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CLASSIFICATION OF FOLIATIONS OF DEGREE THREE ON $\mathbb{P}^2_{\mathbb{C}}$ WITH A FLAT LEGENDRE TRANSFORM

by Samir BEDROUNI & David MARÍN (*)

Abstract. — The set $\mathbf{F}(3)$ of foliations of degree three on the complex projective plane can be identified with a Zariski's open set of a projective space of dimension 23 on which acts $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$. The subset $\mathbf{FP}(3)$ of $\mathbf{F}(3)$ consisting of foliations of $\mathbf{F}(3)$ with a flat Legendre transform (dual web) is a Zariski closed subset of $\mathbf{F}(3)$. We classify up to automorphism of $\mathbb{P}^2_{\mathbb{C}}$ the elements of $\mathbf{FP}(3)$. More precisely, we show that up to an automorphism there are 16 foliations of degree three with a flat Legendre transform. From this classification we deduce that $\mathbf{FP}(3)$ has exactly 12 irreducible components. We also deduce that up to an automorphism there are 4 convex foliations of degree three on $\mathbb{P}^2$.

Résumé. — L'ensemble $\mathbf{F}(3)$ des feuilletages de degré trois du plan projectif complexe s'identifie à un ouvert de Zariski dans un espace projectif de dimension 23 sur lequel agit le groupe $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$. Le sous-ensemble $\mathbf{FP}(3)$ de $\mathbf{F}(3)$ formé des feuilletages de $\mathbf{F}(3)$ ayant une transformée de Legendre (tissu dual) plate est un fermé de Zariski de $\mathbf{F}(3)$. Nous classifions à automorphisme de $\mathbb{P}^2_{\mathbb{C}}$ près les éléments de $\mathbf{F}(3)$; plus précisément, nous montrons qu'à automorphisme près il y a 16 feuilletages de degré 3 ayant une transformée de Legendre plate. De cette classification nous obtenons la décomposition de $\mathbf{F}(3)$ en ses composantes irréductibles. Nous en déduisons aussi la classification à automorphisme près des feuilletages convexes de degré 3 de $\mathbb{P}^2_{\mathbb{C}}$.

Introduction

A (regular) d-web on $(\mathbb{C}^2, 0)$ is the data of a family $\{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_d\}$ of regular holomorphic foliations on $(\mathbb{C}^2, 0)$ which are pairwise transverse at the origin. The first significant result in the study of webs was obtained

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by W. Blaschke and J. Dubourdieu around 1920. They have introduced, for any regular 3-web $\mathcal{W}$ on $(\mathbb{C}^2, 0)$, a differential 2-form $K(\mathcal{W})$ known as the Blaschke curvature of $\mathcal{W}$, whose vanishing implies ([5]) that $\mathcal{W}$ is analytically equivalent to the trivial 3-web defined by $\mathrm{d}x.\mathrm{d}y.\mathrm{d}(x+y)$. The curvature of a d-web $\mathcal{W}$ with $d > 3$ is defined as the sum of the Blaschke curvatures of all 3-subwebs of $\mathcal{W}$. A web with zero curvature is called flat. This notion is useful for the classification of maximal rank webs. Indeed, a result of N. Mihăileanu shows that the flatness is a necessary condition for the maximality of the rank, see for instance [16, 22].

Recently, the study of global holomorphic webs defined on complex surfaces has been updated, see for instance [10, 21, 18]. In the sequel we will focus on webs of the complex projective plane. A (global) d-web on $\mathbb{P}^2_{\mathbb{C}}$ is given in an affine chart (x, y) by an implicit differential equation $F(x, y, y') = 0$, where $F(x, y, p) = \sum_{i=0}^{d} a_i(x, y)p^{d-i} \in \mathbb{C}[x, y, p]$ is a reduced polynomial whose coefficient a_0 is not identically zero. In a neighborhood of a point $z_0 = (x_0, y_0)$ such that $a_0(x_0, y_0)\Delta(x_0, y_0) \neq 0$, being $\Delta(x, y)$ the p-discriminant of F, the integral curves of this equation define a regular d-web on $(\mathbb{C}^2, z_0)$.

The curvature of a web $\mathcal{W}$ on $\mathbb{P}^2_{\mathbb{C}}$ is a meromorphic 2-form with poles along the discriminant $\Delta(\mathcal{W})$, see Section 1.2.

D. Marín and J.V. Pereira have shown, in [18], how to associate to every degree d foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$, a global d-web on the dual projective plane $\check{\mathbb{P}}^2_{\mathbb{C}}$, called *Legendre transform* of $\mathcal{F}$ and denoted by $\operatorname{Leg}\mathcal{F}$. The leaves of $\operatorname{Leg}\mathcal{F}$ are the dual curves of the leaves of $\mathcal{F}$, see Section 1.1.

The set $\mathbf{F}(d)$ of degree d foliation on $\mathbb{P}^2_{\mathbb{C}}$ can be naturally identified with a Zariski open subset of the projective space $\mathbb{P}^{(d+2)^2-2}_{\mathbb{C}}$. The automorphism group of $\mathbb{P}^2_{\mathbb{C}}$ acts on $\mathbf{F}(d)$; the orbit of an element $\mathcal{F} \in \mathbf{F}(d)$ under the action of $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}}) = \mathrm{PGL}_3(\mathbb{C})$ will be denoted by $\mathcal{O}(\mathcal{F})$. The subset $\mathbf{FP}(d)$ of $\mathbf{F}(d)$ consisting of $\mathcal{F} \in \mathbf{F}(d)$ such that $\operatorname{Leg}\mathcal{F}$ is flat is Zariski closed in $\mathbf{F}(d)$ and saturated by the action of $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$.

In [18] the authors pose a problem concerning the geometry of webs on $\mathbb{P}^2_{\mathbb{C}}$ which, in the framework of the foliations on $\mathbb{P}^2_{\mathbb{C}}$, consists in the description of certain irreducible components of $\mathbf{FP}(d)$. The first nontrivial case that we encounter is the one where $d = 3$. In this paper we describe the decomposition of $\mathbf{FP}(3)$ into its irreducible components. In order to do this, we begin by establishing the classification, up to isomorphism, of the foliations of $\mathbf{FP}(3)$. In a previous work [3], we have shown ([3, Theorem 5.1]) that up to isomorphism there are eleven *homogeneous* foliations (i.e. invariant by homotheties) of degree 3, denoted $\mathcal{H}_1, \ldots, \mathcal{H}_{11}$, with a flat Legendre transform. On the other hand, we have also proved ([3, Theorem 6.1]) that if a foliation of $\mathbf{FP}(3)$ has only non-degenerate singularities (i.e. singularities with Milnor number 1), then it is linearly conjugated to the Fermat foliation $\mathcal{F}_3$ defined by the 1-form $(x^3-x)\mathrm{d}y-(y^3-y)\mathrm{d}x$. In Section 2 by studying the flatness of the dual web of a foliation $\mathcal{F}\in\mathbf{F}(3)$ having at least one degenerate singularity, we obtain the classification, up to automorphism of $\mathbb{P}^2_{\mathbb{C}}$, of the elements of $\mathbf{FP}(3)$.

Theorem A. — *Up to automorphism of $\mathbb{P}^2_{\mathbb{C}}$ there are sixteen foliations of degree three $\mathcal{H}_1,\ldots,\mathcal{H}_{11},\mathcal{F}_1,\ldots,\mathcal{F}_5$ on the complex projective plane having a flat Legendre transform. They are respectively described in affine chart by the following* 1*-forms*

(1) $\omega_1=y^3\mathrm{d}x-x^3\mathrm{d}y$;
(2) $\omega_2=x^3\mathrm{d}x-y^3\mathrm{d}y$;
(3) $\omega_3=y^2(3x+y)\mathrm{d}x-x^2(x+3y)\mathrm{d}y$;
(4) $\omega_4=y^2(3x+y)\mathrm{d}x+x^2(x+3y)\mathrm{d}y$;
(5) $\omega_5=2y^3\mathrm{d}x+x^2(3y-2x)\mathrm{d}y$;
(6) $\omega_6=(4x^3-6x^2y+4y^3)\mathrm{d}x+x^2(3y-2x)\mathrm{d}y$;
(7) $\omega_7=y^3\mathrm{d}x+x(3y^2-x^2)\mathrm{d}y$;
(8) $\omega_8=x(x^2-3y^2)\mathrm{d}x-4y^3\mathrm{d}y$;
(9) $\omega_9=y^2\left((-3+\mathrm{i}\sqrt{3})x+2y\right)\mathrm{d}x+x^2\left((1+\mathrm{i}\sqrt{3})x-2\mathrm{i}\sqrt{3}y\right)\mathrm{d}y$;
(10) $\omega_{10}=(3x+\sqrt{3}y)y^2\mathrm{d}x+(3y-\sqrt{3}x)x^2\mathrm{d}y$;
(11) $\omega_{11}=(3x^3+3\sqrt{3}x^2y+3xy^2+\sqrt{3}y^3)\mathrm{d}x+(\sqrt{3}x^3+3x^2y+3\sqrt{3}xy^2+3y^3)\mathrm{d}y$;
(12) $\overline{\omega}_1=y^3\mathrm{d}x+x^3(x\mathrm{d}y-y\mathrm{d}x)$;
(13) $\overline{\omega}_2=x^3\mathrm{d}x+y^3(x\mathrm{d}y-y\mathrm{d}x)$;
(14) $\overline{\omega}_3=(x^3-x)\mathrm{d}y-(y^3-y)\mathrm{d}x$;
(15) $\overline{\omega}_4=(x^3+y^3)\mathrm{d}x+x^3(x\mathrm{d}y-y\mathrm{d}x)$;
(16) $\overline{\omega}_5=y^2(y\mathrm{d}x+2x\mathrm{d}y)+x^3(x\mathrm{d}y-y\mathrm{d}x)$.

The orbits of $\mathcal{F}_1$ and $\mathcal{F}_2$ are both of dimension 6 which is the minimal dimension possible, and this in any degree greater than or equal to 2 ([11, Proposition 2.3]). D. Cerveau, J. Déserti, D. Garba Belko and R. Meziani have shown that in degree 2 there are exactly two orbits of dimension 6 ([11, Proposition 2.7]). Theorem A allows us to establish a similar result in degree 3:

Corollary B. — *Up to automorphism of $\mathbb{P}^2_{\mathbb{C}}$ the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are the only foliations that realize the minimal dimension of orbits in degree* 3.

A. Beltrán, M. Falla Luza and D. Marín have shown in [4] that $\mathbf{FP}(3)$ contains the set of foliations $\mathcal{F} \in \mathbf{F}(3)$ whose leaves which are not straight lines do not have inflection points. These foliations are called *convex*. From these works ([4, Corollary 4.7]) and from Theorem A we deduce the classification, up to automorphism of $\mathbb{P}^2_{\mathbb{C}}$, of convex foliations of degree 3 on $\mathbb{P}^2_{\mathbb{C}}$.

Corollary C. — *Up to automorphism of $\mathbb{P}^2_{\mathbb{C}}$ there are four convex foliations of degree three on the complex projective plane, namely the foliations $\mathcal{H}_1, \mathcal{H}_3, \mathcal{F}_1$ and $\mathcal{F}_3$.*

This corollary is an analog in degree 3 of a result on the foliations of degree 2 due to C. Favre and J. V. Pereira ([13, Proposition 7.4]).

According to [18, Theorem 3], we know that the closure in $\mathbf{F}(3)$ of the orbit $\mathcal{O}(\mathcal{F}_3)$ of the Fermat foliation $\mathcal{F}_3$ is an irreducible component of $\mathbf{FP}(3)$. To our knowledge, at present, this is the only explicit example of an irreducible component of $\mathbf{FP}(3)$ appearing in the literature. By analyzing the incidence relations between the closures of the orbits of $\mathcal{H}_i$ and $\mathcal{F}_j$, we obtain the decomposition of $\mathbf{FP}(3)$ into its irreducible components.

Theorem D. — *The closures being taken in $\mathbf{F}(3)$ we have*

$$\begin{aligned}
&\overline{\mathcal{O}(\mathcal{F}_1)} = \mathcal{O}(\mathcal{F}_1), && \overline{\mathcal{O}(\mathcal{F}_2)} = \mathcal{O}(\mathcal{F}_2),\\
&\overline{\mathcal{O}(\mathcal{F}_3)} = \mathcal{O}(\mathcal{F}_1) \cup \mathcal{O}(\mathcal{H}_1) \cup \mathcal{O}(\mathcal{H}_3) \cup \mathcal{O}(\mathcal{F}_3), &&\\
&\overline{\mathcal{O}(\mathcal{F}_4)} = \mathcal{O}(\mathcal{F}_1) \cup \mathcal{O}(\mathcal{F}_2) \cup \mathcal{O}(\mathcal{F}_4), &&\\
&\overline{\mathcal{O}(\mathcal{H}_1)} = \mathcal{O}(\mathcal{F}_1) \cup \mathcal{O}(\mathcal{H}_1), && \overline{\mathcal{O}(\mathcal{H}_2)} = \mathcal{O}(\mathcal{F}_2) \cup \mathcal{O}(\mathcal{H}_2),\\
&\overline{\mathcal{O}(\mathcal{H}_3)} = \mathcal{O}(\mathcal{F}_1) \cup \mathcal{O}(\mathcal{H}_3), && \overline{\mathcal{O}(\mathcal{H}_8)} = \mathcal{O}(\mathcal{F}_2) \cup \mathcal{O}(\mathcal{H}_8),\\
&\overline{\mathcal{O}(\mathcal{H}_5)} = \mathcal{O}(\mathcal{F}_1) \cup \mathcal{O}(\mathcal{H}_5), && \overline{\mathcal{O}(\mathcal{H}_4)} \supset \mathcal{O}(\mathcal{F}_2) \cup \mathcal{O}(\mathcal{H}_4),\\
&\overline{\mathcal{O}(\mathcal{H}_7)} \supset \mathcal{O}(\mathcal{F}_1) \cup \mathcal{O}(\mathcal{H}_7), && \overline{\mathcal{O}(\mathcal{H}_6)} \supset \mathcal{O}(\mathcal{F}_2) \cup \mathcal{O}(\mathcal{H}_6),\\
&\overline{\mathcal{O}(\mathcal{H}_9)} \subset \mathcal{O}(\mathcal{F}_1) \cup \mathcal{O}(\mathcal{H}_9), && \overline{\mathcal{O}(\mathcal{H}_{11})} \subset \mathcal{O}(\mathcal{F}_2) \cup \mathcal{O}(\mathcal{H}_{11}),\\
&\overline{\mathcal{O}(\mathcal{H}_{10})} \subset \mathcal{O}(\mathcal{F}_1) \cup \mathcal{O}(\mathcal{F}_2) \cup \mathcal{O}(\mathcal{H}_{10}), && \overline{\mathcal{O}(\mathcal{F}_5)} \subset \mathcal{O}(\mathcal{F}_2) \cup \mathcal{O}(\mathcal{F}_5)
\end{aligned}$$

with

$$\begin{aligned}
&\dim \mathcal{O}(\mathcal{F}_1) = 6, \qquad && \dim \mathcal{O}(\mathcal{F}_2) = 6, \qquad && \dim \mathcal{O}(\mathcal{H}_i) = 7, i = 1, \ldots, 11,\\
&\dim \mathcal{O}(\mathcal{F}_4) = 7, && \dim \mathcal{O}(\mathcal{F}_5) = 7, && \dim \mathcal{O}(\mathcal{F}_3) = 8.
\end{aligned}$$

In particular,

- *the set $\mathbf{FP}(3)$ has exactly twelve irreducible components, namely $\overline{\mathcal{O}(\mathcal{F}_3)}$, $\overline{\mathcal{O}(\mathcal{F}_4)}$, $\overline{\mathcal{O}(\mathcal{F}_5)}$, $\overline{\mathcal{O}(\mathcal{H}_2)}$, $\overline{\mathcal{O}(\mathcal{H}_k)}$, $k = 4, 5, \ldots, 11$;*

- *the set of convex foliations of degree three in* $\mathbb{P}^2_{\mathbb{C}}$ *is exactly the closure* $\overline{\mathcal{O}(\mathcal{F}_3)}$ *of* $\mathcal{O}(\mathcal{F}_3)$ *(it is therefore an irreducible closed subset of* $\mathbf{F}(3)$*).*

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1. Preliminaries

1.1. Webs

Let $k \geqslant 1$ be an integer. A *(global) k-web $\mathcal{W}$* on a complex surface S is given by an open covering $(U_i)_{i\in I}$ of S and a collection of symmetric k-forms $\omega_i \in \operatorname{Sym}^k \Omega^1_S(U_i)$ with isolated zeroes satisfying:

(a) there exists $g_{ij} \in \mathcal{O}^*_S(U_i \cap U_j)$ such that ω_i coincides with $g_{ij}\omega_j$ on $U_i \cap U_j$;

(b) at every generic point m of U_i, $\omega_i(m)$ factorizes as the product of k pairwise non collinear 1-forms.

The subset of points of S not satisfying condition (b) is called the *discriminant* of $\mathcal{W}$ and it is denoted by $\Delta(\mathcal{W})$. When $k = 1$ this condition is always satisfied and we recover the usual definition of a holomorphic foliation on S. The cocycle (g_{ij}) defines a line bundle N on S, which is called *normal bundle* of $\mathcal{W}$, and the local k-forms ω_i patch together to form a global section $\omega \in \mathrm{H}^0(S, \operatorname{Sym}^k \Omega^1_S \otimes N)$.

A global k-web $\mathcal{W}$ on S is said *decomposable* if there are global webs $\mathcal{W}_1, \mathcal{W}_2$ on S with no common subweb such that $\mathcal{W}$ is the superposition of $\mathcal{W}_1$ and $\mathcal{W}_2$. In this case we will write $\mathcal{W} = \mathcal{W}_1 \boxtimes \mathcal{W}_2$. Otherwise it is said that $\mathcal{W}$ is *irreducible*. We will say that $\mathcal{W}$ is *completely decomposable* if there exist global foliations $\mathcal{F}_1, \ldots, \mathcal{F}_k$ on S such that $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$. For more details see [21].

In this work we restrict ourselves to the case $S = \mathbb{P}^2_{\mathbb{C}}$. In this case, every k-web $\mathcal{W}$ on $\mathbb{P}^2_{\mathbb{C}}$ can be defined in a given affine chart (x, y) by a polynomial k-symmetric form $\omega = \sum_{i+j=k} a_{ij}(x, y)\mathrm{d}x^i\mathrm{d}y^j$, with isolated zeroes and whose discriminant is not identically zero. Thus, $\mathcal{W}$ is defined by a polynomial differential equation $F(x, y, y') = 0$ of degree k in y'. A

k-web $\mathcal{W}$ on $\mathbb{P}^2_{\mathbb{C}}$ is said of *degree* d if the number of points where a generic line of $\mathbb{P}^2_{\mathbb{C}}$ is tangent to $\mathcal{W}$ is equal to d, it is equivalent to require that $\mathcal{W}$ has normal bundle $N = \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(d+2k)$. It is well known, see for instance [21, Proposition 1.4.2], that the webs of degree 0 are the algebraic webs (whose leaves are the tangent lines of a given reduced algebraic curve).

The authors in [18] associate to every k-web of degree $d \geqslant 1$ on $\mathbb{P}^2_{\mathbb{C}}$, a d-web of degree k on the dual projective plane $\check{\mathbb{P}}^2_{\mathbb{C}}$, called the *Legendre transform* of $\mathcal{W}$ and denoted by $\operatorname{Leg}\mathcal{W}$. The leaves of $\operatorname{Leg}\mathcal{W}$ are of the form $\check{L} = \{\mathrm{T}_m L\colon m \in L\} \subset \check{\mathbb{P}}^2_{\mathbb{C}}$ where $L \subset \mathbb{P}^2_{\mathbb{C}}$ is a leaf of $\mathcal{W}$. More explicitly, let (x, y) be an affine chart of $\mathbb{P}^2_{\mathbb{C}}$ and consider the affine chart (p, q) of $\check{\mathbb{P}}^2_{\mathbb{C}}$ associated to the line $\{y = px - q\} \subset \mathbb{P}^2_{\mathbb{C}}$. Let $F(x, y; p) = 0$, $p = \frac{\mathrm{d}y}{\mathrm{d}x}$, be an implicit differential equation defining $\mathcal{W}$. Then $\operatorname{Leg}\mathcal{W}$ is given in the affine chart (p, q) of $\check{\mathbb{P}}^2_{\mathbb{C}}$ by the implicit differential equation

$$\check{F}(p, q; x) := F(x, px - q; p) = 0, \quad \text{with} \quad x = \frac{\mathrm{d}q}{\mathrm{d}p}.$$

In particular, if $\mathcal{F}$ is a foliation of degree $d \geqslant 1$ on $\mathbb{P}^2_{\mathbb{C}}$ defined by a 1-form $\omega = A(x, y)\mathrm{d}x + B(x, y)\mathrm{d}y$, where $A, B \in \mathbb{C}[x, y]$, $\gcd(A, B) = 1$, then $\operatorname{Leg}\mathcal{F}$ is the irreducible d-web of degree 1 on $\check{\mathbb{P}}^2_{\mathbb{C}}$ defined by

$$A(x, px - q) + pB(x, px - q) = 0, \quad \text{with} \quad x = \frac{\mathrm{d}q}{\mathrm{d}p}.$$

Conversely, every irreducible d-web of degree 1 on $\check{\mathbb{P}}^2_{\mathbb{C}}$ is necessarily the Legendre transform of a certain foliation of degree d on $\mathbb{P}^2_{\mathbb{C}}$ (see [18]).

1.2. Curvature and flatness

We recall here the definition of the curvature of a k- web $\mathcal{W}$. We assume first that $\mathcal{W}$ is a germ of completely decomposable k-web on $(\mathbb{C}^2, 0)$, $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$. For each $1 \leqslant i \leqslant k$, let ω_i be a 1-form defining the foliation $\mathcal{F}_i$ with isolated singularity at 0. After [20], for each triple (r, s, t) with $1 \leqslant r < s < t \leqslant k$, we define $\eta_{rst} = \eta(\mathcal{F}_r \boxtimes \mathcal{F}_s \boxtimes \mathcal{F}_t)$ as the unique meromorphic 1-form satisfying the following equalities:

$$\begin{cases} \mathrm{d}(\delta_{st}\,\omega_r) = \eta_{rst} \wedge \delta_{st}\,\omega_r \\ \mathrm{d}(\delta_{tr}\,\omega_s) = \eta_{rst} \wedge \delta_{tr}\,\omega_s \\ \mathrm{d}(\delta_{rs}\,\omega_t) = \eta_{rst} \wedge \delta_{rs}\,\omega_t \end{cases}$$

where δ_{ij} denotes the function defined by $\omega_i \wedge \omega_j = \delta_{ij}\,\mathrm{d}x \wedge \mathrm{d}y$. Since each 1-form ω_i is well defined up to multiplication by an invertible element of

$\mathcal{O}(\mathbb{C}^2, 0)$, it follows that each 1-form η_{rst} is well defined up to addition of a closed holomorphic 1-form. Thus, the 1-form

$$\eta(\mathcal{W}) = \eta(\mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k) = \sum_{1 \leqslant r<s<t \leqslant k} \eta_{rst}$$

is well defined up to addition of a closed holomorphic 1-form. The *curvature* of the web $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$ is by definition the 2-form

$$K(\mathcal{W}) = K(\mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k) = \mathrm{d}\,\eta(\mathcal{W}).$$

It can be checked that $K(\mathcal{W})$ is a meromorphic 2-form with poles along the discriminant $\Delta(\mathcal{W})$ of $\mathcal{W}$, canonically associated to $\mathcal{W}$. More precisely, for every dominant holomorphic map φ, we have $K(\varphi^*\mathcal{W}) = \varphi^* K(\mathcal{W})$.

Now, if $\mathcal{W}$ is a (not necessarily completely decomposable) k-web on a complex surface S then its pull-back by a suitable Galoisian branched covering is totally decomposable. The invariance of the curvature of this new web by the action of the Galois group of the covering allows to bring it down in a global meromorphic 2-form on S, with poles along the discriminant of $\mathcal{W}$ (see [18]).

A k-web $\mathcal{W}$ is called *flat* if its curvature $K(\mathcal{W})$ vanishes identically.

We recall a formula due to A. Hénaut [15] which gives the curvature of a planar 3-web $\mathcal{W}$ given by an implicit differential equation

$$F(x,y,p) := a_0(x,y)p^3 + a_1(x,y)p^2 + a_2(x,y)p + a_3(x,y) = 0, \quad p = \frac{\mathrm{d}y}{\mathrm{d}x}.$$

Putting

$$R := \operatorname{Result}(F, \partial_p(F)) = \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 \\ 3a_0 & 2a_1 & a_2 & 0 & 0 \\ 0 & 3a_0 & 2a_1 & a_2 & 0 \\ 0 & 0 & 3a_0 & 2a_1 & a_2 \end{vmatrix} \not\equiv 0$$

$$\alpha_1 = \begin{vmatrix} \partial_y(a_0) & a_0 & -a_0 & 0 & 0 \\ \partial_x(a_0) + \partial_y(a_1) & a_1 & 0 & -2a_0 & 0 \\ \partial_x(a_1) + \partial_y(a_2) & a_2 & a_2 & -a_1 & -3a_0 \\ \partial_x(a_2) + \partial_y(a_3) & a_3 & 2a_3 & 0 & -2a_1 \\ \partial_x(a_3) & 0 & 0 & a_3 & -a_2 \end{vmatrix}$$

and

$$\alpha_2 = \begin{vmatrix} 0 & \partial_y(a_0) & -a_0 & 0 & 0 \\ a_0 & \partial_x(a_0)+\partial_y(a_1) & 0 & -2a_0 & 0 \\ a_1 & \partial_x(a_1)+\partial_y(a_2) & a_2 & -a_1 & -3a_0 \\ a_2 & \partial_x(a_2)+\partial_y(a_3) & 2a_3 & 0 & -2a_1 \\ a_3 & \partial_x(a_3) & 0 & a_3 & -a_2 \end{vmatrix},$$

we have that the curvature of the 3-web $\mathcal{W}$ is given by [15]

$$K(\mathcal{W}) = \left(\partial_y\left(\frac{\alpha_1}{R}\right) - \partial_x\left(\frac{\alpha_2}{R}\right)\right) \mathrm{d}x \wedge \mathrm{d}y. \tag{1.1}$$

1.3. Singularities and inflection divisor of a foliation on the projective plane

A degree d holomorphic foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$ is defined in homogeneous coordinates (x,y,z) by a 1-form

$$\omega = a(x,y,z)\mathrm{d}x + b(x,y,z)\mathrm{d}y + c(x,y,z)\mathrm{d}z,$$

where a, b and c are homogeneous polynomials of degree $d+1$ without common factor and satisfying the Euler condition $i_{\mathrm{R}}\omega = 0$, where $\mathrm{R} = x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}$ denotes the radial vector field and i_{R} is the interior product by R. The *singular locus* $\operatorname{Sing}\mathcal{F}$ of $\mathcal{F}$ is the projectivization of the singular locus of ω $\operatorname{Sing}\omega = \{(x,y,z) \in \mathbb{C}^3 \mid a(x,y,z) = b(x,y,z) = c(x,y,z) = 0\}$.

Let $\mathcal{C} \subset \mathbb{P}^2_{\mathbb{C}}$ be an algebraic curve with homogeneous equation $F(x,y,z) = 0$. We say that $\mathcal{C}$ is an *invariant curve* by $\mathcal{F}$ if $\mathcal{C} \smallsetminus \operatorname{Sing}\mathcal{F}$ is a union of (ordinary) leaves of the regular foliation $\mathcal{F}|_{\mathbb{P}^2_{\mathbb{C}}\smallsetminus \operatorname{Sing}\mathcal{F}}$. In algebraic terms, this is equivalent to require that the 2-form $\omega \wedge \mathrm{d}F$ is divisible by F, i.e. it vanishes along each irreducible component of $\mathcal{C}$.

When each irreducible component of $\mathcal{C}$ is not $\mathcal{F}$-invariant, for every point p of $\mathcal{C}$ we define the *tangency order* $\operatorname{Tang}(\mathcal{F},\mathcal{C},p)$ of $\mathcal{F}$ with $\mathcal{C}$ at p as follows. We fix a local chart (u,v) such that $p=(0,0)$; let $f(\mathrm{u},\mathrm{v}) = 0$ be a reduced local equation of $\mathcal{C}$ at a neighborhood of p and let X be a vector field defining the germ of $\mathcal{F}$ at p. We denote by $\mathrm{X}(f)$ the Lie derivative of f with respect to X and by $\langle f, \mathrm{X}(f)\rangle$ the ideal of $\mathbb{C}\{\mathrm{u},\mathrm{v}\}$ generated by f and $\mathrm{X}(f)$. Then

$$\operatorname{Tang}(\mathcal{F},\mathcal{C},p) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{\mathrm{u},\mathrm{v}\}}{\langle f, \mathrm{X}(f)\rangle}.$$

It is easy to see that this definition is well-posed, and that $\operatorname{Tang}(\mathcal{F},\mathcal{C},p) < +\infty$ because $\mathcal{C}$ is not $\mathcal{F}$-invariant.

Let us recall some local notions attached to the pair $(\mathcal{F}, s)$, where $s \in \operatorname{Sing}\mathcal{F}$. The germ of $\mathcal{F}$ at s is defined, up to multiplication by a unity in the local ring $\mathcal{O}_s$ at s, by a vector field $\mathrm{X} = A(\mathrm{u},\mathrm{v})\frac{\partial}{\partial \mathrm{u}} + B(\mathrm{u},\mathrm{v})\frac{\partial}{\partial \mathrm{v}}$. The *algebraic multiplicity* $\nu(\mathcal{F}, s)$ of $\mathcal{F}$ at s is given by

$$\nu(\mathcal{F}, s) = \min\{\nu(A, s), \nu(B, s)\},$$

where $\nu(g, s)$ denotes the algebraic multiplicity of the algebraic function g at s. Let us denote by $\mathfrak{L}_s$ the family of straight lines through s which are not invariant by $\mathcal{F}$. For every line ℓ_s of $\mathfrak{L}_s$, we have the inequalities $1 \leqslant \operatorname{Tang}(\mathcal{F}, \ell_s, s) \leqslant d$. This allows us to associate to the pair $(\mathcal{F}, s)$ the following natural (invariant) integers

$$\tau(\mathcal{F}, s) = \min\{\operatorname{Tang}(\mathcal{F}, \ell_s, s) \mid \ell_s \in \mathfrak{L}_s\},$$
$$\kappa(\mathcal{F}, s) = \max\{\operatorname{Tang}(\mathcal{F}, \ell_s, s) \mid \ell_s \in \mathfrak{L}_s\}.$$

The invariant $\tau(\mathcal{F}, s)$ represents the tangency order of $\mathcal{F}$ with a generic line passing through s. It is easy to see that

$$\tau(\mathcal{F}, s) = \min\{k \geqslant 1 \colon \det(J^k_s \mathrm{X}, \mathrm{R}_s) \neq 0\} \geqslant \nu(\mathcal{F}, s),$$

where $J^k_s \mathrm{X}$ denotes the k-jet of X at s and R_s is the radial vector field centered at s.

The singularity s is called *radial of order* $n-1$ if $\nu(\mathcal{F}, s) = 1$ and $\tau(\mathcal{F}, s) = n$.

The *Milnor number* of $\mathcal{F}$ at s is the integer

$$\mu(\mathcal{F}, s) = \dim_{\mathbb{C}} \mathcal{O}_s / \langle A, B\rangle,$$

where $\langle A, B\rangle$ denotes the ideal of $\mathcal{O}_s$ generated by A and B.

The singularity s is called *non-degenerate* if $\mu(\mathcal{F}, s) = 1$, or equivalently if the linear part $J^1_s\mathrm{X}$ of X possesses two non-zero eigenvalues λ, μ. In this case, the quantity $\operatorname{BB}(\mathcal{F}, s) = \frac{\lambda}{\mu} + \frac{\mu}{\lambda} + 2$ is called the *Baum–Bott invariant* of $\mathcal{F}$ at s (see [1]). By Briot–Bouquet's Theorem (see [8] for a generalization to any singularity) there is at least a germ of curve $\mathcal{C}$ at s which is invariant by $\mathcal{F}$. Up to local diffeomorphism we can assume that $s = (0, 0)$, $\mathrm{T}_s\mathcal{C} = \{\mathrm{u} = 0\}$ and $J^1_s\mathrm{X} = \lambda \mathrm{u}\frac{\partial}{\partial \mathrm{u}} + (\varepsilon \mathrm{u} + \mu \mathrm{v})\frac{\partial}{\partial \mathrm{v}}$, where we can take $\varepsilon = 0$ if $\lambda \neq \mu$. The quantity $\operatorname{CS}(\mathcal{F}, \mathcal{C}, s) = \frac{\lambda}{\mu}$ is called the *Camacho–Sad index* of $\mathcal{F}$ at s along $\mathcal{C}$.

Finally, let us recall the notion of inflection divisor of $\mathcal{F}$. Let $\mathrm{Z} = E\frac{\partial}{\partial x} + F\frac{\partial}{\partial y} + G\frac{\partial}{\partial z}$ be a homogeneous vector field of degree d on $\mathbb{C}^3$ non collinear to the radial vector field describing $\mathcal{F}$, i.e. such that $\omega = i_{\mathrm{R}} i_{\mathrm{Z}} \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$. The *inflection divisor* of $\mathcal{F}$, denoted by $\mathrm{I}_{\mathcal{F}}$, is the divisor of $\mathbb{P}^2_{\mathbb{C}}$ defined by the homogeneous equation

$$\begin{vmatrix} x & E & \mathrm{Z}(E) \\ y & F & \mathrm{Z}(F) \\ z & G & \mathrm{Z}(G) \end{vmatrix} = 0. \tag{1.2}$$

This divisor has been studied in [19] in a more general context. In particular, the following properties has been proved.

(1) On $\mathbb{P}^2_{\mathbb{C}} \smallsetminus \operatorname{Sing}\mathcal{F}$, $\mathrm{I}_{\mathcal{F}}$ coincides with the curve described by the inflection points of the leaves of $\mathcal{F}$;

(2) If $\mathcal{C}$ is an irreducible algebraic curve invariant by $\mathcal{F}$ then $\mathcal{C} \subset \mathrm{I}_{\mathcal{F}}$ if and only if $\mathcal{C}$ is an invariant line;

(3) $\mathrm{I}_{\mathcal{F}}$ can be decomposed into $\mathrm{I}_{\mathcal{F}} = \mathrm{I}_{\mathcal{F}}^{\mathrm{inv}} + \mathrm{I}_{\mathcal{F}}^{\mathrm{tr}}$, where the support of $\mathrm{I}_{\mathcal{F}}^{\mathrm{inv}}$ consists in the set of invariant lines of $\mathcal{F}$ and the support of $\mathrm{I}_{\mathcal{F}}^{\mathrm{tr}}$ is the closure of the inflection points along the leaves of $\mathcal{F}$ which are not lines;

(4) The degree of the divisor $\mathrm{I}_{\mathcal{F}}$ is $3d$.

The foliation $\mathcal{F}$ will be called *convex* if its inflection divisor $\mathrm{I}_{\mathcal{F}}$ is totally invariant by $\mathcal{F}$, i.e. if $\mathrm{I}_{\mathcal{F}}$ is a product of invariant lines.

2. Foliations of $\mathbf{FP}(3)$ having at least one degenerate singularity

2.1. Case of a degenerate singularity of algebraic multiplicity at most 2

In [2, Appendix A] the first author gives a computational proof of the following statement.

Proposition 2.1. — *Let $\mathcal{F}$ be a degree three foliation on $\mathbb{P}^2_{\mathbb{C}}$ having a degenerate singularity of algebraic multiplicity at most 2. Then the dual 3-web $\operatorname{Leg}\mathcal{F}$ of $\mathcal{F}$ is not flat.*

In this appendix the matter is about a proof by contradiction: first, the author assumes that there is a foliation $\mathcal{F}$ of degree 3 on $\mathbb{P}^2_{\mathbb{C}}$ such that the 3-web $\operatorname{Leg}\mathcal{F}$ is flat and such that the singular locus $\operatorname{Sing}\mathcal{F}$ contains a point m satisfying $\mu(\mathcal{F},m) \geqslant 2$ and $\nu(\mathcal{F},m) \leqslant 2$; then, he explicitly calculates the curvature of $\operatorname{Leg}\mathcal{F}$ by Formula (1.1) and he shows that the condition $K(\operatorname{Leg}\mathcal{F}) \equiv 0$ contradicts the hypothesis $\deg\mathcal{F} = 3$.

Problem 2.2. — *Give a non-computational proof of Proposition 2.1.*

2.2. Case of a degenerate singularity of algebraic multiplicity 3

In this paragraph we are interested in the foliations $\mathcal{F} \in \mathbf{FP}(3)$ which have a degenerate singularity m of algebraic multiplicity 3. We distinguish two cases according to whether $\operatorname{Sing}\mathcal{F} = \{m\}$ or $\{m\} \subsetneq \operatorname{Sing}\mathcal{F}$.

2.2.1. The singular locus is reduced to a point of algebraic multiplicity 3

We start by establishing the following statement classifying the foliations of $\mathbf{F}(3)$ whose singular locus is reduced to a point of algebraic multiplicity 3.

Proposition 2.3. — *Let $\mathcal{F}$ be a foliation of degree 3 on $\mathbb{P}^2_{\mathbb{C}}$ with exactly one singularity. Let ω be a 1-form defining $\mathcal{F}$. If this singularity is of algebraic multiplicity 3, then up to isomorphism ω is of one of the following types*

(1) $x^3\mathrm{d}x + y^2(cx+y)(x\mathrm{d}y - y\mathrm{d}x)$, $c \in \mathbb{C}$;
(2) $x^3\mathrm{d}x + y(x+cxy+y^2)(x\mathrm{d}y - y\mathrm{d}x)$, $c \in \mathbb{C}$;
(3) $x^3\mathrm{d}x + (x^2+cxy^2+y^3)(x\mathrm{d}y - y\mathrm{d}x)$, $c \in \mathbb{C}$;
(4) $x^2y\mathrm{d}x + (x^3+cxy^2+y^3)(x\mathrm{d}y - y\mathrm{d}x)$, $c \in \mathbb{C}$;
(5) $x^2y\mathrm{d}x + (x^3+\delta\, xy+y^3)(x\mathrm{d}y - y\mathrm{d}x)$, $\delta \in \mathbb{C}^*$;
(6) $x^2y\mathrm{d}y + (x^3+cxy^2+y^3)(x\mathrm{d}y - y\mathrm{d}x)$, $c \in \mathbb{C}$;
(7) $xy(x\mathrm{d}y - \lambda\, y\mathrm{d}x) + (x^3+y^3)(x\mathrm{d}y - y\mathrm{d}x)$, $\lambda \in \mathbb{C} \setminus \{0,1\}$;
(8) $xy(y-x)\mathrm{d}x + (c_0\,x^3+c_1x^2y+y^3)(x\mathrm{d}y - y\mathrm{d}x)$, $c_0(c_0+c_1+1) \neq 0$.

These eight 1-forms are not linearly conjugated with each other.

This proposition is an analog in degree 3 of a result on the foliations of degree 2 due to D. Cerveau, J. Déserti, D. Garba Belko and R. Meziani ([11, Proposition 1.8]). The proof that we will give is very close to that of [11]; it will result from Lemmas 2.4, 2.5, 2.6 and 2.7 stated below. In these four lemmas $\mathcal{F}$ denotes a foliation of degree three on $\mathbb{P}^2_{\mathbb{C}}$ defined by a 1-form ω and such that

(1) the unique singularity of $\mathcal{F}$ is $O = [0:0:1]$;
(2) the jets of order 1 and 2 of ω at $(0,0)$ are zero, i.e. $\nu(\mathcal{F}, O) = 3$.

In this case

$$\omega = A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y + C(x,y)(x\mathrm{d}y - y\mathrm{d}x),$$

where A, B and C are homogeneous polynomials of degree 3. The foliation $\mathcal{F}$ being of degree three, the tangent cone $xA+yB$ of ω at $(0,0)$ can not be identically zero. The polynomial C is also not identically zero, otherwise the line at infinity would be invariant by $\mathcal{F}$ which would therefore have a singularity on this line, which is excluded. We will reason according to the nature of the tangent cone which, a priori, can be four lines, three lines, two lines or a single line.

Lemma 2.4. — *Every irreducible factor L of $xA+yB$ divides* $\gcd(A,B)$ *and does not divide C. In particular, the tangent cone of ω at* $(0,0)$ *is not the union of four distinct lines.*

Proof. — Up to isomorphism, we can assume that $L = x$; then x divides B. Thus on the line $x = 0$ the form ω writes as

$$A(0,y)\mathrm{d}x - y\,C(0,y)\mathrm{d}x = y^3\left(A(0,1) - y\,C(0,1)\right)\mathrm{d}x.$$

Since O is the unique singularity of $\mathcal{F}$, the product $A(0,1)C(0,1)$ is zero. The point $[0:1:0]$ being non-singular, $C(0,1)$ is non-zero and as a result $A(0,1) = 0$; hence x divides A but not C. □

Lemma 2.5. — *If the tangent cone of ω at* $(0,0)$ *is the union of three distinct lines, then up to automorphisms of $\mathbb{P}^2_{\mathbb{C}}$, ω is of type*

$$xy(y-x)\mathrm{d}x + (c_0\,x^3 + c_1x^2y + y^3)(x\mathrm{d}y - y\mathrm{d}x), \quad c_0(c_0+c_1+1) \neq 0.$$

Proof. — We can assume that $xA+yB = *x^2y(y-x), * \in \mathbb{C}^*$; it follows that ω writes ([12])

$$x^2y(y-x)\left(\lambda_0\frac{\mathrm{d}x}{x} + \lambda_1\frac{\mathrm{d}y}{y} + \lambda_2\frac{\mathrm{d}(y-x)}{y-x} + \delta\mathrm{d}\left(\frac{y}{x}\right)\right)$$
$$+ (c_0\,x^3 + c_1x^2y + c_2xy^2 + c_3y^3)(x\mathrm{d}y - y\mathrm{d}x), \quad \delta, \lambda_i, c_i \in \mathbb{C}.$$

Therefore we have

$$\begin{aligned}
A(x,y) &= y\Big((y-x)(\lambda_0\,x - \delta y) - \lambda_2 x^2\Big),\\
B(x,y) &= x\Big((y-x)(\lambda_1 x + \delta y) + \lambda_2 xy\Big),\\
C(x,y) &= \sum_{i=0}^{3} c_i x^{3-i} y^i.
\end{aligned}$$

According to Lemma 2.4, the polynomial $xy(y-x)$ divides A and B but not C, which means that

$$A(0,1) = A(1,0) = A(1,1) = B(0,1) = B(1,0) = B(1,1) = 0$$
$$\text{and} \quad C(0,1)C(1,0)C(1,1) \neq 0.$$

It follows that $\delta = \lambda_1 = \lambda_2 = 0$ and that $c_0c_3(c_0+c_1+c_2+c_3) \neq 0$. The foliation $\mathcal{F}$ being of degree three λ_0 is non-zero; we can therefore assume that $\lambda_0 = 1$, hence

$$\omega = xy(y-x)\mathrm{d}x + (c_0\,x^3 + c_1x^2y + c_2xy^2 + c_3y^3)(x\mathrm{d}y - y\mathrm{d}x).$$

After conjugating ω by the homothety $\left(\frac{1}{c_3}x, \frac{1}{c_3}y\right)$, we can normalize the coefficient c_3 to 1; as a consequence

$$\omega = xy(y-x)\mathrm{d}x + (c_0\,x^3 + c_1x^2y + c_2xy^2 + y^3)(x\mathrm{d}y - y\mathrm{d}x), \qquad c_0(c_0+c_1+c_2+1) \neq 0.$$

The conjugation by the automorphism $\left(\frac{x}{1+c_2y}, \frac{y}{1+c_2y}\right)$ of $\mathbb{P}^2_{\mathbb{C}}$ allows us to cancel c_2. Hence the statement holds. □

LEMMA 2.6. — *If the tangent cone of ω at $(0,0)$ is composed of two distinct lines, then up to isomorphism ω is of one of the following types*

(1) $x^2y\mathrm{d}x + (x^3 + cxy^2 + y^3)(x\mathrm{d}y - y\mathrm{d}x)$, $c \in \mathbb{C}$;
(2) $x^2y\mathrm{d}x + (x^3 + \delta\,xy + y^3)(x\mathrm{d}y - y\mathrm{d}x)$, $\delta \in \mathbb{C}^*$;
(3) $x^2y\mathrm{d}y + (x^3 + cxy^2 + y^3)(x\mathrm{d}y - y\mathrm{d}x)$, $c \in \mathbb{C}$;
(4) $xy(x\mathrm{d}y - \lambda\,y\mathrm{d}x) + (x^3 + y^3)(x\mathrm{d}y - y\mathrm{d}x)$, $\lambda \in \mathbb{C} \setminus \{0,1\}$.

Proof. — Up to linear conjugation we are in one of the two following situations

(a) $xA + yB = *x^3y, \quad * \in \mathbb{C}^*$;
(b) $xA + yB = *x^2y^2, \quad * \in \mathbb{C}^*$.

Let us start by studying the eventuality (a). In this case the 1-form ω writes ([12])

$$x^3y\left(\lambda_0\frac{\mathrm{d}x}{x} + \lambda_1\frac{\mathrm{d}y}{y} + \mathrm{d}\left(\frac{\delta_1xy + \delta_2y^2}{x^2}\right)\right) + (c_0\,x^3 + c_1x^2y + c_2xy^2 + c_3y^3)(x\mathrm{d}y - y\mathrm{d}x), \quad \lambda_i, \delta_i, c_i \in \mathbb{C}.$$

Then we have

$$A(x,y) = y(\lambda_0x^2 - \delta_1xy - 2\delta_2y^2),$$
$$B(x,y) = x(\lambda_1x^2 + \delta_1xy + 2\delta_2y^2),$$
$$C(x,y) = \sum_{i=0}^{3} c_ix^{3-i}y^i.$$

According to Lemma 2.4, the polynomial xy divides A and B but not C. As a result $\delta_2 = \lambda_1 = 0$ and $c_0c_3 \neq 0$. The foliation $\mathcal{F}$ being of degree three the coefficient λ_0 is non-zero and we can assume it equals 1. Thus $\mathcal{F}$

is described by

$$\omega = x^2y\mathrm{d}x + \delta_1 xy(x\mathrm{d}y - y\mathrm{d}x) \\ + (c_0\, x^3 + c_1x^2y + c_2xy^2 + c_3y^3)(x\mathrm{d}y - y\mathrm{d}x).$$

The diagonal linear transformation $\left(\frac{1}{c_0}x,\, \sqrt[3]{\frac{1}{c_3c_0^2}}y\right)$ allows us to assume that $c_0 = c_3 = 1$; as a consequence

$$\omega = x^2y\mathrm{d}x + (x^3 + \delta_1 xy + c_1x^2y + c_2xy^2 + y^3)(x\mathrm{d}y - y\mathrm{d}x).$$

If $\delta_1 = 0$, resp. $\delta_1 \neq 0$, by conjugating ω by

$$\left(\frac{x}{1+c_1y}, \frac{y}{1+c_1y}\right),$$

$$\text{resp.} \quad \left(\frac{x}{1-\left(\frac{c_2}{\delta_1}\right)y - \left(\frac{\delta_1c_1+c_2}{\delta_1^2}\right)x}, \frac{y}{1-\left(\frac{c_2}{\delta_1}\right)y - \left(\frac{\delta_1c_1+c_2}{\delta_1^2}\right)x}\right),$$

we reduce ourselves to $c_1 = 0$, resp. $c_1 = c_2 = 0$, that is, to

$$\omega = x^2y\mathrm{d}x + (x^3 + c_2xy^2 + y^3)(x\mathrm{d}y - y\mathrm{d}x),$$
$$\text{resp.} \quad \omega = x^2y\mathrm{d}x + (x^3 + \delta_1xy + y^3)(x\mathrm{d}y - y\mathrm{d}x);$$

hence the two first announced models.

Let us now consider the possibility (b). In this case ω writes ([12])

$$x^2y^2\left(\lambda_0\frac{\mathrm{d}x}{x} + \lambda_1\frac{\mathrm{d}y}{y} + \mathrm{d}\left(\frac{\delta_1x^2+\delta_2y^2}{xy}\right)\right) \\ + (c_0\, x^3 + c_1x^2y + c_2xy^2 + c_3y^3)(x\mathrm{d}y - y\mathrm{d}x), \qquad \lambda_i, \delta_i, c_i \in \mathbb{C}.$$

Here $A(x,y) = y(\delta_1x^2 + \lambda_0\, xy - \delta_2y^2)$ and $B(x,y) = x(\delta_2y^2 + \lambda_1xy - \delta_1x^2)$. According to Lemma 2.4 again, xy divides $\gcd(A,B)$ and does not divide C, which is equivalent to $\delta_1 = \delta_2 = 0$ and $c_0c_3 \neq 0$.

The foliation $\mathcal{F}$ being of degree three the sum $\lambda_0 + \lambda_1$ is non-zero; then one of the coefficients λ_i is non-zero and we can obviously normalize it to 1. Since the lines of the tangent cone (i.e. $x = 0$ and $y = 0$) play a symmetrical role, it suffices to treat the eventuality $\lambda_1 = 1$. Thus $\mathcal{F}$ is given by

$$\omega = xy(x\mathrm{d}y + \lambda_0y\mathrm{d}x) + (c_0\, x^3 + c_1x^2y + c_2xy^2 + c_3y^3)(x\mathrm{d}y - y\mathrm{d}x), \\ (\lambda_0 + 1)c_0c_3 \neq 0.$$

Let α be in $\mathbb{C}$ such that $\alpha^3 = \frac{1}{c_3 c_0^2}$; let us put $\beta = c_0\alpha^2$. After conjugating ω by $(\alpha\, x, \beta y)$, we can assume that $c_0 = c_3 = 1$; as a result

$$\omega = xy(x\mathrm{d}y + \lambda_0 y\mathrm{d}x) + (x^3 + c_1x^2y + c_2xy^2 + y^3)(x\mathrm{d}y - y\mathrm{d}x), \qquad \lambda_0 + 1 \neq 0.$$

If $\lambda_0 = 0$, resp. $\lambda_0 \neq 0$, by conjugating ω by

$$\left(\frac{x}{1-c_1x}, \frac{y}{1-c_1x}\right), \quad \text{resp.} \left(\frac{x}{1+\left(\frac{c_2}{\lambda_0}\right)y - c_1x}, \frac{y}{1+\left(\frac{c_2}{\lambda_0}\right)y - c_1x}\right),$$

we reduce ourselves to $c_1 = 0$, resp. $c_1 = c_2 = 0$, that is, to

$$\omega = x^2y\mathrm{d}y + (x^3 + c_2xy^2 + y^3)(x\mathrm{d}y - y\mathrm{d}x),$$
$$\text{resp.} \quad \omega = xy(x\mathrm{d}y + \lambda_0 y\mathrm{d}x) + (x^3 + y^3)(x\mathrm{d}y - y\mathrm{d}x), \quad \lambda_0(\lambda_0+1) \neq 0,$$

which are the two last announced models. □

LEMMA 2.7. — *If the tangent cone of ω at $(0,0)$ is reduced to a single line, then up to isomorphism ω is of one of the following types*

(1) $x^3\mathrm{d}x + y^2(cx + y)(x\mathrm{d}y - y\mathrm{d}x)$, $c \in \mathbb{C}$;
(2) $x^3\mathrm{d}x + y(x + cxy + y^2)(x\mathrm{d}y - y\mathrm{d}x)$, $c \in \mathbb{C}$;
(3) $x^3\mathrm{d}x + (x^2 + cxy^2 + y^3)(x\mathrm{d}y - y\mathrm{d}x)$, $c \in \mathbb{C}$.

Proof. — We can assume that the tangent cone is the line $x = 0$; then ω writes as

$$x^4\left(\lambda\frac{\mathrm{d}x}{x} + \mathrm{d}\left(\frac{\delta_1x^2y + \delta_2xy^2 + \delta_3y^3}{x^3}\right)\right) + (c_0\, x^3 + c_1x^2y + c_2xy^2 + c_3y^3)(x\mathrm{d}y - y\mathrm{d}x), \quad \lambda, \delta_i, c_i \in \mathbb{C}.$$

Then we have

$$A(x,y) = \lambda\, x^3 - \delta_1x^2y - 2\delta_2xy^2 - 3\delta_3y^3,$$
$$B(x,y) = x(\delta_1x^2 + 2\delta_2xy + 3\delta_3y^2),$$
$$C(x,y) = \sum_{i=0}^{3} c_ix^{3-i}y^i.$$

According to Lemma 2.4, x divides A and B but not C; as a result $\delta_3 = 0$ and $c_3 \neq 0$. The foliation $\mathcal{F}$ being of degree three the coefficient λ is non-zero and we can assume that $\lambda = 1$. The conjugation by the homothety $\left(\frac{1}{c_3}x, \frac{1}{c_3}y\right)$ allows us to assume that $c_3 = 1$. Thus $\mathcal{F}$ is described by

$$\omega = x^3\mathrm{d}x + (\delta_1x^2 + 2\delta_2xy + c_0\, x^3 + c_1x^2y + c_2xy^2 + y^3)(x\mathrm{d}y - y\mathrm{d}x).$$

We have the following three possibilities to study

- $\delta_2 \neq 0$;
- $\delta_1 = \delta_2 = 0$;
- $\delta_2 = 0, \delta_1 \neq 0$.

(1). — If $\delta_2 \neq 0$, then by conjugating ω by $\big(\alpha^2 x, \alpha^{3/2} y - \alpha\delta_1 x\big)$, where $\alpha = 2\delta_2$, we reduce ourselves to $\delta_1 = 0$ and $\delta_2 = \frac{1}{2}$. As a result $\mathcal{F}$ is given by

$$\omega = x^3 \mathrm{d}x + (xy + c_0\, x^3 + c_1 x^2 y + c_2 x y^2 + y^3)(x\mathrm{d}y - y\mathrm{d}x).$$

The conjugation by the diffeomorphism $\big(\frac{x}{1+c_0 y - c_1 x}, \frac{y}{1+c_0 y - c_1 x}\big)$ allows us to assume that $c_0 = c_1 = 0$; as a consequence

$$\omega = x^3 \mathrm{d}x + y(x + c_2 xy + y^2)(x\mathrm{d}y - y\mathrm{d}x),$$

hence the second announced model.

(2). — If $\delta_1 = \delta_2 = 0$ the 1-form ω writes

$$\omega = x^3 \mathrm{d}x + (c_0\, x^3 + c_1 x^2 y + c_2 x y^2 + y^3)(x\mathrm{d}y - y\mathrm{d}x).$$

Let α be in $\mathbb{C}$ such that $3\alpha^2 + 2c_2\alpha + c_1 = 0$. After conjugating ω by $(x, y + \alpha\, x)$, we can assume that $c_1 = 0$. Then the conjugation by the diffeomorphism

$$\left(\frac{x}{1 + c_0 y}, \frac{y}{1 + c_0 y}\right)$$

allows us to cancel c_0; hence the first announced model.

(3). — When $\delta_2 = 0$ and $\delta_1 \neq 0$, the form ω writes

$$\omega = x^3 \mathrm{d}x + (\delta_1 x^2 + c_0\, x^3 + c_1 x^2 y + c_2 x y^2 + y^3)(x\mathrm{d}y - y\mathrm{d}x).$$

By conjugating ω by $\big(\delta_1^4 x, \delta_1^3 y\big)$, we can assume that $\delta_1 = 1$. Then by conjugating by

$$\left(\frac{x}{1 - c_1 y - (c_0 + c_1)x}, \frac{y}{1 - c_1 y - (c_0 + c_1)x}\right),$$

we reduce ourselves to $c_0 = c_1 = 0$, that is, to the third announced model. $\square$

Proof of Proposition 2.3. — It suffices to choose affine coordinates (x, y) such that the point $(0, 0)$ is singular of $\mathcal{F}$ and to use Lemmas 2.4, 2.5, 2.6 and 2.7. $\square$

We are now ready to describe up to isomorphism the foliations of $\mathbf{FP}(3)$ whose singular locus is reduced to a point of algebraic multiplicity 3.

PROPOSITION 2.8. — *Let $\mathcal{F}$ be a foliation of degree 3 on $\mathbb{P}^2_{\mathbb{C}}$ with exactly one singularity. Assume that this singularity is of algebraic multiplicity 3 and that the 3-web $\operatorname{Leg}\mathcal{F}$ is flat. Then $\mathcal{F}$ is linearly conjugated to the foliation $\mathcal{F}_2$ described by the 1-form*

$$\overline{\omega}_2 = x^3\mathrm{d}x + y^3(x\mathrm{d}y - y\mathrm{d}x).$$

Proof. — Let ω be a 1-form describing $\mathcal{F}$ in an affine chart (x,y) and let (p,q) be the affine chart of $\check{\mathbb{P}}^2_{\mathbb{C}}$ corresponding to the line $\{px-qy=1\}\subset\mathbb{P}^2_{\mathbb{C}}$. Up to linear conjugation ω is of one of the eight types of Proposition 2.3.

- If $\omega = x^3\mathrm{d}x + y^2(cx+y)(x\mathrm{d}y - y\mathrm{d}x)$, $c\in\mathbb{C}$, then the 3-web $\operatorname{Leg}\mathcal{F}$ is given by the differential equation $q(q')^3+cq'+1=0$, where $q'=\frac{\mathrm{d}q}{\mathrm{d}p}$. The explicit computation of $K(\operatorname{Leg}\mathcal{F})$ leads to
$$K(\operatorname{Leg}\mathcal{F}) = -\frac{4c^2(2c^3+27q)}{q^2(4c^3+27q)^2}\mathrm{d}p\wedge\mathrm{d}q;$$
as a result $\operatorname{Leg}\mathcal{F}$ is flat if and only if $c=0$, in which case
$$\omega = \overline{\omega}_2 = x^3\mathrm{d}x + y^3(x\mathrm{d}y - y\mathrm{d}x).$$
- If $\omega = x^3\mathrm{d}x + y(x+cxy+y^2)(x\mathrm{d}y - y\mathrm{d}x)$, $c\in\mathbb{C}$, then $\operatorname{Leg}\mathcal{F}$ is described by the differential equation $F(p,q,w) := qw^3+pw^2+(c-q)w+1=0$, where $w=\frac{\mathrm{d}q}{\mathrm{d}p}$. The explicit computation of $K(\operatorname{Leg}\mathcal{F})$ shows that it has the form
$$K(\operatorname{Leg}\mathcal{F}) = \frac{\sum_{i+j\leqslant 6}\rho_i^j(c)p^iq^j}{\Delta(p,q)^2}\mathrm{d}p\wedge\mathrm{d}q,$$
where Δ is the w-discriminant of F and the ρ_i^j's are polynomials in c with $\rho_1^5(c)=4\neq 0$; hence $K(\operatorname{Leg}\mathcal{F})\not\equiv 0$.

  Similarly, we verify that $\operatorname{Leg}\mathcal{F}$ can not be flat when $\mathcal{F}$ is given by one of the last six 1-forms of Proposition 2.3. □

2.2.2. The singular locus contains a point of algebraic multiplicity 3 and is not reduced to this point

We begin by proving four lemmas.

LEMMA 2.9. — *Let $\mathcal{F}$ be a foliation of degree three on $\mathbb{P}^2_{\mathbb{C}}$, let m be a singular point of $\mathcal{F}$ and let ω be a 1-form describing $\mathcal{F}$. Assume that this singularity is of algebraic multiplicity 3 and that the 3-web $\operatorname{Leg}\mathcal{F}$ is flat. Then*

- *either $\mathcal{F}$ is homogeneous;*
- *or the 3-jet of ω at m is not saturated.*

Remark 2.10. — Let us note that a foliation of degree d on $\mathbb{P}^2_{\mathbb{C}}$ is homogeneous if and only if it has a singularity of maximal algebraic multiplicity (i.e. equal to d) and an invariant line not passing through this singularity.

Proof. — Let us choose a system of homogeneous coordinates $[x:y:z] \in \mathbb{P}^2_{\mathbb{C}}$ in which $m = [0:0:1]$. The condition $\nu(\mathcal{F}, m) = 3$ assures that every 1-form ω defining $\mathcal{F}$ in the affine chart (x, y) is of type $\omega = \theta_3 + C_3(x, y)(x\mathrm{d}y - y\mathrm{d}x)$, where θ_3 (resp. C_3) is a homogeneous 1-form (resp. a homogeneous polynomial) of degree 3; the 1-form θ_3 represents the 3-jet of ω at $(0, 0)$.

Let us assume that θ_3 is saturated; we will prove that $\mathcal{F}$ is necessarily homogeneous. Let us denote by $\mathcal{H}$ the homogeneous foliation of degree three on $\mathbb{P}^2_{\mathbb{C}}$ defined by θ_3; $\mathcal{H}$ is well defined thanks to the hypothesis on θ_3. Let us consider the family of homotheties $\varphi = \varphi_\varepsilon = (\varepsilon\, x, \varepsilon y)$. We have

$$\frac{1}{\varepsilon^4}\varphi^*\omega = \theta_3 + \varepsilon\, C_3(x, y)(x\mathrm{d}y - y\mathrm{d}x)$$

which tends to θ_3 as ε tends to 0; it follows that $\mathcal{H} \in \overline{\mathcal{O}(\mathcal{F})}$. The 3-web $\operatorname{Leg}\mathcal{F}$ is by hypothesis flat; it is therefore the same for the 3-web $\operatorname{Leg}\mathcal{H}$. The foliation $\mathcal{H}$ is then linearly conjugated to one of the eleven homogeneous foliations given by Theorem 5.1 of [3]. Thus, according to [3, Table 1], $\mathcal{H}$ has at least one non-degenerate singularity m_0 satisfying $\mathrm{BB}(\mathcal{H}, m_0) \notin \{4, \frac{16}{3}\}$. Let $(\mathcal{F}_\varepsilon)_{\varepsilon\in\mathbb{C}}$ be the family of foliations defined by $\omega_\varepsilon = \theta_3 + \varepsilon\, C_3(x, y)(x\mathrm{d}y - y\mathrm{d}x)$. From what precedes, for $\varepsilon \neq 0$ the foliation $\mathcal{F}_\varepsilon$ belongs to $\mathcal{O}(\mathcal{F})$ and for $\varepsilon = 0$ we have $\mathcal{F}_{\varepsilon=0} = \mathcal{H}$. The singularity m_0 of $\mathcal{H}$ is "stable"; there is a family $(m_\varepsilon)_{\varepsilon\in\mathbb{C}}$ of non-degenerate singularities of $\mathcal{F}_\varepsilon$ such that $m_{\varepsilon=0} = m_0$. The $\mathcal{F}_\varepsilon$'s being conjugated for $\varepsilon \neq 0$, $\mathrm{BB}(\mathcal{F}_\varepsilon, m_\varepsilon)$ is locally constant; as a result $\mathrm{BB}(\mathcal{F}_\varepsilon, m_\varepsilon) = \mathrm{BB}(\mathcal{H}, m_0)$ for ε small. In particular $\mathcal{F}$ has a non-degenerate singularity m' verifying $\mathrm{BB}(\mathcal{F}, m') = \mathrm{BB}(\mathcal{H}, m_0)$ so that $\mathrm{BB}(\mathcal{F}, m') \notin \{4, \frac{16}{3}\}$. According to [3, Lemma 6.7] through the point m' pass exactly two lines invariant by $\mathcal{F}$, of which at least one is necessarily distinct from (mm'); this implies, according to Remark 2.10, that $\mathcal{F}$ is homogeneous. □

Lemma 2.11. — *Let $\mathcal{F}$ be a foliation of degree three on $\mathbb{P}^2_{\mathbb{C}}$ with a singular point m of algebraic multiplicity 3. Let ω be a 1-form describing $\mathcal{F}$. Assume that the singular locus of $\mathcal{F}$ is not reduced to m and that the 3-jet of ω at m is not saturated. Then up to isomorphism ω is of the following type*

$$y(a_0\, x^2 + a_1 xy + y^2)\mathrm{d}x + xy(b_0\, x + b_1 y)\mathrm{d}y + x(x^2 + c_1 xy + c_2 y^2)(x\mathrm{d}y - y\mathrm{d}x),$$

where $a_0, a_1, b_0, b_1, c_1, c_2$ are complex numbers such that the degree of the associated foliation is 3.

Proof. — The condition $\nu(\mathcal{F},m)=3$ assures the existence of a system of homogeneous coordinates $[x:y:z]\in\mathbb{P}^2_{\mathbb{C}}$ in which $m=[0:0:1]$ and $\mathcal{F}$ is defined by a 1-form ω of type

$$\omega = A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y + C(x,y)(x\mathrm{d}y - y\mathrm{d}x),$$

where A, B and C are homogeneous polynomials of degree 3. Since $J^3_{(0,0)}\omega$ is by hypothesis not saturated, we can write

$$\begin{aligned} &A(x,y) = (h_0\,x + h_1 y)(a_0\,x^2 + a_1 xy + a_2 y^2)\\ \text{and}\quad &B(x,y) = (h_0\,x + h_1 y)(b_0\,x^2 + b_1 xy + b_2 y^2).\end{aligned}$$

Let us write $C(x,y)=\sum_{i=0}^{3} c_i x^{3-i} y^i$. The hypothesis $\operatorname{Sing}\mathcal{F}\neq\{m\}$ allows us to assume that the point $m'=[0:1:0]$ is singular of $\mathcal{F}$, which amounts to assuming that $c_3=h_1 b_2=0$. The foliation $\mathcal{F}$ being of degree three the product $h_1 a_2$ is non-zero and as a result $b_2=0$; replacing $h_0=h_0' h_1$, $a_i=a_i' a_2$, $b_i=b_i' a_2$, $c_j=c_j' h_1 a_2$, with $i\in\{0,1\}$ and $j\in\{0,1,2\}$, we can assume that $h_1=a_2=1$. Thus ω writes

$$\begin{aligned}\omega = (h_0\,x + y)\big((a_0\,x^2 + a_1 xy + y^2)\mathrm{d}x + x(b_0\,x + b_1 y)\mathrm{d}y\big)\\ + x(c_0\,x^2 + c_1 xy + c_2 y^2)(x\mathrm{d}y - y\mathrm{d}x).\end{aligned}$$

The conjugation by the diffeomorphism $(x, y-h_0\,x)$ allows us to cancel h_0; as a consequence

$$\begin{aligned}\omega = y\big((a_0\,x^2 + a_1 xy + y^2)\mathrm{d}x + x(b_0\,x + b_1 y)\mathrm{d}y\big)\\ + x(c_0\,x^2 + c_1 xy + c_2 y^2)(x\mathrm{d}y - y\mathrm{d}x).\end{aligned}$$

The equality $\deg\mathcal{F}=3$ implies that $c_0\neq 0$. By conjugating ω by the homothety $\left(\frac{1}{c_0}x, \frac{1}{c_0}y\right)$, we reduce ourselves to $c_0=1$, that is, to the announced model. $\square$

LEMMA 2.12. — *Let $\mathcal{F}$ be a foliation of degree $d\geqslant 2$ on $\mathbb{P}^2_{\mathbb{C}}$. If $m\in\operatorname{Sing}\mathcal{F}$ is such that $\nu(\mathcal{F},m)=d$, then for any $m'\in\operatorname{Sing}\mathcal{F}\setminus\{m\}$ we have $\nu(\mathcal{F},m')\leqslant d-1$.*

Proof. — We know (see for instance [14, p. 158]) that if s is a singularity of $\mathcal{F}$ and if $\mathrm{X}=A(\mathrm{u},\mathrm{v})\frac{\partial}{\partial\mathrm{u}}+B(\mathrm{u},\mathrm{v})\frac{\partial}{\partial\mathrm{v}}$ is a vector field defining the germ of $\mathcal{F}$ at s, then $\nu(\mathcal{F},s)^2\leqslant\nu(A,s)\cdot\nu(B,s)\leqslant\mu(\mathcal{F},s)$. We also know (see [6]) that $\sum_{s\in\operatorname{Sing}\mathcal{F}}\mu(\mathcal{F},s)=d^2+d+1$. Let us assume now that there is $m\in\operatorname{Sing}\mathcal{F}$ such that $\nu(\mathcal{F},m)=d$; let m' be a point of $\operatorname{Sing}\mathcal{F}\setminus\{m\}$. It follows that

$$\begin{aligned}\nu(\mathcal{F},m)^2+\nu(\mathcal{F},m')^2\leqslant d^2+d+1 &\Longrightarrow \nu(\mathcal{F},m')\leqslant\sqrt{d+1}\\ &\Longrightarrow \nu(\mathcal{F},m')\leqslant d-1,\end{aligned}$$

because $d\geqslant 2$. □

Lemma 2.13. — *Let $\mathcal{F}$ be a foliation of degree three on $\mathbb{P}^2_{\mathbb{C}}$ with a singular point m of algebraic multiplicity* 3*. Assume that the singular locus of $\mathcal{F}$ is not reduced to m and that the* 3*-web* $\operatorname{Leg}\mathcal{F}$ *is flat. Then any singularity m' distinct from m is non-degenerate, the line (mm') is $\mathcal{F}$-invariant and*

- *either $\mathcal{F}$ is homogeneous;*
- *or* $\mathrm{CS}(\mathcal{F},(mm'),m')\in\{1,3\}$ *for any* $m'\in\operatorname{Sing}\mathcal{F}\smallsetminus\{m\}$.

Proof. — Let m' be a singular point of $\mathcal{F}$ distinct from m. According to Lemma 2.12 the equalities $\deg\mathcal{F}=3$ and $\nu(\mathcal{F},m)=3$ imply that $\nu(\mathcal{F},m')\leqslant 2$. If the singularity m' were degenerate, then, according to Proposition 2.1, the 3-web $\operatorname{Leg}\mathcal{F}$ would not be flat, which is impossible by hypothesis. Therefore $\mu(\mathcal{F},m')=1$.

Since $\deg\mathcal{F}=3$, $\tau(\mathcal{F},m)=3$ and $\tau(\mathcal{F},m')\geqslant 1$, the line (mm') is invariant by $\mathcal{F}$ (otherwise we would have $3=\deg\mathcal{F}=\sum_{p\in(mm')}\operatorname{Tang}(\mathcal{F},(mm'),p)\geqslant\tau(\mathcal{F},m)+\tau(\mathcal{F},m')\geqslant 4$).

Let us assume that it is possible to choose m' in such a way that $\mathrm{CS}(\mathcal{F},(mm'),m')\notin\{1,3\}$; we will show that $\mathcal{F}$ is necessarily homogeneous. The equality $\mu(\mathcal{F},m')=1$ and the condition $\mathrm{CS}(\mathcal{F},(mm'),m')\neq 1$ imply that $\mathrm{BB}(\mathcal{F},m')\neq 4$.

- If $\mathrm{BB}(\mathcal{F},m')\neq\frac{16}{3}$, then, according to [3, Lemma 6.7], through the point m' passes a line invariant by $\mathcal{F}$ and distinct from the line (mm'), which implies, according to Remark 2.10, that $\mathcal{F}$ is homogeneous;
- If $\mathrm{BB}(\mathcal{F},m')=\frac{16}{3}$, then, according to [2, Lemma 3.12], through the singularity m' passes a line ℓ invariant by $\mathcal{F}$ and such that $\mathrm{CS}(\mathcal{F},\ell,m')=3$; as we have assumed that $\mathrm{CS}(\mathcal{F},(mm'),m')\neq 3$, we deduce that $\ell\neq(mm')$, which implies (Remark 2.10) that $\mathcal{F}$ is homogeneous. □

We are now able to describe up to isomorphism the foliations of $\mathbf{FP}(3)$ whose singular locus contains a point of algebraic multiplicity 3 and is not reduced to this point.

Proposition 2.14. — *Let $\mathcal{F}$ be a foliation of degree three on $\mathbb{P}^2_{\mathbb{C}}$. Assume that $\mathcal{F}$ has a singularity of algebraic multiplicity* 3 *and that* $\operatorname{Sing}\mathcal{F}$

is not reduced to this singularity. Assume moreover that the 3*-web* $\operatorname{Leg}\mathcal{F}$ *is flat. Then either* $\mathcal{F}$ *is homogeneous, or* $\mathcal{F}$ *is, up to the action of an automorphism of* $\mathbb{P}^2_{\mathbb{C}}$*, defined by one of the following* 1*-forms*

(1) $\overline{\omega}_1 = y^3\mathrm{d}x + x^3(x\mathrm{d}y - y\mathrm{d}x)$;
(2) $\overline{\omega}_4 = (x^3 + y^3)\mathrm{d}x + x^3(x\mathrm{d}y - y\mathrm{d}x)$;
(3) $\overline{\omega}_5 = y^2(y\mathrm{d}x + 2x\mathrm{d}y) + x^3(x\mathrm{d}y - y\mathrm{d}x)$.

Proof. — Let us assume that $\mathcal{F}$ is not homogeneous; we must show that up to linear conjugation $\mathcal{F}$ is described by one of the three 1-forms $\overline{\omega}_1, \overline{\omega}_4, \overline{\omega}_5$.

Let us denote by m the singularity of $\mathcal{F}$ of algebraic multiplicity 3. Let ω be a 1-form describing $\mathcal{F}$ in an affine chart (x, y) of $\mathbb{P}^2_{\mathbb{C}}$. Since by hypothesis $\operatorname{Leg}\mathcal{F}$ is flat, it follows, according to Lemma 2.9, that the 3-jet of ω at m is not saturated. By hypothesis we have $\operatorname{Sing}\mathcal{F} \neq \{m\}$. As a result, Lemma 2.11 assures us that ω is, up to isomorphism, of the following type

$$y(a_0\, x^2 + a_1 xy + y^2)\mathrm{d}x + xy(b_0\, x + b_1 y)\mathrm{d}y \\ + x(x^2 + c_1 xy + c_2 y^2)(x\mathrm{d}y - y\mathrm{d}x), \quad a_i, b_i, c_j \in \mathbb{C}.$$

In this situation, $m = [0 : 0 : 1]$, $m' := [0 : 1 : 0] \in \operatorname{Sing}\mathcal{F}$ and the line $(mm') = (x = 0)$ is invariant by $\mathcal{F}$; moreover, a straightforward computation shows that $\operatorname{CS}(\mathcal{F}, (mm'), m') = 1 + b_1$. Lemma 2.13 then implies that $b_1 \in \{0, 2\}$.

If $b_0 \neq 0$, resp. $(b_0, b_1) = (0, 2)$, resp. $b_0 = b_1 = 0, c_2 \neq 0$, resp. $b_0 = b_1 = c_2 = 0, c_1 \neq 0$, then by conjugating ω by

$$\left(\frac{b_0^2\, x}{1 - c_1 b_0\, x}, \frac{b_0^3 y}{1 - c_1 b_0\, x}\right), \qquad \text{resp.} \left(\frac{x}{1 - \left(\frac{c_2}{2}\right) x}, \frac{y}{1 - \left(\frac{c_2}{2}\right) x}\right), \\ \text{resp.} \left(c_2^{-1} x, c_2^{-3/2}\, y\right), \qquad \text{resp.} \left(c_1^{-2} x, c_1^{-3} y\right),$$

we reduce ourselves to $(b_0, c_1) = (1, 0)$, resp. $c_2 = 0$, resp. $c_2 = 1$, resp. $c_1 = 1$. Therefore, it suffices us to treat the following possibilities

$$\begin{aligned} &(b_0, b_1, c_1) = (1, 0, 0), && (b_0, b_1, c_1) = (1, 2, 0), \\ &(b_0, b_1, c_2) = (0, 2, 0), && (b_0, b_1, c_2) = (0, 0, 1), \\ &(b_0, b_1, c_1, c_2) = (0, 0, 1, 0), && (b_0, b_1, c_1, c_2) = (0, 0, 0, 0). \end{aligned}$$

Let us place ourselves in the affine chart (p, q) of $\check{\mathbb{P}}^2_{\mathbb{C}}$ associated to the line $\{py - qx = 1\} \subset \mathbb{P}^2_{\mathbb{C}}$; the 3-web $\operatorname{Leg}\mathcal{F}$ is described by the differential equation

$$F(p,q,w) := pw^3 + (a_1p + b_1q - c_2)w^2 + (a_0p + b_0q - c_1)w - 1 = 0, \quad \text{with} \quad w = \frac{\mathrm{d}q}{\mathrm{d}p}.$$

The explicit computation of $K(\operatorname{Leg}\mathcal{F})$ shows that it has the form

$$K(\operatorname{Leg}\mathcal{F}) = \frac{\sum_{i+j\leqslant 6}\rho_i^j p^i q^j}{\Delta(p,q)^2}\mathrm{d}p\wedge\mathrm{d}q,$$

where Δ is the w-discriminant of F and the ρ_i^j's are polynomials in the parameters a_i, b_i, c_j.

Step 1. — If $(b_0, b_1, c_1) = (1, 0, 0)$, then the explicit computation of $K(\operatorname{Leg}\mathcal{F})$ leads to $\rho_0^5 = 4c_2$ and

$$\rho_2^1 = a_0^2(a_0 + 8a_1)c_2^3 - (177a_0^2 - 60a_0 - 24a_0a_1 - 84a_0a_1^2 + 24a_1^2 + 24a_1^3)c_2^2 \\ +(108a_1^2 - 36a_1 - 81a_0)c_2 + 81,$$

so that the system $\rho_0^5 = \rho_2^1 = 0$ has no solutions. So this first case does not happen.

Step 2. — When $(b_0, b_1, c_1) = (1, 2, 0)$, the explicit computation of $K(\operatorname{Leg}\mathcal{F})$ gives us:

$$\begin{aligned} \rho_0^6 &= -4(6a_0 - 5a_1 + 4),\\ \rho_1^5 &= -4(12a_0^2 + 20a_0 - 10a_0a_1 - 3a_1^2),\\ \rho_5^0 &= 32a_0^5c_2 - 8(a_1^2c_2 + 2a_1c_2 + 12)a_0^4 + 4(a_1^3c_2 + 4a_1^2 + a_1 - 12)a_0^3\\ &\quad + (4a_1^2 - 5a_1 + 30)a_0^2a_1^2 - 4a_0a_1^4; \end{aligned}$$

it is easy to see that the system $\rho_0^6 = \rho_1^5 = \rho_5^0 = 0$ has no solutions. So this second case is not possible.

Step 3. — If $(b_0, b_1, c_2) = (0, 2, 0)$, then the explicit computation of $K(\operatorname{Leg}\mathcal{F})$ shows that:

$$\rho_1^0 = 24c_1^4, \qquad \rho_1^4 = -256a_0^2, \qquad \rho_0^4 = 64(14a_1 + 3a_0c_1),$$

so that $a_0 = a_1 = c_1 = 0$. As a consequence, in this third case, $\mathcal{F}$ is given by

$$\overline{\omega}_5 = y^2(y\mathrm{d}x + 2x\mathrm{d}y) + x^3(x\mathrm{d}y - y\mathrm{d}x);$$

we verify by computation that its Legendre transform is flat.

Step 4. — When $(b_0, b_1, c_2) = (0, 0, 1)$, the explicit computation of $K(\operatorname{Leg}\mathcal{F})$ gives us:

$$\begin{aligned}
\rho_5^0 &= 2a_0^2(4a_0 - a_1^2)(2a_0^2 - 6a_0 - a_0a_1c_1 + 2a_1^2),\\
\rho_1^0 &= 2(c_1^2 - 4)\big((12a_0 - a_1^2)c_1^2 - (5a_0 + 18)a_1c_1 + a_0^2 - 24a_0 + 14a_1^2 + 27\big),\\
\rho_4^0 &= 2a_1(2a_0^4 + 24a_0^3 - 54a_0^2 - 5a_0^2a_1^2 + 30a_0a_1^2 - 4a_1^4)\\
&\quad - (60a_0^3 - 36a_0^2 - 11a_0^2a_1^2 + 45a_0a_1^2 - 8a_1^4)a_0c_1 + (24a_0 - 5a_1^2)a_0^2a_1c_1^2,\\
\rho_2^0 &= 3(5a_0^2 + 52a_0 - 2a_1^2 + 27)a_1c_1^2 - (64a_0^2 + 60a_0 - 7a_0a_1^2 + 7a_1^2)c_1^3\\
&\quad - 2a_1(a_0^2 + 162a_0 - 48a_1^2 - 27) + (8a_0 - a_1^2)a_1c_1^4\\
&\quad - (a_0^3 - 177a_0^2 - 81a_0 + 84a_0a_1^2 + 108a_1^2 + 81)c_1;
\end{aligned}$$

the system $\rho_1^0 = \rho_2^0 = \rho_4^0 = \rho_5^0 = 0$ is equivalent to $(a_0, a_1, c_1) \in \{(1, 2, 2), (1, -2, -2)\}$, as an explicit computation shows. If $(a_0, a_1, c_1) = (1, 2, 2)$, resp. $(a_0, a_1, c_1) = (1, -2, -2)$, then ω writes

$$\begin{aligned}
&\omega = (x + y)^2(y\mathrm{d}x + x(x\mathrm{d}y - y\mathrm{d}x)),\\
\text{resp.}\quad &\omega = (x - y)^2(y\mathrm{d}x + x(x\mathrm{d}y - y\mathrm{d}x)),
\end{aligned}$$

which contradicts the equality $\deg\mathcal{F} = 3$.

Step 5. — When $(b_0, b_1, c_1, c_2) = (0, 0, 1, 0)$, the explicit computation of $K(\operatorname{Leg}\mathcal{F})$ shows that:

$$\begin{aligned}
\rho_3^0 &= -2a_0^2(6a_0 + a_0a_1 - 2a_1^2)(4a_0 - a_1^2),\\
\rho_0^0 &= -81 - 60a_0 + 8a_0a_1 + 81a_1 - 7a_1^2 - a_1^3,\\
\rho_1^0 &= -4(a_1 - 3)(6a_0^2 - 9a_0a_1 - a_0a_1^2 + 2a_1^3);
\end{aligned}$$

the system $\rho_0^0 = \rho_1^0 = \rho_3^0 = 0$ is verified if and only if $(a_0, a_1) = (2, 3)$, in which case

$$\omega = (x + y)\big(y(y + 2x)\mathrm{d}x + x^2(x\mathrm{d}y - y\mathrm{d}x)\big),$$

but this contradicts the equality $\deg\mathcal{F} = 3$.

Step 6. — If $(b_0, b_1, c_1, c_2) = (0, 0, 0, 0)$, then $\omega = y(a_0x^2 + a_1xy + y^2)\mathrm{d}x + x^3(x\mathrm{d}y - y\mathrm{d}x)$; the differential equation describing $\operatorname{Leg}\mathcal{F}$ writes

$$F(p, q, q') = p(q')^3 + a_1p(q')^2 + a_0pq' - 1 = 0, \quad \text{with} \quad q' = \frac{\mathrm{d}q}{\mathrm{d}p}.$$

We study two eventualities according to whether a_1 is zero or not.

Substep 6.1. — When $a_1 = 0$ the explicit computation of $K(\operatorname{Leg}\mathcal{F})$ gives us

$$K(\operatorname{Leg}\mathcal{F}) = -\frac{48a_0^4p}{(4a_0^3p^2 + 27)^2}\mathrm{d}p \wedge \mathrm{d}q;$$

as a result $\operatorname{Leg}\mathcal{F}$ is flat if and only if $a_0 = 0$, in which case $\mathcal{F}$ is described by

$$\overline{\omega}_1 = y^3\mathrm{d}x + x^3(x\mathrm{d}y - y\mathrm{d}x).$$

Subtep 6.2. — If $a_1 \neq 0$, then by conjugating ω by $\left(\alpha^2 x, \alpha^3 y\right)$, where $\alpha = \frac{1}{3}a_1$, we can assume that $a_1 = 3$. In this case the explicit computation of $K(\operatorname{Leg}\mathcal{F})$ shows that

$$K(\operatorname{Leg}\mathcal{F}) = -\frac{12\,(a_0-3)\left(a_0^2(4a_0-9)p+27(a_0-2)\right)}{\left(a_0^2(4a_0-9)p^2+54(a_0-2)p+27\right)^2}\mathrm{d}p\wedge\mathrm{d}q;$$

as a consequence $\operatorname{Leg}\mathcal{F}$ is flat if and only if $a_0 = 3$, in which case

$$\omega = y(3x^2+3xy+y^2)\mathrm{d}x + x^3(x\mathrm{d}y - y\mathrm{d}x).$$

After replacing ω by $\varphi^*\omega$, where $\varphi(x,y) = (x, -x-y)$, the foliation $\mathcal{F}$ is given in the affine coordinates (x,y) by the 1-form

$$\overline{\omega}_4 = (x^3+y^3)\mathrm{d}x + x^3(x\mathrm{d}y - y\mathrm{d}x). \qquad \square$$

Remark 2.15. — The five foliations $\mathcal{F}_1, \ldots, \mathcal{F}_5$ have the following properties:

(i) $\#\operatorname{Sing}\mathcal{F}_2 = 1$, $\#\operatorname{Sing}\mathcal{F}_3 = 13$ and $\#\operatorname{Sing}\mathcal{F}_j = 2$ for $j = 1,4,5$;
(ii) $\mathcal{F}_j$ is convex if and only if $j \in \{1,3\}$;
(iii) $\mathcal{F}_j$ has a radial singularity of order 2 if and only if $j \in \{1,3,4\}$;
(iv) $\mathcal{F}_j$ admits a double inflection point if and only if $j \in \{2,4\}$.

The verifications of these properties are easy and left to the reader.

Remark 2.16. — The sixteen foliations $\mathcal{H}_1, \ldots, \mathcal{H}_{11}, \mathcal{F}_1, \ldots, \mathcal{F}_5$ are not linearly conjugated. Indeed, by construction, the $\mathcal{F}_j$'s are not homogeneous and are therefore not conjugated to the homogeneous foliations $\mathcal{H}_i$. The $\mathcal{H}_i$'s are not linearly conjugated ([3, Theorem 5.1]). Finally, the fact that the $\mathcal{F}_j$'s are not linearly conjugated follows from the properties (i), (ii) and (iii) above.

Theorem A follows from [3, Theorems 5.1, 6.1], Propositions 2.1, 2.8, 2.14 and Remark 2.16.

Proof of Corollary C. — According to [4, Corollary 4.7] every convex foliation of degree three on $\mathbb{P}^2_{\mathbb{C}}$ has a flat Legendre transform and is therefore linearly conjugated to one of the sixteen foliations given by Theorem A. The statement then follows from the fact that the only convex foliations appearing in this theorem are $\mathcal{H}_1, \mathcal{H}_3, \mathcal{F}_1$ and $\mathcal{F}_3$. $\square$

3. Orbits under the action of $\mathrm{PGL}_3(\mathbb{C})$

In this section, we describe the irreducible components of $\mathbf{FP}(3)$. We start by determining the dimensions of the orbits $\mathcal{O}(\mathcal{H}_i), \mathcal{O}(\mathcal{F}_j)$ under the action of $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}}) = \mathrm{PGL}_3(\mathbb{C})$. Next we classify up to isomorphism the foliations of $\mathbf{F}(3)$ which realize the minimal dimension of the orbits in degree 3. Finally, we study the closure of the orbits $\mathcal{O}(\mathcal{H}_i), \mathcal{O}(\mathcal{F}_j)$ in $\mathbf{F}(3)$ and we prove the Theorem D describing the irreducible components of $\mathbf{FP}(3)$.

3.1. Isotropy groups and dimensions of the orbits $\mathcal{O}(\mathcal{H}_i)$ and $\mathcal{O}(\mathcal{F}_j)$

Definition 3.1. — *Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_{\mathbb{C}}$. The subgroup of* $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$ *(resp.* $\mathrm{Aut}(\check{\mathbb{P}}^2_{\mathbb{C}})$*) which preserves $\mathcal{F}$ (resp.* $\mathrm{Leg}\,\mathcal{F}$*) is called the* isotropy group *of $\mathcal{F}$ (resp.* $\mathrm{Leg}\,\mathcal{F}$*) and is denoted by* $\mathrm{Iso}(\mathcal{F})$ *(resp.* $\mathrm{Iso}(\mathrm{Leg}\,\mathcal{F})$*);* $\mathrm{Iso}(\mathcal{F})$ *and* $\mathrm{Iso}(\mathrm{Leg}\,\mathcal{F})$ *are algebraic groups.*

Remark 3.2. — Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_{\mathbb{C}}$. If $[a:b:c]$ are the homogeneous coordinates in $\check{\mathbb{P}}^2_{\mathbb{C}}$ associated to the line $\{ax+by+cz=0\} \subset \mathbb{P}^2_{\mathbb{C}}$, then

$$\mathrm{Iso}(\mathrm{Leg}\,\mathcal{F}) = \left\{[a:b:c]\cdot A^{-1} \,\middle|\, A \in \mathrm{PGL}_3(\mathbb{C}), [x:y:z]\cdot {}^{\mathrm{t}}A \in \mathrm{Iso}(\mathcal{F})\right\}.$$

More precisely, the isomorphism $\tau\colon\ \mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}}) \to \mathrm{Aut}(\check{\mathbb{P}}^2_{\mathbb{C}})$ which, for A in $\mathrm{PGL}_3(\mathbb{C})$, sends $[x:y:z]\cdot {}^{\mathrm{t}}A$ into $[a:b:c]\cdot A^{-1}$ induces an isomorphism from $\mathrm{Iso}(\mathcal{F})$ onto $\mathrm{Iso}(\mathrm{Leg}\,\mathcal{F})$.

The following result is elementary and its proof is left to the reader.

Proposition 3.3. — *The groups* $\mathrm{Iso}(\mathcal{H}_i)$ *and* $\mathrm{Iso}(\mathcal{F}_j)$ *are given by*

(1) $\mathrm{Iso}(\mathcal{H}_1) = \left\{[\pm x : y : \alpha z], [\pm y : x : \alpha z] \,\middle|\, \alpha \in \mathbb{C}^*\right\}$;

(2) $\mathrm{Iso}(\mathcal{H}_2) = \left\{[\pm x : y : \alpha z], [\pm y : x : \alpha z], [\pm \mathrm{i}x : y : \alpha z], [\pm \mathrm{i}y : x : \alpha z] \,\middle|\, \alpha \in \mathbb{C}^*\right\}$;

(3) $\mathrm{Iso}(\mathcal{H}_3) = \left\{[x : y : \alpha z], [y : x : \alpha z] \,\middle|\, \alpha \in \mathbb{C}^*\right\}$;

(4) $\mathrm{Iso}(\mathcal{H}_4) = \left\{[x : y : \alpha z], [y : x : \alpha z] \,\middle|\, \alpha \in \mathbb{C}^*\right\}$;

(5) $\mathrm{Iso}(\mathcal{H}_5) = \left\{[x : y : \alpha z] \,\middle|\, \alpha \in \mathbb{C}^*\right\}$;

(6) $\mathrm{Iso}(\mathcal{H}_6) = \left\{[x : y : \alpha z] \,\middle|\, \alpha \in \mathbb{C}^*\right\}$;

(7) $\mathrm{Iso}(\mathcal{H}_7) = \left\{[\pm x : y : \alpha z] \,\middle|\, \alpha \in \mathbb{C}^*\right\}$;

(8) $\mathrm{Iso}(\mathcal{H}_8) = \Big\{[\,x : y : \alpha z], [4y - x : y : \alpha z] \,\Big|\, \alpha \in \mathbb{C}^*\Big\}$;

(9) $\mathrm{Iso}(\mathcal{H}_9) = \Big\{[\,x : y : \alpha z], [\,x - y : x : \alpha z], [y : y - x : \alpha z] \,\Big|\, \alpha \in \mathbb{C}^*\Big\}$;

(10) $\mathrm{Iso}(\mathcal{H}_{10}) = \Big\{[\,x : y : \alpha z], [-y : x : \alpha z] \,\Big|\, \alpha \in \mathbb{C}^*\Big\}$;

(11) $\mathrm{Iso}(\mathcal{H}_{11}) = \Big\{[\,x : y : \alpha z], [y : x : \alpha z], [\xi^5\, x : x + \xi\, y : \alpha z], [\xi^{-5}\, x : x + \xi^{-1}\, y : \alpha z], [\xi^5\, y : y + \xi\, x : \alpha z], [\xi^{-5}\, y : y + \xi^{-1}\, x : \alpha z], [\xi^5\, x - y : x + \xi^{-1}\, y : \alpha z], [\xi^{-5}\, x - y : x + \xi\, y : \alpha z], [\xi^5\, x + \xi^4\, y : x : \alpha z], [\xi^{-5}\, x + \xi^{-4}\, y : x : \alpha z], [\xi^5\, y + \xi^4\, x : y : \alpha z], [\xi^{-5}\, y + \xi^{-4}\, x : y : \alpha z] \,\Big|\, \alpha \in \mathbb{C}^*\Big\}$ *where* $\xi = \mathrm{e}^{\mathrm{i}\pi/6}$;

(12) $\mathrm{Iso}(\mathcal{F}_1) = \Big\{[\alpha^2 x : \alpha^3 y : z + \beta\, x] \,\Big|\, \alpha \in \mathbb{C}^*, \beta \in \mathbb{C}\Big\}$;

(13) $\mathrm{Iso}(\mathcal{F}_2) = \Big\{[\alpha^4 x : \alpha^3 y : z + \beta\, x] \,\Big|\, \alpha \in \mathbb{C}^*, \beta \in \mathbb{C}\Big\}$;

(14) $\mathrm{Iso}(\mathcal{F}_3) = \Big\{[\pm\, x : \pm\, y : z], [\pm\, y : \pm\, x : z], [\pm\, x : \pm\, z : y], [\pm\, z : \pm\, x : y], [\pm\, y : \pm\, z : x], [\pm\, z : \pm\, y : x]\Big\}$;

(15) $\mathrm{Iso}(\mathcal{F}_4) = \Big\{[\,x : y : z + \alpha x], [\,\mathrm{j}\, x : y : z + \alpha\, x], [\,\mathrm{j}^2 x : y : z + \alpha\, x] \,\Big|\, \alpha \in \mathbb{C}\Big\}$ *where* $\mathrm{j} = \mathrm{e}^{2\mathrm{i}\pi/3}$;

(16) $\mathrm{Iso}(\mathcal{F}_5) = \Big\{[\alpha^2 x : \alpha^3 y : z] \,\Big|\, \alpha \in \mathbb{C}^*\Big\}$.

In particular, the dimensions of the orbits $\mathcal{O}(\mathcal{H}_i)$ *and* $\mathcal{O}(\mathcal{F}_j)$ *are the following*

$$\begin{aligned} &\dim \mathcal{O}(\mathcal{F}_1) = 6, \quad \dim \mathcal{O}(\mathcal{F}_2) = 6, \quad \dim \mathcal{O}(\mathcal{H}_i) = 7, \quad i = 1, \ldots, 11,\\ &\dim \mathcal{O}(\mathcal{F}_4) = 7, \quad \dim \mathcal{O}(\mathcal{F}_5) = 7, \quad \dim \mathcal{O}(\mathcal{F}_3) = 8. \end{aligned}$$

3.2. Description of degree three foliations $\mathcal{F}$ such that $\dim \mathcal{O}(\mathcal{F}) = 6$

Proposition 2.3 of [11] asserts that if $\mathcal{F}$ is a foliation of degree $d \geqslant 2$ on $\mathbb{P}^2_{\mathbb{C}}$, then the dimension of $\mathcal{O}(\mathcal{F})$ is at least 6, or equivalently, the dimension of $\mathrm{Iso}(\mathcal{F})$ is at most 2. Notice that these bounds are attained by the foliations $\mathcal{F}_1^{(d)}$ and $\mathcal{F}_2^{(d)}$ defined in the affine chart $z = 1$ respectively by the 1-forms

$$\overline{\omega}_1^{(d)} = y^d \mathrm{d}x + x^d (x\mathrm{d}y - y\mathrm{d}x) \quad \text{and} \quad \overline{\omega}_2^{(d)} = x^d \mathrm{d}x + y^d (x\mathrm{d}y - y\mathrm{d}x).$$

Indeed, it is easy to check that

$$\left\{\left(\frac{\alpha^{d-1}x}{1+\beta x}, \frac{\alpha^d y}{1+\beta x}\right) \,\middle|\, \alpha \in \mathbb{C}^*, \beta \in \mathbb{C}\right\} \subset \mathrm{Iso}(\mathcal{F}_1^{(d)})$$

and

$$\left\{\left(\frac{\alpha^{d+1}x}{1+\beta x}, \frac{\alpha^d y}{1+\beta x}\right) \middle| \, \alpha \in \mathbb{C}^*, \beta \in \mathbb{C}\right\} \subset \mathrm{Iso}(\mathcal{F}_2^{(d)}),$$

so that $\dim \mathrm{Iso}(\mathcal{F}_i^{(d)}) \geqslant 2, i = 1, 2$, and so $\dim \mathrm{Iso}(\mathcal{F}_i^{(d)}) = 2$.

Remark 3.4. — By construction, we have $\mathcal{F}_1^{(3)} = \mathcal{F}_1$ and $\mathcal{F}_2^{(3)} = \mathcal{F}_2$.

D. Cerveau, J. Déserti, D. Garba Belko and R. Meziani have shown that up to isomorphism of $\mathbb{P}^2_{\mathbb{C}}$ the quadratic foliations $\mathcal{F}_1^{(2)}$ and $\mathcal{F}_2^{(2)}$ are the only foliations realizing the minimal dimension of the orbits in degree 2 ([11, Proposition 2.7]). Corollary B stated in the Introduction is a similar result in degree 3.

Proof of Corollary B. — Let $\mathcal{F}$ be a degree three foliation on $\mathbb{P}^2_{\mathbb{C}}$ such that $\dim \mathcal{O}(\mathcal{F}) = 6$. Since $\mathrm{Iso}(\mathrm{Leg}\,\mathcal{F})$ is isomorphic to $\mathrm{Iso}(\mathcal{F})$, we have that

$$\dim \mathrm{Iso}(\mathrm{Leg}\,\mathcal{F}) = \dim \mathrm{Iso}(\mathcal{F}) = 8 - 6 = 2.$$

Let us fix $m \in \check{\mathbb{P}}^2_{\mathbb{C}} \smallsetminus \Delta(\mathrm{Leg}\,\mathcal{F})$ and let $\mathcal{W}_m$ be the germ of the 3-web $\mathrm{Leg}\,\mathcal{F}$ at m. After É. Cartan [9] the equality $\dim \mathrm{Iso}(\mathrm{Leg}\,\mathcal{F}) = 2$ implies that $\mathcal{W}_m$ is parallelizable and so flat. Since the curvature $\mathrm{Leg}\,\mathcal{F}$ is holomorphic on $\check{\mathbb{P}}^2_{\mathbb{C}} \smallsetminus \Delta(\mathrm{Leg}\,\mathcal{F})$, we deduce that $\mathrm{Leg}\,\mathcal{F}$ is flat. Therefore $\mathcal{F}$ is linearly conjugate to one of the 16 foliations given by Theorem A. Proposition 3.3 and the hypothesis $\dim \mathcal{O}(\mathcal{F}) = 6$ allows us to conclude. □

3.3. Closure of the orbits and irreducible components of $\mathbf{FP}(3)$

We begin by studying the closure of the orbits $\mathcal{O}(\mathcal{H}_i)$ and $\mathcal{O}(\mathcal{F}_j)$ in $\mathbf{F}(3)$, then we prove Theorem D describing the irreducible components of $\mathbf{FP}(3)$.

The following definition will be useful.

Definition 3.5 ([11]). — *Let $\mathcal{F}$ and $\mathcal{F}'$ be two foliations of $\mathbf{F}(3)$. We say that $\mathcal{F}$* degenerates *onto $\mathcal{F}'$ if the closure $\overline{\mathcal{O}(\mathcal{F})}$ (inside $\mathbf{F}(3)$) of $\mathcal{O}(\mathcal{F})$ contains $\mathcal{O}(\mathcal{F}')$ and $\mathcal{O}(\mathcal{F}) \neq \mathcal{O}(\mathcal{F}')$.*

Remarks 3.6. — Let $\mathcal{F}$ and $\mathcal{F}'$ be two foliations such that $\mathcal{F}$ degenerates onto $\mathcal{F}'$. Then

(i) $\dim \mathcal{O}(\mathcal{F}') < \dim \mathcal{O}(\mathcal{F})$;
(ii) if $\mathrm{Leg}\,\mathcal{F}$ is flat then $\mathrm{Leg}\,\mathcal{F}'$ is also flat;
(iii) $\deg \mathrm{I}^{\mathrm{inv}}_{\mathcal{F}} \leqslant \deg \mathrm{I}^{\mathrm{inv}}_{\mathcal{F}'}$, equivalently $\deg \mathrm{I}^{\mathrm{tr}}_{\mathcal{F}} \geqslant \deg \mathrm{I}^{\mathrm{tr}}_{\mathcal{F}'}$. In particular, if $\mathcal{F}$ is convex then $\mathcal{F}'$ is also convex.

As we have already noted in the Introduction, D. Marín and J. V. Pereira have shown in [18] that the closure of the orbit $\mathcal{O}(\mathcal{F}_3)$ of $\mathcal{F}_3$ is an irreducible component of $\mathbf{FP}(3)$. Assertion (2) in the proposition below gives a more precise description.

PROPOSITION 3.7.
(1) *The orbits $\mathcal{O}(\mathcal{F}_1)$ and $\mathcal{O}(\mathcal{F}_2)$ are closed.*
(2) $\overline{\mathcal{O}(\mathcal{F}_3)} = \mathcal{O}(\mathcal{F}_1) \cup \mathcal{O}(\mathcal{H}_1) \cup \mathcal{O}(\mathcal{H}_3) \cup \mathcal{O}(\mathcal{F}_3)$.

Proof. — First assertion follows from Corollary B and Remark 3.6 (i).

By Corollary C and Remark 3.6 (iii), $\mathcal{F}_3$ can degenerate only onto $\mathcal{F}_1, \mathcal{H}_1$ or $\mathcal{H}_3$. Let us show that this is the case. Consider the family of homotheties $\varphi = \varphi_\varepsilon = \left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)$. We have that

$$-\varepsilon^4 \varphi^* \overline{\omega}_3 = (y^3 - \varepsilon^2 y)\mathrm{d}x + (\varepsilon^2 x - x^3)\mathrm{d}y$$

tends to ω_1 as ε goes to 0. Thus, the foliation $\mathcal{F}_3$ degenerates onto $\mathcal{H}_1$.

In the affine chart $x = 1$, $\mathcal{F}_1$, resp. $\mathcal{F}_3$, is given by

$$\bar{\theta}_1 = \mathrm{d}y - y^3\mathrm{d}z, \qquad \text{resp. } \bar{\theta}_3 = (y^3 - y)\mathrm{d}z - (z^3 - z)\mathrm{d}y;$$

consider the family of automorphisms $\sigma = \left(\frac{y}{\varepsilon}, 2 + 6\varepsilon^2 z\right)$. A direct computation shows that

$$-\frac{\varepsilon}{6}\sigma^*\bar{\theta}_3 = (1 + 11\varepsilon^2 z + 36\varepsilon^4 z^2 + 36\varepsilon^6 z^3)\mathrm{d}y + (\varepsilon^2 y - y^3)\mathrm{d}z$$

which tends to $\bar{\theta}_1$ as ε tends to 0. Thus $\mathcal{F}_3$ degenerates onto $\mathcal{F}_1$.

In homogeneous coordinates $\mathcal{H}_3$, resp. $\mathcal{F}_3$, is given by

$$\Omega_3 = z\,y^2(3x + y)\mathrm{d}x - z\,x^2(x + 3y)\mathrm{d}y + xy(x^2 - y^2)\mathrm{d}z,$$
$$\text{resp. } \overline{\Omega}_3 = x^3(y\mathrm{d}z - z\mathrm{d}y) + y^3(z\mathrm{d}x - x\mathrm{d}z) + z^3(x\mathrm{d}y - y\mathrm{d}x);$$

by putting $\psi = [x - y : 2\varepsilon\, z - x - y : x + y]$ we obtain

$$\begin{aligned}\frac{1}{8\varepsilon}\psi^*\overline{\Omega}_3 = {} & z\,y(y - \varepsilon\, z)(3x + y - 2\varepsilon\, z)\mathrm{d}x \\ & - z\,x(x - \varepsilon\, z)(x + 3y - 2\varepsilon\, z)\mathrm{d}y + xy(x^2 - y^2)\mathrm{d}z\end{aligned}$$

which tends to Ω_3 as ε goes to 0. As a consequence $\mathcal{F}_3$ degenerates onto $\mathcal{H}_3$. $\square$

Remark 3.8. — By combining Assertion **2.** of Proposition 3.7 and Corollary C, we deduce that the set of convex foliations of degree three on $\mathbb{P}^2_{\mathbb{C}}$ is exactly the closure $\overline{\mathcal{O}(\mathcal{F}_3)}$ of $\mathcal{O}(\mathcal{F}_3)$ and is therefore an irreducible closed subset of $\mathbf{F}(3)$.

Next result is an immediate consequence of Corollary B and Remark 3.6 (i).

Corollary 3.9. — *Let $\mathcal{F}$ be an element of* $\mathbf{F}(3)$ *such that* $\dim \mathcal{O}(\mathcal{F}) \leqslant 7$. *Then*

$$\overline{\mathcal{O}(\mathcal{F})} \subset \mathcal{O}(\mathcal{F}) \cup \mathcal{O}(\mathcal{F}_1) \cup \mathcal{O}(\mathcal{F}_2).$$

The following result provides a necessary condition for a degree three foliation on $\mathbb{P}^2_{\mathbb{C}}$ degenerates onto the foliation $\mathcal{F}_1$.

Proposition 3.10. — *Let $\mathcal{F}$ be an element of* $\mathbf{F}(3)$*. If $\mathcal{F}$ degenerates onto $\mathcal{F}_1$, then $\mathcal{F}$ possesses a non-degenerate singular point m satisfying* $\mathrm{BB}(\mathcal{F}, m) = 4$.

Proof. — Assume that $\mathcal{F}$ degenerates onto $\mathcal{F}_1$. Then there exists an analytic family $(\mathcal{F}_\varepsilon)$ of foliations defined by 1-forms ω_ε such that $\mathcal{F}_\varepsilon \in \mathcal{O}(\mathcal{F})$ for $\varepsilon \neq 0$ and $\mathcal{F}_{\varepsilon=0} = \mathcal{F}_1$. The non-degenerate singular point m_0 of $\mathcal{F}_1$ is "stable", i.e. there is an analytic family (m_ε) of non-degenerate singular points of $\mathcal{F}_\varepsilon$ such that $m_{\varepsilon=0} = m_0$. The $\mathcal{F}_\varepsilon$'s being conjugated to $\mathcal{F}$ for $\varepsilon \neq 0$, the foliation $\mathcal{F}$ admits a non-degenerate singular point m such that

$$\forall\, \varepsilon \in \mathbb{C}^*, \quad \mathrm{BB}(\mathcal{F}_\varepsilon, m_\varepsilon) = \mathrm{BB}(\mathcal{F}, m).$$

Since $\mu(\mathcal{F}_\varepsilon, m_\varepsilon) = 1$ for every ε in $\mathbb{C}$, the function $\varepsilon \mapsto \mathrm{BB}(\mathcal{F}_\varepsilon, m_\varepsilon)$ is continuous, hence constant on $\mathbb{C}$. As a result

$$\mathrm{BB}(\mathcal{F}, m) = \mathrm{BB}(\mathcal{F}_{\varepsilon=0}, m_{\varepsilon=0}) = \mathrm{BB}(\mathcal{F}_1, m_0) = 4. \qquad \square$$

Corollary 3.11. — *The foliations $\mathcal{H}_2, \mathcal{H}_8, \mathcal{H}_{11}$ and $\mathcal{F}_5$ do not degenerate onto $\mathcal{F}_1$.*

A sufficient condition for the degeneration of a degree three foliation into $\mathcal{F}_1$ is the following:

Proposition 3.12. — *Let $\mathcal{F}$ be an element of* $\mathbf{F}(3)$ *such that $\mathcal{F}_1 \notin \mathcal{O}(\mathcal{F})$. If $\mathcal{F}$ possesses a non-degenerate singular point m satisfying*

$$\mathrm{BB}(\mathcal{F}, m) = 4 \qquad \textit{and} \qquad \kappa(\mathcal{F}, m) = 3,$$

then $\mathcal{F}$ degenerates onto $\mathcal{F}_1$.

Proof. — Assume that $\mathcal{F}$ has a such singular point m. The equality $\kappa(\mathcal{F}, m) = 3$ assures the existence of a line ℓ_m through m which is not invariant by $\mathcal{F}$ and such that $\mathrm{Tang}(\mathcal{F}, \ell_m, m) = 3$. Taking an affine coordinate system (x, y) such that $m = (0, 0)$ and $\ell_m = (x = 0)$, the foliation

$\mathcal{F}$ is defined by a 1-form ω of the following type

$$\begin{aligned}&(*x+\beta y+*x^2+*xy+*y^2+*x^3+*x^2y+*xy^2+*y^3)\mathrm{d}x\\&+(\alpha\,x+ry+*x^2+*xy+sy^2+*x^3+*x^2y+*xy^2+\gamma\,y^3)\mathrm{d}y\\&+(*x^3+*x^2y+*xy^2+*y^3)(x\mathrm{d}y-y\mathrm{d}x),\\&\text{with } *,r,s,\alpha,\beta,\gamma\in\mathbb{C}.\end{aligned}$$

Along the line $x=0$ the 2-form $\omega\wedge\mathrm{d}x$ writes as $(ry+sy^2+\gamma\,y^3)\mathrm{d}y\wedge\mathrm{d}x$. The equality $\mathrm{Tang}(\mathcal{F},\ell_m,m)=3$ is equivalent to $r=s=0$ and $\gamma\neq0$. The equalities $r=0$, $\mu(\mathcal{F},m)=1$ and $\mathrm{BB}(\mathcal{F},m)=4$ imply that $\beta=-\alpha\neq0$. Thus ω writes as

$$\begin{aligned}&(*x-\alpha y+*x^2+*xy+*y^2+*x^3+*x^2y+*xy^2+*y^3)\mathrm{d}x\\&+(\alpha\,x+*x^2+*xy+*x^3+*x^2y+*xy^2+\gamma\,y^3)\mathrm{d}y\\&+(*x^3+*x^2y+*xy^2+*y^3)(x\mathrm{d}y-y\mathrm{d}x),\\&\text{where } *\in\mathbb{C},\alpha,\gamma\in\mathbb{C}^*.\end{aligned}$$

Put $\varphi=\big(\varepsilon^3x,\varepsilon y\big)$ and fix $(i,j)\in\mathbb{Z}_+^2\setminus\{(0,0)\}$. Notice that

(1) $\varphi^*(x^iy^j\mathrm{d}x)=\varepsilon^{3i+j+3}x^iy^j\mathrm{d}x$ is divisible by ε^4 and $\frac{1}{\varepsilon^4}\varphi^*(x^iy^j\mathrm{d}x)$ tends to 0 as ε tends to 0 except for $(i,j)=(0,1)$;

(2) $\varphi^*(x^iy^j\mathrm{d}y)=\varepsilon^{3i+j+1}x^iy^j\mathrm{d}y$ is divisible by ε^4 except for $(i,j)=(0,1)$ and $(i,j)=(0,2)$. If $(i,j)\notin\{(0,1),(0,2),(0,3),(1,0)\}$, then the 1-form $\frac{1}{\varepsilon^4}\varphi^*(x^iy^j\mathrm{d}y)$ tends to 0 as ε goes to 0.

Therefore

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon^4}\varphi^*\omega=\alpha(x\mathrm{d}y-y\mathrm{d}x)+\gamma\,y^3\mathrm{d}y.$$

The foliation defined by $\alpha(x\mathrm{d}y-y\mathrm{d}x)+\gamma\,y^3\mathrm{d}y$ is conjugated to $\mathcal{F}_1$ because, as a straightforward computation shows, it is a convex foliation whose singular locus is formed of two points. As a result $\mathcal{F}$ degenerates onto $\mathcal{F}_1$. □

Corollary 3.13. — *The foliations $\mathcal{H}_1,\mathcal{H}_3,\mathcal{H}_5,\mathcal{H}_7$ and $\mathcal{F}_4$ degenerate onto $\mathcal{F}_1$.*

The converse of Proposition 3.12 is false as the following example shows.

Example 3.14. — Let $\mathcal{F}$ be the degree 3 foliation on $\mathbb{P}^2_{\mathbb{C}}$ defined in the affine chart $z=1$ by

$$\omega=x\mathrm{d}y-y\mathrm{d}x+(y^2+y^3)\mathrm{d}y.$$

The singular locus of $\mathcal{F}$ consists of the two points $m = [0:0:1]$ and $m' = [1:0:0]$; moreover

$$\mu(\mathcal{F}, m) = 1, \qquad \mathrm{BB}(\mathcal{F}, m) = 4, \qquad \kappa(\mathcal{F}, m) = 2, \qquad \mu(\mathcal{F}, m') > 1.$$

The foliation $\mathcal{F}$ degenerates onto $\mathcal{F}_1$; indeed, putting $\varphi = \left(\frac{1}{\varepsilon^3}x, \frac{1}{\varepsilon}y\right)$, we have that

$$\lim_{\varepsilon \to 0} \varepsilon^4 \varphi^* \omega = x\mathrm{d}y - y\mathrm{d}x + y^3\mathrm{d}y.$$

Next, we give a necessary condition for a degree three foliation on $\mathbb{P}^2_{\mathbb{C}}$ degenerates onto $\mathcal{F}_2$:

Proposition 3.15. — *Let $\mathcal{F}$ be an element of $\mathbf{F}(3)$. If $\mathcal{F}$ degenerates onto $\mathcal{F}_2$, then $\deg \mathrm{I}^{\mathrm{tr}}_{\mathcal{F}} \geqslant 2$.*

Proof. — If $\mathcal{F}$ degenerates onto $\mathcal{F}_2$ then $\deg \mathrm{I}^{\mathrm{tr}}_{\mathcal{F}} \geqslant \deg \mathrm{I}^{\mathrm{tr}}_{\mathcal{F}_2}$. A straightforward computation shows that $\mathrm{I}^{\mathrm{tr}}_{\mathcal{F}_2} = y^2$ so that $\deg \mathrm{I}^{\mathrm{tr}}_{\mathcal{F}_2} = 2$. □

Corollary 3.16. — *The foliations $\mathcal{H}_5$ and $\mathcal{H}_9$ do not degenerate onto $\mathcal{F}_2$.*

A sufficient condition for that a degree three foliations on $\mathbb{P}^2_{\mathbb{C}}$ degenerates onto $\mathcal{F}_2$ is the following:

Proposition 3.17. — *Let $\mathcal{F}$ be an element of $\mathbf{F}(3)$ such that $\mathcal{F}_2 \notin \mathcal{O}(\mathcal{F})$. If $\mathcal{F}$ possesses a double inflection point, then $\mathcal{F}$ degenerates onto $\mathcal{F}_2$.*

Proof. — Assume that $\mathcal{F}$ possesses such a point m. We take an affine coordinate system (x, y) such that $m = (0, 0)$ is a double inflection point of $\mathcal{F}$ and $x = 0$ is the tangent line to the leaf of $\mathcal{F}$ passing through m. Let ω be a 1-form defining $\mathcal{F}$ in these coordinates. Since $\mathrm{T}_m\mathcal{F} = (x = 0)$, ω has the following type

$$\begin{aligned}
&(\alpha + *x + *y + *x^2 + *xy + *y^2 + *x^3 + *x^2y + *xy^2 + *y^3)\mathrm{d}x \\
&+ (*x + ry + *x^2 + *xy + sy^2 + *x^3 + *x^2y + *xy^2 + \beta y^3)\mathrm{d}y \\
&+ (*x^3 + *x^2y + *xy^2 + *y^3)(x\mathrm{d}y - y\mathrm{d}x), \\
&\text{with } *, r, s, \beta, \in \mathbb{C}, \alpha \in \mathbb{C}^*.
\end{aligned}$$

Along the line $x = 0$, the 2-form $\omega \wedge \mathrm{d}x$ writes as $(ry + sy^2 + \beta\, y^3)\mathrm{d}y \wedge \mathrm{d}x$. The fact that $(0, 0)$ is a double inflection point is equivalent to $r = s = 0$

and $\beta \neq 0$. Thus ω writes as

$$\begin{aligned}&(\alpha + *x + *y + *x^2 + *xy + *y^2 + *x^3 + *x^2y + *xy^2 + *y^3)\mathrm{d}x\\&+ (*x + *x^2 + *xy + *x^3 + *x^2y + *xy^2 + \beta y^3)\mathrm{d}y\\&+ (*x^3 + *x^2y + *xy^2 + *y^3)(x\mathrm{d}y - y\mathrm{d}x),\\&\text{where } * \in \mathbb{C}, \alpha, \beta \in \mathbb{C}^*.\end{aligned}$$

We consider the following family of automorphisms $\varphi_\varepsilon = \varphi = (\varepsilon^4 x, \varepsilon y)$. Fix $(i, j) \in \mathbb{Z}_+^2$ and notice that

(1) $\varphi^*(x^iy^j\mathrm{d}x) = \varepsilon^{4i+j+4}x^iy^j\mathrm{d}x$ is divisible by ε^4 and $\frac{1}{\varepsilon^4}\varphi^*(x^iy^j\mathrm{d}x)$ tends to 0 as ε tends to 0 except for $i = j = 0$;

(2) $\varphi^*(x^iy^j\mathrm{d}y) = \varepsilon^{4i+j+1}x^iy^j\mathrm{d}y$ is divisible by ε^4 except for $(i, j) \in \{(0, 0), (0, 1), (0, 2)\}$. If $(i, j) \notin \{(0, 0), (0, 1), (0, 2), (0, 3)\}$, then the 1-form $\frac{1}{\varepsilon^4}\varphi^*(x^iy^j\mathrm{d}y)$ tends to 0 as ε goes to 0.

We obtain that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^4}\varphi^*\omega = \alpha\mathrm{d}x + \beta y^3\mathrm{d}y.$$

Clearly $\alpha\mathrm{d}x + \beta y^3\mathrm{d}y$ defines a foliation which is conjugated to $\mathcal{F}_2$; as a result $\mathcal{F}$ degenerates onto $\mathcal{F}_2$. □

Corollary 3.18. — *The foliations $\mathcal{H}_2, \mathcal{H}_4, \mathcal{H}_6, \mathcal{H}_8$ and $\mathcal{F}_4$ degenerate onto $\mathcal{F}_2$.*

Example 3.19 (Jouanolou). — Consider the degree three foliation $\mathcal{F}_J$ on $\mathbb{P}^2_{\mathbb{C}}$ defined in the affine chart $z = 1$ by

$$\omega_J = (x^3y - 1)\mathrm{d}x + (y^3 - x^4)\mathrm{d}y;$$

this example is due to Jouanolou ([17]). Historically it is the first explicit example of foliation without invariant algebraic curves ([17]); it is also a foliation without non-trivial minimal set ([7]). The point $m = (0, 0)$ is a double inflection point of $\mathcal{F}_J$ because $\mathrm{T}_m\mathcal{F}_J = (x = 0)$ and $\omega_J \wedge \mathrm{d}x\big|_{x=0} = y^3\mathrm{d}y \wedge \mathrm{d}x$; thus $\mathcal{F}_J$ degenerates onto $\mathcal{F}_2$.

The converse of Proposition 3.17 is false as the following example shows.

Example 3.20. — Let $\mathcal{F}$ be the degree 3 foliation on $\mathbb{P}^2_{\mathbb{C}}$ defined in the affine chart $z = 1$ by

$$\omega = \mathrm{d}x + (y^2 + y^3)\mathrm{d}y.$$

A straightforward computation shows that $\mathcal{F}$ has no double inflection point. This foliation degenerates onto $\mathcal{F}_3$ in the following way. Putting

$\varphi = \left(\frac{1}{\varepsilon^4}x, \frac{1}{\varepsilon}y\right)$, we obtain that

$$\lim_{\varepsilon \to 0} \varepsilon^4 \varphi^* \omega = \mathrm{d}x + y^3 \mathrm{d}y.$$

Theorem D follows directly from Theorem A, Propositions 3.3, 3.7 and Corollaries C, 3.9, 3.11, 3.13, 3.16, 3.18.

PROBLEM 3.21. — *Give a criterion for deciding whether or not a degree three foliation on $\mathbb{P}^2_{\mathbb{C}}$ degenerates onto $\mathcal{F}_1$ or $\mathcal{F}_2$.*

Thanks to Corollary 3.9, an affirmative answer to this problem would allows us to decide whether or not an orbit of dimension 7 in $\mathbf{F}(3)$ is closed.

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