Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

Nguyen Cong Phuc & Igor E. Verbitsky

BMO solutions to quasilinear equations of p-Laplace type

Tome 72, n° 5 (2022), p. 1911-1939.

https://doi.org/10.5802/aif.3485

Les *Annales de l'Institut Fourier* sont membres du Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org e-ISSN : 1777-5310

Ann. Inst. Fourier, Grenoble
72, 5 (2022) 1911-1939

BMO SOLUTIONS TO QUASILINEAR EQUATIONS OF p-LAPLACE TYPE

by Nguyen Cong PHUC & Igor E. VERBITSKY (*)

Abstract. — We give necessary and sufficient conditions for the existence of a BMO solution to the quasilinear equation $-\Delta_p u = \mu$ in $\mathbb{R}^n$, $u \geqslant 0$, where μ is a locally finite Radon measure, and $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian ($p > 1$).

We also characterize BMO solutions to equations $-\Delta_p u = \sigma u^q + \mu$ in $\mathbb{R}^n$, $u \geqslant 0$, with $q > 0$, where both μ and σ are locally finite Radon measures. Our main results hold for a class of more general quasilinear operators $\mathrm{div}(\mathcal{A}(x, \nabla\cdot))$ in place of Δ_p.

Résumé. — Nous donnons les conditions nécessaires et suffisantes pour l'existence d'une solution BMO de l'équation quasi-linéaire $-\Delta_p u = \mu$ dans $\mathbb{R}^n$, $u \geqslant 0$, où μ est une mesure de Radon localement finie, et $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ est le p-Laplacien ($p > 1$).

Nous caractérisons également les solutions BMO de l'équation $-\Delta_p u = \sigma u^q + \mu$ dans $\mathbb{R}^n$, $u \geqslant 0$, avec $q > 0$, où μ et σ sont des mesures de Radon localement finies. Nos principaux résultats sont valables pour la classe plus générale des opérateurs quasi-linéaires plus généraux $\mathrm{div}(\mathcal{A}(x, \nabla\cdot))$ à la place de Δ_p.

1. Introduction

Let $M^+(\mathbb{R}^n)$ denote the class of all (locally finite) positive Radon measures in $\mathbb{R}^n$, $n \geqslant 2$. Let $\mu \in M^+(\mathbb{R}^n)$. In this paper, the following quasilinear equation with measure data is considered:

$$\begin{cases} -\Delta_p u = \mu, \quad u \geqslant 0 \quad \text{in } \mathbb{R}^n, \\ \liminf\limits_{|x|\to\infty} u = 0. \end{cases} \tag{1.1}$$

Keywords: BMO spaces, Wolff potentials, p-Laplacian.
2010 *Mathematics Subject Classification:* 35J92, 42B37, 31B15, 42B35.
(*) N. C. Phuc is supported in part by Simons Foundation, award number 426071.

Here $\Delta_p u$ is the p-Laplacian of u defined by $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$. All solutions to (1.1) are understood to be p-superharmonic solutions, or equivalently local renormalized solutions (see [10, 12]). Since nontrivial nonnegative p-superharmonic functions on $\mathbb{R}^n$ do not exist for $p \geqslant n$, it will be our standing assumption that solutions to (1.1) are considered for $1 < p < n$.

It is known that a necessary and sufficient condition for (1.1) to admit a solution is the finiteness condition (see, e.g., [24])

$$\int_1^\infty \left(\frac{\mu(B(0,\rho))}{\rho^{n-p}}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} < +\infty. \tag{1.2}$$

This is equivalent to the condition $\mathbf{W}_p\mu(x) < +\infty$ for some $x \in \mathbb{R}^n$ (or equivalently quasi-everywhere in $\mathbb{R}^n$ with respect to the p-capacity), where

$$\mathbf{W}_p\mu(x) := \int_0^\infty \left(\frac{\mu(B(x,\rho))}{\rho^{n-p}}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho}$$

is the Havin–Maz'ya–Wolff potential of μ, often called the Wolff potential (see [9, 16, 17]).

By the important result of Kilpeläinen and Malý [14], any solution u to (1.1) satisfies the pointwise estimates

$$C_1\mathbf{W}_p\mu(x) \leqslant u(x) \leqslant C_2\mathbf{W}_p\mu(x), \quad x \in \mathbb{R}^n, \tag{1.3}$$

where C_1, C_2 are positive constants that depend only on p and n.

The p-capacity $\mathrm{cap}_p(\cdot)$ is a natural capacity associated with the p-Laplace operator defined for each compact set K of $\mathbb{R}^n$ by

$$\mathrm{cap}_p(K) = \inf\left\{\int_{\mathbb{R}^n} |\nabla h|^p\, dx : \ h \in C_0^\infty(\mathbb{R}^n),\ h \geqslant 1 \text{ on } K\right\}.$$

In [28, Lemma 3.1], it is shown that, for $1 < p < n$, if (1.1) has a solution $u \in \mathrm{BMO}(\mathbb{R}^n)$ then (1.2) holds along with the following bound for μ:

$$\mu(B(x,R)) \leqslant CR^{n-p}, \quad \forall\, x \in \mathbb{R}^n,\, R > 0. \tag{1.4}$$

Here $\mathrm{BMO}(\mathbb{R}^n)$ is the space of functions u of bounded mean oscillation in $\mathbb{R}^n$, i.e., $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\frac{1}{|B|}\int_B |u - \bar{u}_B|\, dx \leqslant C,$$

for all balls B in $\mathbb{R}^n$, where $\bar{u}_B = \frac{1}{|B|}\int_B u\, dx$.

It is also known that, conversely, if μ satisfies (1.2) and (1.4), then (1.1) has a solution $u \in \mathrm{BMO}(\mathbb{R}^n)$ provided $2 - \frac{1}{n} < p < n$ (see [28]). A local version of this result was established in [18] for $p > 2$. The linear case $p = 2$ is due to D. Adams [1].

One of the main goals of this paper is to extend this existence criterion to the full range $1 < p < n$.

Theorem 1.1. — *Let* $\mu \in M^+(\mathbb{R}^n)$ *and* $1 < p < n$. *Then equation* (1.1) *has a solution* $u \in \mathrm{BMO}(\mathbb{R}^n)$ *if and only if* μ *satisfies conditions* (1.2) *and* (1.4).

Moreover, any solution u *to* (1.1) *lies in* $\mathrm{BMO}(\mathbb{R}^n)$ *if and only if* μ *satisfies* (1.4).

Remark 1.2. — If μ satisfies (1.4), then actually any solution u to (1.6) satisfies the Morrey condition

$$\int_{B(x,R)} |\nabla u|^s \, dy \leqslant C\, R^{n-s}, \quad \forall\, x \in \mathbb{R}^n,\ R > 0,$$

provided $0 < s < p$, which yields $u \in \mathrm{BMO}(\mathbb{R}^n)$ for any $s \geqslant 1$ by Poincaré's inequality.

A sharper local estimate for the end-point weak L^p norm

$$\|\nabla u\|_{L^{p,\infty}(B(x,R))}$$

in place of Morrey's norm is obtained in Theorem 2.6 below.

We observe that Theorem 1.1 can be regarded as the end-point case $\alpha = 0$ of the corresponding criterion for $u \in C^\alpha(\mathbb{R}^n)$ $(0 \leqslant \alpha < 1)$, where $C^\alpha(\mathbb{R}^n)$ is the Campanato space of functions $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\frac{1}{|B|} \int_B |u - \bar{u}_B| \, dx \leqslant C\, |B|^{\frac{\alpha}{n}},$$

for all balls B in $\mathbb{R}^n$. Then $u \in C^\alpha(\mathbb{R}^n)$ if and only if $u \in \mathrm{BMO}(\mathbb{R}^n)$ when $\alpha = 0$, and u is α-Hölder continuous when $\alpha \in (0,1]$ (see [7, Section 2.3]).

A local version of the following result was obtained in [14, Theorem 4.18] and [15, Theorem 1.14]:

Let $0 < \alpha < 1$. *A solution* u *to* (1.1) *is in* $C^\alpha(\mathbb{R}^n)$ *if and only if* μ *satisfies the condition*

$$\mu(B(x,R)) \leqslant C R^{n-p+\alpha(p-1)}, \quad \forall\, x \in \mathbb{R}^n,\ R > 0. \tag{1.5}$$

Notice that condition (1.5) combined with (1.2) is necessary and sufficient for the existence of a solution $u \in C^\alpha(\mathbb{R}^n)$ to (1.1). The proof is similar to the proof of Theorem 1.1 given below for $\alpha = 0$.

Using Theorem 1.1, we obtain criteria for the existence of BMO solutions to the equation

$$\begin{cases} -\Delta_p u = \sigma u^q + \mu, \quad u \geqslant 0 \quad \text{in } \mathbb{R}^n, \\ \liminf\limits_{|x| \to \infty} u = 0, \end{cases} \tag{1.6}$$

where $q > 0$ and $\mu, \sigma \in M^+(\mathbb{R}^n)$ $(\sigma \neq 0)$. Here we assume without loss of generality that $\sigma \neq 0$, since the case $\sigma = 0$ is covered by Theorem 1.1.

The corresponding results are more complicated due to the possible interaction between the datum μ and the source term σu^q on the right-hand side, as well as competition with $-\Delta_p u$ on the left-hand side.

We first consider the super-natural growth case $q > p - 1$.

Theorem 1.3. — *Let $\mu, \sigma \in M^+(\mathbb{R}^n)$ $(\sigma \neq 0)$. Let $q > p - 1$ and $1 < p < n$. Then equation* (1.6) *has a solution $u \in \mathrm{BMO}(\mathbb{R}^n)$ if μ satisfies* (1.4), *and*

(a) $$\mathbf{W}_p\big[\,(\mathbf{W}_p\mu)^q\,d\sigma\big](x) \leqslant c\,\mathbf{W}_p\mu(x), \quad \forall\, x \in \mathbb{R}^n,$$

(b) $$\sigma(B(x,R)) \times \left[\int_R^\infty \left(\frac{\mu(B(x,r))}{r^{n-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}\right]^q \leqslant C\,R^{n-p},\ \forall\, x \in \mathbb{R}^n,\ R > 0,$$

where c, C are positive constants, and $c = c(p,q,n)$ is sufficiently small.

Conversely, if there exists a solution $u \in \mathrm{BMO}(\mathbb{R}^n)$ to (1.6), *then* (1.4), *and conditions (a), (b) hold for some positive constants c, C.*

In the sub-natural growth case $0 < q < p - 1$, we denote by κ the least constant in the weighted norm inequality for Wolff potentials (see [5]),

$$\|\mathbf{W}_p\nu\|_{L^q(d\sigma)} \leqslant \kappa\,\|\nu\|^{\frac{1}{p-1}}, \quad \forall\, \nu \in M^+\left(\mathbb{R}^n\right), \tag{1.7}$$

where $\|\nu\| = \nu(\mathbb{R}^n)$ stands for the total variation of ν.

Using (1.3), it is easy to see that κ is equivalent to the least constant $\varkappa$ in the inequality

$$\|\phi\|_{L^q(d\sigma)} \leqslant \varkappa\,\|\Delta_p\phi\|^{\frac{1}{p-1}}, \tag{1.8}$$

for all positive test functions ϕ that are p-superharmonic in $\mathbb{R}^n$ such that $\liminf_{|x|\to\infty} \phi(x) = 0$.

For $\sigma \in M^+(\mathbb{R}^n)$ and a ball B in $\mathbb{R}^n$, let $\sigma_B = \sigma|_B$ be the restriction of σ to B. We denote by $\kappa(B)$ the least constant in the *localized* version of (1.7), namely,

$$\|\mathbf{W}_p\nu\|_{L^q(d\sigma_B)} \leqslant \boldsymbol{\kappa}(B)\,\|\nu\|^{\frac{1}{p-1}}, \quad \forall\, \nu \in M^+\left(\mathbb{R}^n\right). \tag{1.9}$$

As mentioned above, equivalent constants $\varkappa(B)$, associated with σ_B in place of σ in (1.8), can be used in place of $\kappa(B)$. Various lower and upper estimates of $\kappa(B)$ can be found in [5].

Theorem 1.4. — *Let* $\mu, \sigma \in M^+(\mathbb{R}^n)$ $(\sigma \neq 0)$*. Let* $0 < q < p-1$ *and* $1 < p < n$*. Then equation* (1.6) *has a nontrivial solution* $u \in \mathrm{BMO}(\mathbb{R}^n)$ *if and only if* μ *satisfies condition* (1.4)*, and there exists a constant* C *such that*

$$\text{(a)} \qquad \kappa(B(x,R))^{\frac{q(p-1)}{p-1-q}} \leqslant C\, R^{n-p},$$

$$\text{(b)} \qquad \sigma(B(x,R)) \left[\int_R^\infty \left(\frac{\mu(B(x,r))}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right]^q \leqslant C\, R^{n-p},$$

$$\text{(c)} \qquad \sigma(B(x,R)) \left[\int_R^\infty \left(\frac{\sigma(B(x,r))}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right]^{\frac{q(p-1)}{p-1-q}} \leqslant C\, R^{n-p},$$

$$\text{(d)} \qquad \sigma(B(x,R)) \left[\int_R^\infty \left(\frac{\kappa(B(x,r))^{\frac{q(p-1)}{p-1-q}}}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right]^q \leqslant C\, R^{n-p},$$

for all $x \in \mathbb{R}^n$ *and* $R > 0$*.*

Moreover, under the above conditions on μ *and* σ *any solution* u *to* (1.6) *lies in* $\mathrm{BMO}(\mathbb{R}^n)$*, and satisfies the Morrey estimates of Remark 1.2.*

We remark that Theorem 1.4 (except for the last statement) was proved in [28] for $2 - \frac{1}{n} < p < n$ in the special case $\mu = 0$. Conditions on μ and σ in Theorem 1.4 can be simplified substantially under the assumption

$$\sigma(K) \leqslant C\, \mathrm{cap}_p(K), \quad \forall \text{ compact sets } K \subset \mathbb{R}^n. \tag{1.10}$$

Corollary 1.5. — *Let* $\mu, \sigma \in M^+(\mathbb{R}^n)$*, where* $\sigma \neq 0$ *satisfies condition* (1.10)*. Let* $0 < q < p-1$ *and* $1 < p < n$*. Then equation* (1.6) *has a nontrivial solution* $u \in \mathrm{BMO}(\mathbb{R}^n)$ *if and only if* μ *satisfies* (1.4)*, and, for all* $x \in \mathbb{R}^n$*,* $R > 0$*,*

$$\text{(a)} \qquad \sigma(B(x,R)) \left[\int_R^\infty \left(\frac{\mu(B(x,r))}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right]^q \leqslant C\, R^{n-p},$$

$$\text{(b)} \qquad \sigma(B(x,R)) \left[\int_R^\infty \left(\frac{\sigma(B(x,r))}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{\rho} \right]^{\frac{q(p-1)}{p-1-q}} \leqslant C\, R^{n-p}.$$

Moreover, under the above conditions on μ *and* σ *any solution* u *to* (1.6) *lies in* $\mathrm{BMO}(\mathbb{R}^n)$*, and satisfies the Morrey estimates of Remark 1.2.*

Similar criteria for the existence of BMO solutions in the natural growth case $q = p - 1$ are obtained in Section 5 below (see Theorem 5.1) under some additional assumptions on σ stronger than (1.10), which is necessary in that case.

Remark 1.6. — Theorems 1.1, 1.3, 1.4 and Corollary 1.5 can be extended to equations with more general quasilinear elliptic operators div $(\mathcal{A}(x,\nabla\cdot))$ in place of Δ_p, as long as the nonlinearity $\mathcal{A}(x,\xi)$ satisfies conditions (2.28) and (2.30) below with $1<p<n$. This remark also applies to Theorem 5.1.

2. Proof of Theorem 1.1

As mentioned above, the "only if" part of Theorem 1.1 is proved in [28, Lemma 3.1].

To prove the "if" part, we construct a solution $u \in \mathrm{BMO}(\mathbb{R}^n)$ to (1.1) under conditions (1.2) and (1.4) on μ.Our construction below is based on an a priori estimate of [4]. Alternatively, it is possible to use the gradient estimates of [20, 21] for the construction, but that will not be implemented in this paper.

We first observe that under (1.4),

$$\|\mathbf{I}_1\mu\|_{L^{\frac{p}{p-1},\infty}(B(x,R))} \leqslant CMR^{\frac{(n-p)(p-1)}{p}}, \quad \forall\, x\in\mathbb{R}^n,\ R>0, \tag{2.1}$$

where C depends only on p and n, and

$$M = \sup_{x\in\mathbb{R}^n, R>0} \frac{\mu(B(x,R))}{R^{n-p}}.$$

A proof of (2.1) can be found in [1]. Here $\mathbf{I}_\alpha$, $\alpha\in(0,n)$, is the Riesz potential of order α, defined for a measure $\mu\in M^+(\mathbb{R}^n)$ by

$$\mathbf{I}_\alpha\mu(x) := \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}}d\mu(y), \qquad x\in\mathbb{R}^n. \tag{2.2}$$

Also, the space $L^{q,\infty}(B(x,R))$, $q>0$, is the weak L^q space over the ball $B(x,R)$ with

$$\|f\|_{L^{q,\infty}(B(x,R))} := \sup_{\lambda>0} \lambda\Big|\,\{y\in B(x,R)\,:\,|f(y)|>\lambda\}\,\Big|^{\frac{1}{q}}.$$

Let μ_k $(k=1,2,\dots)$ be a standard regularization of μ by the convolution

$$\mu_k(x) = \rho_k * \mu(x), \quad \rho_k(\cdot) = k^n\rho(k\cdot),$$

where $0\leqslant\rho\in C_0^\infty(B(0,1))$, $\int_{\mathbb{R}^n}\rho dx = 1$, and ρ is radial. Then it follows from Fubini's Theorem and (1.4) that

$$\mu_k(B(x,R)) \leqslant CR^{n-p}, \quad \forall\, x\in\mathbb{R}^n,\ R>0,$$

where C is independent of k. Thus, we also have

$$\|\mathbf{I}_1\mu_k\|_{L^{\frac{p}{p-1},\infty}(B(x,R))} \leqslant CR^{\frac{(n-p)(p-1)}{p}}, \quad \forall\, x\in\mathbb{R}^n,\ R>0, \tag{2.3}$$

for a constant C independent of k.

Next, for each positive integer N we let $\mu_{B(0,N)}$ be the restriction of the measure μ to the open ball $B(0,N)$. Consider now the unique p-superharmonic solution $u_{N,k} \in W_0^{1,p}(B(0,N)) \subset W_0^{1,p}(\mathbb{R}^n)$ to the equation

$$-\Delta_p u_{N,k} = \rho_k * \mu_{B(0,N)} \quad \text{in } B(0,N). \tag{2.4}$$

Note that $\rho_k * \mu_{B(0,N)} \leqslant \mu_k$ and we can write $\rho_k * \mu_{B(0,N)} = -\mathrm{div}\, \nabla v_{N,k}$ in the sense of distributions in $B(0,N)$, where

$$v_{N,k}(x) = \int_{B(0,N)} G_N(x,y) \left[\rho_k * \mu_{B(0,N)}(y)\right] dy, \qquad x \in B(0,N),$$

with $G_N(x,y)$ being the Green function associated with $-\Delta$ in $B(0,N)$. Moreover, we have

$$|\nabla v_{N,k}| \leqslant C\, \mathbf{I}_1 \left(\rho_k * \mu_{B(0,N)}\right) \leqslant C\, \mathbf{I}_1 \mu_k. \tag{2.5}$$

Now using (2.3), (2.5) and applying [4, Theorem 1.2] we find that

$$\|\nabla u_{N,k}\|_{L^{p,\infty}(B(x,R))} \leqslant C R^{\frac{n-p}{p}} \tag{2.6}$$

for all $x \in B(0,N)$ and $0 < R \leqslant 2N$. The constant C is independent of x, R, N, and k. Since $|\nabla u_{N,k}| = 0$ outside $B(0,N)$, it is obvious that (2.6) also holds for all $x \in \mathbb{R}^n$ and $R > 0$.

When restricted to the ball $B(0,N+1)$, $\mu_{B(0,N)}$ is a nonnegative finite measure and thus we can write

$$\mu_{B(0,N)} = f - \mathrm{div} F + \mu_s$$

as distributions in $B(0,N+1)$ (see, e.g., [6]). Here $f \in L^1(B(0,N+1))$, $F \in L^{\frac{p}{p-1}}(B(0,N+1), \mathbb{R}^n)$, and μ_s is a nonnegative measure concentrated on a set of zero p-capacity in $B(0,N)$.

For any $\varphi \in C_0^\infty(B(0,N))$ and $k \geqslant 1$ we have $\rho_k * \varphi \in C_0^\infty(B(0,N+1))$. Thus it follows that

$$\begin{aligned}
&\int_{B(0,N)} \rho_k * \mu_{B(0,N)} \varphi dx \\
&\quad = \int_{B(0,N+1)} \rho_k * \varphi d\mu_{B(0,N)} \\
&\quad = \int_{B(0,N+1)} f \rho_k * \varphi dx + \int_{B(0,N+1)} F \cdot \nabla \left(\rho_k * \varphi\right) dx \\
&\qquad\qquad + \int_{B(0,N+1)} \rho_k * \varphi d\mu_s \\
&\quad = \int_{B(0,N)} \rho_k * f \varphi dx + \int_{B(0,N)} \rho_k * F \cdot \nabla \varphi dx + \int_{B(0,N)} \rho_k * \mu_s \varphi dx.
\end{aligned}$$

That is,

$$\rho_k * \mu_{B(0,N)} = \rho_k * f - \operatorname{div}(\rho_k * F) + \rho_k * \mu_s$$

pointwise everywhere and as distributions in $B(0,N)$.

As $\mu_s(\mathbb{R}^n \backslash B(0,N)) = 0$, by the Lebesgue Dominated Convergence Theorem, we see that $\rho_k * \mu_s \to \mu_s$ in the narrow topology of measures in $B(0,N)$ (see [6, Definition 2.2]). Moreover, this and the above equality yield

$$\int_{B(0,N)} |\operatorname{div}(\rho_k * F)|\, dx \leqslant M,$$

where M is independent of k.

At this point, in view of (2.4), we apply [6, Theorem 3.2] (see also [23, Remark 6.6]) to find a subsequence $\{u_{N,k_j}\}$ of $\{u_{N,k}\}$ and a p-superharmonic function u_N in $B(0,N)$ such that $u_{N,k_j} \to u_N$ a.e., $\nabla u_{N,k_j} \to \nabla u_N$ a.e. as $j \to \infty$, and u_N solves the equation

$$\begin{cases} -\Delta_p u_N &= \mu_{B(0,N)} \quad \text{in } B(0,N), \\ \quad\;\; u_N &= 0 \quad \text{on } \partial B(0,N), \end{cases}$$

in the renormalized sense (see [6] for the notion of renormalized solutions).

Moreover, by (2.6) we have $u_N \in W_0^{1,s}(B(0,N)) \subset W_0^{1,s}(\mathbb{R}^n)$ for all $1 \leqslant s < p$ and Fatou's Lemma yields that

$$\|\nabla u_N\|_{L^{p,\infty}(B(x,R))} \leqslant C R^{\frac{n-p}{p}}, \quad \forall\ x \in \mathbb{R}^n,\ R > 0. \tag{2.7}$$

By [23, Theorem 2.1] we have

$$u_N(x) \leqslant C\,\mathbf{W}_p\mu(x), \qquad \forall\, x \in \mathbb{R}^n. \tag{2.8}$$

By [13, Theorem 1.17] we can find a subsequence $\{u_{N_j}\}$ of $\{u_N\}$ and a p-superharmonic function u in $\mathbb{R}^n$ such that $u_{N_j} \to u$ a.e. and $\nabla u_{N_j} \to \nabla u$ a.e. as $j \to \infty$. Note that by condition (1.2) and (2.8), u must be finite a.e. (or q.e.) and

$$u(x) \leqslant C\,\mathbf{W}_p\mu(x), \qquad \forall\, x \in \mathbb{R}^n.$$

The weak continuity result of [26] yields that u is a p-superharmonic solution of (1.1). That $\liminf_{|x|\to\infty} u = 0$ follows from the fact that $\inf_{\mathbb{R}^n} u = 0$ and the latter is a direct consequence of the pointwise bound (see [13, 14]):

$$u(x) \leqslant C\,\mathbf{W}_p\mu(x) \leqslant C\left[u(x) - \inf_{\mathbb{R}^n} u\right], \qquad \forall\, x \in \mathbb{R}^n.$$

Applying (2.7) with ∇u_{N_j} in place of ∇u_N, and using Fatou's Lemma as $N_j \to \infty$, we obtain

$$\|\nabla u\|_{L^{p,\infty}(B(x,R))} \leqslant C R^{\frac{n-p}{p}}, \quad \forall\ x \in \mathbb{R}^n,\ R > 0. \tag{2.9}$$

It is worth mentioning here that for $p \geqslant 2$ estimate (2.9) can also be inferred from the work [19]. Thus, for any ball $B = B(x,R)$, by Poincaré's inequality and Hölder's inequality we find

$$\begin{aligned}\frac{1}{|B|}\int_B \left|u(x) - \frac{1}{|B|}\int_B u(y)dy\right| dx &\leqslant CR\frac{1}{|B|}\int_B |\nabla u|\, dx \\ &\leqslant CR|B|^{-\frac{1}{p}} \|\nabla u\|_{L^{p,\infty}(B)} \\ &\leqslant C.\end{aligned}$$

This shows that $u \in \mathrm{BMO}(\mathbb{R}^n)$ as desired.

It remains to show that any solution u to (1.1) lies in $\mathrm{BMO}(\mathbb{R}^n)$ provided μ satisfies condition (1.4). We will actually prove the stronger estimate (2.9) (see Remark 1.2).

Lemma 2.1. — *Let u be a nonnegative p-superharmonic solution of $-\Delta_p u = \mu$ with μ satisfying condition* (1.4)*. Then $|\nabla u| \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ provided $0 < q < p$. Moreover, for any $0 < q < p$, $0 < \epsilon < p-1$ and any ball $B(x,R)$ we have*

$$\left(R^{q-n}\int_{B(x,R)} |\nabla u|^q\, dy\right)^{\frac{1}{q}} \leqslant C\left(\left[\inf_{B(x,2R)} u\right]^{\frac{p-1-\epsilon}{p}} + \inf_{B(x,2R)} u\right), \tag{2.10}$$

where the constant C depends on p, q, ϵ, n and the constant in condition (1.4)*. In particular,*

$$\lim_{R\to+\infty} R^{q-n}\int_{B(0,R)} |\nabla u|^q\, dy = 0. \tag{2.11}$$

Proof. — Let $u_k = \min\{u,k\}$, $k = 1,2,\ldots$ Then $u_k \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ is a supersolution in $\mathbb{R}^n$ and hence the weak Harnack inequality [25] implies that

$$\left(\frac{1}{|B(x,R)|}\int_{B(x,R)} u_k^s dy\right)^{\frac{1}{s}} \leqslant C \inf_{B(x,R)} u_k \leqslant C \inf_{B(x,R)} u$$

for $0 < s < \frac{n(p-1)}{n-p}$. Thus letting $k \to \infty$ we obtain

$$\left(\frac{1}{|B(x,R)|}\int_{B(x,R)} u^s dy\right)^{\frac{1}{s}} \leqslant C \inf_{B(x,R)} u. \tag{2.12}$$

To continue, we recall the following result for p-supersolutions from [10, Lemma 3.57]:

For any nonnegative p-supersolution v in an open set $\Omega\subset\mathbb{R}^n$, any $\epsilon>0$, and any function $\varphi\in C_0^\infty(\Omega)$, it holds that

$$\int_\Omega |\nabla v|^p\, v^{-1-\epsilon}\, |\varphi|^p\, dx \leqslant (p/\epsilon)^p \int_\Omega v^{\,p-1-\epsilon}\, |\nabla\varphi|^p\, dx. \tag{2.13}$$

Now let $0<q<p$ and fix an ϵ such that $0<\epsilon<p-1$. Applying Hölder's inequality, and then using (2.13) with $v=u_k$ and an appropriate cut-off function φ supported in $B(x,2R)$ such that $\varphi=1$ on $B(x,R)$ and $|\nabla\varphi|\leqslant CR^{-1}$, we estimate

$$\begin{aligned}\int_{B(x,R)} |\nabla u_k|^q\, dy &= \int_{B(x,R)} |\nabla u_k|^q\, u_k^{-(1+\epsilon)q/p} u_k^{(1+\epsilon)q/p} dy\\ &\leqslant \left(\int_{B(x,R)} |\nabla u_k|^p\, u_k^{-1-\epsilon} dy\right)^{q/p} \left(\int_{B(x,R)} u_k^{(1+\epsilon)q/(p-q)} dy\right)^{(p-q)/p}\\ &\leqslant CR^{-q}\left(\int_{B(x,2R)} u^{p-1-\epsilon} dy\right)^{q/p} \left(\int_{B(x,R)} u^{(1+\epsilon)q/(p-q)} dy\right)^{(p-q)/p}.\end{aligned}$$

Thus it follows from (2.12) that

$$\begin{aligned}\int_{B(x,R)} |\nabla u_k|^q\, dy \leqslant CR^{-q}&\left(R^n\left[\inf_{B(x,2R)} u\right]^{p-1-\epsilon}\right)^{q/p}\\ &\times\left(\int_{B(x,R)} u^{(1+\epsilon)q/(p-q)} dy\right)^{(p-q)/p}.\end{aligned} \tag{2.14}$$

On the other hand, by [14, Theorem 1.6] we have

$$\begin{aligned}u(y) &\leqslant C\inf_{B(y,3R)} u + C\int_0^{6R}\left(\frac{\mu(B(y,t))}{t^{n-p}}\right)^{\frac{1}{p-1}}\frac{dt}{t}\\ &\leqslant C\inf_{B(x,2R)} u + C\int_0^{6R}\left(\frac{\mu(B(y,t))}{t^{n-p}}\right)^{\frac{1}{p-1}}\frac{dt}{t},\end{aligned}$$

provided $y\in B(x,R)$. Thus,

$$\begin{aligned}\int_{B(x,R)} u^{(1+\epsilon)q/(p-q)} dy &\leqslant CR^n\left[\inf_{B(x,2R)} u\right]^{(1+\epsilon)q/(p-q)}\\ &+ C\int_{B(x,R)}\left[\int_0^{6R}\left(\frac{\mu(B(y,t))}{t^{n-p}}\right)^{\frac{1}{p-1}}\frac{dt}{t}\right]^{(1+\epsilon)q/(p-q)} dy.\end{aligned} \tag{2.15}$$

Note that for $y \in B(x,R)$ by a Hedberg type inequality (see [2, Section 3.1]) we have

$$\begin{aligned}\int_0^{6R}\left(\frac{\mu\left(B(y,t)\right)}{t^{n-p}}\right)^{\frac{1}{p-1}}\frac{dt}{t} &= \int_0^{6R}\left(\frac{\mu_{B(x,7R)}(B(y,t))}{t^{n-p}}\right)^{\frac{1}{p-1}}\frac{dt}{t}\\ &\leqslant C\mu\left(B(x,7R)\right)^{\frac{p-\alpha}{(n-\alpha)(p-1)}}\ \mathbf{M}_\alpha\left(\mu_{B(x,7R)}\right)^{\frac{n-p}{(n-\alpha)(p-1)}}\\ &\leqslant C\mu\left(B(x,7R)\right)^{\frac{p-\alpha}{(n-\alpha)(p-1)}}\ \mathbf{I}_\alpha\left(\mu_{B(x,7R)}\right)^{\frac{n-p}{(n-\alpha)(p-1)}},\end{aligned}$$

provided $0<\alpha<p$. Here $\mathbf{M}_\alpha$, $\alpha\in(0,n)$, is the fractional maximal function of order α defined for a measure $\nu\in M^+(\mathbb{R}^n)$ by

$$\mathbf{M}_\alpha\nu(x) := \sup_{r>0}\frac{\nu(B(x,r))}{r^{n-\alpha}}, \qquad x\in\mathbb{R}^n.$$

We now set $\theta=(1+\epsilon)q/(p-q)$. Then the above bound and (1.4) gives

$$\int_{B(x,R)}\left[\int_0^{6R}\left(\frac{\mu\left(B(y,t)\right)}{t^{n-p}}\right)^{\frac{1}{p-1}}\frac{dt}{t}\right]^\theta dy \leqslant CR^{\frac{(n-p)(p-\alpha)\theta}{(n-\alpha)(p-1)}}\int_{B(x,R)}\mathbf{I}_\alpha(\mu)^{\frac{\theta(n-p)}{(n-\alpha)(p-1)}}dy.$$

We next choose $0<\alpha<p$ such that

$$\frac{p}{p-\alpha} > \frac{\theta(n-p)}{(n-\alpha)(p-1)},$$

and apply Hölder's inequality to get

$$\begin{aligned}\int_{B(x,R)}\left[\int_0^{6R}\left(\frac{\mu\left(B(y,t)\right)}{t^{n-p}}\right)^{\frac{1}{p-1}}\frac{dt}{t}\right]^\theta dy &\leqslant CR^{\frac{(n-p)(p-\alpha)\theta}{(n-\alpha)(p-1)}}R^{n-\frac{n\theta(n-p)(p-\alpha)}{p(n-\alpha)(p-1)}}\left\|\mathbf{I}_\alpha\mu\right\|_{L^{\frac{p}{p-\alpha},\infty}(B(x,R))}^{\frac{\theta(n-p)}{(p-1)(n-\alpha)}}\\ &\leqslant CR^n,\end{aligned}\tag{2.16}$$

where in the last bound we used a result of [1]:

$$\left\|\mathbf{I}_\alpha\mu\right\|_{L^{\frac{p}{p-\alpha},\infty}(B(x,R))} \leqslant CR^{(n-p)\frac{p-\alpha}{p}}.$$

At this point, we plug estimate (2.16) into (2.15) to obtain

$$\int_{B(x,R)}u^{(1+\epsilon)q/(p-q)}dy \leqslant CR^n\left(1+\left[\inf_{B(x,2R)}u\right]^{(1+\epsilon)q/(p-q)}\right).$$

In view of (2.14), this yields

$$R^{q-n}\int_{B(x,R)}|\nabla u_k|^q\,dy \leqslant C\left[\inf_{B(x,2R)} u\right]^{(p-1-\epsilon)q/p}\left(1+\left[\inf_{B(x,2R)} u\right]^{(1+\epsilon)q/p}\right).$$

Now letting $k\to\infty$ we obtain estimate (2.10).

Finally, to obtain the decay (2.11), we observe that estimate (2.10) also holds if u is replaced by $\widetilde{u} := u - \inf_{\mathbb{R}^n} u$ and that $\inf_{\mathbb{R}^n}\widetilde{u} = 0$. □

Remark 2.2. — For $0 < q < \frac{n(p-1)}{n-1}$, Lemma 2.1 holds without assuming condition (1.4) on μ (see [10, Theorem 7.46]). Moreover, the first term on the right-hand side of (2.10) can be dropped in this case.

The next lemma is a local interior version of an analogous result obtained in [4, Proposition 4.4]. We use a modification of its proof based mainly on [4, Theorem 2.3] and [4, Lemma 2.8]. Henceforth, we denote by $\mathbf{M}$ the Hardy-Littlewood maximal operator.

Lemma 2.3. — *There exist constants $A = A(n,p) > 1$ sufficiently large and $\delta_0 = \delta_0(n,p) \in (0, p-1)$ sufficiently small such that the following holds for any $T > 1$, $\lambda > 0$, and $\delta \in (0,\delta_0)$. Fix a ball $B_0 = B(z_0,R_0)$ and let u be a solution of $-\Delta_p u = \operatorname{div}(|\mathbf{f}|^{p-2}\mathbf{f})$ in $2B_0$. Assume that for some ball $B(y,\rho)$ with $\rho \leqslant R_0/8$, we have*

$$B(y,\rho)\cap B_0 \cap \left\{x\in\mathbb{R}^n : \mathbf{M}\left(\chi_{2B_0}|\nabla u|^{p-\delta}\right)^{\frac{1}{p-\delta}}(x) \leqslant \lambda\right\} \cap \left\{\mathbf{M}\left(\chi_{2B_0}|\mathbf{f}|^{p-\delta}\right)^{\frac{1}{p-\delta}} \leqslant \epsilon(T)\lambda\right\} \neq \emptyset,$$

with $\epsilon(T) = T^{\frac{-2\delta_0}{p-\delta}\max\left\{1,\frac{1}{p-1}\right\}}$. Then

$$\left|\left\{x\in\mathbb{R}^n : \mathbf{M}\left(\chi_{2B_0}|\nabla u|^{p-\delta}\right)^{\frac{1}{p-\delta}}(x) > AT\lambda\right\}\cap B(y,\rho)\right| < H\,|B(y,\rho)|,$$

where

$$H = H(T,\delta) = T^{-(p+\delta_0)} + \delta^{(p-\delta)\min\left\{1,\frac{1}{p-1}\right\}}.$$

With this, we can now apply [4, Lemma 4.1] and Lemma 2.3 above to get the following result. Its proof is similar to that of [4, Lemma 4.6].

Lemma 2.4. — *Let $A = A(n,p)$ and $\delta_0 = \delta_0(n,p)$ be as in Lemma 2.3. The following holds for any $T > 2$ and $\delta \in (0,\delta_0)$. Fix a ball $B_0 = B(z_0,R_0)$*

and let u be a solution of $-\Delta_p u = \mathrm{div}\,(|\mathbf{f}|^{p-2}\mathbf{f})$ in $2B_0$. Suppose that there exists $N > 0$ such that

$$\left|\left\{x \in \mathbb{R}^n : \mathbf{M}\left(\chi_{2B_0}|\nabla u|^{p-\delta}\right)^{\frac{1}{p-\delta}}(x) > N\right\}\right| < H\left|\frac{1}{8}B_0\right|.$$

Then for any integer $k \geqslant 0$ we have

$$\begin{aligned}\left|\left\{x \in B_0 : \mathbf{M}\left(\chi_{2B_0}|\nabla u|^{p-\delta}\right)^{\frac{1}{p-\delta}}(x) > N(AT)^{k+1}\right\}\right| \\ \leqslant c(n)\,H\left|\left\{x \in B_0 : \mathbf{M}\left(\chi_{2B_0}|\nabla u|^{p-\delta}\right)^{\frac{1}{p-\delta}}(x) > N(AT)^k\right\}\right| \\ +\ c(n)\left|\left\{x \in B_0 : \mathbf{M}\left(\chi_{2B_0}|\mathbf{f}|^{p-\delta}\right)^{\frac{1}{p-\delta}}(x) > \epsilon(T)N(AT)^k\right\}\right|.\end{aligned}$$

Here $\epsilon(T)$ and $H = H(T,\delta)$ are as defined in Lemma 2.3.

We are now in a position to obtain a local $L^{p,\infty}$ estimate for the gradient.

THEOREM 2.5. — *Let $\mu \in M^+(\mathbb{R}^n)$ and let u be a solution of $-\Delta_p u = \mu$ in $\mathbb{R}^n$. Then for any ball $B_0 = B(z_0,R_0) \subset \mathbb{R}^n$ we have*

$$\begin{aligned}(2.17)\quad \|\nabla u\|_{L^{p,\infty}(B_0)} \leqslant C\,|B_0|^{\frac{1}{p}}\left(\frac{1}{|2B_0|}\int_{2B_0}|\nabla u|^{p-\delta}\,dx\right)^{\frac{1}{p-\delta}} \\ +\ C\left\|\left[\mathbf{I}_1\left(\chi_{2B_0}\mu\right)\right]^{\frac{1}{p-1}}\right\|_{L^{p,\infty}(B_0)},\end{aligned}$$

with a constant $C = C(n,p) > 0$ and a constant $\delta = \delta(n,p) \in (0, p-1)$.

Proof. — Let $B_0 = B(z_0,R_0)$ and δ_0 be as in Lemma 2.3. For $T > 2$ and $\delta \in (0,\delta_0)$ to be determined, we claim that there exists $N > 0$ such that

$$\left|\left\{x \in \mathbb{R}^n : \mathbf{M}\left(\chi_{2B_0}|\nabla u|^{p-\delta}\right)^{\frac{1}{2-\delta}}(x) > N\right\}\right| < H\left|\frac{1}{8}B_0\right|,$$

where $H = H(T,\delta) = T^{-(p-\delta_0)} + \delta^{(p-\delta)\min\{1,\frac{1}{p-1}\}}$ (as in Lemma 2.3).

This can be done by using the weak type $(1,1)$ estimate for the maximal function and choosing $N > 0$ such that

$$(2.18)\qquad \frac{C(n)}{N^{p-\delta}}\int_{2B_0}|\nabla u|^{p-\delta}\,dx = H\left|\frac{1}{8}B_0\right|$$

provided the integral above is non-zero, which we may assume.

Let $A > 1$ and $\epsilon(T) > 0$ be as in Lemma 2.3. Set

$$L = \sup_{k\geqslant 1}(AT)^k\left|\left\{x \in B_0 : \mathbf{M}\left(\chi_{2B_0}|\nabla u|^{p-\delta}\right)^{\frac{1}{p-\delta}}(x) > N(AT)^k\right\}\right|^{\frac{1}{p}}.$$

We have

$$\left\| \mathbf{M}\left(\chi_{2B_0}\,|\nabla u/N|^{p-\delta}\right)^{\frac{1}{p-\delta}}\right\|_{L^{p,\infty}(B_0)} \leqslant AT\,\left(|B_0|^{\frac{1}{p}}+L\right). \tag{2.19}$$

We now set, for $m=1,2,\ldots,$

$$L_m = \sup_{1\leqslant k\leqslant m}(AT)^k\left|\left\{x\in B_0\,:\,\mathbf{M}\left(\chi_{2B_0}\,|\nabla u|^{p-\delta}\right)^{\frac{1}{p-\delta}}(x)>N(AT)^k\right\}\right|^{\frac{1}{p}},$$

and note that

$$\lim_{m\to\infty} L_m = L. \tag{2.20}$$

For any vector field $\mathbf{f}$ such that $\operatorname{div}(|\mathbf{f}|^{p-2}\mathbf{f})=\mu$ in $2B_0$, by Lemma 2.4 we find

$$\begin{aligned} L_m \leqslant C &\sup_{1\leqslant k\leqslant m}(AT)^k H(T,\delta)^{\frac{1}{p}}\times\\ &\times\left|\left\{x\in B_0\,:\,\mathbf{M}\left(\chi_{2B_0}\,|\nabla u|^{p-\delta}\right)^{\frac{1}{p-\delta}}(x)>N(AT)^{k-1}\right\}\right|^{\frac{1}{p}}\\ &+C\sup_{1\leqslant k\leqslant m}(AT)^k\times\\ &\times\left|\left\{x\in B_0\,:\,\left[\mathbf{M}\left(\chi_{2B_0}\,|\mathbf{f}|^{p-\delta}\right)\right]^{\frac{1}{p-\delta}}>\epsilon(T)N(AT)^{k-1}\right\}\right|^{\frac{1}{p}}\\ &\leqslant C\,(AT)H(T,\delta)^{\frac{1}{p}}\left(L_m+|B_0|^{\frac{1}{p}}\right)\\ &\quad + C_1(T,\delta)\left\|\left[\mathbf{M}\left(\chi_{2B_0}\,(|\mathbf{f}|/N)^{p-\delta}\right)\right]^{\frac{1}{p-\delta}}\right\|_{L^{p,\infty}(B_0)}. \end{aligned}$$

By the boundedness property of $\mathbf{M}$, this yields

$$L_m \leqslant C\,(AT)H(T,\delta)^{\frac{1}{p}}\left(L_m+|B_0|^{\frac{1}{p}}\right)+CC_1(T,\delta)\,\|\mathbf{f}/N\|_{L^{p,\infty}(2B_0)}.$$

We next choose T sufficiently large and δ sufficiently small so that

$$C\,(AT)H(T,\delta)^{\frac{1}{p}}\leqslant 1/2$$

and thus deduce from the above bound and (2.20) that

$$\frac{1}{2}L\leqslant\frac{1}{2}|B_0|^{\frac{1}{p}}+C\,\|\mathbf{f}/N\|_{L^{p,\infty}(2B_0)}.$$

In view of (2.19) and (2.18) this gives

$$\begin{aligned}\|\nabla u\|_{L^{p,\infty}(B_0)} &\leqslant C\,|B_0|^{\frac{1}{p}}\,N + C\,\|\mathbf{f}\|_{L^{p,\infty}(2B_0)} \\ &\leqslant C\,|B_0|^{\frac{1}{p}}\left(\frac{1}{|2B_0|}\int_{2B_0}|\nabla u|^{p-\delta}\,dx\right)^{\frac{1}{p-\delta}} + C\,\|\mathbf{f}\|_{L^{p,\infty}(2B_0)}\,.\end{aligned} \tag{2.21}$$

Finally, we write $\mu = \operatorname{div}\mathbf{g}$ in $2B_0$, where

$$\mathbf{g} = -\nabla \int_{2B_0} G(x,y)d\mu(y)$$

and $G(x,y)$ is the Green function associated with $-\Delta$ in $2B_0$. Note then that

$$|\mathbf{g}| \leqslant C\,\mathbf{I}_1\left(\chi_{2B_0}\mu\right)$$

and with $\mathbf{f} = \mathbf{g}|\mathbf{g}|^{\frac{2-p}{p-1}}$ we have $|\mathbf{f}|^{p-2}\mathbf{f} = \mathbf{g}$. Thus $\operatorname{div}(|\mathbf{f}|^{p-2}\mathbf{f}) = \mu$ in $2B_0$ and

$$|\mathbf{f}| \leqslant C\,\left[\mathbf{I}_1\left(\chi_{2B_0}\mu\right)\right]^{\frac{1}{p-1}}\,. \tag{2.22}$$

By (2.21), this completes the proof of the Theorem 2.5. □

We next prove a gradient estimate for solutions of (1.1) under condition (1.4).

Theorem 2.6. — *Let $1<p<n$, and let u be a nonnegative p-superharmonic solution of $-\Delta_p u = \mu$, where μ satisfies condition* (1.4). *Then we have*

$$\|\nabla u\|_{L^{p,\infty}(B(x,R))} \leqslant CM^{\frac{1}{p-1}}R^{\frac{n-p}{p}}, \quad \forall\, x\in\mathbb{R}^n,\ R>0,$$

where

$$M = \sup_{x\in\mathbb{R}^n,\,R>0}\frac{\mu(B(x,R))}{R^{n-p}}. \tag{2.23}$$

Proof. — Let $B_0 = B(z_0,R_0)$ be any fixed ball. By Theorem 2.5 we have

$$\begin{aligned}\|\nabla u\|_{L^{p,\infty}(B_0))} \leqslant C\,|B_0|^{\frac{1}{p}}&\left(\frac{1}{|2B_0|}\int_{2B_0}|\nabla u|^{p-\delta}\,dx\right)^{\frac{1}{p-\delta}} \\ &+ C\,\left\|\left[\mathbf{I}_1\left(\chi_{2B_0}\mu\right)\right]^{\frac{1}{p-1}}\right\|_{L^{p,\infty}(B_0)}\end{aligned} \tag{2.24}$$

for a constant $\delta=\delta(n,p)\in(0,p-1)$ and we may assume that δ is sufficiently small. For any $r_0 > 4R_0 + |z_0|$ and any $r\in(0,r_0]$, let $w\in u + W_0^{1,\,p-\delta}(B(z_0,r))$ solve

$$\begin{cases}\Delta_p w &= 0 \quad \text{in } B(z_0,r),\\ w &= u \quad \text{on } \partial B(z_0,r).\end{cases}$$

By [4, Lemma 2.7] for any $0 < \rho \leqslant r$ we have

$$\int_{B(z_0,\rho)} |\nabla w|^{p-\delta}\, dy \leqslant C(\rho/r)^{n+(p-\delta)(\beta_0-1)} \int_{B(z_0,r)} |\nabla w|^{p-\delta}\, dy,$$

for some $\beta_0 = \beta_0(n,p) \in (0, 1/2]$. Then by using (2.22) and arguing as in the proof of [4, Equation (5.4)] we have

$$\begin{aligned}(2.25)\quad \phi(\rho) \leqslant C\left[\left(\frac{\rho}{r}\right)^{n+(p-\delta)(\beta_0-1)} + \delta^{(p-\delta)\min\left\{1,\frac{1}{p-1}\right\}} + \epsilon\right]\phi(r)\\ + C(\epsilon)\int_{B(z_0,r)} \mathbf{I}_1\left(\chi_{B(z_0,r_0)}\mu\right)^{\frac{p-\delta}{p-1}}\, dx,\end{aligned}$$

which holds for all $\epsilon > 0$ and $\rho \in (0, r]$. In (2.25), we set

$$\phi(\rho) = \int_{B(z_0,\rho)} |\nabla u|^{p-\delta}\, dx.$$

Now by Hölder's inequality and (2.1) we have

$$\begin{aligned}\int_{B(z_0,r)} \mathbf{I}_1\left(\chi_{B(z_0,r_0)}\mu\right)^{\frac{p-\delta}{p-1}}\, dx &\leqslant C\, \|\mathbf{I}_1\mu\|^{\frac{p-\delta}{p-1}}_{L^{\frac{p}{p-1},\infty}(B(z_0,r))}\, r^{\frac{n\delta}{p}}\\ &\leqslant CM^{\frac{p-\delta}{p-1}} r^{\frac{(n-p)(p-\delta)+n\delta}{p}}\\ &\leqslant CM^{\frac{p-\delta}{p-1}} r^{n-p+\delta},\end{aligned}$$

where M is defined in (2.23). Thus it follows from (2.25) that

$$\begin{aligned}\phi(\rho) \leqslant C\left[\left(\frac{\rho}{r}\right)^{n+(p-\delta)(\beta_0-1)} + \delta^{(p-\delta)\min\left\{1,\frac{1}{p-1}\right\}} + \epsilon\right]\phi(r)\\ + C(\epsilon)\, M^{\frac{p-\delta}{p-1}} r^{n-p+\delta},\end{aligned}$$

which holds for all $\epsilon > 0$ and $\rho \in (0, r]$. As $n - p + \delta < n + (p - \delta)(\beta_0 - 1)$, we can apply [8, Lemma 3.4] to obtain

$$\phi(\rho) \leqslant C\left(\frac{\rho}{r}\right)^{n-p+\delta}\phi(r) + CM^{\frac{p-\delta}{p-1}}\rho^{n-p+\delta}$$

provided δ is sufficiently small. Since this estimate holds for all $0 < \rho \leqslant r \leqslant r_0$, we may choose $\rho = 2R_0$ and $r = r_0$ to deduce

$$\begin{aligned}(2.26)\quad \int_{B(z_0,2R_0)} |\nabla u|^{p-\delta}\, dx &\leqslant C\left(\frac{R_0}{r_0}\right)^{n-p+\delta}\int_{B(z_0,r_0)} |\nabla u|^{p-\delta}\, dx\\ &\quad + CM^{\frac{p-\delta}{p-1}} R_0^{n-p+\delta}\\ &\leqslant C\left(\frac{R_0}{r_0}\right)^{n-p+\delta}\int_{B(0,2r_0)} |\nabla u|^{p-\delta}\, dx\\ &\quad + CM^{\frac{p-\delta}{p-1}} R_0^{n-p+\delta},\end{aligned}$$

where we used that $B(z_0, r_0) \subset B(0, 2r_0)$.

At this point we combine (2.24), (2.26), and (2.1) to arrive at

$$\|\nabla u\|_{L^{p,\infty}(B(z_0,R_0)))} \leqslant C\, R_0^{\frac{n-p}{p}} \left(r_0^{-n+p-\delta} \int_{B(0,2r_0)} |\nabla u|^{p-\delta}\, dx \right)^{\frac{1}{p-\delta}} + C\, M^{\frac{1}{p-1}} R_0^{\frac{n-p}{p}}.$$

Finally, letting $r_0 \to \infty$ and applying Lemma 2.1 we complete the proof of the Theorem 2.6. □

We conclude this section with the following remarks regarding quasilinear equations with more general nonlinear structure.

Remark 2.7. — Theorems 2.5 and 2.6 also hold for more general equations of the form

$$-\operatorname{div} \mathcal{A}\,(x, \nabla u) = \mu, \tag{2.27}$$

where $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable in x for every ξ, continuous in ξ for a.e. x, and $\mathcal{A}(x, 0) = 0$ for a.e. $x \in \mathbb{R}^n$. Moreover, $\mathcal{A}$ is assumed to satisfy that

$$\big\langle \mathcal{A}(x,\xi) - \mathcal{A}(x,\zeta), \xi - \zeta \big\rangle \geqslant \Lambda_0 \left(|\xi|^2 + |\zeta|^2\right)^{\frac{p-2}{2}} |\xi - \zeta|^2 \tag{2.28}$$

and for some $\gamma \in (0, 1)$,

$$\big|\mathcal{A}(x,\xi) - \mathcal{A}(x,\zeta)\big| \leqslant \Lambda_1\, |\xi - \zeta|^{\gamma} \left(|\xi|^2 + |\zeta|^2\right)^{\frac{p-1-\gamma}{2}} \tag{2.29}$$

for every $(\xi, \zeta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0, 0)\}$ and a.e. $x \in \mathbb{R}^n$. Here Λ_0 and Λ_1 are positive constants.

Remark 2.8. — Condition (2.29) above can be replaced with the weaker condition

$$|\mathcal{A}(x, \xi)| \leqslant \Lambda_1 |\xi|^{p-1}. \tag{2.30}$$

Indeed, for $\frac{3n-2}{2n-1} < p < n$, this can be done similarly using the method of [22] and the comparison estimate of [20, Lemma 2.2] (see also [19], where this method was first utilized in the case $p \geqslant 2$). For $1 < p \leqslant \frac{3n-2}{2n-1}$, using the method of [22] and the recent comparison estimate of [21, Lemma 2.1], one can obtain the following version of (2.17):

There exists $\epsilon_0 = \epsilon_0(n, p, \Lambda_0, \Lambda_1) \in (0, 2(p-1))$ *such that for* $2-p+\epsilon_0 < q < p+\epsilon_0$,

$$\begin{aligned} \|\nabla u\|_{L^{q,\infty}(B_0))} \leqslant C(\epsilon)\ |B_0|^{\frac{1}{q}} \left(\frac{1}{|2B_0|}\int_{2B_0} |\nabla u|^{2-p}\, dx\right)^{\frac{1}{2-p}} \\ + C(\epsilon)\ \left\| \left[\mathbf{I}_1\left(\chi_{2B_0}\mu\right)\right]^{\frac{1}{p-1}} \right\|_{L^{q,\infty}(B_0)} \\ + \epsilon\ \|\nabla u\|_{L^{q,\infty}(2B_0)} \end{aligned} \tag{2.31}$$

for all balls B_0 *and all* $\epsilon > 0$.

(This estimate still holds if the weak L^q norms are replaced with the L^q norms). The constant $C(\epsilon)$ is independent of q. Thus, for $2-p+\epsilon_0 < q < p$, by Lemma 2.1 and a covering/iteration argument (see, e.g., [3]) the term $\epsilon\,\|\nabla u\|_{L^{q,\infty}(2B_0)}$ on the right-hand side can be absorbed yielding that

$$\begin{aligned} \|\nabla u\|_{L^{q,\infty}(B_0))} \leqslant C\ |B_0|^{\frac{1}{q}} \left(\frac{1}{|2B_0|}\int_{2B_0} |\nabla u|^{2-p}\, dx\right)^{\frac{1}{2-p}} \\ + C\ \left\| \left[\mathbf{I}_1\left(\chi_{2B_0}\mu\right)\right]^{\frac{1}{p-1}} \right\|_{L^{q,\infty}(B_0)} \end{aligned} \tag{2.32}$$

for all balls B_0. Thus letting $q \uparrow p$ we see that (2.32) holds with $q = p$ as well. From this we obtain analogues of Theorems 2.5 and 2.6 under the above assumptions on $\mathcal{A}$.

Using Poincaré's inequality we deduce the following BMO estimate.

Corollary 2.9. — *Let* $1 < p < n$, *and let* μ *satisfy condition* (1.4). *Under assumptions* (2.28) *and* (2.30) *on* $\mathcal{A}$, *for any nonnegative* $\mathcal{A}$*-superharmonic solution* u *to* (2.27) *we have*

$$\|u\|_{\mathrm{BMO}(\mathbb{R}^n)} \leqslant CM^{\frac{1}{p-1}},$$

where M *is the constant in* (2.23), *and* C *depends only on* $p, n, \Lambda_0, \Lambda_1$.

Remark 2.10. — In the case $0 < q < p-1$, Theorem 1.4 and Corollary 1.5 are deduced exactly as in [28] using Theorems 1.1, 2.5 and 2.6 in place of the corresponding statements of [28, Lemma 3.1].

3. Proof of Theorem 1.3

Proof. — In this section we treat the case $q > p-1$ in (1.6). Let $1 < p < n$. As was shown in [24], the existence of a solution u to (1.6) is equivalent to condition (a) of Theorem 1.3, with the small constant $0 < c \leqslant c(n, p, q)$ in the sufficiency part, and some $c > 0$ in the necessity part, where $\mathbf{W}_p\mu \not\equiv \infty$.

Let $d\omega = u^q d\sigma + d\mu$. By Theorem 1.1, any solution u to (1.6) lies in $\mathrm{BMO}(\mathbb{R}^n)$ if and only if

$$\omega\,(B(x,R)) \leqslant C\,R^{n-p}, \quad \forall\, x\in\mathbb{R}^n,\ R>0. \tag{3.1}$$

In particular, μ satisfies (1.4). Also, by the lower bound in (1.3), we have $u \geqslant C\,\mathbf{W}_p\mu$, so that by (1.4)

$$\int_{B(x,R)} \big(\mathbf{W}_p\mu\big)^q d\sigma \leqslant C\,R^{n-p}, \quad \forall\, x\in\mathbb{R}^n,\ R>0. \tag{3.2}$$

This yields the necessity of condition (b) in Theorem 1.3, since for all $y\in B(x,R)$ and $r>R$, we have $B(x,r)\subset B(y,2r)$, and consequently

$$\begin{aligned}\mathbf{W}_p\mu(y) &= 2^{-\frac{n-p}{p-1}} \int_0^\infty \left(\frac{\mu\,(B(y,2r))}{r^{n-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}\\ &\geqslant 2^{-\frac{n-p}{p-1}} \int_R^\infty \left(\frac{\mu\,(B(x,r))}{r^{n-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}.\end{aligned}$$

Conversely, suppose that (1.4), and both condition (a) with the small constant $c\leqslant c(n,p,q)$, and condition (b) of Theorem 1.3 hold. Then the solution u constructed in [24, Theorem 3.10] admits the upper bound $u\leqslant C\,\mathbf{W}_p\mu$. Hence, to verify (3.1), it remains to show that (3.2) holds.

For $B=B(x,R)$, we write $\mu=\mu_{2B}+\mu_{(2B)^c}$. Then clearly

$$\mathbf{W}_p\mu \leqslant c\Big(\mathbf{W}_p\mu_{2B} + \mathbf{W}_p\mu_{(2B)^c}\Big),$$

where c depends only on p. Arguing as above, for all $y\in B(x,R)$, we have $B(y,r)\cap(2B)^c=\emptyset$ if $0<r<R$, and $B(y,r)\cap(2B)^c\subset B(x,2r))$ for $r\geqslant R$, so that

$$\mathbf{W}_p\mu_{(2B)^c}(y) \leqslant C\int_R^\infty \left(\frac{\mu(B(x,r))}{r^{n-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}.$$

Hence by condition (b) of Theorem 1.3, we see that (3.2) holds for $\mu_{(2B)^c}$ in place of μ. Also, as was shown in [24, Theorem 3.1] (with $g=\chi_{2B}$), condition (a) of Theorem 1.3 yields

$$\int_B \big(\mathbf{W}_p\mu_{2B}\big)^q d\sigma \leqslant C\,\mu(2B).$$

Since $\mu(2B)\leqslant C\,R^{n-p}$ by (1.4), combining the preceding estimates we deduce (3.2). This completes the proof of Theorem 1.3. □

4. Proof of Theorem 1.4

In this section we treat the case $0 < q < p-1$ in (1.6). Let $1 < p < n$. It was proved in [5] (see also [27]) that a nontrivial solution to (1.6) exists if and only if (1.2) holds, i.e., $\mathbf{W}_p\mu \not\equiv \infty$, and

$$\int_1^\infty \left(\frac{\kappa\,(B(0,r))^{\frac{q(p-1)}{p-1-q}}}{r^{n-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty. \tag{4.1}$$

Condition (4.1) ensures that $\mathbf{K}_{p,q}\sigma \not\equiv \infty$, where $\mathbf{K}_{p,q}$ is the so-called *intrinsic* nonlinear potential introduced in [5],

$$\mathbf{K}_{p,q}\sigma(x) = \int_0^\infty \left(\frac{\kappa\,(B(x,r))^{\frac{q(p-1)}{p-1-q}}}{r^{n-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}, \quad x \in \mathbb{R}^n.$$

Moreover, as was proved recently in [27], any nontrivial solution u to (1.6) satisfies the bilateral estimates

$$\begin{aligned} c^{-1}\left[\big(\mathbf{W}_p\sigma(x)\big)^{\frac{p-1}{p-1-q}} + \mathbf{K}_{p,q}\sigma(x) + \mathbf{W}_p\mu(x)\right] &\leqslant u(x) \\ \leqslant c\left[\big(\mathbf{W}_p\sigma(x)\big)^{\frac{p-1}{p-1-q}} + \mathbf{K}_{p,q}\sigma(x) + \mathbf{W}_p\mu(x)\right]&, \quad x \in \mathbb{R}^n, \end{aligned} \tag{4.2}$$

where $c > 0$ is a constant which depends only on p, q, and n.

As in the case $q > p-1$, by Theorem 1.1, any solution u to (1.6) lies in $\mathrm{BMO}(\mathbb{R}^n)$ if and only if (3.1) holds, where $d\omega = u^q d\sigma + d\mu$. In view of (4.2), $u \in \mathrm{BMO}(\mathbb{R}^n)$ if and only if both conditions (1.4) and (3.2) hold, and also the following two conditions hold for all $x \in \mathbb{R}^n$ and $R > 0$:

$$\int_{B(x,R)} \big(\mathbf{W}_p\sigma\big)^{\frac{q(p-1)}{p-1-q}} d\sigma \leqslant C\,R^{n-p}, \tag{4.3}$$

$$\int_{B(x,R)} \big(\mathbf{K}_{p,q}\sigma\big)^q d\sigma \leqslant C\,R^{n-p}. \tag{4.4}$$

We first show that (1.4) together with conditions (a)–(d) of Theorem 1.4 yield (3.2), (4.3), and (4.4).

As in the case $q > p-1$ above, (3.2) splits into two parts: condition(b) of Theorem 1.4, and

$$\int_B \big(\mathbf{W}_p\mu_{2B}\big)^q d\sigma \leqslant C\,R^{n-p}, \tag{4.5}$$

where $B = B(x,R)$. To prove the preceding estimate, notice that by (1.9) applied to $\nu = \mu_{2B}$, we have

$$\int_B \big(\mathbf{W}_p\mu_{2B}\big)^q d\sigma \leqslant \kappa(B)^q\,\mu(2B)^{\frac{q}{p-1}}.$$

By (1.4), it follows that $\mu(2B) \leqslant C\, R^{n-p}$, and by condition (a), we have

$$\kappa(B)^q \leqslant C\, R^{\frac{(n-p)(p-1-q)}{p-1}}.$$

Hence, (4.5) follows from (1.4)& (a), and consequently (3.2) follows from (a) & (b) & (1.4).

To prove (4.3), for $B = B(x,R)$, we write $\sigma = \sigma_{2B} + \sigma_{(2B)^c}$. Again, (4.3) splits into two parts. Arguing as above in the case $q > p-1$ we have

$$\begin{aligned}\int_B \left(\mathbf{W}_p \sigma_{(2B)^c}\right)^{\frac{q(p-1)}{p-1-q}} d\sigma \\ &\leqslant C\, \sigma(B) \left[\int_R^\infty \left(\frac{\sigma\left(B(x,r)\right)^{\frac{q(p-1)}{p-1-q}}}{r^{n-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}\right]^{\frac{q(p-1)}{p-1-q}} \\ &\leqslant C\, R^{n-p},\end{aligned}$$

by condition (c) of Theorem 1.4.

Next, denote by $v_{2B} \in L^q(\sigma_{2B})$ the nontrivial solution to the equation

$$v_{2B} = \mathbf{W}_p \left(v_{2B}^q \sigma_{2B}\right) \quad \text{in } \mathbb{R}^n \tag{4.6}$$

constructed in [5], which exists since $\kappa(2B) < \infty$ by condition (a) of Theorem 1.4. By [5, Corollary 4.3],

$$\int_{2B} (v_{2B})^q \, d\sigma \leqslant \kappa(2B)^{\frac{q(p-1)}{p-1-q}}. \tag{4.7}$$

On the other hand, $v_{2B} \geqslant C\, \mathbf{W}_p \sigma_{2B})^{\frac{p-1}{p-1-q}}$ by the lower estimate in (4.2). Combining these estimates yields

$$\int_B \left(\mathbf{W}_p \sigma_{2B}\right)^{\frac{q(p-1)}{p-1-q}} d\sigma \leqslant C\kappa(2B)^{\frac{q(p-1)}{p-1-q}} \leqslant C R^{n-p}$$

by condition (a).

We now prove (4.4). For $y \in B = B(x,R)$, we split $\mathbf{K}_{p,q}\sigma(y)$ into two parts,

$$\begin{aligned}&\mathbf{K}_{p,q}\sigma(y) \\ &\quad = I + II \\ &\quad = \int_0^R \left(\frac{\kappa(B(y,r))^{\frac{q(p-1)}{p-1-q}}}{r^{n-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r} + \int_R^\infty \left(\frac{\kappa(B(y,r))^{\frac{q(p-1)}{p-1-q}}}{r^{n-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}.\end{aligned}$$

To estimate the term involving I, notice that $B(y,r) \subset 2B$ for $0 < r \leqslant R$. Hence by the lower estimate in (4.2) with $\mu = 0$ and σ_{2B} in place of σ, we have $I \leqslant C\, v_{2B}$, where v_{2B} is defined by (4.6). It follows that

$$\int_B I^q d\sigma \leqslant C \int_B v_{2B}^q d\sigma.$$

By the preceding estimate, (4.7), and condition (a), we deduce

$$\int_B I^q d\sigma \leqslant C\, \kappa(2B)^{\frac{q(p-1)}{p-1-q}} \leqslant C\, R^{n-p}.$$

For $r > R$ and $y \in B$, we obviously have $B(y,r) \subset B(x,2r)$, so that $\kappa(B(y,r)) \leqslant \kappa(B(x,2r))$, and consequently, for all $y \in B$,

$$II \leqslant 2^{\frac{n-p}{p-1}} \int_{2R}^{\infty} \left(\frac{\kappa\left(B(x,r)\right)^{\frac{q(p-1)}{p-1-q}}}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}.$$

It follows that

$$\int_B II^q d\sigma \leqslant C\, \sigma(B) \left[\int_{2R}^{\infty} \left(\frac{\kappa(B(x,r))^{\frac{q(p-1)}{p-1-q}}}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right]^q \leqslant C\, R^{n-p}$$

by condition (d) of Theorem 1.4. This proves that (4.4) follows from conditions (a) & (d). Thus, (3.1) holds, so that $u \in \mathrm{BMO}(\mathbb{R}^n)$.

Conversely, suppose that $u \in \mathrm{BMO}(\mathbb{R}^n)$ is a solution to (1.6). Then as was mentioned above (3.1) holds, which obviously yields (1.4). Since (3.1) also yields

$$\int_B u^q d\sigma \leqslant C\, R^{n-p}, \tag{4.8}$$

and by [5, Lemma 4.2],

$$\kappa(B)^{\frac{q(p-1)}{p-1-q}} \leqslant C \int_B u^q d\sigma, \tag{4.9}$$

we combine (4.8) and (4.9) to obtain (a).

Next, by (4.8) and the lower estimate in (4.2) we deduce that (3.2), (4.3), and (4.4) hold.

Notice that condition (b) follows from (3.2) exactly as in the case $q > p-1$ above. Similarly, for all $y \in B = B(x,R)$, we have

$$\mathbf{W}_p\mu(y) \geqslant 2^{-\frac{n-p}{p-1}} \int_R^{\infty} \left(\frac{\mu(B(x,r))}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}.$$

Hence, (4.3) yields condition (c). In the same way, for all $y \in B = B(x,R)$ and $r > R$, we have $B(y,2r) \supset B(x,r)$, and consequently

$$\begin{aligned}\mathbf{K}_{p,q}\sigma(y) &= 2^{-\frac{n-p}{p-1}} \int_0^\infty \left(\frac{\kappa\left(B(y,2r)\right)^{\frac{q(p-1)}{p-1-q}}}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\ &\geqslant 2^{-\frac{n-p}{p-1}} \int_R^\infty \left(\frac{\kappa\left(B(x,r)\right)^{\frac{q(p-1)}{p-1-q}}}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}.\end{aligned}$$

This shows that (4.4) yields condition (d). The proof of Theorem 1.4 is complete. □

The proof of Corollary 1.5 is based on the following pointwise estimate for all solutions u to (1.6) in the case $0 < q < p-1$ [27, Corollary 1.2],

$$\begin{aligned}c^{-1}\left[\left(\mathbf{W}_p\sigma(x)\right)^{\frac{p-1}{p-1-q}} + \mathbf{W}_p\mu(x)\right] &\leqslant u(x) \\ &\leqslant c\left[\left(\mathbf{W}_p\sigma(x)\right)^{\frac{p-1}{p-1-q}} + \mathbf{W}_p\sigma(x) + \mathbf{W}_p\mu(x)\right], \quad x \in \mathbb{R}^n,\end{aligned}$$

provided σ satisfies condition (1.10). The argument is similar to that of [28, Corollary 1.5] in the case $\mu = 0$; we omit the details.

5. The natural growth case

In this section we suppose that $1 < p < n$ and $\mu, \sigma \in M^+(\mathbb{R}^n)$, where both $\sigma \neq 0$ and $\mu \neq 0$. It is well known (see, for instance, [11]) that the capacity condition (1.10) with $C = 1$ is *necessary* for the existence of a nontrivial solution u to the inequality

$$-\Delta_p u \geqslant \sigma\, u^{p-1}, \quad u \geqslant 0 \quad \text{in } \mathbb{R}^n.$$

We have to distinguish between the cases $p > 2$ and $p \leqslant 2$. We recall that $\mathbf{I}_p$ stands for the Riesz potential of order p defined by (2.2) with $\alpha = p$. It is easy to see that

$$\begin{aligned}&\mathbf{I}_p\sigma \leqslant C\left(\mathbf{W}_p\sigma\right)^{p-1} \quad \text{if} \quad p > 2, \\ \text{and} \quad &\left(\mathbf{W}_p\sigma\right)^{p-1} \leqslant C\,\mathbf{I}_p\sigma \quad \text{if} \quad p \leqslant 2,\end{aligned} \tag{5.1}$$

where C is a constant which depends only on p and n.

Theorem 5.1. — *Let $1 < p < n$ and $q = p-1$. Suppose $\mu, \sigma \in M^+(\mathbb{R}^n)$, and*

(a) $$\mathbf{W}_p\sigma \leqslant C_1 \quad \text{if} \quad p > 2,$$

(b) $$\mathbf{I}_p\sigma \leqslant C_2 \quad \text{if} \quad p \leqslant 2.$$

Then there exists a solution $u \in \mathrm{BMO}(\mathbb{R}^n)$ to (1.6) if and only if μ satisfies condition (1.4), and for all $x \in \mathbb{R}^n$, $R > 0$,

$$\sigma\left(B(x,R)\right)\left[\int_R^\infty \left(\frac{\mu\left(B(x,r)\right)}{r^{n-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}\right]^{p-1} \leqslant C\, R^{n-p}, \tag{5.2}$$

where the "if" part requires the smallness of the constant $c = c(p,n)$ in the condition

$$\sigma(K) \leqslant c\, \mathrm{cap}_p(K), \quad \forall \text{ compact sets } K \subset \mathbb{R}^n. \tag{5.3}$$

Remark 5.2. — Assumptions (a) and (b) in Theorem 5.1 are stronger than the necessary condition (5.3) for some constant c. Without these assumptions, estimates of solutions are substantially more complicated (see [11]).

Proof. — It is known ([11, Remark 1.3 and Section 2] that conditions (a) & (b) of Theorem 5.1, together with (5.3) for some small constant $c = c(p,n)$, ensure that (1.6) has a solution u such that

$$c_1\, \mathbf{W}_p\mu(x) \leqslant u(x) \leqslant c_2\, \mathbf{W}_p\mu(x), \quad x \in \mathbb{R}^n. \tag{5.4}$$

The lower bound obviously holds for all solutions u.

As above, by Theorem 1.1, $u \in \mathrm{BMO}(\mathbb{R}^n)$ if and only if μ satisfies (1.4), and (4.8) holds (with $q = p-1$). By the lower estimate in (5.4), we see that (4.8) yields

$$\int_{B(x,R)} \left(\mathbf{W}_p\mu\right)^{p-1} d\sigma \leqslant C\, R^{n-p}, \quad \forall\ x \in \mathbb{R}^n,\ R > 0. \tag{5.5}$$

Exactly as in the cases $q > p-1$ and $q < p-1$, this estimate yields (5.2), which completes the proof of the "only if" part of Theorem 5.1.

To prove the "if" part, as above we split (5.5) into two parts, condition (5.2) and

$$\int_B \left(\mathbf{W}_p\mu_{2B}\right)^{p-1} d\sigma \leqslant C\ |B|^{\frac{n-p}{n}}, \tag{5.6}$$

where $B = B(x,R)$.

We first prove (5.6) in the easier case $1 < p \leqslant 2$. It follows from (5.1) that $(\mathbf{W}_p\mu_{2B})^{p-1} \leqslant C\,\mathbf{I}_p\mu_{2B}$, and by Fubini's Theorem,

$$\int_B \left(\mathbf{W}_p\mu_{2B}\right)^{p-1} d\sigma \leqslant C\int_B \mathbf{I}_p\mu_{2B}\,d\sigma = C\int_{2B} \mathbf{I}_p\sigma_B\,d\mu.$$

Since $\mathbf{I}_p\sigma_B \leqslant C_2$ by assumption (b), we deduce

$$\int_B \left(\mathbf{W}_p\mu_{2B}\right)^{p-1} d\sigma \leqslant C\,C_2\mu(2B),$$

and (5.6) follows in view of condition (1.4).

We now consider the case $p > 2$. Then (5.6) can be deduced from [11, Lemma 4.4], but we give here a simplified proof based on the following lemma.

Lemma 5.3. — *Let $2 < p < n$, and let $\mu, \sigma \in M^+(\mathbb{R}^n)$, where σ satisfies* (5.3)*. Then*

$$\int_{\mathbb{R}^n} \left(\mathbf{W}_p\mu\right)^{p-1} d\sigma \leqslant C\,c^{\frac{p-2}{p-1}}\int_{\mathbb{R}^n} \left(\mathbf{W}_p\sigma\right) d\mu, \tag{5.7}$$

where c is the constant in (5.3)*, and C is a constant which depends only on p, n.*

Proof. — It is more convenient to use dyadic Wolff potentials introduced originally in [9], in place of $\mathbf{W}_p\mu$,

$$\mathbf{W}^d_p\mu(x) = \sum_{Q\,\in\,\mathcal{D}} \left(\frac{\mu(Q)}{\ell(Q)^{n-p}}\right)^{\frac{1}{p-1}} \chi_Q(x), \quad x \in \mathbb{R}^n,$$

where $\mathcal{D} = \{Q\}$ is the family of all dyadic cubes in $\mathbb{R}^n$, and $\ell(Q)$ stands for the side length of Q.

For $Q \in \mathcal{D}$, we denote by Q^* the concentric cube with side length $\ell(Q^*) = 3\,\ell(Q)$. Clearly, the family of cubes $\{Q^*\}_{Q\,\in\,\mathcal{D}}$ has the finite intersection property

$$\sum_{\ell(Q)\,=\,2^k} \chi_{Q^*}(x) \leqslant \beta(n), \qquad x \in \mathbb{R}^n,\ k \in \mathbb{Z}, \tag{5.8}$$

where $\beta(n)$ is a constant which depends only on n. We will actually need a modified version of $\mathbf{W}^d_p$ defined by

$$\widetilde{\mathbf{W}}^d_p\mu = \sum_{Q\,\in\,\mathcal{D}} \left(\frac{\mu(Q^*)}{\ell(Q)^{n-p}}\right)^{\frac{1}{p-1}} \chi_Q(x), \quad x \in \mathbb{R}^n.$$

It is easy to verify (see [9, p. 170]) that

$$a\,\mathbf{W}^d_p\mu \leqslant \mathbf{W}_p\mu \leqslant A\,\widetilde{\mathbf{W}}^d_p\mu, \tag{5.9}$$

where the constants a, A depend only on p and n.

In view of (5.9), it is enough to prove the following version of (5.7),

$$\int_{\mathbb{R}^n} \left(\widetilde{\mathbf{W}}_p^d \mu\right)^{p-1} d\sigma \leqslant C\, c^{\frac{p-2}{p-1}} \int_{\mathbb{R}^n} \left(\mathbf{W}_p \sigma\right) d\mu. \tag{5.10}$$

Since $p > 2$, we can use duality to rewrite (5.10) in the equivalent form

$$\int_{\mathbb{R}^n} \left(\widetilde{\mathbf{W}}_p^d \mu\right) g\, d\sigma \leqslant C\, c^{\frac{p-2}{(p-1)^2}} \left[\int_{\mathbb{R}^n} \left(\mathbf{W}_p \sigma\right) d\mu\right]^{\frac{1}{p-1}}, \tag{5.11}$$

for all

$$g \in L^{\frac{p-1}{p-2}}(\mathbb{R}^n, \sigma) \quad \text{such that} \quad \|g\|_{L^{\frac{p-1}{p-2}}(\mathbb{R}^n,\sigma)} \leqslant 1.$$

Interchanging the order of integration and summation on the left-hand side of (5.11), we see that (5.10) is equivalent to

$$I = \sum_{Q\in\mathcal{D}} \left(\frac{\mu(Q^*)}{\ell(Q)^{n-p}}\right)^{\frac{1}{p-1}} \int_Q g\, d\sigma \leqslant C\, c^{\frac{p-2}{(p-1)^2}} \left[\int_{\mathbb{R}^n} \left(\mathbf{W}_p \sigma\right) d\mu\right]^{\frac{1}{p-1}}.$$

Using Hölder's inequality with exponents $p-1$ and $\frac{p-1}{p-2}$, we estimate

$$\begin{aligned} I \leqslant & \left[\sum_{Q\in\mathcal{D}} \left(\frac{\sigma(Q)}{\ell(Q)^{n-p}}\right)^{\frac{1}{p-1}} \mu(Q^*)\right]^{\frac{1}{p-1}} \\ & \times \left[\sum_{Q\in\mathcal{D}} \frac{\sigma(Q)^{p'}}{\ell(Q)^{\frac{n-p}{p-1}}} \left(\frac{1}{\sigma(Q)} \int_Q g\, d\sigma\right)^{\frac{p-1}{p-2}}\right]^{\frac{p-2}{p-1}}, \end{aligned} \tag{5.12}$$

where $p' = \frac{p}{p-1}$. Notice that

$$\sum_{Q\in\mathcal{D}} \left(\frac{\sigma(Q)}{\ell(Q)^{n-p}}\right)^{\frac{1}{p-1}} \mu(Q^*) = \int_{\mathbb{R}^n} \sum_{Q\in\mathcal{D}} \left(\frac{\sigma(Q)}{\ell(Q)^{n-p}}\right)^{\frac{1}{p-1}} \chi_{Q^*}\, d\mu.$$

If $x \in Q^*$, then obviously $Q \subset B(x, \alpha(n)\, \ell(Q))$, where $\alpha(n)$ is a constant which depends only on n. We estimate

$$\begin{aligned} \sum_{Q\in\mathcal{D}} \left(\frac{\sigma(Q)}{\ell(Q)^{n-p}}\right)^{\frac{1}{p-1}} \chi_{Q^*}(x) &= \sum_{k\in\mathbb{Z}} \sum_{\ell(Q)=2^k} \left(\frac{\sigma(Q)}{\ell(Q)^{n-p}}\right)^{\frac{1}{p-1}} \chi_{Q^*}(x) \\ &\leqslant \sum_{k\in\mathbb{Z}} \left(\frac{\sigma\left(B(x,\alpha(n)2^k)\right)}{2^{k(n-p)}}\right)^{\frac{1}{p-1}} \sum_{\ell(Q)=2^k} \chi_{Q^*}(x). \end{aligned}$$

Clearly,

$$\sum_{k\in\mathbb{Z}}\left(\frac{\sigma\left(B\left(x,\alpha(n)2^k\right)\right)}{2^{k(n-p)}}\right)^{\frac{1}{p-1}}\leqslant C\int_0^\infty\left(\frac{\sigma(B(x,\alpha(n)r)}{r^{n-p}}\right)^{\frac{1}{p-1}}\frac{dr}{r}$$
$$=C\,\alpha(n)^{\frac{n-p}{p-1}}\mathbf{W}_p\sigma(x),$$

where C depends only on p and n. Hence, by the finite intersection property (5.8),

$$\sum_{Q\in\mathcal{D}}\left(\frac{\sigma(Q)}{\ell(Q)^{n-p}}\right)^{\frac{1}{p-1}}\chi_{Q^*}(x)\leqslant C\,\alpha(n)^{\frac{n-p}{p-1}}\,\beta(n)\mathbf{W}_p\sigma(x).$$

Integration of both sides of the preceding inequality with respect to $d\mu$ gives

$$\sum_{Q\in\mathcal{D}}\left(\frac{\sigma(Q)}{\ell(Q)^{n-p}}\right)^{\frac{1}{p-1}}\mu\left(Q^*\right)\leqslant C\,\alpha(n)^{\frac{n-p}{p-1}}\,\beta(n)\int_{\mathbb{R}^n}\left(\mathbf{W}_p\sigma\right)d\mu.$$

We estimate the second factor in (5.12) using the dyadic Carleson measure theorem. We observe that Theorem 5.1 (a) yields the capacity condition (5.3). It is known (see [11, Theorem 3.9]) that (5.3) is equivalent to the dyadic Carleson measure condition

$$\sum_{Q\subseteq P}\frac{\sigma(Q)^{p'}}{\ell(Q)^{\frac{n-p}{p-1}}}\leqslant C\,c^{p'-1}\sigma(P),$$

for all dyadic cubes P, where c is the constant in (5.3), and C depends only on p,n. Hence by the dyadic Carleson measure theorem,

$$\sum_{Q\in\mathcal{D}}\frac{\sigma(Q)^{p'}}{\ell(Q)^{\frac{n-p}{p-1}}}\left(\frac{1}{\sigma(Q)}\int_Q g\,d\sigma\right)^{\frac{p-1}{p-2}}\leqslant C\,c^{p'-1}\,\|g\|^{\frac{p-1}{p-2}}_{L^{\frac{p-1}{p-2}}(\mathbb{R}^n,\sigma)}\leqslant C\,c^{p'-1},$$

since $\|g\|_{L^{\frac{p-1}{p-2}}(\mathbb{R}^n,\sigma)}\leqslant 1$. Combining the preceding estimates proves (5.7). □

Applying Lemma 5.3 with μ_{2B} and σ_B in place of μ and σ, respectively, we obtain

$$\int_B\left(\mathbf{W}_p\mu_{2B}\right)^{p-1}d\sigma\leqslant C\,c^{\frac{p-2}{p-1}}\int_{2B}\left(\mathbf{W}_p\sigma_B\right)d\mu.$$

Invoking Theorem 5.1 (a) and condition (1.4) yields

$$\int_{2B}\left(\mathbf{W}_p\sigma_B\right)d\mu\leqslant C\,C_1\,c^{\frac{p-2}{p-1}}\mu(2B)\leqslant C\,|B|^{\frac{n-p}{n}}.$$

Thus, (5.6) holds for all $1<p<n$, and consequently $u\in\mathrm{BMO}(\mathbb{R}^n)$. □

BIBLIOGRAPHY

[1] D. R. Adams, "A note on Riesz potentials", *Duke Math. J.* **4** (1975), p. 765-778.

[2] D. R. Adams & L. I. Hedberg, *Function Spaces and Potential Theory*, Grundlehren der Mathematischen Wissenschaften, vol. 314, Springer, 1996.

[3] K. Adimurthi, T. Mengesha & N. C. Phuc, "Gradient weighted norm inequalities for linear elliptic equations with discontinuous coefficients", *Appl. Math. Optim.* **83** (2021), no. 1, p. 327-371.

[4] K. Adimurthi & N. C. Phuc, "Global Lorentz and Lorentz–Morrey estimates below the natural exponent for quasilinear equations", *Calc. Var. Partial Differ. Equ.* **54** (2015), no. 3, p. 3107-3139.

[5] D. T. Cao & I. E. Verbitsky, "Nonlinear elliptic equations and intrinsic potentials of Wolff type", *J. Funct. Anal.* **272** (2017), no. 1, p. 112-165.

[6] G. Dal Maso, F. Murat, L. Orsina & A. Prignet, "Renormalized solutions of elliptic equations with general measure data", *Ann. Sc. Norm. Super. Pisa, Cl. Sci.* **28** (1999), no. 4, p. 741-808.

[7] E. Giusti, *Direct Methods in the Calculus of Variations*, World Scientific, 2003.

[8] Q. Han & F. Lin, *Elliptic Partial Differential Equations*, second ed., Courant Lecture Notes in Mathematics, vol. 1, American Mathematical Society, 2011.

[9] L. I. Hedberg & T. H. Wolff, "Thin sets in nonlinear potential theory", *Ann. Inst. Fourier* **33** (1983), no. 4, p. 161-187.

[10] J. Heinonen, T. Kilpeläinen & O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Mathmatical Monographs, Oxford University Press, 1993.

[11] B. J. Jaye & I. E. Verbitsky, "Local and global behaviour of solutions to nonlinear equations with natural growth terms", *Arch. Ration. Mech. Anal.* **204** (2012), no. 2, p. 627-681.

[12] T. Kilpeläinen, T. Kuusi & A. Tuhola-Kujanpää, "Superharmonic functions are locally renormalized solutions", *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **28** (2011), no. 6, p. 775-795.

[13] T. Kilpeläinen & J. Malý, "Degenerate elliptic equations with measure data and nonlinear potentials", *Ann. Sc. Norm. Super. Pisa, Cl. Sci.* **19** (1992), no. 4, p. 591-613.

[14] ———, "The Wiener test and potential estimates for quasilinear elliptic equations", *Acta Math.* **172** (1994), no. 1, p. 137-161.

[15] T. Kilpeläinen & X. Zhong, "Removable sets for continuous solutions of quasilinear elliptic equations", *Proc. Am. Math. Soc.* **130** (2002), no. 6, p. 1681-1688.

[16] T. Kuusi & G. Mingione, "Guide to nonlinear potential estimates", *Bull. Math. Sci.* **4** (2014), no. 1, p. 1-82.

[17] V. G. Maz'ya, *Sobolev Spaces, with Applications to Elliptic Partial Differential Equations*, second, revised and augmented ed., Grundlehren der Mathematischen Wissenschaften, vol. 342, Springer, 2011.

[18] G. Mingione, "The Calderón–Zygmund theory for elliptic problems with measure data", *Ann. Sc. Norm. Super. Pisa, Cl. Sci.* **6** (2007), no. 2, p. 195-261.

[19] ———, "Gradient estimates below the duality exponent", *Math. Ann.* **346** (2010), no. 3, p. 571-627.

[20] Q.-H. Nguyen & N. C. Phuc, "Good-λ and Muckenhoupt–Wheeden type bounds in quasilinear measure datum problems, with applications", *Math. Ann.* **374** (2019), no. 1-2, p. 67-98.

[21] ———, "Existence and regularity estimates for quasilinear equations with measure data: the case $1 < p \leqslant \frac{3n-2}{2n-1}$" (to appear in *Analysis & PDE*), available at `https://arxiv.org/abs/2003.03725`, 2020.

[22] N. C. Phuc, "Morrey global bounds and quasilinear Riccati type equations below the natural exponent", *J. Math. Pures Appl.* **102** (2014), no. 1, p. 99-123.

[23] N. C. Phuc & I. E. Verbitsky, "Quasilinear and Hessian equations of Lane–Emden type", *Ann. Math.* **168** (2008), no. 3, p. 859-914.

[24] ———, "Singular quasilinear and Hessian equations and inequalities", *J. Funct. Anal.* **256** (2009), no. 6, p. 1875-1906.

[25] N. S. Trudinger, "On Harnack type inequalities and their applications to quasilinear elliptic equations", *Commun. Pure Appl. Math.* **20** (1967), p. 721-747.

[26] N. S. Trudinger & X.-J. Wang, "On the weak continuity of elliptic operators and applications to potential theory", *Am. J. Math.* **124** (2002), no. 2, p. 369-410.

[27] I. E. Verbitsky, "Bilateral estimates of solutions to quasilinear elliptic equations with sub-natural growth terms" (to appear in *Adv. Calc. Var.*), available at `https://arxiv.org/abs/arXiv:2101.02368`.

[28] ———, "Quasilinear elliptic equations with sub-natural growth terms and nonlinear potential theory", *Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat., IX. Ser., Rend. Lincei, Mat. Appl.* **30** (2019), no. 4, p. 733-758.

Manuscrit reçu le 19 janvier 2021,
révisé le 9 avril 2021,
accepté le 6 mai 2021.

Nguyen Cong PHUC
Department of Mathematics,
Louisiana State University,
303 Lockett Hall, Baton Rouge,
LA 70803 (USA)
pcnguyen@math.lsu.edu

Igor E. VERBITSKY
Department of Mathematics,
University of Missouri,
Columbia, MO 65211 (USA)
verbitskyi@missouri.edu