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On the anti-canonical geometry of weak ℚ-Fano threefolds II

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ON THE ANTI-CANONICAL GEOMETRY OF WEAK ℚ-FANO THREEFOLDS II

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ABSTRACT. — By a canonical (resp. terminal) weak ℚ-Fano 3-fold we mean a normal projective one with at worst canonical (resp. terminal) singularities on which the anti-canonical divisor is ℚ-Cartier, nef and big. For a canonical weak ℚ-Fano 3-fold V, we show that there exists a terminal weak ℚ-Fano 3-fold X, being birational to V, such that the m-th anti-canonical map defined by $|-mK_X|$ is birational for all $m \geqslant 52$. As an intermediate result, we show that for any K-Mori fiber space Y of a canonical weak ℚ-Fano 3-fold, the m-th anti-canonical map defined by $|-mK_Y|$ is birational for all $m \geqslant 52$.

RÉSUMÉ. — Par variété de dimension trois canonique (resp. terminale) faiblement ℚ-Fano, nous entendons une variété normale projective avec au plus des singularités canoniques (resp. terminales) sur laquelle le diviseur anticanonique est ℚ-Cartier, nef et big. Pour une variété de dimension trois canonique faiblement ℚ-Fano V, nous montrons qu'il existe une variété de dimension trois terminale faiblement ℚ-Fano X birationnelle à V, de sorte que l'application pluri-anticanonique définie par $|-mK_X|$ est birationnelle sur son image pour tous les $m \geqslant 52$. Comme un résultat intermédiaire, nous montrons que pour n'importe quelle K-fibration de Mori Y d'une variété de dimension trois canonique faiblement ℚ-Fano, l'application pluri-anticanonique définie par $|-mK_Y|$ est birationnelle sur son image pour tous les $m \geqslant 52$.

1. Introduction

Throughout this paper, we work over an algebraically closed field k of characteristic 0 (for instance, $k = \mathbb{C}$). We adopt the standard notation in [20] and will freely use them.

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A normal projective variety X is called a *weak $\mathbb{Q}$-Fano variety* (resp. $\mathbb{Q}$-*Fano variety*) if the anti-canonical divisor $-K_X$ is nef and big (resp. ample). A *canonical* (resp. *terminal)* weak $\mathbb{Q}$-Fano variety is a weak $\mathbb{Q}$-Fano variety with at worst canonical (resp. terminal) singularities.

According to Minimal Model Program, weak $\mathbb{Q}$-Fano varieties form a fundamental class among research objects of birational geometry. There are a lot of works (for instance, [1, 4, 5, 10, 12, 16, 19, 22, 23, 24, 25, 26, 29]) which study the explicit geometry of canonical or terminal (weak) $\mathbb{Q}$-Fano 3-folds.

Given a canonical weak $\mathbb{Q}$-Fano n-fold X, the *m-th anti-canonical map* $\varphi_{-m,X}$ (or simply φ_{-m}) is the rational map defined by the linear system $|-mK_X|$. Since $-K_X$ is big, $\varphi_{-m,X}$ is a birational map onto its image when m is sufficiently large. Therefore it is interesting to find such a practical number m_n, independent of X, which guarantees the stable birationality of φ_{-m_n}. Such a number m_3 exists for canonical weak $\mathbb{Q}$-Fano 3-folds due to the boundedness result proved by Kollár, Miyaoka, Mori, and Takagi [19]. In general, m_n exists for canonical weak $\mathbb{Q}$-Fano n-folds due to the recent work of Birkar [2]. It is still mysterious to the authors whether m_n can be explicitly computed or effectively estimated for $n \geqslant 4$.

In this paper we are interested in the explicit birational geometry of canonical weak $\mathbb{Q}$-Fano 3-folds, so it is natural to consider the following problem.

PROBLEM 1.1. — *Find the optimal constant c such that φ_{-m} is birational (onto its image) for all $m \geqslant c$ and for all canonical weak $\mathbb{Q}$-Fano 3-folds.*

The following example suggests that $c \geqslant 33$.

Example 1.2 ([14, List 16.6, No. 95]*).* — The general weighted hypersurface $X_{66} \subset \mathbb{P}(1, 5, 6, 22, 33)$ is a $\mathbb{Q}$-factorial terminal $\mathbb{Q}$-Fano 3-fold of Picard number one. It is clear that φ_{-m} is birational for $m \geqslant 33$, but φ_{-32} is non-birational.

In our previous article [12][(1)] (see also [15] for related results on generic finiteness), we showed the following theorems.

THEOREM 1.3 ([12, Theorem 1.6]). — *Let X be a $\mathbb{Q}$-factorial terminal $\mathbb{Q}$-Fano 3-fold of Picard number one. Then $\varphi_{-m,X}$ is birational for all $m \geqslant 39$.*

[(1)] In [12], a $\mathbb{Q}$-*Fano 3-fold* means a $\mathbb{Q}$-factorial terminal $\mathbb{Q}$-Fano 3-fold of Picard number one, and a *weak $\mathbb{Q}$-Fano 3-fold* means a $\mathbb{Q}$-factorial terminal weak $\mathbb{Q}$-Fano 3-fold.

THEOREM 1.4 ([12, Theorem 1.8, Remark 1.9]). — *Let X be a canonical weak $\mathbb{Q}$-Fano 3-fold. Then $\varphi_{-m,X}$ is birational for all $m \geqslant 97$.*

One intuitively feels that the numerical bound "97" obtained in Theorem 1.4 might be far from optimal. As being indicated in [10, 12], the birationality problem is closely related to the following problem.

PROBLEM 1.5. — *Given a canonical weak $\mathbb{Q}$-Fano 3-fold X, what is the smallest positive integer $\delta_1 = \delta_1(X)$ satisfying $\dim \overline{\varphi_{-\delta_1}(X)} > 1$?*

In our previous paper we have shown the following theorems.

THEOREM 1.6 ([12, Theorem 1.4]). — *Let X be a $\mathbb{Q}$-factorial terminal $\mathbb{Q}$-Fano 3-fold of Picard number one. Then there exists an integer $n_1 \leqslant 10$ such that $\dim \overline{\varphi_{-n_1}(X)} > 1$.*

THEOREM 1.7 ([12, Theorem 1.7, Remark 1.9]). — *Let X be a canonical weak $\mathbb{Q}$-Fano 3-fold. Then $\dim \overline{\varphi_{-n_2}(X)} > 1$ for all $n_2 \geqslant 71$.*

Note that Theorems 1.3 and 1.6 are very close to be optimal due to Example 1.2 and the following example.

Example 1.8 ([14, List 16.7, No. 85]). — Consider the general codimension 2 weighted complete intersection $X = X_{24,30} \subset \mathbb{P}(1, 8, 9, 10, 12, 15)$ which is a $\mathbb{Q}$-factorial terminal $\mathbb{Q}$-Fano 3-fold of Picard number one. Then $\dim \overline{\varphi_{-9}(X)} > 1$ while $\dim \overline{\varphi_{-8}(X)} = 1$ since $P_{-8} = 2$. Clearly $\delta_1(X) = 9$.

The aim of this paper is to intensively study Problem 1.5. It turns out that, modulo birational equivalence, the following theorem considerably improves Theorems 1.4 and 1.7:

THEOREM 1.9. — *Let V be a canonical weak $\mathbb{Q}$-Fano 3-fold. Then there exists a terminal weak $\mathbb{Q}$-Fano 3-fold X birational to V such that*

(1) $\dim \overline{\varphi_{-m}(X)} > 1$ *for all $m \geqslant 37$;*
(2) $\varphi_{-m,X}$ *is birational for all $m \geqslant 52$.*

According to Minimal model program, Mori fiber spaces are fundamental classes in birational geometry, so it is also interesting to consider the relation of a $\mathbb{Q}$-Fano variety and its Mori fiber spaces. As an intermediate result, we show the following theorem.

THEOREM 1.10. — *Let V be a canonical weak $\mathbb{Q}$-Fano 3-fold. Then for any K-Mori fiber space Y of V (see Definition 3.5),*

(1) $\dim \overline{\varphi_{-m}(Y)} > 1$ *for all $m \geqslant 37$;*
(2) $\varphi_{-m,Y}$ *is birational for all $m \geqslant 52$.*

Note that here $-K_Y$ is big, but not necessarily nef. But we can still consider the map $\varphi_{-m,Y}$ defined by $|-mK_Y|$.

Remark 1.11. — The birationality of anti-pluricanonical maps of weak $\mathbb{Q}$-Fano varieties is not necessarily invariant under birational equivalence. This may account for the dilemma while the notion "birational geometry" is used to understand the geometry of $\mathbb{Q}$-Fanos.

The main idea in this paper is to show that, after a birational modification, a canonical weak $\mathbb{Q}$-Fano 3-fold is birationally equivalent to another terminal weak $\mathbb{Q}$-Fano 3-fold which takes over one vertex of an interesting triangle (i.e. so-called *Fano–Mori triple*, see Definition 3.7). Instead of dealing with the given $\mathbb{Q}$-Fano 3-fold, we treat the Fano–Mori triple which exhibits richer geometry. More precisely, we prove Theorems 1.9 and 1.10 by showing the following two theorems.

THEOREM 1.12 ($=$ Proposition 3.9). — *Let V be a canonical weak $\mathbb{Q}$-Fano 3-fold and Y a K-Mori fiber space of V. Then there exists a Fano–Mori triple (X, Y, Z) containing Y as the second term.*

THEOREM 1.13. — *Let (X, Y, Z) be a Fano–Mori triple in which X is a terminal weak $\mathbb{Q}$-Fano 3-fold. Then*

(1) $\dim \overline{\varphi_{-m}(X)} > 1$ *for all $m \geqslant 37$;*
(2) $\varphi_{-m,X}$ *is birational for all $m \geqslant 52$.*

Remark 1.14. — In Theorem 1.13, we can show moreover that φ_{-51} is not birational only if $-K_X^3 = 1/330$ and X has the Reid basket

$$\{(1, 2), (2, 5), (1, 3), (2, 11)\},$$

that is, X has exactly the same anti-canonical degree and singularities as X_{66} in Example 1.2.

The key step is to establish an effective method to tell when the anti-pluricanonical system of the terminal weak $\mathbb{Q}$-Fano 3-fold in a Fano–Mori triple is not composed with a pencil, which settles Problem 1.5 in this case. Then we can use the birationality criterion established in [12] to prove the effective birationality as stated in the main theorem.

This paper is organized as follows. In Section 2, we recall some basic knowledge. In Section 3, we introduce the concept of Fano–Mori triples and their basic properties, in particular, we show that any canonical weak $\mathbb{Q}$-Fano 3-fold is birational to a Fano–Mori triple containing certain K-Mori fiber space. In Section 4, we prove an important geometric inequality for Fano–Mori triples on which there is an anti-pluricanonical pencil. In

Section 5, we collect some criteria for $|-mK|$ being not composed with a pencil and for birationality of $|-mK|$. In Section 6, we apply those criteria to prove the main theorems and, meanwhile, the structure of weighted baskets revealed in [9] is effectively used to classify $\mathbb{Q}$-Fano 3-folds with small invariants.

Notation

For the convenience of readers, we list here the notation that will be frequently used in this paper. Let X be a terminal weak $\mathbb{Q}$-Fano 3-fold.

φ_{-m}	the m-th anti-canonical map corresponding to $|-mK_X|$
$P_{-m}(X) = h^0(X, \mathcal{O}_X(-mK_X))$	the m-th anti-plurigenus of X
$\sim_{\mathbb{Q}}$	$\mathbb{Q}$-linear equivalence
$B = B_X = \{(b_i, r_i)\}$	the Reid basket of orbifold points of X
$r_X = \mathrm{lcm}\{r_i \,|\, r_i \in B_X\}$	the Gorenstein index of X, i.e., the Cartier index of K_X
$r_{\max} = \max\{r_i \,|\, r_i \in B_X\}$	the maximal local index
$M_X = r_X(-K_X^3)$	see Section 5
$\lambda(M_X)$	see Section 5
$\sigma(B) = \sum_i b_i$	see Subsection 2.3
$\sigma'(B) = \sum_i \frac{b_i^2}{r_i}$	see Subsection 2.3
$\mathbb{B} = (B, \widetilde{P}_{-1})$	a weighted basket
$-K^3(\mathbb{B})$	the volume of $\mathbb{B}$
$\widetilde{P}_{-m}(\mathbb{B})$	the m-th anti-plurigenus of $\mathbb{B}$
$\{B^{(m)}\}$	the canonical sequence of B
$B^{(m)} = \{n_{b,r}^m \times (b, r)\}$	expression of $B^{(m)}$
$\epsilon_m(B)$	the number of prime packings from $B^{(m-1)}$ to $B^{(m)}$
$\sigma_5 = \sum_{r \geqslant 5} n_{1,r}^0$	see Subsection 2.3
$\epsilon = 2\sigma_5 - n_{1,5}^0$	see Subsection 2.3
$\gamma(B) = \sum_i \frac{1}{r_i} - \sum_i r_i + 24$	see (2.3)

2. Preliminaries

Let X be a canonical weak $\mathbb{Q}$-Fano 3-fold. Denote by r_X the Gorenstein index of X (= the Cartier index of K_X). For any positive integer m, the number $P_{-m}(X) = h^0(X, \mathcal{O}_X(-mK_X))$ is called the *m-th anti-plurigenus*

of X. Clearly, since $-K_X$ is nef and big, Kawamata–Viehweg vanishing theorem [17, Theorem 1-2-5] implies that

$$h^i(X, -mK_X) = h^i(X, K_X - (m+1)K_X) = 0$$

for all $i > 0$ and $m \geqslant 0$. Hence $\chi(\mathcal{O}_X) = 1$.

For two linear systems $|A|$ and $|B|$, we write $|A| \preceq |B|$ if there exists an effective divisor F such that

$$|B| \supset |A| + F.$$

In particular, if $A \leqslant B$ as divisors, then $|A| \preceq |B|$.

2.1. Rational maps defined by Weil divisors

Consider an effective $\mathbb{Q}$-Cartier Weil divisor D on X with $h^0(X, D) \geqslant 2$. We study the rational map defined by $|D|$, say

$$X \overset{\Phi_D}{\dashrightarrow} \mathbb{P}^{h^0(D)-1}$$

which is not necessarily well-defined everywhere. By Hironaka's big theorem, we can take successive blow-ups $\pi : W \to X$ such that:

  (i) W is smooth projective;
 (ii) the movable part $|M|$ of the linear system $|\lfloor \pi^*(D) \rfloor|$ is base point free and, consequently, the rational map $\gamma = \Phi_D \circ \pi$ is a morphism;
(iii) the support of the union of $\pi_*^{-1}(D)$ and the exceptional divisors of π is simple normal crossings.

Let $W \overset{f}{\to} \Gamma \overset{s}{\to} Z$ be the Stein factorization of γ with $Z = \gamma(W) \subset \mathbb{P}^{h^0(D)-1}$. We have the following commutative diagram:

$$
\begin{array}{ccc}
W & \overset{f}{\longrightarrow} & \Gamma \\
\pi \downarrow & \overset{\gamma}{\searrow} & \downarrow s \\
X & \dashrightarrow[\Phi_D] & Z
\end{array}
$$

Case (f_{np}). — If $\dim(\Gamma) \geqslant 2$, a general member S of $|M|$ is a smooth projective surface by Bertini's theorem. In this case, we say that $|D|$ *is not composed with a pencil of surfaces* (not composed with a pencil, for short).

Case (f_{p}). — If $\dim(\Gamma) = 1$, then $\Gamma \cong \mathbb{P}^1$ since $g(\Gamma) \leqslant q(W) = q(X) = 0$. Furthermore, a general fiber S of f is an irreducible smooth projective surface by Bertini's theorem. We may write

$$M = \sum_{i=1}^{l} S_i \sim lS$$

where S_i is a smooth fiber of f for each i and $l = h^0(D) - 1$. We can write

$$|D| = |lS'| + E,$$

where $|S'| = |\pi_* S|$ is an irreducible rational pencil, $|lS'|$ is the movable part and E the fixed part. In this case, $|D|$ is said to *be composed with a (rational) pencil of surfaces* (composed with a pencil, for short).

DEFINITION 2.1. — *For another $\mathbb{Q}$-Cartier Weil divisor D' on X satisfying $h^0(X, D') \geqslant 2$, we say that $|D|$ and $|D'|$ are composed with the same pencil (of surfaces) if $|D|$ and $|D'|$ are composed with pencils and, through the Stein factorization, they define the same fibration structure $W \to \mathbb{P}^1$ on some smooth model W.*

DEFINITION 2.2. — *In the above setting, S is called a generic irreducible element of $|M|$. By abuse of notion, we also say that $S' = \pi_*(S)$ is a generic irreducible element of $\operatorname{Mov}|D|$ or $|D|$.*

2.2. Reid's Riemann–Roch formula

A *basket B* is a collection of pairs of integers (permitting weights), say $\{(b_i, r_i) \mid i = 1, \ldots, s; b_i \text{ is coprime to } r_i\}$. For simplicity, we will alternatively write a basket as a set of pairs with weights, say for example,

$$B = \{(1, 2), (1, 2), (2, 5)\} = \{2 \times (1, 2), (2, 5)\}.$$

Assume X to be a terminal weak $\mathbb{Q}$-Fano 3-fold. According to Reid [28], there is a basket of orbifold points (called *Reid basket*)

$$B_X = \left\{ (b_i, r_i) \,\middle|\, i = 1, \ldots, s; 0 < b_i \leqslant \frac{r_i}{2}; b_i \text{ is coprime to } r_i \right\}$$

associated to X, where a pair (b_i, r_i) corresponds to an orbifold point Q_i of type $\frac{1}{r_i}(1, -1, b_i)$. Moreover, by Reid's Riemann–Roch formula, Kawamata–Viehweg vanishing theorem and Serre duality, one has, for any $n > 0$,

$$P_{-n}(X) = -\chi(\mathcal{O}_X((n+1)K_X))$$

$$= \frac{1}{12} n(n+1)(2n+1)(-K_X^3) + (2n+1) - l(-n)$$

where $l(-n) = l(n+1) = \sum_i \sum_{j=1}^n \frac{\overline{jb_i}(r_i - \overline{jb_i})}{2r_i}$ and the first sum runs over all orbifold points in Reid basket.

The above formula can be rewritten as:

$$P_{-1} = \frac{1}{2}\left(-K_X^3 + \sum_i \frac{b_i^2}{r_i}\right) - \frac{1}{2}\sum_i b_i + 3,$$

(2.1)

$$P_{-m} - P_{-(m-1)} = \frac{m^2}{2}\left(-K_X^3 + \sum_i \frac{b_i^2}{r_i}\right) - \frac{m}{2}\sum_i b_i + 2 - \Delta^m$$

where $\Delta^m = \sum_i\left(\frac{\overline{b_i m}(r_i - \overline{b_i m})}{2r_i} - \frac{b_i m(r_i - b_i m)}{2r_i}\right)$ for any $m \geqslant 2$.

2.3. Weighted baskets according to Chen–Chen

All contents of this subsection are mainly from Chen–Chen [7, 9]. We list them as follows:

(i) Let $B = \{(b_i, r_i) \,|\, i = 1, \ldots, s; 0 < b_i \leqslant \frac{r_i}{2}; b_i \text{ is coprime to } r_i\}$ be a basket. We set $\sigma(B) = \sum_i b_i$, $\sigma'(B) = \sum_i \frac{b_i^2}{r_i}$, and $\Delta^n(B) = \sum_i\left(\frac{\overline{b_i n}(r_i - \overline{b_i n})}{2r_i} - \frac{b_i n(r_i - b_i n)}{2r_i}\right)$ for any integer $n > 1$.

(ii) The new (generalized) basket

$$B' = \{(b_1 + b_2, r_1 + r_2), (b_3, r_3), \ldots, (b_s, r_s)\}$$

is called a *packing* of B, denoted as $B \succeq B'$. We call $B \succ B'$ a *prime packing* if $b_1 r_2 - b_2 r_1 = 1$. A composition of finite packings is also called a packing. So the relation "$\succeq$" is a partial ordering on the set of baskets.

(iii) Note that for a terminal weak $\mathbb{Q}$-Fano 3-fold X, all the anti-pluri-genera P_{-n} can be determined by Reid basket B_X and $P_{-1}(X)$. This leads to the notion of "weighted basket". We call a pair $\mathbb{B} = (B, \widetilde{P}_{-1})$ a *weighted basket* if B is a basket and $\widetilde{P}_{-1}$ is a non-negative integer. We write $(B, \widetilde{P}_{-1}) \succeq (B', \widetilde{P}_{-1})$ if $B \succeq B'$.

(iv) Given a weighted basket $\mathbb{B} = (B, \widetilde{P}_{-1})$, define $\widetilde{P}_{-1}(\mathbb{B}) = \widetilde{P}_{-1}$ and the volume

$$-K^3(\mathbb{B}) = 2\widetilde{P}_{-1} + \sigma(B) - \sigma'(B) - 6.$$

For all $m \geqslant 1$, we define the "anti-plurigenus" in the following inductive way:

$$\widetilde{P}_{-(m+1)} - \widetilde{P}_{-m}$$
$$= \frac{1}{2}(m+1)^2(-K^3(\mathbb{B}) + \sigma'(B)) + 2 - \frac{m+1}{2}\sigma - \Delta^{m+1}(B).$$

Note that, if we set $\mathbb{B} = (B_X, P_{-1}(X))$ for a given terminal weak $\mathbb{Q}$-Fano 3-fold X, then one can verify directly that $-K^3(\mathbb{B}) = -K_X^3$ and $\widetilde{P}_{-m}(\mathbb{B}) = P_{-m}(X)$ for all $m \geqslant 1$.

PROPOSITION 2.3 ([9, Section 3]). — Assume that $\mathbb{B} = (B, \widetilde{P}_{-1}) \succeq \mathbb{B}' = (B', \widetilde{P}_{-1})$. Then

(1) $\sigma(B) = \sigma(B')$ and $\sigma'(B) \geqslant \sigma'(B')$;
(2) for all integer $n \geqslant 1$, $\Delta^n(B) \geqslant \Delta^n(B')$;
(3) $-K^3(\mathbb{B}) + \sigma'(B) = -K^3(\mathbb{B}') + \sigma'(B')$;
(4) $-K^3(\mathbb{B}) \leqslant -K^3(\mathbb{B}')$;
(5) $\widetilde{P}_{-m}(\mathbb{B}) \leqslant \widetilde{P}_{-m}(\mathbb{B}')$ for all $m \geqslant 2$.

Next we recall the "canonical" sequence of a basket B. Set $S^{(0)} = \{\frac{1}{n} \mid n \geqslant 2\}$, $S^{(5)} = S^{(0)} \cup \{\frac{2}{5}\}$, and inductively for all $n \geqslant 5$,

$$S^{(n)} = S^{(n-1)} \cup \left\{ \frac{b}{n} \,\middle|\, 0 < b < \frac{n}{2}, \ b \text{ is coprime to } n \right\}.$$

Each set $S^{(n)}$ gives a division of the interval $(0, \frac{1}{2}] = \bigcup_i [\omega_{i+1}^{(n)}, \omega_i^{(n)}]$ with $\omega_i^{(n)}, \omega_{i+1}^{(n)} \in S^{(n)}$. Let $\omega_{i+1}^{(n)} = \frac{q_{i+1}}{p_{i+1}}$ and $\omega_i^{(n)} = \frac{q_i}{p_i}$ with $\gcd(q_l, p_l) = 1$ for $l = i, i+1$. Then it is easy to see that $q_i p_{i+1} - p_i q_{i+1} = 1$ for all n and i (cf. [9, Claim A]).

Now given a basket $B = \{(b_i, r_i) \mid i = 1, \ldots, s\}$, we define new baskets $\mathcal{B}^{(n)}(B)$, where $\mathcal{B}^{(n)}(\cdot)$ can be regarded as an operator on the set of baskets. For each $(b_i, r_i) \in B$, if $\frac{b_i}{r_i} \in S^{(n)}$, then we set $\mathcal{B}_i^{(n)} = \{(b_i, r_i)\}$. If $\frac{b_i}{r_i} \notin S^{(n)}$, then $\omega_{l+1}^{(n)} < \frac{b_i}{r_i} < \omega_l^{(n)}$ for some l. We write $\omega_l^{(n)} = \frac{q_l}{p_l}$ and $\omega_{l+1}^{(n)} = \frac{q_{l+1}}{p_{l+1}}$ respectively. In this situation, we can unpack (b_i, r_i) to $\mathcal{B}_i^{(n)} = \{(r_i q_l - b_i p_l) \times (q_{l+1}, p_{l+1}), (-r_i q_{l+1} + b_i p_{l+1}) \times (q_l, p_l)\}$. Adding up those $\mathcal{B}_i^{(n)}$, we get a new basket $\mathcal{B}^{(n)}(B)$, which is uniquely defined according to the construction and $\mathcal{B}^{(n)}(B) \succeq B$ for all n. Note that, by the definition, $B = \mathcal{B}^{(n)}(B)$ for sufficiently large n.

Moreover, we have

$$\mathcal{B}^{(n-1)}(B) = \mathcal{B}^{(n-1)}(\mathcal{B}^{(n)}(B)) \succeq \mathcal{B}^{(n)}(B)$$

for all $n \geqslant 1$ (cf. [9, Claim B]). Therefore we have a chain of baskets

$$\mathcal{B}^{(0)}(B) \succeq \mathcal{B}^{(5)}(B) \succeq \cdots \succeq \mathcal{B}^{(n)}(B) \succeq \cdots \succeq B.$$

The step $\mathcal{B}^{(n-1)}(B) \succeq \mathcal{B}^{(n)}(B)$ can be achieved by a number of successive prime packings. Let $\epsilon_n(B)$ be the number of such prime packings. For any $n > 0$, set $B^{(n)} = \mathcal{B}^{(n)}(B)$.

The following properties are essential in representing $B^{(n)}$.

LEMMA 2.4 ([9, Lemma 2.16]). — *For the above sequence* $\{B^{(n)}\}$, *the following statements hold:*

(1) $\Delta^j(B^{(0)}) = \Delta^j(B)$ *for* $j = 3, 4$;
(2) $\Delta^j(B^{(n-1)}) = \Delta^j(B^{(n)})$ *for all* $j < n$;
(3) $\Delta^n(B^{(n-1)}) = \Delta^n(B^{(n)}) + \epsilon_n(B)$.

It follows that $\Delta^j(B^{(n)}) = \Delta^j(B)$ for all $j \leqslant n$ and

$$\epsilon_n(B) = \Delta^n(B^{(n-1)}) - \Delta^n(B^{(n)}) = \Delta^n(B^{(n-1)}) - \Delta^n(B).$$

Moreover, given a weighted basket $\mathbb{B} = (B, \widetilde{P}_{-1})$, we can similarly consider $\mathcal{B}^{(n)}(\mathbb{B}) = (B^{(n)}, \widetilde{P}_{-1})$. It follows that

$$\widetilde{P}_{-j}(\mathcal{B}^{(n)}(\mathbb{B})) = \widetilde{P}_{-j}(\mathbb{B})$$

for all $j \leqslant n$. Therefore we can realize the canonical sequence of weighted baskets as an approximation of weighted baskets via anti-plurigenera.

We now recall the relation between weighted baskets and anti-plurigenera more closely. For a given weighted basket $\mathbb{B} = (B, \widetilde{P}_{-1})$, we start by computing the non-negative number ϵ_n and $B^{(0)}$, $B^{(5)}$ in terms of $\widetilde{P}_{-m}$. From the definition of $\widetilde{P}_{-m}$ we get

$$\sigma(B) = 10 - 5\widetilde{P}_{-1} + \widetilde{P}_{-2},$$

$$(2.2) \qquad \Delta^{m+1} = (2 - 5(m+1) + 2(m+1)^2) + \frac{1}{2}(m+1)(2 - 3m)\widetilde{P}_{-1}$$

$$+ \frac{1}{2}m(m+1)\widetilde{P}_{-2} + \widetilde{P}_{-m} - \widetilde{P}_{-(m+1)}.$$

In particular, we have

$$\Delta^3 = 5 - 6\widetilde{P}_{-1} + 4\widetilde{P}_{-2} - \widetilde{P}_{-3};$$

$$\Delta^4 = 14 - 14\widetilde{P}_{-1} + 6\widetilde{P}_{-2} + \widetilde{P}_{-3} - \widetilde{P}_{-4}.$$

Assume that $B^{(0)} = \{n_{1,r}^0 \times (1, r) \, | \, r \geqslant 2\}$. By Lemma 2.4, we have

$$\sigma(B) = \sigma(B^{(0)}) = \sum n_{1,r}^0;$$

$$\Delta^3(B) = \Delta^3(B^{(0)}) = n_{1,2}^0;$$

$$\Delta^4(B) = \Delta^4(B^{(0)}) = 2n_{1,2}^0 + n_{1,3}^0.$$

Thus we get $B^{(0)}$ as follows:

$$\begin{cases} n_{1,2}^0 = 5 - 6\widetilde{P}_{-1} + 4\widetilde{P}_{-2} - \widetilde{P}_{-3}; \\ n_{1,3}^0 = 4 - 2\widetilde{P}_{-1} - 2\widetilde{P}_{-2} + 3\widetilde{P}_{-3} - \widetilde{P}_{-4}; \\ n_{1,4}^0 = 1 + 3\widetilde{P}_{-1} - \widetilde{P}_{-2} - 2\widetilde{P}_{-3} + \widetilde{P}_{-4} - \sigma_5; \\ n_{1,r}^0 = n_{1,r}^0, r \geqslant 5, \end{cases}$$

where $\sigma_5 = \sum_{r \geqslant 5} n^0_{1,r}$. A computation gives

$$\epsilon_5 = 2 + \widetilde{P}_{-2} - 2\widetilde{P}_{-4} + \widetilde{P}_{-5} - \sigma_5.$$

Therefore we get $B^{(5)} = \{n^5_{1,r} \times (1,r), n^5_{2,5} \times (2,5) \,|\, r \geqslant 2\}$ as follows:

$$\begin{cases} n^5_{1,2} = 3 - 6\widetilde{P}_{-1} + 3\widetilde{P}_{-2} - \widetilde{P}_{-3} + 2\widetilde{P}_{-4} - \widetilde{P}_{-5} + \sigma_5; \\ n^5_{2,5} = 2 + \widetilde{P}_{-2} - 2\widetilde{P}_{-4} + \widetilde{P}_{-5} - \sigma_5; \\ n^5_{1,3} = 2 - 2\widetilde{P}_{-1} - 3\widetilde{P}_{-2} + 3\widetilde{P}_{-3} + \widetilde{P}_{-4} - \widetilde{P}_{-5} + \sigma_5; \\ n^5_{1,4} = 1 + 3\widetilde{P}_{-1} - \widetilde{P}_{-2} - 2\widetilde{P}_{-3} + \widetilde{P}_{-4} - \sigma_5; \\ n^5_{1,r} = n^0_{1,r}, r \geqslant 5. \end{cases}$$

Because $B^{(5)} = B^{(6)}$, we see $\epsilon_6 = 0$ and on the other hand

$$\epsilon_6 = 3\widetilde{P}_{-1} + \widetilde{P}_{-2} - \widetilde{P}_{-3} - \widetilde{P}_{-4} - \widetilde{P}_{-5} + \widetilde{P}_{-6} - \epsilon = 0$$

where $\epsilon = 2\sigma_5 - n^0_{1,5} \geqslant 0$.

Going on a similar calculation, we get

$$\epsilon_7 = 1 + \widetilde{P}_{-1} + \widetilde{P}_{-2} - \widetilde{P}_{-5} - \widetilde{P}_{-6} + \widetilde{P}_{-7} - 2\sigma_5 + 2n^0_{1,5} + n^0_{1,6};$$

$$\epsilon_8 = 2\widetilde{P}_{-1} + \widetilde{P}_{-2} + \widetilde{P}_{-3} - \widetilde{P}_{-4} - \widetilde{P}_{-5} - \widetilde{P}_{-7} + \widetilde{P}_{-8}$$
$$- 3\sigma_5 + 3n^0_{1,5} + 2n^0_{1,6} + n^0_{1,7}.$$

A weighted basket $\mathbb{B} = (B, \widetilde{P}_{-1})$ is said to be *geometric* if $\mathbb{B} = (B_X, P_{-1}(X))$ for some terminal weak $\mathbb{Q}$-Fano 3-fold X. Geometric baskets are subject to some geometric properties. By [19], we have that $(-K_X \cdot c_2(X)) \geqslant 0$. Therefore [28, 10.3] gives the inequality

$$(2.3) \qquad \gamma(B) := \sum_i \frac{1}{r_i} - \sum_i r_i + 24 \geqslant 0.$$

For packings, it is easy to see the following lemma.

LEMMA 2.5. — *Given a packing of baskets $B_1 \succeq B_2$, we have $\gamma(B_1) \geqslant \gamma(B_2)$. In particular, if inequality (2.3) does not hold for B_1, then it does not hold for B_2.*

Furthermore, $-K^3(\mathbb{B}) = -K^3_X > 0$ gives the inequality

$$(2.4) \qquad \sigma'(B) < 2P_{-1} + \sigma(B) - 6.$$

Finally, by [18, Lemma 15.6.2], if $P_{-m} > 0$ and $P_{-n} > 0$, then

$$(2.5) \qquad P_{-m-n} \geqslant P_{-m} + P_{-n} - 1.$$

2.4. Effective results on terminal weak $\mathbb{Q}$-Fano 3-folds

We collect here some known facts about terminal weak $\mathbb{Q}$-Fano 3-folds.

PROPOSITION 2.6. — *Let X be a terminal weak $\mathbb{Q}$-Fano 3-folds. Then*
  (1) $P_{-m} > 0$ for $m \geqslant 6$ ([7, Theorem 1.1] and [10, Corollary 5.1]);
  (2) $P_{-8} \geqslant 2$ ([7, Theorem 1.1]);
  (3) $-K_X^3 \geqslant 1/330$ ([7, Theorem 1.1]);
  (4) $r_X \leqslant 660$ or $r_X = 840$, moreover, if $r_X = 840$, then $r_{\max} = 8$ ([12, Proposition 2.4]);
  (5) $r_{\max} \leqslant 24$ by inequality (2.3).

3. Fano–Mori triples

In this section, we introduce the concept of Fano–Mori triples and prove some basic properties.

DEFINITION 3.1. — *We say that a normal projective variety Y admits a Mori fiber structure if the following conditions hold:*
  (1) $g : Y \to T$ is a surjective morphism onto a normal projective variety T;
  (2) Y is $\mathbb{Q}$-factorial with at worst terminal singularities;
  (3) $g_* \mathcal{O}_Y = \mathcal{O}_T$;
  (4) $-K_Y$ is g-ample;
  (5) $\rho(Y/T) = 1$;
  (6) $\dim Y > \dim T$.

Example 3.2. — Note that, in dimension 3, there are three kinds of Mori fiber structures according to the value of $\dim T$:
  (1) $\dim T = 0$, Y is a $\mathbb{Q}$-factorial terminal $\mathbb{Q}$-Fano 3-folds with $\rho(Y) = 1$;
  (2) $\dim T = 1$, $Y \to T$ is a *del Pezzo fibration*, of which a general fiber is a smooth del Pezzo surface;
  (3) $\dim T = 2$, $Y \to T$ is a *conic bundle*, of which a general fiber is a smooth rational curve.

DEFINITION 3.3 (cf. [3, Definition 3.6.1]). — *Let $\phi : X \dashrightarrow Y$ be a proper birational contraction (i.e., a birational map extracting no divisors) between normal quasi-projective varieties and D a $\mathbb{Q}$-Cartier divisor on X such that $D' = \phi_* D$ is also $\mathbb{Q}$-Cartier. We say that ϕ is a D-non-positive*

contraction *if, for some common resolution $p : W \to X$ and $q : W \to Y$, one may write*

$$p^*D = q^*D' + E$$

where E is an effective q-exceptional $\mathbb{Q}$-divisor.

LEMMA 3.4. — *Let $\phi : X \dashrightarrow Y$ be a birational contraction between normal projective varieties and D a $\mathbb{Q}$-Cartier divisor on X such that $D' = \phi_*D$ is also $\mathbb{Q}$-Cartier. If $-D$ is nef, then ϕ is D-non-positive.*

Proof. — Taking a common resolution $p : W \to V'$ and $q : W \to Y$, one may write

$$p^*D = q^*D' + E$$

where E is a q-exceptional $\mathbb{Q}$-divisor. Note that $-E = q^*D' - p^*D$ is q-nef since $-D$ is nef, hence $E \geqslant 0$ by Negativity Lemma (see [20, Corollary 3.39 (1)]). $\qquad\square$

Note that the definition of D-non-positive contraction is independent of the choice of common resolutions. The most standard example of a D-non-positive contraction might be a composition of steps of D-MMP.

DEFINITION 3.5. — *Let V be a normal projective variety with at worst canonical singularities and V' be a terminalization of V. Then we say Y is a K-Mori fiber space of V if the following conditions are satisfied:*

(1) *Y has a Mori fiber structure $Y \to T$;*
(2) *there is a birational contraction $\sigma : V' \dashrightarrow Y$;*
(3) *σ is a $K_{V'}$-non-positive birational contraction.*

Note that if $-K_V$ is nef (or equivalently, $-K_{V'}$ is nef), then condition (3) automatically holds by Lemma 3.4.

Remark 3.6. — Note that the definition of K-Mori fiber spaces is independent of the choice of the terminalization. By [3], if K_V is not pseudo-effective, then K-Mori fiber spaces of V always exist by running K-MMP on V'.

The following is the key object that we are interested in.

DEFINITION 3.7. — *We say that (X, Y, Z) is a Fano–Mori triple if there is a commutative diagram*

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & Y \\ & {}_{p}\searrow \quad \swarrow{}_{q} & \\ & Z & \end{array}$$
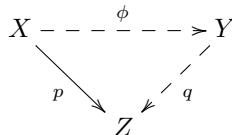
satisfying the following conditions:

    (1) X is a $\mathbb{Q}$-factorial terminal weak $\mathbb{Q}$-Fano 3-fold;
    (2) Y has a Mori fiber structure $Y \to T$;
    (3) Z is a $\mathbb{Q}$-factorial canonical weak $\mathbb{Q}$-Fano 3-fold;
    (4) $\phi : X \dashrightarrow Y$ is a birational contraction;
    (5) $q : Y \dashrightarrow Z$ is a birational contraction which is isomorphic in codimension one;
    (6) $p : X \to Z$ is a terminalization of Z, or equivalently, $K_X = p^*(K_Z)$.

Remark 3.8. — For a Fano–Mori triple (X, Y, Z), if $\rho(Y) = 1$ (or equivalently, $\dim T = 0$), then X, Y, Z are isomorphic to each other, and they are $\mathbb{Q}$-factorial terminal $\mathbb{Q}$-Fano 3-folds with $\rho = 1$. In fact, in this case, since $\rho(Y) = 1$ and Y admits a Mori fiber structure, Y is a $\mathbb{Q}$-factorial terminal $\mathbb{Q}$-Fano 3-fold of Picard number one. Since Y and Z are isomorphic in codimension one, it turns out that $\rho(Z) = 1$ and $Y \cong Z$. In particular, Z is terminal and hence $X \cong Z$.

By Remark 3.8, the concept of Fano–Mori triples can be viewed as a natural generalization of the concept of $\mathbb{Q}$-factorial terminal $\mathbb{Q}$-Fano 3-folds with $\rho = 1$. Moreover, the following proposition suggests that Fano–Mori triples appear naturally in the study of birational geometry of canonical weak $\mathbb{Q}$-Fano 3-folds.

PROPOSITION 3.9. — *Let V be a canonical weak $\mathbb{Q}$-Fano 3-fold and Y a K-Mori fiber space of V. Then there exists a Fano–Mori triple (X, Y, Z) containing Y as the second term.*

Proof. — Let V be a canonical weak $\mathbb{Q}$-Fano 3-fold and Y a K-Mori fiber space of V. After replacing V by its terminalization, we may assume that V is $\mathbb{Q}$-factorial and terminal and there exists a birational contraction $\sigma : V \dashrightarrow Y$.

Note that $-K_V$ is semi-ample by Basepoint-free Theorem (see [20, Theorem 3.3]), we may choose an effective $\mathbb{Q}$-divisor $B \sim_{\mathbb{Q}} -K_V$ such that (V, B) is terminal and $\mathrm{Supp}(B)$ does not contain any exceptional divisor on V over Y. Take $B_Y = \sigma_* B$. Then $K_Y + B_Y \sim_{\mathbb{Q}} 0$. Taking a common resolution $f : W \to V$ and $g : W \to Y$. Then $f^*(K_V + B) - g^*(K_Y + B_Y)$ is g-exceptional. By Negativity Lemma (see [20, Corollary 3.39 (1)]), we see that $f^*(K_V + B) = g^*(K_Y + B_Y)$. Hence (Y, B_Y) is canonical by the construction of (V, B).

Noting that $-K_Y = \sigma_*(-K_V)$ is big, we may write $-K_Y \sim_{\mathbb{Q}} A_Y + E_Y$ where A_Y is an ample $\mathbb{Q}$-divisor and E_Y is an effective $\mathbb{Q}$-divisor. Now take

$t > 0$ to be sufficiently small such that $(Y, B_Y + tE_Y)$ is klt. Then the pair $(Y, (1-t)B_Y + tE_Y)$ is also klt and

$$-(K_Y + (1-t)B_Y + tE_Y) \sim_\mathbb{Q} tA_Y$$

is ample. By [3, Corollary 1.3.2], Y is a Mori dream space, which means that we can run MMP for any divisor on Y.

Running the $(-K)$-MMP on Y, we end up with a minimal model Z which is $\mathbb{Q}$-factorial such that $-K_Z$ is nef and big. Since $-K_V$ is semi-ample, the stable base locus of $-K_Y$ does not contain any divisors, hence the $(-K)$-MMP on Y does not contract any divisor, which means that the birational contraction $q : Y \dashrightarrow Z$ is an isomorphism in codimension one. Moreover, take $B_Z = q_* B_Y$, then (Z, B_Z) is canonical also by Negativity Lemma since (Y, B_Y) is canonical and $K_Y + B_Y \sim_\mathbb{Q} 0$, which implies that Z is also canonical. In summery, Z is a $\mathbb{Q}$-factorial canonical weak $\mathbb{Q}$-Fano 3-fold.

Finally, we take $p : X \to Z$ to be a terminalization of Z, then X is a $\mathbb{Q}$-factorial terminal weak $\mathbb{Q}$-Fano 3-fold. Clearly the induced map $\phi = q^{-1} \circ p : X \dashrightarrow Y$ is a birational contraction. $\qquad\square$

PROPOSITION 3.10. — Let (X, Y, Z) be a Fano–Mori triple. Then

(1) $\phi : X \dashrightarrow Y$ is K_X-non-positive, in particular, Y is a K-Mori fiber space of X;
(2) $q : Y \dashrightarrow Z$ is $(-K_Y)$-non-positive;
(3) for any integer $m \geqslant 0$,

$$h^0(X, -mK_X) = h^0(Y, -mK_Y) = h^0(Z, -mK_Z).$$

In particular, $\varphi_{-m,X}$, $\varphi_{-m,Y}$, and $\varphi_{-m,Z}$ factor through each other by the birational maps ϕ, p, and q, and therefore share the same birational properties.

Proof. — Since $-K_X$ is nef, by Lemma 3.4, ϕ is K_X-non-positive, which proves statement (1).

Since q is an isomorphism in codimension one, $q^{-1} : Z \dashrightarrow Y$ is also a birational contraction. Since $-K_Z$ is nef, again by Lemma 3.4, q^{-1} is K_Z-non-positive. Since q is an isomorphism in codimension one, it is easy to check by definition that $q : Y \dashrightarrow Z$ is $(-K_Y)$-non-positive, which proves statement (2).

For statement (3), fix an integer $m \geqslant 0$. Since ϕ is K_X-non-positive and q is $(-K_Y)$-non-positive, we have

$$h^0(X, -mK_X) \leqslant h^0(Y, -mK_Y) = h^0(Z, -mK_Z).$$

On the other hand, since $K_X = p^*(K_Z)$, we have

$$h^0(X, -mK_X) = h^0(Z, -mK_Z).$$

We complete the proof.   □

LEMMA 3.11. — *Let V be a canonical weak $\mathbb{Q}$-Fano 3-fold and Y a K-Mori fiber space of V. Let $Y \to T$ be the Mori fiber structure, then T is of Fano type, that is, there exists an effective $\mathbb{Q}$-divisor B_T such that (T, B_T) is klt and $-(K_T + B_T)$ is ample. In particular, if (X, Y, Z) is a Fano–Mori triple and $Y \to T$ be the Mori fiber structure, then T is of Fano type.*

Proof. — In the proof of Proposition 3.9, we have seen that the pair $(Y, (1-t)B_Y + tE_Y)$ is klt and

$$-(K_Y + (1-t)B_Y + tE_Y) \sim_{\mathbb{Q}} tA_Y$$

is ample. Hence Y is of Fano type. Then we may apply [27, Lemma 2.8] or [13, Corollary 3.3] to conclude that T is of Fano type as well.

The last statement follows from Proposition 3.10 (1).   □

4. A geometric inequality

In our previous paper, we used the following proposition to get a number $m_1 > 0$ so that $|-m_1 K|$ is not composed with a pencil.

PROPOSITION 4.1 ([12, Proof of Corollary 4.2]). — *Let X be a canonical weak $\mathbb{Q}$-Fano 3-fold and $m > 0$ an integer. Assume that $|-mK_X|$ is composed with a pencil. Keep the notation in Subsection 2.1. Then*

$$\frac{P_{-m} - 1}{m} \leqslant \frac{-K_X^3}{(\pi^*(K_X)^2 \cdot S)}.$$

Proposition 4.1 is, however, too weak for applications. The main goal of this section is to present an alternative inequality (see Proposition 4.5) which turns out to be the key ingredient of this paper.

To start up we recall the setting. Let (X, Y, Z) be a Fano–Mori triple and assume that $|-mK_X|$ is composed with a pencil for an integer $m > 0$. By the notation in Subsection 2.1, we may take W to be a common log resolution of X and Y, and $f : W \to \mathbb{P}^1$ is the induced fibration by $|-mK_X|$. A general fiber of f is denoted by S. We have the following

commutative diagram:
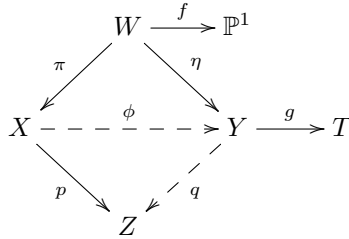
LEMMA 4.2. — Let (X, Y, Z) be a Fano–Mori triple and $m > 0$ an integer. Assume that $|-mK_X|$ is composed with a pencil. Keep the notation in Subsection 2.1. Assume that there exists an effective η-exceptional $\mathbb{Q}$-divisor F on W such that, $(-\pi^*(K_X)|_S \cdot F|_S) > 0$ for a general S, then

$$\frac{P_{-m} - 1}{m} \leqslant 2r_X(\pi^*(K_X)^2 \cdot S).$$

Proof. — Since $-\pi^*(K_X)|_S$ is nef, there exists a component F_0 of F such that $(-\pi^*(K_X)|_S \cdot F_0|_S) > 0$, which means that

$$(-\pi_Z^*(K_Z)|_S \cdot F_0|_S) = (-\pi^*(K_X)|_S \cdot F_0|_S) > 0$$

where we denote by π_Z the composition $p \circ \pi : W \to Z$. In particular, $F_0|_S$ is not contracted by π_Z. Hence there exists a curve $C_Z \subset Z$ such that

$$C_Z \subset \pi_Z(F_0 \cap S) \subset \pi_Z(F_0).$$

On the other hand, since Y and Z are isomorphic in codimension one and F_0 is η-exceptional, F_0 is also π_Z-exceptional. This implies that $\pi_Z(F_0)$ is a subvariety of codimension at least 2. Hence $C_Z = \pi_Z(F_0)$. In particular, C_Z is independent of S, and for any general S, $C_Z = \pi_Z(F_0 \cap S)$. Moreover,

$$(-K_Z \cdot C_Z) = \frac{1}{n}(\pi_Z^*(-K_Z) \cdot F_0 \cdot S) > 0$$

by the projection formula, where $\pi_*(F_0|_S) = nC_Z$ as 1-cycles. Hence $(-K_Z \cdot C_Z) \geqslant \frac{1}{r_X}$ since $r_X K_Z$ is Cartier.

By assumption, we have

$$-\pi^*(mK_X) \sim (P_{-m} - 1)S + D_m$$

where D_m is an effective $\mathbb{Q}$-divisor. Set $w = \frac{P_{-m} - 1}{m}$. Then

(4.1) $$-\frac{1}{w}\pi^*(K_X) - S \sim_{\mathbb{Q}} D$$

for some effective $\mathbb{Q}$-divisor D. Write

$$K_W = \pi^*(K_X) + E_\pi,$$

where $E_\pi \geqslant 0$ is π-exceptional. Pick two general fibers S_1 and S_2 of f and consider the pair

$$(W, -E_\pi + 2D + S_1 + S_2),$$

which can be assumed to have simple normal crossing support modulo a further birational modification of W. Note that

$$-(K_W - E_\pi + 2D + S_1 + S_2) \sim_\mathbb{Q} -\left(1 - \frac{2}{w}\right)\pi^*(K_X)$$

$$\sim_\mathbb{Q} -\left(1 - \frac{2}{w}\right)\pi_Z^*(K_Z)$$

is π_Z-nef and $\pi_{Z*}(-E_\pi + 2D + S_1 + S_2) \geqslant 0$ since E_π is π_Z-exceptional. Denote by G the support of the effective part of $\lfloor -E_\pi + 2D \rfloor + S_1 + S_2$. By Connectedness Lemma (see [20, Theorem 5.48]),

$$G \cap \pi_Z^{-1}(z)$$

is connected for any point $z \in Z$.

We claim that there exists a prime divisor F_1 on W such that $2D \geqslant F_1$ and $\pi_Z(F_1 \cap S)$ contains C_Z for a general S. Consider a point $z \in C_Z \subset Z$, then $S \cap \pi_Z^{-1}(z) \neq \emptyset$ for a general S since $C_Z = \pi_Z(F_0 \cap S) \subset \pi_Z(S)$. In particular, $S_1 \cap \pi_Z^{-1}(z) \neq \emptyset$ and $S_2 \cap \pi_Z^{-1}(z) \neq \emptyset$, which are two disconnected sets in $G \cap \pi_Z^{-1}(z)$. Since $G \cap \pi_Z^{-1}(z)$ is connected, there exists a curve $B_z \subset G \cap \pi_Z^{-1}(z)$ such that $B_z \cap S_1 \cap \pi_Z^{-1}(z) \neq \emptyset$ and $B_z \not\subset S_1 \cap \pi_Z^{-1}(z)$. Moving z in C_Z, B_z deforms to a prime divisor $F_1 \subset G$, to be more precise, there exists a prime divisor $F_1 \subset G$ such that $B_z \subset F_1 \cap \pi_Z^{-1}(z)$ for infinitely many $z \in C_Z$. Hence

$$z \in \pi_Z(B_z \cap S_1 \cap \pi_Z^{-1}(z)) \subset \pi_Z(F_1 \cap S_1 \cap \pi_Z^{-1}(z)) \subset \pi_Z(F_1 \cap S_1)$$

for infinitely many $z \in C_Z$. This means that $C_Z \subset \pi_Z(F_1 \cap S_1)$. By the construction of F_1, $F_1 \subset G$ and it is clear that F_1 is different from S_1 and S_2, hence $\mathrm{coeff}_{F_1}(-E_\pi + 2D) \geqslant 1$ and in particular, $2D \geqslant F_1$. By the generality of S_1, $C_Z \subset \pi_Z(F_1 \cap S)$ for general S.

By equation (4.1),

$$\frac{2}{w}(\pi^*(K_X)^2 \cdot S) = \left(-\pi^*(K_X)|_S \cdot (2S + 2D)|_S\right)$$

$$= \left(\pi^*(-K_X)|_S \cdot 2D|_S\right) \geqslant \left(\pi^*(-K_X)|_S \cdot F_1|_S\right)$$

$$= \left(\pi_Z^*(-K_Z)|_S \cdot F_1|_S\right) = \left(-K_Z \cdot \pi_{Z*}(F_1|_S)\right)$$

$$\geqslant \left(-K_Z \cdot C_Z\right) \geqslant \frac{1}{r_X}.$$

In other words, we have

$$\frac{P_{-m} - 1}{m} = w \leqslant 2r_X(\pi^*(K_X)^2 \cdot S). \qquad \square$$

LEMMA 4.3. — Let (X, Y, Z) be a Fano–Mori triple and $m > 0$ an integer. Assume that $|-mK_X|$ is composed with a pencil. Keep the notation in Subsection 2.1. Assume that $f : W \to \mathbb{P}^1$ factors through Y. Then

$$\frac{P_{-m} - 1}{m} \leqslant \max\{-K_X^3, 2r_X(\pi^*(K_X)^2 \cdot S)\}.$$

Proof. — Let S_Y be a general fiber of $Y \to \mathbb{P}^1$. Write

$$\pi^*(K_X) = \eta^*(K_Y) + E$$

where E is an effective η-exceptional $\mathbb{Q}$-divisor by Proposition 3.10(1). Restricting to a general fiber S of f, we have

$$(4.2) \qquad \pi^*(K_X)|_S = \eta^*(K_Y)|_S + E|_S = \eta_S^*(K_{S_Y}) + E|_S,$$

where $\eta_S : S \to S_Y$ is the restriction of η on S.

If $E|_S = 0$, then

$$(\pi^*(K_X)^2 \cdot S) = (\pi^*(K_X)|_S)^2 = \eta_S^*(K_{S_Y})^2$$

is an integer since S_Y is smooth. Since $-\pi^*(K_X)|_S$ is nef and big, $(\pi^*(K_X)^2 \cdot S) \geqslant 1$ and, by Proposition 4.1,

$$\frac{P_{-m} - 1}{m} \leqslant -K_X^3.$$

Now we may assume that $E|_S \neq 0$. Since E is exceptional over Y and S is general, $E|_S$ is an effective $\mathbb{Q}$-divisor on S exceptional over S_Y. By Hodge Index Theorem, $(E|_S)^2 < 0$. By (4.2),

$$\left(\pi^*(K_X)|_S \cdot E|_S\right) = \left(\eta_S^*(K_{S_Y}) \cdot E|_S\right) + (E|_S)^2 = (E|_S)^2 < 0.$$

Hence we may apply Lemma 4.2 to get

$$\frac{P_{-m} - 1}{m} \leqslant 2r_X(\pi^*(K_X)^2 \cdot S).$$

So we have completed the proof. $\qquad \square$

LEMMA 4.4. — Let S be a smooth del Pezzo surface and D a non-zero integral effective divisor on S such that $-K_S - aD$ is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{Q}$-divisor for some positive rational number a. Then $a \leqslant 3$.

Proof. — By assumption, there is an effective $\mathbb{Q}$-divisor B such that

$$aD + B \sim_{\mathbb{Q}} -K_S.$$

Since $(S, D + \frac{1}{a}B)$ is not klt, the log canonical threshold $\mathrm{lct}(S; aD+B) \leqslant \frac{1}{a}$. Recall that the global log-canonical threshold is defined as:

$$\mathrm{lct}(S) = \inf\{\mathrm{lct}(S; L) \mid L \sim_{\mathbb{Q}} -K_S, \ L \text{ is an effective } \mathbb{Q}\text{-divisor}\}.$$

Clearly we have $\mathrm{lct}(S) \leqslant \frac{1}{a}$. On the other hand, $\mathrm{lct}(S) \geqslant \frac{1}{3}$ by [6, Theorem 1.7], which implies that $a \leqslant 3$.   □

PROPOSITION 4.5. — *Let (X, Y, Z) be a Fano–Mori triple such that $\rho(Y) > 1$ and $m > 0$ an integer. Assume that $|-mK_X|$ is composed with a pencil. Keep the same notation as in Subsection 2.1. Then*

$$\frac{P_{-m} - 1}{m} \leqslant \max\{3, -K_X^3, 2r_X(\pi^*(K_X)^2 \cdot S)\}.$$

Proof. — Note that we have

$$-\pi^*(mK_X) \sim (P_{-m} - 1)S + D_m,$$

where D_m is an effective $\mathbb{Q}$-divisor (the fixed part). Pushing forward to Y, one has

(4.3) $-mK_Y = -\eta_*\pi^*(mK_X) \sim (P_{-m} - 1)\eta_*S + \eta_*D_m.$

Consider the Mori fiber structure $g : Y \to T$. Since $\rho(Y) > 1$, $\dim T > 0$. We argue by discussing the value of $\dim T$.

If $\dim T = 1$, then $T \simeq \mathbb{P}^1$ since $g(T) \leqslant q(Y) = q(X) = 0$. Take S_Y to be a general fiber of g. If $\eta_*S|_{S_Y} = 0$, then $|\eta_*S|$ coincides with the pencil $|S_Y|$, and hence f factors through g. By Lemma 4.3 we have

$$\frac{P_{-m} - 1}{m} \leqslant \max\{-K_X^3, 2r_X(\pi^*(K_X)^2 \cdot S)\}.$$

If $\eta_*S|_{S_Y} \neq 0$, then $\eta_*S|_{S_Y} = D$ is an effective non-zero integral divisor on S_Y and by equation (4.3),

$$-mK_{S_Y} = -mK_Y|_{S_Y} \sim (P_{-m} - 1)\eta_*S|_{S_Y} + \eta_*D_m|_{S_Y} \geqslant (P_{-m} - 1)D,$$

which means that $\frac{P_{-m}-1}{m} \leqslant 3$ by Lemma 4.4 since S_Y is a smooth del Pezzo surface.

Now we consider the case when $\dim T = 2$. Note that for a general fiber $C \simeq \mathbb{P}^1$ of g, $(\eta_*S \cdot C)$ is a non-negative integer. If $(\eta_*S \cdot C) \geqslant 1$ for a general fiber C of g, then by equation (4.3) intersecting with C,

$$2m = -m(K_Y \cdot C) \geqslant (P_{-m} - 1)(\eta_*S \cdot C) \geqslant P_{-m} - 1,$$

which means that

$$\frac{P_{-m} - 1}{m} \leqslant 2.$$

Hence we may assume that $(\eta_*S \cdot C) = 0$ for a general fiber C of g. By Cone Theorem (see [20, Theorem 3.7]), there exists a $\mathbb{Q}$-divisor H on T

such that $\eta_* S \sim_{\mathbb{Q}} g^* H$. In particular, $(\eta_* S)^2 \equiv (H^2)C$ as 1-cycles. Note that, if we fix a general S, then we may just take $H = g(\eta_* S)$. Since $|\eta_* S|$ is a movable linear system, H is nef and $H^2 \geqslant 0$. Moreover, H is semi-ample by Basepoint-free Theorem and Lemma 3.11.

First we consider the case $H^2 = 0$. In this case $|nH|$ defines a contraction $T \to \Lambda$ to a curve Λ (after taking Stein factorization) for a sufficiently large n. Moreover, it is easy to see that the induced morphism $W \to Y \to T \to \Lambda$ contracts S, which means that this map coincides with $f : W \to \mathbb{P}^1$. Hence $f : W \to \mathbb{P}^1$ factors through Y and we may apply Lemma 4.3 to get

$$\frac{P_{-m} - 1}{m} \leqslant \max\{-K_X^3, 2r_X \pi^*(K_X)^2 \cdot S\}.$$

Next we consider the case $H^2 > 0$. We have the following claim.

CLAIM 4.6. — For any η-exceptional prime divisor E_0 on W,

$$(E_0 \cdot \eta^* \eta_* S \cdot S) = 0.$$

Proof. — By projection formula, we have

$$(E_0 \cdot \eta^* \eta_* S \cdot S) = (E_0|_S \cdot \eta^* \eta_* S) = (\eta_*(E_0|_S) \cdot \eta_* S).$$

We may assume that $\eta_*(E_0|_S) \neq 0$ for a general S since, otherwise, there is nothing to prove. Then there is a curve $G \subset \eta(E_0 \cap S) \subset \eta(E_0)$. Note that E_0 is an η-exceptional prime divisor, we have $G = \eta(E_0 \cap S) = \eta(E_0)$. In particular, G does not depend on S and $G = \eta(E_0 \cap S) \subset \eta_* S$ for general S. Recall that $\eta_* S \sim_{\mathbb{Q}} g^* H$, which means that the intersection of two general $\eta_* S$ lies in fibers of g, hence G lies in a fiber of g. In particular, since $\text{Supp}(\eta_*(E_0|_S)) = G$,

$$(\eta_*(E_0|_S) \cdot \eta_* S) = (\eta_*(E_0|_S) \cdot g^* H) = 0.$$

We have proved the claim. $\qquad\square$

Write

$$\pi^*(K_X) = \eta^*(K_Y) + E$$

where $E \geqslant 0$ is an η-exceptional $\mathbb{Q}$-divisor. By Claim 4.6,

$$(-\pi^*(K_X) \cdot \eta^* \eta_* S \cdot S) = (-\eta^*(K_Y) \cdot \eta^* \eta_* S \cdot S)$$
$$= (-K_Y \cdot \eta_* S \cdot \eta_* S)$$
$$= (-K_Y \cdot (H^2)C) = 2(H^2) > 0.$$

On the other hand, we may write

$$\eta^* \eta_* S = S + F$$

where $F \geqslant 0$ is an η-exceptional effective $\mathbb{Q}$-divisor on W, and we have

$$0 < (-\pi^*(K_X) \cdot \eta^* \eta_* S \cdot S) = (-\pi^*(K_X) \cdot (S+F) \cdot S) = (-\pi^*(K_X)|_S \cdot F|_S).$$

Hence we may apply Lemma 4.2 to get

$$\frac{P_{-m} - 1}{m} \leqslant 2 r_X (\pi^*(K_X)^2 \cdot S).$$

Combining the above cases, we complete the proof. $\qquad\square$

5. Criteria for birationality

Based on Section 4, firstly we give criteria for $|-mK|$ being not composed with pencils (Propositions 5.2 and 5.4) and then recall criteria for birationality of $|-mK|$ established in our previous paper [12] (Theorems 5.7 and 5.9).

We always assume that X is a terminal weak $\mathbb{Q}$-Fano 3-fold.

Set $M_X = r_X(-K_X^3)$, which is a positive integer. For any positive integer N, define the following functions

$$\theta(M_X, N) = \min\{M_X/N, \max\{3, M_X/r_X, 2N\}\},$$
$$\theta(M_X) = \max_{N \in \mathbb{Z}_{>0}} \theta(M_X, N),$$

and

$$\lambda(M_X) = \begin{cases} M_X, & \text{if } M_X \leqslant 3; \\ \max\left\{3, M_X/r_X, 2\left\lfloor \sqrt{M_X/2} \right\rfloor, \dfrac{M_X}{\left\lceil \sqrt{M_X/2} \right\rceil}\right\} & \text{if } M_X \geqslant 4. \end{cases}$$

We have the following relation between $\lambda(M_X)$ and $\theta(M_X)$.

Lemma 5.1. — $\lambda(M_X) \geqslant \theta(M_X)$.

Proof. — It suffices to show that $\lambda(M_X) \geqslant \theta(M_X, N)$ holds for any positive integer N.

If $M_X \leqslant 3$, then

$$\lambda(M_X) = M_X \geqslant M_X/N \geqslant \theta(M_X, N).$$

If $M_X \geqslant 4$ and $N \geqslant \left\lceil \sqrt{M_X/2} \right\rceil$, then

$$\lambda(M_X) \geqslant \frac{M_X}{\left\lceil \sqrt{M_X/2} \right\rceil} \geqslant M_X/N \geqslant \theta(M_X, N).$$

If $M_X \geqslant 4$ and $N \leqslant \left\lfloor \sqrt{M_X/2} \right\rfloor$, then

$$\lambda(M_X) = \max \left\{ 3, M_X/r_X, 2 \left\lfloor \sqrt{M_X/2} \right\rfloor, \frac{M_X}{\left\lceil \sqrt{M_X/2} \right\rceil} \right\}$$
$$\geqslant \max \left\{ 3, M_X/r_X, 2 \left\lfloor \sqrt{M_X/2} \right\rfloor \right\}$$
$$\geqslant \max\{3, M_X/r_X, 2N\}$$
$$\geqslant \theta(M_X, N). \qquad \square$$

PROPOSITION 5.2. — Let (X, Y, Z) be a Fano–Mori triple with $\rho(Y) > 1$ and $m > 0$ an integer. If

$$P_{-m} > \lambda(M_X)m + 1,$$

then $|-mK_X|$ is not composed with a pencil.

Proof. — First the assumption implies that $P_{-m} \geqslant 2$. Hence $\varphi_{-m,X}$ is non-trivial. Assume that $|-mK_X|$ is composed with a pencil. Keep the notation in Subsection 2.1. Take $N_0 = r_X(\pi^*(K_X)^2 \cdot S)$, which is a positive integer (cf. [10, Lemma 2.2]). Then, by Propositions 4.1 and 4.5,

$$\frac{P_{-m} - 1}{m} \leqslant \frac{M_X}{N_0}$$

and

$$\frac{P_{-m} - 1}{m} \leqslant \max\{3, M_X/r_X, 2N_0\}.$$

That is,

$$\frac{P_{-m} - 1}{m} \leqslant \min\{M_X/N_0, \max\{3, M_X/r_X, 2N_0\}\}$$
$$= \theta(M_X, N_0) \leqslant \lambda(M_X),$$

a contradiction. $\qquad \square$

In practice, we need to know the lower bound of P_{-m} which is fairly computable by virtue of Reid's Riemann–Roch formula. Let us recall the following proposition from [12], which we will use to estimate the anti-plurigenus.

PROPOSITION 5.3 (cf. [12, Proof of Proposition 4.3]). — Let X be a terminal weak $\mathbb{Q}$-Fano 3-fold and $t > 0$ a real number. Then, for any integer $n \geqslant t$ and $n \geqslant r_{\max}t/3$, one has

$$P_{-n} \geqslant \frac{1}{12}n(n+1)(2n+1)(-K_X^3) + 1 - \frac{2n}{t}.$$

Proof. — From [12, Proof of Proposition 4.3 (p. 90)], one has

$$l(-n) \leqslant \frac{2(t+1)n}{t}$$

provided that $n \geqslant t$ and $n \geqslant r_{\max}t/3$. It follows from Reid's Riemann–Roch formula that

$$
\begin{aligned}
P_{-n} &= \frac{1}{12}n(n+1)(2n+1)(-K_X^3) + 2n + 1 - l(-n) \\
&\geqslant \frac{1}{12}n(n+1)(2n+1)(-K_X^3) + 1 - \frac{2n}{t}.
\end{aligned}
$$ $\square$

PROPOSITION 5.4. — *Let (X, Y, Z) be a Fano–Mori triple such that $\rho(Y) > 1$. Let $t > 0$ be a real number and m an integer. If $m \geqslant t$, $m \geqslant \frac{r_{\max}t}{3}$, and*

$$m > -\frac{3}{4} + \sqrt{\frac{12}{t \cdot (-K_X^3)} + \frac{6\lambda(M_X)}{-K_X^3} + \frac{1}{16}},$$

then $|-mK_X|$ is not composed with a pencil.

Proof. — Under the assumptions, one has

$$\frac{1}{12}(m+1)(2m+1) > \frac{2}{t \cdot (-K_X^3)} + \frac{\lambda(M_X)}{-K_X^3},$$

and, by Proposition 5.3,

$$P_{-m} > \lambda(M_X)m + 1.$$

Hence $|-mK_X|$ is not composed with a pencil by Proposition 5.2. $\square$

To apply Proposition 5.4, we always use the following lemma to estimate the value of $\lambda(M_X)/(-K_X^3)$.

LEMMA 5.5. — *The following inequalities hold:*

(1) $\dfrac{\lambda(M_X)}{-K_X^3} \leqslant \max\left\{1, \dfrac{3}{-K_X^3}, \sqrt{\dfrac{2r_X}{-K_X^3}}\right\};$

(2) $\lambda(M_X) \leqslant M_X$ *and* $\lambda(M_X)/(-K_X^3) \leqslant r_X.$

Proof. — If $M_X \leqslant 3$, then it is clear. We may assume that $M_X \geqslant 4$. Note that $2 \left\lfloor \sqrt{M_X/2} \right\rfloor \leqslant \sqrt{2M_X}$ and $\frac{M_X}{\left\lceil \sqrt{M_X/2} \right\rceil} \leqslant \sqrt{2M_X}$. Hence

$$\frac{\lambda(M_X)}{-K_X^3} = \frac{1}{-K_X^3} \max \left\{ 3, M_X/r_X, 2 \left\lfloor \sqrt{M_X/2} \right\rfloor, \frac{M_X}{\left\lceil \sqrt{M_X/2} \right\rceil} \right\}$$

$$\leqslant \max \left\{ \frac{3}{-K_X^3}, 1, \frac{\sqrt{2M_X}}{-K_X^3} \right\}$$

$$= \max \left\{ \frac{3}{-K_X^3}, 1, \sqrt{\frac{2r_X}{-K_X^3}} \right\},$$

which implies (1).

Similarly, since $2 \left\lfloor \sqrt{M_X/2} \right\rfloor \leqslant M_X$ and $\frac{M_X}{\left\lceil \sqrt{M_X/2} \right\rceil} \leqslant M_X$, it is clear that $\lambda(M_X) \leqslant M_X$ which implies (2). □

As the last part of this section, we recall the established birationality criterion in [12] along with a newly developed criterion.

ASSUMPTION 5.6. — *Let X be a terminal weak $\mathbb{Q}$-Fano 3-fold and $m_0 > 0$ an integer with $P_{-m_0} \geqslant 2$. Let $m_1 \geqslant m_0$ be another integer with $P_{-m_1} \geqslant 2$ such that $|-m_1 K_X|$ and $|-m_0 K_X|$ are not composed with the same pencil. Pick a generic irreducible element S of $|M_{-m_0}| = \mathrm{Mov}|\lfloor \pi^*(-m_0 K_X) \rfloor|$. Define the real number*

$$\mu_0 = \inf\{t \in \mathbb{Q}^+ \mid t\pi^*(-K_X) - S \sim_{\mathbb{Q}} \text{effective } \mathbb{Q}\text{-divisor}\},$$

and in practice we may choose a suitable rational number $\mu_0' \geqslant \mu_0$ such that

$$\mu_0' \pi^*(-K_X) - S \sim_{\mathbb{Q}} \text{effective } \mathbb{Q}\text{-divisor}.$$

Let ν_0 be an integer such that $h^0(-\nu_0 K_X) > 0$. Set

$$a(m_0) = \begin{cases} 6, & \text{if } m_0 \geqslant 2; \\ 1, & \text{if } m_0 = 1, \end{cases}$$

and the positive integer $N_0 = r_X(\pi^*(K_X)^2 \cdot S)$.

THEOREM 5.7 ([12, Theorem 5.11]). — *Let X be a terminal weak $\mathbb{Q}$-Fano 3-fold and keep Assumption 5.6. Then $\varphi_{-m,X}$ is birational if one of the following holds:*

(1) $m \geqslant \max\{m_0 + m_1 + a(m_0), \lfloor 3\mu_0 \rfloor + 3m_1\}$;
(2) $m \geqslant \max\{m_0 + m_1 + a(m_0), \lfloor \frac{5}{3}\mu_0 + \frac{5}{3}m_1 \rfloor, \lfloor \mu_0 \rfloor + m_1 + 2r_{\max}\}$;
(3) $m \geqslant \max\{m_0 + m_1 + a(m_0), \lfloor \mu_0 \rfloor + m_1 + 2\nu_0 r_{\max}\}$.

Remark 5.8 (cf. [12, Remark 5.3]). — Most of the time in practice, we will take m_0 to be the minimal positive integer with $P_{-m_0} \geqslant 2$. By definition one can always choose $\mu'_0 = m_0$ and have $\mu_0 \leqslant m_0$. If $|-m_0 K_X|$ is composed with a pencil, then we can choose $\mu'_0 = \frac{m_0}{P_{-m_0}-1}$ and $\mu_0 \leqslant \frac{m_0}{P_{-m_0}-1}$. Moreover, if for some integer k, $|-k K_X|$ and $|-m_0 K_X|$ are composed with the same pencil, then we can choose $\mu'_0 = \frac{k}{P_{-k}-1}$ and $\mu_0 \leqslant \frac{k}{P_{-k}-1}$.

When both r_X and $r_{\max}$ are small, we feel that the following new criterion is very helpful.

THEOREM 5.9. — *Let X be a terminal weak $\mathbb{Q}$-Fano 3-fold and keep Assumption 5.6. Then $\varphi_{-m,X}$ is birational if*

$$m \geqslant \max \left\{ m_0 + a(m_0), \lceil \mu'_0 \rceil + 4\nu_0 r_{\max} - 1, \left\lfloor \mu'_0 + \sqrt{8 r_X/N_0} \right\rfloor \right\}.$$

LEMMA 5.10 (see [21, Proposition 4] or [11, Lemma 2.6]). — *Let S be a smooth projective surface. Let L be a nef and big $\mathbb{Q}$-divisor on S satisfying the following conditions:*

(1) *$L^2 > 8$;*
(2) *$(L \cdot C_P) \geqslant 4$ for all irreducible curves C_P passing through any very general point $P \in S$.*

Then the linear system $|K_S + \lceil L \rceil|$ separates two distinct points in very general positions. Consequently, $|K_S + \lceil L \rceil|$ gives a birational map.

Proof of Theorem 5.9. — By [12, Lemma 5.2], the birationality of $\varphi_{-m,X}$ is equivalent to the birationality of Φ_{Λ_m} defined by the linear system $\Lambda_m = |K_W + \lceil (m+1)\pi^*(-K_X) \rceil|$ on W.

By [12, Proposition 5.8], Λ_m can distinguish different generic irreducible elements of $|M_{-m_0}|$ since $m \geqslant m_0 + a(m_0)$. Hence the usual birationality principle (see, for instance, [8, 2.7]) reduces the birationality of Φ_{Λ_m} to that of $\Phi_{\Lambda_m}|_S$ for a generic irreducible element S of $|M_{-m_0}|$.

Now we will show that $\Phi_{\Lambda_m}|_S$ is birational under the assumption of the theorem.

By the definition of μ'_0, we have

$$\mu'_0 \pi^*(-K_X) \sim_{\mathbb{Q}} S + E$$

for an effective $\mathbb{Q}$-divisor E. Re-modify the resolution π in Subsection 2.1 so that E has simple normal crossing support.

For the given integer $m > 0$, we have

$$(5.1) \quad |K_Y + \lceil (m+1)\pi^*(-K_X) - E \rceil| \preceq |K_Y + \lceil (m+1)\pi^*(-K_X) \rceil|.$$

Since by assumption $m \geqslant \mu'_0$, the $\mathbb{Q}$-divisor

$$(m+1)\pi^*(-K_X) - E - S \equiv (m+1-\mu'_0)\pi^*(-K_X)$$

is nef and big and thus

$$H^1(Y, K_Y + \lceil (m+1)\pi^*(-K_X) - E \rceil - S) = 0$$

by Kawamata–Viehweg vanishing theorem. Hence we have the surjective map

(5.2) $\qquad H^0(Y, K_Y + \lceil (m+1)\pi^*(-K_X) - E \rceil) \longrightarrow H^0(S, K_S + L_m)$

where

(5.3) $\qquad L_m := (\lceil (m+1)\pi^*(-K_X) - E \rceil - S)|_S \geqslant \lceil \mathcal{L}_m \rceil$

and $\mathcal{L}_m := ((m+1)\pi^*(-K_X) - E - S)|_S$ which is a nef and big $\mathbb{Q}$-divisor on S.

By relations (5.1)–(5.3), to show that $\Phi_{\Lambda_m}|_S$ is birational, it suffices to show that $|K_S + \lceil \mathcal{L}_m \rceil|$ gives a birational map. Note that

$$\mathcal{L}_m^2 = (m+1-\mu'_0)^2(\pi^*(K_X)|_S)^2 = \frac{(m+1-\mu'_0)^2 N_0}{r_X} > 8$$

since $m \geqslant \left\lfloor \mu'_0 + \sqrt{8r_X/N_0} \right\rfloor$. Now consider an irreducible curve C_P passing through a very general point $P \in S$. Recall that $-\nu_0 K_X \sim D$ for some effective Weil divisor D. Take $q : Y \to X$ to be the resolution of isolated singularities and we may assume that W dominates Y by $p : W \to Y$. Then we write

$$q^*D = D_Y + \sum \frac{a_i}{r_i} E_i.$$

Here D_Y is the strict transform of D on Y and E_i is the exceptional divisor over an isolated singular point of index r_i for some $r_i \in B_X$ and a_i is a positive integer. Then

$$\pi^*D = p^*D_Y + \sum \frac{a_i}{r_i} p^*E_i.$$

Since P is a very general point, $(p^*D_Y \cdot C_P)$ and $(p^*E_i \cdot C_P)$ are non-negative integers, and at least one of them is positive since $(\pi^*D \cdot C_P) > 0$ by the fact that D is nef and big. Hence $(\pi^*D \cdot C_P) \geqslant \frac{1}{r_{\max}}$. Consequently,

$$(\mathcal{L}_m \cdot C_P) = (m+1-\mu'_0)(\pi^*(-K_X) \cdot C_P) \geqslant \frac{m+1-\mu'_0}{\nu_0 r_{\max}} \geqslant 4.$$

Hence, by Lemma 5.10, $|K_S + \lceil \mathcal{L}_m \rceil|$ gives a birational map and so does $\varphi_{-m,X}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Finally we explain the strategy to assert the birationality. Given a certain set of terminal weak $\mathbb{Q}$-Fano 3-folds, or a set of geometric weighted baskets, we can find a number $m_0 > 0$ such that $P_{-m_0} \geqslant 2$. Then we can find another number $m_1 > 0$, by using either the explicit computation of Reid's Riemann–Roch formula or Proposition 5.4, so that $|-m_1 K|$ is not composed with a pencil. Meanwhile we have the estimate of μ_0 according to Remark 5.8. With all the information, we may apply either Theorem 5.7 or 5.9 to obtain the anti-pluricanonical birationality except the fact that to classify weighted baskets with pre-assigned invariants might be a tedious work.

6. Proof of the main theorems

In this section, we prove the main theorems.

6.1. Proof of Theorems 1.9 and 1.10

Let V be a canonical weak $\mathbb{Q}$-Fano 3-fold and Y a K-Mori fiber space of V (by Remark 3.6, such Y always exists). By Theorem 1.12, V is birational to a Fano–Mori triple (X, Y, Z).

By Theorem 1.13, the terminal weak $\mathbb{Q}$-Fano 3-fold X in the Fano–Mori triple (X, Y, Z) satisfies the properties we require in the Theorem 1.9.

By Proposition 3.10 (3), $\varphi_{-m,X}$ and $\varphi_{-m,Y}$ share the same birational properties. Hence Theorem 1.10 also follows from Theorem 1.13. $\qquad\square$

6.2. Proof of Theorem 1.13

Let (X, Y, Z) be a Fano–Mori triple.

If $\rho(Y) = 1$, then by Remark 3.8, $X \cong Y \cong Z$ is a $\mathbb{Q}$-factorial terminal $\mathbb{Q}$-Fano 3-fold of Picard number one. Then statement (1) follows from Theorem 1.6 and Proposition 2.6 (1). Statement (2) follows from Theorem 1.3.

From now on we may assume that $\rho(Y) > 1$. Consider the terminal weak $\mathbb{Q}$-Fano 3-fold X. If $P_{-2} = 0$ or $r_X = 840$, then the theorem follows from Theorem 6.4 or 6.6. Hence we may assume that $P_{-2} > 0$ and $r_X \leqslant 660$ by Proposition 2.6.

For statement (1), if $-K_X^3 \geqslant 1/30$, it follows from Lemma 6.9; if $-K_X^3 < 1/30$, it follows from Theorem 6.12 (1).

For statement (2), if $-K_X^3 \geqslant 0.21$, it follows from Theorem 6.7; if $P_{-1} = 0$ and $-K_X^3 < 0.21$, it follows from Theorem 6.8; if $P_{-1} > 0$ and $1/30 \leqslant -K_X^3 < 0.21$, it follows from Theorem 6.11; if $P_{-1} > 0$ and $-K_X^3 < 1/30$, it follows from Theorem 6.12.

In summary, the theorem is proved.

Finally, we remark that from the proof of statement (2), φ_{-51} is not birational only if when $-K_X^3 = 1/330$ and X has Reid basket $\{(1,2),(2,5),(1,3),(2,11)\}$, which appears in the proof of Theorem 6.12. $\qquad\square$

6.3. The case $P_{-8} = 2$ and two lemmas

THEOREM 6.1. — *Let (X, Y, Z) be a Fano–Mori triple such that $\rho(Y) > 1$. Assume that $P_{-8} = 2$. Then φ_{-m} is birational for all $m \geqslant 51$.*

Proof. — Note that $P_{-2} \leqslant 1$ since $P_{-8} \geqslant 4P_{-2} - 3$.

If $P_{-2} = 0$, then By [7, Theorem 3.5], any geometric basket of a terminal weak $\mathbb{Q}$-Fano 3-fold with $P_{-1} = P_{-2} = 0$ and $P_{-8} = 2$ is among the following list:

Table 6.1

B_X	$-K^3$	M_X	$\lambda(M_X)$	n_1	m_0	$r_{\max}$	n_2
$\{2 \times (1,2), 3 \times (2,5), (1,3), (1,4)\}$	1/60	1	1	20	8	5	46
$\{5 \times (1,2), 2 \times (1,3), (2,7), (1,4)\}$	1/84	1	1	22	8	7	50
$\{5 \times (1,2), 2 \times (1,3), (3,11)\}$	1/66	1	1	20	8	11	50
$\{5 \times (1,2), (1,3), (3,10), (1,4)\}$	1/60	1	1	20	8	10	48

In Table 6.1, for each basket B_X, we can compute M_X and $\lambda(M_X)$, then find n_1 such that $P_{-n_1} > \lambda(M_X)n_1 + 1$ where P_{-n_1} is computed by Reid's Riemann–Roch formula. Hence, by Proposition 5.2, $\dim \overline{\varphi_{-n_1}(X)} > 1$. Again by Reid's Riemann–Roch formula, we may find m_0 such that $P_{-m_0} \geqslant 2$. Then, by Proposition 5.7 (2), take $m_1 = n_1$ and $\mu_0 \leqslant m_0$, we get the integer n_2 such that φ_{-m} is birational for all $m \geqslant n_2$.

We consider the case $P_{-2} = 1$ from now on. We follow the argument of [7, Proof of Theorem 3.10] to classify all possible geometric baskets with $P_{-2} = 1$ and $P_{-8} = 2$. Note that in this case $P_{-1} \leqslant 1$ and $P_{-4} = 1$.

If $P_{-1} = 0$, then by [7, Proof of Theorem 3.10, Case 1], we have $P_{-3} = 0$ and $\sigma_5 = 1$. Since we know the values of P_{-n} for $1 \leqslant n \leqslant 4$ and σ_5, $B^{(0)} = \{9 \times (1,2), (1,3), (1,s)\}$ for some $s \geqslant 5$. If $s \geqslant 7$, then $P_{-8} \geqslant \widetilde{P}_{-8}(B^{(0)}) \geqslant 4$, a contradiction. If $s < 7$, then $-K^3(B^{(0)}) \leqslant 0$. It is easy to see that B_X is dominated by either $B' = \{8 \times (1,2), (2,5), (1,6)\}$ or

$\{7\times(1,2),(3,7),(1,5)\}$. Note that for both cases, we have $P_{-8} \geqslant \widetilde{P}_{-8}(B') \geqslant 3$, a contradiction.

If $P_{-1} = 1$, then $P_{-3} = P_{-4} = 1$ and

$$1 \leqslant P_{-5} \leqslant P_{-6} \leqslant P_{-7} \leqslant 2.$$

We have $(n_{1,2}^0, n_{1,3}^0, n_{1,4}^0) = (2, 2, 2 - \sigma_5)$. Hence $\sigma_5 \leqslant 2$. On the other hand, $\epsilon_6 = 0$ gives $\epsilon = P_{-6} - P_{-5} + 2 \geqslant 2$, which implies that $\sigma_5 > 0$.

If $(\sigma_5, P_{-5}) = (2, 2)$, then $\epsilon_5 = 1$, $P_{-6} = 2$, and $\epsilon = 2$, which implies that $n_{1,5}^0 = 2$. Hence $B^{(5)} = \{(1,2), (2,5), (1,3), 2 \times (1,5)\}$. But all the packings have $-K^3 < 0$, which is absurd.

If $(\sigma_5, P_{-5}) = (2, 1)$, then $\epsilon_5 = 0$ and

$$n_{1,5}^0 = 2\sigma_5 - \epsilon = 2 - P_{-6} + P_{-5} > 0.$$

Hence $B^{(5)} = \{2 \times (1,2), 2 \times (1,3), (1,5), (1,s)\}$ for some $s \geqslant 5$. If $s \leqslant 7$, then all the further packings have $-K^3 < 0$, which is absurd. Hence $s \geqslant 8$ and $B_X = B^{(5)}$. Note that in this case $s = 8, 9, 10$ by $\gamma \geqslant 0$. It is easy to check that $P_{-8} = 3$, a contradiction.

If $(\sigma_5, P_{-5}) = (1, 2)$, then $\epsilon_5 = 2$. Hence $B^{(5)} = \{2 \times (2,5), (1,4), (1,s)\}$ for some $s \geqslant 5$. Then $\gamma \geqslant 0$ implies that $s \leqslant 10$. If $s \leqslant 6$, then all the further packings have $-K^3 < 0$, which is absurd. Hence $s \geqslant 7$ and $B_X = B^{(5)}$. It is easy to check that $P_{-8} \geqslant 4$, a contradiction.

If $(\sigma_5, P_{-5}) = (1, 1)$, then $\epsilon = P_{-6} - P_{-5} + 2$ implies that $P_{-6} = 1 - n_{1,5}^0$, which means that $P_{-6} = 1$ and $n_{1,5}^0 = 0$. Hence $B^{(5)} = \{(1,2), (2,5), (1,3), (1,4), (1,s)\}$ for some $s \geqslant 6$. Note that $s = 6, 7$ is impossible since all further packings have $-K^3 < 0$. Hence we have $8 \leqslant s \leqslant 11$ by $\gamma \geqslant 0$. Also we have $\epsilon_7 = P_{-7} - 1$ and $\epsilon_8 = 1 - P_{-7}$, which means that $\epsilon_7 = \epsilon_8 = 0$. Hence $B_X = B^{(8)} = B^{(5)} = \{(1,2), (2,5), (1,3), (1,4), (1,s)\}$ for $8 \leqslant s \leqslant 11$. Note that $s = 8$ is absurd since in this case $-K^3 < 0$.

In summary, all possible geometric baskets with $P_{-2} = 1$ and $P_{-8} = 2$ are

$$B_X = \{(1,2), (2,5), (1,3), (1,4), (1,s)\}$$

for $9 \leqslant s \leqslant 11$.

If $B_X = \{(1,2), (2,5), (1,3), (1,4), (1,9)\}$, then $-K_X^3 = 1/180$, $M_X = \lambda(M_X) = 1$. Then $P_{-30} = 33 > 30\lambda(M_X) + 1$ where P_{-30} is computed by Reid's Riemann–Roch formula. Hence by Proposition 5.2, $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant 30$ since $P_{-1} > 0$. Recall that $P_{-8} = 2$. Take $m_0 = 8$ and $\nu_0 = 1$. Note that $P_{-19} = 11$ by Reid's Riemann–Roch formula. If $|-19K_X|$ and $|-8K_X|$ are not composed with the same pencil, then we may take $\mu_0 \leqslant 8$ and $m_1 = 19$, and by Proposition 5.7(3), φ_{-m} is birational for $m \geqslant 45$. If $|-19K_X|$ and $|-8K_X|$ are composed with the same pencil, then we may take

$\mu_0 \leqslant \frac{19}{P_{-19}-1} = \frac{19}{10}$ by Remark 5.8 and $m_1 = 30$, then by Proposition 5.7 (3), φ_{-m} is birational for $m \geqslant 49$.

If $B_X = \{(1,2),(2,5),(1,3),(1,4),(1,10)\}$, then $-K_X^3 = 1/60$, $M_X = \lambda(M_X) = 1$. Then $P_{-19} = 22 > 19\lambda(M_X) + 1$ where P_{-19} is computed by Reid's Riemann–Roch formula. Hence by Proposition 5.2, $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant 19$ since $P_{-1} > 0$. Recall that $P_{-8} = 2$. Then by Proposition 5.7 (3), take $m_1 = 19$, $\mu_0 \leqslant m_0 = 8$, and $\nu_0 = 1$, we get the integer n_2 such that φ_{-m} is birational for all $m \geqslant 47$.

If $B_X = \{(1,2),(2,5),(1,3),(1,4),(1,11)\}$, then $-K_X^3 = 17/660$. Recall that $P_{-8} = 2$. We may take $m_0 = 8$ and $\nu_0 = 1$. Note that $P_{-21} = 43$. If $|-21K_X|$ and $|-8K_X|$ are not composed with the same pencil, then we may take $\mu_0 \leqslant 8$ and $m_1 = 21$, and by Proposition 5.7 (3), φ_{-m} is birational for $m \geqslant 51$. Hence we may assume that $|-21K_X|$ and $|-8K_X|$ are composed with the same pencil, and we may take $\mu_0 \leqslant \mu_0' = \frac{21}{P_{-21}-1} = \frac{1}{2}$ by Remark 5.8. If $|-26K_X|$ is not composed with a pencil, then we may take $m_1 = 26$ and by Proposition 5.7 (3), φ_{-m} is birational for $m \geqslant 48$. If $|-26K_X|$ is composed with a pencil, by Proposition 4.5 and keep the same notation as in Subsection 2.1, one has

$$3 < \frac{80}{26} = \frac{P_{-26}-1}{26} \leqslant \max\{3, -K_X^3, 2N_0\},$$

where $N_0 = r_X(\pi^*(K_X)^2 \cdot S)$. This implies that $N_0 \geqslant 2$, and by Proposition 5.9, φ_{-m} is birational for $m \geqslant 51$.

Combining all above discussions, the proof is completed. $\qquad\square$

LEMMA 6.2. — *Let (X, Y, Z) be a Fano–Mori triple such that $\rho(Y) > 1$. Assume that $r_X \leqslant 165$. Then $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant 37$;*

Proof. — Note that in this case $-K_X^3 \geqslant 1/165$. By Lemma 5.5,

$$\frac{\lambda(M_X)}{-K_X^3} \leqslant r_X \leqslant 165.$$

Take $t = 37/8$, then

$$-\frac{3}{4} + \sqrt{\frac{12}{t(-K_X^3)} + \frac{6\lambda(M_X)}{-K_X^3} + \frac{1}{16}} < 37.$$

By Proposition 5.4, $|-mK_X|$ is not composed with a pencil for $m \geqslant 37$. $\quad\square$

LEMMA 6.3. — *Let (X, Y, Z) be a Fano–Mori triple such that $\rho(Y) > 1$. Assume that one of the following conditions holds:*

(1) $r_X \leqslant 69$ and $r_{\max} \leqslant 12$; or
(2) $P_{-1} > 0$, $r_X \leqslant 287$, and $r_{\max} \leqslant 12$.

Then $\varphi_{-m,X}$ is birational for all $m \geqslant 51$.

Proof. — Recall that $P_{-8} \geqslant 2$ by Proposition 2.6. If $P_{-8} = 2$, then we are done by Theorem 6.1. Hence we may assume that $P_{-8} \geqslant 3$.

(1). — Note that in this case $-K_X^3 \geqslant 1/69$. By Lemma 5.5,

$$\frac{\lambda(M_X)}{-K_X^3} \leqslant r_X \leqslant 69.$$

Take $t = 23/4$, then

$$-\frac{3}{4} + \sqrt{\frac{12}{t(-K_X^3)} + \frac{6\lambda(M_X)}{-K_X^3} + \frac{1}{16}} < 23.$$

By Proposition 5.4, $|-mK_X|$ is not composed with a pencil for $m \geqslant 23$.

If $|-8K_X|$ is not composed with a pencil, then we may take $\mu_0 \leqslant 8$ and $m_1 = 8$, and by Proposition 5.7(2), φ_{-m} is birational for $m \geqslant 40$. If $|-8K_X|$ is composed with a pencil, then we may take $\mu_0 \leqslant \frac{8}{P_{-8}-1} \leqslant 4$ by Remark 5.8 and $m_1 = 23$, then by Proposition 5.7(2), φ_{-m} is birational for $m \geqslant 51$.

(2). — In this case, we may take $\nu_0 = 1$ and $m_0 = 8$. If $|-8K_X|$ is not composed with a pencil, then we may take $\mu_0 \leqslant 8$ and $m_1 = 8$, and by Proposition 5.7(3), φ_{-m} is birational for $m \geqslant 40$. If $|-8K_X|$ is composed with a pencil, then we may take $\mu_0' = \frac{8}{P_{-8}-1} \leqslant 4$ by Remark 5.8 and $N_0 \geqslant 1$, and by Proposition 5.9, φ_{-m} is birational for $m \geqslant 51$. $\qquad\square$

6.4. The case $P_{-2} = 0$

THEOREM 6.4. — *Let (X, Y, Z) be a Fano–Mori triple such that $\rho(Y) > 1$. Assume that $P_{-2} = 0$. Then*

(1) $\dim \overline{\varphi_{-m}(X)} > 1$ *for all $m \geqslant 37$;*
(2) $\varphi_{-m,X}$ *is birational for all $m \geqslant 51$.*

Proof. — Since $P_{-2} = 0$, $P_{-1} = 0$. If $r_X \leqslant 69$ and $r_{\max} \leqslant 12$, then we are done by Lemmas 6.2 and 6.3(1). By [7, Theorem 3.5], any geometric basket of a terminal weak $\mathbb{Q}$-Fano 3-fold with $P_{-1} = P_{-2} = 0$ and with either $r_X > 69$ or $r_{\max} > 12$ is among the following list:

Table 6.2

B_X	$-K^3$	M_X	$\lambda(M_X)$	n_1	m_0	$r_{\max}$	n_2
$\{5 \times (1,2), 2 \times (1,3), (2,7), (1,4)\}$	1/84	1	1	22	8	7	50
$\{3 \times (1,2), (5,14), 2 \times (1,3)\}$	1/21	2	2	16	6	14	50
$\{(1,2), (3,7), (2,5), 4 \times (1,3)\}$	17/210	17	17/3	20	5	7	41
$\{2 \times (1,2), (2,5), (3,8), 3 \times (1,3)\}$	3/40	9	4	18	5	8	39
$\{2 \times (1,2), (5,13), 3 \times (1,3)\}$	1/13	6	3	15	5	13	46

In Table 6.2, for each basket B_X, we can compute M_X and $\lambda(M_X)$, then find n_1 such that $P_{-n} > \lambda(M_X)n + 1$ for $n_1 \leqslant n \leqslant n_1 + 5$ where P_{-n} is computed by Reid's Riemann–Roch formula. Hence by Proposition 5.2, $\dim \overline{\varphi_{-n}(X)} > 1$ for all $n_1 \leqslant n \leqslant n_1 + 5$. Since $P_{-m} > 0$ for all $m \geqslant 6$ by Proposition 2.6, this implies that $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant n_1$. Again by Reid's Riemann–Roch formula, we may find m_0 such that $P_{-m_0} \geqslant 2$. Then, by Proposition 5.7 (2), take $m_1 = n_1$ and $\mu_0 \leqslant m_0$, we get the integer n_2 such that φ_{-m} is birational for all $m \geqslant n_2$. □

6.5. The case $r_X = 840$

LEMMA 6.5. — Let X be a terminal weak $\mathbb{Q}$-Fano 3-fold with $r_X = 840$. Then

(1) $P_{-1} \geqslant 1$;
(2) $-K_X^3 \geqslant \frac{47}{840}$;
(3) $P_{-2} \geqslant 2$ unless B_X is in Table 6.3.

Table 6.3

No.	B_X
1	$\{(1,2), (1,3), (2,5), (1,7), (1,8)\}$
2	$\{(1,2), (1,3), (1,5), (2,7), (1,8)\}$
3	$\{(1,3), (1,5), (1,7), (3,8)\}$
4	$\{(1,3), (1,5), (3,7), (1,8)\}$
5	$\{(1,3), (2,5), (2,7), (1,8)\}$

Proof. — By [12, Proof of Proposition 2.4 (p. 67)], we know that the set of local indices $\{r_i\}$ of X is either $\{2,3,5,7,8\}$ or $\{3,5,7,8\}$, that is, $B_X = \{(1,2), (1,3), (a,5), (b,7), (c,8)\}$ or $\{(1,3), (a,5), (b,7), (c,8)\}$ for some $a \in \{1,2\}$, $b \in \{1,2,3\}$, $c \in \{1,3\}$. To unify the notation, we write $B_X = \{k \times (1,2), (1,3), (a,5), (b,7), (c,8)\}$, where $k \in \{0,1\}$.

By inequality (2.4),

$$2P_{-1} > \sigma'(B_X) - \sigma(B_X) + 6$$

$$= -\frac{k}{2} - \frac{2}{3} - \frac{a(5-a)}{5} - \frac{b(7-b)}{7} - \frac{c(8-c)}{8} + 6$$

$$\geqslant -\frac{1}{2} - \frac{2}{3} - \frac{6}{5} - \frac{12}{7} - \frac{15}{8} + 6 > 0.$$

Hence $P_{-1} \geqslant 1$, which proves statement (1).

If $P_{-1} \geqslant 2$, then

$$-K_X^3 = 2P_{-1} - \sigma'(B_X) + \sigma(B_X) - 6$$

$$\geqslant 4 + \frac{k}{2} + \frac{2}{3} + \frac{a(5-a)}{5} + \frac{b(7-b)}{7} + \frac{c(8-c)}{8} - 6$$

$$\geqslant \frac{2}{3} + \frac{4}{5} + \frac{6}{7} + \frac{7}{8} - 2 > 1.$$

If $P_{-1} = 1$, then by equality (2.2),

$$k + 1 + a + b + c = \sigma(B_X) = 10 - 5P_{-1} + P_{-2} \geqslant 6.$$

When $k = 1$, then at least one of a, b, c is greater than 1. In particular, if $a \geqslant 2$, then

$$-K_X^3 = 2P_{-1} - \sigma'(B_X) + \sigma(B_X) - 6$$

$$\geqslant 2 + \frac{k}{2} + \frac{2}{3} + \frac{a(5-a)}{5} + \frac{b(7-b)}{7} + \frac{c(8-c)}{8} - 6$$

$$\geqslant \frac{1}{2} + \frac{2}{3} + \frac{6}{5} + \frac{6}{7} + \frac{7}{8} - 4 = \frac{83}{840}.$$

Similarly we see $-K^3 \geqslant \frac{227}{840}$ when $b \geqslant 2$ or $c = 3$. When $k = 0$, we see $-K_X^3 \geqslant \frac{47}{840}$ in the similar way if $b = 3$ or $c = 3$. The remaining cases are $(a, b, c) = (1, 2, 1), (2, 1, 1), (2, 2, 1)$ and we get either $-K_X^3 = \frac{143}{840}$ or $-K_X^3 < 0$ (impossible cases). This asserts statement (2).

Finally, we consider the case $P_{-2} < 2$ which forces $P_{-1} = P_{-2} = 1$. By equality (2.2),

$$k + 1 + a + b + c = \sigma(B_X) = 10 - 5P_{-1} + P_{-2} = 6.$$

Hence B_X can only be in Table 6.3. □

THEOREM 6.6. — *Let (X, Y, Z) be a Fano–Mori triple such that $\rho(Y) > 1$. Assume that $r_X = 840$. Then*

(1) *$\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant 32$;*

(2) *$\varphi_{-m,X}$ is birational for all $m \geqslant 50$.*

Proof. — Recall that in this case $r_{\max} = 8$. By Lemma 6.5, $P_{-1} \geqslant 1$ and $-K_X^3 \geqslant 47/840$.

By Lemma 5.5,

$$\frac{\lambda(M_X)}{-K_X^3} \leqslant \max\left\{1, \frac{3}{47/840}, \sqrt{\frac{2 \times 840}{47/840}}\right\} < 174.$$

Take $t = 12$, then

$$-\frac{3}{4} + \sqrt{\frac{12}{t(-K_X^3)} + \frac{6\lambda(M_X)}{-K_X^3} + \frac{1}{16}} < 32.$$

By Proposition 5.4, $|-mK_X|$ is not composed with a pencil for $m \geqslant 32$.

For statement (2), first we consider the case $P_{-2} \geqslant 2$. In this case, we may take $\mu_0 \leqslant m_0 = 2$, $m_1 = 32$, and $\nu_0 = 1$. Then by Proposition 5.7(3), φ_{-m} is birational for $m \geqslant 50$.

Then we consider the case $P_{-2} < 2$, that is, $P_{-1} = P_{-2} = 1$. In this case we are dealing with baskets listed in Table 6.3. In the following Table 6.4, for each basket B_X in Table 6.3, we can compute M_X and $\lambda(M_X)$, then find n_1 such that $P_{-n_1} > \lambda(M_X)n_1 + 1$ where P_{-n_1} is computed by Reid's Riemann–Roch formula. Hence by Proposition 5.2, $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant n_1$ since $P_{-1} > 0$. Again by Reid's Riemann–Roch formula, we may find m_0 such that $P_{-m_0} \geqslant 2$. Then by Proposition 5.7(3), take $m_1 = n_1$, $\mu_0 \leqslant m_0$, and $\nu_0 = 1$, we get the integer n_2 such that φ_{-m} is birational for all $m \geqslant n_2$.

Table 6.4

No.	B_X	$-K^3$	M_X	$\lambda(M_X)$	n_1	m_0	$r_{\max}$	n_2
1	$\{(1,2),(1,3),(2,5),(1,7),(1,8)\}$	83/840	83	12	27	5	8	48
2	$\{(1,2),(1,3),(1,5),(2,7),(1,8)\}$	227/840	227	227/11	21	3	8	40
3	$\{(1,3),(1,5),(1,7),(3,8)\}$	167/840	167	18	23	3	8	42
4	$\{(1,3),(1,5),(3,7),(1,8)\}$	47/840	47	47/5	31	5	8	52
5	$\{(1,3),(2,5),(2,7),(1,8)\}$	143/840	143	16	23	3	8	42

Note that, for case No. 4, we need further discussion. We know $P_{-5} = 2$ and $P_{-11} = 16$. Take $m_0 = 5$ and $\nu_0 = 1$. If $|-11K_X|$ and $|-5K_X|$ are not composed with the same pencil, then we may take $m_1 = 11$ and $\mu_0 \leqslant 5$, and by Proposition 5.7(3), φ_{-m} is birational for all $m \geqslant 32$; if $|-11K_X|$ and $|-5K_X|$ are composed with the same pencil, then we may take $m_1 = 31$ and $\mu_0 \leqslant \frac{11}{15}$ by Remark 5.8, and by Proposition 5.7(3), φ_{-m} is birational for all $m \geqslant 47$.

Combining all above cases, the proof is completed. □

6.6. The case $-K_X^3 \geqslant 0.21$

THEOREM 6.7. — *Let (X,Y,Z) be a Fano–Mori triple such that $\rho(Y)>1$. Assume that $r_X \leqslant 660$. Assume that one of the following holds*

(1) $r_{\max} \leqslant 13$ *and* $-K_X^3 \geqslant 0.12$; *or*
(2) $r_{\max} \geqslant 14$ *and* $-K_X^3 \geqslant 0.21$.

Then $\varphi_{-m,X}$ is birational for all $m \geqslant 51$.

Proof.

(1). — If $r_{\max} \leqslant 13$ and $-K_X^3 \geqslant 0.12$, recall that $r_X \leqslant 660$ by assumption. By Lemma 5.5,

$$\frac{\lambda(M_X)}{-K_X^3} \leqslant \max\left\{1, \frac{3}{0.12}, \sqrt{\frac{2 \times 660}{0.12}}\right\} < 105.$$

Take $t = 75/13$, then

$$-\frac{3}{4} + \sqrt{\frac{12}{t(-K_X^3)} + \frac{6\lambda(M_X)}{-K_X^3} + \frac{1}{16}} < 25.$$

By Proposition 5.4, $|-mK_X|$ is not composed with a pencil for $m \geqslant 25$.

Now by Proposition 5.3, taking $t = 30/13$ and using the fact that $(-K_X)^3 \geqslant 0.12$, $r_X \leqslant 660$, and $r_{\max} \leqslant 13$, we have $P_{-10} \geqslant 16$. We may take $m_0 = 8$ since $P_{-8} \geqslant 2$. If $|-8K_X|$ and $|-10K_X|$ are not composed with the same pencil, take $m_1 = 10$ and $\mu_0 \leqslant 8$; if $|-8K_X|$ and $|-10K_X|$ are composed with the same pencil, take $m_1 = 25$ and $\mu_0 \leqslant \frac{10}{15}$ by Remark 5.8. Then by Proposition 5.7 (2), φ_{-m} is birational for $m \geqslant 51$.

(2). — If $r_{\max} \geqslant 14$ and $-K_X^3 \geqslant 0.21$, then by [12, Proof of Theorem 1.8, Case II (p. 105)], $r_X \leqslant 240$. By Lemma 5.5,

$$\frac{\lambda(M_X)}{-K_X^3} \leqslant \max\left\{1, \frac{3}{0.21}, \sqrt{\frac{2 \times 240}{0.21}}\right\} < 48.$$

Note that $r_{\max} \leqslant 24$. Take $t = 17/8$, then

$$-\frac{3}{4} + \sqrt{\frac{12}{t(-K_X^3)} + \frac{6\lambda(M_X)}{-K_X^3} + \frac{1}{16}} < 17.$$

By Proposition 5.4, $|-mK_X|$ is not composed with a pencil for $m \geqslant 17$.

Now by Proposition 5.3, taking $t = 1$ and using the fact that $(-K_X)^3 \geqslant 0.21$, $r_X \leqslant 240$, and $r_{\max} \leqslant 24$, we have $P_{-8} \geqslant 7$. Again by Proposition 5.3, taking $t = 11/8$, we have $P_{-11} \geqslant 39$. We may take $m_0 = 8$. If $|-8K_X|$ is not composed with a pencil, take $m_1 = 8$ and $\mu_0 \leqslant 8$; if $|-8K_X|$ is composed with a pencil, but $|-8K_X|$ and $|-11K_X|$ are not composed with

the same pencil, take $m_1 = 11$ and $\mu_0 \leqslant \frac{8}{6}$ by Remark 5.8; if $|-8K_X|$ and $|-11K_X|$ are composed with the same pencil, take $m_1 = 17$ and $\mu_0 \leqslant \frac{11}{38}$ by Remark 5.8. Then by Proposition 5.7 (1), $\varphi_{-m,X}$ is birational for $m \geqslant 51$. □

6.7. The case $P_{-1} = 0$, $P_{-2} > 0$ and $-K_X^3 < 0.21$

We mainly apply Chen–Chen's method to classify all possible baskets.

THEOREM 6.8. — Let (X, Y, Z) be a Fano–Mori triple such that $\rho(Y) > 1$. Assume that $P_{-1} = 0$, $P_{-2} > 0$, and $-K_X^3 < 0.21$. Then $\varphi_{-m,X}$ is birational for all $m \geqslant 51$.

Proof. — By [7, Inequality (4.1)],

$$0.21 > -K_X^3 \geqslant \frac{1}{12}(-1 - P_{-1} - P_{-2} + P_{-4}),$$

which means that

$$(6.1) \qquad P_{-4} \leqslant P_{-2} + 3.$$

Since $P_{-1} = 0$, the basket $B^{(0)}$ has datum

$$\begin{cases} n_{1,2}^0 = 5 + 4P_{-2} - P_{-3}; \\ n_{1,3}^0 = 4 - 2P_{-2} + 3P_{-3} - P_{-4}; \\ n_{1,4}^0 = 1 - P_{-2} - 2P_{-3} + P_{-4} - \sigma_5. \end{cases}$$

By Lemma 2.5, $B^{(0)}$ satisfies inequality (2.3) and thus

$$0 \leqslant \gamma(B^{(0)}) = \sum_{r \geqslant 2} \left(\frac{1}{r} - r \right) n_{1,r}^0 + 24$$

$$\leqslant \sum_{r=2,3,4} \left(\frac{1}{r} - r \right) n_{1,r}^0 - \frac{24}{5}\sigma_5 + 24$$

$$= \frac{25}{12} + \frac{37}{12}P_{-2} + P_{-3} - \frac{13}{12}P_{-4} - \frac{21}{20}\sigma_5.$$

Hence, by $n_{1,3}^0 \geqslant 0$ and $n_{1,4}^0 \geqslant 0$, we have

$$(6.2) \qquad \begin{cases} \frac{25}{12} + \frac{37}{12}P_{-2} + P_{-3} - \frac{13}{12}P_{-4} - \frac{21}{20}\sigma_5 \geqslant 0; \\ 4 - 2P_{-2} + 3P_{-3} - P_{-4} \geqslant 0; \\ 1 - P_{-2} - 2P_{-3} + P_{-4} - \sigma_5 \geqslant 0. \end{cases}$$

(6.3)

(6.4)

Considering the inequality "(6.2) + (6.3) + 2 × (6.4)":

$$(6.5) \qquad \frac{97}{12} - \frac{11}{12}P_{-2} - \frac{1}{12}P_{-4} - \frac{61}{20}\sigma_5 \geqslant 0,$$

we obtain $\sigma_5 \leqslant 2$.

Case 1: $\sigma_5 = 0$. — In this case, $B^{(0)} = \{n^0_{1,2} \times (1,2), n^0_{1,3} \times (1,3), n^0_{1,4} \times (1,4)\}$. By [12, Proof of Theorem 3.12, Subcase II-1], $P_{-3} \geqslant 1$. By inequalities (6.4) and (6.1),

$$1 + P_{-4} \geqslant P_{-2} + 2P_{-3} \geqslant P_{-4} - 3 + 2P_{-3},$$

which means that $P_{-3} \leqslant 2$. By inequality (6.3) and the fact that $P_{-4} \geqslant 2P_{-2} - 1$,

$$10 \geqslant 2P_{-2} + P_{-4} \geqslant 4P_{-2} - 1,$$

which means that $P_{-2} \leqslant 2$.

Subcase 1(i): $P_{-2} = 2$. — In this subcase, inequalities (6.3) and (6.4) imply that $3P_{-3} \geqslant P_{-4} \geqslant 2P_{-3} + 1$. Combining with the fact that $1 \leqslant P_{-3} \leqslant 2$ and inequality (6.1), (P_{-3}, P_{-4}) can only be $(1,3)$ or $(2,5)$. Hence we may compute corresponding values of $(n^0_{1,2}, n^0_{1,3}, n^0_{1,4})$, which is $(12, 0, 0)$ or $(11, 1, 0)$.

Hence we may get all possible baskets B_X as packings of corresponding $B^{(0)}$ with $\gamma \geqslant 0$, which are listed in Table 6.5. In Table 6.5, for each basket B_X, if $r_X \leqslant 69$ and $r_{\max} \leqslant 12$, then we apply Lemma 6.3(1) (such baskets are marked with ✓ in the last column), otherwise we can compute M_X and $\lambda(M_X)$, then find n_1 such that $P_{-n_1} > \lambda(M_X)n_1 + 1$ where P_{-n_1} is computed by Reid's Riemann–Roch formula. Hence by Proposition 5.2, $\dim \overline{\varphi_{-n_1}(X)} > 1$. By assumption, $P_{-2} = 2$, we may take $m_0 = 2$. Then by Proposition 5.7(1), take $m_1 = n_1$ and $\mu_0 \leqslant m_0$, we get the integer n_2 such that φ_{-m} is birational for all $m \geqslant n_2$.

Table 6.5

B_X	$-K^3$	M_X	$\lambda(M_X)$	n_1	m_0	$r_{\max}$	n_2
$\{12 \times (1,2)\}$	0						
$\{11 \times (1,2), (1,3)\}$							✓
$\{10 \times (1,2), (2,5)\}$							✓
$\{9 \times (1,2), (3,7)\}$							✓
$\{8 \times (1,2), (4,9)\}$							✓
$\{7 \times (1,2), (5,11)\}$							✓
$\{6 \times (1,2), (6,13)\}$	3/13	6	3	9	2	13	33
$\{5 \times (1,2), (7,15)\}$	7/30	7	7/2	9	2	15	33
$\{4 \times (1,2), (8,17)\}$	4/17	8	4	10	2	17	36
$\{3 \times (1,2), (9,19)\}$	9/38	9	4	10	2	19	36
$\{2 \times (1,2), (10,21)\}$	5/21	10	4	10	2	21	36

Subcase 1(ii): $P_{-2} = 1$. — In this subcase, inequalities (6.3) and (6.4) imply that $3P_{-3} + 2 \geqslant P_{-4} \geqslant 2P_{-3}$. Combining with the fact that $1 \leqslant P_{-3} \leqslant 2$ and inequality (6.1), (P_{-3}, P_{-4}) can only take the following values: $(1, 2)$, $(1, 3)$, $(1, 4)$, $(2, 4)$. Hence we may compute corresponding values of $B^{(0)}$ with $\gamma(B^{(0)}) \geqslant 0$ and list those possible $B^{(0)}$ in the following table.

Table 6.6

$B^{(0)}$
$\{8 \times (1, 2), 3 \times (1, 3)\}$
$\{8 \times (1, 2), 2 \times (1, 3), (1, 4)\}$
$\{8 \times (1, 2), (1, 3), 2 \times (1, 4)\}$
$\{7 \times (1, 2), 4 \times (1, 3)\}$

Hence we may get all possible baskets B_X as packings of corresponding $B^{(0)}$ with $\gamma \geqslant 0$, which are listed in Table 6.7. In Table 6.7, for each basket B_X, if $r_X \leqslant 69$ and $r_{\max} \leqslant 12$, then we apply Lemma 6.3(1) (such baskets are marked with ✓ in the last column), otherwise we can compute M_X and $\lambda(M_X)$, then find n_1 such that $P_{-n_1} > \lambda(M_X)n_1 + 1$ where P_{-n_1} is computed by Reid's Riemann–Roch formula. Hence by Proposition 5.2, $\dim \overline{\varphi_{-n_1}(X)} > 1$. Again by Reid's Riemann–Roch formula, we may find m_0 such that $P_{-m_0} \geqslant 2$. Then by Proposition 5.7(2), take $m_1 = n_1$ and $\mu_0 \leqslant m_0$, we get the integer n_2 such that φ_{-m} is birational for all $m \geqslant n_2$. (For the values of n_2 with the mark "$*$", we apply Proposition 5.7(1) instead).

Table 6.7

B_X	$-K^3$	M_X	$\lambda(M_X)$	n_1	m_0	$r_{\max}$	n_2
$\{8 \times (1, 2), 3 \times (1, 3)\}$	0						
$\{7 \times (1, 2), (2, 5), 2 \times (1, 3)\}$							✓
$\{6 \times (1, 2), (3, 7), 2 \times (1, 3)\}$							✓
$\{5 \times (1, 2), (4, 9), 2 \times (1, 3)\}$							✓
$\{4 \times (1, 2), (5, 11), 2 \times (1, 3)\}$							✓
$\{3 \times (1, 2), (6, 13), 2 \times (1, 3)\}$	5/78	5	3	16	4	13	46
$\{2 \times (1, 2), (7, 15), 2 \times (1, 3)\}$	1/15	2	2	13	4	15	47
$\{(1, 2), (8, 17), 2 \times (1, 3)\}$	7/102	7	7/2	17	4	17	55?
$\{7 \times (1, 2), (3, 8), (1, 3)\}$							✓
$\{7 \times (1, 2), (4, 11)\}$							✓
$\{6 \times (1, 2), 2 \times (2, 5), (1, 3)\}$							✓
$\{5 \times (1, 2), (3, 7), (2, 5), (1, 3)\}$	17/210	17	17/3	20	4	7	40
$\{4 \times (1, 2), (4, 9), (2, 5), (1, 3)\}$	4/45	8	4	16	4	9	38
$\{3 \times (1, 2), (5, 11), (2, 5), (1, 3)\}$	31/330	31	31/4	22	4	11	48
$\{2 \times (1, 2), (6, 13), (2, 5), (1, 3)\}$	19/195	38	8	22	4	13	52?
$\{(1, 2), (7, 15), (2, 5), (1, 3)\}$	1/10	3	3	13	4	15	47
$\{5 \times (1, 2), (5, 12), (1, 3)\}$							✓
Continued on next page							

Table 6.7 – continued from previous page

B_X	$-K^3$	M_X	$\lambda(M_X)$	n_1	m_0	$r_{\max}$	n_2
$\{6 \times (1,2), (2,5), (3,8)\}$							✓
$\{5 \times (1,2), (3,7), (3,8)\}$							✓
$\{4 \times (1,2), (4,9), (3,8)\}$	7/72	7	7/2	14	4	9	36
$\{3 \times (1,2), (5,11), (3,8)\}$	9/88	9	4	15	4	11	41
$\{2 \times (1,2), (6,13), (3,8)\}$	11/104	11	4	15	4	13	45
$\{4 \times (1,2), 2 \times (3,7), (1,3)\}$							✓
$\{3 \times (1,2), (4,9), (3,7), (1,3)\}$	13/126	13	13/3	16	4	9	38
$\{2 \times (1,2), (5,11), (3,7), (1,3)\}$	25/231	50	10	23	4	11	49
$\{(1,2), (6,13), (3,7), (1,3)\}$	61/546	61	61/6	23	4	13	53?
$\{3 \times (1,2), (7,16), (1,3)\}$	5/48	5	3	13	4	16	49
$\{2 \times (1,2), 2 \times (4,9), (1,3)\}$							✓
$\{(1,2), (5,11), (4,9), (1,3)\}$	23/198	23	6	18	4	11	44
$\{6 \times (1,2), (5,13)\}$	1/13	2	2	12	4	13	42
$\{5 \times (1,2), 3 \times (2,5)\}$							✓
$\{4 \times (1,2), (3,7), 2 \times (2,5)\}$	4/35	8	4	14	4	7	32
$\{3 \times (1,2), (4,9), 2 \times (2,5)\}$	11/90	11	4	14	4	9	36
$\{2 \times (1,2), (5,11), 2 \times (2,5)\}$	7/55	14	14/3	15	4	11	41
$\{4 \times (1,2), (5,12), (2,5)\}$							✓
$\{4 \times (1,2), (7,17)\}$	2/17	4	3	12	4	17	48*
$\{3 \times (1,2), 2 \times (3,7), (2,5)\}$	9/70	9	4	14	4	7	32
$\{2 \times (1,2), (4,9), (3,7), (2,5)\}$	43/315	86	86/7	23	4	9	45
$\{2 \times (1,2), (7,16), (2,5)\}$	11/80	11	4	13	4	16	49
$\{2 \times (1,2), (4,9), (5,12)\}$							✓
$\{3 \times (1,2), (3,7), (5,12)\}$	11/84	11	4	13	4	12	41
$\{3 \times (1,2), (8,19)\}$	5/38	5	3	12	4	19	48*
$\{2 \times (1,2), 3 \times (3,7)\}$							✓
$\{8 \times (1,2), 2 \times (1,3), (1,4)\}$							✓
$\{7 \times (1,2), (2,5), (1,3), (1,4)\}$							✓
$\{7 \times (1,2), (3,8), (1,4)\}$							✓
$\{6 \times (1,2), (3,7), (1,3), (1,4)\}$	11/84	11	4	13	4	7	31
$\{5 \times (1,2), (4,9), (1,3), (1,4)\}$							✓
$\{4 \times (1,2), (5,11), (1,3), (1,4)\}$	19/132	19	6	16	4	11	42
$\{3 \times (1,2), (6,13), (1,3), (1,4)\}$	23/156	23	6	15	4	13	45
$\{8 \times (1,2), (1,3), (2,7)\}$							✓
$\{7 \times (1,2), (2,5), (2,7)\}$	9/70	9	4	13	4	7	31
$\{6 \times (1,2), (3,7), (2,7)\}$							✓
$\{5 \times (1,2), (4,9), (2,7)\}$	19/126	19	6	15	4	9	37
$\{4 \times (1,2), (5,11), (2,7)\}$	12/77	24	6	15	4	11	41
$\{6 \times (1,2), 2 \times (2,5), (1,4)\}$							✓
$\{5 \times (1,2), (3,7), (2,5), (1,4)\}$	23/140	23	6	15	4	7	33
$\{5 \times (1,2), (5,12), (1,4)\}$							✓
$\{4 \times (1,2), (4,9), (2,5), (1,4)\}$	31/180	31	31/4	16	4	9	38
$\{3 \times (1,2), (5,11), (2,5), (1,4)\}$	39/220	39	8	16	4	11	42
$\{4 \times (1,2), 2 \times (3,7), (1,4)\}$							✓
$\{3 \times (1,2), (4,9), (3,7), (1,4)\}$	47/252	47	47/5	17	4	9	39
$\{8 \times (1,2), (3,10)\}$							✓
$\{8 \times (1,2), (1,3), 2 \times (1,4)\}$							✓
$\{7 \times (1,2), (2,5), 2 \times (1,4)\}$							✓
$\{6 \times (1,2), (3,7), 2 \times (1,4)\}$							✓
$\{5 \times (1,2), (4,9), 2 \times (1,4)\}$							✓
$\{8 \times (1,2), (2,7), (1,4)\}$							✓
$\{8 \times (1,2), (3,11)\}$							✓
$\{7 \times (1,2), 4 \times (1,3)\}$							✓
$\{6 \times (1,2), (2,5), 3 \times (1,3)\}$							✓

Continued on next page

Table 6.7 – continued from previous page

B_X	$-K^3$	M_X	$\lambda(M_X)$	n_1	m_0	$r_{\max}$	n_2
$\{5 \times (1,2), (3,7), 3 \times (1,3)\}$							✓
$\{4 \times (1,2), (4,9), 3 \times (1,3)\}$							✓
$\{3 \times (1,2), (5,11), 3 \times (1,3)\}$							✓
$\{2 \times (1,2), (6,13), 3 \times (1,3)\}$	3/13	18	6	12	3	13	41
$\{6 \times (1,2), (3,8), 2 \times (1,3)\}$							✓
$\{6 \times (1,2), (4,11), (1,3)\}$							✓
$\{6 \times (1,2), (5,14)\}$	3/14	3	3	9	3	14	36*
$\{5 \times (1,2), 2 \times (2,5), 2 \times (1,3)\}$							✓
$\{4 \times (1,2), (3,7), (2,5), 2 \times (1,3)\}$	26/105	52	10	15	3	7	32
$\{3 \times (1,2), (4,9), (2,5), 2 \times (1,3)\}$	23/90	23	6	12	3	9	33
$\{3 \times (1,2), (4,9), (3,8), (1,3)\}$	19/72	19	6	12	3	9	33
$\{4 \times (1,2), (5,12), 2 \times (1,3)\}$							✓
$\{4 \times (1,2), (3,7), (3,8), (1,3)\}$	43/168	43	43/5	14	3	8	33
$\{4 \times (1,2), (3,7), (4,11)\}$	20/77	40	8	13	3	11	38
$\{3 \times (1,2), 2 \times (3,7), 2 \times (1,3)\}$							✓
$\{5 \times (1,2), (2,5), (3,8), (1,3)\}$	29/120	29	29/4	13	3	8	32
$\{5 \times (1,2), (5,13), (1,3)\}$	19/78	19	6	12	3	13	41
$\{5 \times (1,2), (2,5), (4,11)\}$	27/110	27	27/4	13	3	11	38
$\{5 \times (1,2), 2 \times (3,8)\}$							✓
$\{4 \times (1,2), 3 \times (2,5), (1,3)\}$							✓
$\{3 \times (1,2), (3,7), 2 \times (2,5), (1,3)\}$	59/210	59	10	14	3	7	31
$\{3 \times (1,2), (5,12), (2,5), (1,3)\}$							✓
$\{4 \times (1,2), 2 \times (2,5), (3,8)\}$							✓
$\{4 \times (1,2), (2,5), (5,13)\}$	18/65	36	8	13	3	13	42
$\{4 \times (1,2), (7,18)\}$	5/18	5	3	8	3	18	33*
$\{3 \times (1,2), 4 \times (2,5)\}$							✓

Note that there are 3 cases where the values in the n_2 column are marked with "?", of which $n_2 = 55, 52, 53$ respectively. We shall discuss them in more details to prove our statement.

If $B_X = \{(1,2), (8,17), 2 \times (1,3)\}$, we know that $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant 17$ from the list. Note that $P_{-10} = 13$. Take $m_0 = 4$. If $|-10K_X|$ and $|-4K_X|$ are not composed with the same pencil, then we may take $m_1 = 10$ and $\mu_0 \leqslant 4$, and by Proposition 5.7(1), φ_{-m} is birational for all $m \geqslant 42$; if $|-10K_X|$ and $|-4K_X|$ are composed with the same pencil, then we may take $m_1 = 17$ and $\mu_0 \leqslant \frac{10}{12}$ by Remark 5.8, and by Proposition 5.7(2), φ_{-m} is birational for all $m \geqslant 51$.

If $B_X = \{2 \times (1,2), (6,13), (2,5), (1,3)\}$, we know that $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant 22$ from the list. Note that $P_{-8} = 10$. Take $m_0 = 4$. If $|-8K_X|$ and $|-4K_X|$ are not composed with the same pencil, then we may take $m_1 = 8$ and $\mu_0 \leqslant 4$, and by Proposition 5.7(1), $\varphi_{-m,X}$ is birational for all $m \geqslant 36$; if $|-8K_X|$ and $|-4K_X|$ are composed with the same pencil, then we may take $m_1 = 22$ and $\mu_0 \leqslant \frac{8}{9}$ by Remark 5.8, and by Proposition 5.7(2), φ_{-m} is birational for all $m \geqslant 48$.

If $B_X = \{(1,2), (6,13), (3,7), (1,3)\}$, we know that $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant 23$ from the list. Note that $P_{-8} = 11$. Take $m_0 = 4$. If $|-8K_X|$ and

$|-4K_X|$ are not composed with the same pencil, then we may take $m_1 = 8$ and $\mu_0 \leqslant 4$, and by Proposition 5.7(1), $\varphi_{-m,X}$ is birational for all $m \geqslant 36$; if $|-8K_X|$ and $|-4K_X|$ are composed with the same pencil, then we may take $m_1 = 23$ and $\mu_0 \leqslant \frac{8}{10}$ by Remark 5.8, and by Proposition 5.7(2), φ_{-m} is birational for all $m \geqslant 49$.

Case 2: $\sigma_5 = 2$. — In this case, by [12, Proof of Theorem 3.12, Subcase II-2], we have $(P_{-2}, P_{-3}, P_{-4}) = (1, 0, 2)$ and

$$B_X = B^{(0)} = \{9 \times (1, 2), 2 \times (1, 5)\}.$$

We may apply Lemma 6.3(1).

Case 3: $\sigma_5 = 1$. — In this case, all possible baskets B_X are classified in [12, Proof of Theorem 3.12, Subcase II-3] under the strict inequality $\gamma(B^{(0)}) > 0$. The output is however the same even under the weaker condition $\gamma(B^{(0)}) \geqslant 0$, which is concluded after checking the original proof word by word. So we simply list all possible baskets in Table 6.8. In Table 6.8, for each basket B_X, if $r_X \leqslant 69$ and $r_{\max} \leqslant 12$, then we may directly apply Lemma 6.3(1) (such baskets are marked with ✓ in the last column); if the condition $0 < -K_X^3 < 0.21$ is not satisfied, we mark it with × in the last column (such baskets are irrelevant to the proof); otherwise we can compute M_X and $\lambda(M_X)$, then we may find n_1 such that $P_{-n_1} > \lambda(M_X)n_1 + 1$ where P_{-n_1} is computed by Reid's Riemann–Roch formula. Hence by Proposition 5.2, $\dim \overline{\varphi_{-n_1}(X)} > 1$. Again by Reid's Riemann–Roch formula, we may find m_0 such that $P_{-m_0} \geqslant 2$. Then by Proposition 5.7(2), take $m_1 = n_1$ and $\mu_0 \leqslant m_0$, we get the integer n_2 such that φ_{-m} is birational for all $m \geqslant n_2$.

Table 6.8

B_X	$-K^3$	M_X	$\lambda(M_X)$	n_1	m_0	$r_{\max}$	n_2
$\{11 \times (1, 2), (1, 3), (1, 5)\}$							✓
$\{12 \times (1, 2), (1, 5)\}$							✓
$\{12 \times (1, 2), (1, 6)\}$							✓
$\{9 \times (1, 2), 2 \times (1, 3), (1, 5)\}$							✓
$\{10 \times (1, 2), (1, 4), (1, 5)\}$							✓
$\{10 \times (1, 2), (2, 9)\}$							✓
$\{10 \times (1, 2), (1, 3), (1, 6)\}$							✓
$\{10 \times (1, 2), (1, 3), (1, 5)\}$							✓
$\{9 \times (1, 2), (2, 5), (1, 5)\}$							✓
$\{8 \times (1, 2), (3, 7), (1, 5)\}$	18/35						×
$\{11 \times (1, 2), (1, 5)\}$							✓
$\{11 \times (1, 2), (1, 6)\}$							✓
$\{11 \times (1, 2), (1, 7)\}$							✓
$\{7 \times (1, 2), 3 \times (1, 3), (1, 5)\}$							✓
$\{6 \times (1, 2), (2, 5), 2 \times (1, 3), (1, 5)\}$							✓
Continued on next page							

Table 6.8 – continued from previous page

B_X	$-K^3$	M_X	$\lambda(M_X)$	n_1	m_0	$r_{\max}$	n_2
$\{8 \times (1,2),(1,3),(1,4),(1,5)\}$							✓
$\{8 \times (1,2),(2,7),(1,5)\}$	$8/35$						✗
$\{8 \times (1,2),(1,3),(2,9)\}$							✓
$\{7 \times (1,2),(2,5),(1,4),(1,5)\}$							✓
$\{8 \times (1,2),2 \times (1,3),(1,6)\}$							✓
$\{7 \times (1,2),(2,5),(1,3),(1,6)\}$							✓
$\{8 \times (1,2),2 \times (1,3),(1,5)\}$							✓
$\{7 \times (1,2),(2,5),(1,3),(1,5)\}$							✓
$\{6 \times (1,2),2 \times (2,5),(1,5)\}$							✓
$\{6 \times (1,2),(3,7),(1,3),(1,5)\}$	$19/105$	38	8	16	4	7	34
$\{5 \times (1,2),(4,9),(1,3),(1,5)\}$	$17/90$	17	$17/3$	13	4	9	35
$\{7 \times (1,2),(3,8),(1,5)\}$							✓
$\{5 \times (1,2),(3,7),(2,5),(1,5)\}$	$3/14$						✗
$\{9 \times (1,2),(1,4),(1,6)\}$							✓
$\{9 \times (1,2),(1,4),(1,5)\}$							✓
$\{9 \times (1,2),(2,9)\}$							✓
$\{9 \times (1,2),(1,3),(1,7)\}$							✓
$\{8 \times (1,2),(2,5),(1,7)\}$	$2/35$	4	3	18	6	7	40
$\{8 \times (1,2),(2,5),(1,6)\}$							✓
$\{7 \times (1,2),(3,7),(1,6)\}$							✓
$\{6 \times (1,2),(4,9),(1,6)\}$							✓
$\{7 \times (1,2),(3,7),(1,5)\}$	$1/70$	1	1	20	6	7	43
$\{6 \times (1,2),(4,9),(1,5)\}$	$1/45$	2	2	23	6	9	48
$\{5 \times (1,2),(5,11),(1,5)\}$	$3/110$	3	3	25	6	11	$53?$
$\{4 \times (1,2),(6,13),(1,5)\}$	$2/65$	4	3	24	6	13	$56?$

Note that there are 2 cases with "?" marked in the n_2 column of Table 6.8, for which we discuss them in more details as follows.

If $B_X = \{5 \times (1,2),(5,11),(1,5)\}$, we know that $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant 25$ from the list. Note that $P_{-15} = 18$. Take $m_0 = 6$. If $|-15K_X|$ and $|-6K_X|$ are not composed with the same pencil, then we may take $m_1 = 15$ and $\mu_0 \leqslant 6$, and by Proposition 5.7(2), φ_{-m} is birational for all $m \geqslant 43$; if $|-15K_X|$ and $|-6K_X|$ are composed with the same pencil, then we may take $m_1 = 25$ and $\mu_0 \leqslant \frac{15}{17}$ by Remark 5.8, and by Proposition 5.7(2), φ_{-m} is birational for all $m \geqslant 47$.

If $B_X = \{4 \times (1,2),(6,13),(1,5)\}$, we know that $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant 24$ from the list. Note that $P_{-14} = 16$. Take $m_0 = 6$. If $|-14K_X|$ and $|-6K_X|$ are not composed with the same pencil, then we may take $m_1 = 14$ and $\mu_0 \leqslant 6$, and by Proposition 5.7(2), φ_{-m} is birational for all $m \geqslant 46$; if $|-14K_X|$ and $|-6K_X|$ are composed with the same pencil, then we may take $m_1 = 24$ and $\mu_0 \leqslant \frac{14}{15}$ by Remark 5.8, and by Proposition 5.7(2), φ_{-m} is birational for all $m \geqslant 50$.

Combining all above cases, we complete the proof. □

6.8. The case $P_{-1} > 0$ and $1/30 \leqslant -K_X^3 < 0.21$

LEMMA 6.9. — *Let* (X, Y, Z) *be a Fano–Mori triple such that* $\rho(Y) > 1$. *Assume that* $-K_X^3 \geqslant 1/30$ *and* $r_X \leqslant 660$. *Then*

(1) $P_{-18} \geqslant 21$;

(2) $\dim \overline{\varphi_{-m}(X)} > 1$ *for all* $m \geqslant 35$.

Proof.

(1). — By Proposition 5.3, take $t = 18/8$ and use the fact $-K_X^3 \geqslant 1/30$, we have $P_{-18} > 20$.

(2). — By Lemma 5.5,

$$\frac{\lambda(M_X)}{-K_X^3} \leqslant \max\left\{1, \frac{3}{1/30}, \sqrt{\frac{2 \times 660}{1/30}}\right\} < 199.$$

Take $t = 35/8$, then

$$-\frac{3}{4} + \sqrt{\frac{12}{t(-K_X^3)} + \frac{6\lambda(M_X)}{-K_X^3} + \frac{1}{16}} < 35.$$

By Proposition 5.4, $|-mK_X|$ is not composed with a pencil for $m \geqslant 35$. $\quad\square$

LEMMA 6.10. — *Let* (X, Y, Z) *be a Fano–Mori triple such that* $\rho(Y) > 1$. *Assume that* $P_{-1} > 0$ *and* $1/30 \leqslant -K_X^3 < 0.21$. *Assume that one of the following conditions holds:*

(1) $11 \leqslant r_{\max} \leqslant 13$ *and* $-K_X^3 < 0.12$; *or*

(2) $r_{\max} \geqslant 14$.

Then $\varphi_{-m,X}$ *is birational for all* $m \geqslant 51$.

Proof. — We will use Chen–Chen's method to classify all possible baskets.

Consider $B_X = \{(b_i, r_i)\}$, we may always assume that $r_1 = r_{\max}$.

By equality (2.1),

$$2P_{-1} = -K_X^3 + 6 - \sum_i \frac{b_i(r_i - b_i)}{r_i}$$

(6.6)
$$\leqslant -K_X^3 + 6 - \frac{b_1(r_1 - b_1)}{r_1}$$

$$\leqslant -K_X^3 + 6 - \frac{r_1 - 1}{r_1}$$

$$< 6.$$

Hence $P_{-1} \leqslant 2$.

Case 1: $P_{-1} = 2$. — By [7, Inequality (4.1)],

$$0.21 > -K_X^3 \geqslant \frac{1}{12}(-1 - P_{-1} - P_{-2} + P_{-4}),$$

which means that

$$(6.7) \qquad\qquad P_{-4} \leqslant P_{-2} + 5.$$

By inequality (6.6),

$$\frac{b_1(r_1 - b_1)}{r_1} \leqslant 2 - K_X^3,$$

which implies that $b_1 \leqslant 2$. If $b_1 = 1$, then $(1, r_1) \in B_X$, which means that $(1, r_1) \in B^{(0)}$. If $b_1 = 2$, then r_1 is an odd number, we may write $r_1 = 2s+1$ for some $s \geqslant 5$, and $(2, r_1) \in B_X$ implies that $\{(1, s), (1, s+1)\} \subset B^{(0)}$.

Recall that $B^{(0)} = \{n_{1,r}^0 \times (1, r)\}$, we have

$$(6.8) \qquad \sum_r \frac{n_{1,r}^0 \times (r-1)}{r} = 2 - K^3(B^{(0)}) \leqslant 2 - K_X^3 < 2.21.$$

Note that either $n_{1,r_1}^0 \geqslant 1$, or $n_{1,s}^0 \geqslant 1$ and $n_{1,s+1}^0 \geqslant 1$ where $r_1 = 2s + 1$. Also we have

$$\sigma(B^{(0)}) = \sigma(B_X) = 10 - 5P_{-1} + P_{-2} = P_{-2} \geqslant 2P_{-1} - 1 = 3.$$

If $\sigma \geqslant 4$, then

$$\sum_r \frac{n_{1,r}^0 \times (r-1)}{r} \geqslant \begin{cases} 3 \times \frac{1}{2} + \frac{10}{11} > 2.21, & \text{or} \\ 2 \times \frac{1}{2} + \frac{4}{5} + \frac{5}{6} > 2.21, \end{cases}$$

a contradiction. Hence $\sigma = 3$. By easy computation, we know that $B^{(0)}$ satisfying inequality (6.8) is one of the following:

Table 6.9

$B^{(0)}$
$\{2 \times (1, 2), (1, r_1)\}$
$\{(1, 2), (1, 3), (1, r_1)\}$
$\{(1, 2), (1, 4), (1, r_1)\}$
$\{(1, 2), (1, s), (1, s+1)\}, r_1 = 2s + 1$

Note that the first case is absurd since it admits no packing and $-K^3(B^{(0)}) < 0$. From Table 6.9, we may get all possible packings with $\gamma \geqslant 0$ and $r_{\max} = r_1 \geqslant 11$, which are listed in Table 6.10. In Table 6.10, for each basket B_X, if $r_X \leqslant 287$ and $r_{\max} \leqslant 12$, then we apply Lemma 6.3 (2)

(such baskets are marked with ✓ in the last column); if $-K_X^3$ does not satisfy the assumption in the lemma, we mark it with the symbol "×" in the last column; otherwise we can compute M_X and $\lambda(M_X)$, then find n_1 such that $P_{-n_1} > \lambda(M_X)n_1 + 1$ where P_{-n_1} is computed by Reid's Riemann–Roch formula. Hence by Proposition 5.2, $\dim\overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant n_1$ since $P_{-1} > 0$. By assumption, $P_{-1} = 2$, we may take $m_0 = \nu_0 = 1$. Then by Proposition 5.7(1), take $m_1 = n_1$ and $\mu_0 \leqslant m_0$, we get the integer n_2 such that φ_{-m} is birational for all $m \geqslant n_2$. (For the value of n_2 with a "*" marked, we have applied Proposition 5.7(3)).

Table 6.10

B_X	$-K^3$	M_X	$\lambda(M_X)$	n_1	m_0	$r_{\max}$	n_2
$\{(1,2),(1,3),(1,11)\}$							✓
$\{(1,2),(1,3),(1,12)\}$							✓
$\{(1,2),(1,3),(1,13)\}$	7/78	7	7/2	14	1	13	41*
$\{(1,2),(1,3),(1,14)\}$	2/21	4	3	13	1	14	42
$\{(1,2),(1,3),(1,15)\}$	1/10	3	3	13	1	15	42
$\{(1,2),(1,3),(1,16)\}$	5/48	5	3	13	1	16	42
$\{(1,2),(1,3),(1,17)\}$	11/102	11	4	15	1	17	48
$\{(1,2),(1,3),(1,18)\}$	1/9	2	2	10	1	18	33
$\{(1,2),(1,3),(1,19)\}$	13/114	13	13/3	15	1	19	48
$\{(2,5),(1,11)\}$							✓
$\{(2,5),(1,12)\}$							✓
$\{(2,5),(1,13)\}$	8/65						×
$\{(2,5),(1,14)\}$	9/70	9	4	13	1	14	42
$\{(2,5),(1,15)\}$	2/15	2	2	9	1	15	30
$\{(2,5),(1,16)\}$	11/80	11	4	13	1	16	42
$\{(2,5),(1,17)\}$	12/85	12	4	13	1	17	42
$\{(2,5),(1,18)\}$	13/90	13	13/3	13	1	18	42
$\{(2,5),(1,19)\}$	14/95	14	14/3	14	1	19	45
$\{(1,2),(1,4),(1,11)\}$							✓
$\{(1,2),(1,4),(1,12)\}$							✓
$\{(1,2),(1,4),(1,13)\}$	9/52						×
$\{(1,2),(1,4),(1,14)\}$	5/28	5	3	10	1	14	33
$\{(1,2),(1,4),(1,15)\}$	11/60	11	4	11	1	15	36
$\{(1,2),(1,4),(1,16)\}$	3/16	3	3	10	1	16	33
$\{(1,2),(1,4),(1,17)\}$	13/68	13	13/3	12	1	17	39
$\{(1,2),(1,4),(1,18)\}$	7/36	7	7/2	10	1	18	33
$\{(1,2),(2,11)\}$							✓
$\{(1,2),(2,13)\}$	5/26						×
$\{(1,2),(2,15)\}$	7/30						×
$\{(1,2),(2,17)\}$	9/34						×
$\{(1,2),(2,19)\}$	11/38						×
$\{(1,2),(2,21)\}$	13/42						×

Case 2: $P_{-1} = 1$. — By [7, Inequality (4.1)],

$$0.21 > -K_X^3 \geqslant \frac{1}{12}(-1 - P_{-1} - P_{-2} + P_{-4}),$$

which means that

(6.9) $$P_{-4} \leqslant P_{-2} + 4.$$

The basket $B^{(0)}$ has datum

$$\begin{cases} n^0_{1,2} = -1 + 4P_{-2} - P_{-3}; \\ n^0_{1,3} = 2 - 2P_{-2} + 3P_{-3} - P_{-4}; \\ n^0_{1,4} = 4 - P_{-2} - 2P_{-3} + P_{-4} - \sigma_5. \end{cases}$$

By $n^0_{1,4} \geqslant 0$ and inequality (6.9), we get $P_{-3} \leqslant 4$ and $P_{-3} = 4$ only if $P_{-4} = P_{-2} + 4$ and $\sigma_5 = 0$. By $n^0_{1,3} \geqslant 0$ and $P_{-4} \geqslant 2P_{-2} - 1$, we have $P_{-2} \leqslant 3$. Recall that we also have $n^0_{1,2} \geqslant 0$ and $P_{-3} \geqslant P_{-2}$ since $P_{-1} = 1$. Hence the possible values of (P_{-2}, P_{-3}) are $(3, 4)$, $(2, 4)$, $(3, 3)$, $(2, 3)$, $(1, 3)$, $(2, 2)$, $(1, 2)$, $(1, 1)$.

Subcase 2 (i): $(P_{-2}, P_{-3}) = (3, 4)$. — In this case, $P_{-4} = 7$ and $\sigma_5 = 0$. Hence

$$B^{(0)} = \{7 \times (1, 2), (1, 3)\},$$

and we may get all possible packings with $\gamma \geqslant 0$ and $r_{\max} \geqslant 11$, which are the following:

$\{3 \times (1, 2), (5, 11)\},$	$- K^3 = 5/22;$
$\{2 \times (1, 2), (6, 13)\},$	$- K^3 = 3/13;$
$\{(1, 2), (7, 15)\},$	$- K^3 = 7/30;$
$\{(8, 17)\},$	$- K^3 = 4/17.$

Since $-K^3 > 0.21$, none of the baskets satisfy the assumption of the lemma.

Subcase 2 (ii): $(P_{-2}, P_{-3}) = (2, 4)$. — In this case, $P_{-4} = 6$ and $\sigma_5 = 0$. Hence

$$B^{(0)} = \{3 \times (1, 2), 4 \times (1, 3)\},$$

and we may get all possible packings with $\gamma \geqslant 0$ and $r_{\max} \geqslant 11$, which are the following:

$\{2 \times (1, 2), (5, 14)\},$	$- K^3 = 3/14;$
$\{(1, 2), (5, 13), (1, 3)\},$	$- K^3 = 19/78;$
$\{(2, 5), (5, 13)\},$	$- K^3 = 18/65;$
$\{(5, 12), 2 \times (1, 3)\},$	$- K^3 = 1/4;$
$\{(7, 18)\},$	$- K^3 = 5/18.$

It is clear that, for each above basket, $-K^3 > 0.21$, which does not satisfy the assumption of the lemma.

Subcase 2(iii): $(P_{-2}, P_{-3}) = (3, 3)$. — In this case,

$$\begin{cases} n_{1,2}^0 = 8; \\ n_{1,3}^0 = 5 - P_{-4}; \\ n_{1,4}^0 = -5 + P_{-4} - \sigma_5. \end{cases}$$

Hence $P_{-4} = 5$ and $\sigma_5 = 0$, and $B^{(0)} = \{8 \times (1, 2)\}$. It is clear that $B_X = B^{(0)}$, contradicting to assumption of the local index.

Subcase 2(iv): $(P_{-2}, P_{-3}) = (2, 3)$. — In this case,

$$\begin{cases} n_{1,2}^0 = 4; \\ n_{1,3}^0 = 7 - P_{-4}; \\ n_{1,4}^0 = -4 + P_{-4} - \sigma_5. \end{cases}$$

By inequality (6.9), $P_{-4} \leqslant 6$. Hence $(P_{-4}, \sigma_5) = (6, 0), (6, 1), (6, 2), (5, 0), (5, 1), (4, 0)$. Hence the corresponding $B^{(0)}$ is in the following list:

Table 6.11

$B^{(0)}$
$\{4 \times (1, 2), (1, 3), 2 \times (1, 4)\}$
$\{4 \times (1, 2), (1, 3), (1, 4), (1, s)\}, s \geqslant 5$
$\{4 \times (1, 2), (1, 3), (1, s_1), (1, s_2)\}, 5 \leqslant s_1 \leqslant s_2$
$\{4 \times (1, 2), 2 \times (1, 3), (1, 4)\}$
$\{4 \times (1, 2), 2 \times (1, 3), (1, s)\}, s \geqslant 5$
$\{4 \times (1, 2), 3 \times (1, 3)\}$

Hence all possible packings B_X of $B^{(0)}$ with $\gamma \geqslant 0$ and $r_{\max} \geqslant 11$ are dominated by one of the baskets B' listed in Table 6.12.

Table 6.12

No.	B'
1	$\{(5, 11), 2 \times (1, 4)\}$
2	$\{4 \times (1, 2), (3, 11)\}$
3	$\{(5, 11), (1, 4), (1, s)\}, 5 \leqslant s \leqslant 9$
4	$\{4 \times (1, 2), (1, 3), (1, 4), (1, 11)\}$
5	$\{(5, 11), (1, s_1), (1, s_2)\}, 5 \leqslant s_1 \leqslant s_2 \leqslant 7$
6	$\{4 \times (1, 2), (1, 3), (2, r_1)\}, r_1 = 11, 13, 15$
7	$\{(5, 11), (1, 3), (1, 4)\}$
8	$\{(1, 2), (5, 12), (1, 4)\}$
9	$\{4 \times (1, 2), 2 \times (1, 3), (1, r_1)\}, r_1 = 11, 12$
10	$\{(5, 11), (1, 3), (1, 5)\}$
11	$\{(5, 11), (1, 3), (1, s)\}, 6 \leqslant s \leqslant 10$
12	$\{(1, 2), (5, 12), (1, s)\}, 5 \leqslant s \leqslant 10$
13	$\{(7, 17)\}$
14	$\{(5, 11), 2 \times (1, 3)\}$
15	$\{(1, 2), (5, 12), (1, 3)\}$
16	$\{3 \times (1, 2), (4, 11)\}$

In Table 6.12, for cases No. 1, 3–6, 9, 11–12, one has $-K^3(B') > 0.21$ by direct calculation. Hence B_X can not be dominated by these baskets. For cases No. 2, 8, and 10, they are minimal and all satisfy $-K^3 > 0.12$, which must be excluded. For case No. 7, B' has only one possible packing and both have $-K^3 > 0.12$ and $r_{\max} = 11$. Hence case No. 7 must be excluded. Hence all possible B_X appear as packings of B' in No. 13–16, which are the followings:

$$\{(7, 17)\},$$
$$\{(5, 11), 2 \times (1, 3)\},$$
$$\{(1, 2), (5, 12), (1, 3)\},$$
$$\{2 \times (1, 2), (5, 13)\},$$
$$\{3 \times (1, 2), (4, 11)\}.$$

If $B_X = \{(7, 17)\}$, then $-K_X^3 = 2/17$, $M_X = \lambda(M_X) = 2$. Note that $P_{-9} > 9\lambda(M_X) + 1$ where P_{-9} is computed by Reid's Riemann–Roch formula. Hence by Proposition 5.2, $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant 9$ since $P_{-1} > 0$. By assumption, $P_{-2} = 2$, we may take $m_0 = 2$. Then by Proposition 5.7(1), take $m_1 = 9$, $\mu_0 \leqslant 2$, we get that φ_{-m} is birational for all $m \geqslant 33$.

If $B_X = \{2 \times (1, 2), (5, 13)\}$, then $-K_X^3 = 1/13$, $M_X = \lambda(M_X) = 2$. Note that $P_{-10} > 10\lambda(M_X) + 1$ where P_{-10} is computed by Reid's Riemann–Roch formula. Hence by Proposition 5.2, $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant 10$ since $P_{-1} > 0$. By assumption, $P_{-2} = 2$, we may take $m_0 = 2$. Then by Proposition 5.7(1), take $m_1 = 10$, $\mu_0 \leqslant 2$, we get that φ_{-m} is birational for all $m \geqslant 36$.

For the remaining 3 baskets, we may apply Lemma 6.3(2) since $r_{\max} \leqslant 12$.

Subcase 2(v): $(P_{-2}, P_{-3}) = (1, 3)$. — In this case,

$$\begin{cases} n_{1,2}^0 = 0; \\ n_{1,3}^0 = 9 - P_{-4}; \\ n_{1,4}^0 = -3 + P_{-4} - \sigma_5. \end{cases}$$

By inequality (6.9), $P_{-4} \leqslant 5$. Hence $(P_{-4}, \sigma_5) = (5, 0), (5, 1), (5, 2), (4, 0), (4, 1), (3, 0)$. Hence the corresponding $B^{(0)}$ is in the following list:

Table 6.13

No.	$B^{(0)}$
1	$\{4 \times (1,3), 2 \times (1,4)\}$
2	$\{4 \times (1,3), (1,4), (1,s)\}, s \geqslant 5$
3	$\{4 \times (1,3), (1,s_1), (1,s_2)\}, 5 \leqslant s_1 \leqslant s_2$
4	$\{5 \times (1,3), (1,4)\}$
5	$\{5 \times (1,3), (1,s)\}, s \geqslant 5$
6	$\{6 \times (1,3)\}$

Clearly cases No. 5 and 6 are absurd since they admit no packings with $r_{\max} \geqslant 11$.

For case No. 2, B_X with $\gamma \geqslant 0$ and $r_{\max} \geqslant 11$ is dominated by $B' = \{(1,3), (4,13), (1,s)\}$ for some $5 \leqslant s \leqslant 8$. But then $-K_X^3 \geqslant -K^3(B') > 0.21$, a contradiction.

For case No. 3, B_X with $\gamma \geqslant 0$ and $r_{\max} \geqslant 11$ is $B_X = \{4 \times (1,3), (2,r_1)\}$ for $r_1 = 11, 13$. But then $-K_X^3 > 0.21$, a contradiction.

For cases No. 1 and 4, we may get all possible packings with $\gamma \geqslant 0$ and $r_{\max} \geqslant 11$, which are listed in Table 6.14. In Table 6.14, for each basket B_X, if $r_X \leqslant 287$ and $r_{\max} \leqslant 12$, then we apply Lemma 6.3(2) (such baskets are marked with ✓ in the last column); if $-K_X^3$ does not satisfy the assumption in the lemma, we mark it with × in the last column; otherwise we can compute M_X and $\lambda(M_X)$, then find n_1 such that $P_{-n_1} > \lambda(M_X)n_1 + 1$ where P_{-n_1} is computed by Reid's Riemann–Roch formula. Hence by Proposition 5.2, $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant n_1$ since $P_{-1} > 0$. By assumption, $P_{-3} = 3$, we may take $m_0 = 3$. Then by Proposition 5.7(1), take $m_1 = n_1$, $\mu_0 \leqslant m_0$, and $\nu_0 = 1$, we get the integer n_2 such that φ_{-m} is birational for all $m \geqslant n_2$. (For the value of n_2 with a $*$ mark, we apply Proposition 5.7(3)).

Table 6.14

B_X	$-K^3$	M_X	$\lambda(M_X)$	n_1	m_0	$r_{\max}$	n_2
$\{(5,16), (1,4)\}$	3/16	3	3	9	3	16	36
$\{(1,3), (5,17)\}$	10/51	10	4	10	3	17	39
$\{(1,3), (4,13), (1,4)\}$	29/156						×
$\{(4,13), (2,7)\}$	18/91						×
$\{3 \times (1,3), (3,11)\}$							✓
$\{2 \times (1,3), (4,13)\}$	4/39	4	3	12	3	13	41*
$\{(1,3), (5,16)\}$	5/48	5	3	12	3	16	45
$\{(6,19)\}$	2/19	2	2	10	3	19	39

Subcase 2 (vi): $(P_{-2}, P_{-3}) = (2, 2)$. — In this case,

$$\begin{cases} n_{1,2}^0 = 5; \\ n_{1,3}^0 = 4 - P_{-4}; \\ n_{1,4}^0 = -2 + P_{-4} - \sigma_5. \end{cases}$$

Hence $(P_{-4}, \sigma_5) = (4, 0), (4, 1), (4, 2), (3, 0), (3, 1), (2, 0)$. Hence the corresponding $B^{(0)}$ is in the following list:

Table 6.15

No.	$B^{(0)}$
1	$\{5 \times (1, 2), 2 \times (1, 4)\}$
2	$\{5 \times (1, 2), (1, 4), (1, s)\}, s \geqslant 5$
3	$\{5 \times (1, 2), (1, s_1), (1, s_2)\}, 5 \leqslant s_1 \leqslant s_2$
4	$\{5 \times (1, 2), (1, 3), (1, 4)\}$
5	$\{5 \times (1, 2), (1, 3), (1, s)\}, s \geqslant 5$
6	$\{5 \times (1, 2), 2 \times (1, 3)\}$

Clearly No.1 is absurd since there is no further packing.

Case No. 6 is also absurd since all its packing has

$$-K^3 \leqslant -K^3(\{(7, 16)\}) < 0.$$

Similarly, case No. 4 is also absurd since all its packing has $-K^3 < 0$.

For case No. 2, the only possible baskets are $B_X = B^{(0)} = \{5 \times (1, 2), (1, 4), (1, s)\}$ for $s = 11, 12$ and we may apply Lemma 6.3 (2).

For case No. 3, possible B_X with $\gamma \geqslant 0$ and $r_{\max} \geqslant 11$ are

$$\{5 \times (1, 2), (1, 5), (1, 11)\},$$
$$\{5 \times (1, 2), (2, 11)\},$$
$$\{5 \times (1, 2), (2, 13)\},$$
$$\{5 \times (1, 2), (2, 15)\}.$$

For the first two cases, we may apply Lemma 6.3 (2). The last two cases are absurd since $-K_X^3$ does not satisfy the assumption in the lemma.

For case No. 5, we may get all possible packings with $\gamma \geqslant 0$ and $r_{\max} \geqslant 11$, which are listed in Table 6.16. In Table 6.16, for each basket B_X, if $r_X \leqslant 287$ and $r_{\max} \leqslant 12$, then we apply Lemma 6.3 (2) (such baskets are marked with $\checkmark$ in the last column); if $-K_X^3$ does not satisfy the assumption in the lemma, we mark it with $\times$ in the last column; otherwise we can compute M_X and $\lambda(M_X)$, then find n_1 such that $P_{-n_1} > \lambda(M_X)n_1 + 1$ where P_{-n_1} is computed by Reid's Riemann–Roch formula. Hence by Proposition 5.2, $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant n_1$ since $P_{-1} > 0$. By assumption, $P_{-2} = 2$,

we may take $m_0 = 2$. Then by Proposition 5.7 (3), take $m_1 = n_1$, $\mu_0 \leqslant m_0$, and $\nu_0 = 1$, we get the integer n_2 such that φ_{-m} is birational for all $m \geqslant n_2$.

Table 6.16

B_X	$-K^3$	M_X	$\lambda(M_X)$	n_1	m_0	$r_{\max}$	n_2
$\{5 \times (1,2), (1,3), (1,11)\}$							✓
$\{4 \times (1,2), (2,5), (1,11)\}$							✓
$\{3 \times (1,2), (3,7), (1,11)\}$							✓
$\{2 \times (1,2), (4,9), (1,11)\}$							✓
$\{(1,2), (5,11), (1,11)\}$							✓
$\{(6,13), (1,11)\}$	$20/143$						×
$\{5 \times (1,2), (1,3), (1,12)\}$							✓
$\{4 \times (1,2), (2,5), (1,12)\}$							✓
$\{3 \times (1,2), (3,7), (1,12)\}$							✓
$\{2 \times (1,2), (4,9), (1,12)\}$							✓
$\{5 \times (1,2), (1,3), (1,13)\}$	$7/78$	7	$7/2$	15	2	13	43
$\{4 \times (1,2), (2,5), (1,13)\}$	$8/65$						×
$\{(1,2), (5,11), (1,5)\}$							✓
$\{(1,2), (5,11), (1,6)\}$							✓
$\{(1,2), (5,11), (1,7)\}$							✓
$\{(1,2), (5,11), (1,8)\}$							✓
$\{(1,2), (5,11), (1,9)\}$							✓
$\{(1,2), (5,11), (1,10)\}$							✓
$\{(6,13), (1,5)\}$	$2/65$						×
$\{(6,13), (1,6)\}$	$5/78$	5	3	15	2	13	43
$\{(6,13), (1,7)\}$	$8/91$	8	4	16	2	13	44
$\{(6,13), (1,8)\}$	$11/104$	11	4		2	13	43
$\{(6,13), (1,9)\}$	$14/117$	14	$14/3$	15	2	13	43
$\{(6,13), (1,10)\}$	$17/130$						×

Subcase 2 (vii): $(P_{-2}, P_{-3}) = (1,2)$. — In this case,

$$
\begin{cases}
n_{1,2}^0 = 1; \\
n_{1,3}^0 = 6 - P_{-4}; \\
n_{1,4}^0 = -1 + P_{-4} - \sigma_5.
\end{cases}
$$

By inequality (6.9) and $P_{-4} \geqslant P_{-3}$, $2 \leqslant P_{-4} \leqslant 5$. Hence $(P_{-4}, \sigma_5) = (5,0)$, $(5,1)$, $(5,2)$, $(5,3)$, $(5,4)$, $(4,0)$, $(4,1)$, $(4,2)$, $(4,3)$, $(3,0)$, $(3,1)$, $(3,2)$, $(2,0)$, $(2,1)$. Hence the corresponding $B^{(0)}$ is in the following list:

Table 6.17

No.	$B^{(0)}$
1	$\{(1,2),(1,3),4\times(1,4)\}$
2	$\{(1,2),(1,3),3\times(1,4),(1,s)\}, s\geqslant 5$
3	$\{(1,2),(1,3),2\times(1,4),(1,s_1),(1,s_2)\}, 5\leqslant s_1\leqslant s_2$
4	$\{(1,2),(1,3),(1,4),(1,s_1),(1,s_2),(1,s_3)\}, 5\leqslant s_1\leqslant s_2\leqslant s_3$
5	$\{(1,2),(1,3),(1,s_1),(1,s_2),(1,s_3),(1,s_4)\}, 5\leqslant s_1\leqslant s_2\leqslant s_3$
6	$\{(1,2),2\times(1,3),3\times(1,4)\}$
7	$\{(1,2),2\times(1,3),2\times(1,4),(1,s)\}, s\geqslant 5$
8	$\{(1,2),2\times(1,3),(1,4),(1,s_1),(1,s_2)\}, 5\leqslant s_1\leqslant s_2$
9	$\{(1,2),2\times(1,3),(1,s_1),(1,s_2),(1,s_3)\}, 5\leqslant s_1\leqslant s_2\leqslant s_3$
10	$\{(1,2),3\times(1,3),2\times(1,4)\}$
11	$\{(1,2),3\times(1,3),(1,4),(1,s)\}, s\geqslant 5$
12	$\{(1,2),3\times(1,3),(1,s_1),(1,s_2)\}, 5\leqslant s_1\leqslant s_2$
13	$\{(1,2),4\times(1,3),(1,4)\}$
14	$\{(1,2),4\times(1,3),(1,s)\}, s\geqslant 5$

For cases No. 2–5 and 9, it is easy to compute that

$$-K_X^3 \geqslant -K^3(B^{(0)}) > 0.21,$$

which is absurd.

For case No. 7, all possible packings B_X with $\gamma \geqslant 0$ and $r_{\max} \geqslant 11$ are $\{(1,2),(1,3),(3,11),(1,s)\}$ or $\{(2,5),(3,11),(1,s)\}$ for some $5 \leqslant s \leqslant 9$. But in this case, $r_{\max} = 11$ and $-K^3 > 0.12$, which is absurd.

For case No. 10, all possible packings B_X with $r_{\max} \geqslant 11$ are the following:

$$\{(4,11),2\times(1,4)\},$$
$$\{(1,2),(5,17)\},$$
$$\{(1,2),(4,13),(1,4)\},$$
$$\{(1,2),2\times(1,3),(3,11)\},$$
$$\{(2,5),(1,3),(3,11)\},$$
$$\{(3,8),(3,11)\}.$$

The second and third baskets have $-K^3 < 1/30$, which is absurd. For other baskets we may apply Lemma 6.3 (2).

For case No. 12, all possible packings B_X with $\gamma \geqslant 0$ and $r_{\max} \geqslant 11$ are the following:

$$\{(4,11),(1,s_1),(1,s_2)\}, 5 \leqslant s_1 \leqslant s_2 \leqslant 8;$$
$$\{(1,2), 3 \times (1,3),(2,r_1)\}, r_1 = 11, 13;$$
$$\{(2,5), 2 \times (1,3),(2,r_1)\}, r_1 = 11, 13;$$
$$\{(3,8),(1,3),(2,r_1)\}, r_1 = 11, 13;$$
$$\{(4,11),(2,r_1)\}, r_1 = 11, 13.$$

Since $r_{\max} \leqslant 13$ and $-K^3 > 0.12$, all above baskets should be excluded.

For case No. 13, all packings dominate either $B_{\min} = \{(5,14),(1,4)\}$ or $B_{\min} = \{(1,2),(5,16)\}$. But then $-K_X^3 \leqslant -K^3(B_{\min}) < 0$, which is absurd.

Finally, for cases No. 1, 6, 8, 11, 14, we may get all possible packings with $\gamma \geqslant 0$ and $r_{\max} \geqslant 11$, which are listed in Table 6.18. In Table 6.18, for each basket B_X, if $r_X \leqslant 287$ and $r_{\max} \leqslant 12$, then we apply Lemma 6.3(2) (such baskets are marked with $\checkmark$ in the last column); if $-K_X^3$ does not satisfy the assumption in the lemma, we mark it with $\times$ in the last column; otherwise we can compute M_X and $\lambda(M_X)$, then find n_1 such that $P_{-n_1} > \lambda(M_X)n_1 + 1$ where P_{-n_1} is computed by Reid's Riemann–Roch formula. Hence by Proposition 5.2, $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant n_1$ since $P_{-1} > 0$. By assumption, $P_{-3} = 2$, we may take $m_0 = 3$. Then by Proposition 5.7(1), take $m_1 = n_1$, $\mu_0 \leqslant m_0$, and $\nu_0 = 1$, we get the integer n_2 such that φ_{-m} is birational for all $m \geqslant n_2$. (For the value of n_2 with a $*$ mark, we apply Proposition 5.7(3)).

Table 6.18

B_X	$-K^3$	M_X	$\lambda(M_X)$	n_1	m_0	$r_{\max}$	n_2
$\{(1,2),(3,11), 2\times(1,4)\}$							$\checkmark$
$\{(1,2),(4,15),(1,4)\}$	11/60	11	4	11	3	15	42
$\{(1,2),(5,19)\}$	7/38	7	7/2	10	3	19	39
$\{(1,2),(5,18)\}$	1/9	2	2	10	3	18	39
$\{(1,2),(1,3),(4,15)\}$	1/10	3	3	12	3	15	45
$\{(2,5),(4,15)\}$	2/15	2	2	9	3	15	36
$\{(1,2),(1,3),(3,11),(1,4)\}$							$\checkmark$
$\{(2,5),(3,11),(1,4)\}$							$\checkmark$
$\{(1,2),(3,11),(2,7)\}$							$\checkmark$
$\{(1,2),(3,11), 2\times(1,5)\}$							$\checkmark$
$\{(1,2), 2\times(1,3),(3,14)\}$	4/21	8	4	11	3	14	42
$\{(1,2),(3,11),(1,5),(1,6)\}$	52/165						$\times$
$\{(1,2), 2\times(1,3),(1,4),(2,11)\}$							$\checkmark$
$\{(2,5),(1,3),(1,4),(2,11)\}$	167/660						$\times$
$\{(3,8),(1,4),(2,11)\}$							$\checkmark$
Continued on next page							

Table 6.18 – continued from previous page

B_X	$-K^3$	M_X	$\lambda(M_X)$	n_1	m_0	$r_{\max}$	n_2
$\{(1,2),(1,3),(2,7),(2,11)\}$	107/462						×
$\{(1,2),(3,11),(2,11)\}$							✓
$\{(2,5),(2,7),(2,11)\}$	102/385						×
$\{(1,2),2\times(1,3),(1,4),(2,13)\}$	43/156						×
$\{(1,2),(1,3),(2,7),(2,13)\}$	157/546						×
$\{(4,11),(1,4),(1,5)\}$							✓
$\{(4,11),(2,9)\}$							✓
$\{(1,2),(4,13),(1,5)\}$	9/130	9	4	18	3	13	47*
$\{(4,11),(1,4),(1,6)\}$							✓
$\{(1,2),(4,13),(1,6)\}$	4/39	8	4	15	3	13	44*
$\{(4,11),(1,4),(1,7)\}$	47/308						×
$\{(1,2),(4,13),(1,7)\}$	23/182						×
$\{(4,11),(1,4),(1,8)\}$							✓
$\{(1,2),(4,13),(1,8)\}$	15/104						×
$\{(4,11),(1,4),(1,9)\}$	73/396						×
$\{(1,2),(4,13),(1,9)\}$	37/234						×
$\{(5,14),(1,5)\}$	1/70						×
$\{(4,11),(1,3),(1,5)\}$							✓
$\{(5,14),(1,6)\}$	1/21	2	2	14	3	14	45*
$\{(4,11),(1,3),(1,6)\}$							✓
$\{(5,14),(1,7)\}$	1/14	1	1	8	3	14	33
$\{(4,11),(1,3),(1,7)\}$							✓
$\{(5,14),(1,8)\}$	5/56	5	3	14	3	14	45*
$\{(4,11),(1,3),(1,8)\}$							✓
$\{(5,14),(1,9)\}$	13/126	13	13/3	16	3	14	47*
$\{(4,11),(1,3),(1,9)\}$							✓
$\{(5,14),(1,10)\}$	4/35	8	4	14	3	14	45*
$\{(4,11),(1,3),(1,10)\}$	37/330	37	8	20	3	11	45*
$\{(1,2),4\times(1,3),(1,11)\}$							✓
$\{(2,5),3\times(1,3),(1,11)\}$							✓

Subcase 2(viii): $(P_{-2}, P_{-3}) = (1,1)$. — In this case,

$$\begin{cases} n_{1,2}^0 = 2; \\ n_{1,3}^0 = 3 - P_{-4}; \\ n_{1,4}^0 = 1 + P_{-4} - \sigma_5. \end{cases}$$

Hence $(P_{-4}, \sigma_5) = (3,0),\ (3,1),\ (3,2),\ (3,3),\ (3,4),\ (2,0),\ (2,1),\ (2,2),$ $(2,3),\ (1,0),\ (1,1),\ (1,2)$. Hence the corresponding $B^{(0)}$ is in the following list:

Table 6.19

$B^{(0)}$
$\{2\times(1,2),4\times(1,4)\}$
$\{2\times(1,2),3\times(1,4),(1,s)\}, s \geqslant 5$
$\{2\times(1,2),2\times(1,4),(1,s_1),(1,s_2)\}, 5 \leqslant s_1 \leqslant s_2$
$\{2\times(1,2),(1,4),(1,s_1),(1,s_2),(1,s_3)\}, 5 \leqslant s_1 \leqslant s_2 \leqslant s_3$
$\{2\times(1,2),(1,s_1),(1,s_2),(1,s_3),(1,s_4)\}, 5 \leqslant s_1 \leqslant s_2 \leqslant s_3$
Continued on next page

Table 6.19 – continued from previous page

$B^{(0)}$
$\{2 \times (1,2), (1,3), 3 \times (1,4)\}$
$\{2 \times (1,2), (1,3), 2 \times (1,4), (1,s)\}, s \geqslant 5$
$\{2 \times (1,2), (1,3), (1,4), (1,s_1), (1,s_2)\}, 5 \leqslant s_1 \leqslant s_2$
$\{2 \times (1,2), (1,3), (1,s_1), (1,s_2), (1,s_3)\}, 5 \leqslant s_1 \leqslant s_2 \leqslant s_3$
$\{2 \times (1,2), 2 \times (1,3), 2 \times (1,4)\}$
$\{2 \times (1,2), 2 \times (1,3), (1,4), (1,s)\}, s \geqslant 5$
$\{2 \times (1,2), 2 \times (1,3), (1,s_1), (1,s_2)\}, 5 \leqslant s_1 \leqslant s_2$

Hence we may get all possible packings with $\gamma \geqslant 0$ and $r_{\max} \geqslant 11$, which are listed in Table 6.20. In Table 6.20, for each basket B_X, if $r_X \leqslant 287$ and $r_{\max} \leqslant 12$, then we apply Lemma 6.3(2) (such baskets are marked with $\checkmark$ in the last column); if $-K_X^3$ does not satisfy the assumption in the lemma, we mark it with $\times$ in the last column; otherwise we can compute M_X and $\lambda(M_X)$, then find n_1 such that $P_{-n_1} > \lambda(M_X)n_1 + 1$ where P_{-n_1} is computed by Reid's Riemann–Roch formula. Hence by Proposition 5.2, $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant n_1$ since $P_{-1} > 0$. Again by Reid's Riemann–Roch formula, we may take m_0 such that $P_{-m_0} \geqslant 2$. Then by Proposition 5.7(3), take $m_1 = n_1$, $\mu_0 \leqslant m_0$, and $\nu_0 = 1$, we get the integer n_2 such that φ_{-m} is birational for all $m \geqslant n_2$. (For the value of n_2 with a $*$ mark, we apply Proposition 5.7(1)).

Table 6.20

B_X	$-K^3$	M_X	$\lambda(M_X)$	n_1	m_0	$r_{\max}$	n_2
$\{2 \times (1,2), (1,4), (3,13)\}$	3/52	3	3	17	4	13	47
$\{2 \times (1,2), (4,17)\}$	1/17	2	2	13	4	17	51
$\{2 \times (1,2), (1,4), (3,14)\}$	3/28	3	3	12	4	14	44
$\{2 \times (1,2), (3,13), (1,5)\}$	7/65	14	14/3	16	4	13	46
$\{2 \times (1,2), (3,13), (1,6)\}$	11/78						$\times$
$\{2 \times (1,2), 2 \times (1,4), (2,11)\}$							$\checkmark$
$\{2 \times (1,2), (3,13), (1,7)\}$	15/91						$\times$
$\{2 \times (1,2), (3,13), (1,8)\}$	19/104						$\times$
$\{2 \times (1,2), 2 \times (1,4), (2,13)\}$	5/26						$\times$
$\{2 \times (1,2), (3,14), (1,5)\}$	11/70	11	4	12	4	14	44
$\{2 \times (1,2), (4,19)\}$	3/19	6	3	10	4	19	42*
$\{2 \times (1,2), (3,14), (1,6)\}$	4/21	8	4	11	4	14	43
$\{2 \times (1,2), (1,4), (1,5), (2,11)\}$							$\checkmark$
$\{2 \times (1,2), (2,9), (2,11)\}$							$\checkmark$
$\{2 \times (1,2), (1,4), (3,16)\}$	3/16	3	3	10	4	16	42*
$\{2 \times (1,2), (3,14), (1,7)\}$	3/14						$\times$
$\{2 \times (1,2), (1,4), (2,11), (1,6)\}$							$\checkmark$
$\{2 \times (1,2), (1,4), (3,17)\}$	15/68						$\times$
$\{2 \times (1,2), (1,5), (3,16)\}$	19/80						$\times$
$\{2 \times (1,2), (4,21)\}$	5/21						$\times$
Continued on next page							

Table 6.20 – continued from previous page

B_X	$-K^3$	M_X	$\lambda(M_X)$	n_1	m_0	$r_{\max}$	n_2
$\{2\times(1,2),(3,11),(1,4)\}$							✓
$\{2\times(1,2),(4,15)\}$	<0						×
$\{2\times(1,2),(3,11),(1,5)\}$							✓
$\{2\times(1,2),(1,3),(3,13)\}$	<0						×
$\{(1,2),(2,5),(3,13)\}$	$1/130$						×
$\{(3,7),(3,13)\}$	$2/91$						×
$\{2\times(1,2),(3,11),(1,6)\}$							✓
$\{2\times(1,2),(3,11),(1,7)\}$							✓
$\{2\times(1,2),(3,11),(1,8)\}$							✓
$\{2\times(1,2),(3,11),(1,9)\}$							✓
$\{2\times(1,2),(3,11),(1,10)\}$							✓
$\{2\times(1,2),(1,3),(3,14)\}$	$1/42$						×
$\{(1,2),(2,5),(3,14)\}$	$2/35$	4	3	17	4	14	49
$\{(3,7),(3,14)\}$	$1/14$	1	1	9	4	14	39*
$\{2\times(1,2),(1,3),(1,4),(2,11)\}$							✓
$\{2\times(1,2),(2,7),(2,11)\}$							✓
$\{(1,2),(2,5),(1,4),(2,11)\}$							✓
$\{(3,7),(1,4),(2,11)\}$	$31/308$	31	$31/4$	21	4	11	47
$\{2\times(1,2),(1,3),(1,4),(2,13)\}$	$17/156$	17	$17/3$	17	4	13	47
$\{2\times(1,2),(2,7),(2,13)\}$	$11/91$						×
$\{(1,2),(2,5),(1,4),(2,13)\}$	$37/260$						×
$\{(3,7),(1,4),(2,13)\}$	$57/364$						×
$\{2\times(1,2),(1,3),(1,5),(2,11)\}$	$17/165$	34	8	21	2	11	47
$\{(1,2),(2,5),(1,5),(2,11)\}$							✓
$\{(3,7),(1,5),(2,11)\}$	$58/385$						×
$\{2\times(1,2),(1,3),(3,16)\}$	$5/48$	5	3	13	4	16	49
$\{(1,2),(2,5),(3,16)\}$	$11/80$	11	4	13	4	16	49
$\{(3,7),(3,16)\}$	$17/112$	17	3	15	4	16	51
$\{2\times(1,2),(1,3),(2,11),(1,6)\}$							✓
$\{(1,2),(2,5),(2,11),(1,6)\}$	$28/165$						×
$\{(3,7),(2,11),(1,6)\}$	$85/462$						×
$\{2\times(1,2),(1,3),(3,17)\}$	$7/51$	14	$14/3$	14	4	17	52?
$\{(1,2),(2,5),(3,17)\}$	$29/170$	29	$29/4$	16	4	17	54?
$\{(3,7),(3,17)\}$	$22/119$	22	6	14	4	17	52?
$\{2\times(1,2),(1,3),(2,11),(1,7)\}$	$37/231$						×
$\{2\times(1,2),(1,3),(1,5),(2,13)\}$	$31/195$						×
$\{2\times(1,2),(1,3),(3,11)\}$							✓
$\{(1,2),(2,5),(3,11)\}$							✓
$\{(3,7),(3,11)\}$							✓
$\{2\times(1,2),2\times(1,3),(1,4),(1,11)\}$							✓
$\{(1,2),(2,5),(1,3),(1,4),(1,11)\}$	$17/660$						×
$\{2\times(1,2),2\times(1,3),(1,4),(1,12)\}$							✓
$\{2\times(1,2),2\times(1,3),(2,11)\}$							✓
$\{(1,2),(2,5),(1,3),(2,11)\}$	$1/330$						×
$\{(3,7),(1,3),(2,11)\}$							✓
$\{(1,2),(3,8),(2,11)\}$							✓
$\{2\times(2,5),(2,11)\}$							✓
$\{2\times(1,2),2\times(1,3),(2,13)\}$	$1/39$						×
$\{(1,2),(2,5),(1,3),(2,13)\}$	$23/390$	23	6	24	5	13	55?
$\{(3,7),(1,3),(2,13)\}$	$20/273$	20	6	22	5	13	53
$\{(1,2),(3,8),(2,13)\}$	$7/104$	7	$7/2$	17	5	13	48
$\{2\times(2,5),(2,13)\}$	$6/65$	6	3	14	5	13	45
$\{2\times(1,2),2\times(1,3),(2,15)\}$	$1/15$	2	2	14	6	15	50
$\{(1,2),(2,5),(1,3),(2,15)\}$	$1/10$	3	3	14	5	15	49

There are 4 cases with "?" marked in the n_2 column of Table 6.20, where the values of n_2 are larger than what we expect. So we discuss them in details in the following.

If B_X is among $\{2 \times (1,2), (1,3), (3,17)\}$, $\{(1,2), (2,5), (3,17)\}$, and $\{(3,7), (3,17)\}$, we know that $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant 16$ from the list. Note that $P_{-7} \geqslant 11$. Take $m_0 = 4$. If $|-7K_X|$ and $|-4K_X|$ are not composed with the same pencil, then we may take $m_1 = 7$ and $\mu_0 \leqslant 4$, and by Proposition 5.7(1), φ_{-m} is birational for all $m \geqslant 33$; if $|-7K_X|$ and $|-4K_X|$ are composed with the same pencil, then we may take $m_1 = 16$ and $\mu_0 \leqslant \frac{7}{10}$ by Remark 5.8, and by Proposition 5.7(3), φ_{-m} is birational for all $m \geqslant 50$.

If $B_X = \{(1,2), (2,5), (1,3), (2,13)\}$, we know that $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant 24$ from the list. Note that $P_{-10} \geqslant 13$. Take $m_0 = 5$. If $|-10K_X|$ and $|-5K_X|$ are not composed with the same pencil, then we may take $m_1 = 10$ and $\mu_0 \leqslant 5$, and by Proposition 5.7(3), φ_{-m} is birational for all $m \geqslant 41$; if $|-10K_X|$ and $|-5K_X|$ are composed with the same pencil, then we may take $m_1 = 24$ and $\mu_0 \leqslant \frac{10}{12}$ by Remark 5.8, and by Proposition 5.7(3), φ_{-m} is birational for all $m \geqslant 50$.

Combining all above cases, the proof is completed. $\qquad\square$

THEOREM 6.11. — *Let (X, Y, Z) be a Fano–Mori triple such that $\rho(Y) > 1$. Assume that $P_{-1} > 0$ and $1/30 \leqslant -K_X^3 < 0.21$. Then φ_{-m} is birational for all $m \geqslant 51$.*

Proof. — If $r_X = 840$, then we are done by Theorem 6.6(2). Hence we may always assume that $r_X \leqslant 660$. Since $P_{-1} > 0$, we may always take $\nu_0 = 1$.

We discuss by the value of $r_{\max}$.

Case 1: $r_{\max} \leqslant 8$. — Recall that $P_{-8} \geqslant 2$ by Proposition 2.6. We may take $m_0 = 8$. Note that by Lemma 6.9(1), $P_{-18} \geqslant 21$. If $|-18K_X|$ and $|-8K_X|$ are not composed with the same pencil, we may take $m_1 = 18$ and $\mu_0 \leqslant 8$. By Proposition 5.7(3), φ_{-m} is birational for all $m \geqslant 42$. If $|-18K_X|$ and $|-8K_X|$ are composed with the same pencil, we may take $m_0 = 35$ by Lemma 6.9(2) and $\mu_0 \leqslant \frac{18}{20}$ by Remark 5.8. By Proposition 5.7(3), φ_{-m} is birational for all $m \geqslant 51$.

Case 2: $r_{\max} = 9$. — In this case, by inequality (2.3), arguing as [12, Proof of Proposition 2.4], one can show that either $r_X \leqslant 504 = 9 \times 8 \times 7$ or $r_X = 630$, moreover, in the latter case, the set of local indices is either $\{2, 5, 7, 9\}$ or $\{2, 2, 5, 7, 9\}$. (We leave this to interested readers as an exercise!)

If $r_X \leqslant 504$, then by Lemma 5.5,

$$\frac{\lambda(M_X)}{-K_X^3} \leqslant \max\left\{1, \frac{3}{1/30}, \sqrt{\frac{2 \times 504}{1/30}}\right\} < 174.$$

Take $t = 11$, then

$$-\frac{3}{4} + \sqrt{\frac{12}{t(-K_X^3)} + \frac{6\lambda(M_X)}{-K_X^3} + \frac{1}{16}} < 33.$$

By Proposition 5.4, $|-mK_X|$ is not composed with a pencil for $m \geqslant 33$. Recall that $P_{-8} \geqslant 2$ by Proposition 2.6. We may take $m_0 = 8$. Note that by Lemma 6.9(1), $P_{-18} \geqslant 21$. If $|-18K_X|$ and $|-8K_X|$ are not composed with the same pencil, we may take $m_1 = 18$ and $\mu_0 \leqslant 8$. By Proposition 5.7(3), φ_{-m} is birational for all $m \geqslant 44$. If $|-18K_X|$ and $|-8K_X|$ are composed with the same pencil, we may take $m_1 = 33$ by Lemma 6.9(2) and $\mu_0 \leqslant \frac{18}{20}$ by Remark 5.8. By Proposition 5.7(3), φ_{-m} is birational for all $m \geqslant 51$.

Now we consider the case $r_X = 630$. In this case, B_X is either $\{(1,2), (a,5), (b,7), (c,9)\}$ or $\{2 \times (1,2), (a,5), (b,7), (c,9)\}$ for some $a \in \{1,2\}$, $b \in \{1,2,3\}$, and $c \in \{1,2,4\}$. Note that by Theorem 6.7(1), we may assume that $1/30 \leqslant -K_X^3 < 0.12$.

First we consider $B_X = \{(1,2), (a,5), (b,7), (c,9)\}$. If $P_{-1} \geqslant 2$, then by the equality (2.1),

$$-K_X^3 = 2P_{-1} + \frac{1}{2} + \frac{a(5-a)}{5} + \frac{b(7-b)}{7} + \frac{c(9-c)}{9} - 6$$
$$\geqslant \frac{1}{2} + \frac{4}{5} + \frac{6}{7} + \frac{8}{9} - 2 > 1,$$

a contradiction. Hence $P_{-1} = 1$ and

$$-K_X^3 = \frac{1}{2} + \frac{a(5-a)}{5} + \frac{b(7-b)}{7} + \frac{c(9-c)}{9} - 4.$$

It is easy to check that $1/30 \leqslant -K_X^3 < 0.12$ if and only if $(a,b,c) = (2,1,2)$, that is, $B_X = \{(1,2), (2,5), (1,7), (2,9)\}$, $-K_X^3 = 71/630$, $M_X = 71$, and $\lambda(M_X) = 71/5$. Note that $P_{-25} > 25\lambda(M_X)+1$ where P_{-25} is computed by Reid's Riemann–Roch formula. Hence by Proposition 5.2, $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant 25$ since $P_{-1} > 0$. Again by Reid's Riemann–Roch formula, we may take $m_0 = 4$ since $P_{-4} = 2$. Then by Proposition 5.7(3), take $m_1 = 25$, $\mu_0 \leqslant 4$, and $\nu_0 = 1$, we get that φ_{-m} is birational for all $m \geqslant 47$.

Then we consider $B_X = \{2 \times (1,2), (a,5), (b,7), (c,9)\}$. If $P_{-1} \geqslant 2$, then by the equality (2.1),

$$-K_X^3 = 2P_{-1} + 2 \times \frac{1}{2} + \frac{a(5-a)}{5} + \frac{b(7-b)}{7} + \frac{c(9-c)}{9} - 6$$
$$\geqslant 2 \times \frac{1}{2} + \frac{4}{5} + \frac{6}{7} + \frac{8}{9} - 2 > 1,$$

a contradiction. Hence $P_{-1} = 1$ and

$$-K_X^3 = 1 + \frac{a(5-a)}{5} + \frac{b(7-b)}{7} + \frac{c(9-c)}{9} - 4.$$

It is easy to check that $1/30 \leqslant -K_X^3 < 0.12$ if and only if $(a,b,c) = (1,2,1)$, that is, $B_X = \{2 \times (1,2), (1,5), (2,7), (1,9)\}$, $-K_X^3 = 37/315$, $M_X = 74$, and $\lambda(M_X) = 12$. Note that $P_{-25} > 25\lambda(M_X) + 1$ where P_{-25} is computed by Reid's Riemann–Roch formula. Hence by Proposition 5.2, $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant 25$ since $P_{-1} > 0$. Again by Reid's Riemann–Roch formula, we may take $m_0 = 4$ since $P_{-4} = 2$. Then by Proposition 5.7(3), take $m_1 = 25$, $\mu_0 \leqslant 4$, and $\nu_0 = 1$, we get that φ_{-m} is birational for all $m \geqslant 47$.

Case 3: $r_{\max} = 10$. — In this case, by inequality (2.3), arguing as [12, Proof of Proposition 2.4], one can show that either $r_X \leqslant 210 = 10 \times 7 \times 3$ or $r_X = 420$, moreover, in the latter case, the set of local indices is $\{3, 4, 7, 10\}$. (We leave this again as an exercise.)

If $r_X \leqslant 210$, then we may apply Lemma 6.3(2). Hence we only consider the case $r_X = 420$. Note that by Theorem 6.7(1), we may assume that $1/30 \leqslant -K_X^3 < 0.12$.

In this case, B_X is $\{(1,3), (1,4), (a,7), (b,10)\}$ for some $a \in \{1,2,3\}$, $b \in \{1,3\}$. If $P_{-1} \geqslant 2$, then by the equality (2.1),

$$-K_X^3 = 2P_{-1} + \frac{2}{3} + \frac{3}{4} + \frac{a(7-a)}{7} + \frac{b(10-b)}{10} - 6$$
$$\geqslant \frac{2}{3} + \frac{3}{4} + \frac{6}{7} + \frac{9}{10} - 2 > 1,$$

a contradiction. Hence $P_{-1} = 1$ and

$$-K_X^3 = \frac{2}{3} + \frac{3}{4} + \frac{a(7-a)}{7} + \frac{b(10-b)}{10} - 4.$$

One can directly check that $1/30 \leqslant -K_X^3 < 0.12$ could never happen.

Case 4: $r_{\max} \geqslant 11$. — This case is proved by Lemma 6.10 and Theorem 6.7.

Combining all above discussions, the proof is completed.   □

6.9. The case $P_{-2} > 0$ and $-K_X^3 < 1/30$

THEOREM 6.12. — *Let (X, Y, Z) be a Fano–Mori triple such that $\rho(Y) > 1$. Assume that $P_{-2} > 0$ and $-K_X^3 < 1/30$. Then*

   (1) $\dim \overline{\varphi_{-m}(X)} > 1$ *for all $m \geqslant 37$;*

   (2) φ_{-m} *is birational for all $m \geqslant 52$.*

Proof. — We follow the argument of [7, Proof of Theorem 4.4] to classify all possible geometric baskets with $-K^3 < 1/30$.

If $P_{-1} = 0$, by [7, Proof of Theorem 4.4, Case I], since $P_{-2} > 0$ and $-K^3 < 1/30$, all possible geometric baskets are dominated by $\{7 \times (1, 2), (3, 7), (1, 5)\}$, which we already treated as the last four cases in Table 6.8 in the proof of Theorem 6.8.

Now we consider $P_{-1} \geqslant 1$. By [7, Proof of Theorem 4.4, Case IV], if $P_{-1} \geqslant 3$, then $-K_X^3 \geqslant 1/2$, a contradiction. Hence $P_{-1} = 1$ or 2.

Case 1: $P_{-1} = 1$. — By [7, Proof of Theorem 4.4, Subcase II-1, Subcase II-2]), one has $-K_X^3 \geqslant 1/12$ when $P_{-2} \geqslant 3$. Hence we have $P_{-2} \leqslant 2$.

Subcase 1(i): $P_{-2} = 2$. — Note that by [7, Proof of Theorem 4.4, Subcase II-3], either

$$B^{(0)} = \{5 \times (1, 2), (1, 3), (1, s)\}$$

for some $s \geqslant 7$, or B_X is dominated by $\{3 \times (1, 2), (3, 7), (1, 5)\}$.

In the former case, if $s \geqslant 8$, then

$$-K^3(B_X) \geqslant -K^3(B^{(0)}) \geqslant \frac{1}{24},$$

which is absurd. So $s = 7$, and one-step packing has $-K^3 > 1/30$, hence $B_X = B^{(0)} = \{5 \times (1, 2), (1, 3), (1, 7)\}$. In the latter case, all packings of $\{3 \times (1, 2), (3, 7), (1, 5)\}$ has $-K^3 < 1/30$. Hence we may get all possible packings, which are listed in Table 6.21.

Table 6.21

B_X	$-K^3$	M_X	$\lambda(M_X)$	n_1	m_0	$r_{\max}$	n_2
$\{5 \times (1, 2), (1, 3), (1, 7)\}$	$1/42$						✓
$\{3 \times (1, 2), (3, 7), (1, 5)\}$	$1/70$						✓
$\{2 \times (1, 2), (4, 9), (1, 5)\}$	$1/45$						✓
$\{(1, 2), (5, 11), (1, 5)\}$	$3/110$						✓
$\{(6, 13), (1, 5)\}$	$2/65$	2	2	17	2	13	45

In Table 6.21, for each basket B_X, if $r_X \leqslant 165$ and $r_{\max} \leqslant 12$, then we apply Lemmas 6.2 and 6.3 (2) (such baskets are marked with ✓ in the last

column), otherwise we can compute M_X and $\lambda(M_X)$, then find n_1 such that $P_{-n_1} > \lambda(M_X)n_1 + 1$ where P_{-n_1} is computed by Reid's Riemann–Roch formula. Hence by Proposition 5.2, $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant n_1$ since $P_{-1} > 0$. By assumption, $P_{-2} = 2$, we may take $m_0 = 2$. Then by Proposition 5.7(3), take $m_1 = n_1$, $\mu_0 \leqslant m_0$, and $\nu_0 = 1$, we get the integer n_2 such that φ_{-m} is birational for all $m \geqslant n_2$.

Subcase 1(ii): $P_{-2} = 1$. — By [7, Proof of Theorem 4.4, Subcase II-4], since $-K_X^3 < 1/30$, (P_{-3}, P_{-4}) can only take the values $(2, 3)$, $(2, 2)$, $(1, 2)$ and $(1, 1)$.

If $(P_{-3}, P_{-4}) = (2, 3)$, then B_X is dominated by

$$\{(1, 2), 2 \times (1, 3), (2, 7), (1, 4)\}$$

by [7, Proof of Theorem 4.4, Case II-4b]. Hence we may get all possible packings with $0 < -K^3(B_X) < 1/30$, which are listed in Table 6.22. In Table 6.22, for each basket B_X, if $r_X \leqslant 165$ and $r_{\max} \leqslant 12$, then we apply Lemmas 6.2 and 6.3(2) (such baskets are marked with $\checkmark$ in the last column), otherwise we can compute M_X and $\lambda(M_X)$, then find n_1 such that $P_{-n_1} > \lambda(M_X)n_1 + 1$ where P_{-n_1} is computed by Reid's Riemann–Roch formula. Hence by Proposition 5.2, $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant n_1$ since $P_{-1} > 0$. By assumption, $P_{-3} = 2$, we may take $m_0 = 3$. Then by Proposition 5.7(3), take $m_1 = n_1$, $\mu_0 \leqslant m_0$, and $\nu_0 = 1$, we get the integer n_2 such that φ_{-m} is birational for all $m \geqslant n_2$. (For the values of n_2 with a $*$ mark, we apply Proposition 5.7(1)).

Table 6.22

B_X	$-K^3$	M_X	$\lambda(M_X)$	n_1	m_0	$r_{\max}$	n_2
$\{(1, 2), 2 \times (1, 3), (2, 7), (1, 4)\}$	1/84						$\checkmark$
$\{(1, 2), (1, 3), (3, 10), (1, 4)\}$	1/60						$\checkmark$
$\{(1, 2), (4, 13), (1, 4)\}$	1/52	1	1	13	3	13	42
$\{(1, 2), 2 \times (1, 3), (3, 11)\}$	1/66						$\checkmark$
$\{(1, 2), (1, 3), 2 \times (2, 7)\}$	1/42						$\checkmark$
$\{(1, 2), (3, 10), (2, 7)\}$	1/35						$\checkmark$
$\{(1, 2), (5, 17)\}$	1/34	1	1	11	3	17	42$*$

If $(P_{-3}, P_{-4}) = (2, 2)$, then by [7, Proof of Theorem 4.4, Case II-4d], either

$$B^{(0)} = \{(1, 2), 4 \times (1, 3), (1, s)\}$$

for some $s \geqslant 7$, or B_X is dominated by $\{(3,8), 2 \times (1,3), (1,5)\}$. In the former case, if $s \geqslant 8$, then

$$-K^3(B_X) \geqslant -K^3(B^{(0)}) \geqslant \frac{1}{24},$$

which is absurd. So $s = 7$, and since its one-step packing has $-K^3 > 1/30$, $B_X = B^{(0)} = \{(1,2), 4 \times (1,3), (1,7)\}$. Hence we may get all possible packings with $0 < -K^3(B_X) < 1/30$, which are listed in Table 6.23. In Table 6.23, for each basket B_X, if $r_X \leqslant 165$ and $r_{\max} \leqslant 12$, then we apply Lemmas 6.2 and 6.3 (2) (such baskets are marked with ✓ in the last column), otherwise we can compute M_X and $\lambda(M_X)$, then find n_1 such that $P_{-n_1} > \lambda(M_X)n_1 + 1$ where P_{-n_1} is computed by Reid's Riemann–Roch formula. Hence by Proposition 5.2, $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant n_1$ since $P_{-1} > 0$. By assumption, $P_{-3} = 2$, we may take $m_0 = 3$. Then by Proposition 5.7 (3), take $m_1 = n_1$, $\mu_0 \leqslant m_0$, and $\nu_0 = 1$, we get the integer n_2 such that φ_{-m} is birational for all $m \geqslant n_2$.

Table 6.23

B_X	$-K^3$	M_X	$\lambda(M_X)$	n_1	m_0	$r_{\max}$	n_2
$\{(1,2), 4 \times (1,3), (1,7)\}$	$1/42$						✓
$\{(3,8), 2 \times (1,3), (1,5)\}$	$1/120$						✓
$\{(4,11), (1,3), (1,5)\}$	$2/165$						✓
$\{(5,14), (1,5)\}$	$1/70$	1	1	16	3	14	47

If $(P_{-3}, P_{-4}) = (1,2)$, then by [7, Proof of Theorem 4.4, Case II-4e], $1 \leqslant \sigma_5 \leqslant 3$.

If $\sigma_5 \geqslant 2$, then $B^{(0)}$ is either

$$B^{(0)} = \{2 \times (1,2), (1,3), (1,4), (1,s_1), (1,s_2)\}$$

for some $s_2 \geqslant s_1 \geqslant 5$ or

$$B^{(0)} = \{2 \times (1,2), (1,3), (1,s_1), (1,s_2), (1,s_3)\}$$

for some $s_3 \geqslant s_2 \geqslant s_1 \geqslant 5$. For the former case, if $s_2 \geqslant 6$, then

$$-K^3(B_X) \geqslant -K^3(B^{(0)}) \geqslant \frac{1}{20},$$

hence $s_1 = s_2 = 5$ and

$$B^{(0)} = \{2 \times (1,2), (1,3), (1,4), 2 \times (1,5)\}.$$

For the latter one, we always have

$$-K^3(B_X) \geqslant -K^3(B^{(0)}) \geqslant \frac{1}{15}.$$

If $\sigma_5 = 1$, then by [7, Proof of Theorem 4.4, Case II-4e], either

$$B^{(0)} = \{2 \times (1,2), (1,3), 2 \times (1,4), (1,s)\}$$

for some $s \geqslant 7$, or B_X is dominated by one of the following baskets:

$$\{2 \times (1,2), (2,7), (1,4), (1,6)\},$$
$$\{(3,7), 2 \times (1,4), (1,5)\},$$
$$\{(1,2), (2,5), (1,4), (2,9)\}.$$

In the former case, if $s \geqslant 8$, then

$$-K^3(B_X) \geqslant -K^3(B^{(0)}) \geqslant \frac{1}{24},$$

which is absurd. So $s = 7$, and since its one-step packing has $-K^3 > 1/30$, $B_X = B^{(0)} = \{2 \times (1,2), (1,3), 2 \times (1,4), (1,7)\}$. Hence we may get all possible packings with $0 < -K^3(B_X) < 1/30$, which are listed in Table 6.24. In Table 6.24, for each basket B_X, if $r_X \leqslant 165$ and $r_{\max} \leqslant 12$, then we apply Lemmas 6.2 and 6.3(2) (such baskets are marked with $\checkmark$ in the last column), otherwise we can compute M_X and $\lambda(M_X)$, then find n_1 such that $P_{-n_1} > \lambda(M_X)n_1 + 1$ where P_{-n_1} is computed by Reid's Riemann–Roch formula. Hence by Proposition 5.2, $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant n_1$ since $P_{-1} > 0$. By assumption, $P_{-4} = 2$, we may take $m_0 = 4$. Then by Proposition 5.7(3), take $m_1 = n_1$, $\mu_0 \leqslant m_0$, and $\nu_0 = 1$, we get the integer n_2 such that φ_{-m} is birational for all $m \geqslant n_2$.

Table 6.24

B_X	$-K^3$	M_X	$\lambda(M_X)$	n_1	m_0	$r_{\max}$	n_2
$\{2 \times (1,2), (1,3), (1,4), 2 \times (1,5)\}$	1/60						$\checkmark$
$\{2 \times (1,2), (2,7), 2 \times (1,5)\}$	1/35						$\checkmark$
$\{2 \times (1,2), (1,3), (2,9), (1,5)\}$	1/45						$\checkmark$
$\{2 \times (1,2), (1,3), (3,14)\}$	1/42	1	1	14	4	14	46
$\{2 \times (1,2), (1,3), 2 \times (1,4), (1,7)\}$	1/42						$\checkmark$
$\{2 \times (1,2), (2,7), (1,4), (1,6))\}$	1/84						$\checkmark$
$\{2 \times (1,2), (3,11), (1,6))\}$	1/66						$\checkmark$
$\{(3,7), 2 \times (1,4), (1,5)\}$	1/70						$\checkmark$
$\{(1,2), (2,5), (1,4), (2,9)\}$	1/180	1	1	26	4	9	48
$\{(3,7), (1,4), (2,9)\}$	5/252	5	3	28	4	9	50
$\{(3,7), (3,13)\}$	2/91	2	2	21	4	13	51
$\{(1,2), (2,5), (3,13)\}$	1/130	1	1	22	4	13	52?

Note that there is one case with "?" mark in the n_2 value column of Table 6.24 where we get bigger n_2 value than we expect. So we discuss it in details. If $B_X = \{(1,2), (2,5), (3,13)\}$, we know that $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant 22$ from the list. Note that $P_{-12} = 7$. Take $m_0 = 4$. If $|-12K_X|$ and

$|-4K_X|$ are not composed with the same pencil, then we may take $m_1 = 12$ and $\mu_0 \leqslant 4$, and by Proposition 5.7 (3), φ_{-m} is birational for all $m \geqslant 42$; if $|-12K_X|$ and $|-4K_X|$ are composed with the same pencil, then we may take $m_1 = 22$ and $\mu_0 \leqslant 2$ by Remark 5.8, and by Proposition 5.7 (3), φ_{-m} is birational for all $m \geqslant 50$.

If $(P_{-3}, P_{-4}) = (1, 1)$, then by [7, Proof of Theorem 4.4, Case II-4f], any B_X satisfying $-K^3(B_X) < 1/30$ and $\gamma(B_X) \geqslant 0$ is either dominated by one of the following baskets:

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Table 6.25

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B
$\{2 \times (1,2), (1,3), (2,7), (1,11)\}$
$\{(1,2), (2,5), (1,3), (1,4), (1,s)\}, s = 9, 10, 11$
$\{(3,7), (1,3), (1,4), (1,8)\}$
$\{(1,2), (2,5), (2,7), (1,8)\}$
$\{2 \times (2,5), (1,4), (1,s)\}, s = 7, 8$
$\{(1,2), (2,5), (1,3), (1,5), (1,7)\}$
$\{(3,7), (1,3), (1,5), (1,6)\}$
$\{(1,2), (3,8), (1,5), (1,6)\}$
$\{(1,2), (2,5), (1,3), (2,11)\}$

or its initial basket

$$B^{(0)} = \{2 \times (1,2), 2 \times (1,3), (1, s_1), (1, s_2)\}$$

for some $s_2 \geqslant s_1 \geqslant 5$ and $s_1 + s_2 \geqslant 13$. Note that in the latter case, by $-K^3(B^{(0)}) < 1/30$, the possible values of (s_1, s_2) are $(5,8), (5,9), (6,7)$.

Hence we may get all possible packings with $0 < -K^3(B_X) < 1/30$ with $\gamma \geqslant 0$, which are listed in Table 6.26. In Table 6.26, for each basket B_X, if $r_X \leqslant 165$ and $r_{\max} \leqslant 12$, then we apply Lemmas 6.2 and 6.3 (2) (such baskets are marked with ✓ in the last column), otherwise we can compute M_X and $\lambda(M_X)$, then find n_1 such that $P_{-n_1} > \lambda(M_X)n_1 + 1$ where P_{-n_1} is computed by Reid's Riemann–Roch formula. Hence by Proposition 5.2, $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant n_1$ since $P_{-1} > 0$. Again by Reid's Riemann–Roch formula, we may find m_0 such that $P_{-m_0} \geqslant 2$. Then by Proposition 5.7 (3), take $m_1 = n_1$, $\mu_0 \leqslant m_0$, and $\nu_0 = 1$, we get the integer n_2 such that φ_{-m} is birational for all $m \geqslant n_2$.

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Table 6.26

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B_X	$-K^3$	M_X	$\lambda(M_X)$	n_1	m_0	$r_{\max}$	n_2
$\{2 \times (1,2), (1,3), (2,7), (1,11)\}$	1/231						?
$\{2 \times (1,2), (3,10), (1,11)\}$	1/110						?
Continued on next page							

Table 6.26 – continued from previous page

B_X	$-K^3$	M_X	$\lambda(M_X)$	n_1	m_0	$r_{\max}$	n_2
$\{(1,2),(2,5),(1,3),(1,4),(1,9)\}$	$1/180$	1	1	30	8	9	56?
$\{(1,2),(2,5),(2,7),(1,9)\}$	$11/630$	11	4	36	7	9	61?
$\{(3,7),(2,7),(1,9)\}$	$2/63$						✓
$\{(3,7),(1,3),(1,4),(1,9)\}$	$5/252$	5	3	29	7	9	54?
$\{(1,2),(3,8),(1,4),(1,9)\}$	$1/72$						✓
$\{(1,2),(2,5),(1,3),(1,4),(1,10)\}$	$1/60$						✓
$\{(3,7),(1,3),(1,4),(1,10)\}$	$13/420$	13	$13/3$	29	7	10	56?
$\{(1,2),(3,8),(1,4),(1,10)\}$	$1/40$						✓
$\{(1,2),(2,5),(2,7),(1,10)\}$	$1/35$						✓
$\{(1,2),(2,5),(1,3),(1,4),(1,11)\}$	$17/660$	17	$17/3$	36	8	11	66?
$\{(3,7),(1,3),(1,4),(1,8)\}$	$1/168$	1	1	28	7	8	51
$\{(1,2),(2,5),(2,7),(1,8)\}$	$1/280$	1	1	36	7	8	59?
$\{(3,7),(2,7),(1,8)\}$	$1/56$						✓
$\{2\times(2,5),(1,4),(1,7)\}$	$1/140$						✓
$\{2\times(2,5),(1,4),(1,8)\}$	$1/40$						✓
$\{(1,2),(2,5),(1,3),(1,5),(1,7)\}$	$1/42$	5	3	26	5	7	45
$\{(1,2),(3,8),(1,5),(1,7)\}$	$9/280$	9	4	26	5	8	47
$\{(3,7),(1,3),(1,5),(1,6)\}$	$1/70$	3	3	34	5	7	53?
$\{(3,7),(1,3),(2,11)\}$	$4/231$	4	3	31	5	11	58?
$\{(1,2),(3,8),(1,5),(1,6)\}$	$1/120$						✓
$\{(1,2),(3,8),(2,11)\}$	$1/88$						✓
$\{(1,2),(2,5),(1,3),(2,11)\}$	$1/330$	1	1	37	5	11	64?
$\{2\times(1,2),2\times(1,3),(1,5),(1,8)\}$	$1/120$						✓
$\{2\times(1,2),2\times(1,3),(1,5),(1,9)\}$	$1/45$						✓
$\{2\times(1,2),2\times(1,3),(1,6),(1,7)\}$	$1/42$						✓
$\{2\times(1,2),2\times(1,3),(2,13)\}$	$1/39$	2	2	20	6	13	52?

There are 12 cases with "?" marked in the n_2 column of Table 6.26. The first two baskets are non-geometric since they have $P_{-1} = 1$ but $P_{-5} = 0$. For the rest 10 baskets with "?" marked, we discuss them in more details as followings.

If $B_X = \{(1,2),(2,5),(1,3),(1,4),(1,9)\}$, then $P_{-8} = 2$ and we dealt it in Theorem 6.1.

If $B_X = \{(1,2),(2,5),(2,7),(1,9)\}$, we know that $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant 36$ from the list. Note that $P_{-18} = 22$. Take $m_0 = 7$. If $|-18K_X|$ and $|-7K_X|$ are not composed with the same pencil, then we may take $m_1 = 18$ and $\mu_0 \leqslant 7$, and by Proposition 5.7(3), $\varphi_{-m,X}$ is birational for all $m \geqslant 43$. Hence we may assume that $|-18K_X|$ and $|-7K_X|$ are composed with the same pencil, then we may take $\mu_0 \leqslant \mu_0' = \frac{18}{21}$ by Remark 5.8. We go on studying this situation by considering $|-31K_X|$. We have $P_{-31} = 96$. If $|-31K_X|$ is not composed with a pencil, then we may take $m_1 = 31$ and by Proposition 5.7(3), φ_{-m} is birational for $m \geqslant 49$; if $|-31K_X|$ is composed with a pencil, then by Proposition 4.5, keep the same notation as in Subsection 2.1,

$$3 < \frac{96}{31} = \frac{P_{-31} - 1}{31} \leqslant \max\{3, -K_X^3, 2N_0\} \leqslant 2N_0,$$

where $N_0 = r_X(\pi^*(K_X)^2 \cdot S)$. This implies that $N_0 \geqslant 2$, and by Proposition 5.9, $\varphi_{-m,X}$ is birational for $m \geqslant 51$.

If $B_X = \{(3,7),(1,3),(1,4),(1,9)\}$, we know that $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant 29$ from the list. Note that $P_{-17} = 20$. Take $m_0 = 7$. If $|-17K_X|$ and $|-7K_X|$ are not composed with the same pencil, then we may take $m_1 = 17$ and $\mu_0 \leqslant 7$, and by Proposition 5.7 (3), φ_{-m} is birational for all $m \geqslant 42$; if $|-17K_X|$ and $|-7K_X|$ are composed with the same pencil, then we may take $m_1 = 29$ and $\mu_0 \leqslant \frac{17}{19}$ by Remark 5.8, and by Proposition 5.7 (3), φ_{-m} is birational for all $m \geqslant 47$.

If $B_X = \{(3,7),(1,3),(1,4),(1,10)\}$, we know that $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant 29$ from the list. Note that $P_{-14} = 17$. Take $m_0 = 7$. If $|-14K_X|$ and $|-7K_X|$ are not composed with the same pencil, then we may take $m_1 = 14$ and $\mu_0 \leqslant 7$, and by Proposition 5.7 (3), φ_{-m} is birational for all $m \geqslant 41$; if $|-14K_X|$ and $|-7K_X|$ are composed with the same pencil, then we may take $m_1 = 29$ and $\mu_0 \leqslant \frac{14}{16}$ by Remark 5.8, and by Proposition 5.7 (3), φ_{-m} is birational for all $m \geqslant 49$.

If $B_X = \{(1,2),(2,5),(1,3),(1,4),(1,11)\}$, then $P_{-8} = 2$ and we dealt it in Theorem 6.1.

If $B_X = \{(1,2),(2,5),(2,7),(1,8)\}$, take $m_0 = 7$. Note that $P_{-22} = 12$. If $|-22K_X|$ and $|-7K_X|$ are not composed with the same pencil, then we may take $m_1 = 22$ and $\mu_0 \leqslant 7$, and by Proposition 5.7 (3), φ_{-m} is birational for all $m \geqslant 45$; if $|-22K_X|$ and $|-7K_X|$ are composed with the same pencil, then we may take $\mu_0' = \frac{22}{P_{-22}-1} = 2$ by Remark 5.8, and by Proposition 5.9, φ_{-m} is birational for all $m \geqslant 49$.

If $B_X = \{(3,7),(1,3),(1,5),(1,6)\}$, take $m_0 = 5$. Then we may take $\mu_0' \leqslant m_0 = 5$ by Remark 5.8, and by Proposition 5.9, φ_{-m} is birational for all $m \geqslant 45$.

If $B_X = \{(3,7),(1,3),(2,11)\}$, take $m_0 = 5$. Then we may take $\mu_0' \leqslant m_0 = 5$ by Remark 5.8, and by Proposition 5.9, φ_{-m} is birational for all $m \geqslant 48$.

If $B_X = \{(1,2),(2,5),(1,3),(2,11)\}$, take $m_0 = 5$. Note that $P_{-24} = 16$. If $|-24K_X|$ and $|-5K_X|$ are not composed with the same pencil, then we may take $m_1 = 24$ and $\mu_0 \leqslant 5$, and by Proposition 5.7 (3), φ_{-m} is birational for all $m \geqslant 51$; if $|-24K_X|$ and $|-5K_X|$ are composed with the same pencil, then we may take $\mu_0' = \frac{24}{P_{-24}-1} = \frac{24}{15}$ by Remark 5.8, and by Proposition 5.9, φ_{-m} is birational for all $m \geqslant 52$. Note that this is in fact the only case that φ_{-51} might not be birational.

If $B_X = \{2 \times (1,2), 2 \times (1,3), (2,13)\}$, we know that $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geqslant 20$ from the list. Note that $P_{-13} = 14$. Take $m_0 = 6$. If $|-13K_X|$

and $|-6K_X|$ are not composed with the same pencil, then we may take $m_1 = 13$ and $\mu_0 \leqslant 6$, and by Proposition 5.7(3), φ_{-m} is birational for all $m \geqslant 45$; if $|-13K_X|$ and $|-6K_X|$ are composed with the same pencil, then we may take $m_1 = 20$ and $\mu_0 \leqslant 1$ by Remark 5.8, and by Proposition 5.7(3), φ_{-m} is birational for all $m \geqslant 47$.

Case 2: $P_{-1} = 2$. — In this case, by [7, Proof of Theorem 4.4, Case III],

$$B^{(0)} = \{(1,2),(1,3),(1,s)\}$$

for some $s \geqslant 7$. Note that only

$$B_X = B^{(0)} = \{(1,2),(1,3),(1,7)\}$$

satisfies $-K^3(B_X) < 1/30$. In this case, we may apply Lemmas 6.2 and 6.3(2).

Combining all above discussions, the proof is completed. $\qquad\square$

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