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CANONICALITY OF MAKANIN–RAZBOROV DIAGRAMS – COUNTEREXAMPLE

by Gili BERK

Abstract. — Sets of solutions to systems of equations with finitely many variables in a free group, are equivalent to sets of homomorphisms from a fixed finitely generated group into a free group. The latter can be encoded in a diagram, which is known to be canonical for a fixed finitely generated group with a fixed generating set. In this paper we prove that the construction depends on the chosen generating set of the given finitely generated group.

Résumé. — Des ensembles de solutions à des systèmes d'équations à nombre fini de variables dans un groupe libre, sont équivalents à des ensembles d'homomorphismes d'un groupe de type fini fixé en un groupe libre. Chacun de ces ensembles peut être codé dans un diagramme, qui est connu pour être canonique pour un groupe de type fini fixé avec une partie génératrice fixée. Dans cet article, nous montrons que la construction dépend de la partie génératrice choisie pour le groupe de type fini donné.

Introduction

For a given system of equations Φ over a free group $\mathbb{F}_k$, there is a natural associated finitely generated group $G(\Phi)$. If the system Φ is defined by the coefficients $a_1,\dots,a_k$, the unknowns $x_1,\dots,x_n$ and the equations $\{w_i(a_1,\dots,a_k,x_1,\dots,x_n)=1\}_{i=1}^{s}$ (possibly $s=\infty$), then $G(\Phi)=\langle a_1,\dots,a_k,x_1,\dots,x_n|\{w_i\}_{i=1}^{s}\rangle$. If there are no coefficients then $G(\Phi)=\langle x_1,\dots,x_n|\{w_i\}_{i=1}^{s}\rangle$. There is a correspondence between solutions of the system Φ and homomorphisms $h:G(\Phi)\to\mathbb{F}_k$ (for which the restriction $h(a_j)=a_j$ holds $\forall\ 1\leqslant j\leqslant k$, in the case with coefficients). Therefore, the study of one is equivalent to the study of the other (see [1]). In addition, every finitely generated group G has a canonical finite (restricted) factor set

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$\{q_i : G \to L_i\}_{i=1}^m$ through which any (restricted) homomorphism $h : G \to \mathbb{F}_k$ factors, where the L_i are (restricted) limit groups [11, Section 7]. So in order to understand the set of (restricted) homomorphisms from a finitely generated group into the a free group, it is sufficient to study the set of (restricted) homomorphisms from a (restricted) limit group into a free group.

For a given (restricted) limit group G with a finite generating set $\boldsymbol{g}$, the set of (restricted) homomorphisms from G to $\mathbb{F}_k$ is encoded in the canonical (restricted) Makanin–Razborov (MR) diagram (see Section 5 and Section 8 in [11] for the non-restricted and restricted case, respectively). The diagram is constructed iteratively, so that each level is comprised of (restricted) maximal shortening quotients of freely indecomposable components of groups from the previous level. These maximal shortening quotients are taken with respect to generating sets inherited from $\boldsymbol{g}$. Therefore, to conclude whether or not a (restricted) MR-diagram is dependent on the generating set, it is sufficient to examine the canonicality of the set of (restricted) maximal shortening quotients of a freely indecomposable (restricted) limit group (up to isomorphism of shortening quotients).

It transpires that in the restricted case a counterexample exists:

Theorem. — 2.8 *There exists a restricted limit group H with generating sets $\boldsymbol{s}, \boldsymbol{t}$ and a maximal restricted shortening quotient (R, ρ) with respect to $\boldsymbol{s}$, which is not isomorphic to any restricted shortening quotient with respect to $\boldsymbol{t}$.*

The proof of Theorem 2.8 is constructive – we give a particular restricted limit group and two generating sets $\boldsymbol{g}, \boldsymbol{u}$. These generating sets $\boldsymbol{g}$ and $\boldsymbol{u}$ are chosen such that there cannot be an isomorphism of shortening quotients between a strict restricted $\boldsymbol{g}$-shortening quotient and a strict restricted $\boldsymbol{u}$-shortening quotient, because they have different abelianizations.

The paper is organised as follows: Section 1 provides some terminology, notation and facts. Section 2 is devoted to the description of the counterexample, thereby proving Theorem 2.8. That section uses a particular word w studied by S. V. Ivanov [6] and results of S. Heil [5] regarding JSJ forms of doubles of free groups of rank 2.

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1. Preliminaries

Throughout this paper, $\mathbb{F}$ will denote some finitely generated free group with a fixed free generating set. Let $X(\mathbb{F})$ be the Cayley graph with respect to this generating set. For $g \in \mathbb{F}$, denote $|g| = d(1_{\mathbb{F}}, g.1_{\mathbb{F}})$ the *displacement length* of $g \in \mathbb{F}$, where d is the simplicial metric on $X(\mathbb{F})$. The *translation length* of an element $g \in \mathbb{F}$ in $X(\mathbb{F})$ is $\mathrm{tr}(g) = \min_{h \in G}|hgh^{-1}|$.

We refer to JSJ decompositions in the sense of Rips and Sela [9].

Shortness with respect to a generating set

It will be helpful to use terminology and notation which keep track of the generating set with respect to which the shortening is done.

Definition 1.1 ([11, Section 5]). — *For a freely indecomposable limit group G with a finite generating set $\boldsymbol{g} = (g_1, \ldots, g_k)$:*

- *A homomorphism $h \in \mathrm{Hom}(G, \mathbb{F})$ is called $\boldsymbol{g}$-*shortest *if* $\max_i |h(g_i)| \leqslant \max_i |\iota_c \circ h \circ \varphi(g_i)|$ *for all $c \in \mathbb{F}$ and all $\varphi \in \mathrm{Mod}(G)$, where ι_c is the conjugation by c (see also* [3, Definition 4.2]*).*
- *Let L be a limit group derived from G, $\boldsymbol{g}$ and a stable sequence $\{h_n\}_{n\in\mathbb{N}} \subseteq \mathrm{Hom}(G, \mathbb{F})$ (as described in* [11, Section 1]*), and let $\eta : G \to L$ be the associated quotient map. If $\{h_n\}_{n\in\mathbb{N}}$ are $\boldsymbol{g}$-shortest morphisms then the pair (L, η) is called a $\boldsymbol{g}$-*shortening quotient. *Unless the image of h_n is abelian for all but finitely many $n \in \mathbb{N}$ (and this occurs iff L is abelian), then in fact $L = G/\underrightarrow{\mathrm{Ker}}\, h_n$* [12, Theorem 4.8]. *Denote the set of $\boldsymbol{g}$-shortening quotients of G as* $\mathrm{SQ}(G, \boldsymbol{g})$.
- *The set of $\boldsymbol{g}$-shortening quotients of G is partially ordered by the relation $\leqslant_{\boldsymbol{g}}$ given by $(L, \eta) \leqslant_{\boldsymbol{g}} (M, \pi)$ if there exists an epimorphism $\sigma : M \twoheadrightarrow L$ such that $\eta = \sigma \circ \pi$.* Maximal $\boldsymbol{g}$-shortening quotients *are maximal elements in* $\mathrm{SQ}(G, \boldsymbol{g})$ *with respect to the partial order $\leqslant_{\boldsymbol{g}}$. Denote the set of maximal $\boldsymbol{g}$-shortening quotients as* $\mathrm{MSQ}(G, \boldsymbol{g})$.

Remark 1.2. — One of the properties of $\mathrm{MSQ}(G, \boldsymbol{g})$ is that any $h \in \mathrm{Hom}(G, \mathbb{F})$ factors through some $(L, \eta) \in \mathrm{MSQ}(G, \boldsymbol{g})$, i.e. there exist $\varphi \in \mathrm{Mod}(G)$ and $h' \in \mathrm{Hom}(L, \mathbb{F})$ such that $h = h' \circ \eta \circ \varphi$. We will soon define a subset $\widetilde{\mathrm{MSQ}}(G, \boldsymbol{g}) \subseteq \mathrm{MSQ}(G, \boldsymbol{g})$ for which this property remains.

Definition 1.3 ([11, Section 5]). — *An* isomorphism of shortening quotients*, or an* SQ-isomorphism*, is a group isomorphism between two shortening quotients (not necessarily with respect to the same generating set), which in addition respects the quotient maps up to some modular automorphism of G. In other words, it is some isomorphism $\sigma : M \underset{Groups}{\xrightarrow{\sim}} L$ for*

$(L,\eta) \in \mathrm{SQ}(G,\boldsymbol{g})$, $(M,\pi) \in \mathrm{SQ}(G,\boldsymbol{u})$, *such that there exists* $\varphi \in \mathrm{Mod}(G)$ *for which the diagram*

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\pi"'] & G \arrow[d, "\eta"] \\ M \arrow[r, "\sigma"'] & L \end{tikzcd}$$

commutes, i.e. $\eta \circ \varphi = \sigma \circ \pi$. *In particular, if both shortening quotients are* $\boldsymbol{g}$*-shortening quotients, then this is an* isomorphism of $\boldsymbol{g}$-shortening quotients.

Notice that it is possible for two $\boldsymbol{g}$-shortening quotients to be isomorphic as groups but not as shortening quotients. Denote $\mathrm{SQ}(G,\boldsymbol{g})/\sim$ the set of equivalence classes of $\boldsymbol{g}$-shortening quotients of G, where $(L,\eta),(M,\pi) \in \mathrm{SQ}(G,\boldsymbol{g})$ are equivalent iff they are SQ-isomorphic. (Note that this is indeed an equivalence relation.) Likewise denote the set of equivalence classes of maximal $\boldsymbol{g}$-shortening quotients as $\mathrm{MSQ}(G,\boldsymbol{g})/\sim$.

Properly maximal shortening quotients

For a freely indecomposable limit group G with a given generating set $\boldsymbol{g}$, let $\widetilde{\mathrm{MSQ}}(G,\boldsymbol{g})$ denote the set of *properly maximal* $\boldsymbol{g}$*-shortening quotients*, i.e. the subset of $\mathrm{MSQ}(G,\boldsymbol{g})$ consisting of the elements whose equivalence classes are maximal with respect to the partial order defined on $\mathrm{MSQ}(G,\boldsymbol{g})/\sim$ by $[(N,\eta)] \leqslant [(Q,q)]$ iff there exist an epimorphism $\tau : Q \twoheadrightarrow N$ and some $\varphi \in \mathrm{Mod}(G)$ such that the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "q"'] & G \arrow[d, "\eta"] \\ Q \arrow[r, two heads, "\tau"'] & N \end{tikzcd}$$

It is easy to check that this relation is well defined on the equivalence classes. Additionally, this is indeed a partial order: it is clearly reflexive and transitive. As for antisymmetry, if $[(N,\eta)] \leqslant [(Q,q)]$ but also $[(Q,q)] \leqslant [(N,\eta)]$, then in particular there exist two epimorphisms $\tau_2 : Q \twoheadrightarrow N$, $\tau_1 : N \twoheadrightarrow Q$. Their composition is an epimorphism $\tau_2 \circ \tau_1 : N \twoheadrightarrow N$, and by the Hopf property for limit groups ([11, end of Section 4]) it follows that $\tau_2 \circ \tau_1$ is a group isomorphism. Consequently, also τ_1 is a group isomorphism, so indeed $[(N,\eta)] = [(Q,q)]$.

In a manner of speaking, $\widetilde{\mathrm{MSQ}}(G,\boldsymbol{g})$ is sufficient for the sake of studying $\mathrm{Hom}(G,\mathbb{F})$, as the property that every $h \in \mathrm{Hom}(G,\mathbb{F})$ factors through some element of $\mathrm{MSQ}(G,\boldsymbol{g})$ (up to composition with some $\varphi \in \mathrm{Mod}(G)$), is preserved by the subset $\widetilde{\mathrm{MSQ}}(G,\boldsymbol{g})$.

It is worth noting the following observation regarding properly maximal $\boldsymbol{g}$-shortening quotients:

Lemma 1.4. — *For a limit group G, TFAE:*

(i) $\widetilde{\mathrm{MSQ}}(G,\boldsymbol{g}) = \widetilde{\mathrm{MSQ}}(G,\boldsymbol{u})$ *(up to isomorphism of shortening quotients between the elements of both sets) for any two generating sets $\boldsymbol{g},\boldsymbol{u}$ of G.*

(ii) $\widetilde{\mathrm{MSQ}}(G,\boldsymbol{g}) \subseteq \mathrm{SQ}(G,\boldsymbol{u})$ *(up to SQ-isomorphism of the elements) for any two generating sets $\boldsymbol{g},\boldsymbol{u}$ of G.*

Proof. — The direction 1. $\Rightarrow$ 2. is trivial. In the other direction, let $\boldsymbol{g},\boldsymbol{u}$ be two generating sets of G, and let $(Q_{\boldsymbol{g}},q_{\boldsymbol{g}}) \in \widetilde{\mathrm{MSQ}}(G,\boldsymbol{g})$. By assumption, for every element of $\widetilde{\mathrm{MSQ}}(G,\boldsymbol{g})$ there exists an element of $\mathrm{SQ}(G,\boldsymbol{u})$ which is SQ-isomorphic to it. So there exist $(Q_{\boldsymbol{u}},q_{\boldsymbol{u}}) \in \mathrm{SQ}(G,\boldsymbol{u})$, a group isomorphism $\sigma_1 : Q_{\boldsymbol{u}} \to Q_{\boldsymbol{g}}$ and $\varphi_1 \in \mathrm{Mod}(G)$ such that $q_{\boldsymbol{g}} \circ \varphi_1 = \sigma_1 \circ q_{\boldsymbol{u}}$. Since $(Q_{\boldsymbol{u}},q_{\boldsymbol{u}}) \in \mathrm{SQ}(G,\boldsymbol{u})$, there exist some $(M_{\boldsymbol{u}},\mu_{\boldsymbol{u}}) \in \mathrm{MSQ}(G,\boldsymbol{u})$ and an epimorphism $\sigma_2 : M_{\boldsymbol{u}} \twoheadrightarrow Q_{\boldsymbol{u}}$ such that $q_{\boldsymbol{u}} = \sigma_2 \circ \mu_{\boldsymbol{u}}$. There exists some maximal element $(\widetilde{M}_{\boldsymbol{u}},\widetilde{\mu}_{\boldsymbol{u}}) \in \widetilde{\mathrm{MSQ}}(G,\boldsymbol{u})$ with an epimorphism $\sigma_3 : \widetilde{M}_{\boldsymbol{u}} \twoheadrightarrow M_{\boldsymbol{u}}$ and some $\varphi_3 \in \mathrm{Mod}(G)$ such that $\mu_{\boldsymbol{u}} \circ \varphi_3 = \sigma_3 \circ \widetilde{\mu}_{\boldsymbol{u}}$. We get the following commutative diagram:

$$\begin{tikzcd} G \arrow[r, "\varphi_1\circ\varphi_3=\varphi"] \arrow[d, "\tilde{\mu}_{\boldsymbol{u}}"'] & G \arrow[d, "q_{\boldsymbol{g}}"] \\ \widetilde{M}_{\boldsymbol{u}} \arrow[r, two heads, "\sigma_1\circ\sigma_2\circ\sigma_3=\sigma"'] & Q_{\boldsymbol{g}} \end{tikzcd}$$

Notice that by assumption also $\widetilde{\mathrm{MSQ}}(G,\boldsymbol{u}) \subseteq \mathrm{SQ}(G,\boldsymbol{g})$ (up to SQ-isomorphism of the elements), so by symmetric argument there exist $(N_{\boldsymbol{g}},\eta_{\boldsymbol{g}}) \in \widetilde{\mathrm{MSQ}}(G,\boldsymbol{g})$, $\psi \in \mathrm{Mod}(G)$ and an epimorphism $\tau : N_{\boldsymbol{g}} \twoheadrightarrow \widetilde{M}_{\boldsymbol{u}}$ such that $\widetilde{\mu}_{\boldsymbol{u}} \circ \psi = \tau \circ \eta_{\boldsymbol{g}}$.

By adding this information to the previous diagram, the resulting commutative diagram

$$\begin{tikzcd} G \arrow[r, "\varphi\circ\psi"] \arrow[d, "\eta_{\boldsymbol{g}}"'] & G \arrow[d, "q_{\boldsymbol{g}}"] \\ N_{\boldsymbol{g}} \arrow[r, "\sigma\circ\tau"'] & Q_{\boldsymbol{g}} \end{tikzcd}$$

shows that $[(Q_{\boldsymbol{g}}, q_{\boldsymbol{g}})] \leqslant [(N_{\boldsymbol{g}}, \eta_{\boldsymbol{g}})]$. Since $(Q_{\boldsymbol{g}}, q_{\boldsymbol{g}}) \in \widetilde{\mathrm{MSQ}}(G, \boldsymbol{g})$, $[(Q_{\boldsymbol{g}}, q_{\boldsymbol{g}})]$ is maximal in this partial order, so $[(Q_{\boldsymbol{g}}, q_{\boldsymbol{g}})] = [(N_{\boldsymbol{g}}, \eta_{\boldsymbol{g}})]$. In particular, σ is a group isomorphism, and by the first diagram $(Q_{\boldsymbol{u}}, q_{\boldsymbol{u}})$ and $(\widetilde{M}_{\boldsymbol{u}}, \widetilde{\mu}_{\boldsymbol{u}})$ are SQ-isomorphic. Therefore $\widetilde{\mathrm{MSQ}}(G, \boldsymbol{g}) \subseteq \widetilde{\mathrm{MSQ}}(G, \boldsymbol{u})$ (up to SQ-isomorphism of the elements). By symmetric argument $\widetilde{\mathrm{MSQ}}(G, \boldsymbol{g}) \supseteq \widetilde{\mathrm{MSQ}}(G, \boldsymbol{u})$ (up to SQ-isomorphism of the elements), hence the equality. $\square$

Remark 1.5. — The same argument holds with restriction to strict maximal shortening quotients, since (using the above notation) strictness of $(Q_{\boldsymbol{g}}, q_{\boldsymbol{g}})$ passes to $(Q_{\boldsymbol{u}}, q_{\boldsymbol{u}})$, so $(M_{\boldsymbol{u}}, \mu_{\boldsymbol{u}})$ can be chosen from among the strict elements of $\mathrm{MSQ}(G, \boldsymbol{u})$. The strictness passes on to $(\widetilde{M}_{\boldsymbol{u}}, \widetilde{\mu}_{\boldsymbol{u}})$, and by symmetric argument also $(N_{\boldsymbol{g}}, \eta_{\boldsymbol{g}})$ is strict. Likewise, the argument holds with reduction to restricted shortening quotients.

Strict shortening quotients

Definition 1.6 ([11, Definition 5.9]). — *A $\boldsymbol{g}$-shortening quotient (L, η) of a freely indecomposable limit group G is called* a strict shortening quotient *if:*

(i) *η is monomorphic on the each cyclic edge group in the cyclic JSJ decomposition of G. In addition, for every non-CMQ, non-abelian vertex group G_v in the cyclic JSJ decomposition of G, obtain a new graph of groups Λ_v from the cyclic JSJ decomposition of G by replacing each abelian vertex group by the direct summand containing the edge groups connected to it. The subgroup $\widetilde{G}_v$ of G is generated by G_v together with centralisers of edge groups connected to it in the new graph of groups Λ_v. η is monomorphic on every such $\widetilde{G}_v$.*

(ii) *For every CMQ subgroup S of G, $\eta(S)$ is non-abelian, and boundary elements of S have non-trivial images.*

(iii) *For every abelian vertex group A in the cyclic JSJ decomposition of L, let $\widetilde{A} < A$ be the subgroup generated by all edge groups connected to the vertex stabilised by A. Then $\eta|_{\tilde{A}}$ is a monomorphism.*

Among the elements of $\mathrm{MSQ}(G, \boldsymbol{g})$ there is at least one strict maximal $\boldsymbol{g}$-shortening quotient [11, Lemma 5.10].

It also worth noting that strictness is a property that is preserved under SQ-isomorphisms, and consequently a strict shortening quotient cannot be SQ-isomorphic to a non-strict shortening quotient.

Makanin–Razborov diagrams

The *Makanin–Razborov (MR) diagram* of a limit group G with a finite generating set $\boldsymbol{g}$ is constructed iteratively, so that each level is comprised of maximal shortening quotients of freely indecomposable components of groups from the previous level. These maximal shortening quotients are taken with respect to generating sets inherited from $\boldsymbol{g}$. For full detail see [11, Section 5]. Due to the factorisation property of maximal shortening quotients, noted in Remark 1.2, this diagram encodes all the elements of $\mathrm{Hom}(G, \mathbb{F})$.

Restricted MR-diagrams

Section 8 of [11] is devoted to the restricted case. Let $2 \leqslant k \in \mathbb{N}$, and fix an ordered generating set $(y_1, \ldots, y_k)$ for $\mathbb{F}_k$. For a finitely generated group G and a finite ordered subset $(\gamma_1, \ldots, \gamma_k) \subseteq G$ which generates a proper subgroup $\Gamma < G$, denote $\mathrm{Hom}_\Gamma(G, \mathbb{F}_k) = \{h \in \mathrm{Hom}(G, \mathbb{F}_k) : \forall\ 1 \leqslant i \leqslant k\ h_n(\gamma_i) = y_i\}$ the set of *restricted homomorphisms*. A *restricted limit group relative to* $(\gamma_1, \ldots, \gamma_k)$ is a limit group $L = G/\underrightarrow{\mathrm{Ker}}\, h_n$ for a stable sequence $\{h_n\}_{n\in\mathbb{N}} \subseteq \mathrm{Hom}_\Gamma(G, \mathbb{F}_k)$.

Many of the definitions in the non-restricted case can be modified to suit the restricted case, via the notion of relative group splitting. Let H be a subgroup of L. A *splitting of L relative to H* is a presentation of L as a finite graph of groups in which H is conjugate into one of the vertex groups. L is *freely indecomposable relative to H* if L does not split into a non-trivial free product such that H is contained in one of the free factors. The canonical*cyclic JSJ decomposition* of *Lrelative to H* is a cyclic splitting of L relative to H, which encodes all of L's cyclic splittings relative to H. This gives rise to the *modular group of L relative to H*, which in the current context, when taking H to be $\eta(\Gamma)$, is also called the *restricted modular group of L with respect to* Γ:

Definition 1.7. — *Let $L = G/\underrightarrow{\mathrm{Ker}}\, h_n$ be a restricted limit group relative to* Γ*, and let $\eta : G \to L$ such that* $\mathrm{Ker}\,\eta = \underrightarrow{\mathrm{Ker}}\, h_n$*. Assume that L is freely indecomposable relative to $\eta(\Gamma)$, and that $\eta(\Gamma)$ is a proper subgroup of L. The* restricted modular group of L with respect to Γ, *denoted* $\mathrm{RMod}(L)$*, is the subgroup of* $\mathrm{Aut}(L)$ *generated by:*

(i) *Dehn twists along the edges of the cyclic JSJ decomposition of L relative to $\eta(\Gamma)$, which fix $\eta(\Gamma)$ elementwise.*

(ii) *Dehn twists along simple closed curves in surface vertex groups of the cyclic JSJ decomposition of L relative to $\eta(\Gamma)$, which fix $\eta(\Gamma)$ elementwise.*
(iii) *Generalised Dehn twists, which are the natural extension of automorphisms on an Abelian vertex group A in the cyclic JSJ decomposition of L relative to $\eta(\Gamma)$, which fix elementwise the subgroup of L generated by all the edge groups connecting A to the other vertex groups in the cyclic JSJ decomposition of L relative to $\eta(\Gamma)$, and which fix the vertex group of the vertex stabilised by $\eta(\Gamma)$.*

While bearing in mind the similarities between this definitions of $\mathrm{RMod}(L)$ and the definition of $\mathrm{Mod}(L)$ in [11, Definition 5.2], one noticeable difference is with regard to $\mathrm{Inn}(L)$. The elements of $\mathrm{RMod}(L)$ are required to stabilise $(\eta(\gamma_1), \dots, \eta(\gamma_k))$, and consequently $\mathrm{Inn}(L) \nsubseteq \mathrm{RMod}(L)$. Hence for a finite generating set $\boldsymbol{g} = (g_1, \dots, g_k)$ of L, $h \in \mathrm{Hom}_{\eta(\Gamma)}(L, \mathbb{F})$ is a *$\boldsymbol{g}$-shortest restricted morphism relative to* $\eta(\Gamma)$ if $\max_i |h(g_i)| \leqslant \max_i |h \circ \varphi(g_i)|$ for all $\varphi \in \mathrm{RMod}(L)$. Here too, preservation of the restriction requirement relative to Γ necessiates excluding left-composition with elements of $\mathrm{Inn}(\mathbb{F})$, which appeared in the non-restricted definition of $\boldsymbol{g}$-shortest morphisms.

The set of *restricted $\boldsymbol{g}$-shortening quotients*, denoted $\mathrm{RSQ}(L, \boldsymbol{g})$, is defined by substituting $\boldsymbol{g}$-shortest morphisms with $\boldsymbol{g}$-shortest restricted morphisms, in the definition of $\mathrm{SQ}(L, \boldsymbol{g})$. The set of *restricted maximal $\boldsymbol{g}$-shortening quotients*, denoted $\mathrm{RMSQ}(L, \boldsymbol{g})$, is the set of maximal elements in a partial order on $\mathrm{RSQ}(L, \boldsymbol{g})$ (identical to the partial order defined on $\mathrm{SQ}(L, \boldsymbol{g})$, with respect to which $\mathrm{MSQ}(L, \boldsymbol{g})$ is defined). Similarly to the non-restricted case, $\mathrm{RMSQ}(L, \boldsymbol{g})$ is used to define the *restricted MR-diagram with respect to $\boldsymbol{g}$*.

Just as in the non-restricted case, there exists at least one *strict* element of $\mathrm{RMSQ}(L, \boldsymbol{g})$.

Essential JSJ decompositions

Definition 1.8. — *For freely indecomposable hyperbolic limit groups, it is possible to modify the cyclic JSJ decomposition to a canonical essential splitting, called the* essential JSJ decomposition *[10, Theorem 1.8]. An* essential $\mathbb{Z}$-splitting *of a group is a splitting whose edge groups are all maximal cyclic subgroups.*

In general, the JSJ of a freely indecomposable limit group G is needed to define $\mathrm{Mod}(G)$. However, non-essential splittings of the form $G = A \underset{C}{*} \mathbb{Z}$

do not truly add Dehn twists to the modular group [4], for such Dehn twists are in fact inner automorphisms. Hence it is possible to understand $\mathrm{Mod}(G)$ from its essential JSJ decomposition, when such a decomposition exists.

The concept of essential JSJ decompositions will be useful in the analysis of the counterexample group $D_{w,z}$ described in the next section. Once ascertaining the essential JSJ decomposition of $D_{w,z}$ (Lemma 2.2), its restricted modular group becomes apparent (Corollary 2.3) and it is then possible to find its shortening quotients (Remark 2.5 and Proposition 2.7).

Ivanov words

Definition 1.9 ([6, 8]). — *A* C-test word in n letters *is a non-trivial word $w(x_1,\dots,x_n) \in \mathbb{F}_n = \langle x_1,\dots,x_n\rangle$ such that for any finitely generated free group F and n-tuples $(A_1,\dots,A_n),(B_1,\dots,B_n) \in F^n$, if $w(A_1,\dots,A_n) = w(B_1,\dots,B_n) \neq 1$ then there exists some $S \in F$ such that $SA_iS^{-1} = B_i$ for all $1 \leqslant i \leqslant n$.*

An Ivanov word *is a C-test word in n letters which is not a proper power, and with the additional property that for elements $A_1,\dots,A_n$ in any free group F, $w(A_1,\dots,A_n) = 1$ iff $\langle A_1,\dots,A_n\rangle$ is a cyclic subgroup of F.*

Lemma 1.10 ([8, Corollary 1]). — *Let $\varphi,\psi \in \mathrm{End}(\mathbb{F}_n)$ such that ψ is a monomorphism and $\varphi(w) = \psi(w)$ for some w which is an Ivanov word in n letters. Then $\varphi = \tau_S \circ \psi$ for some $S \in \mathbb{F}_n$ such that $\langle S,\psi(w)\rangle \leqslant \mathbb{F}_n$ is cyclic, where τ_S is the conjugation by S.*

If, in addition, ψ is surjective, then $S \in \langle\psi(w)\rangle$.

Proof. — Let $w(x_1,\dots,x_n)$ be an Ivanov word in n letters. Denote $A_i = \psi(x_i)$, $B_i = \varphi(x_i)$ for $1 \leqslant i \leqslant n$. Then

$$\varphi(w(x_1,\dots,x_n)) = w(\varphi(x_1),\dots,\varphi(x_n)) = w(B_1,\dots,B_n)$$

and likewise

$$\psi(w(x_1,\dots,x_n)) = w(A_1,\dots,A_n)$$

Since $\psi(w(x_1,\dots,x_n)) = \varphi(w(x_1,\dots,x_n))$, it follows that $w(A_1,\dots,A_n) = w(B_1,\dots,B_n)$, and since ψ is a monomorphism and $w(x_1,\dots,x_n) \neq 1_{\mathbb{F}_n}$, it follows also that $w(A_1,\dots,A_n) \neq 1_{\mathbb{F}_n}$. Consequently, there exists some $S \in \mathbb{F}_n$ such that $SA_iS^{-1} = B_i$ for all $1 \leqslant i \leqslant n$ (because w is a C-test word). This can also be written as $\tau_S \circ \psi(x_i) = \varphi(x_i)$ for all $1 \leqslant i \leqslant n$. $(x_1,\dots,x_n)$ is a generating set of $\mathbb{F}_n$, so in fact $\tau_S \circ \psi = \varphi$.

In particular, $Sw(A_1,\dots,A_n)S^{-1} = w(B_1,\dots,B_n) = w(A_1,\dots,A_n)$. In other words, $w(A_1,\dots,A_n)$ and S commute in $\mathbb{F}_n$. This is only possible if there exists a cyclic subgroup $\langle c\rangle \leqslant \mathbb{F}_n$ to which both $w(A_1,\dots,A_n)$ and S belong.

Now assume in addition that ψ is surjective, then $S \in \mathrm{Im}(\psi)$. Let $p, q \in \mathbb{Z}$ s.t. $S = c^p$ and $\psi(w) = c^q$. Without loss of generality, assume $\gcd(p,q) = 1$ (else take $c^{\gcd(p,q)}$ instead of c). By Bézout's lemma, there exist $k, \ell \in \mathbb{Z}$ with $p\cdot k + q\cdot \ell = 1$. It follows that

$$c = c^{p\cdot k+q\cdot\ell} = c^{p\cdot k}\cdot c^{q\cdot\ell} = S^k\cdot(\psi(w))^{\ell} \in \mathrm{Im}(\psi)$$

Since ψ is a monomorphism, there exists a unique element $v = \psi^{-1}(c)$. This element is in fact a root of $w(x_1,\dots,x_n)$: $\psi(v^q) = c^q = \psi(w(x_1,\dots,x_n))$, but ψ is a monomorphism, so $v^q = w(x_1,\dots,x_n)$. But as an Ivanov word, w cannot be a proper power, hence $q = 1$. Consequently, $(\psi(w))^{\,p} = (\psi(v))^{\,p} = c^p = S$, as required. □

Notice that in the lemma above, if ψ is surjective then $S = \psi(w)^p$ for some $p \in \mathbb{Z}$, so φ can be written as $\varphi = \tau^p_{\psi(w)} \circ \psi = \psi \circ \tau^p_w$.

Lemma 1.11 ([6, 8]).

$$\begin{aligned} w(x_1,x_2) = {} & [x_1^8,x_2^8]^{100}x_1[x_1^8,x_2^8]^{200}x_1[x_1^8,x_2^8]^{300}x_1^{-1}[x_1^8,x_2^8]^{400}x_1^{-1} \\ & \cdot [x_1^8,x_2^8]^{500}x_2[x_1^8,x_2^8]^{600}x_2[x_1^8,x_2^8]^{700}x_2^{-1}[x_1^8,x_2^8]^{800}x_2^{-1} \end{aligned}$$

is an Ivanov word in 2 letters (where $[x_1,x_2] = x_1x_2x_1^{-1}x_2^{-1}$).

This particular Ivanov word will be extremely useful in the construction of the group in Section 2.

2. Counterexample to canonicality of MR-diagrams

The *construction* of the (restricted) MR-diagram of a (restricted) limit group L depends on the chosen generating set $\boldsymbol{g}$, for it is with respect to generating sets inherited from $\boldsymbol{g}$ that the (restricted) maximal shortening quotients are taken. However, this does not automatically mean that the *resulting diagram* likewise depends on the choice of generating set.

To examine the canonicality of (restricted) MR-diagrams with dependence only on the limit group, and not also on the generating set, it is enough to examine the canonicality of the first level, since all other levels are built iteratively.

For a (restricted) limit group L with two different generating sets $\boldsymbol{g} = (g_1,\dots,g_t)$ and $\boldsymbol{u} = (u_1,\dots,u_r)$, is the (restricted) MR-diagram of L with respect to $\boldsymbol{g}$ the same as the (restricted) MR-diagram of L with respect to $\boldsymbol{u}$, up to isomorphism of shortening quotients? In the restricted case the answer is negative, and this section is devoted to constructing a counterexample.

For a word $1 \neq v \in \mathbb{F}_k$ which is not primitive and has no roots, the group $\mathbb{F}_k *_{\langle v \rangle} \mathbb{F}_k$ is a limit group. S. Heil [5] has described all the possible forms of cyclic JSJ decompositions of a double of free groups of rank 2 (along a word which is not necessarily Ivanov). Some of those forms can be eliminated when taking such a double along an Ivanov word $w(a,b)$ for some generating set $\{a,b\}$ of $\mathbb{F}_2$. For example, this eliminates all forms that are possible iff $w(a,b) \in \langle xyx^{-1}, y\rangle$ for some (other) generating set $\{x,y\}$ of $\mathbb{F}_2 = \langle a,b\rangle$. Suppose otherwise, then $w(a,b) \in \langle xyx^{-1}, y\rangle$ for some generating set $\{x,y\}$ of $\mathbb{F}_2$. There exists $\varphi \in \mathrm{Aut}(\mathbb{F}_2)$ which is not an inner automorphism but fixes y and xyx^{-1} (for example, φ that is defined by $\varphi(x) = xy, \varphi(y) = y$). So this φ fixes the group generated by y and xyx^{-1}. In particular, φ fixes $w(a,b)$, so $w(a,b) = \varphi(w(a,b)) = w(\varphi(a), \varphi(b))$. Since w is an Ivanov word, it follows that $\varphi(b) = SbS^{-1}$ and $\varphi(a) = SaS^{-1}$. But this means that φ is an inner automorphism, a contradiction. By similar argument, it can be shown that w does not correspond to a simple closed curve in a surface whose surface group appears in the JSJ decomposition (because such words can be fixed by non-inner automorphisms, whereas Ivanov words cannot). The three remaining cyclic JSJ forms, described in Figure 2.1, all share the same essential JSJ form, identical to the full JSJ form of (I), which coincides with the double decomposition.

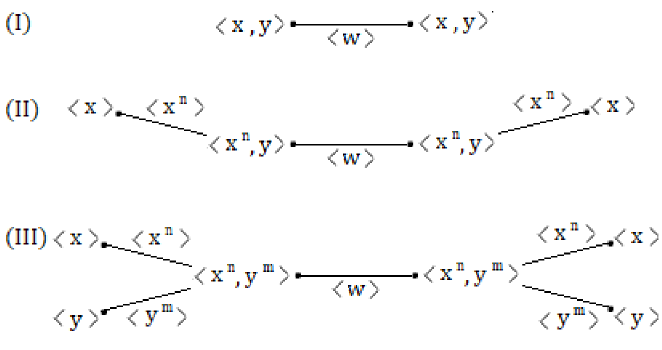

Figure 2.1. Possible JSJ forms of the double of $\mathbb{F}_2$ along an Ivanov word w

Let $w(y_1, y_2)$ be the particular Ivanov word given in Lemma 1.11, and let

$$G_w(a_1,a_2,b_1,b_2) = F(a_1,a_2) \underset{w(a_1,a_2)=w(b_1,b_2)}{*} F(b_1,b_2)$$

be the double of $\mathbb{F}_2$ over that word. Take the graph of groups associated with the double

$$D_w = G_w(a_1, a_2, b_1, b_2) \underset{w(b_1,b_2)=w(c_1,c_2)}{*} G_w(d_1, d_2, c_1, c_2)$$

and add an edge between the two vertices. Let the edge group of the new edge be the group generated by the element $z = x_1y_2x_1y_1x_1y_1$ in $G_w(x_1, x_2, y_1, y_2)$.

The resulting double-edged double will be denoted $D_{w,z}$ (see Figure 2.2). Notice that by adding the second edge, the underlying graph of the graph of groups is no longer a tree. Therefore, z can be embedded by the identity function into only one of the vertex groups, whereas the embedding into the other vertex group must be by conjugation with a Bass–Serre element γ. This gives the equation

$$a_1b_2a_1b_1a_1b_1 = z(a_1, a_2, b_1, b_2) = \gamma z(d_1, d_2, c_1, c_2)\gamma^{-1} = \gamma d_1c_2d_1c_1d_1c_1\gamma^{-1}.$$

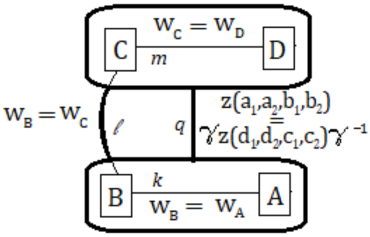

Figure 2.2. *The group $D_{w,z}$. The bold lines represent the essential JSJ decomposition. k, ℓ, m and q are the twisting parameters for their edge groups, as they appear in the proof of Lemma 2.4. As Lemma 2.4 shows, in fact $m = k$.*

For simplicity, assume the notation $A = F(a_1, a_2)$, $B = F(b_1, b_2)$, $C = F(c_1, c_2)$, $D = F(d_1, d_2)$.

LEMMA 2.1. — *$D_{w,z}$ is a restricted limit group with respect to the coefficients $\{b_1, b_2\}$.*

Proof. — To show that $D_{w,z}$ is a restricted limit group, it is enough to find a restricted strict MR resolution from $D_{w,z}$ to $\mathbb{F}_2 = \langle b_1, b_2\rangle$. (This is due to the modification of Theorem 5.12 in [11] to the restricted case. See Definition 5.11 there of a strict MR resolution of a finitely generated group which is not necessarily a limit group.) The suggested resolution is $D_{w,z} \xrightarrow{\eta} G_w \xrightarrow{\pi} \mathbb{F}_2$, where $\eta(\gamma) = 1$, $\eta(c_i) = \eta(b_i) = b_i$, $\eta(d_i) = \eta(a_i) = a_i$

and $\pi(a_i) = \pi(b_i) = b_i$ for $i \in \{1, 2\}$. It is clearly restricted with respect to the coefficients $\{b_1, b_2\}$. It remains to check that it is indeed a strict MR resolution. There are four criteria in [11, Definition 5.11], examine each in turn.

(i) By way of contradiction, assume that $\langle z(a_1, a_2, b_1, b_2)\rangle$ is not maximal abelian in G_w, then $z(a_1, a_2, b_1, b_2) = x^j$ for some $x \in G_w$ and $j \in \mathbb{Z}\backslash\{\pm 1, 0\}$. Then $b_1 b_2 b_1^4 = \pi(z(a_1, a_2, b_1, b_2)) = y^j$ for $\pi(x) = y \in \mathbb{F}_2 = \langle b_1, b_2\rangle$. This is not possible, as $b_1 b_2 b_1^4$ clearly has no roots in $\mathbb{F}_2 = \langle b_1, b_2\rangle$. A similar argument can be used for w, bearing in mind that $w(b_1, b_2)$ is known to have no roots in $\mathbb{F}_2 = \langle b_1, b_2\rangle$.

(ii) η is indeed monomorphic on $\langle A *_{\langle w\rangle} B,\, w,\, z(a_1, a_2, b_1, b_2)\rangle$ and on $\langle C *_{\langle w\rangle} D,\, w,\, z(d_1, d_2, c_1, c_2)\rangle$, and π is monomorphic on $\langle A,\, w\rangle$ and on $\langle B,\, w\rangle$.

There are no QH vertices, nor abelian vertices, in the given splittings, so criteria (iii) and (iv) hold vacuously.

Hence the suggested resolution is indeed a restricted strict MR resolution of $D_{w,z}$. □

$D_{w,z}$ is also a hyperbolic group, as the free group is hyprbolic, and hyperbolicity is preserved under amalgamations and HNN over maximal cyclic subgroups ([2], see also Corollaries 1, 2 in [7]). Hence $D_{w,z}$ has an essential JSJ decomposition.

Lemma 2.2. — *The double-edged double decomposition*

$$D_{w,z} = G_w(a_1, a_2, b_1, b_2) \underset{z(a_1,a_2,b_1,b_2)=\gamma z(d_1,d_2,c_1,c_2)\gamma^{-1}}{\overset{w(b_1,b_2)=w(c_1,c_2)}{*}} G_w(d_1, d_2, c_1, c_2)$$

is also the essential JSJ decomposition of $D_{w,z}$.

Proof. — The double decomposition of

$$G_w = F(a_1, a_2) \underset{w(a_1,a_2)=w(b_1,b_2)}{*} F(b_1, b_2)$$

is also the essential JSJ decomposition of G_w. The double

$$D_w = G_w(a_1, a_2, b_1, b_2) \underset{w(b_1,b_2)=w(c_1,c_2)}{*} G_w(d_1, d_2, c_1, c_2)$$

is a limit group, but the double decomposition of D_w is not its essential JSJ decomposition, as both vertex groups can be further split with respect to the edge group. However, the double-edged double decomposition

$$D_{w,z} = G_w(a_1, a_2, b_1, b_2) \underset{z(a_1,a_2,b_1,b_2)=\gamma z(d_1,d_2,c_1,c_2)\gamma^{-1}}{\overset{w(b_1,b_2)=w(c_1,c_2)}{*}} G_w(d_1, d_2, c_1, c_2)$$

is the essential JSJ decomposition of $D_{w,z}$. This is due to the fact that the vertex groups are both G_w, whose only possible essential splitting is the double decomposition. But this splitting is not compatible with the incident edges. □

Lemma 2.2 immediately gives the following:

Corollary 2.3. — *The restriction with respect to the set of coefficients* $\{b_1, b_2\}$ *ensures that* $\mathrm{RMod}(D_{w,z})$ *is generated solely by the Dehn twists along* z *or along* w*, which fix* $G_w(a_1, a_2, b_1, b_2)$*.*

Lemma 2.4. — *Let* $h \in \mathrm{Hom}_B(D_{w,z}, B)$*, then* h *is of the form*

$$h|_B = \mathrm{id}_B, \qquad h|_A = (\tau_{w_B})^k \circ \{a_i \mapsto b_i\},$$
$$h|_C = (\tau_{w_B})^\ell \circ \{c_i \mapsto b_i\}, \qquad h|_D = (\tau_{w_B})^{k+\ell} \circ \{d_i \mapsto b_i\}$$

and $h(\gamma) = h(z^q)w^{-\ell}$ *for some* $k, \ell, q \in \mathbb{Z}$ *(see* Figure 2.2*), where* τ_{w_B} *is the conjugation by* w_B*.*

Proof. — Let $h \in \mathrm{Hom}_B(D_{w,z}, B)$, so in particular $h(w) \neq 1$. $h(A)$ is a 2-generated subgroup of $\mathbb{F}_2$, and as such $h(A) \in \{\{1\}, \mathbb{Z}, \mathbb{F}_2\}$. Likewise $h(C), h(D) \in \{\{1\}, \mathbb{Z}, \mathbb{F}_2\}$. But if $h(A)$ is cyclic then $h(w) = 1$, a contradiction. Therefore $h(A) \cong \mathbb{F}_2$, and by similar argument $h(C) \cong \mathbb{F}_2 \cong h(D)$ (and by assumption $h(B) = B$).

$h|_A, h|_B, h|_C, h|_D \in \mathrm{End}(\mathbb{F}_2)$ all agree on the word w in their respective generating sets, and are all monomorphic (by the Hopf property for finitely generated free groups, since they are all epimorphisms from $\mathbb{F}_2$ to itself). Because w is an Ivanov word, it follows from Lemma 1.10 that

$$\begin{aligned} h|_A &= h|_B \circ (\tau_{w_B})^k \circ \{a_i \mapsto b_i\} = (\tau_{w_B})^k \circ \{a_i \mapsto b_i\} \\ h|_C &= h|_B \circ (\tau_{w_B})^\ell \circ \{c_i \mapsto b_i\} = (\tau_{w_B})^\ell \circ \{c_i \mapsto b_i\} \\ h|_D &= h|_C \circ (\tau_{w_C})^m \circ \{d_i \mapsto c_i\} = (\tau_{w_B})^{\ell+m} \circ \{d_i \mapsto b_i\} \end{aligned}$$

for some $k, \ell, m \in \mathbb{Z}$. The choice $a_1b_2a_1b_1a_1b_1 = z = \gamma d_1c_2d_1c_1d_1c_1\gamma^{-1}$ gives rise to the equation

$$\begin{aligned} &w^k b_1 w^{-k} b_2 w^k b_1 w^{-k} b_1 w^k b_1 w^{-k} b_1 \\ &\quad = h(a_1b_2a_1b_1a_1b_1) \\ &\quad = h(\gamma d_1c_2d_1c_1d_1c_1\gamma^{-1}) \\ &\quad = h(\gamma) w^{\ell+m} b_1 w^{-(\ell+m)} w^\ell b_2 w^{-\ell} w^{\ell+m} b_1 w^{-(\ell+m)} \\ &\qquad \cdot w^\ell b_1 w^{-\ell} w^{\ell+m} b_1 w^{-(\ell+m)} w^\ell b_1 w^{-\ell} (h(\gamma))^{-1} \\ &\quad = \left(h(\gamma)w^\ell\right) w^m b_1 w^{-m} b_2 w^m b_1 w^{-m} b_1 w^m b_1 w^{-m} b_1 \left(w^{-\ell}(h(\gamma))^{-1}\right) \end{aligned}$$

Because $w(b_1,b_2)$ is the specific Ivanov word given in Lemma 1.11, it is possible to know what the beginning and the end of the word on the left hand side of this equation look like in terms of b_1 and b_2, depending on the sign of k. If $0<k$, this word begins with $b_1^8b_2^8$ (which is the beginning of w) and ends with $b_2^{-8}b_1^{-8}b_1$. If $k<0$, it begins with $b_2b_2^8b_1^8$ (which is the beginning of w^{-1}) and ends with $b_1^{-8}b_2^{-8}b_2^{-1}b_1$. If $k=0$, it begins and ends with b_1. By taking

$$\varepsilon(r)=\begin{cases}1, & \text{if } 0<r\\ 0, & \text{if } r\leqslant 0,\end{cases}$$

$(b_1)^{-7\varepsilon(k)}\,w^kb_1w^{-k}b_2w^kb_1w^{-k}b_1w^kb_1w^{-k}b_1\,(b_1)^{7\varepsilon(k)}$ is therefore a cyclically reduced word in $\mathbb{F}_2$. Thus by conjugating both sides of the above equation by $b_1^{-7\varepsilon(k)}$, the modified equation

$$\begin{aligned}&(b_1)^{-7\varepsilon(k)}\,w^kb_1w^{-k}b_2w^kb_1w^{-k}b_1w^kb_1w^{-k}b_1\,(b_1)^{7\varepsilon(k)}\\ &\qquad=\left((b_1)^{-7\varepsilon(k)}\,h(\gamma)w^{\ell}\,(b_1)^{7\varepsilon(m)}\right)\\ &\qquad\qquad\cdot(b_1)^{-7\varepsilon(m)}\,w^mb_1w^{-m}b_2w^mb_1w^{-m}b_1w^mb_1w^{-m}b_1\,(b_1)^{7\varepsilon(m)}\\ &\qquad\qquad\cdot\left((b_1)^{-7\varepsilon(m)}\,w^{-\ell}(h(\gamma))^{-1}\,(b_1)^{7\varepsilon(k)}\right)\end{aligned}$$

is between a cyclically reduced word and a conjugation of a cyclically reduced word in $\mathbb{F}_2$. The only solutions are when the first cyclically reduced word is equal to some cyclic permutation of the second cyclically reduced word. This ensures that $m=k$; in addition the conjugating element must commute with the cyclically reduced word, hence $h(\gamma)\in\langle h(z)\rangle w^{-\ell}$. □

Remark 2.5. — In light of Corollary 2.3, it becomes apparent that the only way to shorten $h\in\mathrm{Hom}_B(D_{w,z},\mathbb{F}_2)$ parameterised by k,ℓ and q as in Lemma 2.4, is by right-composition with some power of the Dehn twist along z, which affects the value of q, or along w, which affects the value of ℓ (but not of k).

Lemma 2.6. — *Let $\boldsymbol{g}$ be a generating set of $D_{w,z}$. Let $\{h_n\}_{n\in\mathbb{N}}\subseteq\mathrm{Hom}_B(D_{w,z},\mathbb{F}_2)$ be a stable sequence of $\boldsymbol{g}$-shortest morphisms, with $k_n,\ell_n,q_n\in\mathbb{Z}$ as in Lemma 2.4 for each h_n. If the associated restricted $\boldsymbol{g}$-shortening quotient $(\widetilde{L},\widetilde{\eta})$ is strict, then $|k_n|\underset{n\to\infty}{\to}\infty$.*

Proof. — Assume otherwise, then by extraction of subsequence, without loss of generality the original sequence, $\{k_n\}_{n\in\mathbb{N}}$ is the constant sequence $k_n=k_0$. Hence, for any word $v(x_1,x_2)\in F(x_1,x_2)$, the h_n-image of the element $v(a_1,a_2)\in A$ is identified with the h_n-image of $w_B^{k_0}v(b_1,b_2)w_B^{-k_0}$

for all $n \in \mathbb{N}$. Therefore also

$$\widetilde{\eta}(v(a_1,a_2)) = w^{k_0}\widetilde{\eta}(v(b_1,b_2))w^{-k_0} = \widetilde{\eta}(w^{k_0}v(b_1,b_2)w^{-k_0}),$$

contradicting strictness. $\square$

We now examine the set of restricted strict shortening quotients of $D_{w,z}$ with respect to two generating sets:

$$\boldsymbol{g} = (a_1,a_2,b_1,b_2,c_1,c_2,d_1,d_2,\gamma,a_1d_1a_1d_1)$$

and

$$\boldsymbol{u} = (a_1,a_2,b_1,b_2,c_1,c_2,d_1,d_2,\gamma,a_1c_1a_1c_1)$$

Proposition 2.7. — *Let $(\widetilde{L},\widetilde{\eta})$ be a restricted strict $\boldsymbol{g}$-shortening quotient of $D_{w,z}$ and $(\widetilde{M},\widetilde{\pi})$ be a restricted strict $\boldsymbol{u}$-shortening quotient of $D_{w,z}$. So:*

(i) *$\widetilde{L} = G_w$, $\widetilde{\eta}|_{A*_{\langle w\rangle}B} = \mathrm{id}$, $\widetilde{\eta}|_{C*_{\langle w\rangle}D}$ sends c_1,c_2,d_1,d_2 to b_1,b_2,a_1,a_2 respectively (up to conjugation by some bounded power ℓ of w), and $\widetilde{\eta}(\gamma) = 1_{G_w}$ (up to multiplication by $w^{-\ell}$).*

(ii) *$\widetilde{M}$ is $\langle a_1,a_2,b_1,b_2,\gamma|\gamma^2a_i\gamma^{-2} = w^{\varepsilon}b_iw^{-\epsilon},\ i=1,2\rangle$ for some ε, or a quotient thereof.*

Proof. — First consider the generating set

$$\boldsymbol{g} = (a_1,a_2,b_1,b_2,c_1,c_2,d_1,d_2,\gamma,a_1d_1a_1d_1).$$

Let $(\widetilde{L},\widetilde{\eta})$ be a restricted strict $\boldsymbol{g}$-shortening quotient of $D_{w,z}$ and let $\{h_n\}_{n\in\mathbb{N}} \subseteq \mathrm{Hom}_B(D_{w,z},\mathbb{F}_2)$ be a stable sequence of restricted $\boldsymbol{g}$-shortest morphisms with $\operatorname{Ker}\widetilde{\eta} = \underrightarrow{\operatorname{Ker}}\, h_n$ and with $k_n,\ell_n,q_n \in \mathbb{Z}$ as in Lemma 2.4 for each h_n.

For each $n \in \mathbb{N}$,

$$\max_{g\in\boldsymbol{g}}|h_n(g)| = \max\left\{\begin{array}{c}|b_i|,|w^{k_n}b_iw^{-k_n}|,|w^{\ell_n}b_iw^{-\ell_n}|,\\ |w^{\ell_n+k_n}b_iw^{-(\ell_n+k_n)}|,|h_n(z^{q_n})w^{-\ell_n}|,:i\in\{1,2\}\\ |w^{k_n}b_1w^{\ell_n}b_1w^{-\ell_n}b_1w^{\ell_n}b_1w^{-(k_n+\ell_n)}|\end{array}\right\}$$

It will be helpful to understand the asymptotic behaviour of these distances (as $n\to\infty$) after normalisation by $\mathrm{tr}(w^{k_n})$. (Since the free group B acts freely on its Cayley graph, it follows that $\mathrm{tr}(w) \neq 0$ and likewise $\mathrm{tr}(w^{k_n}) \neq 0$.)

By Lemma 2.6 $|k_n| \underset{n\to\infty}{\to} \infty$, and therefore

$$\frac{O(1)}{\mathrm{tr}(w^{k_n})} = \frac{O(1)}{|k_n|\cdot\mathrm{tr}(w)} \underset{n\to\infty}{\to} 0$$

Also notice that $|w^t| - \operatorname{tr}(w^t) = \text{const}$ for any $t \in \mathbb{Z}$, and in particular

$$\frac{|w^{k_n}|}{\operatorname{tr}(w^{k_n})} = 1 + \frac{O(1)}{\operatorname{tr}(w^{k_n})} \underset{n\to\infty}{\to} 1$$

It also follows that

$$\frac{|w^{\ell_n}|}{\operatorname{tr}(w^{k_n})} - \frac{|\ell_n|}{|k_n|} = \frac{|w^{\ell_n}|}{\operatorname{tr}(w^{k_n})} - \frac{\operatorname{tr}(w^{\ell_n})}{\operatorname{tr}(w^{k_n})} = \frac{O(1)}{\operatorname{tr}(w^{k_n})} \underset{n\to\infty}{\to} 0$$

The approximations of $\frac{|w^{\ell_n} b_i w^{-\ell_n}|}{\operatorname{tr}(w^{k_n})}$ ($i \in \{1,2\}$) are given by counting the number of appearances of w and taking into account small cancellations:

$$|w^{\ell_n} b_i w^{-\ell_n}| = 2\operatorname{tr}(w) \cdot |\ell_n| + \widetilde{s_n}$$

where $\widetilde{s_n} \in \mathbb{Z}$ accounts for the bounded cancellation between the factors. Hence

$$\frac{|w^{\ell_n} b_i w^{-\ell_n}|}{\operatorname{tr}(w^{k_n})} = 2\frac{|\ell_n|}{|k_n|} + \frac{O(1)}{\operatorname{tr}(w^{k_n})}$$

and by the same reasoning,

$$\frac{|w^{k_n} b_i w^{-k_n}|}{\operatorname{tr}(w^{k_n})} = 2 + \frac{O(1)}{\operatorname{tr}(w^{k_n})}$$

$$\frac{|w^{k_n+\ell_n} b_i w^{-(k_n+\ell_n)}|}{\operatorname{tr}(w^{k_n})} = 2\frac{|\ell_n + k_n|}{|k_n|} + \frac{O(1)}{\operatorname{tr}(w^{k_n})}$$

$$\frac{|w^{k_n} b_1 w^{\ell_n} b_1 w^{-\ell_n} b_1 w^{\ell_n} b_1 w^{-(k_n+\ell_n)}|}{\operatorname{tr}(w^{k_n})} = \left(1 + 3\frac{|\ell_n|}{|k_n|} + \frac{|\ell_n + k_n|}{|k_n|}\right) + \frac{O(1)}{\operatorname{tr}(w^{k_n})}$$

With the approximation of $\frac{|h_n(z^{q_n})w^{-l_n}|}{\operatorname{tr}(w^{k_n})}$ there is more need for care, because of the contribution of q_n.

Let us calculated the numerator in stages - first $|h_n(z)|$, then $|h_n(z^{q_n})|$, and finally $|h_n(z^{q_n})w^{-l_n}|$. Recall that z is mapped by h_n to

$$w^{k_n} b_1 w^{-k_n} b_2 w^{k_n} b_1 w^{-k_n} b_1 w^{k_n} b_1 w^{-k_n} b_1.$$

Therefore,

$$|h_n(z)| = 6\operatorname{tr}(w) \cdot |k_n| + s_n$$

where $s_n \in \mathbb{Z}$ accounts for the bounded cancellation between the factors and (for large enough $|k_n|$, which by Lemma 2.6 must occur for almost every n) is in fact dependent only on the sign of k_n.

$$|h_n(z^{q_n})| = |h_n(z)| \cdot q_n$$

because there is no additional cancellation, as (the reduced word in b_1 and b_2 that is equal to) $h_n(z)$ is cyclically reduced. Recall the form of the word

w, which is given in Lemma 1.11. If $q_n \geqslant 0$ then $w^{-\ell_n}$ has no cancellations with $h_n(z^{q_n})$ in the multiplication $h_n(z^{q_n})w^{-\ell_n}$, so

$$|h_n(z^{q_n})w^{-\ell_n}| = |h_n(z^{q_n})| + |w^{-\ell_n}| = |h_n(z)| \cdot q_n + |w^{-\ell_n}|$$

If $q_n < 0$, then $h_n(z^{q_n})w^{-\ell_n} = h_n(z^{q_n+1}) \cdot \big(h_n(z^{-1})w^{-\ell_n}\big)$, and $h_n(z^{-1})w^{-\ell_n}$ has no cancellations with $h_n(z^{q_n+1})$, so

$$\begin{aligned}|h_n(z^{q_n})w^{-\ell_n}| &= |h_n(z^{q_n+1})| + |h_n(z^{-1})w^{-\ell_n}| \\ &= |h_n(z)| \cdot |q_n + 1| + |h_n(z^{-1})w^{-\ell_n}| \\ &= |h_n(z)| \cdot (|q_n| - 1) + |h_n(z^{-1})w^{-\ell_n}|\end{aligned}$$

Examine $h_n(z^{-1})w^{-\ell_n}$: it is of the form

$$(b_1^{-1}w^{k_n}b_1^{-1}w^{-k_n}b_1^{-1}w^{k_n}b_1^{-1}w^{-k_n}b_2^{-1}w^{k_n}b_1^{-1}w^{-k_n})w^{-\ell_n}$$

Therefore,

$$|h_n(z^{-1})w^{-\ell_n}| = 5\,\mathrm{tr}(w) \cdot |k_n| + \mathrm{tr}(w) \cdot |k_n + \ell_n| + s'_n$$

where $s'_n \in \mathbb{Z}$ accounts for the bounded cancellation between the factors (and the value of s'_n depends on the sign of $(k_n + \ell_n)$ when $|k_n + \ell_n|$ is large enough). So

$$|h_n(z^{q_n})w^{-\ell_n}| = |h_n(z)| \cdot (|q_n| - 1) + (5\,\mathrm{tr}(w) \cdot |k_n| + \mathrm{tr}(w) \cdot |k_n + \ell_n| + s'_n)$$

Now it is possible to approximate $\frac{|h_n(z^{q_n})w^{-l_n}|}{\mathrm{tr}(w^{k_n})}$. Using $\delta_n = \begin{cases} 1, & \text{if } q_n < 0 \\ 0, & \text{if } 0 \leqslant q_n \end{cases}$ and combining both cases, it follows that

$$\frac{|h_n(z^{q_n})w^{-l_n}|}{\mathrm{tr}(w^{k_n})} = \left(6 + \frac{s_n}{|k_n| \cdot \mathrm{tr}(w)}\right) \cdot (|q_n| - \delta_n) + 5\delta_n + |\delta_n + \frac{\ell_n}{k_n}| + \frac{\delta_n s'_n}{|k_n| \cdot \mathrm{tr}(w)}.$$

(Notice that $\frac{s'_n}{|k_n|} = \frac{O(1)}{|k_n|} \underset{n\to\infty}{\to} 0$, but $\frac{s_n}{|k_n|} \cdot (|q_n| - \delta_n)$ cannot be similarly dismissed. However, the term $(6 + \frac{s_n}{|k_n| \cdot \mathrm{tr}(w)}) \cdot (|q_n| - \delta_n) + 5\delta_n$ can be lower-bounded by $5 \cdot (|q_n| - \delta_n) + 5\delta_n = 5 \cdot |q_n|$. Therefore

$$\frac{|h_n(z^{q_n})w^{-l_n}|}{\mathrm{tr}(w^{k_n})} \geqslant 5 \cdot |q_n| + |\delta_n + \frac{\ell_n}{k_n}| + \frac{O(1)}{\mathrm{tr}(w^{k_n})} \geqslant 4 \cdot |q_n|$$

for large enough $n \in \mathbb{N}$, i.e. when $\frac{O(1)}{\mathrm{tr}(w^{k_n})}$ is small enough, and this bound will be useful in analysing the case $q_n \neq 0$.)

So the distances can indeed be estimated in units of $\mathrm{tr}(w^{k_n}) = |k_n| \cdot |\mathrm{tr}(w)|$. First assume $q_n = 0$.

$$\max_{g \in \boldsymbol{g}} \frac{|h_n(g)|}{\mathrm{tr}(w^{k_n})} = \max \left\{ \begin{matrix} \frac{O(1)}{\mathrm{tr}(w^{k_n})},\ 2 + \frac{O(1)}{\mathrm{tr}(w^{k_n})},\ 2\frac{|\ell_n|}{|k_n|} + \frac{O(1)}{\mathrm{tr}(w^{k_n})} \\ 2\frac{|\ell_n + k_n|}{|k_n|} + \frac{O(1)}{\mathrm{tr}(w^{k_n})},\ \frac{|\ell_n|}{|k_n|} + \frac{O(1)}{\mathrm{tr}(w^{k_n})}, \\ 1 + 3\frac{|\ell_n|}{|k_n|} + \frac{|k_n + \ell_n|}{|k_n|} + \frac{O(1)}{\mathrm{tr}(w^{k_n})} \end{matrix} \ : i \in \{1, 2\} \right\}$$

As mentioned in Remark 2.5, k_n is given and cannot be changed by $\mathrm{RMod}(D_{w,z})$, but ℓ_n can be changed by $\mathrm{RMod}(D_{w,z})$. Since $\{h_n\}_{n\in\mathbb{N}}$ are restricted $\boldsymbol{g}$-shortest homomorphisms, ℓ_n must be such that $\max_{g\in\boldsymbol{g}} \frac{|h_n(g)|}{\mathrm{tr}(w^{k_n})}$ is minimal. Denote $x_n = \frac{\ell_n}{k_n}$, so ℓ_n must be chosen such that

$$x_n = \underset{\{x=j/k_n : j\in\mathbb{Z}\}}{\operatorname{argmin}} \max\{2, 2|x|, 2|x+1|, |x|, 1+3|x|+|1+x|\}$$

But

$$\min_{x\in\mathbb{R}} \max\{2, 2|x|, 2|x+1|, |x|, 1+3|x|+|1+x|\} = 2$$

is realised at $x_0 = 0$, and therefore also $x_n = 0 + \frac{O(1)}{\mathrm{tr}(w^{k_n})}$, i.e. $\ell_n = 0 + O(1)$. By taking $q_n \neq 0$, $|h_n(\gamma)|$ would raise the value of $\max_{g\in\boldsymbol{g}} |h_n(g)|$ (for in this case $\frac{|h_n(z^{q_n})w^{-l_n}|}{\mathrm{tr}(w^{k_n})} > 4 \cdot |q_n| \geqslant 4$, which already exceeds the minimal value 2 which is obtained in the case $q_n = 0$), hence $q_n = 0$. It follows that the only restricted strict $\boldsymbol{g}$-shortening quotient of $D_{w,z}$ is $(G_w, \widetilde{\eta})$ where $\widetilde{\eta}|_{A*_{\langle w\rangle}B} = \mathrm{id}$, $\widetilde{\eta}|_{C*_{\langle w\rangle}D}$ sends c_1, c_2, d_1, d_2 to b_1, b_2, a_1, a_2 respectively (up to conjugation by some bounded power ℓ of w), and $\widetilde{\eta}(\gamma) = 1_{G_w}$ (up to multiplication by $w^{-\ell}$).

Next consider the generating set

$$\boldsymbol{u} = (a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, \gamma, a_1c_1a_1c_1).$$

Let $(\widetilde{M}, \widetilde{\pi})$ be a restricted strict $\boldsymbol{u}$-shortening quotient of $D_{w,z}$ and let $\{h_n\}_{n\in\mathbb{N}} \subseteq \mathrm{Hom}_B(D_{w,z}, \mathbb{F}_2)$ be a stable sequence of restricted $\boldsymbol{u}$-shortest morphisms with $\operatorname{Ker}\widetilde{\pi} = \underrightarrow{\operatorname{Ker}}\, h_n$ and with $k_n, \ell_n, q_n \in \mathbb{Z}$ as in Lemma 2.4. By similar analysis, while initially assuming $q_n = 0$, ensuring h_n is restricted $\boldsymbol{u}$-shortest means finding $x_n = \frac{\ell_n}{k_n}$ which equals

$$\underset{\{x=j/k_n : j\in\mathbb{Z}\}}{\operatorname{argmin}} \max\{2, 2|x|, 2|x+1|, |x|, 1+|x|+3|x-1|\}$$

Here

$$\min_{x\in\mathbb{R}} \max\{2, 2|x|, 2|x+1|, |x|, 1+|x|+3|x-1|\} = 3$$

and this value is realised at $x = \frac{1}{2}$, so

$$\ell_n \in \left\{\frac{k_n}{2} + O(1), \frac{k_n+1}{2} + O(1), \frac{k_n-1}{2} + O(1)\right\}$$

(since $\max\{2, 2|x|, 2|x+1|, |x|, 1+|x|+3|x-1|\}$ is monotonically decreasing before $x = \frac{1}{2}$ and monotonically increasing afterwards). Again, taking $q_n \neq 0$ raises the value of $\max_{u\in\boldsymbol{u}} |h_n(u)|$, so indeed $q_n = 0$. For every $n \in \mathbb{N}$ and every $1 \leqslant i \leqslant 2$,

$$h_n(\gamma c_i \gamma^{-1}) = w^{-\ell_n} w^{\ell_n} b_i w^{-\ell_n} w^{\ell_n} = b_i = h_n(b_i)$$

and

$$h_n(\gamma d_i \gamma^{-1}) = w^{-\ell_n} w^{k_n+\ell_n} b_i w^{-k_n-\ell_n} w^{\ell_n} = w^{k_n} b_i w^{-k_n} = h_n(a_i)$$

Denote $\varepsilon_n = k_n - 2\ell_n$, so $\varepsilon_n = O(1)$. By extraction of subsequence, ε_n is a constant sequence ε and $h_n(\gamma^2 a_i \gamma^{-2}) = h_n(w^\varepsilon b_i w^{-\varepsilon})$. The limit group $\widetilde{M} = G/\underrightarrow{\operatorname{Ker}}\, h_n$ is therefore $\langle a_1, a_2, b_1, b_2, \gamma \mid \gamma^2 a_i \gamma^{-2} = w^\varepsilon b_i w^{-\epsilon},\, i = 1, 2\rangle$ or a quotient thereof. □

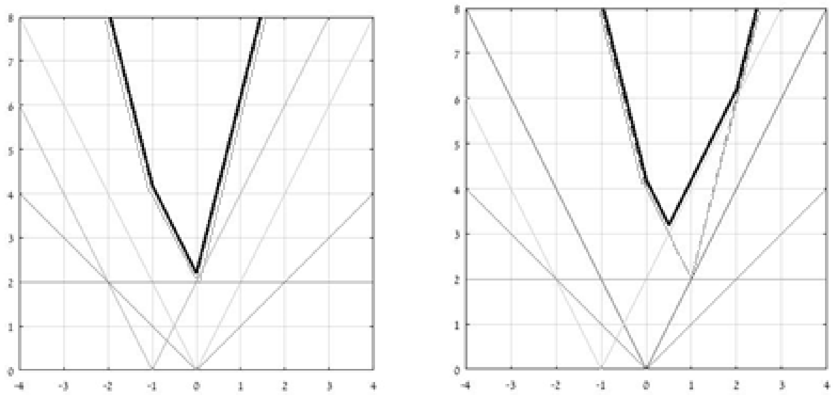

Figure 2.3. $\max\{2, 2|x|, 2|x+1|, |x|, 1+3|x|+|1+x|\}$ *for* $\boldsymbol{g}$ *(left), and* $\max\{2, 2|x|, 2|x+1|, |x|, 1+|x|+3|x-1|\}$ *for* $\boldsymbol{u}$ *(right)*

Finally, we prove the main result:

Theorem 2.8. — *There exists a restricted limit group H with generating sets $\boldsymbol{s}, \boldsymbol{t}$ and a maximal restricted shortening quotient (R, ρ) with respect to $\boldsymbol{s}$, which is not isomorphic to any restricted shortening quotient with respect to $\boldsymbol{t}$.*

Proof. — We have seen that $D_{w,z}$ is a restricted limit group with respect to $\{b_1, b_2\}$ (Lemma 2.1). We will now see that, together with the generating sets $\boldsymbol{g}$ and $\boldsymbol{u}$, it does indeed satisfy the theorem, i.e. it has a maximal restricted shortening quotient with respect to $\boldsymbol{g}$ which is not isomorphic to any restricted shortening quotient with respect to $\boldsymbol{u}$.

For every generating set of a restricted limit group, there exists a restricted strict maximal shortening quotient. As seen in Proposition 2.7, $(G_w, \widetilde{\eta})$ is the sole strict element of $\mathrm{RSQ}(D_{w,z}, \boldsymbol{g})$, and is therefore the only strict element of $\mathrm{RMSQ}(D_{w,z}, \boldsymbol{g})$. Likewise, for $\boldsymbol{u}$, any strict restricted shortening quotient, and in particular any strict restricted maximal shortening quotient $(\widetilde{M}, \widetilde{\pi})$, is a (possibly not proper) quotient of $\langle a_1, a_2, b_1, b_2, \gamma \mid$

$\gamma^2 a_i \gamma^{-2} = w^{\varepsilon} b_i w^{-\epsilon}$, $i = 1, 2\rangle$ for some ε. However, no such group can be isomorphic to $(G_w, \widetilde{\eta})$; for example, the homology group of $(\widetilde{M}, \widetilde{\pi})$ is $\mathbb{Z}^t$, $t \leqslant 3$, whereas the homology group of $(G_w, \widetilde{\eta})$ is $\mathbb{Z}^4$. Since $(G_w, \widetilde{\eta})$ and $(\widetilde{M}, \widetilde{\pi})$ are not isomorphic as groups, they are in particular not SQ-isomorphic. A strict restricted shortening quotient cannot be SQ-isomorphic to a non-strict shortening quotient, so in fact the only restricted strict $\boldsymbol{g}$-shortening quotient is not SQ-isomorphic to any restricted strict maximal $\boldsymbol{u}$-shortening quotient. □

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