note Series, vol. 181, Cambridge University Press, 1993, p. 97-119.

[20] W. Jaco & P. B. Shalen, "A new decomposition theorem for irreducible sufficiently-large 3-manifolds", in *Algebraic and geometric topology (Stanford, 1976), Part 2*, Proceedings of Symposia in Pure Mathematics, vol. 32, American Mathematical Society, 1978, p. 71-84.

[21] K. Johannson, *Homotopy equivalences of 3-manifolds with boundaries*, Lecture Notes in Mathematics, vol. 761, Springer, 1979, ii+303 pages.

[22] A. Katok & B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and Its Applications, vol. 54, Cambridge University Press, 1995, xviii+802 pages.

[23] A. Manning, "Topological entropy for geodesic flows", *Ann. Math.* **110** (1979), no. 3, p. 567-573.

[24] L. Marquis, "Around groups in Hilbert geometry", in *Handbook of Hilbert geometry*, IRMA Lectures in Mathematics and Theoretical Physics, vol. 22, European Mathematical Society, 2014, p. 207-261.

[25] J. D. Masters & X. Zhang, "Closed quasi-Fuchsian surfaces in hyperbolic knot complements", *Geom. Topol.* **12** (2008), no. 4, p. 2095-2171.

[26] A. Papadopoulos & M. Troyanov (eds.), *Handbook of Hilbert geometry*, IRMA Lectures in Mathematics and Theoretical Physics, vol. 22, European Mathematical Society, 2014, viii+460 pages.

[27] J.-F. Quint, "An overview of Patterson–Sullivan theory", `https://www.math.u-bordeaux.fr/~jquint/publications/courszurich.pdf`.

[28] J. Vey, "Sur les automorphismes affines des ouverts convexes saillants", *Ann. Sc. Norm. Super. Pisa, Cl. Sci.* **24** (1970), p. 641-665.

[29] C. Walsh, "Gauge-reversing maps on cones, and Hilbert and Thompson isometries", *Geom. Topol.* **22** (2018), no. 1, p. 55-104.

[30] P. Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics, vol. 79, Springer, 1982, ix+250 pages.

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