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ANTIFLIPS, MUTATIONS, AND UNBOUNDED SYMPLECTIC EMBEDDINGS OF RATIONAL HOMOLOGY BALLS

by Jonathan D. EVANS & Giancarlo URZÚA (*)

Abstract. — The Milnor fibre of a $\mathbb{Q}$-Gorenstein smoothing of a Wahl singularity is a rational homology ball $B_{p,q}$. For a canonically polarised surface of general type X, it is known that there are bounds on the number p for which $B_{p,q}$ admits a symplectic embedding into X. In this paper, we give a recipe to construct unbounded sequences of symplectically embedded $B_{p,q}$ into surfaces of general type equipped with *non-canonical* symplectic forms. Ultimately, these symplectic embeddings come from Mori's theory of flips, but we give an interpretation in terms of almost toric structures and mutations of polygons. The key point is that a flip of surfaces, as studied by Hacking, Tevelev and Urzúa, can be formulated as a combination of mutations of an almost toric structure and deformation of the symplectic form.

Résumé. — La fibre de Milnor d'un lissage $\mathbb{Q}$-Gorenstein d'une singularité de Wahl est une boule d'homologie rationelle $B_{p,q}$. Si X est une surface de type général polarisée canoniquement, l'ensemble des entiers p pour lesquels il existe un plongement symplectique de $B_{p,q}$ dans X est borné. Dans cet article, nous montrons comment construire une suite non-bornée de boules d'homologie rationnelles plongées symplectiquement dans des surfaces de type général munies de formes symplectiques non-canoniques. Ces plongements proviennent de la théorie de Mori sur les flips, mais nous les interprétons en termes de structures presque toriques et de mutations de polygones. Un flip de surfaces tel que ceux étudiés par Hacking, Tevelev et Urzúa peut être décomposé en une succession de mutations de structure presque torique et de déformations de la forme symplectique.

1. Introduction

1.1. Setting and results

Wahl singularities are the cyclic quotient surface singularities admitting a $\mathbb{Q}$-Gorenstein smoothing whose Milnor fibre is a rational homology ball [12,

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25]. The rational homology balls $B_{p,q}$ arising this way are Stein manifolds whose Lagrangian skeleton is a certain cell complex called a *Lagrangian pinwheel* $L_{p,q}$, with one 1-cell and one 2-cell [2, 7, 10]. If X is an algebraic surface, one can hope to understand which Wahl singularities can appear in degenerations of X by studying the symplectic embeddings of rational homology balls $B_{p,q}$ (or, equivalently, Lagrangian embeddings of pinwheels $L_{p,q}$) in X.

In [3], it was proved that for a symplectic 4-manifold (X,ω), with $b^+ > 1$ and $[\omega] = K_X$ (which one can think of as a surface of general type with positive geometric genus), there is a bound on the integers p for which there is a symplectic embedding of the rational homology ball $B_{p,q}$ into X (equivalently, by [7, Lemmas 3.3 and 3.4]), a Lagrangian pinwheel of type $L_{p,q}$). Namely, if ℓ denotes the length of the continued fraction expansion of $\frac{p^2}{pq-1}$, we have

$$\ell \leqslant 4K^2 + 7.$$

This implies a bound on p. (Compare with the similar proof of the better bound $\ell \leqslant 4K^2 + 1$ in the context of algebraic geometry in [19].)

In the current paper, we will show that the hypothesis $[\omega] = K_X$ in this result is necessary. We do this by exhibiting symplectic 4-manifolds which admit sequences of embedded Lagrangian pinwheels $\{L_{p_i,q_i}\}_{i=1}^{\infty}$ where $p_i \to \infty$.

The sequences (p_i, q_i) in question all satisfy a certain recursion relation which arises in Mori's theory of flips; we call them *Mori sequences*. A Mori sequence is determined by its first two terms; we therefore write $M(p_1, q_1; p_2, q_2)$ to specify a Mori sequence. See Section 3.4 for the definition.

Our construction applies very widely and yields unbounded Lagrangian pinwheels in any surface of general type which arises as a smoothing of a suitable KSBA-stable surface. The only requirement is that the KSBA-stable surface has at worst Wahl singularities and contains a suitable rational curve passing through at most two of these singularities (see Theorem 5.4 for a precise statement). We illustrate the applicability of the construction with two examples, one with $b^+ > 1$ and one with $b^+ = 1$:

Theorem 1.1. — *In each of the cases listed below, X carries a symplectic form ω for which there is a sequence of Lagrangian pinwheels $L_{p_i,q_i} \subset (X,\omega)$, for the given Mori sequence $\{(p_i,q_i)\}_{i=1}^{\infty}$:*

- *X is a quintic surface ($b^+ = 9$), with Mori sequence*

$$M(1,0;5,3) = \{(1,1),(5,3),(14,9),(37,24),(97,63),(254,165),\ldots\}.$$

- *X is a simply-connected Godeaux surface*[(1)] *($b^+ = 1$), with Mori sequence*

$$M(5,2;39,17) = \{(5,2),(39,17),(268,49),(1837,326),(12591,2233),\ldots\}.$$

Remark 1.2. — In fact, with essentially no extra work, we can also find a symplectic form on the quintic containing the Mori sequence

$$M(2,1;7,5) = \{(2,1),(7,5),(19,14),(50,37),(131,97),(343,254),\ldots\}$$

of Lagrangian pinwheels, and a symplectic form on the same Godeaux surface with the Mori sequence

$$M(4,1;33,10) = \{(4,1),(33,10),(227,69),(1556,473),(10665,3242),\ldots\}$$

of Lagrangian pinwheels. In the proof of Theorem 1.1, we will focus for convenience on *right mutations* and *right initial antiflips*, but running the same arguments with left mutations and left initial antiflips gives these other sequences.

Remark 1.3. — Our construction is a generalisation of the constructions by Khodorovskiy [8], Park–Park–Shin [16], Owens [15] and Park–Shin [17]; we additionally keep track of the symplectic form.

Remark 1.4. — It follows from the proof that the symplectic forms ω are deformation equivalent to the forms representing the canonical class K coming from the canonical embedding, however our forms have $[\omega] \neq K$. Since forms in the class K admit only bounded Lagrangian pinwheels, it is an interesting question to determine how far one needs to deform ω away from the class K before one sees Mori sequences of pinwheels. We will discuss this in Section 4.3, where we observe that our construction produces unbounded pinwheels when the symplectic form crosses an affine distance δ from the canonical class, where $\delta \geqslant 2$ is an integer which shows up in the recursion formula for the Mori sequence. It is not clear if this gap is an artefact of our construction, and that there are unbounded pinwheels closer to the canonical class, or if boundedness for pinwheels really persists in some neighbourhood of the canonical class.

1.2. Idea of proof

The idea of the proof is to deform the symplectic form along a compact codimension zero submanifold $U \subset X$. The submanifold U has the rational

(1) A *Godeaux surface* is a minimal surface of general type with $K^2 = 1$; the simply-connected ones are homeomorphic to $\mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$.

homology of $\mathbb{CP}^1$ and ∂U is a lens space. We will exhibit a 1-parameter family of symplectic forms ω_t on U such that (U, ω_0) is negatively monotone and (U, ω_1) is positively monotone. The symplectic manifolds (U, ω_t) are all symplectomorphic in a neighbourhood of ∂U, so the deformation ω_t extends to a deformation of symplectic structures on X which is constant outside U. We call this deformation an *initial antiflip* of the symplectic form.

We will then show that (U, ω_1) contains Mori sequences of Lagrangian pinwheels. We prove this by giving an almost toric structure on (U, ω_1) in which the pinwheels L_{p_1,q_1} and L_{p_2,q_2} are visible surfaces, then performing an infinite sequence of mutations(2) to get different almost toric structures on U in which the pinwheels L_{p_i,q_i} and $L_{p_{i+1},q_{i+1}}$ are visible. We need to be careful with our deformation of symplectic forms to ensure that there is "enough room" in U for an infinite sequence of mutations to be performed.

This initial antiflip is related to the k2A 3-fold flip discovered by Mori [13] and further studied in [6]. Roughly speaking, the total space $\mathcal{X}$ of a $\mathbb{Q}$-Gorenstein smoothing $\mathcal{X} \to \mathbb{C}$ of a singular algebraic surface $\mathcal{X}_0$ can sometimes be flipped to give a new $\mathbb{Q}$-Gorenstein smoothing $\mathcal{X}^+ \to \mathbb{C}$ of a different singular surface $\mathcal{X}_0^+$ without affecting any of the smooth fibres: $\mathcal{X}_z \cong \mathcal{X}_z^+$ for $z \neq 0$. Since $\mathcal{X}_z$ and $\mathcal{X}_z^+$ arise from smoothing different singularities, they contain the Milnor fibres of those singularities. The same singular surface $\mathcal{X}_0^+$ can arise when performing the flip of many different $\mathbb{Q}$-Gorenstein smoothings $\mathcal{X}$ of different singular surfaces $\mathcal{X}_0$ (indeed, a whole Mori sequence of them).

This whole paper can be read as a symplectic topologist's guide to [6], presenting those parts of that paper which can be cast purely in terms of symplectic topology.

1.3. Outline

In Section 2, we define *rational homology projective lines* (QHPs) and construct toric orbifold QHPs, V_Π, from polygons Π which we call truncated wedges. We then construct smooth QHPs, U_Π, as symplectic smoothings of these toric QHPs. These manifolds are equipped with an almost toric fibration with visible Lagrangian pinwheels.

(2) In the language of [22], a mutation is a *branch move* which switches one of the branch cuts in the almost toric structure for one pointing in the opposite direction. The terminology *mutation* comes from the paper of Galkin and Usnich [5]; the definition there is given for the fan (rather than polytope) side of toric geometry.

In Section 3, we study when the almost toric fibrations on U_Π can be mutated to give new almost toric fibrations. This allows us to construct infinite sequences of visible Lagrangian pinwheels corresponding to Mori sequences. In Section 3.4, we define Mori sequences and summarise their asymptotic behaviour. In Section 3.5 also discuss when infinitely many mutations can be performed in a bounded region of a truncated wedge.

In Section 4, we study those truncated wedges which cannot be mutated and introduce a new operation which involves a deformation of the symplectic form followed by a mutation. This leads us to the *initial antiflip* of a symplectic form and its inverse, the *flip*. The initial antiflip is a deformation of the symplectic form, and, in Section 4.3, we discuss how the cohomology class of ω varies along this deformation. In Section 4.4, we explain the link to Mori theory; in Section 4.5, we give the interpretation of k1A flips in our setting; and, in Section 4.6, we give a summary of how to view the flip and antiflips topologically.

Finally, in Section 5, we give an algebro-geometric recipe for constructing examples to which the theory applies and we explain the examples stated in Theorem 1.1.

1.4. Notation

We will write $[b_1, \ldots, b_r]$ to mean both:

- a chain of spheres $C_1, \ldots, C_r$ which intersect according to the graph 

with self-intersections $C_i^2 = -b_i$.
- the continued fraction

$$[b_1, \ldots, b_r] = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\cdots - \frac{1}{b_r}}}.$$

If we write $[b_{1,1}, \ldots, b_{1,r_1}] - c - [b_{2,1}, \ldots, b_{2,r_2}]$ we mean the chain

$$[b_{1,1}, \ldots, b_{1,r_1}, c, b_{2,1}, \ldots, b_{2,r_2}],$$

but where we group together certain spheres which we wish to collapse down to a singular point (or which have just arisen from resolving a singular point).

1.5. Acknowledgements

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2. Rational homology projective lines

Definition 2.1. — *A* rational homology projective line *(QHP) will mean a 4-dimensional manifold or orbifold X with $H_*(X;\mathbb{Q}) \cong H_*(\mathbb{CP}^1;\mathbb{Q})$.*

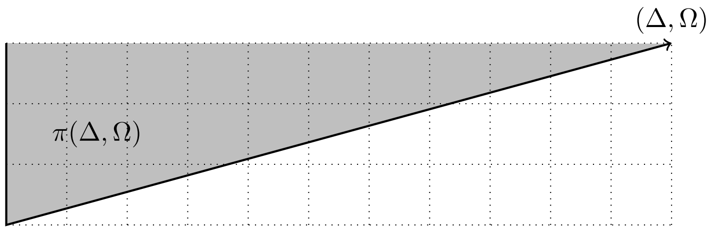
We will give a recipe for constructing symplectic QHPs as smoothings of symplectic orbifold QHPs.

2.1. Toric QHP-orbifolds: V_Π

2.1.1. Truncated wedges

Given coprime integers Δ, Ω with $0 \leqslant \Omega < \Delta$, let $\pi(\Delta,\Omega)$ denote the wedge

$$\pi(\Delta,\Omega) := \{(x,y) \in \mathbb{R}^2 \ : \ x \geqslant 0,\ \Delta y \geqslant \Omega x\}.$$

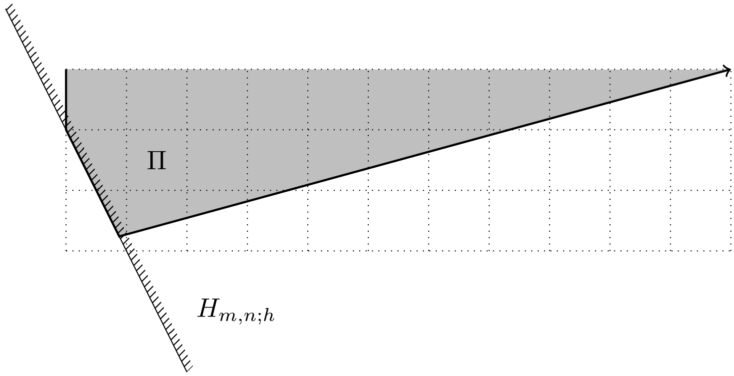


This is the moment polygon for a Hamiltonian torus action on the cyclic quotient singularity(3) $\frac{1}{\Delta}(1,\Omega)$.

Let m, n be coprime integers with $n > 0$ and let $h > 0$ be a real number. Consider the half-space $H_{m,n;h} = \{(x,y) \in \mathbb{R}^2 \ : \ mx + ny \geqslant h\}$ and the

(3) The cyclic quotient singularity $\frac{1}{\Delta}(1,\Omega)$ is the quotient of $\mathbb{C}^2$ by the action of the group of Δ^{th} roots of unity given by $\mu \cdot (x,y) = (\mu x, \mu^\Omega y)$.

truncation $\Pi = H_{m,n;h} \cap \pi(\Delta, \Omega)$.



This truncated wedge is the moment image of a partial resolution V_Π of the cyclic quotient singularity. The vertices x_1 and x_2 of Π are the images under the moment map of cyclic quotient singularities (abusively, also called x_1, x_2) in V_Π; if x_i has type $\frac{1}{P_i}(1, Q_i)$ then:

- $P_1 = n$, $Q_1 = -m \mod P_1$,
- $P_2 = m\Delta + n\Omega$, $Q_2 = k\Delta + \ell\Omega \mod P_2$, where $kn - \ell m = 1$.

We will also abusively say that the vertices x_i have type $\frac{1}{P_i}(1, Q_i)$.

Definition 2.2. — *We will say that a vertex of a polygon is a* Wahl vertex *if it has type* $\frac{1}{p^2}(1, pq-1)$ *for some coprime integers* $0 \leqslant q \leqslant p \neq 0$ *(Wahl singularities are precisely the cyclic quotient surface singularities of this type, see* [12, Remark 5.10]*). Below,* x_i *will be a Wahl vertex of type* $\frac{1}{p_i^2}(1, p_i q_i - 1)$.

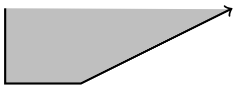
Remark 2.3. — Note that we allow $(p,q) = (1,1)$ and $(p,q) = (1,0)$, both of which represent a smooth point in V_Π. In order for our formulae below to work out, we must only ever use $(1,0)$ for a smooth point x_1 and $(1,1)$ for a smooth point x_2. If you accidentally plug in $p_1 = q_1 = 1$ or $p_2 = 1$, $q_2 = 0$, then you will get the wrong answers.

2.1.2. Shear invariant

Let Π be a truncated wedge. Let E_Π denote the edge between x_1 and x_2 and let $C_\Pi \subset V_\Pi$ denote the corresponding component of the toric boundary; C_Π is a rational curve which generates $H_2(V_\Pi; \mathbb{Q})$.

Definition 2.4. — *The* shear invariant *of* Π *is defined to be the integer* c *such that* $\widetilde{C}_\Pi^2 = -c$*, where* $\widetilde{C}_\Pi$ *is the proper transform of* C_Π *in the minimal resolution* $\widetilde{V}_\Pi \to V_\Pi$.

The reason for the name is visible in the standard moment polygon for the total space of the line bundle $\mathcal{O}(-c) \to \mathbb{CP}^1$, which is a truncated wedge with shear invariant c, with the zero-section (self-intersection $-c$) living over the compact edge:



2.1.3. Constructing polygons

Definition 2.5. — *Given a real number $a > 0$ and integers p_1, q_1, p_2, q_2, c such that $0 \leqslant q_1 < p_1$, $0 < q_2 \leqslant p_2$ and $\gcd(p_i, q_i) = 1$ for $i = 1, 2$, define the polygon*

$$\Pi(p_1, q_1, p_2, q_2, c, a) := \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, \begin{array}{l} y \geqslant 0, \\ p_1^2 x \geqslant y(p_1(p_1 - q_1) - 1) \\ p_2^2(x - a) \leqslant y(cp_2^2 - p_2 q_2 + 1) \end{array} \right\}.$$

Remark 2.6. — As mentioned in Remark 2.3, this definition does not allow $(p_1, q_1) = (1, 1)$ and $(p_2, q_2) = (1, 0)$; rather, you should use $(p_1, q_1) = (1, 0)$ and $(p_2, q_2) = (1, 1)$).

The polygon $\Pi := \Pi(p_1, q_1, p_2, q_2, c, a)$ has:

- one horizontal compact edge E_Π of affine length a,
- two noncompact edges: $+R_1$, emanating from the origin and pointing in the direction

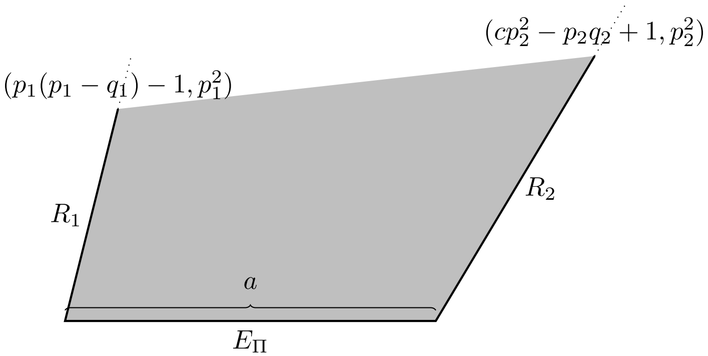
$$\begin{pmatrix} p_1(p_1 - q_1) - 1 \\ p_1^2 \end{pmatrix}$$

  $+R_2$, emanating from the point $(a, 0)$ and pointing in the direction

$$\begin{pmatrix} cp_2^2 - p_2 q_2 + 1 \\ p_2^2 \end{pmatrix}.$$

- Wahl vertices x_i of type $\frac{1}{p_i^2}(1, p_i q_i - 1)$,
- shear invariant c.

We illustrate the polygon Π below.



This relates to our earlier description of polygons as truncations of $\pi(\Delta, \Omega)$ in the following way:

Lemma 2.7. — *Given a polygon* $\Pi := \Pi(p_1, q_1, p_2, q_2, c, a)$, *define*

$$\begin{aligned}\sigma(\Pi) &:= (c-1)p_1p_2 + p_2q_1 - p_1q_2,\\ \Delta(\Pi) &= p_1^2 + p_2^2 + \sigma(\Pi)p_1p_2,\\ \Omega(\Pi) &= p_1q_1 + p_2q_2 - 1 + \sigma(\Pi)p_2q_1 - (c-1)p_2^2 \mod \Delta(\Pi).\end{aligned}$$

If $\Delta(\Pi) > 0$ *then* Π *is* $\mathbb{Z}$*-affine isomorphic to a truncation of* $\pi(\Delta(\Pi), \Omega(\Pi))$.

Proof. — We may apply the matrix

$$\begin{pmatrix} p_1^2 & 1 - p_1^2 + p_1q_1 \\ p_1q_1 - 1 & 1 - p_1q_1 + q_1^2 \end{pmatrix}$$

to Π to move the edge R_1 so that it points in the direction $(0,1)$; this moves R_2 into the direction

$$\begin{pmatrix} \Delta \\ \Omega' \end{pmatrix} = \begin{pmatrix} p_1^2 + p_2^2 + \sigma p_1p_2 \\ p_1q_1 + p_2q_2 - 1 + \sigma p_2q_1 - (c-1)p_2^2 \end{pmatrix}$$

where $\sigma = (c-1)p_1p_2 + p_2q_1 - p_1q_2$.

If $\Omega' = k\Delta + \Omega$, where $0 \leqslant \Omega < \Delta$, then shearing using the matrix $\left(\begin{smallmatrix} 1 & 0 \\ -k & 1 \end{smallmatrix}\right)$ allows us to see Π as a truncation of the wedge $\pi(\Delta, \Omega)$. ☐

Remark 2.8. — The number $\sigma(\Pi)$ from Lemma 2.7 has geometric meaning: it is equal to $p_1p_2K_{V_\Pi} \cdot C_\Pi$. This means that V_Π is K-positive or K-negative if $\sigma(\Pi)$ is positive or negative respectively. We will say that our polygon Π is *K-positive* or *K-negative* according to the sign of $\sigma(\Pi)$.

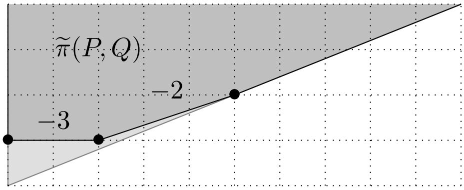


Figure 2.1. The moment polygon $\widetilde{\pi}(5,2)$ for the minimal resolution of $\frac{1}{5}(1,2)$ superimposed on the moment polygon $\pi(5,2)$ for the singularity. We have labelled the self-intersections of the curves in the exceptional locus. The continued fraction expansion of $\frac{5}{2}$ is $3-\frac{1}{2}$, and we see a -3-sphere and a -2-sphere as we move around the boundary of $\widetilde{\pi}(5,2)$ anticlockwise.

Remark 2.9. — The condition $\Delta(\Pi)>0$ is equivalent to requiring that the rays R_1 and R_2 do not intersect, which is necessary for Π to be $\mathbb{Z}$-affine equivalent to a truncated wedge.

Instead of specifying p_1,q_1,p_2,q_2,c, we can equivalently specify the chain $[b_{1,1},\dots,b_{1,r_1}]-c-[b_{2,1},\dots,b_{2,r_2}]$ where:

$$\widetilde{C}_\Pi^2=-c,$$
$$\frac{p_i^2}{p_iq_i-1}=[b_{i,1},\dots,b_{i,r_i}]$$

Remark 2.10. — This chain of spheres $[b_{1,1},\dots,b_{1,r_1}]-c-[b_{2,1},\dots,b_{2,r_2}]$ arises in the minimal resolution of $\widetilde{V}_\Pi\to V_\Pi$ as the preimage of C_Π. Note that $\widetilde{V}_\Pi$ is also toric; its moment polygon $\widetilde{\Pi}$ is obtained from Π by a sequence of truncations at non-Delzant vertices (see Figure 2.1). With our conventions, in the minimal resolution of a vertex of type $\frac{1}{P}(1,Q)$, the exceptional spheres with self-intersections $-b_1,\dots,-b_r$ with $P/Q=[b_1,\dots,b_r]$ are encountered in that order as one moves *anticlockwise* around the boundary of $\widetilde{\Pi}$. Reversing the order corresponds to replacing Q by its multiplicative inverse modulo P (if $P=p^2$, $Q=pq-1$, this means replacing q by $p-q$).

In terms of this chain there is a simple way to compute Δ and Ω:

$$\frac{\Delta}{\Omega}=[b_{1,1},\dots,b_{1,r_1},c,b_{2,1},\dots,b_{2,r_2}].$$

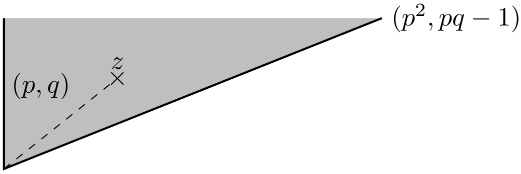


Figure 2.2. The wedge $\pi(p^2, pq-1)$ together with the branch cut for performing a nodal trade.

2.2. Smooth, almost toric QHPs: U_Π

Since x_1 and x_2 are Wahl singularities, we may symplectically smooth these points, replacing them with symplectic rational homology balls B_{p_i,q_i}, see [21]. This operation gives a smooth symplectic QHP which we denote by U_Π.

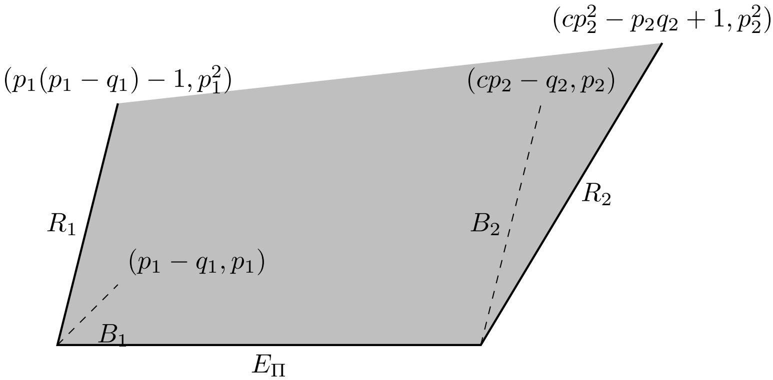
2.2.1. Almost toric structure

The operation of passing from V_Π to U_Π can be visualised by means of an almost toric structure on Π: we perform *nodal trades* at the two vertices of Π, introducing a branch cut at each vertex.

Remark 2.11. — We briefly recall Symington's nodal trades [22]. We can modify the affine structure on $\pi(p^2, pq-1)$ by cutting from the origin along a branch cut in the (p,q)-direction to an interior terminus z, and regluing the two sides using the affine monodromy matrix $\begin{pmatrix} 1+pq & -p^2 \\ q^2 & 1-pq \end{pmatrix}$. More precisely, we choose a coorientation $(q,-p)$ of the branch cut and apply the affine monodromy to tangent vectors as we cross the branch cut in the direction of the coorientation (and its inverse if we cross in the opposite direction). This modification is called a *nodal trade*. The toric fibration on the Wahl singularity $\frac{1}{p^2}(1, pq-1)$ deforms to give an *almost toric fibration* on $B_{p,q}$. This almost toric fibration is a map from $B_{p,q}$ to this modified affine surface whose general fibres are Lagrangian tori; moreover, the affine structure on the base agrees with the natural one given by local action-angle coordinates on a Lagrangian fibration. Over the singular point z, there is a *focus-focus* singular fibre (nodal torus); living over points in the boundary there are circles.

To extend this construction to Π, we perform nodal trades at both vertices, introducing two branch cuts, B_1 and B_2, joining the vertices to interior points z_1, z_2. To determine in which direction the branch cut B_i is to be taken, one must use a $\mathbb{Z}$-affine transformation to take a neighbourhood of the vertex in the model $\pi(p_i^2, p_iq_i - 1)$ to a neighbourhood of the vertex $x_i \in \Pi$; the branch cut should be taken along the image of (p, q) under this transformation. In our model for $\Pi(p_1, q_1, p_2, q_2, c, a)$ from Definition 2.5, the branch cuts B_1 and B_2 for the nodal trades of are made in the directions:

- $(p_1 - q_1, p_1)$ at vertex x_1;
- $(cp_2 - q_2, p_2)$ at vertex x_2.



The manifold U_Π admits an almost toric fibration to this new singular $\mathbb{Z}$-affine surface:

- over the points of the interior of $\Pi \setminus \{z_1, z_2\}$, we have a Lagrangian torus fibre;
- over z_1, z_2 there are a singular Lagrangian fibres (pinched tori);
- over each point of the boundary of Π we have a circle; the preimage of the whole boundary is a symplectic cylinder;
- over the branch cut B_i there lives[4] a Lagrangian disc L_i which becomes immersed p_i-to-1 along its boundary; this is called a *Lagrangian pinwheel*. A neighbourhood of this pinwheel is the symplectic rational homology ball B_{p_i,q_i}.

Remark 2.12. — In general, an almost toric structure can be specified by drawing an *almost toric base diagram*, which is a decorated polygon with focus-focus singularities and branch cuts indicated. The symplectic

[4] Surfaces like this which project to lines in the base of an almost toric fibration are called *visible surfaces* in [22].

4-manifold on which the almost toric structure lives is determined(5) by an almost toric base diagram [22].

2.2.2. The homology of U_Π

It is easy to see that U_Π is a QHP: since $H_*(B_{p_i,q_i};\mathbb{Q}) \cong H_*(B^4;\mathbb{Q})$, the rational homology of U_Π is isomorphic to that of the orbifold V_Π, which is isomorphic to $H_*(\mathbb{CP}^1;\mathbb{Q})$. A generator for $H_2(U_\Pi;\mathbb{Q})$ can be described explicitly as follows. The preimage of the edge E_Π is a symplectic cylinder C with area equal to the affine length a of E_Π. Consider the singular chain $G_\Pi := p_1p_2C - p_2L_1 - p_1L_2$ (where L_i are the Lagrangian discs living over the branch cuts). This is a cycle because $\partial L_1 = p_1\partial C$ and $\partial L_2 = p_2\partial C$. The evaluation of $[\omega]$ on its homology class can be computed by integrating ω over p_2L_1, p_1L_2, p_1p_2C separately, which yields p_1p_2a (as the L_i are Lagrangian). Therefore G_Π generates $H_2(U_\Pi;\mathbb{Q})$.

3. Mutations

3.1. Mutation of polygons

Definition 3.1. — *Suppose we equip a polygon Π with the data of an almost toric base diagram. Given a branch cut B_z emanating from a focus-focus singularity z, let B'_z be the ray emanating from z in the opposite direction to B_z. We assume that B'_z is also disjoint from the other branch cuts. The line $B_z \cup B'_z$ cuts Π into two pieces Π_{upper} and Π_{lower} (where the coorientation points into Π_{upper}). The* mutation *of Π along B_z is the polygon $\Pi_{upper} \cup A\Pi_{lower}$ (or, $\mathbb{Z}$-affine equivalently, $A^{-1}\Pi_{upper} \cup \Pi_{lower}$), where A is the affine monodromy across the branch cut B_z. The mutated almost toric base data is unchanged on Π_{upper} and transformed by A on Π_{lower}.*

Example 3.2. — In Figure 3.1 we see a mutation of almost toric structures on $\mathbb{CP}^2$. The structure before mutation is obtained from the standard toric structure by performing nodal trades at each corner. The affine

(5) To reconstruct the symplectic manifold *together with its almost toric structure*, one must make extra choices at the singularities to determine the asymptotic behaviour of the period lattice as one approaches the singularity. This was first worked out by Vũ Ngoc [24]; with this extra data, the almost toric fibration is determined up to fibred symplectomorphism [20, Theorem 4.60]. Without this data, the total space is still determined up to symplectomorphism.

monodromy for the branch cut B_z is $A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$, which is the unique $A \in \mathrm{SL}(2,\mathbb{Z})$ which satisfies both $A(1,1) = (1,1)$ (so it has B_z as an eigendirection) and $A(1,0) = (0,-1)$ (which means that, after mutation, the origin is an interior point of a straight edge).

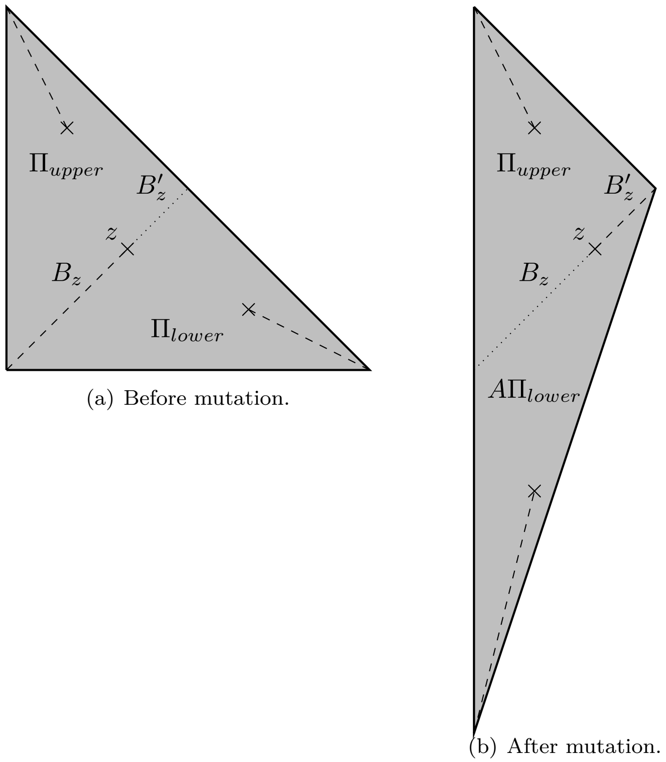


Figure 3.1. An example of mutation between two almost toric base diagrams of $\mathbb{CP}^2$ (coming from the $\mathbb{Q}$-Gorenstein degeneration of $\mathbb{CP}^2$ to $\mathbb{P}(1,1,4)$). The line B_z is dashed in (a) and dotted in (b); the line B'_z is dotted in (a) and dashed in (b). Dashed lines are branch cuts; dotted lines indicate linear continuations of branch cuts and are not part of the almost toric data.

Though the polygons before and after mutation look very different, this operation does not actually change the associated symplectic manifold, as we will now prove:

Lemma 3.3. — *Let X_1, X_2 be almost toric symplectic 4-manifolds with contractible almost toric base diagrams B_1 and B_2. Suppose that B_1 and B_2 are related by a mutation. Then X_1 and X_2 are symplectomorphic.*

Proof. — Let $f\colon X \to A$ be an almost toric fibration with focus-focus fibres over the set $A_{ff} \subset A$. Let $\widetilde{A}$ be the universal cover of $A \setminus A_{ff}$. Action-angle coordinates allow us to define an integral affine structure on $\widetilde{A}$ which descends to an integral affine structure on $A \setminus A_{ff}$. It is this integral affine structure which determines X up to symplectomorphism (this follows by [14, Theorem 1.5] when the bases are contractible because then the Lagrangian Chern class automatically vanishes).

An almost toric base diagram B can be constructed from A as follows. By choosing branch cuts, pick a fundamental domain $B \subset \widetilde{A}$ for the action of the deck group. Let $I\colon \widetilde{A} \to \mathbb{R}^2$ be the developing map for the integral affine structure. The developing map does not descend to A, but its restriction to B can be considered as a "branch" of the developing map on A. The branch cuts form part of the boundary of $I(B) \subset \mathbb{R}^2$; the branch cuts are identified by the action of specified (integral affine) deck transformations of $\widetilde{A}$, so to reconstruct A all we need is the immersed polygon $I(B)$ together with a collection of integral affine transformations ("affine monodromies") which identify pairs of boundary components of $I(B)$. This is precisely the data of an almost toric base diagram.

We often (but not always) pick these branch cuts to point along the eigenvectors of the deck transformations which pair them; if we do this then any two branch cuts which are identified have the property that their images under the developing map coincide, so that $I(B)$ is actually homeomorphic to A. However, the branch cut data is still important for reconstructing the integral affine structure on A: when you cross a branch cut, vectors normal to the cut will still be affected by the affine monodromy. If branch cuts are chosen along eigenlines of the monodromy then we say the diagram is *eigensliced*.

Mutation is the operation on eigensliced almost toric base diagrams which corresponds to changing branch cuts by 180 degrees. This produces another eigensliced almost toric base diagram. Mutation changes the branch cuts, and thereby affects the almost toric base diagram, but does not affect the underlying integral affine manifold A. In particular, it does not affect the symplectomorphism type of X, which depends only on the integral affine structure of A. □

Remark 3.4. — There is another closely related operation on almost toric base diagrams which often accompanies a mutation, namely a *nodal slide*.

This is when a focus-focus point moves in the direction of the eigenvector of its affine monodromy. This also leaves the symplectomorphism type of X unaffected [22, Proposition 6.2], but it does change the Lagrangian torus fibration (whereas a mutation only changes the picture we draw to represent the torus fibration).

3.2. Mutability

Mutation of polygons always makes sense, but it is possible that the mutation of a truncated wedge is no longer a truncated wedge. We therefore make the following definitions:

Definition 3.5. — *Given the polygon $\Pi = \Pi(p_1, q_1, p_2, q_2, c, a)$ and the almost toric structure with branch cuts B_1 and B_2, let $\overline{B}_i$ denote the semi-infinite ray through x_i extending B_i. We say that Π is:*

- right-mutable *if $\overline{B}_1$ intersects the edge R_2,*
- right-borderline *if $\overline{B}_1$ is parallel to R_2,*
- right-immutable *otherwise.*

Left-mutability is defined similarly. The right mutation

$\mathfrak{R}(\Pi)$ is the mutation of Π along B_1. For notational convenience, we will focus entirely on right rather than left mutation in what follows; indeed, one can reflect one's polygon in a vertical line and always work with right mutation.

For our model polygon $\Pi(p_1, q_1, p_2, q_2, c, a)$, the affine monodromy of B_1 (with its coorientation pointing to the left) is

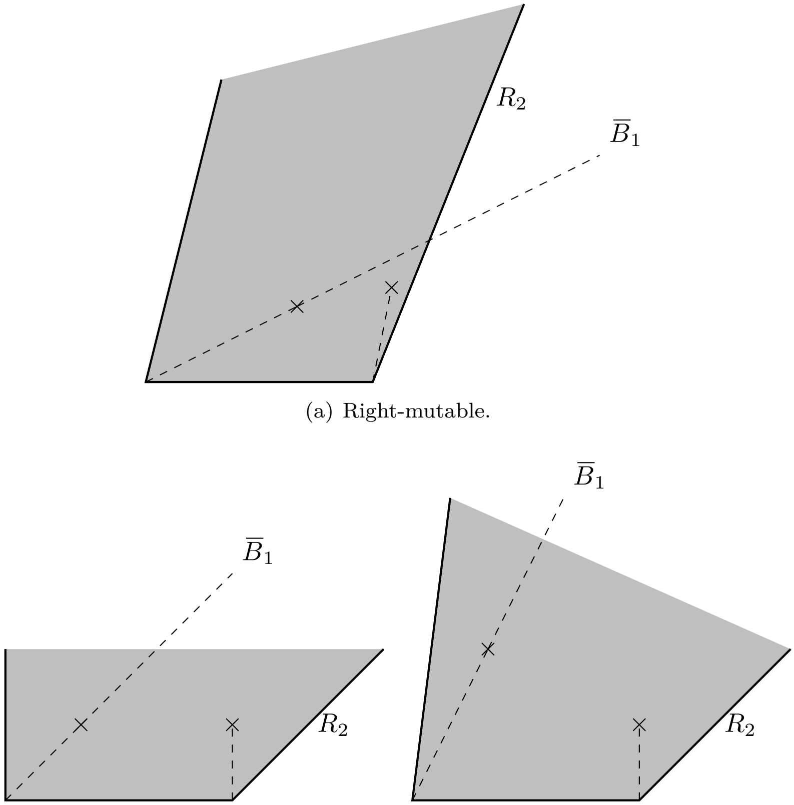
$$A = \begin{pmatrix} 1 + p_1 q_1 - p_1^2 & (p_1 - q_1)^2 \\ -p_1^2 & 1 - p_1 q_1 + p_1^2 \end{pmatrix},$$

and the affine monodromy of B_2 (with its coorientation pointing to the right) is

$$\begin{pmatrix} cp_2^2 - p_2 q_2 + 1 & -(cp_2 - q_2)^2 \\ p_2^2 & 1 + p_2 q_2 - cp_2^2 \end{pmatrix}.$$

Note that, for a right mutation, AE_Π points in the (negative) R_1-direction so $R_1 \cup AE_\Pi$ is now a single edge of $\mathfrak{R}(\Pi)$. Indeed, A is determined by this condition and the condition that it has B_1 as an eigenray.

Lemma 3.6. — *A polygon $\Pi = \Pi(p_1, q_1, p_2, q_2, c, a)$ is right-mutable if and only if $c \leqslant 1$ and $\delta p_2 - p_1 > 0$, where $\delta = -\sigma(\Delta)$. As a consequence, mutability implies $\sigma(\Pi) < 0$. For left-mutability, we use the inequality $\delta p_1 - p_2 > 0$ instead.*



(a) Right-mutable.

(b) Right-borderline.

(c) Right-immutable.

Figure 3.2. Mutability of truncated wedges.

Proof. — Note that if $c \leqslant 0$ then the invariant $\Delta(\Pi)$ from Lemma 2.7 is negative, so we do not consider this case.

If $c \geqslant 2$ then the slope $\frac{p_2^2}{cp_2^2 - p_2q_2+1}$ of R_2 is less than or equal to 1. The slope $\frac{p_1}{p_1-q_1}$ (or 1 if $p_1 = q_1 = 1$) of $\overline{B}_1$ is greater than or equal to 1, so these lines never intersect and the polygon is not right-mutable.

If $c = 1$ then we have right-mutability if and only if the slope of $\overline{B}_1$ is strictly less than the slope of R_2:

$$\frac{p_2^2}{p_2^2 - p_2q_2 + 1} > \frac{p_1}{p_1 - q_1}.$$

This gives

$$p_2(p_1q_2 - p_2q_1) - p_1 > 0,$$

which is again equivalent to $\delta p_2 - p_1 > 0$. In particular, we see that this can only hold if $\sigma(\Pi)$ is negative.

The criterion for left-mutability is proved similarly. □

Remark 3.7. — Mutability also makes sense when $\Delta(\Pi) < 0$, and we always get mutability in both directions. However, these polygons are not truncated wedges, so we ignore them.

3.3. Effect of mutations

The polygon $\mathfrak{R}(\Pi)$ has vertices at $x_1' = Ax_2$ and at x_2', the point of intersection between $\overline{B}_1$ and R_2. Since the type of a vertex is invariant under $\mathbb{Z}$-affine transformations, we see that x_1' has type $\frac{1}{p_2^2}(1, p_2q_2 - 1)$.

Remark 3.8. — Remembering our convention that a smooth point has $(p_1, q_1) = (1, 0)$ or $(p_2, q_2) = (1, 1)$ if it occurs on the left or on the right respectively, one sees that this should switch under a mutation; however, if $(p_2, q_2) = (1, 1)$ then the polygon is not right-mutable, so it is never an issue.

To identify the type of vertex x_2' we need a recognition lemma:

Lemma 3.9. — *Suppose we have an edge R of a polygon and a branch cut B disjoint from R whose semi-infinite extension $\overline{B}$ intersects R. Make a $\mathbb{Z}$-affine transformation M to put R in the vertical direction with the polygon on its right. If $-MB$ points in the direction $(p, q + kp)$ with $0 < q < p$ then the result of mutation along B will have a vertex of type $\frac{1}{p^2}(1, pq - 1)$ at the point of intersection between B and R.*

Proof. — The polygon $\pi(p^2, pq - 1)$ equipped with a branch cut starting at the origin and pointing in the (p, q) direction can be mutated to get the right half-space with a branch cut pointing out to infinity in the (p, q)-direction. Shearing this using matrices $\left(\begin{smallmatrix} 1 & 0 \\ k & 1 \end{smallmatrix}\right)$ gives the local models in the lemma, which will then necessarily give (a shear of) the original polygon $\pi(p^2, pq - 1)$ upon mutation. (The sign in $-MB$ is because we reverse the direction of the branch cut when we mutate). □

Lemma 3.10. — *Let $\mathfrak{R}(\Pi)$ be the right mutation of $\Pi(p_1, q_1, p_2, q_2, 1, a)$. Define:*

$$\delta := -\sigma(\Pi) = p_2q_1 - p_1q_2,$$
$$p_3 := \delta p_2 - p_1, \qquad q_3 := \delta q_2 - q_1.$$

Then:

- *the affine length of* $E_{\mathfrak{R}(\Pi)}$ *is* p_1a/p_3*;*
- *the vertex* x_2' *has type* $\frac{1}{p_3^2}(1, p_3q_3 - 1)$ *where*
- $\sigma(\mathfrak{R}(\Pi)) = \sigma(\Pi) = -\delta$*.*

Proof. — To find x_2', we parametrise B_1 as $(\tau_1(p_1 - q_1), \tau_1 p_1)$ (or (τ_1, τ_1) if $p_1 = q_1 = 1$) and R_2 as $(a + \tau_2(p_2(p_2 - q_2) + 1), \tau_2 p_2^2)$ and we see this intersection occurs when

$$\tau_1 = \frac{p_2^2 \tau_2}{p_1}$$
$$\tau_2 = \frac{p_1 a}{\delta p_2 - p_1},$$

where $\delta = p_1q_2 - p_2q_1$ (since $c = 1$). After mutation, a fraction τ_2 of the affine length of R_2 becomes the edge $E_{\mathfrak{R}(\Pi)}$, so the affine length of this edge in the new polygon is $\tau_2 = ap_1/(\delta p_2 - p_1)$.

To see what kind of vertex we get at x_2' after a left mutation, we can use the affine transformation $M := \left(\begin{smallmatrix} -p_2^2 & p_2(p_2-q_2)+1 \\ -p_2q_2-1 & q_2(p_2-q_2)+1 \end{smallmatrix}\right)$ to put the ray R_2 in the direction $(0, -1)$; this makes R_2 vertical and puts Π to the right of R_2 so we may apply Lemma 3.9 and compute $-MB_1 = (p_3, q_3)$ to get the type $\frac{1}{p_3^3}(1, p_3q_3 - 1)$ of x_2'. Since B_1 points in the direction $(p_1 - q_1, p_1)$, $-MB_1$ points in the direction (p_3, q_3) where $p_3 = \delta p_2 - p_1$ and $q_3 = \delta q_2 - q_1$.

For the final part of the lemma, $\sigma(\Pi) = p_2q_1 - p_1q_2$ and

$$\begin{aligned}\sigma(\mathfrak{R}(\Pi)) &= p_3q_2 - p_2q_3 \\ &= (\delta p_2 - p_1)q_2 - p_2(\delta q_2 - q_1) \\ &= p_2q_1 - p_1q_2.\end{aligned}$$

□

3.4. Mori sequences

3.4.1. Definition of Mori sequences

Definition 3.11. — *Let* (p_1, q_1) *and* (p_2, q_2) *be pairs of positive integers with* $\gcd(p_i, q_i) = 1$*,* $q_i \leqslant p_i$*. Using the notation*

$$[b_1, \ldots, b_r] = b_1 - \frac{1}{b_2 - \frac{1}{\cdots - \frac{1}{b_r}}}$$

for continued fractions, let

$$\frac{p_i^2}{p_iq_i - 1} = [b_{i,1}, \ldots, b_{i,r_i}],$$

and suppose that $\Delta = p_1^2 + p_2^2 + \sigma p_1 p_2 > 0$ *and that the rational number*

$$\frac{\Delta}{\Omega} := [b_{1,1}, \ldots, b_{1,r_1}, 1, b_{2,1}, \ldots, b_{2,r_2}]$$

is well-defined (no division by zero). Let

$$\delta = p_1 q_2 - p_2 q_1$$

and suppose that $\delta > 0$*. The* Mori sequence $M(p_1, q_1; p_2, q_2)$ *is the sequence of pairs* (p_i, q_i) *extending* $(p_1, q_1), (p_2, q_2)$ *and satisfying the recursions*

$$p_{i+2} = \delta p_{i+1} - p_i,$$
$$q_{i+2} = \delta q_{i+1} - q_i.$$

3.4.2. Behaviour of Mori sequences

If (p_i, q_i) is a Mori sequence then we can recast the recursion relation for the p_i as a matrix equation

$$\begin{pmatrix} p_{i+1} \\ p_{i+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \delta \end{pmatrix} \begin{pmatrix} p_i \\ p_{i+1} \end{pmatrix}.$$

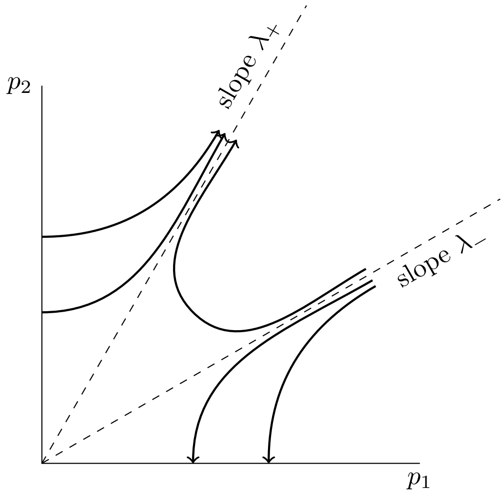
This matrix $M = \left(\begin{smallmatrix} 0 & 1 \\ -1 & \delta \end{smallmatrix}\right)$ has eigenvalues $\lambda_\pm = \frac{\delta \pm \sqrt{\delta^2 - 4}}{2}$; if $\delta \geqslant 2$ then these are real and satisfy $\lambda_- \lambda_+ = 1$. In this case, there are eigenrays with slopes $\lambda_\pm$.

Repeated application of M defines a discrete dynamical system on the plane, and the behaviour of $(p_{i+1}, p_{i+2}) = M^i(p_1, p_2)$ under repeated application of M is indicated by the arrows in the figure. This behaviour separates into three distinct regions, separated by the eigenrays:

- In the region $p_2 > \lambda_+ p_1$, the Mori sequence is increasing and the ratio p_{i+1}/p_i tends to λ_+ from above.
- In the region $p_2 < \lambda_- p_1$, the Mori sequence is decreasing and terminates when $M^i(p_1, p_2)$ leaves the positive quadrant.
- In the region between the two eigenrays, the Mori sequence decreases, reaches a minimum, then increases again. It does not terminate in either direction.

Note that (p_1, p_2) lives in the region between the eigenrays if and only if $\Delta(\Pi) = p_1^2 + p_2^2 + \delta p_1 p_2$ is negative. Recall from Lemma 2.7 and Remark 2.9 that $\Delta(\Pi) > 0$ for all truncated wedges, so we find ourselves automatically in the situation where our Mori sequence is increasing or decreasing (if

$p_1 < p_2$ or $p_2 < p_1$ respectively).



3.5. Infinite mutability

DEFINITION 3.12. — *We say that a K-negative polygon Π is* infinitely right-mutable *if $\mathfrak{R}^j(\Pi)$ is right-mutable for $j = 0, 1, \ldots$. From the previous subsection, this is equivalent to $\delta \geqslant 2$, $p_1 \leqslant p_2$.*

If Π is infinitely right-mutable, then, by Lemma 3.10, we obtain a sequence of mutations $\Pi(p_i, q_i, p_{i+1}, q_{i+1}, 1, a_i)$ where (p_i, q_i) is a Mori sequence $M(p_1, q_1; p_2, q_2)$ ($\delta = -\sigma(\Pi)$).

By construction, the symplectic manifold $U_{\mathfrak{R}^{j-1}(\Pi)}$ contains Lagrangian pinwheels L_{p_j,q_j} and $L_{p_{j+1},q_{j+1}}$ as visible surfaces in its almost toric fibration, see Section 2.2.1. The manifolds $U_{\mathfrak{R}^{j-1}(\Pi)}$ and U_Π are symplectomorphic by Lemma 3.3, since their almost toric structures are related by mutations. We summarise this in the following corollary.

COROLLARY 3.13. — *Let $\Pi = \Pi(p_1, q_1, p_2, q_2, 1, a)$ be an infinitely mutable polygon. The symplectic manifold U_Π contains Lagrangian pinwheels L_{p_i,q_i} where (p_i, q_i) is the Mori sequence $M(p_1, q_1; p_2, q_2)$ with $\delta = -\sigma(\Pi) = p_1q_2 - p_2q_1$.*

In practice, we are looking for these Mori sequences of pinwheels in *compact* symplectic manifolds, so it is important that we can perform the sequence of mutations in a compact subdomain of U_Π. To that end, we introduce some new notation:

Definition 3.14. — *Given a truncated wedge* $\Pi = \Pi(p_1, q_1, p_2, q_2, c, a)$ *and two positive real numbers* ℓ_1, ℓ_2*, let* $y_i \in R_i$ *be the unique point on the ray* R_i *at a distance* ℓ_i *from* x_i*, for* $i = 1, 2$*. Define* Π_{ℓ_1,ℓ_2} *to be the convex hull of* x_1, x_2, y_1, y_2*. Let* $V_\Pi(\ell_1, \ell_2)$ *(respectively* $U_\Pi(\ell_1, \ell_2)$*) be the preimage of* Π_{ℓ_1,ℓ_2} *under the moment map (respectively almost toric fibration).*

The manifold $U_\Pi(\ell_1, \ell_2)$ is a compact symplectic manifold whose boundary is a lens space $L(\Delta, \Omega)$ of contact-type. The diffeomorphism type is independent of the parameters a, ℓ_1, ℓ_2, but these are important for the symplectic structure.

Lemma 3.15. — *Let* $\Pi^- = \Pi(p_1, q_1, p_2, q_2, 1, a^-)$ *be an infinitely right-mutable* K*-negative polygon and let* ℓ_1, ℓ_2 *be positive real numbers. If* $\ell_2 > \frac{a^-}{\lambda_+^2 - 1}$ *then the right mutations of* Π^- *may be performed inside the subpolygon* $\Pi^-_{\ell_1,\ell_2}$*. As a consequence,* $U_{\Pi^-}(\ell_1, \ell_2)$ *contains an infinite Mori sequence* $M(p_1, q_1; p_2, q_2)$ *of Lagrangian pinwheels.*

Proof. — As we can see in Figure 3.3, each mutation we perform "eats up" a certain amount of the affine length ℓ_2 of R_2: by Lemma 3.10, the first mutation uses $a_1 := a^- p_1/p_3$ and the k^{th} mutation uses $a_k := a_{k-1} p_k/p_{k+2}$. Therefore, in total, to perform arbitrarily many mutations of this subpolygon, we need ℓ_2 to be at least

$$a^- \frac{p_1}{p_3}\left(1 + \frac{p_2}{p_4}\left(1 + \frac{p_3}{p_5}\left(1 + \cdots\right)\right)\right).$$

By the discussion in Section 3.4, since $\frac{p_2}{p_1} > \lambda_+$, the sequence of quotients $\frac{p_i}{p_{i+1}}$ is increasing and its limit is λ_-; likewise, the sequence $\frac{p_i}{p_{i+2}} = \frac{p_i}{p_{i+1}}\frac{p_{i+1}}{p_{i+2}}$ is increasing and its limit is λ_-^2. Therefore, the infinite sum is bounded from above by

$$a^- \lambda_-^2 \left(1 + \lambda_-^2\left(1 + \cdots\right)\right) = \frac{a^- \lambda_-^2}{1 - \lambda_-^2} = \frac{a^-}{\lambda_+^2 - 1},$$

as required. □

4. When mutation fails

4.1. Immutability: flips and the initial antiflip

Suppose we have a right-immutable polygon Π. We can make a symplectic deformation (U_t, ω_t) of U_Π and a deformation of the almost toric structure to put us into a situation where the mutation can be performed.

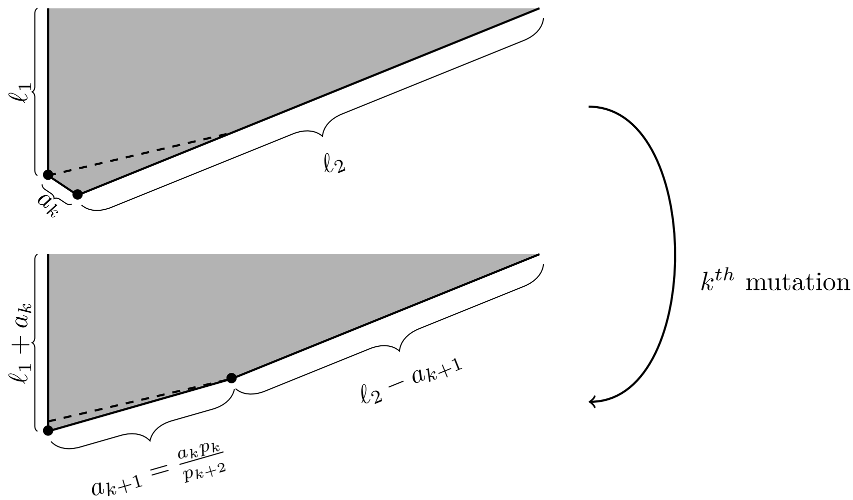


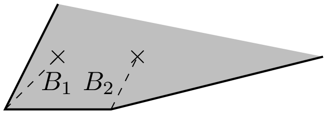
Figure 3.3. A mutation eats up available affine length. Before the mutation, the right-hand edge R_2 has affine length ℓ_2; after mutation, it has lost affine length a_{k+1}.

We will show this by giving a family of almost toric base diagrams Π_t (which determine the symplectic manifolds U_t). See Figure 4.1 for an illustration of this deformation.

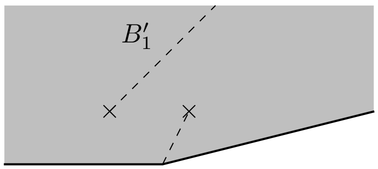
(1) We first perform the right mutation along B_1 (as Π is immutable, this will not be a truncated wedge: see Figure 4.1). This replaces B_1 with an opposite branch cut B_1'.
(2) Pick a smooth path $\gamma\colon [0,1] \to \Pi$ such that:
   - $\gamma(0) = z_1$,
   - $\gamma(t) \notin B_2$ for all $t \in [0,1]$,

   Let $B_1(t)$ (respectively $B_1'(t)$) be the ray pointing the direction of B_1 (respectively B_1') and emanating from $\gamma(t)$. Assume that $\gamma(1)$ is sufficiently far to the right so that $B_1(1)$ intersects R_2 at some point x.
(3) When we perform a mutation along $B_1'(1)$, we therefore obtain a new truncated wedge having x as a vertex.

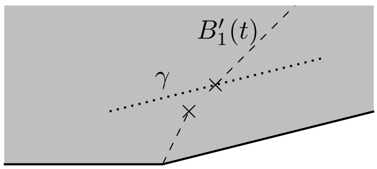
Remark 4.1. — Note that this is a continuous deformation of symplectic manifolds: although it involves steps which look discrete (mutations) these steps do not affect the symplectomorphism type of U_t (see Lemma 3.3). We are simply choosing to draw pictures using different branch cuts at different



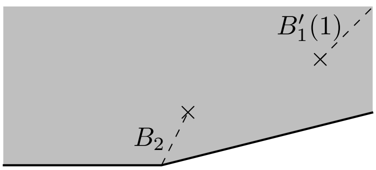
(a) We start with a K-positive polygon Π with branch cuts B_1, B_2.



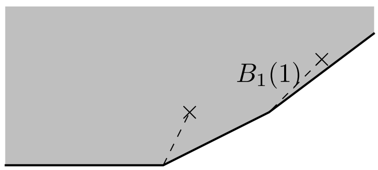
(b) We perform a right mutation, replacing the branch cut B_1 with B_1'.



(c) We parallel-translate B_1' along the path γ...



(d) ...until it is sufficiently far to the right...



(e) ...then we mutate again, switching $B_1'(1)$ to $B_1(1)$.

Figure 4.1. A cartoon of an initial antiflip.

stages of the deformation, as this allows us to highlight different aspects of the geometry.

Definition 4.2. — *Suppose we have a K-positive polygon*

$$\Pi^+(a^+) := \Pi(p_0, q_0, p_1, q_1, c, a^+).$$

Suppose that $\Pi^-(a^-)$ is the result of performing the aforementioned operations to $\Pi^+(a^+)$, where a^- is the affine length of the compact edge in the truncated wedge at the end of the process. We call $\Pi^-(a^-)$ the initial right-antiflip polygon *of $\Pi^+(a^+)$ with parameter a^- (initial left-antiflip is defined in the obvious way). We call the symplectic manifold $(U_{\Pi^-(a^-)}, \omega_1)$ the* initial antiflip of the symplectic form with parameter a^-. *We will omit the $a^\pm$ when it is unimportant to the discussion.*

Remark 4.3. — The reverse procedure, in which we begin with a left-immutable K-negative polygon and follow the same steps to force a left mutation, is called the *flip*.

The parameter a^- may be chosen freely by picking γ suitably; however, when we work with a bounded subset $\Pi^+_{\ell_1,\ell_2} \subset \Pi^+$ as in Definition 3.14, we will not have complete freedom and a^- will need to be chosen sufficiently small. Namely, after an initial antiflip, $\Pi^+(a^+)_{\ell_1,\ell_2}$ is replaced by $\Pi^-(a^-)_{\ell_1+a^+,\ell_2-a^-}$, so we need $a^- < \ell_2$. If we wish additionally to ensure infinite right-mutability within this bounded polygon, we need the stronger inequality

$$\ell_2 - a^- > \frac{a^-}{\lambda_+^2 - 1},$$

by Lemma 3.15. This can also be achieved by picking a^- sufficiently small. We deduce the following corollary:

Corollary 4.4. — *Let $\Pi^+(a^+) = \Pi(p_1, q_1, p_2, q_2, c, a^+)$ be a K-positive truncated wedge whose initial antiflip $\Pi^-(a^-) = \Pi(p_1, q_1', p_2, q_2, 1, a^-)$ is infinitely right-mutable (the numbers q_1', p_2, q_2 will be defined in Lemma 4.5). If we are given $\ell_1, \ell_2 > 0$, then there exists a constant $C > 0$ such that, for all $0 < a^- \leqslant C$, the full Mori sequence of right mutations can be performed on $\Pi^-(a^-)_{\ell_1+a^+,\ell_2-a^-}$. In particular, the initial antiflip of the symplectic form with parameter a^- in the range $(0, C]$ admits a Mori sequence $M(p_1, q_1; p_2, q_2)$ of Lagrangian pinwheels.*

4.2. Numerology of the initial antiflip

The following lemma is proved using Lemma 3.9; its proof is very similar to Lemma 3.10, and we omit it:

Lemma 4.5. — *Suppose we have a K-positive polygon*

$$\Pi^+ := \Pi(p_0, q_0, p_1, q_1, c, a^+).$$

Let

$$\delta = \sigma(\Pi) = (c-1)p_0p_1 + p_1q_0 - p_0q_1.$$

Given a positive real number $a^- > 0$, the initial antiflip polygon $\Pi^-(a^-)$ is $\mathbb{Z}$-affine isomorphic to the polygon

$$\Pi^-(a^-) = \Pi(p_1, q_1', p_2, q_2, 1, a^-),$$

where:

$$q_1' = \begin{cases} 0 \text{ if } p_1 = q_1 = 1 \\ q_1 \text{ otherwise,} \end{cases}$$
$$p_2 := \delta p_1 + p_0,$$
$$q_2 := \frac{\delta + p_2 q_1}{p_1}.$$

The following lemma is easy to check using Lemma 2.7 and the definitions of p_2, q_2:

Lemma 4.6. — *The initial antiflip polygon Π^- is a left-immutable, K-negative polygon with*

$$\sigma(\Pi^-) = -\delta, \quad \Delta(\Pi^+) = \Delta(\Pi^-), \quad \Omega(\Pi^+) = \Omega(\Pi^-).$$

Consequently, both Π^+ and Π^- are truncations of the same wedge $\pi(\Delta, \Omega)$.

4.3. Variation of the cohomology class $[\omega_t]$

Each almost toric base diagram in the family Π_t from Section 4.1 determines a symplectic manifold, so we get a symplectic deformation (U_t, ω_t). The de Rham cohomology group $H^2_{dR}(U_t)$ is one-dimensional, so the cohomology class $[\omega_t]$ is determined by its integral over some fixed[(6)] homology class.

We use as our fixed class the unique class $G_t \in H_2(U_t; \mathbb{R})$ such that $K_{U_t} \cdot G_t = \delta$. Since K_{U_t} is an integral class, this means that G_t is also an integral class, hence constant in the family. Recall the class G_Π from Section 2.2.2:

- When $t = 0$, we know that $K_{U_{\Pi^+}} \cdot G_{\Pi^+} = \sigma(\Pi^+) = \delta$, so take $G_0 = G_{\Pi^+}$.
- When $t = 1$, we know that $K_{U_{\Pi^-}} \cdot G_{\Pi^-} = \sigma(\Pi^-) = -\delta$, so take $G_1 = -G_{\Pi^-}$.

We know that $\int_{G_{\Pi^+}} \omega_0 = p_0 p_1 a^+$ and $\int_{G_{\Pi^-}} \omega_1 = p_1 p_2 a^-$. Therefore, at the level of cohomology classes $[\omega_t]$, the deformation of ω_t gives a path in $H^2_{dR}(U) = \mathbb{R}$ from $a^+ p_0 p_1$ to $-a^- p_1 p_2$. In particular, at some point in this path ω_t is exact (at this point the edge has length zero, so G_Π consists of two multiples of Lagrangian discs sharing a common circle boundary).

(6) i.e. constant with respect to the Gauss–Manin connection.

The cohomology $H^2_{dR}(U)$ inherits a $\mathbb{Z}$-affine structure from its isomorphism with $H^2(U;\mathbb{Z})\otimes\mathbb{R}$, so there is an intrinsic notion of affine distances d_{aff} along lines of rational slope. For surfaces of general type, we use this to give an estimate on how far one needs to deform $[\omega]$ away from the canonical class before one gets unbounded Mori sequences of Lagrangian pinwheels using our antiflip-and-mutate construction:

Lemma 4.7. — *Let $\Pi^+ = \Pi(p_0, q_0, p_1, q_1, c, a^+)$ be a K-positive truncated wedge whose initial antiflip polygon is infinitely right-mutable (so $\delta = -\sigma(\Pi^+) = \sigma(\Pi^-) \geqslant 2$). Suppose that a compact $U_{\Pi^+}(\ell_1,\ell_2)$ embeds symplectically into a symplectic manifold (X,ω) with $[\omega] = K_X$. Let ω_t be the initial antiflip deformation of the symplectic form on X along the submanifold $U_{\Pi^+}(\ell_1,\ell_2)$ with parameter a^-. Then there exists a constant $\epsilon > 0$ such that (X,ω_t) contains a Mori sequence of Lagrangian pinwheels when $d_{aff}([\omega_0],[\omega_t]) \in (\delta, \delta+\epsilon]$.*

Proof. — By Corollary 4.4, there is a constant $C > 0$ such that the initial antiflip with parameter $a^- \in (0, C]$ contains a Mori sequence of Lagrangian pinwheels. Therefore (X,ω_t) contains a Mori sequence of Lagrangian pinwheels whenever $\int_{G_t}\omega_t \in [-Cp_1p_2, 0)$. Let t_0 and t_1 be the times such that $\int_{G_{t_0}}\omega_{t_0} = 0$ and $\int_{G_{t_1}}\omega_{t_1} = -Cp_1p_2$ (we have $t_0 < t_1$ since the ω_t-area of G_t is decreasing in t).

Since $[\omega] = K_X$, the number $a^+p_0p_1$ is integral (it is the canonical class evaluated on the generator $G_{\Pi^+} \in H_2(U;\mathbb{Z})$ from Section 2.2.2). In fact, $a^+p_0p_1 = \delta = -\sigma(\Pi^+) = \sigma(\Pi^-)$. Therefore, $d_{aff}(\omega_0,\omega_{t_0}) = \delta$ and $d_{aff}(\omega_0,\omega_{t_1}) = \delta + Cp_1p_2$, so we take $\epsilon = Cp_1p_2$. □

4.4. Link with Mori theory

Given a K-positive polygon Π^+, we have constructed an initial antiflip Π^- with the property that U_{Π^+} is symplectic deformation equivalent to U_{Π^-}. This whole discussion was inspired by results in Mori theory [6]. Here is an alternative, Mori-theoretic proof that U_{Π^+} and U_{Π^-} are diffeomorphic:

Theorem 4.8. — *Let Π^+ be a K-positive truncated wedge and let Π^- be its initial antiflip. The manifolds U_{Π^+} and U_{Π^-} are diffeomorphic.*

Proof. — The variety V_{Π^-} admits a $\mathbb{Q}$-Gorenstein smoothing $\pi^-\colon \mathcal{V}^- \to \mathbb{C}$. The curve $C_{\Pi^-} \subset V_{\Pi^-} \subset \mathcal{V}^-$ is a $K_{\mathcal{V}_{\Pi^-}}$-negative curve, and, in this situation, Mori theory furnishes us with a *flip* $\pi^+\colon \mathcal{V}^+ \to \mathbb{C}$ such that:

- $\pi^+\colon \mathcal{V}^+ \to \mathbb{C}$ is a $\mathbb{Q}$-Gorenstein smoothing of V_{Π^+};

- there is a biholomorphism $f\colon \mathcal{V}^+ \setminus C^+ \to \mathcal{V}^- \setminus C^-$ such that $\pi^- = \pi^+ \circ f$.

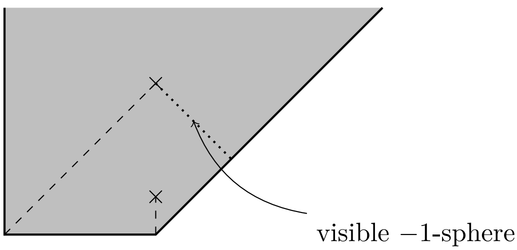
See [6, proof of Corollary 3.23, p. 44 of arXiv version] for a justification of the particular numbers involved in the definitions of the polygons $\Pi^{\pm}$.

The smooth fibre of the $\mathbb{Q}$-Gorenstein smoothing $\pi^{\pm}\colon \mathcal{V}^{\pm} \to \mathbb{C}$ is diffeomorphic to $U_{\Pi^{\pm}}$, and since $\mathcal{V}^-$ and $\mathcal{V}^+$ are fibre-preservingly biholomorphic away from the singular fibre this means that U_{Π^-} and U_{Π^+} are diffeomorphic to one another. $\square$

Of course, in Mori theory, a $\mathbb{Q}$-Gorenstein smoothing with at worst canonical singularities of *any* K-negative V_Π (not necessarily an initial antiflip) admits a flip. In terms of our pictures, the algorithm to find the flip is to perform left mutations *down* the Mori sequence until your K-negative polygon is not longer left-mutable. At that point, one of two things happens:

- the polygon becomes left-immutable, in which case you perform the flip as in Definition 4.2;
- the polygon becomes borderline for left-mutability.

In the borderline case, B_2 is parallel to R_1. In this case, there is a visible surface in the almost toric base Π, connecting the singular point z_2 at the end of B_2 to the edge R_1 (visible surfaces are surfaces which project to paths in the almost toric base; see [22, Definition 7.2]). This visible surface is a symplectic -1-sphere (see Symington [22, Lemma 7.11]). This corresponds to the phenomenon of *divisorial contraction* in the minimal model programme; rather than the 3-fold $\mathbb{Q}$-Gorenstein smoothing of V_Π admitting a flip along C_Π, a whole surface can be contracted; this surface is the union of C_Π and all these visible -1-spheres.



Remark 4.9. — We remark that the term "antiflip" is not always a well-defiend operation in algebraic geometry: not only is there a whole Mori sequence of antiflips, but it is entirely possible for a 3-fold containing a curve C with $K \cdot C > 0$ (e.g. some $\mathbb{Q}$-Gorenstein smoothings of V_Π for a

K-positive Π) not to arise as a flip at all. See [6] for a discussion of when antiflips exist in the algebro-geometric sense.

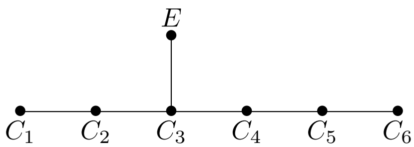
4.5. Flips of type k1A

The paper [6] also discusses flips where the K-negative surface has only one Wahl singularity, obtained by $\mathbb{Q}$-Gorenstein smoothing V_Π for some K-negative Π. We explain by example how this situation arises in our almost toric pictures.

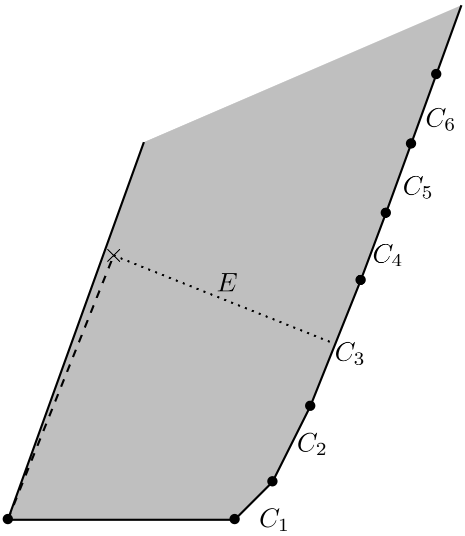
Example 4.10. — The following chain defines a K-negative polygon Π such that the QHP U_Π is a symplectic filling of $L(11,3)$:

$$[2,5,3] - 1 - [2,3,2,2,7,3].$$

If we $\mathbb{Q}$-Gorenstein smooth the singularity $[2,5,3]$ and take the minimal resolution of the other singularity then we find a configuration of spheres $C_1,\ldots,C_6,E$, where $\bigcup C_i$ is the exceptional locus of the minimal resolution ($[-C_1^2,\cdots,-C_6^2] = [2,3,2,2,7,3]$) and E is a -1-sphere, intersecting according to the following graph:



We can also understand this in terms of almost toric pictures. An almost toric picture of the k1A neighbourhood can be obtained by performing a single nodal trade the left-hand vertex of Π. The minimal resolution of the other vertex can also be performed torically. We now see the -1-sphere as a visible surface, since the branch cut is parallel to the edge representing the sphere C_3 in the minimal resolution.



In our picture, the k1A flip is no different from the k2A flip: one simply performs one nodal trade and mutation at a time.

4.6. A topological viewpoint

An almost toric structure on a truncated wedge Π exhibits U_Π as a handlebody obtained by attaching two Lagrangian 2-handles (the pinwheel discs) to $S^1 \times B^3$. The process of performing a flip or initial antiflip is, topologically, a handleslide, from which point of view it is clear that they are diffeomorphic.

On the other hand, if we think of them as smoothings of singular orbifolds then the flip, initial antiflip and all the mutations can be seen as compositions of well-known topological operations:

(1) Find two rational homology balls B_{p_i,q_i}, $i = 1, 2$. Perform generalised rational blow-up in both balls, yielding Hirzebruch–Jung chains of exceptional spheres $[b_{i,1}, \ldots, b_{i,r_i}]$ representing the continued fractions $\frac{p_i^2}{p_i q_i - 1}$.

(2) If you can find another curve C in the rational blow-up with self-intersection $-c$ such that the union of the Hirzebruch–Jung chains and C forms a chain

$$[b_{1,1}, \ldots, b_{1,r_1}] - c - [b_{2,1}, \ldots, b_{2,r_2}],$$

then continue.

(3) Perform blow-up and blow-down on this chain to transform it into another chain of the form

$$[b'_{1,1},\dots,b'_{1,r'_1}] - c' - [b'_{2,1},\dots,b'_{2,r'_2}].$$

with $[b'_{i,1},\dots,b'_{i,r'_i}] = \frac{(p'_i)^2}{p'_i q'_i - 1}$ for some p'_i, q'_i, $i = 1,2$.

(4) Rationally blow down the bracketed Hirzebruch–Jung chains at either end to obtain a new 4-manifold with two new rational homology balls $B_{p'_i,q'_i}$, $i = 1,2$.

Such a string of operations need not yield a result diffeomorphic to the manifold you started with; from this point of view, the fact that the flip, initial antiflip and its mutations are all diffeomorphic is something of a miracle.

Example 4.11. — Suppose we can rationally blow-up a $B_{2,1}$ to get a chain $[4] - 3$ (we will see an example of this in the quintic surface later). Starting with the chain $[4,3]$ we can blow up a point on the -4-sphere (away from its intersection with the -3-sphere) to get $[1,5,3]$, then once on the -1-sphere then rationally blow-down the $[2,5,3]$ to get the initial antiflip. If we want to get the first right mutation of the initial antiflip, we continue blowing up and down:

$$\begin{gathered}[1,2,5,3]\\ [1,5,3]\\ [2,1,6,3]\\ [2,2,1,7,3]\\ [2,3,1,2,7,3]\\ [2,4,1,2,2,7,3]\\ [2,5,1,2,2,2,7,3]\\ [2,5,2,1,3,2,2,7,3]\\ [2,5,3,1,2,3,2,2,7,3]\end{gathered}$$

Finally, we can rationally blow-down $[2,5,3]$ and $[2,3,2,2,7,3]$ to get $B_{5,3}$ and $B_{14,9}$.

5. Examples

Our examples will be built by smoothing certain singular surfaces to find starting configurations of rational homology balls to which we can apply Lemma 3.15.

5.1. Symplectic smoothing

We wish to consider three fillings of the lens space $L(p^2, pq-1)$:

(1) the rational homology ball $B_{p,q}$,
(2) the singularity of type $\frac{1}{p^2}(1, pq-1)$ (an orbifold filling),
(3) the minimal resolution of this singularity.

All three are almost toric (2 and 3 are actually toric) and have the same contact boundary. Symington defines generalised rational blowdown as the surgery of almost toric manifolds going from 3 to 1; in other words, it is a surgery of almost toric symplectic manifolds defined by performing surgery on the almost toric base diagrams. Similarly:

Definition 5.1. — *We define symplectic smoothing as the surgery of almost toric orbifolds going from 2 to 1.*

Example 5.2. — The symplectic smoothing of V_Π is U_Π.

We remark that you do not need a global almost toric structure to perform these surgeries, only one over the region where the surgery is taking place.

Lemma 5.3. — *If V is a surface with Wahl singularities, which admits a $\mathbb{Q}$-Gorenstein smoothing whose total space supports a relatively ample line bundle, then any smooth fibre of this smoothing is a surface symplectomorphic to the symplectic smoothing U of V.*

Proof. — The relatively ample line bundle yields a symplectic form on all the fibres (away from the singular locus) and a symplectic connection on this family of symplectic manifolds. The link of each Wahl singularity in V is a lens space $L(p^2, pq-1)$ of contact type (equipped with a Milnor-fillable contact structure), and we can symplectically parallel transport this link into the smooth fibres. Each smooth fibre X therefore contains a separating lens space Σ of contact type; we will write $X = U \cup_\Sigma (X \setminus U)$ where U is the region in X which has Σ has convex (rather than concave) boundary. This subset U is a symplectic filling of Σ.

A $\mathbb{Q}$-Gorenstein smoothing of a Wahl singularity has Milnor number zero, so U is a rational homology ball. By Lisca's classification [11] of symplectic fillings of lens spaces, U is diffeomorphic to $B_{p,q}$. Bhupal–Ono [1] showed that this is a classification up to symplectic deformation, but $B_{p,q}$ has trivial second cohomology, so in this case it is a classification up to symplectomorphism. Thus U is symplectomorphic to $B_{p,q}$, and X is obtained from V by symplectic smoothing. □

5.2. Strategy

We now explain how to construct examples of symplectically embedded copies of $U_{\Pi^+}(\ell_1, \ell_2)$ in compact complex surfaces of general type (for suitable K-positive polygons Π^+ and real numbers ℓ_1, ℓ_2) using algebraic geometry. Then we will perform the initial antiflip of the symplectic form and obtain an infinitely mutable $U_{\Pi^-}(\ell_1, \ell_2)$ containing a Mori sequence of Lagrangian pinwheels.

Recall that a KSBA-stable surface is a complex projective surface with semi-log canonical singularities and ample dualising sheaf. If V is a KSBA-stable surface with at worst Wahl singularities then it is $\mathbb{Q}$-factorial, so we can replace this condition with having ample canonical bundle; let k be a positive integer such that $K_V^{\otimes k}$ is very ample. Pulling back a Fubini–Study form along the k-canonical embedding $V \to \mathbb{P}((H^0(K_V^{\otimes k})^\vee)$ and rescaling by $1/k$ furnishes V with a Kähler form ω satisfying $[\omega] = K_V$.

We can symplectically smooth the singularities of V to obtain a symplectic manifold U as in Definition 5.1. Suppose that V is $\mathbb{Q}$-Gorenstein smoothable. Since V is KSBA-stable, its canonical bundle is ample, and since amplitude is an open condition, the relative canonical bundle for this smoothing is ample (at least for fibres near the singular fibre). By Lemma 5.3, the smooth fibre, which is necessarily a canonically polarised surface of general type, is symplectomorphic to the symplectic smoothing U.

THEOREM 5.4. — *Let V be a KSBA-stable surface with at worst Wahl singularities. Suppose that V contains a rational curve passing through precisely two of its singularities x_0 and x_1 such that x_i is a Wahl singularity of type $\frac{1}{p_i^2}(1, p_i q_i - 1)$, and the preimage of C in the minimal resolution of X_0 is a chain*

$$[b_{0,1}, \dots, b_{0,r_0}] - c - [b_{1,1}, \dots, b_{1,r_1}],$$

with $\widetilde{C}_0^2 = -c$, $\frac{p_i^2}{p_i q_i - 1} = [b_{i,1}, \dots, b_{i,r_i}]$. Then the symplectic smoothing U contains a symplectically embedded copy of $U_{\Pi^+}(\ell_0, \ell_1)$ for some $\ell_0, \ell_1 > 0$, where $\Pi^+ = \Pi(p_0, q_0, p_1, q_1, c, K_V \cdot C)$.

Let $\Pi^- = \Pi(p_1, q_1, p_2, q_2, 1, a^-)$ be an initial right antiflip of Π^+ with parameter a^- sufficiently small and suppose that Π^- is infinitely right-mutable. Then the symplectic smoothing U admits a family of symplectic forms ω_t such that $[\omega_0] = K$ and such that ω_1 admits an infinite Mori sequence $M(p_1, q_1; p_2, q_2)$ of Lagrangian pinwheels.

Proof. — By the symplectic neighbourhood theorem for symplectic suborbifolds ([4, Theorem 11]), a neighbourhood of C in V is symplectomorphic to $V_{\Pi^+}(\ell_0, \ell_1)$ for some $\ell_0, \ell_1 > 0$, where $\Pi^+ = \Pi(p_0, q_0, p_1, q_1, c, a^+)$ and a^+ is the symplectic area of C. Since $[\omega] = K_V$, this means that $a^+ = K_V \cdot C$.

The symplectic smoothing U of V is therefore obtained by performing the symplectic smoothing on the almost toric region $V_{\Pi^+}(\ell_0, \ell_1)$, which (as in Example 5.2) yields a copy of $U_{\Pi^+}(\ell_0, \ell_1)$ inside U.

The initial right antiflip U' of U along U_{Π^+} with parameter a^- is a symplectically embedded copy of $U_{\Pi^-}(\ell'_1, \ell'_2)$ for some ℓ'_1, ℓ'_2, where Π^- is the initial right antiflip polygon of Π^+ with parameter a^-. By Lemma 3.15, if a^- is sufficiently small then U' admits the required Mori sequence of Lagrangian pinwheels. □

5.3. The quintic surface

Lemma 5.5. — *There exists a KSBA-stable surface V with $K^2 = 5$, $p_g = 4$ with a single singularity of type $\frac{1}{4}(1,1)$ such that its minimal resolution contains a chain of spheres:*

$$[4] - 3.$$

Moreover, V admits a $\mathbb{Q}$-Gorenstein smoothing whose smooth fibre is a quintic surface.

Proof. — Following Rana [18], observe that the minimal resolution of a stable quintic surface with a $\frac{1}{4}(1,1)$ singularity is a Horikawa surface with $K^2 = p_g = 4$ containing a -4-sphere. Moreover, such stable quintic surfaces V are always $\mathbb{Q}$-Gorenstein smoothable, since the local-to-global obstruction group $H^2(V, T_V)$ vanishes by [18, Theorem 4.10]. Let $B \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ be a curve of bidegree $(6,6)$; the branched double cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$ branched over B is a Horikawa surface of the required type.

- If B intersects the diagonal at six points each with multiplicity 2 then the preimage of the diagonal contains two irreducible rational -4-spheres (intersecting at four points).

- If B intersects $\mathbb{CP}^1 \times \{z\}$ at three points each with multiplicity 2 then the preimage of this ruling is a pair of rational -3-spheres (intersecting at three points).

If we have found such a B then we obtain a $[4]-3$ configuration in the minimal resolution of a stable quintic.

One can verify that the curve B given in the affine chart $([x:1],[y:1])$ by $\{1-2y^3+y^6+2x^3-xy^5-2x^5y+x^6y^6=0\}$ has the required properties: it is smooth, it intersects the ruling $\{x=0\}$ at the three points $(0,\mu)$, $\mu^3=1$, each with multiplicity two, and it intersects the diagonal at the six points (μ,μ), $\mu^6=1$, each with multiplicity two. □

By Theorem 5.4, this implies that the smooth quintic surface contains a symplectically embedded $U_{\Pi^+}(\ell_1,\ell_2)$ where $\Pi^+=\Pi(2,1,1,1,3,a^+)$ with $a^+=K_V\cdot C=\frac{\delta}{p_1p_2}=\frac{3}{2}$, and that its initial right antiflip contains a Mori sequence of Lagrangian pinwheels. In this case, we have $\delta=3$ and the initial antiflip polygon is $\Pi^-=\Pi(1,0,5,3,1,a^-)$, so the relevant Mori sequence is $M(1,0;5,3)$.

5.4. A Godeaux surface

Lemma 5.6. — *There exists a KSBA-stable surface V with $K^2=1$, $p_g=0$ with one ordinary double point and four Wahl singularities with continued fractions*

$$[7,2,2,2],\ [3,5,2],\ [6,2,2],\ [4],$$

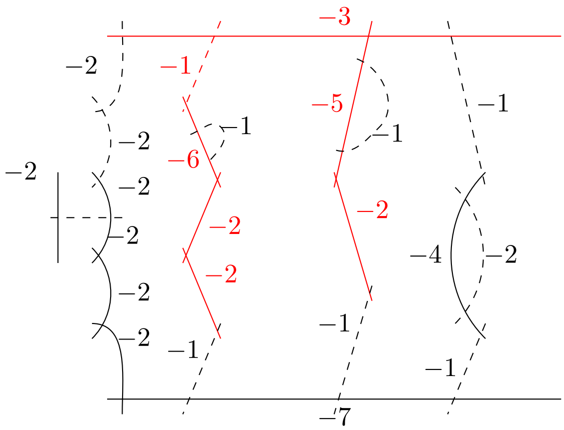
such that its minimal resolution contains a chain of spheres:

$$[2,2,6]-1-[3,5,2]$$

Moreover, V admits a $\mathbb{Q}$-Gorenstein smoothing whose smooth fibre is a simply-connected Godeaux surface.

Proof. — This surface is constructed in [23, Section 5] by flipping an example of Lee and Park [9]. □

Below, we reproduce Figure 5 from [23] which illustrates a configuration of curves in the minimal resolution of V including the chain we want (in red). The solid curves are collapsed by the minimal resolution to give the ordinary double point and four Wahl singularities of V. The dashed curves become rational curves in V.



Theorem 5.4 implies that the simply-connected Godeaux surface obtained by smoothing V contains a symplectically embedded $U_{\Pi^+}(\ell_1, \ell_2)$ where $\Pi^+ = \Pi(4,3,5,2,1,a^+)$ with $a^+ = K_V \cdot C = \frac{\delta}{4\times 5} = \frac{7}{20}$, and that its initial right antiflip contains a Mori sequence of Lagrangian pinwheels. In this case, we have $\delta = 7$ and the initial antiflip polygon is $\Pi^- = \Pi(5,2,39,17,1,a^-)$, so the relevant Mori sequence is $M(5,2;39,17)$.

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