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# NARROW QUANTUM $D$-MODULES AND QUANTUM SERRE DUALITY

**by Mark SHOEMAKER (*)**

Abstract. — Given $\mathcal{Y}$ a non-compact manifold or orbifold, we define a natural subspace of the cohomology of $\mathcal{Y}$ called the narrow cohomology. We show that despite $\mathcal{Y}$ being non-compact, there is a well-defined and non-degenerate pairing on this subspace. The narrow cohomology proves useful for the study of genus zero Gromov–Witten theory. When $\mathcal{Y}$ is a smooth complex variety or Deligne–Mumford stack, one can define a quantum $D$-module on the narrow cohomology of $\mathcal{Y}$. This yields a new formulation of quantum Serre duality.

Résumé. — Étant donnée une variété ou une orbifold non compacte $\mathcal{Y}$, on définit un sous-espace naturel de la cohomologie de $\mathcal{Y}$ que nous appelons cohomologie étroite. On montre qu'en dépit du fait que $\mathcal{Y}$ est non compacte, il existe un couplage non-dégénéré sur ce sous-espace. Cette cohomologie étroite s'avère utile pour l'étude de la théorie de Gromov–Witten en genre 0. Lorsque $\mathcal{Y}$ est une variété complexe lisse ou un champ de Deligne–Mumford lisse, on peut définir un $D$-module quantique sur sa cohomologie étroite. Ceci nous amène à une nouvelle formulation de la dualité de Serre quantique.

## 1. Introduction

Let $\mathcal{X}$ be a smooth complex variety (or orbifold) and let $\mathcal{E} \to \mathcal{X}$ be a vector bundle over $\mathcal{X}$, with a regular section $s \in \Gamma(\mathcal{X}, \mathcal{E})$. We can define the subvarieties

$$\mathcal{Z} := \{s = 0\} \subset \mathcal{X} \quad \text{and} \quad \mathcal{Y} := \mathrm{tot}(\mathcal{E}^\vee),$$

to obtain the following diagram

$$\begin{tikzcd} & \mathcal{Y} \arrow[d, "\pi"] \\ \mathcal{Z} \arrow[r, hook, "j"] & \mathcal{X}. \end{tikzcd} \tag{1.1}$$

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As was originally observed in [17] in the case where $\mathcal{X}$ is projective space $\mathbb{P}^N$, there is a nontrivial relationship between the genus zero Gromov–Witten invariants of $\mathcal{Z}$ and those of $\mathcal{Y}$. This correspondence was given the name *quantum Serre duality*. It was originally proven by showing that (equivariant lifts of) generating functions of Gromov–Witten invariants of $\mathcal{Z}$ and $\mathcal{Y}$ obey very similar recursions.

### 1.1. Other incarnations

The correspondence has since been generalized and rephrased many times. In [14], it was generalized to the case of an arbitrary base $\mathcal{X}$ using Givental's Lagrangian cones. In this formulation, one endows $\mathcal{Y}$ with a $\mathbb{C}^*$-action by scaling fibers. Then the $\mathbb{C}^*$-equivariant genus zero Gromov–Witten theory of $\mathcal{Y}$ can be used to recover the genus zero Gromov–Witten theory of $\mathcal{Z}$. This result forms the basis (and the proof) of later formulations.

Recently in [24], quantum Serre duality was re-expressed as a correspondence between *quantum $D$-modules*. The quantum $D$-module $QDM(\mathcal{X})$ for a (compact) space $\mathcal{X}$ consists of

- the quantum connection:[(1)]
$$\nabla^{\mathcal{X}} : H^*(\mathcal{X}) \longrightarrow H^*(\mathcal{X})[\boldsymbol{t}, z, z^{-1}],$$
a flat connection defined in terms of the quantum product $\bullet_{\boldsymbol{t}}$;
- a pairing:
$$S^{\mathcal{X}}(\,\cdot\,,\cdot\,) : H^*(\mathcal{X})[\boldsymbol{t}, z, z^{-1}] \otimes H^*(\mathcal{X})[\boldsymbol{t}, z, z^{-1}] \longrightarrow \mathbb{C}[z, z^{-1}],$$
which is flat with respect to the connection.

See Definition 3.10 for details. As with Givental's Lagrangian cone, the quantum $D$-module of $\mathcal{X}$ fully determines the genus zero Gromov–Witten theory of $\mathcal{X}$. In [21], Iritani defined a corresponding *integral structure*, a lattice lying inside the kernel of $\nabla^{\mathcal{X}}$ defined as the image of the bounded derived category $D(\mathcal{X})$ under a map
$$\boldsymbol{s}^{\mathcal{X}} : D(\mathcal{X}) \longrightarrow H^*(\mathcal{X})[\boldsymbol{t}, z, z^{-1}].$$
See Section 3.4.1 for details.

(1) In the case where $\mathcal{Z}$ and $\mathcal{Y}$ are orbifolds we should replace cohomology with *Chen–Ruan* cohomology. See Section 2.2 for details.

One benefit to the formulation in terms of quantum $D$-modules [24], is that quantum Serre duality can be phrased non-equivariantly, as the quantum connection $\nabla^{\mathcal{Y}}$ is well-defined without the need for $\mathbb{C}^*$-localization. It is shown ([24, Theorem 3.14, Corollary 3.17]) that the map

$$\pi^* \circ j_* : H^*_{\mathrm{amb}}(\mathcal{Z})[\boldsymbol{t}, z, z^{-1}] \longrightarrow H^*(\mathcal{Y})[\boldsymbol{t}, z, z^{-1}] \tag{1.2}$$

sends $\ker(\nabla^{\mathcal{Z}})$ to $\ker(\nabla^{\mathcal{Y}})$ after a change of variables. Here $H^*_{\mathrm{amb}}(\mathcal{Z})$ denotes the image $j^*(H^*(\mathcal{X})) \subset H^*(\mathcal{Z})$. Furthermore, they prove that this map is compatible with the integral structures and with the functor

$$(-1)^{\mathrm{rk}}(\mathcal{E}) \det(\mathcal{E}) \otimes (\pi^* \circ j_*)(\,\cdot\,) : D(\mathcal{Z}) \longrightarrow D(\mathcal{Y}).$$

It is important to note, however, that (1.2) does not give a map of quantum $D$-modules, because the pairing $S^{\mathcal{Y}}(\,\cdot\,,\cdot\,)$ is not well-defined when $\mathcal{Y}$ is non-compact. Nevertheless, the formulation of quantum Serre duality in terms of the quantum connection seems to be a natural and useful way of viewing the correspondence.

### 1.2. Results

The work described above raises the following interrelated questions:

(1) Is there a well-defined quantum $D$-module associated to $\mathcal{Y}$ when $\mathcal{Y}$ is non-compact? More precisely, can one define a pairing which is flat with respect to the quantum connection?
(2) If (1) holds, can quantum Serre duality be rephrased as a map between quantum $D$-modules, identifying not just the quantum connection but also the pairings?
(3) The map in (1.2) is not an isomorphism; under mild assumptions it is an inclusion. Is there a natural way of describing the image $\pi^* \circ j_*(\ker(\nabla^{\mathcal{Z}}))$ inside $\ker(\nabla^{\mathcal{Y}})$?

This paper answers each of these questions in the affirmative. Given $\mathcal{Y}$ a non-compact smooth variety, we define a subspace $H^*_{\mathrm{nar}}(\mathcal{Y}) \subset H^*(\mathcal{Y})$ which we call the *narrow cohomology* of $\mathcal{Y}$. There is a natural forgetful morphism

$$\phi : H^*_{\mathrm{c}}(\mathcal{Y}) \longrightarrow H^*(\mathcal{Y})$$

from compactly supported cohomology to cohomology. The narrow cohomology $H^*_{\mathrm{nar}}(\mathcal{Y})$ is defined to be the image of $\phi$. This turns out to be the right framework in which to formulate quantum Serre duality as an *isomorphism* of quantum $D$-modules.

One observes that the Poincaré pairing between $H^*_{\mathrm{c}}(\mathcal{Y})$ and $H^*(\mathcal{Y})$ induces a non-degenerate pairing on $H^*_{\mathrm{nar}}(\mathcal{Y})$. We use this to define a quantum $D$-module on the narrow subspace.

Theorem 1.1 (Corollary 4.8). — *The quantum connection*

$$\nabla^{\mathcal{Y}} : H^*(\mathcal{Y}) \longrightarrow H^*(\mathcal{Y})[\![\boldsymbol{t}]\!][z, z^{-1}]$$

*preserves $H^*_{\mathrm{nar}}(\mathcal{Y})$. Furthermore, there is a well-defined and nondegenerate pairing*

$$S^{\mathcal{Y},\mathrm{nar}}(\,\cdot\,,\cdot\,) : H^*_{\mathrm{nar}}(\mathcal{Y})[\![\boldsymbol{t}]\!][z, z^{-1}] \otimes H^*_{\mathrm{nar}}(\mathcal{Y})[\![\boldsymbol{t}]\!][z, z^{-1}] \longrightarrow \mathbb{C}[z, z^{-1}],$$

*which is flat with respect to $\nabla^{\mathcal{Y}}$. We obtain a quantum $D$-module $QDM_{\mathrm{nar}}(\mathcal{Y})$ on $H^*_{\mathrm{nar}}(\mathcal{Y})$.*

There is also a well-defined integral structure for the narrow quantum $D$-module, coming from the derived category $D(\mathcal{Y})_{\mathrm{c}}$ of complexes of coherent sheaves on $\mathcal{Y}$, exact outside a proper subvariety. The narrow quantum $D$-module defined above turns out to be particularly relevant to quantum Serre duality. In the particular case of $\mathcal{Y}$ and $\mathcal{Z}$ from (1.1), we show there is an isomorphism of quantum $D$-modules from $QDM_{\mathrm{nar}}(\mathcal{Y})$ to $QDM_{\mathrm{amb}}(\mathcal{Z})$. This is the main result of the paper.

Theorem 1.2 (Theorem 6.14). — *There is an isomorphism*

$$\overline{\Delta}_+ : H^*_{\mathrm{nar}}(\mathcal{Y})[z, z^{-1}] \longrightarrow H^*_{\mathrm{amb}}(\mathcal{Z})[z, z^{-1}]$$

*which identifies the quantum $D$-module $QDM_{\mathrm{nar}}(\mathcal{Y})$ with $\overline{f}^*(QDM_{\mathrm{amb}}(\mathcal{Z}))$ (where $\overline{f}$ is an explicit change of variables). Furthermore it is compatible with the integral structure and the functor $j^* \circ \pi_*$, i.e., the following diagram commutes;*

$$\begin{tikzcd} D(\mathcal{Y})_{\mathcal{X}} \arrow[r, "j^*\circ\pi_*"] \arrow[d, "s^{\mathcal{Y},\mathrm{nar}}"] & j^*(D(\mathcal{X})) \arrow[d, "s^{\mathcal{Z},\mathrm{amb}}"] \\ QDM_{\mathrm{nar}}(\mathcal{Y}) \arrow[r, "\bar{\Delta}_+"] & \overline{f}^*(QDM_{\mathrm{amb}}(\mathcal{Z})). \end{tikzcd} \tag{1.3}$$

In the course of proving the above theorem we make several interesting connections between the (non-equivariant) quantum connection on $\mathcal{Y}$ and different modifications of the Gromov–Witten theory over $\mathcal{X}$.

First, in Section 5.2 we give an explicit description of a modified quantum product, denoted $\bullet_{\boldsymbol{t}}^{\mathcal{Y}\to\mathcal{X}}$, on $\mathcal{X}$ which pulls back to the usual (non-equivariant) quantum product $\bullet_{\boldsymbol{t}}^{\mathcal{Y}}$ via $\pi^*$. This result is dual in spirit to [28, Section 2] where the "$\bullet_{\mathcal{Z}}$ product induced by a hypersurface," a modified

quantum product on $\mathcal{X}$, is defined and related to the usual quantum product on $\mathcal{Z}$.

Second, we consider the quantum connection with compact support on $\mathcal{Y}$:

$$\nabla^{\mathcal{Y},\mathrm{c}} : H^*_{\mathrm{c}}(\mathcal{Y}) \longrightarrow H^*_{\mathrm{c}}(\mathcal{Y})[\boldsymbol{t}, z, z^{-1}]$$

which is defined identically to $\nabla^{\mathcal{Y}}$ but acts on cohomology with compact support. In a short remark [24, Remark 3.17], it was reasoned by an analogy that (the non-equivariant limit of) the Euler-twisted quantum connection $\nabla^{e(\mathcal{E})}$ on $\mathcal{X}$ (see Definition 3.8) could be thought of as the quantum connection with compact support. In Proposition 6.2 we make this observation precise, showing that the pushforward isomorphism

$$i^{\mathrm{c}}_* : H^*_{\mathrm{CR}}(\mathcal{X}) \longrightarrow H^*_{\mathrm{CR,c}}(\mathcal{Y})$$

identifies $\nabla^{e(\mathcal{E})}$ with $\nabla^{\mathcal{Y},\mathrm{c}}$ up to a change of variables.

The above results imply yet another variation of quantum Serre duality, which states, roughly, that

$$j^* \circ \pi^{\mathrm{c}}_* : H^*_{\mathrm{CR,c}}(\mathcal{Y}) \longrightarrow H^*_{\mathrm{CR,amb}}(\mathcal{Z})$$

maps $\ker(\nabla^{\mathcal{Y},\mathrm{c}})$ to $\ker(\nabla^{\mathcal{Z}})$ and is compatible with integral structures and with the functor

$$j^* \circ \pi_* : D(\mathcal{Y})_{\mathcal{X}} \longrightarrow D(\mathcal{Z}).$$

See Theorem 6.5 for the precise statement. This result is essentially the adjoint of the statement given in Theorem 3.13 of [24], once one observes $\nabla^{\mathcal{Y},\mathrm{c}}$ and $\nabla^{\mathcal{Y}}$ as dual with respect to the pairing between cohomology and compactly supported cohomology on $\mathcal{Y}$. However there is a difference in the change of variables used in [24] versus the theorem above. We give a direct proof of Theorem 6.5 in a slightly more general context than that appearing in [24], however the proof techniques are similar. See Remark 6.6 for more details on the connection.

### 1.3. Applications

We expect the constructions and results of this paper to be useful in formulating and proving new correspondences in genus zero Gromov–Witten theory. For instance, with the narrow quantum $D$-module on hand, one can formulate a crepant transformation conjecture between non-compact $K$-equivalent spaces $\mathcal{Y}$ and $\mathcal{Y}'$. Previously, most results along these lines have assumed that $\mathcal{Y}$ and $\mathcal{Y}'$ be toric varieties and have used equivariant

Gromov–Witten theory for both the statement and proof of the correspondence. In [29] we show how, in a particular case, the equivariant correspondence implies the narrow correspondence. This is generalized to toric wall crossings in [27].

The formulation of quantum Serre duality as in Theorem 1.2 will be useful as a tool for proving other correspondences. In [26], we observed that the LG/CY correspondence is implied by a suitable version of the crepant transformation conjecture. We use the shorthand "CTC implies LG/CY." However the implication was somewhat messy to state in [26], as it required a careful analysis of the non-equivariant limits of certain maps and generating functions. Furthermore it did not involve integral structures.

In [29], we show that Theorem 1.2 together with a sister result involving FJRW theory may be used in tandem to clarify the "CTC implies LG/CY" statement of [26]. We show that the narrow crepant transformation conjecture immediately implies a $D$-module formulation of the LG/CY correspondence (first described in [11]). This result requires no mention of equivariant Gromov–Witten theory, and is compatible with all integral structures. In fact this was the first motivation for the current paper.

Another more recent application appears in [27], where R. Mi and the author us the results of this paper prove a comparison result for the Gromov–Witten theory of extremal transitions.

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## 2. Narrow cohomology

Let $Y$ be a non-compact oriented manifold. Let $H^*(Y;\mathbb{R})$ and $H^*_c(Y;\mathbb{R})$ denote the de Rham cohomology and cohomology with compact support respectively. We assume always that $Y$ has a finite good cover [8, Section 5], so that $\operatorname{rank}(H^*_c(Y;\mathbb{R}))$ and $\operatorname{rank}(H^*(Y;\mathbb{R}))$ are finite. In this paper we will primarily be concerned with the case where $Y$ is the total space of a vector

bundle on a compact manifold $X$. In this case the existence of a finite good cover is automatic.

Let $\Omega^k(Y;\mathbb{R})$ and $\Omega^k_{\mathrm{c}}(Y;\mathbb{R})$ denote the real vector space of $k$-forms and $k$-forms with compact support. We have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^0_{\mathrm{c}}(Y) & \xrightarrow{d} & \Omega^1_{\mathrm{c}}(Y) & \xrightarrow{d} & \Omega^2_{\mathrm{c}}(Y) & \xrightarrow{d} & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^0(Y) & \xrightarrow{d} & \Omega^1(Y) & \xrightarrow{d} & \Omega^2(Y) & \xrightarrow{d} & \cdots \end{array}$$

where the vertical arrows are just the inclusion obtained by forgetting that a $k$-form had compact support. This induces "forgetful" maps $\phi_k : H^k_{\mathrm{c}}(Y;\mathbb{R}) \to H^k(Y;\mathbb{R})$ for $1 \leqslant k \leqslant \dim(Y)$. Let $\phi : H^*_{\mathrm{c}}(Y;\mathbb{R}) \to H^*(Y;\mathbb{R})$ denote the direct sum of these maps.

Definition 2.1. — *We define the* narrow *cohomology of $Y$ to be the image of $\phi$:*

$$H^k_{\mathrm{nar}}(Y;\mathbb{R}) := \mathrm{im}(\phi_k : H^k_{\mathrm{c}}(Y;\mathbb{R}) \longrightarrow H^k(Y;\mathbb{R})),$$

$$H^*_{\mathrm{nar}}(Y;\mathbb{R}) := \bigoplus_{k=0}^{\dim(Y)} H^k_{\mathrm{nar}}(Y;\mathbb{R}) \subseteq H^*(Y;\mathbb{R}).$$

The subspace $\ker(\phi)$ consists of classes which can be represented by differential forms with compact support but which are boundaries of classes with non-compact support.

Definition 2.2. — *Given a class $\alpha \in H^k_{\mathrm{nar}}(Y;\mathbb{R})$, we define a* lift *of $\alpha$ to be any class $\widetilde{\alpha} \in H^k_{\mathrm{c}}(Y;\mathbb{R})$ such that $\phi_k(\widetilde{\alpha}) = \alpha$. Lifts are well defined up to a choice of degree $k$ element in* $\ker(\phi)$.

Using singular cohomology over an arbitrary coefficient ring $R$, a completely analogous definition can be made via the inclusion

$$C^*_{\mathrm{c}}(Y;R) \hookrightarrow C^*(Y;R).$$

of the subchain complex $C^*_{\mathrm{c}}(Y;R)$ of singular cochains with compact support into the chain complex $C^*(Y;R)$ of singular cochains with coefficients in $R$. Via the induced map $\phi : H^*_{\mathrm{c}}(Y;R) \to H^*(Y;R)$, one defines the narrow *singular* cohomology

$$H^*_{\mathrm{nar}}(Y;R) := \mathrm{im}(\phi : H^*_{\mathrm{c}}(Y;R) \longrightarrow H^*(Y;R)).$$

It will be convenient to also introduce narrow homology. This can be defined in terms of Borel–Moore homology (See e.g. [9, Section V] for an exposition consistent with that described below). Assume for simplicity

that $Y$ is $\sigma$-compact to simplify our exposition. Let $R$ be a ring and let $C_k(Y;R)$ denote the set of finite singular $k$-chains, consisting of *finite* $R$-linear combinations

$$\sum a_\sigma \sigma$$

of continuous maps $\sigma : \Delta^k \to Y$ with $a_\sigma \in R$. We define $C_k^{\mathrm{BM}}(Y;R)$ to be the set of *locally finite* singular $k$-chains, consisting of (possibly infinite) $R$-linear combinations

$$\sum a_\sigma \sigma$$

such that for each compact set $C \subset Y$, $a_\sigma$ is zero for all but finitely many of the maps $\sigma$ whose image meets $C$. In reasonable situations [9], Borel–Moore homology can be defined as the homology of the complex $C_*^{\mathrm{BM}}(Y;R)$. Note we have a similar map of complexes as before

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial} & C_{i+1}(Y;R) & \xrightarrow{\partial} & C_i(Y;R) & \xrightarrow{\partial} & C_{i-1}(Y;R) & \xrightarrow{\partial} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \xrightarrow{\partial} & C_{i+1}^{\mathrm{BM}}(Y;R) & \xrightarrow{\partial} & C_i^{\mathrm{BM}}(Y;R) & \xrightarrow{\partial} & C_{i-1}^{\mathrm{BM}}(Y;R) & \xrightarrow{\partial} & \cdots
\end{array}$$

again inducing a "forgetful" map $\underline{\phi} : H_*(Y;R) \to H_*^{\mathrm{BM}}(Y;R)$.

Definition 2.3. — *Define the* narrow homology *to be the image of* $\underline{\phi}$*:*

$$H_*^{\mathrm{nar}}(Y;R) := \mathrm{im}(\underline{\phi} : H_*(Y;R) \longrightarrow H_*^{\mathrm{BM}}(Y;R)).$$

Poincaré duality [9] gives isomorphisms

$$\begin{aligned} H_{\mathrm{c}}^k(Y;\mathbb{Z}) &\cong H_{n-k}(Y;\mathbb{Z}) \\ H^k(Y;\mathbb{Z}) &\cong H_{n-k}^{\mathrm{BM}}(Y;\mathbb{Z}). \end{aligned}$$

The following diagram commutes,

$$\begin{array}{ccc}
H_{\mathrm{c}}^k(Y;\mathbb{Z}) & \xrightarrow{\mathrm{PD}} & H_{n-k}(Y;\mathbb{Z}) \\
\downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\underline{\phi}} \\
H^k(Y;\mathbb{Z}) & \xrightarrow{\mathrm{PD}} & H_{n-k}^{\mathrm{BM}}(Y;\mathbb{Z}),
\end{array} \tag{2.1}$$

as can be seen by observing that both horizontal maps are given by capping with the fundamental class $[Y] \in H_n^{\mathrm{BM}}(Y;\mathbb{Z})$. We immediately deduce the following.

Lemma 2.4. — *The Poincaré duality isomorphism* $H^k(Y;\mathbb{Z}) \cong H_{n-k}^{\mathrm{BM}}(Y;\mathbb{Z})$ *induces an isomorphism* $H_{\mathrm{nar}}^k(Y;\mathbb{Z}) \cong H_{n-k}^{\mathrm{nar}}(Y;\mathbb{Z})$.

Proposition 2.5. — *Let $f : X \to Y$ be a smooth, proper, oriented map between the manifolds $X$ and $Y$. There exist induced pullback and pushforward maps*

$$f^*_{\mathrm{nar}} : H^*_{\mathrm{nar}}(Y;\mathbb{Z}) \longrightarrow H^*_{\mathrm{nar}}(X;\mathbb{Z})$$
$$f^{\mathrm{nar}}_* : H^*_{\mathrm{nar}}(X;\mathbb{Z}) \longrightarrow H^*_{\mathrm{nar}}(Y;\mathbb{Z}).$$

*Proof.* — Because $f$ is proper, there is a pullback with compact support

$$f^*_{\mathrm{c}} : H^*_{\mathrm{c}}(Y;\mathbb{Z}) \longrightarrow H^*_{\mathrm{c}}(X;\mathbb{Z})$$

in addition to the usual pullback on cohomology

$$f^* : H^*(Y;\mathbb{Z}) \longrightarrow H^*(X;\mathbb{Z}).$$

The fact that $f^*$ induces a pullback on the narrow subspace follows from the commutative diagram

$$\begin{array}{ccc} H^*_{\mathrm{c}}(Y;\mathbb{Z}) & \xrightarrow{f^*_{\mathrm{c}}} & H^*_{\mathrm{c}}(X;\mathbb{Z}) \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} \\ H^*(Y;\mathbb{Z}) & \xrightarrow{f^*} & H^*(X;\mathbb{Z}). \end{array}$$

The pushforward map

$$f^{\mathrm{c}}_* : H^*_{\mathrm{c}}(X;\mathbb{Z}) \longrightarrow H^*_{\mathrm{c}}(Y;\mathbb{Z})$$

is defined via Poincaré duality together with the pushforward on homology. Because $f$ is proper, there is a well defined pushforward map

$$f^{\mathrm{BM}}_* : H^{\mathrm{BM}}_*(X;\mathbb{Z}) \longrightarrow H^{\mathrm{BM}}_*(Y;\mathbb{Z}).$$

Composing with Poincaré duality defines a pushforward

$$f_* : H^*(X;\mathbb{Z}) \longrightarrow H^*(Y;\mathbb{Z}).$$

The following diagram commutes,

$$\begin{array}{ccc} H_*(X;\mathbb{Z}) & \xrightarrow{f_*} & H_*(Y;\mathbb{Z}) \\ \downarrow{\scriptstyle\underline{\phi}} & & \downarrow{\scriptstyle\underline{\phi}} \\ H^{\mathrm{BM}}_*(X;\mathbb{Z}) & \xrightarrow{f^{\mathrm{BM}}_*} & H^{\mathrm{BM}}_*(Y;\mathbb{Z}). \end{array}$$

By invoking Poincaré duality and (2.1), we see that $\phi \circ f^{\mathrm{c}}_* = f^{\mathrm{c}}_* \circ \phi$, therefore $f_*$ induces a pushforward on the narrow subspaces. $\square$

### 2.1. Pairing

To simplify proofs, we will focus on the de Rham definition of narrow cohomology in what follows. The wedge product on differential forms induces the cup products

$$\begin{aligned} \cup : H^i(Y;\mathbb{R}) \times H^j(Y;\mathbb{R}) &\longrightarrow H^{i+j}(Y;\mathbb{R}), \\ \cup : H^i_{\mathrm{c}}(Y;\mathbb{R}) \times H^j(Y;\mathbb{R}) &\longrightarrow H^{i+j}_{\mathrm{c}}(Y;\mathbb{R}). \end{aligned} \tag{2.2}$$

Lemma 2.6. — *The cup product is zero when restricted to* $\ker(\phi) \times H^*_{\mathrm{nar}}(Y;\mathbb{R}) \subset H^*_{\mathrm{c}}(Y;\mathbb{R}) \times H^*(Y;\mathbb{R})$.

*Proof.* — Let $\Omega$ denote a closed differential form representing a class $\omega$ in $\ker(\phi)$. Then by definition of $\ker(\phi)$ there exists a form $\Psi$ (potentially with non-compact support) such that $\mathrm{d}\Psi = \Omega$. If $\Theta$ is a closed form with compact support representing $\theta \in H^*_{\mathrm{nar}}(Y;\mathbb{R})$, then

$$\mathrm{d}(\Theta \wedge \Psi) = \Theta \wedge \mathrm{d}\Psi = \Theta \wedge \Omega.$$

Since the support of $\Theta \wedge \Psi$ is contained in the support of $\Theta$ and is therefore compact, we conclude that $\theta \cup \omega = 0 \in H^*_{\mathrm{c}}(Y;\mathbb{R})$. □

Of course the wedge product also induces a cup product

$$\cup : H^i_{\mathrm{c}}(Y;\mathbb{R}) \times H^j_{\mathrm{c}}(Y;\mathbb{R}) \longrightarrow H^{i+j}_{\mathrm{c}}(Y;\mathbb{R}).$$

It is clear from the definitions that for $\alpha, \beta \in H^*_{\mathrm{c}}(Y;\mathbb{R})$,

$$\alpha \cup \beta = \alpha \cup \phi(\beta). \tag{2.3}$$

We let

$$\langle \cdot , \cdot \rangle : H^*(Y;\mathbb{R}) \times H^*_{\mathrm{c}}(Y;\mathbb{R}) \longrightarrow \mathbb{R}$$

denote the pairing defined by

$$\langle \alpha, \beta \rangle := \int_Y \alpha \cup \beta.$$

This is well-defined because $\beta$ and therefore $\alpha \cup \beta$ are compactly supported. It is known to be non-degenerate [8].

Corollary 2.7. — *With respect to the pairing* $\langle \cdot , \cdot \rangle$, $H^*_{\mathrm{nar}}(Y;\mathbb{R}) = \ker(\phi)^{\perp}$.

*Proof.* — By Lemma 2.6, $H^*_{\mathrm{nar}}(Y;\mathbb{R}) \subseteq \ker(\phi)^{\perp}$. However they are the same rank and so must be equal. □

Given two classes $\alpha$ and $\beta$ in $H^*_{\mathrm{nar}}(Y;\mathbb{R})$, *a-priori* the product $\alpha \cup \beta$ lies in $H^*(Y;\mathbb{R})$. It is clear that $\alpha \cup \beta$ in fact lies in $H^*_{\mathrm{nar}}(Y;\mathbb{R})$, so the narrow state space inherits a ring structure from $H^*(Y;\mathbb{R})$.

We can refine this product to obtain a class $\alpha \cup_{\mathrm{c}} \beta$ lying in $H^*_{\mathrm{c}}(Y;\mathbb{R})$ as follows.

Definition 2.8. — *Given $\alpha$ and $\beta$ in $H^*_{\mathrm{nar}}(Y;\mathbb{R})$, define the* compactly supported *cup product of $\alpha$ and $\beta$ to be*

$$\alpha \cup_{\mathrm{c}} \beta := \widetilde{\alpha} \cup \beta \in H^*_{\mathrm{c}}(Y;\mathbb{R}),$$

*where $\widetilde{\alpha}$ is a lift of $\alpha$.*

Corollary 2.9. — *The product described above is well-defined.*

*Proof.* — The class $\widetilde{\alpha}$ is well-defined up to a choice of elements in $\ker(\phi)$. If $\widetilde{\alpha}'$ is a different lift, then

$$\widetilde{\alpha}' = \widetilde{\alpha} + \alpha_k$$

where $\alpha_k \in \ker(\phi)$. By Lemma 2.6, $\alpha_k \cup \beta = 0$ in $H^*_{\mathrm{c}}(Y;\mathbb{R})$, so $\widetilde{\alpha}' \cup \beta = \widetilde{\alpha} \cup \beta$. □

With the above we can equip $H^*_{\mathrm{nar}}(Y;\mathbb{R})$ with a pairing.

Definition 2.10. — *Given $\alpha$ and $\beta$ in $H^*_{\mathrm{nar}}(Y;\mathbb{R})$, define*

$$\langle \alpha, \beta \rangle^{\mathrm{nar}} := \int_Y \alpha \cup_{\mathrm{c}} \beta.$$

Proposition 2.11. — *The pairing $\langle \cdot, \cdot \rangle^{\mathrm{nar}}$ is nondegenerate.*

*Proof.* — Given a nonzero element $\beta \in H^*_{\mathrm{nar}}(Y;\mathbb{R})$, there exists an element $\widetilde{\alpha} \in H^*_{\mathrm{c}}(Y;\mathbb{R})$ which pairs non-trivially with $\beta$. By definition,

$$\langle \phi(\widetilde{\alpha}), \beta \rangle^{\mathrm{nar}} = \int_Y \phi(\widetilde{\alpha}) \cup_{\mathrm{c}} \beta = \int_Y \widetilde{\alpha} \cup \beta = \langle \widetilde{\alpha}, \beta \rangle \neq 0. \qquad \square$$

## 2.2. Orbifolds

More generally, let $\mathcal{Y}$ be an oriented orbifold in the sense of [3]. Assume $\mathcal{Y}$ admits a finite good cover [3]. Let $I\mathcal{Y}$ denote the *inertia stack* of $\mathcal{Y}$, defined by the fiber diagram

$$\begin{tikzcd} I\mathcal{Y} \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d, "\Delta"] \\ \mathcal{Y} \arrow[r, "\Delta"] & \mathcal{Y} \times \mathcal{Y}. \end{tikzcd}$$

The orbifold $I\mathcal{Y}$ is a disjoint union of connected components called twisted sectors. These components are indexed by equivalence classes $\{\gamma\}_T$ of isotropy elements. The connected component indexed by $\gamma$ is referred to as the $\gamma$-twisted sector and will be denoted by $\mathcal{Y}_\gamma$. There is an involution

$$I : I\mathcal{Y} \longrightarrow I\mathcal{Y}$$

exchanging twisted sectors with their inverses. In the case $\mathcal{Y} = [V/\Gamma]$, the involution $I$ maps $\mathcal{Y}_\gamma$ to $\mathcal{Y}_{\gamma^{-1}}$ via the natural isomorphism.

Let $\mathcal{Y}_\gamma$ be a twisted sector. Take a point $(y, \gamma) \in \mathcal{Y}_\gamma$. The tangent space $Ty\mathcal{Y}$ splits as a direct sum of eigenspaces with respect to the action of $\gamma$:

$$T_y\mathcal{Y} = \bigoplus_{0 \leqslant f < 1} (T_y\mathcal{Y})_f$$

where $\gamma$ acts on $(T_y\mathcal{Y})_f$ by multiplication by $e^{2\pi\sqrt{-1}f}$. Define the *age shift* for $\mathcal{Y}_\gamma$ to be

$$\iota_\gamma = \sum_{0 \leqslant f < 1} f \dim_{\mathbb{C}}(T_y\mathcal{Y})_f.$$

Definition 2.12 ([3, 10]). — *The* Chen–Ruan cohomology *of $\mathcal{Y}$ of (Chen–Ruan) degree $k$ is*

$$H^k_{\mathrm{CR}}(\mathcal{Y}) := \bigoplus_{\gamma \in T} H^{k-2\iota_\gamma}(\mathcal{Y}_\gamma; \mathbb{C}),$$

*and*

$$H^*_{\mathrm{CR}}(\mathcal{Y}) := \bigoplus_{k \in \mathbb{Q}_{\geqslant 0}} H^k_{\mathrm{CR}}(\mathcal{Y}).$$

*Note that we will always use complex coefficients in what follows. Define the* compactly supported Chen–Ruan cohomology $H^*_{\mathrm{CR,c}}(\mathcal{Y})$ *analogously. Define the* Chen–Ruan pairing

$$\langle \alpha, \beta \rangle^{\mathcal{Y}} : H^*_{\mathrm{CR}}(\mathcal{Y}) \times H^*_{\mathrm{CR,c}}(\mathcal{Y}) \longrightarrow \mathbb{C}$$

*to be*

$$\langle \alpha, \beta \rangle^{\mathcal{Y}} := \int_{I\mathcal{Y}} \alpha \cup I_*(\beta).$$

As in the previous section, let $\phi : H^*_{\mathrm{CR,c}}(\mathcal{Y}) \to H^*_{\mathrm{CR}}(\mathcal{Y})$ denote the natural map.

Definition 2.13. — *Define the* narrow *subspace of $H^*_{\mathrm{CR}}(\mathcal{Y})$ to be*

$$H^*_{\mathrm{CR,nar}}(\mathcal{Y}) := \mathrm{im}(\phi).$$

*Denote by* $\ker(\phi)$ *the kernel in* $H^*_{\mathrm{CR,c}}(\mathcal{Y})$. *Given* $\alpha$ *and* $\beta$ *in* $H^*_{\mathrm{CR,nar}}(\mathcal{Y})$, *define*

$$\langle \alpha, \beta \rangle^{\mathcal{Y},\mathrm{nar}} := \int_{I\mathcal{Y}} \alpha \cup_{\mathrm{c}} I_*(\beta).$$

By orbifold versions of the same arguments, Lemma 2.4, Proposition 2.5, Corollary 2.7, and Proposition 2.11 also hold for the narrow Chen–Ruan cohomology of an orbifold.

*Remark 2.14.* — The correct cup product to use in this setting is the *Chen–Ruan cup product* [10]. Unless otherwise specified, all products of classes in Chen–Ruan cohomology in this paper are understood to be Chen–Ruan cup products. The definition will be given in Definition 3.7. For now we simply note the following fact [31, Lemma 2.3.8]. For $\alpha \in H^*(\mathcal{Y}) \subset H^*_{\mathrm{CR}}(\mathcal{Y})$ supported in the untwisted sector and any $\beta \in H^*_{\mathrm{CR}}(\mathcal{Y})$,

$$\alpha \cup \beta = q * (\alpha) \cup_{I\mathcal{Y}} \beta \in H^*_{\mathrm{CR}}(\mathcal{Y}),$$

where $q : I\mathcal{Y} \to \mathcal{Y}$ is the natural map and $\cup_{I\mathcal{Y}}$ denotes the usual cup product on $H^*(I\mathcal{Y})$. As a consequence of the above fact, the fundamental class on the untwisted sector $1 \in H^*(\mathcal{Y}) \subset H^*_{\mathrm{CR}}(\mathcal{Y})$ is the multiplicative identity with respect to the Chen–Ruan cup product.

### 2.3. The total space example

Let $\mathcal{X}$ be a compact oriented orbifold of dimension $n$ and let $\mathcal{E} \to \mathcal{X}$ be a complex vector bundle of complex rank $r$ over $\mathcal{X}$. Let $\mathcal{Y}$ denote the total space of $\mathcal{E}^\vee$ over $\mathcal{X}$ (we use $\mathcal{E}^\vee$ only to be consistent with later sections), endowed with the orientation induced by the orientation on $\mathcal{X}$. Let $i : I\mathcal{X} \to I\mathcal{Y}$ denote the inclusion via the zero section. This is a proper, oriented map. Let $\pi : I\mathcal{Y} \to I\mathcal{X}$ denote the projection.

Proposition 2.15. — *The following are equal:*

$$H^*_{\mathrm{CR,nar}}(\mathcal{Y}) = \mathrm{im}(i_* : H^*_{\mathrm{CR}}(\mathcal{X}) \longrightarrow H^*_{\mathrm{CR,nar}}(\mathcal{Y})).$$

*If* $\mathcal{E}$ *is pulled back from a vector bundle* $E \to X$ *on the coarse space of* $\mathcal{X}$, *then this is equal to*

$$\mathrm{im}(e(\pi^*\mathcal{E}^\vee) \cup \cdot\,)$$

*where* $e(\pi^*\mathcal{E}^\vee) \cup \cdot : H^*_{\mathrm{CR}}(\mathcal{Y}) \to H^*_{\mathrm{CR}}(\mathcal{Y})$ *is the (Chen–Ruan) cup product with the Euler class of* $\pi^*\mathcal{E}^\vee \in H^*(\mathcal{Y}) \subset H^*_{\mathrm{CR}}(\mathcal{Y})$.

*Proof.* — By the proof of Proposition 2.5, $\phi \circ i_*^{\mathrm{c}} = i_* \circ \phi = i_*$, so

$$\mathrm{im}(i_* : H^*_{\mathrm{CR}}(\mathcal{X}) \longrightarrow H^*_{\mathrm{CR,nar}}(\mathcal{Y})) \subseteq H^*_{\mathrm{CR,nar}}(\mathcal{Y}).$$

The two are equal if $i_*^{\mathrm{c}} : H^*_{\mathrm{CR}}(\mathcal{X}) \to H^*_{\mathrm{CR,c}}(\mathcal{Y})$ is an isomorphism. This holds because the pushforward $i_* : H_*(I\mathcal{X};\mathbb{R}) \to H_*(I\mathcal{Y};\mathbb{R})$ in homology is an isomorphism.

Assume that $\mathcal{E}$ is pulled back from a vector bundle $E \to X$ on the coarse space of $\mathcal{X}$. In this case all isotropy groups of $\mathcal{X}$ act trivially on the fibers of $\mathcal{E}$. Consequently each component $\mathcal{Y}_\gamma$ of $I\mathcal{Y}$ is given by the total space of $\mathcal{E}^\vee$ restricted to $\mathcal{X}_\gamma$. The base $\mathcal{X}_\gamma \subset \mathcal{Y}_\gamma$ can be viewed as the zero locus of the tautological section of $\pi^*\mathcal{E}^\vee$ restricted to $\mathcal{Y}_\gamma$. On the untwisted sector, we have $e(\pi^*\mathcal{E}^\vee) = i_*(1) \in H^*(\mathcal{Y}) \subset H^*_{\mathrm{CR}}(\mathcal{Y})$. Then for all $\alpha \in H^*_{\mathrm{CR}}(\mathcal{Y})$,

$$e(\pi^*\mathcal{E}^\vee) \cup \alpha = i_*(1) \cup \alpha = i_*(1 \cup i^*(\alpha)) = i_*(i^*(\alpha))$$

where the second equality follows from the projection formula and Remark 2.14 and the last equality is Remark 2.14. The proof concludes by noting that $i^* : H^*_{\mathrm{CR}}(\mathcal{Y}) \to H^*(\mathcal{X})$ is an isomorphism. □

## 3. Quantum $D$-modules for a proper target

This section serves to recall the basic definitions and constructions of Gromov–Witten theory and to set notation. We recall how the genus zero Gromov–Witten theory of a smooth and proper Deligne–Mumford stack $\mathcal{X}$ defines a flat connection known as the Dubrovin connection, which in turn gives a *quantum $D$-module* with integral structure. See [15, 21, 23] for more details.

For the remainder of the paper, $\mathcal{X}$ will be a smooth Deligne–Mumford stack over $\mathbb{C}$. We will use the same notation to denote the corresponding complex orbifold. We always assume further that the coarse moduli space of $\mathcal{X}$, denoted by $X$, is projective.

Definition 3.1. — *Given $\mathcal{X}$ as above, let $\overline{\mathscr{M}}_{h,n}(\mathcal{X}, d)$ denote the moduli space of representable degree $d$ stable maps from orbi-curves of genus $h$ with $n$ marked points (as defined in* [1]*). Here $d$ is an element of the cone* $\mathrm{Eff} = \mathrm{Eff}(\mathcal{X}) \subset H_2(\mathcal{X}, \mathbb{Q})$ *of effective curve classes. Denote by $[\overline{\mathscr{M}}_{h,n}(\mathcal{X}, d)]^{vir}$ the virtual fundamental class of* [1] *and* [6].

Recall that for each marked point $p_i$ there exist evaluation maps

$$ev_i : \overline{\mathscr{M}}_{h,n}(\mathcal{X}, d) \longrightarrow \bar{I}\mathcal{X},$$

where $\bar{I}\mathcal{X}$ is the *rigidified inertia stack* as in [1]. By the discussion in [1, Section 6.1.3], it is convenient to work as if the map $ev_i$ factors through $I\mathcal{X}$. While this is not in fact true, due to the isomorphism $H^*(\bar{I}\mathcal{X};\mathbb{C}) \cong H^*(I\mathcal{X};\mathbb{C})$ it makes no difference in terms of defining Gromov–Witten invariants.

Definition 3.2. — *Given $\alpha_1,\dots,\alpha_n \in H^*_{\mathrm{CR}}(\mathcal{X})$, $d \in \mathrm{Eff}$ and integers $h, b_1,\dots,b_n \geqslant 0$, define the Gromov–Witten invariant*

$$\langle \alpha_1\psi^{b_1},\dots,\alpha_n\psi^{b_n}\rangle^{\mathcal{X}}_{h,n,d} := \int_{[\overline{\mathscr{M}}_{h,n}(\mathcal{X},d)]^{vir}} \prod_{i=1}^{n} ev_i^*(\alpha_i)\cup\psi_i^{b_i}.$$

### 3.1. Twisted invariants

Many of the results of Section 5 and Section 6 are in terms of *twisted invariants*, which we define below. See [14] for details of the theory.

Let $\mathcal{E}$ be a vector bundle on a Deligne–Mumford stack $\mathcal{X}$. Given formal parameters $s_k$ for $k \geqslant 0$, one defines the formal invertible multiplicative characteristic class

$$\boldsymbol{s} : \mathcal{E} \longmapsto \exp\left(\sum_{k\geqslant 0} s_k\,\widetilde{\mathrm{ch}}_k(\mathcal{E})\right).$$

The twisted Gromov–Witten invariants depend on the parameters $\boldsymbol{s} = (s_0, s_1,\dots)$ and take values in $\mathbb{C}[\![\boldsymbol{s}]\!]$. Let $f^*(\mathcal{E})$ denote the pullback of $\mathcal{E}$ to the universal curve $\widetilde{\mathcal{C}}$ over $\overline{\mathscr{M}}_{h,r}(\mathcal{X},d)$.

Definition 3.3. — *Given $\alpha_1,\dots,\alpha_n$ in $H^*_{\mathrm{CR}}(\mathcal{X})$, $d \in \mathrm{Eff}$ and integers $h, b_1,\dots,b_n \geqslant 0$, define the $\boldsymbol{s}$-twisted Gromov–Witten invariant of $\mathcal{X}$ to be*

$$\begin{aligned}&\langle \alpha_1\psi^{b_1},\dots,\alpha_n\psi^{b_n}\rangle^{\mathcal{X},\boldsymbol{s}}_{h,n,d}\\ &\qquad := \int_{[\overline{\mathscr{M}}_{h,n}(\mathcal{X},d)]^{vir}} \exp\left(\sum_{k\geqslant 0} s_k\,\widetilde{\mathrm{ch}}_k(\mathbb{R}\pi_* f^*(\mathcal{E}))\right)\prod_{i=1}^{n} ev_i^*(\alpha_i)\cup\psi_i^{b_i}.\end{aligned}$$

Definition 3.4. — *Define an $\boldsymbol{s}$-twisted Gromov–Witten pairing as follows. Given $\alpha,\beta \in H^*_{\mathrm{CR}}(\mathcal{X})\otimes\mathbb{C}[\![\boldsymbol{s}]\!]$, define*

$$\langle \alpha,\beta\rangle^{\mathcal{X},\boldsymbol{s}} := \left\langle \exp\left(\sum_{k\geqslant 0} s_k\,\widetilde{\mathrm{ch}}_k(\mathcal{E})\right)\alpha,\beta\right\rangle^{\mathcal{X}}.$$

### 3.2. Quantum connections

Choose a basis $\{T_i\}_{i\in I}$ for the $H^*_{\mathrm{CR}}(\mathcal{X})$ state space such that $I = I' \coprod I''$ where $I''$ indexes a basis for the degree two part of the cohomology supported on the untwisted sector, and $I'$ indexes a basis for the cohomology of the twisted sectors together with the degree not equal to two cohomology of the untwisted sector. Let $\boldsymbol{t}' = \sum_{i\in I'} t^i T_i$ and let $\boldsymbol{t} = \sum_{i\in I'\cup I''} t^i T_i$. Let $q^i = e^{t_i}$ for $i \in I''$. Choose $\alpha_1, \dots, \alpha_n$ from $H^*_{\mathrm{CR}}(\mathcal{X})$. For $\square = \mathcal{X}$ or $\square = (\mathcal{X}, \boldsymbol{s})$,

$$\begin{aligned}(3.1)\quad &\langle\!\langle \alpha_1\psi^{b_1}, \dots, \alpha_n\psi^{b_n}\rangle\!\rangle^{\square}(\boldsymbol{t}) \\ &\qquad := \sum_{d\in \mathrm{Eff}} \sum_{k\geqslant 0} \frac{1}{k!} \langle \alpha_1\psi^{b_1}, \dots, \alpha_n\psi^{b_n}, \boldsymbol{t}, \dots, \boldsymbol{t}\rangle^{\square}_{0,n+k,d}\end{aligned}$$

where a summand is implicitly assumed to be zero if $d = 0$ and $n + k < 3$. Let $\mathbb{C}[\![\boldsymbol{t}']\!] := \mathbb{C}[\![t^i]\!]_{i\in I'}$ and $\mathbb{C}[\![\boldsymbol{q}]\!] := \mathbb{C}[\![q^i]\!]_{i\in I''}$. By the divisor equation [15, Section 10.1], if $b_1 = \cdots = b_n = 0$ then (3.1) can be viewed as a formal power series in $\mathbb{C}[\![\boldsymbol{t}', \boldsymbol{q}]\!]$ or $\mathbb{C}[\![\boldsymbol{t}', \boldsymbol{q}, \boldsymbol{s}]\!]$ in the twisted case.

*Notation 3.5.* — Denote by $P^{\square}$ the power series $\mathbb{C}[\![\boldsymbol{t}', \boldsymbol{q}]\!]$ when $\square = \mathcal{X}$ refers to Gromov–Witten theory, or $\mathbb{C}[\![\boldsymbol{t}', \boldsymbol{q}, \boldsymbol{s}]\!]$ when $\square = (\mathcal{X}, \boldsymbol{s})$ refers to a $\boldsymbol{s}$-twisted theory.

Definition 3.6. — *For elements $\alpha, \beta$ in $H^*_{\mathrm{CR}}(\mathcal{X})$ with $\square = \mathcal{X}$ or $(\mathcal{X}, \boldsymbol{s})$, define the* quantum product $\alpha \bullet^{\square}_{\boldsymbol{t}} \beta \in H^*_{\mathrm{CR}}(\mathcal{X}) \otimes P^{\square}$ *by the formula*

$$\langle \alpha \bullet_{\boldsymbol{t}} \beta, \gamma\rangle^{\square} = \langle\!\langle \alpha, \beta, \gamma\rangle\!\rangle^{\square}(\boldsymbol{t})$$

*for all $\gamma \in H^*_{\mathrm{CR}}(\mathcal{X})$.*

In the non-twisted setting ($\boldsymbol{s} = 0$) this is equivalent to

$$\begin{aligned}\alpha \bullet^{\square}_{\boldsymbol{t}} \beta := \sum_{d\in \mathrm{Eff}} \sum_{k\geqslant 0} \frac{1}{k!} I_* \circ ev_{3*}\Bigg( &ev_1^*(\alpha) \cup ev_2^*(\beta) \\ &\cup \prod_{j=4}^{k+3} ev_j^*(\boldsymbol{t}) \cap [\overline{\mathscr{M}}_{0,k+3}(\mathcal{X}, d)]^{vir}\Bigg)\end{aligned}$$

where to simplify notation we suppress the Poincaré duality isomorphism $\mathrm{PD} : H^*_{CR}(\mathcal{X}) \to H^{\mathrm{BM}}_*(I\mathcal{X})$ from (2.1) which identifies the vector spaces on the left and right side of the expression.

Definition 3.7. — *For elements $\alpha, \beta$ in $H^*_{\mathrm{CR}}(\mathcal{X})$ define the* Chen–Ruan cup product $\alpha \cup \beta \in H^*_{\mathrm{CR}}(\mathcal{X})$ *by the formula*

$$\langle \alpha \cup \beta, \gamma \rangle^{\mathcal{X}} = \langle\!\langle \alpha, \beta, \gamma \rangle\!\rangle^{\mathcal{X}}(\mathbf{0})|_{\mathbf{q}=\mathbf{0}}$$

*for all $\gamma \in H^*_{\mathrm{CR}}(\mathcal{X})$.*

### 3.3. Quantum $D$-module

Following [23], define a $z$-sesquilinear pairing $S^{\square}$ on $H^*_{\mathrm{CR}}(\mathcal{X}) \otimes P^{\square}[z, z^{-1}]$ as

$$S^{\square}(u(z), v(z)) := (2\pi\sqrt{-1}z)^{\dim(\mathcal{X})} \langle u(-z), v(z) \rangle^{\square}.$$

Definition 3.8. — *The Dubrovin connection is given by the formula*

$$\nabla^{\square}_i = \partial_i + \frac{1}{z} T_i \bullet^{\square}_{\boldsymbol{t}},$$

*where recall that $\{T_i\}_{i \in I}$ is a basis for $H^*_{\mathrm{CR}}(\mathcal{X})$. These operators act on $H^*_{\mathrm{CR}}(\mathcal{X}) \otimes P^{\square}[z, z^{-1}]$.*

When we are not in the twisted setting, we can define the quantum connection in the $z$-direction as well.[(2)] Define the Euler vector field

$$\mathfrak{E} := \partial_{\rho(\mathcal{X})} + \sum_{i \in I} \left(1 - \frac{1}{2} \deg T_i\right) t^i \partial_i$$

where $\rho(\mathcal{X}) := c_1(T\mathcal{X}) \subset H^2(\mathcal{X}; \mathbb{C})$.

Define the grading operator Gr by

$$\mathrm{Gr}(\alpha) := \frac{\deg \alpha}{2} \alpha$$

for $\alpha$ in $H^*_{\mathrm{CR}}(\mathcal{X})$. Then define

$$\nabla^{\mathcal{X}}_z = \partial_z - \frac{1}{z^2} \mathfrak{E} \bullet^{\mathcal{X}}_{\boldsymbol{t}} + \frac{1}{z} \mathrm{Gr}\,.$$

Define the operator $L^{\square}(\boldsymbol{t}, z)$ by

$$L^{\square}(\boldsymbol{t}, z)\alpha := \alpha + \sum_{i \in I} \left\langle\!\!\left\langle \frac{\alpha}{-z - \psi}, T^i \right\rangle\!\!\right\rangle^{\square}(\boldsymbol{t}) T_i.$$

(2) The definition of $\nabla^{\square}_z$ is based on the fact that (when we are not in the twisted setting) the virtual class is of pure dimension on connected components.

When we are not in the twisted setting this is equivalent to

$$(3.2)\quad L^{\mathcal{X}}(\boldsymbol{t},z)(\alpha)$$
$$:= \alpha + \sum_{d\in \mathrm{Eff}}\sum_{k\geqslant 0}\frac{1}{k!}I_*\circ ev_{2*}\left(\frac{ev_1^*(\alpha)}{-z-\psi}\prod_{j=3}^{k+2} ev_j^*(\boldsymbol{t})\cap[\overline{\mathscr{M}}_{0,k+2}(\mathcal{X},d)]^{vir}\right).$$

Proposition 3.9 ([15, 23]). — *Let $\square = \mathcal{X}$ or $(\mathcal{X},\boldsymbol{s})$. The quantum connection $\nabla^{\square}$ is flat. In the untwisted case,*

$$(3.3)\qquad \nabla_i^{\mathcal{X}}(L^{\mathcal{X}}(\boldsymbol{t},z)z^{-\mathrm{Gr}}z^{\rho(\mathcal{X})}\alpha) = \nabla_z^{\mathcal{X}}(L^{\mathcal{X}}(\boldsymbol{t},z)z^{-\mathrm{Gr}}z^{\rho(\mathcal{X})}\alpha) = 0$$

*for $i\in I$ and $\alpha\in H^*_{\mathrm{CR}}(\mathcal{X})$. In the twisted case,*

$$(3.4)\qquad \nabla_i^{\mathcal{X},\boldsymbol{s}}(L^{\mathcal{X},\boldsymbol{s}}(\boldsymbol{t},z)\alpha) = 0.$$

*In both the twisted and untwisted setting the pairing $S^{\square}$ is flat with respect to $\nabla^{\square}$. For $\alpha,\beta\in H^*_{\mathrm{CR}}(\mathcal{X})$,*

$$(3.5)\qquad \langle L^{\square}(\boldsymbol{t},-z)\alpha, L^{\square}(\boldsymbol{t},z)\beta\rangle^{\square} = \langle\alpha,\beta\rangle^{\square}.$$

Definition 3.10. — *The* quantum $D$-module *for $\mathcal{X}$, $QDM(\mathcal{X})$, is defined to be the triple:*

$$QDM(\mathcal{X}) := (H^*_{\mathrm{CR}}(\mathcal{X})\otimes P^{\mathcal{X}}[z,z^{-1}], \nabla^{\mathcal{X}}, S^{\mathcal{X}}).$$

### 3.3.1. Ambient Gromov–Witten theory

There is a restricted quantum $D$-module for local complete intersection sub-stacks.

Definition 3.11. — *A vector bundle $\mathcal{E}\to\mathcal{X}$ is called* convex *if, for every representable morphism*

$$f:\mathcal{C}\longrightarrow\mathcal{X}$$

*from a genus zero orbi-curve $\mathcal{C}$, the cohomology $H^1(\mathcal{C},f^*(\mathcal{E}))$ is zero.*

*Remark 3.12.* — Note that convexity of $\mathcal{E}$ implies that it is pulled back from a vector bundle $E\to X$ on the coarse space [13, Remark 5.3]. Furthermore one can check that $E$ will itself be convex in this case.

Let $\mathcal{X}$ be as before. Consider $\mathcal{E}$ a convex vector bundle on $\mathcal{X}$, and let $\mathcal{Z}$ be the zero locus of a transverse section $s\in\Gamma(\mathcal{X},\mathcal{E})$. Let $j:\mathcal{Z}\to\mathcal{X}$ denote the inclusion map and define $H^*_{\mathrm{CR,amb}}(\mathcal{Z}) := \mathrm{im}(j^*)$.

Assumption 3.13. — *As in* [24], *we will always assume that the Poincaré pairing on* $H^*_{\mathrm{CR,amb}}(\mathcal{Z})$ *is non-degenerate. This is equivalent to the condition that*

$$H^*_{\mathrm{CR}}(\mathcal{Z}) = \mathrm{im}(j^*) \oplus \ker(j_*). \tag{3.6}$$

Assumption 3.13 holds for instance if $\mathcal{E}$ is the pullback of an ample line bundle on $X$ and $\mathcal{Z}$ intersects each twisted component of $\mathcal{X}$ transversally.

Proposition 3.14 ([23, Corollary 2.5]). — *For* $\bar{\boldsymbol{t}} \in H^*_{\mathrm{CR,amb}}(\mathcal{Z})$, *the quantum product* $\bullet^{\mathcal{Z}}_{\bar{\boldsymbol{t}}}$ *is closed on* $H^*_{\mathrm{CR,amb}}(\mathcal{Z})$. *The quantum connection and solution* $L^{\mathcal{Z}}(\bar{\boldsymbol{t}}, z)$ *preserve* $H^*_{\mathrm{CR,amb}}(\mathcal{Z})$ *for* $\bar{\boldsymbol{t}} \in H^*_{\mathrm{CR,amb}}(\mathcal{Z})$.

*Remark 3.15.* — The proof of the above proposition in [23, Corollary 2.5] follows an argument from [28]. The assumptions on $\mathcal{X}$ and $\mathcal{E}$ above are weaker than in either, but the same argument goes through, as observed in [12, Remark 2.2].

Definition 3.16. — *The* ambient *quantum $D$-module is defined to be*

$$QDM_{\mathrm{amb}}(\mathcal{Z}) := (H^*_{\mathrm{CR,amb}}(\mathcal{Z}) \otimes P^{\mathcal{Z},\mathrm{amb}}[z, z^{-1}], \nabla^{\mathcal{Z}}, S^{\mathcal{Z}})$$

*where* $P^{\mathcal{Z},\mathrm{amb}}$ *denotes the restriction of* $P^{\mathcal{Z}}$ *to* $H^*_{\mathrm{CR,amb}}(\mathcal{Z})$.

### 3.4. Integral structure

In [21], Iritani defines an integral structure for Gromov–Witten theory. We recall the ingredients here.

For $\mathcal{F}$ a vector bundle on $\mathcal{X}$, let $\mathcal{F}_\gamma := \mathcal{F}|_{\mathcal{X}_\gamma}$ denote the restriction of $\mathcal{F}$ to a twisted sector $\mathcal{X}_\gamma$. Recall as in Section 2.2, that $\mathcal{F}_\gamma$ splits into a sum of eigenbundles

$$\mathcal{F}_\gamma = \bigoplus_{0 \leqslant f < 1} \mathcal{F}_{\gamma,f}$$

where the action of $\gamma$ on $\mathcal{F}_{\gamma,f}$ is multiplication by $e^{2\pi\sqrt{-1}f}$.

Definition 3.17. — *Define the* Gamma class $\widehat{\Gamma}(\mathcal{F})$ *to be the class in* $H^*(I\mathcal{X})$:

$$\widehat{\Gamma}(\mathcal{F}) := \bigoplus_{\gamma \in T} \prod_{0 \leqslant f < 1} \prod_{i=1}^{\mathrm{rk}(\mathcal{F}_{b,f})} \Gamma(1 - f + \rho_{b,f,i})$$

*where* $\Gamma(1+x)$ *should be understood in terms of its Taylor expansion at* $x = 0$, *and* $\{\rho_{b,f,i}\}$ *are the Chern roots of* $\mathcal{F}_{b,f}$. *Define* $\widehat{\Gamma}_{\mathcal{X}}$ *to be the class* $\widehat{\Gamma}(T\mathcal{X})$.

### 3.4.1. Flat sections

Definition 3.18. — *Define the operator* $\deg_0$ *to be the degree operator which multiplies a homogeneous class by its unshifted degree. In Gromov–Witten theory it multiplies a class in* $H^n(I\mathcal{X})$ *(with the standard grading) by* $n$.

Denote by $D(\mathcal{X}) := D^b(\mathcal{X})$ the bounded derived category of coherent sheaves on $\mathcal{X}$. We will omit the superscript $b$. Given an object $F$ in $D(\mathcal{X})$, define $\boldsymbol{s}^{\mathcal{X}}(F)(\boldsymbol{t}, z)$ to be

$$\frac{1}{(2\pi\sqrt{-1})^{\dim(\mathcal{X})}} L^{\mathcal{X}}(\boldsymbol{t}, z) z^{-Gr} z^{\rho(\mathcal{X})} \widehat{\Gamma}_{\mathcal{X}} \cup_{I\mathcal{X}} ((2\pi\sqrt{-1})^{\deg_0/2} I^*(\widetilde{\mathrm{ch}}(F))),$$

where $\cup_{I\mathcal{X}}$ denotes the ordinary cup product on $H^*(I\mathcal{X})$.

Proposition 3.19 ([23]). — *The map* $\boldsymbol{s}^{\mathcal{X}}$ *identifies the pairing in the derived category with* $S^{\mathcal{X}}$*:*

$$S^{\mathcal{X}}(\boldsymbol{s}^{\mathcal{X}}(F)(\boldsymbol{t}, z), \boldsymbol{s}^{\mathcal{X}}(F')(\boldsymbol{t}, z)) = e^{\pi\sqrt{-1}\dim(\mathcal{X})}\chi(F', F).$$

Assumption 3.20. — *Assume that* $H^*_{\mathrm{CR}}(\mathcal{X})$ *is spanned by the image of*

$$\widetilde{\mathrm{ch}} : D(\mathcal{X}) \longrightarrow H^*_{\mathrm{CR}}(\mathcal{X}).$$

The set

$$\{\boldsymbol{s}^{\mathcal{X}}(F)(\boldsymbol{t}, z) \mid F \in D(\mathcal{X})\}$$

forms a lattice in $\ker(\nabla^{\mathcal{X}})$. This is the integral structure of the quantum $D$-module $QDM(\mathcal{X})$.

Let $j : \mathcal{Z} \to \mathcal{X}$ be a smooth subvariety defined as in Section 3.3.1 (i.e. by a convex vector bundle). Then again with assumption 3.20 we can define an integral structure for the ambient quantum $D$-module as

$$\{\boldsymbol{s}^{\mathcal{Z}}(F)(\boldsymbol{t}, z) \mid F \in j^*(D(\mathcal{X}))\}.$$

Note that by orbifold Grothendieck–Riemann–Roch, $\widetilde{\mathrm{ch}}(F)$ will lie in $H^*_{\mathrm{CR,amb}}(\mathcal{Z})$ for $F \in j^*(D(\mathcal{X}))$.

## 4. Quantum $D$-modules for a non-proper target

In fact most of the constructions of genus zero Gromov–Witten theory go through in the case of a non-proper target $\mathcal{Y}$. We explore this in this section. The majority of this section is known to the experts, see e.g. [20, 24]. The perspective of Section 4.2 in terms narrow cohomology, however, is new.

Let $\mathcal{Y}$ denote a smooth Deligne–Mumford stack with quasi-projective coarse moduli space $Y$. Although $\mathcal{Y}$ may not be proper, if the evaluation maps

$$ev_i : \overline{\mathscr{M}}_{h,n}(\mathcal{Y}, d) \longrightarrow \bar{I}\mathcal{Y}$$

are proper, one can still define Gromov–Witten invariants in many cases. In particular, there is still a well-defined virtual class $[\overline{\mathscr{M}}_{h,n}(\mathcal{Y}, d)]^{vir}$ over which one can integrate classes of compact support.

Definition 4.1. — *Assume the maps $ev_i : \overline{\mathscr{M}}_{h,n}(\mathcal{Y}, d) \to \bar{I}\mathcal{Y}$ are proper for $1 \leqslant i \leqslant n$. Given $\alpha_1 \in H^*_{\mathrm{CR,c}}(\mathcal{Y})$, $\alpha_2, \ldots, \alpha_n \in H^*_{\mathrm{CR}}(\mathcal{Y})$ and integers $b_1, \ldots, b_n \geqslant 0$ define the Gromov–Witten invariant*

$$\langle \alpha_1 \psi^{b_1}, \ldots, \alpha_n \psi^{b_n} \rangle^{\mathcal{Y}}_{h,n,d} := \int_{[\overline{\mathscr{M}}_{h,n}(\mathcal{Y},d)]^{vir}} \prod_{i=1}^{n} ev_i^*(\alpha_i) \cup \psi_i^{b_i}.$$

*Given $\alpha_1, \alpha_2 \in H^*_{\mathrm{CR,nar}}(\mathcal{Y})$, $\alpha_3, \ldots, \alpha_n \in H^*_{\mathrm{CR}}(\mathcal{Y})$ and integers $b_1, \ldots, b_n \geqslant 0$ define the Gromov–Witten invariant*

$$\langle \alpha_1 \psi^{b_1}, \ldots, \alpha_n \psi^{b_n} \rangle^{\mathcal{Y}}_{h,n,d} := \int_{[\overline{\mathscr{M}}_{h,n}(\mathcal{Y},d)]^{vir}} \alpha_1 \cup_{\mathrm{c}} \alpha_2 \cup \prod_{i=3}^{n} ev_i^*(\alpha_i) \cup \psi_i^{b_i}$$

*where recall the product $\cup_{\mathrm{c}} : H^*_{\mathrm{CR,nar}}(\mathcal{Y}) \times H^*_{\mathrm{CR,nar}}(\mathcal{Y}) \to H^*_{\mathrm{CR,c}}(\mathcal{Y})$ was defined via a lift $\widetilde{\alpha}_1 \in H^*_{\mathrm{CR,c}}(\mathcal{Y})$ as in Definition 2.8.*

The lemma below gives an important scenario when the evaluation maps are in fact proper. Let $\mathcal{X}$ be a smooth proper Deligne–Mumford stack, and $\mathcal{E} \to \mathcal{X}$ a vector bundle on $\mathcal{X}$. Let $\mathcal{Y}$ denote the total space of $\mathcal{E}^\vee$ over $\mathcal{X}$.

Lemma 4.2. — *For $1 \leqslant i \leqslant n$, the evaluation map $ev_i : \overline{\mathscr{M}}_{0,n}(\mathcal{Y}, d) \to \bar{I}\mathcal{Y}$ is proper in the following situations:*

(1) *The degree $d = 0$;*
(2) *The vector bundle $\mathcal{E}$ is convex.*

*Proof.* — In the first case, the evaluation map $ev_i : \overline{\mathscr{M}}_{0,n}(\mathcal{Y}, 0) \to \bar{I}\mathcal{Y}$ factors through a rigidification of $I^n\mathcal{Y} = \coprod_{\mathbf{g}} \mathcal{Y}_{g_1,\ldots,g_n}$, the stack of $n$-multisectors [3, Example 2.5]. The map $\pi_1 : \overline{\mathscr{M}}_{0,n}(\mathcal{Y}, 0) \to \bar{I}^n\mathcal{Y}$ forgets the source curve while remembering the maps $B\mu_{r_i} \to \mathcal{Y}$ at each marked point $p_i$. The map $\overline{ev_i} : \bar{I}^n\mathcal{Y} \to \bar{I}\mathcal{Y}$ sends the component $\mathcal{Y}_{g_1,\ldots,g_n}$ to $\mathcal{Y}_{g_i}$. The evaluation map $ev_i$ is the composition of the proper map $\pi_1$ and the closed immersion $\overline{ev_i}$ and is therefore proper.

In the second case, we note that a map $f_{\mathcal{Y}} : \mathcal{C} \to \mathcal{Y}$ consists of a map to $f : \mathcal{C} \to \mathcal{X}$ together with a section $s \in H^0(\mathcal{C}, f^*(\mathcal{E}^\vee))$. Let $r_{\mathcal{C}} : \mathcal{C} \to C$ be the map to the underlying coarse curve. Let $\bar{i}$ be the map $\bar{I}\mathcal{X} \to X$. By [2, Theorem 1.4.1], the composition $\mathcal{C} \to \mathcal{X} \to X$ factors through a

map $|f| : C \to X$. By Remark 3.12, $\mathcal{E}$ is the pullback of a vector bundle $E \to X$ and $E$ is convex. Therefore on each irreducible component $C_j$ of $C$, $E \cong \sum_{i=1}^{r} \mathcal{O}(k_i)$ with each $k_i \geqslant 0$. Therefore the evaluation map

$$ev_i : H^0(\mathcal{C}, f^*(\mathcal{E}^\vee)) = H^0(C, |f|^*(E^\vee)) \longrightarrow E^\vee|_{|f|(r_{\mathcal{C}}(x_i))} = \bar{i}^* E^\vee|_{f(x_i)}$$

is an injection (see [24, Lemma 3.8]). Then $\overline{\mathscr{M}}_{0,n}(\mathcal{Y}, d)$ is seen to be a substack of $\overline{\mathscr{M}}_{0,n}(\mathcal{X}, d)_{ev_i} \times_{proj} \mathrm{tot}(\bar{i}^* E^\vee)$ via the map $ev_i$. Here $proj$ is the projection from $\mathrm{tot}(\bar{i}^* E^\vee)$ to $\bar{I}\mathcal{X}$. In fact we have the following fiber square

$$\begin{array}{ccc} \overline{\mathscr{M}}_{0,n}(\mathcal{Y}, d) & \xrightarrow{\text{inc}} & \overline{\mathscr{M}}_{0,n}(\mathcal{X}, d)_{ev_i} \times_{proj} \mathrm{tot}(\bar{i}^* E^\vee) \\ \downarrow & & \downarrow \\ \overline{\mathscr{M}}_{0,n}(Y, d) & \longrightarrow & \overline{\mathscr{M}}_{0,n}(X, d)_{ev_i} \times_{proj} \mathrm{tot}(E^\vee). \end{array}$$

The bottom map is proper by [24, Lemma 3.8], and therefore so is the top map. The map

$$\overline{\mathscr{M}}_{0,n}(\mathcal{X}, d)_{ev_i} \times_{proj} \mathrm{tot}(\bar{i}^* E^\vee) \xrightarrow{(ev_i, \mathrm{id}_2)} \bar{I}\mathcal{X}_{\mathrm{id}} \times_{proj} \mathrm{tot}(\bar{i}^* E^\vee) = \bar{I}\mathcal{Y}$$

is proper. The evaluation map $ev_i : \overline{\mathscr{M}}_{0,n}(\mathcal{Y}, d) \to \bar{I}\mathcal{Y}$ is equal to the composition $(ev_i, \mathrm{id}_2) \circ \mathrm{inc}$. □

*Remark 4.3.* — More generally, for a noncompact space $\mathcal{Y}$, the evaluation map $ev_i : \overline{\mathscr{M}}_{g,n}(\mathcal{Y}, d) \to \bar{I}\mathcal{Y}$ will be proper if $\mathcal{Y}$ is projective over an affine variety [27]. However we will not use this in what follows so we omit the proof.

### 4.1. Quantum connections

Let us assume from now on that $\mathcal{Y}$ is not necessarily proper, but that the (genus zero) evaluation maps are proper.

As in Section 3 we may define double brackets. The setup is as before. Choose a basis $\{T_i\}_{i \in I}$ for the $H^*_{\mathrm{CR}}(\mathcal{Y})$ state space such that $I = I' \coprod I''$ where $I''$ indexes a basis for the degree two part of the cohomology supported on the untwisted sector, and $I'$ indexes a basis for the cohomology of the twisted sectors together with the degree not equal to two cohomology of the untwisted sector. Let $\boldsymbol{t}' = \sum_{i \in I'} t^i T_i$ and let $\boldsymbol{t} = \sum_{i \in I' \cup I''} t^i T_i$. Let $q^i = e^{t_i}$ for $i \in I''$. Denote by $P^{\mathcal{Y}}$ the power series $\mathbb{C}[\![\boldsymbol{t}', \boldsymbol{q}]\!]$. Choose $\alpha_1, \ldots, \alpha_n$ from $H^*_{\mathrm{CR}}(\mathcal{Y}) \cup H^*_{\mathrm{CR,c}}(\mathcal{Y})$. Assume that for at least one $i$, $\alpha_i$ is

in $H^*_{\mathrm{CR,c}}(\mathcal{Y})$, or that for some $i<j$, $\alpha_i,\alpha_j\in H^*_{\mathrm{CR,nar}}(\mathcal{Y})$. Then define

$$\begin{aligned}(4.1)\quad &\langle\!\langle \alpha_1\psi^{b_1},\dots,\alpha_n\psi^{b_n}\rangle\!\rangle^{\mathcal{Y}}(\boldsymbol{t})\\ &\qquad:=\sum_{d\in\mathrm{Eff}}\sum_{k\geqslant 0}\frac{1}{k!}\langle\alpha_1\psi^{b_1},\dots,\alpha_n\psi^{b_n},\boldsymbol{t},\dots,\boldsymbol{t}\rangle^{\mathcal{Y}}_{0,n+k,d}\end{aligned}$$

where a summand is implicitly assumed to be zero if $d=0$ and $n+k<3$. In the absence of $\psi$-classes this yields a formal power series in $P^{\mathcal{Y}}$.

In the case of a non-proper target, there are two possible quantum products, in analogy with (2.2). For elements $\alpha,\beta$ in $H^*_{\mathrm{CR}}(\mathcal{Y})$, define $\alpha\bullet^{\mathcal{Y}}_{\boldsymbol{t}}\beta\in H^*_{\mathrm{CR}}(\mathcal{Y})\otimes P^{\mathcal{Y}}$ by the formula

$$\langle\alpha\bullet_{\boldsymbol{t}}\beta,\gamma\rangle^{\mathcal{Y}}=\langle\!\langle\alpha,\beta,\gamma\rangle\!\rangle^{\mathcal{Y}}(\boldsymbol{t})$$

for all $\gamma\in H^*_{\mathrm{CR,c}}(\mathcal{Y})$. Similarly to the cup product, we can also multiply a cohomology class with a cohomology class with compact support. If $\alpha\in H^*_{\mathrm{CR}}(\mathcal{Y})$ and $\beta\in H^*_{\mathrm{CR,c}}(\mathcal{Y})$, define $\alpha\bullet^{\mathcal{Y}}_{\boldsymbol{t}}\beta\in H^*_{\mathrm{CR,c}}(\mathcal{Y})$ by the formula

$$\langle\gamma,\alpha\bullet_{\boldsymbol{t}}\beta\rangle^{\mathcal{Y}}=\langle\!\langle\alpha,\beta,\gamma\rangle\!\rangle^{\mathcal{Y}}(\boldsymbol{t})$$

for all $\gamma\in H^*_{\mathrm{CR}}(\mathcal{Y})$. Alternatively, in the first case the above definition is equivalent to

$$\begin{aligned}(4.2)\quad \alpha\bullet^{\mathcal{Y}}_{\boldsymbol{t}}\beta:=\sum_{d\in\mathrm{Eff}}\sum_{k\geqslant 0}\frac{1}{k!}I_*\circ ev_{3*}\Bigg(&ev_1^*(\alpha)\cup ev_2^*(\beta)\\ &\cup\prod_{j=4}^{k+3}ev_j^*(\boldsymbol{t})\cap[\overline{\mathscr{M}}_{0,k+3}(\mathcal{Y},d)]^{vir}\Bigg).\end{aligned}$$

The second case is the same but replacing $ev_2^*$ and $ev_{3*}$ with $ev_{2_{\mathrm{c}}}^*$ and $ev_{3_*}^{\mathrm{c}}$, the pullback and pushforward in cohomology with compact support.

Exactly as in the proper case, The pairing $\langle\cdot,\cdot\rangle^{\mathcal{Y}}$ can be extended to a $z$-sesquilinear pairing $S^{\mathcal{Y}}$ between $H^*_{\mathrm{CR}}(\mathcal{Y})\otimes P^{\mathcal{Y}}[z,z^{-1}]$ and $H^*_{\mathrm{CR,c}}(\mathcal{Y})\otimes P^{\mathcal{Y}}[z,z^{-1}]$ by defining

$$S^{\mathcal{Y}}(u(z),v(z)):=(2\pi\sqrt{-1}z)^{\dim(\mathcal{Y})}\langle u(-z),v(z)\rangle^{\mathcal{Y}}.$$

Definition 4.4. — *As before, the Dubrovin connection is defined by the formulas*

$$\nabla^{\mathcal{Y}}_i=\partial_i+\frac{1}{z}T_i\bullet^{\mathcal{Y}}_{\boldsymbol{t}}$$

*and*

$$\nabla^{\mathcal{Y}}_z=\partial_z-\frac{1}{z^2}\mathfrak{E}\bullet^{\mathcal{Y}}_{\boldsymbol{t}}+\frac{1}{z}\,\mathrm{Gr}$$

*where recall that $\{T_i\}_{i\in I}$ is a basis for $H^*_{\mathrm{CR}}(\mathcal{Y})$. These operators act on both $H^*_{\mathrm{CR}}(\mathcal{Y})\otimes P^{\mathcal{Y}}[z,z^{-1}]$ and $H^*_{\mathrm{CR,c}}(\mathcal{Y})\otimes P^{\mathcal{Y}}[z,z^{-1}]$. In particular, for*

*$\alpha \in H^*_{\mathrm{CR,c}}(\mathcal{Y}) \otimes P^{\mathcal{Y}}[z, z^{-1}]$, $T_i \bullet^{\mathcal{Y}}_{\boldsymbol{t}} \alpha$ and therefore $\nabla^{\mathcal{Y}}_i \alpha$ lies in $H^*_{\mathrm{CR,c}}(\mathcal{Y}) \otimes P^{\mathcal{Y}}[z, z^{-1}]$. To avoid confusion, we will denote the connection by $\nabla^{\mathcal{Y}}$ when acting on cohomology and by $\nabla^{\mathcal{Y},\mathrm{c}}$ when acting on cohomology with compact support.*

Define $L^{\mathcal{Y}}(\boldsymbol{t}, z)$ by

$$L^{\mathcal{Y}}(\boldsymbol{t}, z)(\alpha) := \alpha + \sum_{d \in \mathrm{Eff}} \sum_{k \geqslant 0} \frac{1}{k!} I_* \circ ev_{2*} \left( \frac{ev_1^*(\alpha)}{-z - \psi} \prod_{j=3}^{k+2} ev_j^*(\boldsymbol{t}) \cap [\overline{\mathscr{M}}_{0,k+2}(\mathcal{Y}, d)]^{vir} \right) \tag{4.3}$$

for $\alpha$ in $H^*_{\mathrm{CR}}(\mathcal{Y})$. Define $L^{\mathcal{Y},\mathrm{c}}(\boldsymbol{t}, z)$ by

$$L^{\mathcal{Y},\mathrm{c}}(\boldsymbol{t}, z)(\alpha) := \alpha + \sum_{d \in \mathrm{Eff}} \sum_{k \geqslant 0} \frac{1}{k!} I_* \circ ev_{2*}^{\mathrm{c}} \left( \frac{ev_{1\mathrm{c}}^*(\alpha)}{-z - \psi} \prod_{j=3}^{k+2} ev_j^*(\boldsymbol{t}) \cap [\overline{\mathscr{M}}_{0,k+2}(\mathcal{Y}, d)]^{vir} \right) \tag{4.4}$$

for $\alpha$ in $H^*_{\mathrm{CR,c}}(\mathcal{Y})$.

There is a completely analogous result to Proposition 3.9 in the non-proper case.

Proposition 4.5. — *Let $\mathcal{Y}$ be a non-proper space. The quantum connection $\nabla^{\square}$ is flat, with fundamental solution $L^{\square}(\boldsymbol{t}, z) z^{-\mathrm{Gr}} z^{\rho(\mathcal{Y})}$. More precisely,*

$$\nabla^{\mathcal{Y}}_i (L^{\mathcal{Y}}(\boldsymbol{t}, z) z^{-\mathrm{Gr}} z^{\rho(\mathcal{Y})} \alpha) = \nabla^{\mathcal{Y}}_z (L^{\mathcal{Y}}(\boldsymbol{t}, z) z^{-\mathrm{Gr}} z^{\rho(\mathcal{Y})} \alpha) = 0 \tag{4.5}$$

*for $\alpha \in H^*_{\mathrm{CR}}(\mathcal{Y})$ and*

$$\nabla^{\mathcal{Y},\mathrm{c}}_i (L^{\mathcal{Y},\mathrm{c}}(\boldsymbol{t}, z) z^{-\mathrm{Gr}} z^{\rho(\mathcal{Y})} \beta) = \nabla^{\mathcal{Y},\mathrm{c}}_z (L^{\mathcal{Y},\mathrm{c}}(\boldsymbol{t}, z) z^{-\mathrm{Gr}} z^{\rho(\mathcal{Y})} \beta) = 0 \tag{4.6}$$

*for $\beta \in H^*_{\mathrm{CR,c}}(\mathcal{Y})$. Furthermore the pairing $S^{\mathcal{Y}}$ satisfies*

$$\partial_i S^{\mathcal{Y}}(u, v) = S^{\mathcal{Y}}(\nabla^{\mathcal{Y}}_i u, v) + S^{\mathcal{Y}}(u, \nabla^{\mathcal{Y},\mathrm{c}}_i v). \tag{4.7}$$

*In other words $\nabla^{\mathcal{Y}}$ and $\nabla^{\mathcal{Y},\mathrm{c}}$ are dual with respect to $S^{\mathcal{Y}}$. Finally, for $\alpha \in H^*_{\mathrm{CR}}(\mathcal{Y})$ and $\beta \in H^*_{\mathrm{CR,c}}(\mathcal{Y})$,*

$$\langle L^{\mathcal{Y}}(\boldsymbol{t}, -z)\alpha, L^{\mathcal{Y},\mathrm{c}}(\boldsymbol{t}, z)\beta \rangle^{\mathcal{Y}} = \langle \alpha, \beta \rangle^{\mathcal{Y}}. \tag{4.8}$$

*Proof.* — The proof of these statements is almost identical to the case of a proper target. The only difference is the precise statement of the topological recursion relation for a non-compact target, which is used to prove (4.5)

and (4.6). In this context the statement is that for $\alpha \in H^*_{\mathrm{CR,c}}(\mathcal{Y})$, $\beta, \gamma \in H^*_{\mathrm{CR}}(\mathcal{Y})$, and $b_1, b_2, b_3 \geqslant 0$,

$$\langle\!\langle \alpha\psi^{b_1+1}, \beta\psi^{b_2}, \gamma\psi^{b_3} \rangle\!\rangle^{\mathcal{Y}} = \sum_{i\in I} \langle\!\langle \alpha\psi^{b_1}, T_i \rangle\!\rangle^{\mathcal{Y}} \langle\!\langle T^i, \beta\psi^{b_2}, \gamma\psi^{b_3} \rangle\!\rangle^{\mathcal{Y}}$$

where recall that $\{T_i\}_{i\in I}$ and $\{T^i\}_{i\in I}$ are bases for $H^*_{\mathrm{CR}}(\mathcal{Y})$ and $H^*_{\mathrm{CR,c}}(\mathcal{Y})$ respectively, so both factors in the right hand sum are well-defined. Similarly, if $\alpha, \beta \in H^*_{\mathrm{CR}}(\mathcal{Y})$ and $\gamma \in H^*_{\mathrm{CR,c}}(\mathcal{Y})$ we have

$$\langle\!\langle \alpha\psi^{b_1+1}, \beta\psi^{b_2}, \gamma\psi^{b_3} \rangle\!\rangle^{\mathcal{Y}} = \sum_{i\in I} \langle\!\langle \alpha\psi^{b_1}, T^i \rangle\!\rangle^{\mathcal{Y}} \langle\!\langle T_i, \beta\psi^{b_2}, \gamma\psi^{b_3} \rangle\!\rangle^{\mathcal{Y}}.$$

The proof of these statements is identical to the proof in the case of a proper target, after using the following version of the Künneth formula

$$H^*_{\mathrm{CR}}(\mathcal{Y}) \otimes H^*_{\mathrm{CR,c}}(\mathcal{Y}) \cong H^*_{CR,c-vert}(\mathcal{Y} \times \mathcal{Y}),$$

where the right-hand side denotes cohomology with compact vertical support (i.e. in the second factor). Under this isomorphism, the class of the diagonal $[\Delta]$ in $H^*_{CR,c-vert}(\mathcal{Y} \times \mathcal{Y})$ is given by $\sum_{i\in I} T_i \otimes T^i$. □

*Remark 4.6.* — The notion above of the compactly supported quantum connection and solution were described previously in [22, Section 2.5]. We expect $\nabla^{\mathcal{Y}}$ and $\nabla^{\mathcal{Y},\mathrm{c}}$ to be related to the conjectures of [7].

### 4.2. Narrow quantum $D$-module

We cannot define a (non-equivariant) quantum $D$-module in the case that $\mathcal{Y}$ is non-proper due to the fact that $H^*_{\mathrm{CR}}(\mathcal{Y})$ does not have a well-defined pairing. In this section we show that there is a well-defined *narrow* quantum $D$-module for $\mathcal{Y}$, defined in terms of the narrow cohomology of Section 2. We will see in Section 6.2 the geometric significance of this construction.

Proposition 4.7. — *The map $\phi : H^*_{\mathrm{CR,c}}(\mathcal{Y}) \to H^*_{\mathrm{CR}}(\mathcal{Y})$ commutes with the quantum product, quantum connection, and the fundamental solution: For $u \in H^*_{\mathrm{CR,c}}(\mathcal{Y}) \otimes P^{\mathcal{Y}}$ and $T_i \in H^*_{\mathrm{CR}}(\mathcal{Y})$*

$$T_i \bullet^{\mathcal{Y}}_{\boldsymbol{t}} \phi(u) = \phi(T_i \bullet^{\mathcal{Y}}_{\boldsymbol{t}} u); \tag{4.9}$$

$$\nabla^{\mathcal{Y}}_i \phi(u) = \phi(\nabla^{\mathcal{Y},\mathrm{c}}_i u); \tag{4.10}$$

$$L^{\mathcal{Y}}(\boldsymbol{t}, z)\phi(u) = \phi(L^{\mathcal{Y},\mathrm{c}}(\boldsymbol{t}, z)u). \tag{4.11}$$

*Proof.* — To prove (4.9), it suffices to show that for $T^j, \beta \in H^*_{\mathrm{CR,c}}(\mathcal{Y})$

$$\langle T_i \bullet^{\mathcal{Y}}_{\boldsymbol{t}} \phi(\beta), T^j\rangle^{\mathcal{Y}} = \langle \phi(T_i \bullet^{\mathcal{Y}}_{\boldsymbol{t}} \beta), T^j\rangle^{\mathcal{Y}}.$$

The left hand side is equal to $\langle\langle T_i, \phi(\beta), T^j\rangle\rangle^{\mathcal{Y}}$. By the fact that $\phi$ commutes with pullback and (2.3), we note that

$$ev_2^*(\phi(\beta)) \cup ev_{3\mathrm{c}}^*(T^j) = \phi(ev_{2\mathrm{c}}^*(\beta)) \cup ev_{3\mathrm{c}}^*(T^j) = ev_{2\mathrm{c}}^*(\beta) \cup ev_{3\mathrm{c}}^*(T^j).$$

This implies that

$$\langle\langle T_i, \phi(\beta), T^j\rangle\rangle^{\mathcal{Y}} = \langle\langle T_i, \beta, T^j\rangle\rangle^{\mathcal{Y}}.$$

Again by (2.3),

$$\langle \phi(T_i \bullet^{\mathcal{Y}}_{\boldsymbol{t}} \beta), T^j\rangle^{\mathcal{Y}} = \langle \phi(T^j), T_i \bullet^{\mathcal{Y}}_{\boldsymbol{t}} \beta\rangle^{\mathcal{Y}},$$

but by an identical argument as above, this is given by

$$\langle\langle T_i, \beta, \phi(T^j)\rangle\rangle^{\mathcal{Y}} = \langle\langle T_i, \beta, T^j\rangle\rangle^{\mathcal{Y}}.$$

Thus the two are equal. Formula (4.10) follows immediately from (4.9), and (4.11) uses a similar argument. □

Define a $z$-sesquilinear pairing $S^{\mathcal{Y},\mathrm{nar}}$ on $H^*_{\mathrm{CR,nar}}(\mathcal{Y}) \otimes P^{\mathcal{Y}}[z, z^{-1}]$ by

$$S^{\mathcal{Y},\mathrm{nar}}(u(z), v(z)) := (2\pi\sqrt{-1}z)^{\dim(\mathcal{Y})}\langle u(-z), v(z)\rangle^{\mathcal{Y},\mathrm{nar}}.$$

Corollary 4.8. — *For any $\boldsymbol{t} \in H^*_{\mathrm{CR}}(\mathcal{Y})$, the narrow state space is closed under the quantum product $\bullet^{\mathcal{Y}}_{\boldsymbol{t}}$. The quantum connection $\nabla^{\mathcal{Y}}$ and solution $L^{\mathcal{Y}}(\boldsymbol{t}, z)$ preserve $H^*_{\mathrm{CR,nar}}(\mathcal{Y})$. The pairing $S^{\mathcal{Y},\mathrm{nar}}$ is flat with respect to $\nabla^{\mathcal{Y}}$, i.e.*

$$\partial_i S^{\mathcal{Y},\mathrm{nar}}(u, v) = S^{\mathcal{Y},\mathrm{nar}}(\nabla^{\mathcal{Y}}_i u, v) + S^{\mathcal{Y}}(u, \nabla^{\mathcal{Y}}_i v).$$

*Finally, for $\alpha, \beta \in H^*_{\mathrm{CR,nar}}(\mathcal{Y})$,*

$$\langle L^{\mathcal{Y}}(\boldsymbol{t}, -z)\alpha, L^{\mathcal{Y}}(\boldsymbol{t}, z)\beta\rangle^{\mathcal{Y},\mathrm{nar}} = \langle \alpha, \beta\rangle^{\mathcal{Y},\mathrm{nar}}.$$

*Proof.* — The first two claims are immediate from (4.9), (4.10), (4.11), and the definition of $H^*_{\mathrm{CR,nar}}(\mathcal{Y})$ as the image of $\phi$. The last two claims follow from the same equations together with Proposition 4.5. □

With this we can define

Definition 4.9. — *The* narrow *quantum $D$-module of $\mathcal{Y}$ is defined to be*

$$QDM_{\mathrm{nar}}(\mathcal{Y}) := (H^*_{\mathrm{CR,nar}}(\mathcal{Y}) \otimes P^{\mathcal{Y}}[z, z^{-1}], \nabla^{\mathcal{Y}}, S^{\mathcal{Y},\mathrm{nar}})$$

*Note that the coefficients ring $P^{\mathcal{Y}}$ is* not *restricted to just the dual coordinates of the narrow cohomology.*

### 4.3. Integral structure

Identical considerations to Section 3.4.1 in the non-proper case allow one to define dual integral lattices in $\ker(\nabla^{\mathcal{Y}})$ and $\ker(\nabla^{\mathcal{Y},c})$, compatible with the Dubrovin connection.

Assumption 4.10. — *The compactly supported Chern character*

$$\widetilde{\mathrm{ch}}^{\mathrm{c}} : D_{\mathrm{c}}(\mathcal{Y}) \longrightarrow H^*_{\mathrm{CR,c}}(\mathcal{Y})$$

*is given as Definition B.4 of the appendix. Assume that $H^*_{\mathrm{CR,c}}(\mathcal{Y})$ is spanned by the image of $\widetilde{\mathrm{ch}}^{\mathrm{c}}$ and that $H^*_{\mathrm{CR}}(\mathcal{Y})$ is spanned by the image of*

$$\widetilde{\mathrm{ch}} : D(\mathcal{Y}) \longrightarrow H^*_{\mathrm{CR}}(\mathcal{Y}).$$

Importantly, Assumption 4.10 holds if $\mathcal{Y}$ is the total space of a vector bundle $\mathcal{E}$ on $\mathcal{X}$ and $H^*_{\mathrm{CR}}(\mathcal{X})$ is spanned by the image of $\widetilde{\mathrm{ch}}$. Let $F$ be an object in $D_{\mathrm{c}}(\mathcal{Y})$, assume $F$ can be represented by a complex $F^\bullet$ which is exact outside a proper substack $\mathcal{X}$. Define $\boldsymbol{s}^{\mathcal{Y},\mathrm{c}}(F)(\boldsymbol{t},z)$ to be

$$\frac{1}{(2\pi\sqrt{-1})^{\dim(\mathcal{Y})}} L^{\mathcal{Y},\mathrm{c}}(\boldsymbol{t},z) z^{-Gr} z^{\rho(\mathcal{Y})} \widehat{\Gamma}_{\mathcal{Y}} \cup_{I\mathcal{Y}} ((2\pi\sqrt{-1})^{\deg_0/2} I^*(\widetilde{\mathrm{ch}}^{\mathrm{c}}(F))),$$

where $\widetilde{\mathrm{ch}}^{\mathrm{c}}(F) = \widetilde{\mathrm{ch}}^{\mathrm{c}}(F^\bullet)$ is given by Definition B.4 and $\cup_{I\mathcal{Y}}$ denotes the ordinary cup product on $H^*(I\mathcal{Y})$. Similarly, for $F \in D(\mathcal{Y})$, define $\boldsymbol{s}^{\mathcal{Y}}(F)(\boldsymbol{t},z)$ to be

$$\frac{1}{(2\pi\sqrt{-1})^{\dim(\mathcal{Y})}} L^{\mathcal{Y}}(\boldsymbol{t},z) z^{-Gr} z^{\rho(\mathcal{Y})} \widehat{\Gamma}_{\mathcal{Y}} \cup_{I\mathcal{Y}} ((2\pi\sqrt{-1})^{\deg_0/2} I^*(\widetilde{\mathrm{ch}}(F))).$$

We obtain lattices

$$\{\boldsymbol{s}^{\mathcal{Y},\mathrm{c}}(F)(\boldsymbol{t},z) \,|\, F \in D_{\mathrm{c}}(\mathcal{Y})\} \quad \text{and} \quad \{\boldsymbol{s}^{\mathcal{Y}}(F)(\boldsymbol{t},z) \,|\, F \in D(\mathcal{Y})\},$$

which are dual with respect to the pairing $S^{\mathcal{Y}}$. Proposition 3.19 holds in this context by the same argument. Namely,

$$S^{\mathcal{Y}}(\boldsymbol{s}^{\mathcal{Y}}(F)(\boldsymbol{t},z), \boldsymbol{s}^{\mathcal{Y},\mathrm{c}}(F')(\boldsymbol{t},z)) = e^{\pi\sqrt{-1}\dim(\mathcal{Y})}\chi(F',F).$$

See [22, Section 2.5] for a similar description.

By Proposition B.5 the orbifold Chern character map $\widetilde{\mathrm{ch}} : D(\mathcal{Y}) \to H^*_{\mathrm{CR}}(\mathcal{Y})$ maps $D_{\mathrm{c}}(\mathcal{Y})$ to $H^*_{\mathrm{CR,nar}}(\mathcal{Y})$. Therefore, given an object $F$ in $D_{\mathrm{c}}(\mathcal{Y})$, define $\boldsymbol{s}^{\mathcal{Y},\mathrm{nar}}(F)(\boldsymbol{t},z)$ as

$$\frac{1}{(2\pi\sqrt{-1})^{\dim(\mathcal{Y})}} L^{\mathcal{Y}}(\boldsymbol{t},z) z^{-Gr} z^{\rho(\mathcal{Y})} \widehat{\Gamma}_{\mathcal{Y}} \cup_{I\mathcal{Y}} ((2\pi\sqrt{-1})^{\deg_0/2} I^*(\widetilde{\mathrm{ch}}(F))).$$

Definition 4.11. — *Define the integral structure for $QDM_{\mathrm{nar}}(\mathcal{Y})$ to be*

$$\{\boldsymbol{s}^{\mathcal{Y},\mathrm{nar}}(F)(\boldsymbol{t},z) | F \in D_{\mathrm{c}}(\mathcal{Y})\}.$$

*This set forms a lattice in* $\ker(\nabla^{\mathcal{Y}}|_{H^*_{\mathrm{CR,nar}}(\mathcal{Y})})$

## 5. Equivariant Euler twistings

In many cases, twisted invariants of a vector bundle $\mathcal{E} \to \mathcal{X}$ are closely related to Gromov–Witten invariants of both the total space and a corresponding complete intersection. In this section we recall the connections to each, with a particular focus on the relationship to the *non-equivariant* Gromov–Witten theory of the total space of $\mathcal{E}^\vee$, which has not been studied as thoroughly as the complete intersection.

Let $\mathcal{E} \to \mathcal{X}$ be a convex vector bundle. Let

$$j : \mathcal{Z} \longrightarrow \mathcal{X}$$

denote a smooth sub-variety of $\mathcal{Z}$ defined by the vanishing of a regular section of $\mathcal{E}$. Let $T = \mathbb{C}^*$ act on $\mathcal{E}$ by scaling in the fiber direction, with equivariant parameter $\lambda$.

### 5.1. Subvarieties

Choose $s_k$ such that $\boldsymbol{s}(\mathcal{E}) = e_\lambda(\mathcal{E})$, the equivariant Euler characteristic:

$$s_0 = \ln(\lambda),\ s_k = (-1)^{k-1}(k-1)!/\lambda^k \text{ for } k > 0. \tag{5.1}$$

In this case, the genus-zero $\boldsymbol{s}$-twisted invariants with respect to $\mathcal{E}$ are related to invariants of the local complete intersection subvariety $\mathcal{Z}$ cut out by a generic section of $\mathcal{E}$ by the so-called *quantum Lefschetz principle* ([14, 31]). One way of phrasing this is the following:

Proposition 5.1 ([23, Proposition 2.4]). — *Let $L^{e_\lambda(\mathcal{E})}(\boldsymbol{t}, z)$ denote the fundamental solution of the equivariant twisted theory of $\mathcal{E}$ after specializing parameters as in* (5.1)*. Then the non-equivariant limit*

$$L^{e(\mathcal{E})}(\boldsymbol{t}, z) := \lim_{\lambda \mapsto 0} L^{e_\lambda(\mathcal{E})}(\boldsymbol{t}, z)$$

*is well defined. Furthermore, for $\alpha \in H^*_{\mathrm{CR}}(\mathcal{X})$,*

$$j^*(L^{e(\mathcal{E})}(\boldsymbol{t}, z)\alpha) = L^{\mathcal{Z}}(j^*(\boldsymbol{t}), z) j^*(\alpha).$$

### 5.2. The total space

On the other hand, let $\mathcal{Y}$ denote the total space of $\mathcal{E}^\vee$. One can consider the equivariant Gromov–Witten invariants of $\mathcal{Y}$ with respect to the torus action described above. We will denote the equivariant Gromov–Witten invariants by $\langle \alpha_1 \psi^{b_1}, \ldots, \alpha_n \psi^{b_n} \rangle^{\mathcal{Y}_T}$. These take values in $H^*_T(\mathrm{pt}) = \mathbb{C}[\lambda]$, the

equivariant cohomology of a point. Let $\pi : \mathcal{Y} \to \mathcal{X}$ denote the projection. If we specialize the twisted parameters to

$$s'_0 = -\ln(-\lambda),\ s'_k = (k-1)!/\lambda^k \text{ for } k > 0 \tag{5.2}$$

then $\boldsymbol{s}'(\mathcal{E}^\vee) = e_\lambda^{-1}(\mathcal{E}^\vee)$. In this case by (virtual) Atiyah–Bott localization [20], the $e_\lambda^{-1}(\mathcal{E}^\vee)$-twisted invariants compute the equivariant invariants of $\mathcal{Y}$ after inverting the equivariant parameter. Given $\alpha_1, \dots, \alpha_n \in H^*_{\mathrm{CR},T}(\mathcal{X})$ and $b_1, \dots, b_n \in \mathbb{Z}_{\geqslant 0}$, we have

$$\langle \alpha_1 \psi^{b_1}, \dots, \alpha_n \psi^{b_n} \rangle_{h,n,d}^{e_\lambda^{-1}(\mathcal{E}^\vee)} := \int_{[\overline{\mathscr{M}}_{h,n}(\mathcal{X},d)]^{vir}} \frac{\prod_{i=1}^n ev_i^*(\alpha_i) \cup \psi_i^{b_i}}{e_\lambda(\mathbb{R}\pi_* f^*(\mathcal{E})^\vee)}.$$

Proposition 5.2. — *Let $L^{e_\lambda^{-1}(\mathcal{E}^\vee)}(\boldsymbol{t}, z)$ denote the fundamental solution of the equivariant twisted theory of $\mathcal{E}^\vee$ after specializing parameters as in* (5.2)*. Then the non-equivariant limit*

$$L^{e^{-1}(\mathcal{E}^\vee)}(\boldsymbol{t}, z) := \lim_{\lambda \mapsto 0} L^{e_\lambda^{-1}(\mathcal{E}^\vee)}(\boldsymbol{t}, z)$$

*is well defined. Furthermore, for $\alpha \in H^*_{\mathrm{CR}}(\mathcal{X})$,*

$$\pi^*(L^{e^{-1}(\mathcal{E}^\vee)}(\boldsymbol{t}, z)\alpha) = L^{\mathcal{Y}}(\pi^*(\boldsymbol{t}), z)\pi^*(\alpha).$$

*Proof.* — By [18, Section 4] using the virtual localization formula of [20], the equivariant Gromov–Witten invariants of $\mathcal{Y}$ may be calculated as twisted invariants of $\mathcal{X}$:

$$\langle \alpha_1 \psi^{b_1}, \dots, \alpha_n \psi^{b_n} \rangle_{h,n,d}^{e_\lambda^{-1}(\mathcal{E}^\vee)} = \langle \alpha_1 \psi^{b_1}, \dots, \alpha_n \psi^{b_n} \rangle_{h,n,d}^{\mathcal{Y}_T} \tag{5.3}$$

where $\mathcal{Y}_T$ denotes the $T$-equivariant Gromov–Witten theory of $\mathcal{Y}$. By the (non-virtual) localization theorem, the $e_\lambda^{-1}(\mathcal{E}^\vee)$-twisted pairing agrees with the equivariant pairing on $\mathcal{Y}$:

$$\langle \alpha, \beta \rangle^{e_\lambda^{-1}(\mathcal{E}^\vee)} = \langle \pi^*(\alpha), \pi^*(\beta) \rangle^{\mathcal{Y}_T}. \tag{5.4}$$

Thus if we are given a basis $\{T_i\}_{i \in I}$ for $\mathcal{X}$ and a dual basis $\{T^i\}_{i \in I}$ with respect to the $e_\lambda^{-1}(\mathcal{E}^\vee)$-twisted pairing, then $\{\pi^*(T_i)\}_{i \in I}$ and $\{\pi^*(T^i)\}_{i \in I}$ will be dual bases with respect to the equivariant pairing on $\mathcal{Y}$.

By comparing term-by-term, we conclude that the fundamental solutions agree:

$$\pi^*(L^{e_\lambda^{-1}(\mathcal{E}^\vee)}(\boldsymbol{t}, z)\alpha) = L^{\mathcal{Y}_T}(\pi^*(\boldsymbol{t}), z)\pi^*(\alpha). \tag{5.5}$$

Because we assume $\mathcal{E}$ is convex, by Lemma 4.2 the genus-zero evaluation maps are proper and the non-equivariant limit of the right hand side is well defined.

If we pull back the left hand side via the map $i : \mathcal{X} \to \mathcal{Y}$ we recover $L^{e_\lambda^{-1}(\mathcal{E}^\vee)}(\boldsymbol{t}, z)\alpha$, which proves the first statement. Taking the non-equivariant limit of (5.5) finishes the proof. □

In Proposition 5.6, we describe the non-equivariant limit of the $e_\lambda^{-1}(\mathcal{E}^\vee)$-twisted quantum product more explicitly. To our knowledge this description is new. It is similar (or more accurately dual) in spirit to the $\bullet_{\boldsymbol{t}}^Z$ product of [28].

Lemma 5.3. — *Assume that $\mathcal{E}$ is convex. For a stable map*

$$f : (\mathcal{C}, p_1, \dots, p_n) \longrightarrow \mathcal{X}$$

*from a genus zero $n$-marked orbi-curve $\mathcal{C}$,*

$$H^0(\mathcal{C}, f^*(\mathcal{E}^\vee)(-p_i)) = 0$$

*for any choice of $1 \leqslant i \leqslant n$.*

*Proof.* — Let $r : \mathcal{C} \to C$ be the map to the underlying coarse curve. By Remark 3.12 $\mathcal{E}$ is the pullback of a convex vector bundle $E \to X$. By [2, Theorem 1.4.1], the map $\mathcal{C} \to \mathcal{X} \to X$ factors through a map $|f| : C \to X$. Therefore $f^*\mathcal{E} = r^*|f|^*E$. We observe that

$$\begin{aligned} H^0(C, r_*(f^*(\mathcal{E}^\vee)(-p_i))) &= H^0(C, r_*(r^*|f|^*(E^\vee)(-p_i))) \\ &= H^0(C, |f|^*E^\vee \otimes r_*(\mathcal{O}_{\mathcal{C}}(-p_i))) \\ &= H^0(C, |f|^*E^\vee \otimes \mathcal{O}_C(-p_i)), \end{aligned}$$

where the second equality is the projection formula and the third can be checked using local coordinates. On each irreducible component $C_j$ of $C$, $E \cong \sum_{l=1}^r \mathcal{O}(k_l)$ with each $k_l \geqslant 0$. From this we see that the only global sections of $|f|^*(E^\vee)(-p_i)|_{C_j}$ are constant sections (possibly just the zero section) if $p_i$ is not on $C_j$, and the zero section if $p_i$ does lie on $C_j$. By an induction argument on the number of components, the only global section of $|f|^*(E^\vee)(-p_i)$ is the zero section. □

Consider the short exact sequence over the universal curve $\widetilde{\mathcal{C}}$ lying over $\overline{\mathscr{M}}_{h,n}(\mathcal{X}, d)$:

$$0 \longrightarrow f^*(\mathcal{E})^\vee(-p_i) \longrightarrow f^*(\mathcal{E})^\vee \longrightarrow f^*(\mathcal{E})^\vee|_{p_i} \longrightarrow 0.$$

Pushing forward we obtain the distinguished triangle

$$\mathbb{R}\pi_* f^*(\mathcal{E})^\vee(-p_i) \longrightarrow \mathbb{R}\pi_* f^*(\mathcal{E})^\vee \longrightarrow ev_i^*(q^*(\mathcal{E}^\vee))[1],$$

where $q : I\mathcal{X} \to \mathcal{X}$ is the natural map.

From this and the previous lemma we can rewrite the twisted invariant $\langle \alpha_1\psi^{b_1},\dots,\alpha_n\psi^{b_n}\rangle_{0,n,d}^{e_\lambda^{-1}(\mathcal{E}^\vee)}$ as

$$\int_{[\overline{\mathscr{M}}_{0,n}(\mathcal{X},d)]^{vir}} \prod_{j=1}^{n}(ev_j^*(\alpha_j)\cup\psi_j^{b_j})\cup\frac{e_\lambda(\mathbb{R}^1\pi_*f^*(\mathcal{E})^\vee(-p_i))}{ev_i^*(q^*(e_\lambda(\mathcal{E}^\vee)))}. \tag{5.6}$$

The above expression motivates the following definition.

Definition 5.4. — *Fix $i$ between 1 and $n$. Given $\alpha_1,\dots,\alpha_n\in H^*_{\mathrm{CR},T}(\mathcal{X})$ and $b_1,\dots,b_n\in\mathbb{Z}_{\geqslant 0}$, define $\langle \alpha_1\psi^{b_1},\dots,\widetilde{\alpha_i\psi^{b_i}},\dots,\alpha_n\psi^{b_n}\rangle_{0,n,d}^{e^{-1}(\mathcal{E}^\vee)}$ to be the integral*

$$\int_{[\overline{\mathscr{M}}_{0,n}(\mathcal{X},d)]^{vir}} \prod_{j=1}^{n}(ev_j^*(\alpha_j)\cup\psi_j^{b_j})\cup e(\mathbb{R}^1\pi_*f^*(\mathcal{E})^\vee(-p_i)).$$

Note that $\mathbb{R}^1\pi_*f^*(\mathcal{E})^\vee(-p_i)$ may be represented by a vector bundle by Lemma 5.3. We now define a new quantum product on $\mathcal{X}$:

Definition 5.5. — *Let $\{\overline{T}_i\}$ be a basis for $H^*_{\mathrm{CR}}(\mathcal{X})$ and let $\{\overline{T}^i\}$ denote the dual basis. For $\alpha$ and $\beta\in H^*_{\mathrm{CR}}(\mathcal{X})$, define*

$$\alpha\bullet_{\boldsymbol{t}}^{\mathcal{Y}\to\mathcal{X}}\beta := \langle\!\langle \alpha,\beta,\widetilde{\overline{T}_i}\rangle\!\rangle^{e^{-1}(\mathcal{E}^\vee)}\overline{T}^i.$$

Proposition 5.6. — *The pullback*

$$\pi^*: H^*_{\mathrm{CR}}(\mathcal{X})\otimes P^{\mathcal{X}}\longrightarrow H^*_{\mathrm{CR}}(\mathcal{Y})\otimes P^{\mathcal{Y}}$$

*is a ring isomorphism from the $\bullet_{\boldsymbol{t}}^{\mathcal{Y}\to\mathcal{X}}$-product on $\mathcal{X}$ to the quantum product $\bullet_{\boldsymbol{t}}^{\mathcal{Y}}$ on $\mathcal{Y}$.*

*Proof.* — We first show that in the non-equivariant limit, the product $\bullet_{\boldsymbol{t}}^{e_\lambda^{-1}(\mathcal{E}^\vee)}$ specializes to $\bullet_{\boldsymbol{t}}^{\mathcal{Y}\to\mathcal{X}}$.

The bases $\{\overline{T}_i\}$ and $\{\overline{T}^i\}$ give dual bases with respect to the equivariant pairing on $H^*_{\mathrm{CR},T}(\mathcal{X})$ via the inclusion $H^*_{\mathrm{CR}}(\mathcal{X})\subset H^*_{\mathrm{CR},T}(\mathcal{X})\cong H^*_{\mathrm{CR}}(\mathcal{X})\otimes\mathbb{C}[\lambda]$. Define $T_i := e_\lambda(\mathcal{E}^\vee)\cup\overline{T}_i$ and $T^i:=\overline{T}^i$. Note that with respect to the $e_\lambda^{-1}(\mathcal{E}^\vee)$-twisted pairing, $\{T_i\}$ and $\{T^i\}$ are dual bases. Therefore, for $\alpha,\beta\in H^*_{\mathrm{CR}}(\mathcal{X})$,

$$\begin{aligned}\alpha\bullet_{\boldsymbol{t}}^{e_\lambda^{-1}(\mathcal{E}^\vee)}\beta &= \sum_{i\in I}\langle\!\langle\alpha,\beta,T_i\rangle\!\rangle^{e_\lambda^{-1}(\mathcal{E}^\vee)}T^i\\ &= \sum_{i\in I}\langle\!\langle\alpha,\beta,e_\lambda(\mathcal{E}^\vee)\cup\overline{T}_i\rangle\!\rangle^{e_\lambda^{-1}(\mathcal{E}^\vee)}\overline{T}^i.\end{aligned} \tag{5.7}$$

By (5.6) and Remark 2.14, the factor of $e_\lambda(\mathcal{E}^\vee)$ in the third insertion cancels with part of the twisted virtual class and this expression becomes

$$\sum_{i\in I}\sum_{d\in\mathrm{Eff}}\sum_{k\geqslant 0}\frac{T^i}{k!}\int_{[\overline{\mathscr{M}}_{0,k+3}(\mathcal{X},d)]^{vir}} ev_1^*(\alpha)\cup ev_2^*(\beta)\cup ev_3^*(T_i) \\ \cup\prod_{j=4}^{k+3} ev_j^*(\boldsymbol{t})\cup e_\lambda(\mathbb{R}^1\pi_* f^*(\mathcal{E})^\vee(-p_3)).$$

In the nonequivariant limit $\lambda\mapsto 0$, this is exactly $\alpha\bullet_{\boldsymbol{t}}^{\mathcal{Y}\to\mathcal{X}}\beta$.

On the other hand, by (5.3) and (5.4), if we pull back (5.7) by $\pi$ we obtain

$$\sum_{i\in I}\langle\!\langle\pi^*(\alpha),\pi^*(\beta),\pi^*(T_i)\rangle\!\rangle^{\mathcal{Y}_T}\pi^*(T^i),$$

which may be rewritten as

$$\sum_{i\in I}\sum_{d\in\mathrm{Eff}}\sum_{k\geqslant 0}\frac{1}{k!}I_*\circ ev_{3*}\Bigg(ev_1^*\pi^*(\alpha)\cup ev_2^*\pi^*(\beta) \\ \cup\prod_{j=4}^{k+3} ev_j^*\pi^*(\boldsymbol{t})\cap[\overline{\mathscr{M}}_{0,k+3}(\mathcal{Y}_T,d)]^{vir}\Bigg),$$

where $[\overline{\mathscr{M}}_{0,k+3}(\mathcal{Y}_T,d)]^{vir}$ is the equivariant virtual fundamental class. In the non-equivariant limit we recover $\pi^*(\alpha)\bullet_{\pi^*(\boldsymbol{t})}^{\mathcal{Y}}\pi^*(\beta)$ by (4.2).

Combining the above results we conclude

$$\begin{aligned}\pi^*(\alpha\bullet_{\boldsymbol{t}}^{\mathcal{Y}\to\mathcal{X}}\beta) &= \pi^*(\lim_{\lambda\mapsto 0}\alpha\bullet_{\boldsymbol{t}}^{e_\lambda^{-1}(\mathcal{E}^\vee)}\beta)\\ &= \lim_{\lambda\mapsto 0}\pi^*(\alpha\bullet_{\boldsymbol{t}}^{e_\lambda^{-1}(\mathcal{E}^\vee)}\beta)\\ &= \lim_{\lambda\mapsto 0}(\pi^*(\alpha)\bullet_{\pi^*(\boldsymbol{t})}^{\mathcal{Y}_T}\pi^*(\beta))\\ &= \pi^*(\alpha)\bullet_{\pi^*(\boldsymbol{t})}^{\mathcal{Y}}\pi^*(\beta).\end{aligned}$$

□

## 6. Quantum Serre duality

In this section we use the definition of the compactly supported quantum connection and the narrow quantum $D$-module to reframe quantum Serre duality in two new ways. First we relate the compactly supported quantum connection of $\mathcal{Y}$ to the quantum connection of $\mathcal{Z}$. Second, we show there is an *isomorphism* between the narrow quantum $D$-module of

$\mathcal{Y}$ and the ambient quantum $D$-module of $\mathcal{Z}$. In both cases we show these correspondences to be compatible with the integral structures.

In all of this section we assume:

- The vector bundle $\mathcal{E} \to \mathcal{X}$ is convex;
- Assumption 3.13;
- Assumption 3.20;
- The stack $\mathcal{X}$ has the resolution property.

### 6.1. Compactly supported quantum Serre duality

In [24], it is observed (Remark 3.17) that the $e(\mathcal{E})$-twisted quantum $D$-module can be viewed as the quantum $D$-module with compact support of the total space $\mathcal{E}^\vee$. We make this observation precise by relating the Euler twisted fundamental solution $L^{e(\mathcal{E})}(\boldsymbol{t}, z)$ with the compactly supported fundamental solution $L^{\mathcal{Y},\mathrm{c}}(\boldsymbol{t}, z)$. This then allows us to directly relate the compactly supported fundamental solution of $\mathcal{Y}$ with the ambient fundamental solution $L^{\mathcal{Z}}(\boldsymbol{t}, z)$, obtaining a new perspective on quantum Serre duality. In Remark 6.6 we explain that the results of this section should be viewed as adjoint to a similar theorem in [24]. The majority of the techniques of this section appeared already in [24], it is mainly the perspective which is new. This particular formulation of quantum Serre duality, described in Theorem 6.5, is convenient for then proving the more refined statement in Section 6.2.

Recall the definition of $P^{\mathcal{X}}$ given in Notation 3.5. Consider the map $\widehat{f}^\lambda(\boldsymbol{t}) : H^*_{\mathrm{CR},T}(\mathcal{X}) \to H^*_{\mathrm{CR},T}(\mathcal{X}) \otimes P^{\mathcal{X}}[\![\lambda]\!]$ given by

$$\widehat{f}^\lambda(\boldsymbol{t}) := \sum_{i \in I} \langle\!\langle e_\lambda(\mathcal{E}^\vee), T_i \rangle\!\rangle^{e_\lambda(\mathcal{E}^\vee)} T^i \tag{6.1}$$

where $\{T_i\}$ and $\{T^i\}$ as dual bases with respect to the $e_\lambda^{-1}(\mathcal{E}^\vee)$-pairing.

Proposition 6.1. — *The map $\widehat{f}^\lambda(\boldsymbol{t})/e_\lambda(\mathcal{E}^\vee)$ has a well defined non-equivariant limit $\widehat{f}^{\mathcal{X}}(\boldsymbol{t})$ given by*

$$\sum_{i \in I} \langle\!\langle \widetilde{1}, \overline{T}_i \rangle\!\rangle^{e^{-1}(\mathcal{E}^\vee)} \overline{T}^i,$$

*where $\{\overline{T}_i\}$ and $\{\overline{T}^i\}$ as dual bases with respect to the usual pairing on $H^*_{\mathrm{CR}}(\mathcal{X})$.*

*Proof.* — Due to the difference in the usual pairing on $H^*_{\mathrm{CR},T}(\mathcal{X})$ and the $e_\lambda^{-1}(\mathcal{E}^\vee)$-twisted pairing, $\widehat{f}^\lambda(\boldsymbol{t})/e_\lambda(\mathcal{E}^\vee)$ can be expressed as

$$\sum_{i\in I}\langle\!\langle e_\lambda(\mathcal{E}^\vee),\overline{T}_i\rangle\!\rangle^{e_\lambda^{-1}(\mathcal{E}^\vee)}\overline{T}^i.$$

The claim then follows by the same argument as in Proposition 5.6. □

Proposition 6.2. — *Let* $\overline{f}^{\mathcal{X}}(\boldsymbol{t}) = \widehat{f}^{\mathcal{X}}(\boldsymbol{t}) - \pi\sqrt{-1}c_1(\mathcal{E})$. *The isomorphism*

$$i^{\mathrm{c}}_*: H^*_{\mathrm{CR}}(\mathcal{X}) \longrightarrow H^*_{\mathrm{CR,c}}(\mathcal{Y})$$

*identifies the connections* $(\overline{f}^{\mathcal{X}}\circ i^*)^*(\nabla^{e(\mathcal{E})})$ *and* $\nabla^{\mathcal{Y},\mathrm{c}}$. *Furthermore,*

$$L^{\mathcal{Y},\mathrm{c}}(\boldsymbol{t},z)i^{\mathrm{c}}_*(\beta) = i^{\mathrm{c}}_*(L^{e(\mathcal{E})}(\overline{f}^{\mathcal{X}}(i^*(\boldsymbol{t})),z)e^{-\pi\sqrt{-1}c_1(\mathcal{E})/z}\beta) \tag{6.2}$$

*for all* $\beta\in H^*_{\mathrm{CR}}(\mathcal{X})$.

*Proof.* — The first claim follows from the second. By Theorem A.3, the symplectic transformation

$$\Delta^\diamond := e^{\pi\sqrt{-1}c_1(\mathcal{E})/z}/e_\lambda(\mathcal{E}^\vee)$$

maps $\mathscr{L}^{e_\lambda^{-1}(\mathcal{E}^\vee)}$ to $\mathscr{L}^{e_\lambda(\mathcal{E})}$.

By (A.3) and (A.5), $\Delta^\diamond z\partial_{-e_\lambda(\mathcal{E}^\vee)}J^{e_\lambda^{-1}(\mathcal{E}^\vee)}(\boldsymbol{t},-z)$ is a $\mathbb{C}[z]$-linear combination of derivatives of $J^{e_\lambda(\mathcal{E})}(\widehat{\boldsymbol{t}},-z)$ at some point $\widehat{\boldsymbol{t}}$. Observe that

$$\begin{aligned}
&\Delta^\diamond z\partial_{-e(\mathcal{E}^\vee)}J^{e_\lambda^{-1}(\mathcal{E}^\vee)}(\boldsymbol{t},-z)\\
&\quad=\frac{1}{e_\lambda(\mathcal{E}^\vee)}(1+\pi\sqrt{-1}c_1(\mathcal{E})/z)(-e_\lambda(\mathcal{E}^\vee)z\\
&\qquad\qquad+\sum_{i\in I}\langle\!\langle e_\lambda(\mathcal{E}^\vee),T_i\rangle\!\rangle^{e_\lambda^{-1}(\mathcal{E}^\vee)}T^i)+\mathcal{O}(1/z)\\
&\quad=-z+\frac{1}{e_\lambda(\mathcal{E}^\vee)}\left(\sum_{i\in I}\langle\!\langle e_\lambda(\mathcal{E}^\vee),T_i\rangle\!\rangle^{e_\lambda^{-1}(\mathcal{E}^\vee)}T^i\right)-\pi\sqrt{-1}c_1(\mathcal{E})+\mathcal{O}(1/z).
\end{aligned}$$

From this we can see that $\Delta^\diamond z\partial_{-e(\mathcal{E}^\vee)}J^{e_\lambda^{-1}(\mathcal{E}^\vee)}(\boldsymbol{t},-z)$ is equal to $J^{e_\lambda(\mathcal{E})}(\overline{f}^\lambda(\boldsymbol{t}),-z)$ where

$$\begin{aligned}
\overline{f}^\lambda(\boldsymbol{t}) &= \frac{1}{e_\lambda(\mathcal{E}^\vee)}\left(\sum_{i\in I}\langle\!\langle e_\lambda(\mathcal{E}^\vee),T_i\rangle\!\rangle^{e_\lambda^{-1}(\mathcal{E}^\vee)}T^i\right)-\pi\sqrt{-1}c_1(\mathcal{E})\\
&= \widehat{f}^\lambda(\boldsymbol{t})/e_\lambda(\mathcal{E}^\vee)-\pi\sqrt{-1}c_1(\mathcal{E}).
\end{aligned}$$

This implies that

$$\Delta^\diamond(T_{J^{e_\lambda^{-1}(\mathcal{E}^\vee)}(\boldsymbol{t},-z)}\mathscr{L}^{e_\lambda^{-1}(\mathcal{E}^\vee)}) = T_{J^{e_\lambda(\mathcal{E})}(\overline{f}^\lambda(\boldsymbol{t}),-z)}\mathscr{L}^{e_\lambda(\mathcal{E})}$$

and so by (A.6), $\Delta_+(\partial_\alpha J^{e_\lambda^{-1}(\mathcal{E}^\vee)}(\boldsymbol{t},-z))$ is a $\mathbb{C}[z]$-linear combination of derivatives of $J^{e_\lambda(\mathcal{E})}(\bar{\boldsymbol{t}},-z)$ evaluated at $\bar{\boldsymbol{t}} = \bar{f}^\lambda(\boldsymbol{t})$. Comparing $z^0$-coefficients, we see

$$\frac{e^{-\pi\sqrt{-1}c_1(\mathcal{E})/z}}{e_\lambda(\mathcal{E}^\vee)}\partial_{e_\lambda(\mathcal{E}^\vee)\cup\alpha}J^{e_\lambda^{-1}(\mathcal{E}^\vee)}(\boldsymbol{t},z) = \partial_\alpha J^{e_\lambda(\mathcal{E})}(\bar{\boldsymbol{t}},z)|_{\hat{\boldsymbol{t}}=\hat{f}(\boldsymbol{t})},$$

where we have replaced $-z$ by $z$. By (A.4), this equation can be written as

$$\frac{e^{-\pi\sqrt{-1}c_1(\mathcal{E})/z}}{e_\lambda(\mathcal{E}^\vee)}(L^{e_\lambda^{-1}(\mathcal{E}^\vee)}(\boldsymbol{t},z)^{-1}e_\lambda(\mathcal{E}^\vee)\cup\alpha) = L^{e_\lambda(\mathcal{E})}(\bar{\boldsymbol{t}},z)^{-1}\alpha$$

or, equivalently,

$$L^{e_\lambda^{-1}(\mathcal{E}^\vee)}(\boldsymbol{t},z)(e_\lambda(\mathcal{E}^\vee)\cup\alpha) = e_\lambda(\mathcal{E}^\vee)\cup L^{e_\lambda(\mathcal{E})}(\bar{\boldsymbol{t}},z)e^{-\pi\sqrt{-1}c_1(\mathcal{E})/z}\alpha. \tag{6.3}$$

By Proposition 6.1, the non-equivariant limit of $\widehat{f}^\lambda(\boldsymbol{t})/e_\lambda(\mathcal{E}^\vee)$ exists. Then by Proposition 5.1 the right side therefore has a non-equivariant limit for $\alpha \in H^*_{\mathrm{CR},T}(\mathcal{X}) \subset H^*_{\mathrm{CR},T}(\mathcal{X}) \otimes_{R_T} S_T$.

To finish the proof, let $\alpha,\beta \in H^*_{\mathrm{CR}}(\mathcal{X})$ and consider the following:

$$\begin{aligned}\langle L^{\mathcal{Y}}(\boldsymbol{t},-z)\pi^*(\alpha), L^{\mathcal{Y},\mathrm{c}}(\boldsymbol{t},z)i^{\mathrm{c}}_*\beta\rangle^{\mathcal{Y}} &= \langle \pi^*(\alpha), i^{\mathrm{c}}_*\beta\rangle^{\mathcal{Y}}\\ &= \langle\alpha,\beta\rangle^{\mathcal{X}}\end{aligned}$$

where the first equality is (4.8) and the second is the projection formula [8]. Note that this equation completely determines $L^{\mathcal{Y},\mathrm{c}}(\boldsymbol{t},z)$ in terms of $L^{\mathcal{Y}}(\boldsymbol{t},z)$. On the other hand, we have

$$\begin{aligned}
&\langle\alpha,\beta\rangle^{\mathcal{X}}\\
&\quad= \lim_{\lambda\mapsto 0}\langle\alpha,\beta\rangle^{\mathcal{X}}\\
&\quad= \lim_{\lambda\mapsto 0}\langle\alpha, e_\lambda(\mathcal{E}^\vee)\cup\beta\rangle^{e_\lambda^{-1}(\mathcal{E}^\vee)}\\
&\quad= \lim_{\lambda\mapsto 0}\big\langle L^{e_\lambda^{-1}(\mathcal{E}^\vee)}(i^*(\boldsymbol{t}),-z)\alpha, L^{e_\lambda^{-1}(\mathcal{E}^\vee)}(i^*(\boldsymbol{t}),z)(e_\lambda(\mathcal{E}^\vee)\cup\beta)\big\rangle^{e_\lambda^{-1}(\mathcal{E}^\vee)}\\
&\quad= \lim_{\lambda\mapsto 0}\big\langle L^{e_\lambda^{-1}(\mathcal{E}^\vee)}(i^*(\boldsymbol{t}),-z)\alpha,\\
&\qquad\qquad e_\lambda(\mathcal{E}^\vee)\cup L^{e_\lambda(\mathcal{E})}(\bar{f}^\lambda(i^*(\boldsymbol{t})),z)e^{-\pi\sqrt{-1}c_1(\mathcal{E})/z}\beta\big\rangle^{e_\lambda^{-1}(\mathcal{E}^\vee)}\\
&\quad= \lim_{\lambda\mapsto 0}\big\langle L^{e_\lambda^{-1}(\mathcal{E}^\vee)}(i^*(\boldsymbol{t}),-z)\alpha, L^{e_\lambda(\mathcal{E})}(\bar{f}^\lambda(i^*(\boldsymbol{t})),z)e^{-\pi\sqrt{-1}c_1(\mathcal{E})/z}\beta\big\rangle^{\mathcal{X}}\\
&\quad= \big\langle L^{e^{-1}(\mathcal{E}^\vee)}(i^*(\boldsymbol{t}),-z)\alpha, L^{e(\mathcal{E})}(\bar{f}^{\mathcal{X}}(i^*(\boldsymbol{t})),z)e^{-\pi\sqrt{-1}c_1(\mathcal{E})/z}\beta\big\rangle^{\mathcal{X}}\\
&\quad= \big\langle \pi^*(L^{e^{-1}(\mathcal{E}^\vee)}(i^*(\boldsymbol{t}),-z)\alpha), i^{\mathrm{c}}_*\big(L^{e(\mathcal{E})}(\bar{f}^{\mathcal{X}}(i^*(\boldsymbol{t})),z)e^{-\pi\sqrt{-1}c_1(\mathcal{E})/z}\beta\big)\big\rangle^{\mathcal{Y}}\\
&\quad= \big\langle L^{\mathcal{Y}}(\boldsymbol{t},-z)\pi^*(\alpha), i^{\mathrm{c}}_*\big(L^{e(\mathcal{E})}(\bar{f}^{\mathcal{X}}(i^*(\boldsymbol{t})),z)e^{-\pi\sqrt{-1}c_1(\mathcal{E})/z}\beta\big)\big\rangle^{\mathcal{Y}}
\end{aligned}$$

The third equality is (3.5), the fourth is (6.3), the fifth is from the difference in the twisted and untwisted pairing, the seventh is again the projection formula and the last is Proposition 5.2.

Combining the above two chains of equalities yields (6.2). □

Definition 6.3. — *Define*

$$\Delta_+^{\mathrm{c}} := j^* \circ \pi_*^{\mathrm{c}} : H^*_{\mathrm{CR,c}}(\mathcal{Y}) \longrightarrow H^*_{\mathrm{CR,amb}}(\mathcal{Z}).$$

*Define*

$$\overline{\Delta}_+^{\mathrm{c}} := (2\pi\sqrt{-1}z)^{\mathrm{rk}(\mathcal{E})}\Delta_+^{\mathrm{c}}.$$

We will show that $\overline{\Delta}_+^{\mathrm{c}}$ is compatible with the quantum connections, integral structures and the functor

$$j^* \circ \pi_* : D(\mathcal{Y})_{\mathcal{X}} \longrightarrow D(\mathcal{Z}).$$

Lemma 6.4. — *Assume that $\mathcal{X}$ has the resolution property. Consider the functor $j^* \circ \pi_* : D(\mathcal{Y})_{\mathcal{X}} \to D(\mathcal{Z})$. The induced map on cohomology from $H^*_{\mathrm{CR,c}}(\mathcal{Y})$ to $H^*_{\mathrm{CR,amb}}(\mathcal{Z})$ is given by $\Delta_+^{\mathrm{c}}(\,\cdot \cup \mathrm{Td}(\pi^*\mathcal{E}^\vee))$, i.e.*

$$\Delta_+^{\mathrm{c}}(\widetilde{\mathrm{ch}}^{\mathrm{c}}(F) \cup \mathrm{Td}(\pi^*\mathcal{E}^\vee)) = \widetilde{\mathrm{ch}} \circ j^* \circ \pi_*(F) \tag{6.4}$$

*for all $F \in D(\mathcal{Y})_{\mathcal{X}}$, where $\widetilde{\mathrm{ch}}^{\mathrm{c}}$ is defined in Definition B.4.*

*Proof.* — By [5, Lemmas 4.6 and 4.8], $D(\mathcal{Y})_{\mathcal{X}}$ is strongly generated by $i_*(D(\mathcal{X}))$. Thus any element of $D(\mathcal{Y})_{\mathcal{X}}$ may be expressed, via a finite sequence of extensions, in terms of elements in $i_*(D(\mathcal{X}))$. It therefore suffices to check the statement when $F = i_*(G)$ for some $G \in D(\mathcal{X})$. By orbifold Grothendieck–Riemann–Roch [30, 31], $\widetilde{\mathrm{ch}}^{\mathrm{c}}(i_*(G)) \cup \mathrm{Td}(\pi^*\mathcal{E}^\vee) = i_*(\widetilde{\mathrm{ch}}(G))$. Then,

$$\begin{aligned}
\Delta_+^{\mathrm{c}}(\widetilde{\mathrm{ch}}^{\mathrm{c}}(i_*(G)) \cup \mathrm{Td}(\pi^*\mathcal{E}^\vee)) &= \Delta_+^{\mathrm{c}}(i_*(\widetilde{\mathrm{ch}}(G))) \\
&= j^*(\widetilde{\mathrm{ch}}(G)) \\
&= j^*(\widetilde{\mathrm{ch}}(\pi_* \circ i_* G)) \\
&= \widetilde{\mathrm{ch}}(j^* \circ \pi_* \circ i_* G)).
\end{aligned}$$

□

Theorem 6.5. — *$\overline{\Delta}_+^{\mathrm{c}}$ maps $(\mathcal{Y}, c)$-flat sections to $\mathcal{Z}$-flat sections. In particular,*

$$\overline{\Delta}_+^{\mathrm{c}} \circ L^{\mathcal{Y},\mathrm{c}}(\boldsymbol{t}, z)(\beta) = L^{\mathcal{Z},\mathrm{amb}}(j^* \circ \overline{f}^{\mathcal{X}} \circ i^*(\boldsymbol{t}), z) \circ \overline{\Delta}_+^{\mathrm{c}} \circ e^{-\pi\sqrt{-1}c_1(\pi^*\mathcal{E})/z}\beta. \tag{6.5}$$

*Furthermore it is compatible with the integral structure and the functor $j^* \circ \pi_*$, i.e., the following diagram commutes;*

$$\begin{tikzcd} D(\mathcal{Y})_{\mathcal{X}} \arrow[r, "j^*\circ\pi_*"] \arrow[d, "s^{\mathcal{Y},\mathrm{c}}"] & j^*(D(\mathcal{X})) \arrow[d, "s^{\mathcal{Z},\mathrm{amb}}"] \\ \ker(\nabla^{\mathcal{Y},\mathrm{c}}) \arrow[r, "\overline{\Delta}^{\mathrm{c}}_+"] & \ker((j^*\circ \overline{f}^{\mathcal{X}} \circ i^*)^*(\nabla^{\mathcal{Z}})). \end{tikzcd} \tag{6.6}$$

*Proof.* — The first statement is an immediate consequence of the previous proposition, the fact that $\pi^{\mathrm{c}}_*$ is the inverse of $i^{\mathrm{c}}_*$, and Proposition 5.1.

To show that

$$\overline{\Delta}^{\mathrm{c}}_+ \circ s^{\mathcal{Y},\mathrm{c}}(\boldsymbol{t}, z)(F) = s^{\mathcal{Z},\mathrm{amb}}(j^* \circ \overline{f}^{\mathcal{X}} \circ i^*(\boldsymbol{t}), z) \circ j^* \circ \pi_*(F),$$

note the following facts. First, recalling the definition of $\widehat{\Gamma}_{\mathcal{Y}}$ from Definition 3.17, we see that

$$\begin{aligned} \widehat{\Gamma}_{\mathcal{Y}} &= \pi^*\big(\widehat{\Gamma}_{\mathcal{X}}\widehat{\Gamma}(\mathcal{E}^\vee)\big) \\ &= \pi^*\left(\frac{\widehat{\Gamma}_{\mathcal{X}}}{\widehat{\Gamma}(\mathcal{E})}\widehat{\Gamma}(\mathcal{E}^\vee)\widehat{\Gamma}(\mathcal{E})\right) \\ &= \pi^*\left(\frac{\widehat{\Gamma}_{\mathcal{X}}}{\widehat{\Gamma}(\mathcal{E})}\prod_{j=1}^{\mathrm{rk}(\mathcal{E})}\Gamma(1-\rho_j)\Gamma(1+\rho_j)\right) \\ &= \pi^*\left(\frac{\widehat{\Gamma}_{\mathcal{X}}}{\widehat{\Gamma}(\mathcal{E})}\prod_{j=1}^{\mathrm{rk}(\mathcal{E})}\frac{(2\pi\sqrt{-1})e^{\pi\sqrt{-1}\rho_j}(-\rho_j)}{1-e^{2\pi\sqrt{-1}\rho_j}}\right) \\ &= \pi^*\left(e^{\pi\sqrt{-1}c_1(\mathcal{E})}\frac{\widehat{\Gamma}_{\mathcal{X}}}{\widehat{\Gamma}(\mathcal{E})}(2\pi\sqrt{-1})^{\deg_0/2}\,\mathrm{Td}(\mathcal{E}^\vee)\right) \end{aligned} \tag{6.7}$$

where $\rho_j$ are the Chern roots of $\mathcal{E}$. Second,

$$j^*\left(\frac{\widehat{\Gamma}_{\mathcal{X}}}{\widehat{\Gamma}(\mathcal{E})}\right) = \widehat{\Gamma}_{\mathcal{Z}}.$$

Observing that

$$\begin{aligned} &\Delta^{\mathrm{c}}_+(z^{-Gr}z^{\rho(\mathcal{Y})}(2\pi\sqrt{-1})^{\deg_0/2}(\,\cdot\,)) \\ &\qquad = \left(\frac{z}{2\pi\sqrt{-1}}\right)^{\mathrm{rk}(\mathcal{E})} z^{-Gr}z^{\rho(\mathcal{Z})}(2\pi\sqrt{-1})^{\deg_0/2}\Delta^{\mathrm{c}}_+(\,\cdot\,), \end{aligned}$$

by Lemma 6.4 we then have that for all $F \in D(\mathcal{Y})_{\mathcal{X}}$,

$$\left(\frac{z}{2\pi\sqrt{-1}}\right)^{\mathrm{rk}(\mathcal{E})} \Delta_+^{\mathrm{c}}\big(e^{-\pi\sqrt{-1}c_1(\pi^*\mathcal{E})/z} z^{-Gr}\widehat{\Gamma}_{\mathcal{Y}}(2\pi\sqrt{-1})^{\deg_0/2} I^*(\widetilde{\mathrm{ch}}^{\mathrm{c}}(F))\big) \tag{6.8}$$
$$= z^{-Gr}\widehat{\Gamma}_{\mathcal{Z}}(2\pi\sqrt{-1})^{\deg_0/2} I^*(\widetilde{\mathrm{ch}}(j^* \circ \pi_*(F))).$$

Finally, note that $\dim(\mathcal{Y}) = \dim(\mathcal{Z}) + 2\,\mathrm{rk}(\mathcal{E})$. The claim follows from this, (6.5), and (6.8). □

*Remark 6.6 (Relation to* [24]*).* — A very similar statement was shown in [24, Theorem 3.13] and the proof above uses the same ingredients. Indeed the statement above may be seen implicitly in the results of [24] as we explain below. Among other things, they show that the ambient quantum connection $\nabla^{\mathcal{Z}}$ is related to $\nabla^{e^{-1}(\mathcal{E}^\vee)}$ by the functor $j_*$ after a twist by $\det(\mathcal{E})[\mathrm{rk}(\mathcal{E})]$. After composing with the pullback $\pi^*$, that result is essentially the adjoint to Theorem 6.5, the observation of which almost gives a second proof of Theorem 6.5, via Proposition 4.5 and the relations in Section 4.5.

Note, however, that the statements differ further in the change of variables. One is the inverse of the other which, to the author's knowledge, is most easily seen *a-posteriori* by comparing the statements of [24, Theorem 3.13] and Theorem 6.5 above. The presentation given above implies more directly the results below involving the narrow quantum $D$-module of $\mathcal{Y}$, and the change of variables given above is designed to be useful in applications such as the LG/CY correspondence of [29].

## 6.2. Narrow quantum Serre duality

In this section we prove a variation of quantum Serre duality which gives an isomorphism between the narrow quantum $D$-module of $\mathcal{Y}$ and the ambient quantum $D$-module of $\mathcal{Z}$. An application of this theorem is given in [29].

Definition 6.7. — *Define the map* $\widehat{f}^{\mathcal{Y}} : H^*_{\mathrm{CR}}(\mathcal{Y}) \to H^*_{\mathrm{CR,nar}}(\mathcal{Y}) \otimes P^{\mathcal{Y}}$ *by*

$$\widehat{f}^{\mathcal{Y}}(\boldsymbol{t}) := \sum_{d\in \mathrm{Eff}} \sum_{k\geqslant 0} \frac{1}{k!} I_* \circ ev_{2*}(ev_1^*(e(\mathcal{E}^\vee)) \cup \prod_{j=3}^{k+2} ev_j^*(\boldsymbol{t}) \cap [\overline{\mathscr{M}}_{0,k+2}(\mathcal{Y}, d)]^{vir}).$$

*Since the evaluation maps are proper and* $e(\mathcal{E}^\vee) \in H^*_{\mathrm{CR,nar}}(\mathcal{Y})$, $\widehat{f}^{\mathcal{Y}}(\boldsymbol{t})$ *will also lie in the narrow cohomology.*

Lemma 6.8. — *In* $H^*_{\mathrm{CR}}(\mathcal{Y}) \otimes P^{\mathcal{Y}}$,

$$\widehat{f}^{\mathcal{Y}}(\boldsymbol{t}) = i_*(\widehat{f}^{\mathcal{X}}(i^*(\boldsymbol{t})).$$

*Proof.* — Using (5.3) and (5.4) and the same analysis as in the proof of Proposition 5.2,

$$\widehat{f}^{\mathcal{Y}}(\boldsymbol{t}) = \lim_{\lambda \mapsto 0} \pi^*(\widehat{f}^{\lambda}(i^*(\boldsymbol{t}))).$$

Next note that

$$\begin{aligned} i_*(\widehat{f}^{\lambda}(\boldsymbol{t})/e_\lambda(\mathcal{E}^\vee)) &= \pi^*(i^*(i_*(\widehat{f}^{\lambda}(\boldsymbol{t})/e_\lambda(\mathcal{E}^\vee)))) \\ &= \pi^*(\widehat{f}^{\lambda}(\boldsymbol{t})) \end{aligned}$$

and $\widehat{f}^{\mathcal{X}}(\boldsymbol{t}) = \lim_{\lambda \mapsto 0} \widehat{f}^{\lambda}(\boldsymbol{t})/e_\lambda(\mathcal{E}^\vee)$. □

Definition 6.9. — *We define the transformation* $\Delta_+ : H^*_{\mathrm{CR,nar}}(\mathcal{Y}) \to H^*_{\mathrm{CR,amb}}(\mathcal{Z})$ *as follows. Given* $\alpha \in H^*_{\mathrm{CR,nar}}(\mathcal{Y})$, *let* $\widetilde{\alpha} \in H^*_{\mathrm{CR,c}}(\mathcal{Y})$ *be a lift of* $\alpha$. *Define*

$$\Delta_+(\alpha) := \Delta^{\mathrm{c}}_+(\widetilde{\alpha}).$$

*Define*

$$\overline{\Delta}_+ := (2\pi\sqrt{-1}z)^{\mathrm{rk}(\mathcal{E})}\Delta_+.$$

Lemma 6.10. — *With assumption 3.13, the map* $\Delta_+ : H^*_{\mathrm{CR,nar}}(\mathcal{Y}) \to H^*_{\mathrm{CR,amb}}(\mathcal{Z})$ *described above is well-defined.*

*Proof.* — The lift $\widetilde{\alpha}$ is only defined up to an element of $\ker(\phi)$. We check that

$$\pi^{\mathrm{c}}_*(\ker(\phi)) \subset \ker(j^*).$$

Given $\widetilde{\alpha} \in \ker(\phi)$, since $i^{\mathrm{c}}_*$ is an isomorphism there exists an element $\beta \in H^*_{\mathrm{CR}}(\mathcal{X})$ such that $\widetilde{\alpha} = i^{\mathrm{c}}_*(\beta)$. Then $\pi^{\mathrm{c}}_*(\alpha) = \pi^{\mathrm{c}}_*(i^{\mathrm{c}}_*(\beta)) = \beta$. We want to show that $j^*(\beta) = 0$. By assumption, $\phi(\widetilde{\alpha}) = \phi \circ i^{\mathrm{c}}_*(\beta) = i_*(\beta) = 0$. Write $\beta$ as $i^*(\gamma)$, for some $\gamma \in H^*_{\mathrm{CR}}(\mathcal{Y})$. Consider the following diagram

$$\begin{tikzcd} \mathcal{Y}|_{\mathcal{Z}} \arrow[r, "\widetilde{j}"] \arrow[d, shift left, "\widetilde{\pi}"] & \mathcal{Y} \arrow[d, shift left, "\pi"] \\ \mathcal{Z} \arrow[r, "j"] \arrow[u, bend left, "\widetilde{i}"] & \mathcal{X}. \arrow[u, bend left, "i"] \end{tikzcd}$$

Then $j^*(\beta) = j^*i^*(\gamma) = \widetilde{i}^*\widetilde{j}^*(\gamma)$ is zero if and only if $\widetilde{j}^*(\gamma) = 0$. By Assumption 3.13, $\widetilde{j}^*(\gamma) = 0$ if and only if $\widetilde{j}_*\widetilde{j}^*(\gamma) = e(\mathcal{E}) \cup \gamma = 0$. Up to a sign, this is equal to $e(\mathcal{E}^\vee) \cup \gamma = i_*i^*(\gamma) = i_*(\beta)$, which is zero by assumption. □

Lemma 6.11. — *Given* $\alpha = e(\mathcal{E}^\vee) \cup \beta \in H^*_{\mathrm{CR}}(\mathcal{X})$,

$$\Delta_+(\pi^*(\alpha)) = j^*\beta.$$

*In particular,* $\Delta_+ : H^*_{\mathrm{CR,nar}}(\mathcal{Y}) \to H^*_{\mathrm{CR,amb}}(\mathcal{Z})$ *is an isomorphism.*

*Proof.* — Observe that

$$\begin{aligned}\pi^*(\alpha) &= e(\mathcal{E}^\vee) \cup \pi^*(\beta)\\ &= i_* \circ i^* \circ \pi^*(\beta)\\ &= i_*(\beta)\\ &= \phi \circ i_*^{\mathrm{c}}(\beta).\end{aligned}$$

Therefore, $\Delta_+(\pi^*(\alpha)) = j^* \circ \pi_*^{\mathrm{c}} \circ i_*^{\mathrm{c}}(\beta) = j^*(\beta)$. The second claim follows from

$$\begin{aligned}H^*_{\mathrm{CR,amb}}(\mathcal{Z}) &= \mathrm{im}(j^*)\\ &\cong H^*_{\mathrm{CR}}(\mathcal{X})/(\ker(\,\cdot\, \cup e(\mathcal{E}^\vee))\\ &\cong \mathrm{im}(\,\cdot\, \cup e(\mathcal{E}^\vee))\\ &= \pi^*(\mathrm{im}(\,\cdot\, \cup e(\mathcal{E}^\vee)))\\ &= H^*_{\mathrm{CR,nar}}(\mathcal{Y}).\end{aligned}$$

where the second and third terms are isomorphic by (3.6) and the final equality is by Proposition 2.15. □

We will need the following lemma.

Lemma 6.12. — $\Delta_+(\widehat{f}^{\mathcal{Y}}(\boldsymbol{t})) = j^* \circ \widehat{f}^{\mathcal{X}} \circ i^*(\boldsymbol{t})$.

*Proof.* — By Lemma 6.8 and the definition of $\Delta_+$,

$$\begin{aligned}\Delta_+(\widehat{f}^{\mathcal{Y}}(\boldsymbol{t})) &= \Delta_+(i_*(\widehat{f}^{\mathcal{X}}(i^*(\boldsymbol{t})))\\ &= \Delta_+(\phi \circ i_*^{\mathrm{c}}(\widehat{f}^{\mathcal{X}}(i^*(\boldsymbol{t}))))\\ &= j^* \circ \pi_*^{\mathrm{c}}(i_*^{\mathrm{c}}(\widehat{f}^{\mathcal{X}}(i^*(\boldsymbol{t}))))\\ &= j^* \circ \widehat{f}^{\mathcal{X}} \circ i^*(\boldsymbol{t}).\end{aligned}$$

□

Proposition 6.13. — *The following operators are equal after a change of variables:*

$$\begin{aligned}\Delta_+ \circ L^{\mathcal{Y}}(\boldsymbol{t}, z) \circ \phi &= \Delta_+^{\mathrm{c}} \circ L^{\mathcal{Y},\mathrm{c}}(\boldsymbol{t}, z)\\ &= L^{\mathcal{Z},\mathrm{amb}}(j^* \circ \overline{f}^{\mathcal{X}} \circ i^*(\boldsymbol{t}), z) \circ \Delta_+^{\mathrm{c}} \circ e^{-\pi\sqrt{-1}c_1(\pi^*\mathcal{E})/z}\\ &= L^{\mathcal{Z},\mathrm{amb}}(\overline{f}^{\mathcal{Y}}(\boldsymbol{t}), z) \circ \Delta_+ \circ e^{-\pi\sqrt{-1}c_1(\pi^*\mathcal{E})/z} \circ \phi,\end{aligned}$$

*where*

$$\begin{aligned}\overline{f}^{\mathcal{Y}}(\boldsymbol{t}) &= \Delta_+(\widehat{f}^{\mathcal{Y}}(\boldsymbol{t})) - \pi\sqrt{-1}c_1(\mathcal{E})\\ &= \Delta_+\left(\sum_{i \in I_{\mathrm{nar}}} \langle\!\langle e(\mathcal{E}^\vee), T_i \rangle\!\rangle^{\mathcal{Y}}(\boldsymbol{t}) T^i\right) - \pi\sqrt{-1}c_1(\mathcal{E}).\end{aligned} \tag{6.9}$$

*Proof.* — This follows almost immediately from the previous section, by applying the map $\phi : H^*_{\mathrm{CR,c}}(\mathcal{Y}) \to H^*_{\mathrm{CR,nar}}(\mathcal{Y})$. Recall (6.5):

$$\begin{aligned}(6.10)\quad & \Delta^{\mathrm{c}}_+ \circ L^{\mathcal{Y},\mathrm{c}}(\boldsymbol{t}, z)(\beta) \\ & \qquad = L^{\mathcal{Z},\mathrm{amb}}(j^* \circ \bar{f}^{\mathcal{X}} \circ i^*(\boldsymbol{t}), z) \circ \Delta^{\mathrm{c}}_+ \circ e^{-\pi\sqrt{-1}c_1(\pi^*\mathcal{E})/z}(\beta).\end{aligned}$$

By Proposition 4.7, the left hand side of (6.10) is equal to

$$\Delta^{\mathrm{c}}_+ \circ \phi(L^{\mathcal{Y},\mathrm{c}}(\boldsymbol{t}, z)(\beta)) = \Delta_+ \circ L^{\mathcal{Y}}(\boldsymbol{t}, z)(\phi(\beta))$$

for all $\beta \in H^*_{\mathrm{CR,c}}(\mathcal{Y})$. By Lemma 6.12 the right hand side of (6.10) is

$$\begin{aligned}& L^{\mathcal{Z},\mathrm{amb}}(\bar{f}^{\mathcal{Y}}(\boldsymbol{t}), z) \circ \Delta^{\mathrm{c}}_+ \circ e^{-\pi\sqrt{-1}c_1(\pi^*\mathcal{E})/z}(\beta) \\ & \qquad = L^{\mathcal{Z},\mathrm{amb}}(\bar{f}^{\mathcal{Y}}(\boldsymbol{t}), z) \circ \Delta_+ \circ e^{-\pi\sqrt{-1}c_1(\pi^*\mathcal{E})/z}(\phi(\beta)). \quad \square\end{aligned}$$

THEOREM 6.14. — *Assume $\mathcal{X}$ has the resolution property. The map $\overline{\Delta}_+$ identifies the quantum $D$-module $QDM_{\mathrm{nar}}(\mathcal{Y})$ with $\bar{f}^*(QDM_{\mathrm{amb}}(\mathcal{Z}))$. Furthermore it is compatible with the integral structure and the functor $j^* \circ \pi_*$, i.e., the following diagram commutes;*

$$(6.11)\qquad \begin{tikzcd} D(\mathcal{Y})_{\mathcal{X}} \arrow[r, "j^*\circ\pi_*"] \arrow[d, "s^{\mathcal{Y},\mathrm{nar}}"] & j^*(D(\mathcal{X})) \arrow[d, "s^{\mathcal{Z},\mathrm{amb}}"] \\ QDM_{\mathrm{nar}}(\mathcal{Y}) \arrow[r, "\bar{\Delta}_+"] & QDM_{\mathrm{amb}}(\mathcal{Z}). \end{tikzcd}$$

*Proof.* — The fact that $\nabla^{\mathcal{Y},\mathrm{nar}}$ is mapped to $\nabla^{\mathcal{Z},\mathrm{amb}}$ follows from Proposition 6.13. To see that the pairings agree, first observe that for $\alpha, \beta \in H^*_{\mathrm{CR}}(\mathcal{X})$,

$$\begin{aligned}\langle \Delta_+ i_*\alpha, \Delta_+ i_*\beta\rangle^{\mathcal{Z}} &= \int_{I\mathcal{Z}} j^*(\alpha) \cup_{I\mathcal{Z}} I_*(j^*\beta) \\ &= \int_{I\mathcal{X}} \alpha \cup_{I\mathcal{X}} I_*(\beta \cup_{I\mathcal{X}} q^*e(\mathcal{E})) \\ &= (-1)^{\mathrm{rk}(\mathcal{E})} \int_{I\mathcal{X}} \alpha \cup_{I\mathcal{X}} I_*(i^{\mathrm{c}*} i^{\mathrm{c}}_*(\beta)) \\ &= (-1)^{\mathrm{rk}(\mathcal{E})} \int_{I\mathcal{Y}} i^{\mathrm{c}}_*\alpha \cup_{I\mathcal{Y}} I_*(i^{\mathrm{c}}_*\beta) \\ &= (-1)^{\mathrm{rk}(\mathcal{E})} \int_{I\mathcal{Y}} i_*\alpha \cup_{I\mathcal{Y},\mathrm{c}} I_*(i_*\beta) \\ &= (-1)^{\mathrm{rk}(\mathcal{E})} \langle i_*\alpha, i_*\beta\rangle^{\mathcal{Y}},\end{aligned}$$

where $\cup_{I\mathcal{Z}}$ (resp. $\cup_{I\mathcal{Y}}$, etc.) indicates we are using the usual cup product on $I\mathcal{Z}$ (resp. $I\mathcal{Y}$, etc.) as opposed to the Chen–Ruan cup product. The fourth equality is the projection formula ([8, Proposition 6.15]). In the fifth

equality we use the fact that $i_*\alpha \cup_{I\mathcal{Y},\mathrm{c}} I_*(i_*\beta) = (\phi \circ i_*^{\mathrm{c}}\alpha) \cup_{I\mathcal{Y},\mathrm{c}} I_*(\phi \circ i_*^{\mathrm{c}}\beta) = i_*^{\mathrm{c}}\alpha \cup_{I\mathcal{Y}} I_*(i_*^{\mathrm{c}}\beta)$ by Definition 2.8 of the compactly supported cup product. Because $\overline{\Delta}_+$ contains the factor of $z^{\mathrm{rk}(\mathcal{E})}$,

$$\begin{aligned} S^{\mathcal{Z}}(\overline{\Delta}_+ i_*\alpha, \overline{\Delta}_+ i_*\beta) &= (-1)^{\mathrm{rk}(\mathcal{E})}(2\pi\sqrt{-1}z)^{\dim(\mathcal{Z})+2\,\mathrm{rk}(\mathcal{E})}\langle \Delta_+ i_*\alpha, \Delta_+ i_*\beta\rangle^{\mathcal{Z}} \\ &= (2\pi\sqrt{-1}z)^{\dim(\mathcal{Y})}\langle i_*\alpha, i_*\beta\rangle^{\mathcal{Y}} \\ &= S^{\mathcal{Y}}(i_*\alpha, i_*\beta). \end{aligned}$$

Because $H^*_{\mathrm{CR,nar}}(\mathcal{Y}) = \mathrm{im}(i_*)$ by Proposition 2.15, we conclude that $\overline{\Delta}_+$ preserves the pairing on all of $H^*_{\mathrm{CR,nar}}(\mathcal{Y})$.

Commutativity of the square (6.11) follows from the equalities of Proposition 6.13 and Theorem 6.5 after observing that $\widetilde{\mathrm{ch}} = \phi \circ \widetilde{\mathrm{ch}}^{\mathrm{c}}$. □

## Appendix A. Givental's formalism

Let □ denote either Gromov–Witten theory of a proper Deligne–Mumford stack $\mathcal{X}$ or an $\boldsymbol{s}$-twisted theory over $\mathcal{X}$. In this section we recall Givental's formalism of an overruled Lagrangian cone for encoding genus-zero Gromov–Witten theory. For more details see [19].

Let $H^{\square}$ be the state space for the theory. Given a basis $\{T_i\}_{i\in I}$, a point in $H^{\square}$ may be written as $\boldsymbol{t} = \sum_{i\in I} t^i T_i$. Denote by $\boldsymbol{t}(z)$ the power series

$$\boldsymbol{t}(z) = \sum_{k\geqslant 0} \boldsymbol{t}_k z^k = \sum_{i\in I}\sum_{k\geqslant 0} t^i_k T_i z^k$$

in $H^{\square}[\![z]\!]$. Define the genus zero descendent potential as the formal function

$$\mathcal{F}_0^{\square}(\boldsymbol{t}(z)) := \sum_{d\in \mathrm{Eff}}\sum_{n\geqslant 0}\frac{1}{n!}\langle \boldsymbol{t}(\psi), \ldots, \boldsymbol{t}(\psi)\rangle^{\square}_{0,n,d}.$$

Let $\mathscr{V}^{\square}$ denote the infinite-dimensional vector space $H^{\square}[z][\![z^{-1}]\!][\![\boldsymbol{s}]\!]$ (where $\boldsymbol{s} = 0$ in the untwisted case). We endow $\mathscr{V}^{\square}$ with a symplectic pairing defined as follows:

$$\Omega^{\square}(f_1(z), f_2(z)) := \mathrm{Res}\langle f_1(-z), f_2(z)\rangle^{\square}.$$

The vector space $\mathscr{V}^{\square}$ may be polarized as

$$\begin{aligned} \mathscr{V}^{\square}_+ &= H^{\square}[z], \\ \mathscr{V}^{\square}_- &= z^{-1}H^{\square}[\![z^{-1}]\!]. \end{aligned}$$

The polarization on $\mathscr{V}^\square$ determines Darboux coordinates $\{q_k^i, p_{k,i}\}$. Each element of $\mathscr{V}^\square$ may be written as

$$\sum_{k\geqslant 0}\sum_{i\in I} q_k^i T_i z^k + \sum_{k\geqslant 0}\sum_{i\in I} p_{k,i} T^i (-z)^{-k-1}$$

We view $\mathcal{F}_0^\square(\boldsymbol{t}(z))$ as a formal function on $\mathscr{V}_+^\square$ via the *dilaton shift*

$$\boldsymbol{q}(z) = \boldsymbol{t}(z) - z.$$

Definition A.1. — *Define the* overruled Lagrangian cone *for* $\square$ *to be*

$$\mathscr{L}^\square := \{\mathbf{p} = d_{\boldsymbol{q}}\mathcal{F}_0^\square\}. \tag{A.1}$$

*Explicitly,* $\mathscr{L}^\square$ *contains the points of the form*

$$\begin{aligned} -z &+ \sum_{\substack{k\geqslant 0\\ i\in I}} t_k^i T_i z^k \\ &+ \sum_{d\in \mathrm{Eff}} \sum_{\substack{a_1,\dots,a_n,a\geqslant 0\\ i_1,\dots,i_n,i\in I}} \frac{t_{a_1}^{i_1}\cdots t_{a_n}^{i_n}}{n!(-z)^{a+1}} \langle \psi^a T_i, \psi^{a_1}T_{i_1},\dots,\psi^{a_n}T_{i_n}\rangle_{0,n+1,d}^\square T^i. \end{aligned} \tag{A.2}$$

As shown in [19], $\mathscr{L}^\square$ takes a special form:

- it is a cone;
- for all $f\in\mathscr{L}^\square$,
$$\mathscr{L}^\square \cap T_f\mathscr{L} = zT_f\mathscr{L}$$
where $T_f\mathscr{L}$ is the tangent space to $\mathscr{L}^\square$ at $f$.

Consider a generic family in $\mathscr{L}^\square$ parameterized by $H^\square$, this will take the form

$$\left\{f(\boldsymbol{t}) \,\middle|\, \boldsymbol{t}\in H^\square\right\} \subset \mathscr{L}^\square,$$

and will be transverse to the ruling. With this, the above properties imply that we can reconstruct $\mathscr{L}^\square$ as

$$\mathscr{L}^\square = \left\{zT_{f(\boldsymbol{t})}\mathscr{L}^\square \,\middle|\, \boldsymbol{t}\in H^\square\right\}. \tag{A.3}$$

Givental's $J$–function is such a family. It is given by the intersection:

$$J^\square(\boldsymbol{t},-z) = \mathscr{L}^\square \cap -z\oplus \boldsymbol{t}\oplus \mathscr{V}^-.$$

More explicitly,

$$J^\square(\boldsymbol{t},-z) = -z + \boldsymbol{t} + \sum_{d\in\mathrm{Eff}}\sum_{n\geqslant 0}\sum_{i\in I}\frac{1}{n!}\left\langle \frac{T_i}{-z-\psi}, \boldsymbol{t},\dots,\boldsymbol{t}\right\rangle_{0,n+1,d}^\square T^i.$$

By (3.5), we see that

$$L^\square(\boldsymbol{t},z)^{-1}\alpha = L^\square(\boldsymbol{t},-z)^T\alpha = \partial_\alpha J^\square(\boldsymbol{t},z). \tag{A.4}$$

In [19] it is shown that the image of $J^{\square}(\boldsymbol{t},-z)$ is transverse to the ruling of $\mathscr{L}^{\square}$, so $J^{\square}(\boldsymbol{t},-z)$ is a function satisfying (A.3). It follows that the ruling at $J^{\square}(\boldsymbol{t},-z)$ is spanned by the derivatives of $J^{\square}$, i.e.

$$\begin{aligned} zT_{J^{\square}(\boldsymbol{t},-z)}\mathscr{L} &= \left\{ J^{\square}(\boldsymbol{t},-z) + z\sum c_i(z)\frac{\partial}{\partial t^i}J^{\square}(\boldsymbol{t},-z) \,\middle|\, c_i(z)\in\mathbb{C}[z] \right\} \\ &= \left\{ z\sum c_i(z)\frac{\partial}{\partial t^i}J^{\square}(\boldsymbol{t},-z) \,\middle|\, c_i(z)\in\mathbb{C}[z] \right\}. \end{aligned} \tag{A.5}$$

where the second equality is by the string equation, $z\frac{\partial}{\partial t^0}J^{\square}(\boldsymbol{t},z) = J^{\square}(\boldsymbol{t},z)$. We note finally that

$$T_{J^{\square}(\boldsymbol{t},-z)}\mathscr{L} = \left\{ \sum c_i(z)\frac{\partial}{\partial t^i}J^{\square}(\boldsymbol{t},-z) \,\middle|\, c_i(z)\in\mathbb{C}[z] \right\}. \tag{A.6}$$

### A.1. Quantum Serre duality with Lagrangian cones

Since its discovery in [17], quantum Serre duality (or non-linear Serre duality) has been formulated in many different ways. Below we recall one of the most general and applicable, in terms of twisted theories and Lagrangian cones. Let $\boldsymbol{s}$ denote the twisting parameters of Section 3.1 and define $\boldsymbol{s}^*$ by

$$s_k^* = (-1)^{k+1}s_k.$$

Note

$$\boldsymbol{s}^*(\mathcal{E}^\vee) = \frac{1}{\boldsymbol{s}(\mathcal{E})}.$$

In this case (genus zero) quantum Serre duality takes the following form.

Theorem A.2 ([14, Corollary 9]). — *The symplectic transformation*

$$\begin{aligned} \mathscr{V}^{\boldsymbol{s}(\mathcal{E})} &\longrightarrow \mathscr{V}^{\boldsymbol{s}^*(\mathcal{E}^\vee)} \\ f(z) &\longmapsto \boldsymbol{s}^*(\mathcal{E}^\vee)f(z) \end{aligned}$$

*identifies* $\mathscr{L}^{\boldsymbol{s}(\mathcal{E})}$ *with* $\mathscr{L}^{\boldsymbol{s}^*(\mathcal{E}^\vee)}$.

See [31] for the orbifold version of this theorem.

Note however that the specializations of the twisting parameters given by (5.1) and (5.2) are not exactly of the form given above, i.e. $\boldsymbol{s}' \neq \boldsymbol{s}^*$. The statement must be modified slightly in this case. We state here the modified statement of the above theorem for the case of the Euler class-twisted theories of Section 5. This specific formulation appears as Theorem 5.17 in [26].

Theorem A.3 ([26, Theorem 5.17]). — *The symplectic transformation*

$$\Delta^\diamond : \mathscr{V}^{e_\lambda^{-1}(\mathcal{E}^\vee)} \longrightarrow \mathscr{V}^{e_\lambda(\mathcal{E})}$$
$$f(z) \longmapsto \frac{e^{\pi\sqrt{-1}c_1(\mathcal{E})/z}}{e_\lambda(\mathcal{E}^\vee)} f(z)$$

*identifies* $\mathscr{L}^{e_\lambda^{-1}(\mathcal{E}^\vee)}$ *with* $\mathscr{L}^{e_\lambda(\mathcal{E})}$.

## Appendix B. Orbifold localized Chern character

Given a complex $F^\bullet \in K_X^0(Y)$ (exact off $X$), there exists a *localized Chern character*

$$\mathrm{ch}_X^Y(F^\bullet) \in H^*(Y, Y - X)$$

as described in [16, Example 19.2.6] (see also [4, 25]).

On the other hand, given a bundle $F$ on a stack $\mathcal{Y}$ with the resolution property, there is an *orbifold Chern character* [30, 31] landing in the cohomology of the inertia stack and defined as follows. Restricting $F$ to a twisted sector $F_\gamma \to \mathcal{Y}_\gamma$, $F_\gamma$ decomposes into eigenbundles according to the action of $\gamma$ on $F$

$$F_\gamma = \bigoplus_{0 \leqslant f < 1} F_{\gamma,f},$$

where the generator $\gamma$ acts as multiplication by $e^{2\pi\sqrt{-1}f}$ on $F_{\gamma,f}$. Define

$$\rho(F_\gamma) := \sum_{0 \leqslant f < 1} e^{2\pi\sqrt{-1}f} F_{\gamma,f}.$$

Observe that for a complex

$$F^\bullet = 0 \longrightarrow F^a \longrightarrow \cdots \longrightarrow F^i \longrightarrow F^{i+1} \longrightarrow \cdots \longrightarrow F^b \longrightarrow 0$$

of vector bundles, the map $d_\gamma^i : F_\gamma^i \to F_\gamma^{i+1}$ is compatible with the splitting into eigenbundles. I.e. $d_\gamma^i$ is a direct sum of the maps

$$d_{\gamma,f}^i : F_{\gamma,f}^i \longrightarrow F_{\gamma,f}^{i+1}.$$

Consequently $\rho$ gives a well-defined map on $K^0(\mathcal{Y}_\gamma)$.

Summing over each twisted sector, this defines a map $\rho : K^0(I\mathcal{Y}) \to K^0(I\mathcal{Y})$.

Definition B.1. — *The orbifold Chern character* $\widetilde{\mathrm{ch}} : K^0(\mathcal{Y}) \to H^*_{\mathrm{CR}}(\mathcal{Y}) = H^*(I\mathcal{Y})$ *is defined as the composition*

$$K^0(\mathcal{Y}) \xrightarrow{q^*} K^0(I\mathcal{Y}) \xrightarrow{\rho} K^0(I\mathcal{Y}) \xrightarrow{\mathrm{ch}} H^*(I\mathcal{Y}),$$

*where $q : I\mathcal{Y} \to \mathcal{Y}$ is the natural union of inclusions and* ch *is the usual Chern character defined by passing to the coarse space.*

One can combine the two notions above to obtain a *localized orbifold Chern character*. Let $\mathcal{X}$ be a closed substack of $\mathcal{Y}$ and let $F^\bullet$ be a complex on $\mathcal{Y}$, exact off of $\mathcal{X}$. Consider the restriction to a twisted sector $\mathcal{Y}_\gamma$, by the observation above, $F^\bullet_\gamma$ splits into eigen-complexes

$$F^\bullet_\gamma = \bigoplus_{0 \leqslant f < 1} F^\bullet_{\gamma,f},$$

with each $F^\bullet_{\gamma,f}$ exact off $\mathcal{X}_\gamma$. This implies that the twisting $\rho$ gives a well defined map on $K^0_{\mathcal{X}_\gamma}(\mathcal{Y}_\gamma)$. Summing over all twisted sectors defines a map $\rho : K^0_{I\mathcal{X}}(I\mathcal{Y}) \to K^0_{I\mathcal{X}}(I\mathcal{Y})$.

Definition B.2. — *Define the* localized orbifold Chern character

$$\widetilde{\mathrm{ch}}^{\mathcal{Y}}_{\mathcal{X}} : K^0_{\mathcal{X}}(\mathcal{Y}) \longrightarrow H^*_{\mathrm{CR}}(\mathcal{Y}, \mathcal{Y} - \mathcal{X})$$

*to be the composition*

$$K^0_{\mathcal{X}}(\mathcal{Y}) \xrightarrow{q^*} K^0_{I\mathcal{X}}(I\mathcal{Y}) \xrightarrow{\rho} K^0_{I\mathcal{X}}(I\mathcal{Y}) \xrightarrow{\mathrm{ch}^{\mathcal{Y}}_{\mathcal{X}}} H^*(I\mathcal{Y}, I\mathcal{Y} - I\mathcal{X}).$$

For $Y$ a non-compact manifold, define $K^0_{\mathrm{c}}(Y)$ to be the direct limit

$$K^0_{\mathrm{c}}(Y) = \varinjlim K^0_X(Y)$$

over all compact subvarieties $X \subset Y$. Assume $X_1 \subset X_2 \subset Y$ and $F^\bullet$ is a complex exact off $X_1$ (and therefore also exact off $X_2$), let $j : X_1 \to X_2$, $i_1 : X_1 \to Y$, and $i_2 : X_2 \to Y$ denote the inclusions. Then the following diagram commutes:

$$\begin{tikzcd}
K^0_{X_1}(Y) \arrow[r, "j_*"] \arrow[d, "\mathrm{ch}^Y_{X_1}"] & K^0_{X_2}(Y) \arrow[d, "\mathrm{ch}^Y_{X_2}"] \\
H^*(Y, Y - X_1) \arrow[d, "\cap[Y]"] & H^*(Y, Y - X_2) \arrow[d, "\cap[Y]"] \\
H_*(X_1) \arrow[r, "j_*"] \arrow[d, "{i_1}^{\mathrm{c}}_*"] & H_*(X_2) \arrow[d, "{i_2}^{\mathrm{c}}_*"] \\
H_*(Y) \cong H^*_{\mathrm{c}}(Y) \arrow[r] & H_*(Y) \cong H^*_{\mathrm{c}}(Y).
\end{tikzcd} \tag{B.1}$$

The commutativity of the top square follows, for instance, from [16, Definition 18.1 and Example 19.2.6].

Definition B.3. — *For $i : X \to Y$ the inclusion of a closed and proper subvariety, define* $\mathrm{ch}^{\mathrm{c}}_X : K^0_X(Y) \to H^*_{\mathrm{c}}(Y)$ *by*

$$\mathrm{ch}^{\mathrm{c}}_X(F^\bullet) = i^{\mathrm{c}}_*(\mathrm{ch}^Y_X(F^\bullet) \cap [Y]).$$

*By diagram* (B.1), *this induces a homomorphism*

$$\mathrm{ch}^{\mathrm{c}} : K^0_{\mathrm{c}}(Y) \longrightarrow H^*_{\mathrm{c}}(Y)$$

*which we will refer to as the* compactly supported Chern character.

The above argument can be extended to the situation where $\mathcal{Y}$ is a smooth Deligne–Mumford stack, we obtain:

Definition B.4. — *The* compactly supported orbifold Chern character

$$\widetilde{\mathrm{ch}}^{\,\mathrm{c}} : K^0_{\mathrm{c}}(\mathcal{Y}) \longrightarrow H^*_{\mathrm{CR},\mathrm{c}}(\mathcal{Y}) = H^*_{\mathrm{c}}(I\mathcal{Y})$$

*is defined to be the composition*

$$K^0_{\mathrm{c}}(\mathcal{Y}) \xrightarrow{q^*} K^0_{\mathrm{c}}(I\mathcal{Y}) \xrightarrow{\rho} K^0_{\mathrm{c}}(I\mathcal{Y}) \xrightarrow{\mathrm{ch}^{\mathrm{c}}} H^*_{\mathrm{c}}(I\mathcal{Y}).$$

Given $i : X \to Y$ the inclusion of a closed and proper subvariety, by [25, Theorem 1.3],

$$i_*(\mathrm{ch}^Y_X(F^\bullet) \cap [Y] = \mathrm{ch}(F^\bullet).$$

This immediately gives the following.

Proposition B.5. — *For $F^\bullet$ a complex of vector bundles on $Y$ with proper support,*

$$\phi(\widetilde{\mathrm{ch}}^{\,\mathrm{c}}(F^\bullet)) = \widetilde{\mathrm{ch}}(F^\bullet)$$

*in $H^*_{\mathrm{CR}}(\mathcal{Y})$.*

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