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K3 SURFACES WITH MAXIMAL FINITE AUTOMORPHISM GROUPS CONTAINING M_{20}

by Cédric BONNAFÉ & Alessandra SARTI (*)

In memory of Laurent Gruson

Abstract. — It was shown by Mukai that the maximum order of a finite group acting faithfully and symplectically on a K3 surface is 960 and that if such a group has order 960, then it is isomorphic to the Mathieu group M_{20}. Then Kondo showed that the maximum order of a finite group acting faithfully on a K3 surface is 3 840 and this group contains M_{20} with index four. Kondo also showed that there is a unique K3 surface on which this group acts faithfully, which is the Kummer surface $\mathrm{Km}(E_i \times E_i)$. In this paper we describe two more K3 surfaces admitting a big finite automorphism group of order 1 920, both groups contains M_{20} as a subgroup of index 2. We show moreover that these two groups and the two K3 surfaces are unique. This result was shown independently by S. Brandhorst and K. Hashimoto in a forthcoming paper, with the aim of classifying all the finite groups acting faithfully on K3 surfaces with maximal symplectic part.

Résumé. — Mukai a montré que l'ordre maximal d'un groupe fini agissant fidèlement et symplectiquement sur une surface K3 est 960 et que, si un tel groupe a pour ordre 960, alors il est isomorphe au groupe de Mathieu M_{20}. Kondo a ensuite montré que l'ordre maximal d'un groupe fini agissant fidèlement sur une K3 surface est 3840 et qu'un tel groupe contient M_{20} comme sous-groupe d'indice 4. Kondo a aussi montré qu'il existe une unique surface K3 sur laquelle ce groupe agit fidèlement: c'est la surface de Kummer $\mathrm{Km}(E_i \times E_i)$. Dans cet article, nous décrivons deux autres surfaces K3 admettant un groupe fini d'automorphismes d'ordre 1920, ces deux groupes et ces deux surfaces K3 étant uniques. Ce résultat a été obtenu indépendamment par S. Brandhorst and K. Hashimoto dans un article à venir, dont le but est de classifier les groupes finis agissant fidèlement sur des K3 surfaces et dont la partie symplectique est maximale.

1. Introduction

A K3 surface is a compact complex surface which is simply connected and has trivial canonical bundle. Given a finite group Γ acting on a K3

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surface X we have an exact sequence

$$1 \longrightarrow \Gamma_0 \longrightarrow \Gamma \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 1$$

where the last map is induced by the action on the nowhere vanishing holomorphic 2-form ω_X. The group Γ_0 is the normal subgroup of maximal order contained in Γ whose automorphisms act trivially on ω_X. The automorphisms of Γ_0 are called *symplectic*. It was shown by Mukai [11, Theorem 0.3] that, if G is a finite group acting faithfully and symplectically on a K3 surface, then $|G| \leqslant 960$ and, if $|G| = 960$, then G is isomorphic to the Mathieu group M_{20}. In his paper Mukai gives the example of a K3 surface with such an action, we recall this example in section 4. More generally, it is an interesting question to classify maximal finite groups Γ acting faithfully on a K3 surface. More precisely we say that Γ is a *maximal finite group* acting faithfully on a K3 surface if the following holds: assume Γ' is another finite group acting faithfully on a K3 surface then Γ is not (isomorphic to) a proper subgroup of Γ'.

In Theorem 6.3 we show that there are only three finite groups Γ containing strictly $\Gamma_0 = M_{20}$ as the normal subgroup of Γ acting faithfully and symplectically and only three K3 surfaces acted on by such a Γ, the main ingredient of the proof is Theorem 2.7. This result is shown also independently in a forthcoming paper of S. Brandhorst and K. Hashimoto [3], where they compute all the finite groups acting faithfully on K3 surfaces with maximal symplectic part. In our situation one of the three K3 surfaces mentioned above was constructed by Kondo [9] (this is the only K3 surface acted on faithfully by a finite group of order $3\,840 = 4 \cdot |M_{20}|$), another one was constructed by Mukai [11], and the existence of the last one was showed by Brandhorst–Hashimoto in loc. cit., we give here explicit equations. In the second and in the third case the order of Γ is equal to $2 \cdot |M_{20}|$. We denote these three surfaces respectively by X_{Ko}, X_{Mu} and X_{BH}. In this note, we compute the transcendental lattice of these three K3 surfaces. This was done by Kondo for the surface X_{Ko}, we recall it here to have a complete picture, and we compute it for X_{Mu} and X_{BH}. Accordingly to [5, Section 3] the transcendental lattice of X_{Mu} was already known by Mukai, but we could not find explicit computations, so we give it here. We give also equations for the three surfaces. Mukai already provided equations for X_{Mu} as a smooth quartic surface in $\mathbb{P}^3(\mathbb{C})$ (which is the *Maschke surface*, see [5, Section 3]) we compute it here in a different way, but we show that up to a projective transformation of $\mathbb{P}^3(\mathbb{C})$, these are equivalent.

The equations for X_{Ko} and X_{BH} are new. In particular one gets easily a (singular) equation for the first one as a complete intersection of two quartics in weighted projective space $\mathbb{P}(1,1,2,2,2)$ by using a result of Inose, [8]. To get the equations for X_{BH} one needs a more careful study of the action of M_{20} on the projective space $\mathbb{P}^5(\mathbb{C})$. It turns out that X_{BH} is a smooth complete intersection of three quadrics and we give here the equations (this answers a question of S. Brandhorst to the authors). All these three K3 surfaces turn out to be Kummer surfaces of abelian surfaces that are the product of two elliptic curves, see Corollary 2.5. By using results of Shioda and Mitani [17] we compute explicitly the two elliptic curves. We have that

$$X_{\mathrm{Ko}} \cong \mathrm{Km}(E_i \times E_i), \qquad X_{\mathrm{Mu}} \cong \mathrm{Km}(E_{i\sqrt{10}} \times E_{i\sqrt{10}}),$$
$$X_{\mathrm{BH}} \cong \mathrm{Km}(E_\tau \times E_{2\tau}), \quad \text{with } \tau = \frac{-1+i\sqrt{5}}{2}.$$

Here, E_z denotes the elliptic curve with complex multiplication given by z. For the example of X_{BH}, we also obtain in Remark 5.8 an explicit Nikulin configuration of 16 disjoint smooth rational curves (we are not able to obtain such an explicit configuration for X_{Mu}: see Remark 4.4).

Notation. — If G is a group, we denote by G' its commutator subgroup (also sometimes called derived subgroup) and by $\mathrm{Z}(G)$ its center. If V is a vector space, we denote by $\mathbb{C}[V]$ the algebra of polynomial functions on V and, if $k \geqslant 0$, we denote by $\mathbb{C}[V]_k$ its homogeneous component of degree k. If $f_1, \ldots, f_r \in \mathbb{C}[V]$ are homogeneous, we denote by $\mathcal{Z}(f_1, \ldots, f_r)$ the associated scheme of $\mathbb{P}(V)$, defined by $f_1 = \cdots = f_r = 0$. If G is a subgroup of $\mathbf{GL}_{\mathbb{C}}(V)$, we denote by PG its image in $\mathbf{PGL}_{\mathbb{C}}(V)$. If $V = \mathbb{C}^n$, we identify naturally $\mathbf{GL}_{\mathbb{C}}(V)$ and $\mathbf{GL}_n(\mathbb{C})$. We denote by M_{20} the Mathieu group of order 960.

If $\tau \in \mathbb{C}$ has a positive imaginary part, we denote by E_τ the elliptic curve $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$. If A is an abelian surface, we denote by $\mathrm{Km}(A)$ its associated Kummer surface. We denote by $\mathbf{L}$ the *K3 lattice* $E_8(-1) \oplus E_8(-1) \oplus U \oplus U \oplus U$, where U is the hyperbolic plane and $E_8(-1)$ is the lattice E_8 endowed with the opposite quadratic form. If X is a K3 surface, we denote by $\mathbf{L}_X$ the lattice $\mathrm{H}^2(X, \mathbb{Z})$ (it turns out that $\mathbf{L}_X \simeq \mathbf{L}$) and by $\mathbf{T}_X$ its transcendental lattice (i.e. the orthogonal, in $\mathbf{L}_X$, of its Néron–Severi group). Finally, we denote by $\mathbf{L}_{20}$ the lattice

$$\mathbf{L}_{20} = \begin{pmatrix} 4 & 0 & -2 \\ 0 & 4 & -2 \\ -2 & -2 & 12 \end{pmatrix}.$$

See the Proposition 2.3 below for the reason for this notation.

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2. K3 surfaces with a faithful action of M_{20}

We gather in this section some properties of the K3 surfaces admitting a faithful action of the finite group M_{20} (since M_{20} is equal to its commutator subgroup, this is necessarily a symplectic action), and we prove the main result of this paper, namely a classification of K3 surfaces admitting a faithful action of a finite group containing strictly M_{20}.

If we consider all the K3 surfaces X that admit a faithful symplectic action of M_{20}, Xiao [18, Nr. 81, Table 2] proved that the minimal resolution of the quotient of X by M_{20} is a K3 surface with Picard number 20. By a result of Inose [8, Corollary 1.2], this means also that X has Picard number 20. This shows the following, with the same notation as before:

Proposition 2.1. — *There are at most countably many K3 surfaces with a faithful symplectic action by M_{20}.*

Proof. — Since the Picard number is 20, then the moduli space of K3 surfaces with a faithful symplectic M_{20}-action is 0-dimensional. □

Remark 2.2. — Observe that the automorphism group of a K3 surface with Picard number 20 is infinite [16, Theorem 5]. Shioda and Inose show it by exhibiting an elliptic fibration with an infinite order section, this gives an automorphism acting symplectically on the K3 surface with infinite order.

Recall the following result [9, proof of Proposition 2.1]:

Proposition 2.3. — *Let X be a K3 surface with a faithful symplectic action by M_{20}. Then the invariant lattice $\mathbf{L}_X^{M_{20}}$ is isometric to $\mathbf{L}_{20}$.*

Remark 2.4. — Note that $\mathbf{L}_{20}$ has signature $(3,0)$, so its isometry group is finite. Let us recall its description. Let

$$\rho_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \rho_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then ρ_1 and ρ_2 belong to the group of isometries of $\mathbf{L}_{20}$ and it is easily checked that the group of isometries of $\mathbf{L}_{20}$ is generated by ρ_1, ρ_2 and $-\operatorname{Id}_{\mathbf{L}_{20}}$ (by using for instance the upcoming Lemma 2.8) and has order 16 (see also [9, Proposition 2.1]).

Corollary 2.5. — *If a K3 surface X admits a faithful action by the group M_{20} then $X = \operatorname{Km}(A)$ for a unique abelian surface A, which is the product of two elliptic curves.*

Proof. — Let (u, v) be a $\mathbb{Z}$-basis of $\mathbf{T}_X \subset \mathbf{L}_X^{M_{20}}$. By Proposition 2.3, we have u^2, $v^2 \in 4\mathbb{Z}$ and $u \cdot v \in 2\mathbb{Z}$. So

$$\mathbf{T}_X \simeq \begin{pmatrix} 4a & 2b \\ 2b & 4c \end{pmatrix}.$$

Following [17, Section 3], we set $A \cong E_{\tau_1} \times E_{\tau_2}$ where

$$\tau_1 = \frac{-b + \sqrt{\Delta}}{2a}, \qquad \tau_2 = \frac{b + \sqrt{\Delta}}{2}$$

and $\Delta = b^2 - 4ac$, so that

$$\mathbf{T}_A := \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}.$$

Hence $\mathbf{T}_X = \mathbf{T}_A(2) = \mathbf{T}_{\operatorname{Km}(A)}$.

The uniqueness follows from [17, Theorem 5.1]. □

Remark 2.6. — Let us prove here that $\mathbf{L}_{20}$ is indecomposable. Assume that it is not indecomposable. Then $\mathbf{L}_{20} = L_1 \oplus^{\perp} L_2$, where L_1 has rank 1 and L_2 has rank 2. By the proof of the Corollary 2.5, we have $L_1 = \langle 4n \rangle$ for some $n \geqslant 0$ and

$$L_2 = \begin{pmatrix} 4a & 2b \\ 2b & 4c \end{pmatrix}$$

for some $a, b, c \in \mathbb{Z}$. Then $160 = \operatorname{disc}(\mathbf{L}_{20}) = \operatorname{disc}(L_1)\operatorname{disc}(L_2) = 16n(4ac - b^2)$. In other words, $10 = n(4ac - b^2)$, which means that $4ac - b^2 \in \{1, 2, 5, 10\}$. But $b^2 \equiv 0$ or $1 \mod 4$, so $4ac - b^2 \equiv 3$ or $4 \mod 4$. This leads to a contradiction.

Our main result in this paper is the following:

THEOREM 2.7. — *Assume that M_{20} acts faithfully on a K3 surface X, and assume moreover that X admits a non–symplectic automorphism ι acting on it, normalizing M_{20} and such that $\iota^2 \in M_{20}$. We set $G = \langle\iota\rangle M_{20}$. Then we have the following three possibilities for the G-invariant Néron–Severi group of X and its transcendental lattice:*

(1) $\langle 40\rangle$, $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

(2) $\langle 4\rangle$, $\begin{pmatrix} 4 & 0 \\ 0 & 40 \end{pmatrix}$

(3) $\langle 8\rangle$, $\begin{pmatrix} 8 & 4 \\ 4 & 12 \end{pmatrix}$

All the three cases are possible and are described in Sections 3, 4, 5.

Proof. — We only prove here the fact that the Néron–Severi group of X and its transcendental lattice is necessarily one of the given three forms: the existence of the three examples will be shown in the upcoming sections (and we will add some geometric features of those examples). We first need two technical lemmas:

LEMMA 2.8. — *Up to isometry, there is a unique embedding of the lattice $\langle 4\rangle$ (resp. $\langle 8\rangle$, resp. $\langle 40\rangle$) as a primitive sublattice of $\mathbf{L}_{20}$.*

Proof of Lemma 2.8. — The uniqueness of the embedding of $\langle 40\rangle$ is shown in [9, Lemma 3.1]. For the two other cases, let (e, f, h) denote the canonical basis of the lattice $\mathbf{L}_{20}$ and let L be a primitive element of $\mathbf{L}_{20}$ such that $L^2 = 4$ (resp. 8). Write $L = \lambda e + \mu f + \delta h$ with λ, μ, $\delta \in \mathbb{Z}$. Then

$$L^2 = (2\lambda - \delta)^2 + (2\mu - \delta)^2 + 10\delta^2,$$

so $\delta = 0$ and $\lambda^2 + \mu^2 = 1$ (resp. $\lambda^2 + \mu^2 = 2$). This gives $(\lambda, \mu) = (\pm 1, 0)$ or $(0, \pm 1)$ (resp. $(\pm 1, \pm 1)$). So $L = \pm e$ or $\pm f$ (resp. $L = \pm e \pm f$), and the four solutions are in the orbit of the group $\langle -\mathrm{Id}_{\mathbf{L}_{20}}, \rho_1\rangle$ (resp. $\langle -\mathrm{Id}_{\mathbf{L}_{20}}, \rho_2\rangle$). □

We choose an isomorphism between $\mathbf{L}_{20}$ and $\mathbf{L}_X^{M_{20}}$. Then the group $G/M_{20} = \langle\iota\rangle$ acts on $\mathbf{L}_{20}$ and ι acts by $-\mathrm{Id}$ on $\mathbf{T}_X$. Also, the lattice $\mathbf{L}_X^G$ has rank 1 because $\mathbf{T}_X$ has rank 2.

LEMMA 2.9. — *The sublattice $\mathbf{L}_X^G \oplus \mathbf{T}_X$ has index 2 in $\mathbf{L}_{20}$.*

Proof of Lemma 2.9. — First, $\mathbf{L}_X^G \oplus \mathbf{T}_X$ is different from $\mathbf{L}_{20}$ since $\mathbf{L}_{20}$ is indecomposable (see Remark 2.6). We have

$$\mathbf{L}_X^G = \{L \in \mathbf{L}_{20} \mid \iota(L) = L\},$$

$$\mathbf{T}_X = \{L \in \mathbf{L}_{20} \mid \iota(L) = -L\}.$$

By [13, Section 5], the projection $\mathbf{L}_{20}/(\mathbf{L}_X^G \oplus \mathbf{T}_X) \longrightarrow (\mathbf{L}_X^G)^\vee/\mathbf{L}_X^G$ is a ι–invariant monomorphism. This shows in particular that $\mathbf{L}_{20}/(\mathbf{L}_X^G \oplus \mathbf{T}_X)$ is cyclic. Also, if $L \in \mathbf{L}_{20}$, then

$$2L = \underbrace{L + \iota(L)}_{\in \mathbf{L}_X^G} + \underbrace{L - \iota(L)}_{\in \mathbf{T}_X} \in \mathbf{L}_X^G \oplus \mathbf{T}_X.$$

So the sublattice $\mathbf{L}_X^G \oplus \mathbf{T}_X$ has index 2 in $\mathbf{L}_{20}$. This completes the proof of the Lemma. □

We now come back to the proof of the theorem. We write $\mathbf{L}_{20}^G = \mathbb{Z}L$. By the proof of Corollary 2.5, we have $L^2 = 4n$ (so that $\mathbf{L}_{20}^G \simeq \langle 4n \rangle$) and the transcendental lattice of X is of the form

$$\mathbf{T}_X = \begin{pmatrix} 4a & 2b \\ 2b & 4c \end{pmatrix}$$

with a, b, c integers such that $d := 4ac - b^2 > 0$, $b^2 \leqslant ac \leqslant \frac{d}{3}$, $-a \leqslant b \leqslant a \leqslant c$, see e.g. [16, p. 128]. We have shown in Lemma 2.9 that $\mathbf{L}_{20}^G \oplus \mathbf{T}_X \simeq \langle 4n \rangle \oplus \mathbf{T}_X$ is a sublattice of index 2 in $\mathbf{L}_{20}$. Hence we have by applying [1, Section 2, Lemma 2.1]

$$4 = [\mathbf{L}_{20} : \langle 4n \rangle \oplus \mathbf{T}_X]^2 = \frac{\det(\langle 4n \rangle \oplus \mathbf{T}_X)}{\det \mathbf{L}_{20}} = \frac{16n(4ac - b^2)}{160}.$$

In conclusion

$$n(4ac - b^2) = 2^3 \cdot 5.$$

We discuss two cases.

Assume that b is odd. — Then $4ac - b^2$ is also odd. This means that it is equal to 1 or 5, but then if $b = 2k + 1$ we get $4ac - 4k^2 - 4k - 1$ equal to 1 or 5 which is clearly impossible.

Assume that b is even. — Then with $b = 2b'$ we get

$$(ac - b'^2)n = 2 \cdot 5$$

We distinguish four cases:

(1) $n = 1$, $ac - b'^2 = 10$,
(2) $n = 2$, $ac - b'^2 = 5$,
(3) $n = 5$, $ac - b'^2 = 2$,
(4) $n = 10$, $ac - b'^2 = 1$.

By Lemma 2.8, the lattices $\langle 4 \rangle$, $\langle 8 \rangle$ and $\langle 40 \rangle$ have a unique primitive embedding in the lattice $\mathbf{L}_{20}$:

(1) If $n=1$, we may assume that $L=e$. We now compute the orthogonal complement of $\mathbb{Z}e$ in the lattice $\mathbf{L}_{20}$. This will give us the transcendental lattice. Let now $\lambda e+\mu f+\delta h$ with $\lambda,\mu,\delta\in\mathbb{Z}$ be such that
$$\langle \lambda e+\mu f+\delta h, e\rangle = 0$$
This gives $4\lambda-2\delta=0$ so that the orthogonal complement is generated by the elements $e+2h$ and f and considering instead the generators $e+f+2h$ and f we get the lattice given in the theorem.

(2) If $n=2$, we may assume that $L=e-f$. We compute the orthogonal complement of $e-f$ in $\mathbf{L}_{20}$ which is generated by $e+f$ and $-h$ which are the generators of the rank two lattice whose bilinear form is as given in the theorem.

(3) If $n=10$, then the orthogonal complement of L has been computed in [9] and one gets the rank two lattice whose bilinear form is given as in the theorem.

We have respectively $(a,b,c)=(1,0,1)$, $(a,b,c)=(1,0,10)$, $(a,b,c)=(2,2,3)$.

We consider now the third case with $ac-b'^2=2$ and we show that it is not possible. The integers a,b,c satisfy $-a\leqslant b\leqslant a\leqslant c$, $ac\leqslant d/3$, $(b')^2\leqslant (ac)/4\leqslant d/3$. By the previous computations, we have that $d=4(ac-b'^2)$ hence in this case $d=8$, we get that $b'^2\leqslant 2$. Hence $b'=0$ or $b'=1$. In the first case we get $a=1$, $c=2$ which gives the matrix
$$M := \begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix}.$$
In the second case we get $a=1, c=3$ but then $ac=3>8/3$ so this is not possible. To make the case $\mathbf{T}_X=M$ possible, we should then find a primitive embedding in $\mathbf{L}_{20}$ with vectors v_1 and v_2 with $v_1^2=4$, $v_2^2=8$, $v_1\cdot v_2=0$ but by the computations in Lemma 2.8 and with the same notations as there we see that we must send v_1 to $\pm e$ or $\pm f$ and v_2 to $\pm e\pm f$, so these never satisfy the condition $v_1\cdot v_2=0$. □

3. Kondo's example

It was shown by Kondo in [9, Theorem 1] that the maximal order of a finite group acting faithfully on a K3 surface is 3 840 and that this bound is reached for a unique K3 surface X_{Ko} and a unique faithful action of a unique finite group G_{Ko} of order 3 840. Kondo shows that $X_{\mathrm{Ko}} = \mathrm{Km}(E_i \times E_i)$. Recall that we have an exact sequence

$$1 \longrightarrow M_{20} \longrightarrow G_{\mathrm{Ko}} \longrightarrow \boldsymbol{\mu}_4 \longrightarrow 1, \tag{3.1}$$

where the last map is induced by the group homomorphism

$$\alpha : G_{\mathrm{Ko}} \longrightarrow \mathbb{C}^*,$$

defined by $g(\omega_{X_{\mathrm{Ko}}}) = \alpha(g)\omega_X$ and $\omega_{X_{\mathrm{Ko}}}$ is the holomorphic 2-form that we have fixed on X_{Ko}. Recall that $X_{\mathrm{Ko}} = \mathrm{Km}(E_i \times E_i)$ (see e.g. [9, Proof of Lemma 1.2]) has transcendental lattice

$$\mathbf{T}_{X_{\mathrm{Ko}}} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}.$$

With the previous notation we have:

Proposition 3.1. — *The invariant Néron–Severi group* $NS(X_{\mathrm{Ko}})^{M_{20}} = \mathbb{Z}\, L_{40}$ *with* $L_{40}^2 = 40$.

Proof. — See [9, Lemma 3.1]. □

Remark 3.2. — In particular this means that we cannot represent X_{Ko} as a quartic surface in $\mathbb{P}^3(\mathbb{C})$ with a faithful action of M_{20} by linear transformations of $\mathbb{P}^3(\mathbb{C})$.

3.1. A geometric model

By using a result of Inose [8, Theorem 2] one can view $X_{\mathrm{Ko}} = \mathrm{Km}(E_i \times E_i)$ as the minimal resolution of a singular surface in $\mathbb{P}(1,1,2,2,2)$. We give here the equation. Inose shows that X_{Ko} is the minimal resolution of the quotient of the Fermat quartic surface

$$F:\ x^4 + y^4 + z^4 + t^4 = 0$$

by the symplectic involution $\iota : (x : y : z : t) \mapsto (x : y : -z : -t)$, which has 8 isolated fixed points [12, Section 5]. Since the automorphism is symplectic, the minimal resolution of the quotient $X_{\mathrm{Ko}} \to F/\langle\iota\rangle$ is again a K3 surface and the Picard number remains unchanged. Moreover, for the transcendental lattices $\mathbf{T}_{X_{\mathrm{Ko}}}(2) = \mathbf{T}_F$ holds. The ring of invariant polynomials for the action of ι is generated by x, y, z^2, t^2, zt. We put $z_0 = x, z_1 = y, z_2 = z^2, z_3 = t^2, z_4 = zt$ and we have then the equations for $F/\langle\iota\rangle$ in $\mathbb{P}(1,1,2,2,2)$:

$$z_0^4 + z_1^4 + z_2^2 + z_3^2 = 0,\ z_4^2 = z_2 z_3.$$

The eight A_1 singularities are determined as follows. First we have singularities coming from the ambient space, these are the intersection with the plane $z_0 = z_1 = 0$. This gives $z_2^2 + z_3^2 = 0$ which together with $z_4^2 = z_2 z_3$ gives four A_1 singularities. The others come from the singularities of the cone $z_4^2 = z_2 z_3$, i.e. with $z_4 = z_2 = z_3 = 0$ we get the four singularities A_1 with equation $z_0^4 + z_1^4 = 0$.

See also [3] for an embedding of X_{Ko} in $\mathbb{P}^{21}(\mathbb{C})$.

4. Mukai's example

Let $G_{\mathrm{Mu}} = \langle s_1, s_2, s_3, s_4 \rangle$, where

$$s_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad s_2 = \frac{1}{2} \begin{pmatrix} 1 & 1 & i & i \\ 1 & 1 & -i & -i \\ -i & i & 1 & -1 \\ -i & i & -1 & 1 \end{pmatrix},$$

$$s_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad s_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then G_{Mu} is the primitive complex reflection group denoted by G_{29} in Shephard–Todd classification [15]. Recall that $|G_{\mathrm{Mu}}| = 7\,680$. We denote by V the vector space $\mathbb{C}^4$, and by $\mathbb{C}[V]$ the algebra of polynomial functions on V, identified naturally with $\mathbb{C}[x, y, z, t]$. If m is a monomial in x, y, z and t, we denote by $\Sigma(m)$ the sum of all monomials obtained by permutation of the variables. For instance,

$$\Sigma(x) = x + y + z + t, \qquad \Sigma(xyzt) = xyzt,$$
$$\Sigma(x^4 y) = x^4(y + z + t) + y^4(x + z + t) + z^4(x + y + t) + t^4(x + y + z)$$
$$= \Sigma(xy^4).$$

Note that the derived subgroup G'_{Mu} of G_{Mu} has index 2, that $G'_{\mathrm{Mu}} = G_{\mathrm{Mu}} \cap \mathbf{SL}_4(\mathbb{C})$, so that $G_{\mathrm{Mu}} = G'_{\mathrm{Mu}} \langle s_1 \rangle$. Note also that $\mathrm{Z}(G_{\mathrm{Mu}}) \simeq \boldsymbol{\mu}_4 \subset G'_{\mathrm{Mu}}$. Moreover, $PG'_{\mathrm{Mu}} \simeq M_{20}$ so that we have a split exact sequence

$$(4.1) \qquad 1 \longrightarrow PG'_{\mathrm{Mu}} \simeq M_{20} \longrightarrow PG_{\mathrm{Mu}} \longrightarrow \boldsymbol{\mu}_2 \longrightarrow 1,$$

where the last map is the determinant.

Now, there exists a unique (up to scalar) homogeneous invariant f of G_{Mu} of degree 4: it is given by

$$f = \Sigma(x^4) - 6\Sigma(x^2 y^2).$$

We set $X_{\mathrm{Mu}} = \mathcal{Z}(f)$. It can easily be checked that X_{Mu} is a smooth and irreducible quartic in $\mathbb{P}^3(\mathbb{C})$, so that it is a K3 surface, endowed with a faithful symplectic action of M_{20} and an extra non-symplectic automorphism of order 2, i.e. one can fix it as $[x : y : z : t] \mapsto [x : y : z : -t]$, the one induced by s_1.

In [11, Nr. 4 on p. 190] Mukai gives the following equation for some M_{20}-invariant quartic polynomial

$$\Sigma(x^4) + 12xyzt,$$

and we denote by X'_{Mu} the zero set of this polynomial which defines a smooth quartic K3 surface. We have

Proposition 4.1. — *There exists $g \in \mathbf{GL}_4(\mathbb{C})$ such that $g(X_{\mathrm{Mu}}) = X'_{\mathrm{Mu}}$.*

Proof. — If one applies to the Mukai's polynomial the change of coordinates:

$$x \mapsto x - y, \;\; y \mapsto x + y, \;\; z \mapsto z - t, \;\; t \mapsto z + t$$

one gets

$$2\Sigma(x^4) + 12x^2y^2 + 12z^2t^2 + 12x^2z^2 - 12x^2t^2 - 12y^2z^2 + 12y^2t^2$$

and by replacing by

$$x \mapsto ix, \;\; t \mapsto it, \;\; y \mapsto y,$$

and dividing by 2 one finds the polynomial f. □

Note the following fact:

(4.2) If $g \in \mathbf{PGL}_4(\mathbb{C})$ leaves invariant X_{Mu} then $g \in PG_{\mathrm{Mu}}$.

Proof. — If $g \in \mathbf{PGL}_4(\mathbb{C})$ leaves X_{Mu} invariant, we may find a representative $\widetilde{g}$ of g in $\mathbf{GL}_4(\mathbb{C})$ which leaves f invariant. Let $\Gamma = \{\gamma \in \mathbf{GL}_4(\mathbb{C}) \mid {}^{\gamma}f = f\}$. We only need to prove that $\Gamma = G_{\mathrm{Mu}}$. By [10] or [14, Theorem 2.1], Γ is finite (because X_{Mu} is smooth), and contains G_{Mu}. Let R denote the set of reflections in G_{Mu} (and recall that $G_{\mathrm{Mu}} = \langle R \rangle$) and let

$$\mathcal{R} = \{\gamma s \gamma^{-1} \mid \gamma \in \Gamma \text{ and } s \in R\},$$

so that $\mathcal{R}$ is a set of reflections contained in Γ. We set $\Gamma_{\mathcal{R}} = \langle \mathcal{R} \rangle$. Then $\Gamma_{\mathcal{R}}$ is a complex reflection group containing G_{Mu}, but it follows from the classification of primitive complex reflection groups that $\Gamma_{\mathcal{R}} = G_{\mathrm{Mu}}$ or (up to conjugacy) the group denoted by G_{31} in Shephard–Todd classification [15]. Since G_{31} has no non-zero invariant of degree 4, this forces $\Gamma_{\mathcal{R}} = G_{\mathrm{Mu}}$. In particular, G_{Mu} is normal in Γ, and so the result follows from [4, Proposition 3.13] (which says that $\mathrm{N}_{\mathbf{GL}_4(\mathbb{C})}(G_{\mathrm{Mu}}) = G_{\mathrm{Mu}} \cdot \mathbb{C}^{\times}$). □

The embedding $X_{\mathrm{Mu}} \hookrightarrow \mathbb{P}^3(\mathbb{C})$ defines the class of a hyperplane section on X_{Mu} that we denote by L_4: then $L_4^2 = 4$ and L_4 is PG_{Mu}-invariant.

Proposition 4.2. — *With the above notation, we have:*

(1) *The transcendental lattice of X_{Mu} is a rank two lattice given by*
$$\mathbf{T}_{X_{\mathrm{Mu}}} := \begin{pmatrix} 4 & 0 \\ 0 & 40 \end{pmatrix}$$
and $NS(X_{\mathrm{Mu}})^{M_{20}} = \mathbb{Z}\, L_4$ with $L_4^2 = 4$.

(2) *The quartic X_{Mu} is the unique invariant quartic for a faithful action of M_{20} on $\mathbb{P}^3$.*

Proof.

(1). — It has been proved in Theorem 2.7, see also [5, Section 3].

(2). — Let $Q \in \mathbb{P}^3(\mathbb{C})$ be a quartic leaved invariant by a faithful action of M_{20}. This means that there exists a representation of M_{20} as a subgroup of $\mathbf{PGL}_4(\mathbb{C})$ which stabilizes Q. Then Q is polarized by the lattice $\langle 4 \rangle$, so that we have an embedding of $\langle 4 \rangle$ in the lattice $\mathbf{L}_Q^{M_{20}}$. Since this embedding is unique by (1), its orthogonal complement $\mathbf{T}_Q$ in $\mathbf{L}_Q^{M_{20}}$ is isometric to $\mathbf{T}_{X_{\mathrm{Mu}}}$. So Q is projectively equivalent to X_{Mu}. ☐

Proposition 4.3. — *The quartic X_{Mu} is the Kummer surface*
$$\mathrm{Km}(E_{i\sqrt{10}} \times E_{i\sqrt{10}}).$$

Proof. — This follows from Corollary 2.5 and its proof. ☐

Remark 4.4. — As X_{Mu} is a Kummer surface, it admits 16 two by two disjoint smooth rational curves (a *Nikulin configuration*). We were not able to find such a set of smooth rational curves, but, using Magma, we have at least found 320 conics in X_{Mu} (from which it is impossible to extract a Nikulin configuration: we can only extract 12 two by two disjoint conics). Let
$$C_+ = \left\{ [x:y:z:t] \in \mathbb{P}^3(\mathbb{C}) \;\middle|\; x+y+z = y^2+yz+z^2+\frac{3+\sqrt{10}}{2}t^2 = 0 \right\}$$
and
$$C_- = \left\{ [x:y:z:t] \in \mathbb{P}^3(\mathbb{C}) \;\middle|\; x+y+z = y^2+yz+z^2+\frac{3-\sqrt{10}}{2}t^2 = 0 \right\}.$$
Then C_+ and C_- are two smooth conics contained in X_{Mu} and, if we denote by $\Omega_\pm$ the G_{Mu}-orbit of $C_\pm$, then $\Omega_+ \neq \Omega_-$, $|\Omega_\pm| = 160$, and all elements of $\Omega_\pm$ are contained in X_{Mu}.

Remark 4.5. — Observe that PG_{Mu} is a maximal finite subgroup of $\operatorname{Aut}(X_{\text{Mu}})$. Indeed, if $PG_{\text{Mu}} \subsetneq \Gamma \subset \operatorname{Aut}(X_{\text{Mu}})$ with Γ finite, then $|\Gamma| \geqslant 2 \cdot |PG_{\text{Mu}}| = 3\,840$ and so by the result of Kondo in [9] the group Γ would be the group G_{Ko} defined in Section 3 and X_{Mu} would be isomorphic to X_{Ko}: this is not the case by Proposition 3.1 and Proposition 4.2.

5. Brandhorst–Hashimoto's example

Let G_{BH} be the subgroup of $\mathbf{GL}_6(\mathbb{C})$ generated by

$$t = \operatorname{diag}(-1,1,1,1,1,1),$$

$$u = \begin{pmatrix} i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

All the numerical facts about G_{BH} stated below can be checked with Magma. Then $|G_{\text{BH}}| = 3\ 840$, $\mathrm{Z}(G) = \boldsymbol{\mu}_2$, $|G_{\text{BH}}/G'_{\text{BH}}| = 2$ and there are two exact sequences

$$1 \longrightarrow \boldsymbol{\mu}_2 \longrightarrow G'_{\text{BH}} \longrightarrow M_{20} \longrightarrow 1$$

and

$$(5.1) \qquad 1 \longrightarrow M_{20} = PG'_{\text{BH}} \longrightarrow PG_{\text{BH}} \longrightarrow \boldsymbol{\mu}_2 \longrightarrow 1.$$

The second exact sequence splits (for instance by sending the non-trivial element of $\boldsymbol{\mu}_2$ to t) and $G'_{\text{BH}} = G_{\text{BH}} \cap \mathbf{SL}_6(\mathbb{C})$. Even though the last exact sequence looks like (4.1),

(5.2) The groups PG_{Mu} and PG_{BH} are not isomorphic.

Note that the group G'_{BH} is isomorphic to the group denoted by $2_3.M_{20}$ in the Atlas of finite groups[(1)]. We denote by $V = \mathbb{C}^6$ the natural representation of G_{BH} and we identify $\mathbb{C}[V]$ with $\mathbb{C}[x_1,x_2,x_3,x_4,x_5,x_6]$. Note that

(5.3) G_{BH} acts doubly transitively on the set of hyperplanes $\{H_1,\ldots,H_6\}$,

where H_i is defined by $x_i = 0$.

(1) http://brauer.maths.qmul.ac.uk/Atlas/v3/group/M20/

S. Brandhorst and K. Hashimoto [3] proved that there is a unique K3 surface admitting a faithful action of PG_{BH} and, in a private communication, they asked the question about the equations of this K3 surface: the aim of this section is to answer the question by exhibiting explicit equations of such a K3 surface.

The group G_{BH} contains the group N of diagonal matrices with coefficients in $\boldsymbol{\mu}_2$ as a normal subgroup (so $N \simeq (\boldsymbol{\mu}_2)^6$) and we have $G_{\mathrm{BH}}/N \simeq \mathfrak{A}_5$. It is easy to see that

$$\mathbb{C}[V]^N = \mathbb{C}[x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2]. \tag{5.4}$$

The following facts are checked with Magma:

(a) As a G_{BH}/N-module, $\mathbb{C}[V]_2^N = S_1 \oplus S_2$, where S_1 and S_2 are the two non-isomorphic irreducible representations of $G_{\mathrm{BH}}/N \simeq \mathfrak{A}_5$ of dimension 3.

(b) Let $\phi = (1+\sqrt{5})/2$ be the golden ratio. If we set

$$\begin{cases} q_1 = x_1^2 + x_4^2 - \phi x_5^2 + \phi x_6^2, \\ q_2 = x_2^2 - \phi x_4^2 + x_5^2 - \phi x_6^2, \\ q_3 = x_3^2 + \phi x_4^2 - \phi x_5^2 + x_6^2, \end{cases}$$

then (q_1, q_2, q_3) is a basis of S_1.

We then define

$$X_{\mathrm{BH}} = \mathscr{Z}(q_1, q_2, q_3).$$

The next proposition can be proved using Magma, but we will provide a proof independent of Magma computations.

Proposition 5.1. — *The scheme X_{BH} is smooth, irreducible, of dimension 2.*

The variety X_{BH} is then an irreducible smooth complete intersection of three quadrics in $\mathbb{P}^5(\mathbb{C})$, so it is a K3 surface. Since the vector space S_k is stable under the action of G_{BH}, the K3 surface X_{BH} is endowed with a faithful action of $PG_{\mathrm{BH}} \simeq \langle t \rangle \ltimes M_{20}$.

Corollary 5.2. — *X_{BH} is a K3 surface endowed with a faithful action of PG_{BH}.*

We show first the following:

Proposition 5.3. — *Let $H = N \cap G'_{BH}$, then the scheme X_{BH}/H is a K3 surface (with A_1 singularities) which is a double cover of $\mathbb{P}^2(\mathbb{C})$ ramified on the union of 6 lines in general position.*

Proof. — Note that

$$\mathbb{C}[x_1,x_2,x_3,x_4,x_5,x_6]^H = \mathbb{C}[x_1^2,x_2^2,x_3^2,x_4^2,x_5^2,x_6^2,x_1x_2\cdots x_6],$$

so that $\mathbb{P}^5(\mathbb{C})/H = \{[y_1:\cdots:y_6:z]\in\mathbb{P}(1,\ldots,1,3)\mid z^2=\prod_{k=1}^6 y_k\}$. Therefore,

$$X_{\mathrm{BH}}/H = \left\{ \begin{matrix}[y_1:\cdots:y_6:z]\\ \in\mathbb{P}(1,\ldots,1,3)\end{matrix} \;\middle|\; z^2=\prod_{k=1}^{6} y_k \text{ and } \begin{cases} y_1+y_4-\phi y_5+\phi y_6=0\\ y_2-\phi y_4+y_5-\phi y_6=0\\ y_3+\phi y_4-\phi y_5+y_6=0\end{cases} \right\}$$

Simplifying the equations, one gets

$$X_{\mathrm{BH}}/H = \left\{ \begin{matrix}[y_4:y_5:y_6:z]\\ \in\mathbb{P}(1,1,1,3)\end{matrix} \;\middle|\; \begin{matrix} z^2 = y_4y_5y_6(-y_4+\phi y_5-\phi y_6)\\ \times(\phi y_4-y_5+\phi y_6)(-\phi y_4+\phi y_5-y_6)\end{matrix} \right\}.$$

So X_{BH}/H is a K3 surface (with A_1 singularities) which is a double cover of $\mathbb{P}^2(\mathbb{C})$ ramified on the union of 6 lines in general position as claimed. □

Another proof of Proposition 5.1. — First, it follows from (5.4) that

$$X_{\mathrm{BH}}/N = \left\{ [y_1:\cdots:y_6]\in\mathbb{P}^5(\mathbb{C}) \;\middle|\; \begin{cases} y_1+y_4-\phi y_5+\phi y_6=0\\ y_2-\phi y_4+y_5-\phi y_6=0\\ y_3+\phi y_4-\phi y_5+y_6=0\end{cases} \right\} \simeq \mathbb{P}^2(\mathbb{C}). \tag{5.5}$$

Hence X_{BH}/N has dimension 2, so X_{BH} has dimension 2. Then one can use [6, Exercice III, 5.5] to see that X_{BH} is connected, so that if it is smooth then it is irreducible. We prove smoothness below, but we can also argue in the way as follows.

By Proposition 5.3 the quotient X_{BH}/H is irreducible. This shows that H acts transitively on the irreducible components of X_{BH}. So G'_{BH} also acts transitively on the irreducible components. Now, let X be an irreducible component of X_{BH} and let K denote its stabilizer in G'_{BH}. Then $8=\deg(X_{\mathrm{BH}})=\deg(X)\cdot|G'_{\mathrm{BH}}/K|$. Since G'_{BH} has no subgroup of index 2, 4 or 8, we conclude that $K=G'_{\mathrm{BH}}$, so that $X=X_{\mathrm{BH}}$, as desired.

We now show that X_{BH} is smooth. Let $p=[x_1:x_2:x_3:x_4:x_5:x_6]\in X_{\mathrm{BH}}$ and assume that p is a singular point of X_{BH}. Since p belongs to X_{BH}, the equations show that at least two of the x_k's are non-zero. By replacing if necessary p by another point in its G_{BH}-orbit, we may assume that $x_1x_2\neq 0$ (thanks to (5.3)). The Jacobian matrix of (q_1,q_2,q_3) at p is given by

$$\mathrm{Jac}_p(q_1,q_2,q_3) = \begin{pmatrix} 2x_1 & 0 & 0 & 2x_4 & -2\phi x_5 & 2\phi x_6 \\ 0 & 2x_2 & 0 & -2\phi x_4 & 2x_5 & -2\phi x_6 \\ 0 & 0 & 2x_3 & 2\phi x_4 & -2\phi x_5 & 2x_6 \end{pmatrix}.$$

Then the rank of $\mathrm{Jac}_p(q_1,q_2,q_3)$ is less than 3, which means that all its minors of size 3 vanish. Therefore,

$$x_{i_1}x_{i_2}x_{i_3} = 0$$

for all $1 \leqslant i_1 < i_2 < i_3 \leqslant 6$. Since $x_1x_2 \neq 0$, we get $x_3 = x_4 = x_5 = x_6 = 0$. But then $q_1(p) \neq 0$, which is impossible. □

Remark 5.4. — Exchanging S_1 and S_2 (whose characters are Galois conjugate under $\sqrt{5} \mapsto -\sqrt{5}$), one gets another K3 surface X'_{BH}, where ϕ is replaced by its Galois conjugate $\phi' = (1-\sqrt{5})/2 = 1-\phi$ in the equations. Let $\sigma \in \mathbf{GL}_6(\mathbb{C})$ be the matrix

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then σ normalizes G_{BH} and $\sigma(X_{\mathrm{BH}}) = X'_{\mathrm{BH}}$, so that X_{BH} and X'_{BH} are isomorphic.

The surface X_{BH} is a K3 surface with polarization L_8 satisfying $L_8^2 = 8$, and as in section 4 this is invariant by the action of M_{20}. We have hence an embedding of $\langle 8\rangle$ in $\mathbf{L}_{X_{\mathrm{BH}}}^{M_{20}}$.

Proposition 5.5. — *With the above notation, we have:*

(1) *The transcendental lattice of X_{BH} is a rank two lattice given by*

$$\mathbf{T}_{X_{\mathrm{BH}}} = \begin{pmatrix} 8 & 4 \\ 4 & 12 \end{pmatrix}$$

and $NS(X_{\mathrm{BH}})^{M_{20}} = \mathbb{Z}\, L_8$ with $L_8^2 = 8$.

(2) *The complete intersection X_{BH} is the unique K3 surface invariant for a faithful action of M_{20} in $\mathbb{P}^5(\mathbb{C})$.*

Proof. — (1) has been proved in Theorem 2.7, and (2) follows from the same argument as in Proposition 4.2. □

Remark 5.6. — Proposition 5.5 gives another proof that $X_{\mathrm{BH}} \cong X'_{\mathrm{BH}}$.

Proposition 5.7. — *The K3 surface* X_{BH} *is the Kummer surface* $\mathrm{Km}(E_\tau \times E_{2\tau})$, *with* $\tau_1 = \frac{-1+i\sqrt{5}}{2}$.

Proof. — This follows from Corollary 2.5 and its proof. □

Remark 5.8 (Smooth rational curves). — Using Magma, one can find an explicit Nikulin configuration in X_{BH} as follows. Let C denote the conic defined by the equations

$$\begin{cases} x_5 = \sqrt{\phi}x_1, \\ x_4 = \sqrt{\phi}x_2, \\ x_3 = \sqrt{\phi}x_6, \\ x_1^2 - x_2^2 - x_6^2 = 0 \end{cases}$$

and let $\mathcal{A}$ denote the subgroup of G_{BH} generated by

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Then C is contained in X_{BH}. It can be checked with Magma that its G_{BH}-orbit contains 80 elements, and that its $\mathcal{A}$-orbit contains 16 elements which are two by two disjoint (note that $|\mathcal{A}| = 32$, that $\boldsymbol{\mu}_2 \subset \mathcal{A}$ and that $\mathcal{A}/\boldsymbol{\mu}_2$ is elementary abelian).

Note also that the conic defined by the equations

$$\begin{cases} x_1 + ix_5 - i\phi x_6 = 0, \\ x_3 - i\phi x_5 + i\phi x_6 = 0, \\ x_4 - \phi x_5 + x_6 = 0, \\ x_2^2 - 2\phi x_5^2 + 2(1+\phi)x_5x_6 - 2\phi x_6^2 = 0, \end{cases}$$

is contained in X_{BH}, and that its G_{BH}-orbit contains 96 elements. However, we can only extract subsets of 12 two by two disjoint conics from this orbit.

6. Final Remarks

Proposition 6.1. — *The K3 surfaces X_{Mu}, X_{BH} and X_{Ko} are two by two non-isomorphic.*

Proof. — Indeed, they do not have the same transcendental lattice (or equivalently they do not admit polarizations of the same degree). □

Proposition 6.2. — *If a K3 surface X admits a faithful action of G_{Ko}, PG_{Mu}, respectively PG_{BH} then X is isomorphic to X_{Ko}, X_{Mu}, respectively X_{BH}.*

Proof. — For G_{Ko} this is shown in [9, Lemma 3.1]. Before going on, note the following fact, which can easily be checked with Magma:

(6.1) The groups PG_{Mu} and PG_{BH} are not isomorphic to subgroups of G_{Ko}.

Consider now the group G_{Mu}, then $PG_{\mathrm{Mu}}/M_{20} = \langle \iota \rangle$ and ι acts non-symplectically, hence X is one of the three surfaces of Theorem 2.7 and PG_{Mu} leaves invariant the polarization, hence it is realized by linear transformations. We only need to show that X_{Ko} and X_{BH} do not admit an automorphism group isomorphic to PG_{Mu}. Assume it is the case, then PG_{Mu} and G_{Ko} leaves invariant the polarization of degree $\langle 40 \rangle$ on X_{Ko}, hence by [7, Proposition 5.3.3] the group that they generate together is finite. By the maximality of G_{Ko} this means that PG_{Mu} is contained in G_{Ko} but by (6.1) the group G_{Ko} does not contain such a subgroup. With a similar argument if PG_{Mu} acts on X_{BH} then we conclude that $PG_{\mathrm{Mu}} \cong PG_{\mathrm{BH}}$ and this is not the case by (5.2). The same argument holds for PG_{BH}. □

Theorem 6.3. — *Let G be a maximal finite group with a faithful and non–symplectic action on a K3 surface X and assume that $M_{20} \subset G$. Then G is isomorphic to G_{Ko}, PG_{Mu} or PG_{BH}.*

Proof. — Since G acts non-symplectically then G/M_{20} is non-trivial and by [9] it has order at most four. If $|G/M_{20}| = 4$ then $G \cong G_{\mathrm{Ko}}$ by [9]. Observe that the group G/M_{20} acts faithfully on $\mathbf{L}_{20}$ since it contains $\mathbf{T}_X$. By Remark 2.4, the group of isometries of $\mathbf{L}_{20}$ has order 2^4 so it is not possible to have $|G/M_{20}| = 3$. We are left with the case $|G/M_{20}| = 2$. By Theorem 2.7 the K3 surface X is isomorphic to X_{Ko}, X_{Mu} or X_{BH}. By the same argument as in Proposition 6.2 and the maximality of G, then G is isomorphic to PG_{Mu} or PG_{BH}. □

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