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GLOBAL WEAK SOLUTIONS FOR QUANTUM ISOTHERMAL FLUIDS

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ABSTRACT. — We construct global weak solutions to isothermal quantum Navier–Stokes equations, with or without Korteweg term, in the whole space of dimension at most three. Instead of working on the initial set of unknown functions, we consider an equivalent reformulation, based on a time-dependent rescaling, that we introduced in a previous paper to study the large time behavior, and which provides suitable a priori estimates, as opposed to the initial formulation where the potential energy is not signed. We proceed by working on tori whose size eventually becomes infinite. On each fixed torus, we consider the equations in the presence of drag force terms. Such equations are solved by regularization, and the limit where the drag force terms vanish is treated by resuming the notion of renormalized solution developed by I. Lacroix-Violet and A. Vasseur. We also establish global existence of weak solutions for the isothermal Korteweg equation (no viscosity), when initial data are well-prepared, in the sense that they stem from a Madelung transform.

RÉSUMÉ. — Nous construisons des solutions faibles globales pour l'équation de Navier-Stokes quantique isotherme, avec ou sans terme de Korteweg, dans tout l'espace en dimension au plus trois. Au lieu de travailler sur les inconnues originales, nous considérons une reformulation équivalente basée sur un changement d'échelle dépendant du temps, introduit précédemment pour étudier le comportement en temps grand, et qui fournit des estimations a priori convenables, par opposition à la formulation originale dans laquelle l'énergie potentielle n'a pas de signe. Nous travaillons sur des tores dont la taille tend vers l'infini. Sur chacun des tores, nous considérons les équations en présence d'une force de traînée. Ces équations sont résolues par régularisation, et on traite la limite où la force de traînée devient nulle en reprenant la notion de solution renormalisée développée par I. Lacroix-Violet et A. Vasseur. Nous montrons également l'existence globale de solutions faibles pour l'équation de Korteweg isotherme (sans viscosité) lorsque les données initiales sont bien préparées, au sens où elles proviennent d'une transformée de Madelung.

Keywords: Weak solutions, Renormalized solutions, Quantum isothermal fluids, Navier–Stokes equation, Korteweg equation.
2020 *Mathematics Subject Classification:* 35D30, 35Q30, 35Q35, 35Q40, 76N15, 76Y05.

1. Introduction

In this paper we consider the isothermal fluid equations in $\mathbb{R}^d$ ($d \leqslant 3$):

$$
\begin{cases}
\partial_t \varrho + \operatorname{div}(\varrho u) = 0, & \text{(1.1a)}\\
\partial_t(\varrho u) + \operatorname{div}\left(\varrho u \otimes u\right) + \nabla \varrho = \dfrac{\epsilon^2}{2}\varrho \nabla\left(\dfrac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}}\right) + \nu \operatorname{div}\left(\varrho \mathbb{D} u\right), & \text{(1.1b)}
\end{cases}
$$

on some time interval $(0,T)$. Here, the unknowns are the density $\varrho : (0,T)\times\mathbb{R}^d \to [0,\infty)$ and the velocity field $u : (0,T)\times\mathbb{R}^d \to \mathbb{R}^d$ of the fluid. We denote by $\mathbb{D}u = \frac{1}{2}(\nabla u + \nabla u^\top)$, the symmetric part of ∇u, and $\epsilon \geqslant 0$, $\nu \geqslant 0$ (with $(\epsilon,\nu) \neq (0,0)$) are given parameters. When $\epsilon = 0$ and $\nu > 0$, the system (1.1) corresponds to the isothermal quantum Navier–Stokes equations; the case $\epsilon, \nu > 0$ corresponds to the isothermal quantum Navier–Stokes–Korteweg equations; the case $\epsilon > 0$ and $\nu = 0$ to the quantum Euler equation. The term $\nabla\rho$ on the left-hand side corresponds to the gradient of the pressure of an isothermal fluid. Analytically, this corresponds to a limiting case of equations for polytropic gases where the pressure is given by a power-law $P(\rho) = a\rho^\gamma$ with $\gamma > 1$ and $a > 0$. Such isothermal models are marginally studied in the literature (see [21] for the quantum Navier–Stokes equations on $\mathbb{T}^d$, $d \leqslant 2$, and [25, 27] for the 2D Newtonian Navier–Stokes case on a bounded domain) whereas they have been derived in a quantum context [10]. We emphasize in the case of the Euler equation ($\epsilon = \nu = 0$), in space dimension $d = 1$, the existence of global weak solution is obtained in [23] by the vanishing viscosity method, under weak assumptions on the initial data: $0 \leqslant \varrho_0 \in L^\infty(\mathbb{R})$ and $|u_0(x)| \lesssim 1 + |\ln \varrho_0(x)|$. In a previous paper [11], we studied the large-time behavior of solutions to (1.1) with $\epsilon, \nu \geqslant 0$, under the assumption that sufficiently integrable solutions do exist globally in time. To our knowledge, the question of the existence of such solutions remains open, specifically in the isothermal case. We answer this question herein by proving that (1.1) admits weak solutions globally in time. The main part of this paper addresses the Navier-Stokes case $\nu > 0$ (with $\epsilon \geqslant 0$) for general initial data, while the Korteweg case $\nu = 0$, $\epsilon > 0$ is considered for well-prepared initial data (stemming from a Madelung transform), and is much more straightforward.

Formally, solutions to (1.1) enjoy the energy equality

$$E(t) + \int_0^t D(s)\mathrm{d}s = E(0), \quad t \geqslant 0,$$

where the energy is defined by

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^d} \left(\varrho|u|^2 + \epsilon^2 \left|\nabla\sqrt{\varrho}\right|^2\right) + \int_{\mathbb{R}^d} \varrho \log \varrho, \tag{1.2}$$

and the dissipation is given by

$$D(t) = \nu \int_{\mathbb{R}^d} \varrho |\mathbb{D}u|^2. \tag{1.3}$$

A feature of the isothermal case is that the pressure part of the energy,

$$\int_{\mathbb{R}^d} \varrho \log \varrho,$$

involves a functional which has no definite sign, as opposed to

$$\frac{1}{\gamma-1}\int_{\mathbb{R}^d} \varrho^\gamma$$

in the polytropic case. This is one of the reasons why there are fewer results regarding the global existence of solutions in the case $\gamma = 1$ than in the case $\gamma > 1$. Also, because we consider the case of an unbounded domain $x \in \mathbb{R}^d$, nonzero constant densities cannot provide finite-energy solutions to (1.1), ruling out natural candidates for an approach based on relative entropy like in e.g. [9].

Following [11], we circumvent this difficulty by considering the auxiliary unknowns (R, U) as defined by

$$\begin{aligned} \varrho(t,x) &= \frac{1}{\tau(t)^d} R\left(t, \frac{x}{\tau(t)}\right) \frac{\|\varrho_0\|_{L^1}}{\|\Gamma\|_{L^1}}, \\ u(t,x) &= \frac{1}{\tau(t)} U\left(t, \frac{x}{\tau(t)}\right) + \frac{\dot\tau(t)}{\tau(t)} x, \end{aligned} \tag{1.4}$$

where $\Gamma(y) = e^{-|y|^2}$ and the function τ is the global solution to the nonlinear ODE

$$\ddot\tau = \frac{2}{\tau}, \quad \tau(0) = 1, \quad \dot\tau(0) = 0.$$

We recall (see [13]) that there exists a unique global solution $\tau \in C^\infty([0,\infty))$ to this system. This solution remains uniformly bounded from below by a strictly positive constant and its large time behavior is known:

$$\tau(t) \underset{t\to\infty}{\sim} 2t\sqrt{\log t}, \quad \dot\tau(t) \underset{t\to\infty}{\sim} 2\sqrt{\log t}.$$

By convention, the space variable for unknowns with capital letters will be denoted by y, in contrast with the initial space variable x. System (1.1) becomes, in the terms of the new unknown $(R, U) = (R(t,y), U(t,y))$,

$$\begin{cases} \partial_t R + \dfrac{1}{\tau^2} \operatorname{div}(RU) = 0 & \text{(1.5a)} \\ \partial_t (RU) + \dfrac{1}{\tau^2} \operatorname{div}(RU \otimes U) + 2yR + \nabla R \\ \qquad = \dfrac{\epsilon^2}{2\tau^2} R \nabla \left(\dfrac{\Delta\sqrt{R}}{\sqrt{R}} \right) + \dfrac{\nu}{\tau^2} \operatorname{div}(R\mathbb{D}U) + \dfrac{\nu\dot{\tau}}{\tau} \nabla R. & \text{(1.5b)} \end{cases}$$

Since the change of unknowns (1.4) preserves the integrability properties of density and velocity unknowns locally in time (we consider velocity *and* space momenta), we focus in the whole paper on system (1.5).

An interesting feature of (1.5) is that it is again associated with a natural energy dissipation estimate, but the new energy involved in this estimate is sign-definite and provides important controls for the unknowns. Indeed, as exploited in [11], the energy associated to (1.5) reads

$$\mathcal{E}(R,U) = \frac{1}{2\tau^2} \int_{\mathbb{R}^d} \left(R|U|^2 + \epsilon^2 \left| \nabla\sqrt{R} \right|^2 \right) + \int_{\mathbb{R}^d} \left(R|y|^2 + R\log R \right), \tag{1.6}$$

so that, formally, solutions to (1.5) satisfy the energy equality

$$\mathcal{E}(R,U)(t) + \int_0^t \mathcal{D}(R,U)(s)\mathrm{d}s = \mathcal{E}\left(R_0, U_0\right) - \nu \int_0^t \frac{\dot{\tau}}{\tau^3} \int_{\mathbb{R}^d} R \operatorname{div} U, \tag{1.7}$$

for $t \geqslant 0$, where the nonnegative dissipation is given by

$$\mathcal{D}(R,U) = \frac{\dot{\tau}}{\tau^3} \int_{\mathbb{R}^d} \left(R|U|^2 + \epsilon^2 \left| \nabla\sqrt{R} \right|^2 \right) + \frac{\nu}{\tau^4} \int_{\mathbb{R}^d} R|\mathbb{D}U|^2. \tag{1.8}$$

In view of the conservation of mass, $\|R(t)\|_{L^1} = \|\Gamma\|_{L^1} = \pi^d$ for all $t \geqslant 0$, we see that the functional $\mathcal{E}$ is positive by writing

$$\int_{\mathbb{R}^d} \left(R|y|^2 + R\log R \right) = \int_{\mathbb{R}^d} R \log \frac{R}{\Gamma} \geqslant \frac{1}{2\pi^d} \|R - \Gamma\|_{L^1}^2,$$

where the last inequality stems from Csiszár-Kullback inequality (see e.g. [1, Theorem 8.2.7]).

The construction of a positive-definite energy which is dissipated with time is a first building-block to construct solutions to (1.5). However, it is classical in compressible fluid mechanics that (1.7) must be completed. For instance, studies on compactness of finite-energy solutions to (1.5) require to handle the viscous stress $R\mathbb{D}U$. Yet, the information provided by (1.7) is insufficient (when $\epsilon = 0$) to pass to the limit in this term (see e.g. [7, 29]), because we lack information on the regularity of the density R. More specifically, in the case of (1.5), with (1.7) alone, it is not clear also how to define the Korteweg term when $\epsilon > 0$. Another important quantity, known as BD-entropy, introduced in [4, 7], is now standard to handle these difficulties. In the case of (1.5), it reads

$$\mathcal{E}_{\mathrm{BD}}(R,U) = \frac{1}{2\tau^2}\int_{\mathbb{R}^d}\left(R\left|U+\nu\nabla\log R\right|^2+\epsilon^2\left|\nabla\sqrt{R}\right|^2\right)+\int_{\mathbb{R}^d}\left(R|y|^2+R\log R\right).$$

Exactly as above, the second integral defines a non-negative functional. The evolution of this BD-entropy is given formally by

$$\begin{aligned}\mathcal{E}_{\mathrm{BD}}(R,U)(t)&+\int_0^t\mathcal{D}_{\mathrm{BD}}(R,U)(s)\mathrm{d}s\\ &=\mathcal{E}_{\mathrm{BD}}(R_0,U_0)+\nu\int_0^t\frac{2d}{\tau^2}\int_{\mathbb{R}^d}R+\nu\int_0^t\frac{\dot\tau}{\tau^3}\int_{\mathbb{R}^d}R\operatorname{div}U,\end{aligned}\tag{1.9}$$

for $t\geqslant 0$, where the above dissipation is defined by

$$\begin{aligned}\mathcal{D}_{\mathrm{BD}}(R,U)=\frac{\dot\tau}{\tau^3}\int\left(R|U|^2+\epsilon^2\left|\nabla\sqrt{R}\right|^2\right)&+\frac{\nu}{\tau^4}\int_{\mathbb{R}^d}R|\mathbb{A}U|^2\\ &+\frac{\nu\epsilon^2}{\tau^4}\int R\left|\nabla^2\log R\right|^2+\frac{4\nu}{\tau^2}\int\left|\nabla\sqrt{R}\right|^2,\end{aligned}\tag{1.10}$$

with $\mathbb{A}U:=\frac{1}{2}(\nabla U-\nabla U^\top)$ the skew-symmetric part of ∇U. Hence putting together the energy and the BD-entropy equalities, it holds

$$\begin{aligned}\mathcal{E}(t)+\mathcal{E}_{\mathrm{BD}}(t)&+\int_0^t\left(\mathcal{D}(s)+\mathcal{D}_{\mathrm{BD}}(s)\right)\mathrm{d}s\\ &=\mathcal{E}(0)+\mathcal{E}_{\mathrm{BD}}(0)+\nu\int_0^t\frac{2d}{\tau^2}\int_{\mathbb{R}^d}R,\end{aligned}\tag{1.11}$$

for $t\geqslant 0$, and thanks to the conservation of mass and the fact that $\int_0^\infty\tau^{-2}(t)\,\mathrm{d}t<\infty$, the last term is uniformly bounded. We note that, in view of (1.9), we gain information on the regularity of R when $\nu>0$ which may help in the compactness issue of weak solutions to (1.5). To define the Korteweg term, we may also apply the classical identity:

$$R\nabla\left(\frac{\Delta\sqrt{R}}{\sqrt{R}}\right)=\operatorname{div}\left(\sqrt{R}\nabla^2\sqrt{R}-\nabla\sqrt{R}\otimes\nabla\sqrt{R}\right),\tag{1.12}$$

in view of

$$\begin{aligned}\int_\Omega\left|\nabla^2\sqrt{R}\right|^2+\int_\Omega\left|\nabla R^{1/4}\right|^4&\lesssim\int_\Omega R\left|\nabla^2\log R\right|^2\\ &\lesssim\int_\Omega\left|\nabla^2\sqrt{R}\right|^2+\int_\Omega\left|\nabla R^{1/4}\right|^4,\end{aligned}\tag{1.13}$$

which holds true for $\Omega=\mathbb{R}^d$ or $\mathbb{T}^d$ (see [21, 29]).

The estimates provided by the above energy and BD-entropy turn out to be fundamental in the construction of a weak solution, and motivate the following definition:

Definition 1.1. — *Assume $\nu > 0$ and $\epsilon \geqslant 0$. Let $(\sqrt{R_0}, \Lambda_0 = (\sqrt{R}U)_0) \in L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. We call global weak solution to* (1.5)*, associated to the initial data $(\sqrt{R_0}, \Lambda_0 = (\sqrt{R}U)_0)$, any pair (R, U) such that there exists a collection $(\sqrt{R}, \sqrt{R}U, \mathbf{S}_K, \mathbf{T}_N)$ satisfying*

(i) *The following regularities:*

$$\begin{aligned}(\langle y\rangle + |U|)\sqrt{R} \in L^\infty_{\mathrm{loc}}\left(0,\infty; L^2\left(\mathbb{R}^d\right)\right), \quad \nabla\sqrt{R} \in L^\infty_{\mathrm{loc}}\left(0,\infty; L^2\left(\mathbb{R}^d\right)\right),\\ \epsilon\nabla^2\sqrt{R} \in L^2_{\mathrm{loc}}\left(0,\infty; L^2\left(\mathbb{R}^d\right)\right), \quad \sqrt{\epsilon}\nabla R^{1/4} \in L^4_{\mathrm{loc}}\left(0,\infty; L^4\left(\mathbb{R}^d\right)\right),\\ \mathbf{T}_N \in L^2_{\mathrm{loc}}\left(0,\infty; L^2\left(\mathbb{R}^d\right)\right),\end{aligned}$$

with the compatibility conditions

$$\sqrt{R} \geqslant 0 \text{ a.e. on } (0,\infty)\times\mathbb{R}^d, \quad \sqrt{R}U = 0 \text{ a.e. on } \left\{\sqrt{R} = 0\right\}.$$

(ii) *The following equations in $\mathcal{D}'((0,\infty)\times\mathbb{R}^d)$*

$$\begin{cases}\partial_t\sqrt{R} + \dfrac{1}{\tau^2}\operatorname{div}\left(\sqrt{R}U\right) = \dfrac{1}{2\tau^2}\operatorname{Trace}\left(\mathbf{T}_N\right),\\ \partial_t(RU) + \dfrac{1}{\tau^2}\operatorname{div}\left(\sqrt{R}U\otimes\sqrt{R}U\right) + 2y\left|\sqrt{R}\right|^2 + \nabla\left(\left|\sqrt{R}\right|^2\right)\\ \qquad\qquad = \operatorname{div}\left(\dfrac{\nu}{\tau^2}\sqrt{R}\mathbf{S}_N + \dfrac{\epsilon^2}{2\tau^2}\mathbf{S}_K\right) + \dfrac{\nu\dot\tau}{\tau}\nabla R,\end{cases} \tag{1.14}$$

with $\mathbf{S}_N$ the symmetric part of $\mathbf{T}_N$ and the compatibility conditions:

$$\sqrt{R}\mathbf{T}_N = \nabla\left(\sqrt{R}\sqrt{R}U\right) - 2\sqrt{R}U\otimes\nabla\sqrt{R}, \tag{1.15}$$

$$\mathbf{S}_K = \sqrt{R}\nabla^2\sqrt{R} - \nabla\sqrt{R}\otimes\nabla\sqrt{R}. \tag{1.16}$$

(iii) *For any $\psi \in C^\infty_0(\mathbb{R}^d)$,*

$$\lim_{t\to 0}\int_{\mathbb{R}^d}\sqrt{R}(t,y)\psi(y)\,\mathrm{d}y = \int_{\mathbb{R}^d}\sqrt{R_0}(y)\psi(y)\,\mathrm{d}y,$$

$$\lim_{t\to 0}\int_{\mathbb{R}^d}\sqrt{R}(t,y)\left(\sqrt{R}U\right)(t,y)\psi(y)\,\mathrm{d}y = \int_{\mathbb{R}^d}\sqrt{R_0}(y)\Lambda_0(y)\psi(y)\,\mathrm{d}y.$$

A specific feature of the previous statement is that we define weak solutions to (1.5) in terms of $\sqrt{R}$ and $\sqrt{R}U$. This is related to the fact that these are the natural quantities that are involved in the energy and entropy estimates. By construction, we shall have $\sqrt{R}U = 0$ where $\sqrt{R} = 0$ so that, whenever U is mentioned, it should be understood as:

$$U = \frac{\sqrt{R}U}{\sqrt{R}} \mathbf{1}_{\sqrt{R} > 0}.$$

Also, thanks to the regularity estimates obtained on the density, the above weak formulation implies the classical continuity equation (see [11, Lemma 2.2]). On the other hand, we mention that a solution $(\sqrt{R}, \sqrt{R}U)$ in the sense of distributions enjoying the regularity of (i) satisfies furthermore that $\sqrt{R} \in C([0,\infty), L^2(\mathbb{R}^d) - w)$ and $RU \in C([0,\infty); L^1(\mathbb{R}^d) - w)$. Consequently, we may require the initial conditions in terms of item (iii)). Finally, we do not claim for an energy estimate in our definition, however we shall derive these solutions from approximate finite-energy, finite-entropy solutions, so that the global weak solutions we construct satisfy: There exist absolute constants C, C' such that, for almost all $t \geqslant 0$, there holds:

$$\mathcal{E}(t) + \int_0^t \mathcal{D}(s)\,\mathrm{d}s \leqslant C(\mathcal{E}(0)), \tag{1.17}$$

$$\mathcal{E}_{\mathrm{BD}}(t) + \int_0^t \mathcal{D}_{\mathrm{BD}}(s)\,\mathrm{d}s \leqslant C'\left(\mathcal{E}(0), \mathcal{E}_{\mathrm{BD}}(0)\right), \tag{1.18}$$

with $\mathcal{E}, \mathcal{D}, \mathcal{E}_{\mathrm{BD}}, \mathcal{D}_{\mathrm{BD}}$ as defined in (1.6)-(1.8)-(1.9)-(1.10). In terms of our weak solutions, the term $R|\mathbb{D}U|^2$ appearing in these estimates must be understood as $|\mathbf{S}_N|^2$ (and, similarly, $R|\mathbb{A}U|^2$ as $|\mathbf{T}_N - \mathbf{S}_N|^2$, and $R|\nabla U|^2$ as $|\mathbf{T}_N|^2$). In addition, item (i) along with (1.14) imply the conservation of mass,

$$\int_{\mathbb{R}^d} R(t,y)\mathrm{d}y = \int_{\mathbb{R}^d} R_0(y)\mathrm{d}y, \quad \forall\, t \geqslant 0,$$

which is hence fixed through all the paper. The extra integral terms present on the right hand side of (1.7) and (1.9) do not appear in the estimates (1.17) and (1.18): thanks to Cauchy-Schwarz inequality, and the conservation of mass, they can be controlled by the dissipation $\mathcal{D}$ (see [11, Remark 2.13] as well as the proof of Proposition 2.6 below). Note that in the previous definition, the entropy of R is not mentioned. The reason is the following lemma.

LEMMA 1.2. — *Let $d \geqslant 1$. For all $M > 0$, there exists $C(M)$ such that for all $f \in H^1 \cap \mathcal{F}(H^1)(\mathbb{R}^d)$ satisfying*

$$\int_{\mathbb{R}^d} \left(1 + |y|^2\right) |f(y)|^2 \mathrm{d}y + \int_{\mathbb{R}^d} |\nabla f(y)|^2\,\mathrm{d}y \leqslant M,$$

the $L\log L$ norm of $|f|^2$ is controlled by

$$\int_{\mathbb{R}^d} |f(y)|^2 \left|\log\left(|f(y)|^2\right)\right| \mathrm{d}y \leqslant C(M).$$

Sketch of proof. — We distinguish the regions where $|f|$ is smaller or larger than one,

$$\begin{aligned}\int_{\mathbb{R}^d} &|f(y)|^2 \left|\log\left(|f(y)|^2\right)\right| \mathrm{d}y\\ &\leqslant \int_{|f|<1} |f(y)|^2 \left|\log\left(|f(y)|^2\right)\right| \mathrm{d}y + \int_{|f|>1} |f(y)|^2 \left|\log\left(|f(y)|^2\right)\right| \mathrm{d}y\\ &\lesssim \int_{\mathbb{R}^d} |f(y)|^{2-\beta} \mathrm{d}y + \int_{\mathbb{R}^d} |f(y)|^{2+\beta} \mathrm{d}y,\end{aligned}$$

where $\beta > 0$ is arbitrarily small. We then invoke the localization estimate in the former region,

$$\int_{\mathbb{R}^d} |f|^{2-\beta} \leqslant C_\beta \|f\|_{L^2}^{2-\beta-d\beta/2} \||y|f\|_{L^2}^{d\beta/2}, \quad 0<\beta<\frac{4}{d+2},$$

which is easily established by distinguishing the regions $|y|<\kappa$ and $|y|>\kappa$, introducing $|y|^2/|y|^2$ in the latter, using Hölder inequality, and eventually optimizing in κ. We may take $\beta = \frac{2}{d+2}$, and the term $\int |f|^{2+\beta}$ is then controlled by the H^1-norm of f thanks to Sobolev embedding. □

Of course if $H^1 \cap \mathcal{F}(H^1)$ is replaced by H^1, the above Lemma 1.2 is no longer true. In view of the above discussion, we will apply this lemma to $\sqrt{R}$. Recalling that the presence of a space momentum is natural when working with the unknown (R,U) (due to (1.5b), implying the definition (1.6)), this yields another motivation for working with (R,U) instead of (ϱ,u): we definitely gain coercivity properties.

With the above definition, the main result of this paper reads:

Theorem 1.3. — *Assume $\nu>0$, $\epsilon \geqslant 0$. Let $(\sqrt{R_0}, \Lambda_0 = (\sqrt{R}U)_0) \in L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d)$ satisfy $\mathcal{E}(0)<\infty$, $\mathcal{E}_{\mathrm{BD}}(0)<\infty$, as well as the compatibility conditions*

$$\sqrt{R_0} \geqslant 0 \text{ a.e. on } \mathbb{R}^d, \quad \left(\sqrt{R}U\right)_0 = 0 \text{ a.e. on } \left\{\sqrt{R_0}=0\right\}.$$

There exists at least one global weak solution to (1.5)*, which satisfies moreover the energy and BD-entropy inequalities* (1.17) *and* (1.18)*.*

In view of [11], we readily infer the following corollary:

Corollary 1.4. — *Under the assumptions of Theorem 1.3, every global weak solution to* (1.5) *enjoying the energy inequality* (1.17) *satisfies*

$$R(t) \rightharpoonup \Gamma \quad \text{in } L^1\left(\mathbb{R}^d\right), \text{ as } t\to\infty.$$

To construct solutions of (1.5), we consider various levels of approximation, by resuming the approach of [28] (summarized in [26]) in the case $\gamma > 1$. The first approximation consists in adding two new terms in the left hand side of (1.5b), leading to more dissipation, hence better a priori estimates,

$$\frac{r_0}{\tau^2}U + \frac{r_1}{\tau^2}R|U|^2U.$$

This yields the following system in $\mathbb{R}^d$, for $r_0, r_1 \geqslant 0$:

$$\begin{cases} \partial_t R + \dfrac{1}{\tau^2}\operatorname{div}(RU) = 0 & (1.19a) \\ \partial_t(RU) + \dfrac{1}{\tau^2}\operatorname{div}(RU\otimes U) + 2yR + \nabla R + \dfrac{r_0}{\tau^2}U + \dfrac{r_1}{\tau^2}R|U|^2U & (1.19b) \\ \qquad = \dfrac{\epsilon^2}{2\tau^2}R\nabla\left(\dfrac{\Delta\sqrt{R}}{\sqrt{R}}\right) + \dfrac{\nu}{\tau^2}\operatorname{div}(R\mathbb{D}U) + \dfrac{\nu\dot{\tau}}{\tau}\nabla R. \end{cases}$$

When $r_0, r_1 > 0$ we call this system the *isothermal fluid system with drag forces*, whereas when $r_0 = r_1 = 0$ we recover the original system (1.5). When the factor $1/\tau^2$ is absent, these terms correspond to physical models; see e.g. [3, 6] and references therein.

The change of unknown functions (1.4) involves a time-dependent spatial rescaling, an aspect which essentially forces us to consider the geometrical framework $x \in \mathbb{R}^d$. On the other hand, construction of weak solutions in the context of compressible fluid mechanics is often performed in the periodic case $x \in \mathbb{T}^d$: this geometry provides compactness in space more easily, and integrations by parts are harmless. The periodic case is also rather convenient for approximating, among others in Lebesgue spaces, the initial density by a density bounded away from zero (see (2.7) below), a step which would be more delicate on $\mathbb{R}^d$. Note also that this property is classically propagated by the flow in a suitable regularized continuity equation (see e.g. [18, 21]), and such a property is needed in the presence of cold pressure and regularizing terms (see e.g. [20, 29]). For these reasons, the second step in our approach consists in replacing $\mathbb{R}^d$ with a box $\mathbb{T}^d_\ell$ of size $\ell > 0$, where ℓ is aimed at going to infinity at the last step of the construction of solutions to the system with drag forces (1.19) with $r_0, r_1 > 0$. The most delicate step turns out to be the adaptation of the initial data, given on $\mathbb{R}^d$, in order to fit in the periodic framework. Details are given in Section 4.

We also emphasize another important difference whether the space variable belongs to $\mathbb{T}^d$ or to $\mathbb{R}^d$. In the former case, it is possible to overcome the lack of positivity in the energy (1.2) by introducing an intermediary constant density, as in e.g. [8, 9, 21]. This strategy cannot be carried out in the case $x \in \mathbb{R}^d$, since no non-zero constant belongs to $L^1(\mathbb{R}^d)$.

To solve (1.19) on the torus $\mathbb{T}^d_\ell$, we proceed as in [29] and introduce regularizing terms in (1.19a) and (1.19b). This procedure consists in adapting to this damped quantum Navier-Stokes system the classical extra-diffusion terms introduced in the classical references [19, 24]. The regularized system hence becomes

$$\left\{\begin{aligned}
&\partial_t R + \frac{1}{\tau^2}\operatorname{div}(RU) = \frac{\delta_1}{\tau^2}\Delta R, && \text{(1.20a)}\\
&\partial_t(RU) + \frac{1}{\tau^2}\operatorname{div}(RU\otimes U) + 2yR + \nabla R - \eta_1 \nabla R^{-\alpha} && \text{(1.20b)}\\
&\qquad + \frac{r_0}{\tau^2}U + \frac{r_1}{\tau^2}R|U|^2 U + \frac{\delta_1}{\tau^2}(\nabla R\cdot\nabla)U\\
&\qquad = \frac{\epsilon^2}{2\tau^2}R\nabla\left(\frac{\Delta\sqrt{R}}{\sqrt{R}}\right) + \frac{\nu}{\tau^2}\operatorname{div}(R\mathbb{D}U) + \frac{\nu\dot\tau}{\tau}\nabla R\\
&\qquad + \frac{\delta_2}{\tau^2}\Delta^2 U + \frac{\eta_2}{\tau^2}R\nabla\Delta^{2s+1}R,
\end{aligned}\right.$$

where the regularization parameters verify $0 < \delta_1, \delta_2, \eta_1, \eta_2 < 1$; $\alpha, s > 0$ are chosen sufficiently large (to be fixed later on); and the drag forces parameters r_0, r_1 as well as the Korteweg parameter ϵ are positive $r_0, r_1, \epsilon > 0$. Such solutions are constructed in Section 2.1. Next, passing to the limit $\delta_1, \delta_2 \to 0$, then $\eta_1, \eta_2 \to 0$, we obtain a solution to the system with drag forces (1.19) with $r_0, r_1, \epsilon > 0$ on the torus $\mathbb{T}^d_\ell$. This is achieved in Section 3.

To pass to the limits $\theta \to 0$, where $\theta > 0$ measures the fact that the initial density is bounded away from zero (see (2.7)), $r_0, r_1 \to 0$ and $\ell \to \infty$ (simultaneously), we proceed as in [22], and consider an adapted notion of renormalized solutions, which is equivalent to our notion of weak solution in the presence of drag forces terms, and provides a weak solution when $r_0 = r_1 = 0$. We thus obtain a solution to (1.5) on the whole space. Note that this step has to be the final one, insofar as the case with drag forces requires to control $r_0 (\log R)_-$ in L^1 (see e.g. [29]), which is inconsistent with the property $\sqrt{R} \in H^1$ in the case $y \in \mathbb{R}^d$. These steps are performed in Section 4.

We note that these final limits, $\theta \to 0$, $r_0, r_1 \to 0$, and $\ell \to \infty$ could be performed in a more independent fashion, by letting first $\theta, r_0, r_1 \to 0$, thus obtaining a global weak solution to (1.5) on $\mathbb{T}^d_\ell$, and then letting $\ell \to \infty$ (recalling that $H^1 \cap \mathcal{F}(H^1)$ provides more compactness than the mere H^1 space). We choose to unify these steps in order to shorten the overall presentation, and also since (1.5) is meaningful on $\mathbb{R}^d$ in view of (1.4), but not necessarily on a (time-independent) torus.

We explain now the outcome of our main theorem in terms of the initial system (1.1). This is the content of the following corollary:

Corollary 1.5. — *Assume $\nu > 0$ and $\epsilon \geqslant 0$. Let $(\sqrt{\varrho_0}, \lambda_0 = (\sqrt{\varrho}u)_0) \in H^1 \cap \mathcal{F}(H^1)(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ satisfy the compatibility conditions*

$$\sqrt{\varrho_0} \geqslant 0 \text{ a.e. on } \mathbb{R}^d, \quad \left(\sqrt{\varrho}u\right)_0 = 0 \text{ a.e. on } \{\sqrt{\varrho_0} = 0\},$$

and assume that the associated functions $(\sqrt{R_0}, \Lambda_0 = (\sqrt{R}U)_0)$ obtained via (1.4) satisfy $\mathcal{E}(0) < \infty$ and $\mathcal{E}_{\mathrm{BD}}(0) < \infty$. Then there exists a global weak solution to (1.1) in the following sense: there exists a collection $(\sqrt{\varrho}, \sqrt{\varrho}u, \mathbb{T}_N, \mathbb{S}_K)$ such that

(i) *The following regularities are satisfied:*

$$\begin{aligned}(\langle x\rangle + |u|)\sqrt{\varrho} \in L^\infty_{\mathrm{loc}}\left(0,\infty; L^2\left(\mathbb{R}^d\right)\right), &\quad \nabla\sqrt{\varrho} \in L^\infty_{\mathrm{loc}}\left(0,\infty; L^2(\mathbb{R}^d)\right),\\ \epsilon\nabla^2\sqrt{\varrho} \in L^2_{\mathrm{loc}}\left(0,\infty; L^2\left(\mathbb{R}^d\right)\right), &\quad \sqrt{\epsilon}\nabla\varrho^{1/4} \in L^4_{\mathrm{loc}}\left(0,\infty; L^4\left(\mathbb{R}^d\right)\right),\\ \mathbb{T}_N \in L^2_{\mathrm{loc}}\left(0,\infty; L^2\left(\mathbb{R}^d\right)\right), &\end{aligned}$$

with the compatibility conditions

$$\sqrt{\varrho} \geqslant 0 \text{ a.e. on } (0,\infty)\times\mathbb{R}^d, \quad \sqrt{\varrho}u = 0 \text{ a.e. on } \left\{\sqrt{\varrho} = 0\right\}.$$

(ii) *The following equations hold in $\mathcal{D}'((0,\infty)\times\mathbb{R}^d)$*

$$\left\{\begin{aligned}&\partial_t\sqrt{\varrho} + \operatorname{div}\left(\sqrt{\varrho}u\right) = \tfrac{1}{2}\operatorname{Trace}\left(\mathbb{T}_N\right),\\ &\partial_t\left(\sqrt{\varrho}\sqrt{\varrho}u\right) + \operatorname{div}\left(\sqrt{\varrho}u\otimes\sqrt{\varrho}u\right) + \nabla\left(\left|\sqrt{\varrho}\right|^2\right)\\ &\qquad\qquad = \operatorname{div}\left(\tfrac{\nu}{\tau^2}\sqrt{\varrho}\mathbb{S}_N + \tfrac{\epsilon^2}{2}\mathbb{S}_K\right),\end{aligned}\right. \tag{1.21}$$

with $\mathbb{S}_N$ the symmetric part of $\mathbb{T}_N$ and the compatibility conditions:

$$\sqrt{\varrho}\mathbb{T}_N = \nabla\left(\sqrt{\varrho}\sqrt{\rho}u\right) - 2\sqrt{\varrho}u\otimes\nabla\sqrt{\varrho}, \tag{1.22}$$

$$\mathbb{S}_K = \sqrt{\varrho}\nabla^2\sqrt{\varrho} - \nabla\sqrt{\varrho}\otimes\nabla\sqrt{\varrho}. \tag{1.23}$$

(iii) *For any $\psi \in C_0^\infty(\mathbb{R}^d)$,*

$$\lim_{t\to 0}\int_{\mathbb{R}^d}\sqrt{\varrho}(t,x)\psi(x)\,\mathrm{d}x = \int_{\mathbb{R}^d}\sqrt{\varrho_0}(x)\psi(x)\,\mathrm{d}x,$$

$$\lim_{t\to 0}\int_{\mathbb{R}^d}\sqrt{\varrho}(t,x)\left(\sqrt{\varrho}u\right)(t,x)\psi(x)\,\mathrm{d}x = \int_{\mathbb{R}^d}\sqrt{\varrho_0}(x)\lambda_0(x)\psi(x)\,\mathrm{d}x.$$

The main shortcoming of this construction is that we do not get the energy inequality corresponding to (1.2) for the initial system (but the regularity obtained ensures that, at any time $t \geqslant 0$, the energy $E(t)$ is well defined). Indeed, we remark that, if U should be going to 0 at infinity, then, our solution u would then be a perturbation of the affine velocity field $(\dot{\tau}/\tau)x$ which increases at infinity. In particular, performing back the change of variable (1.4) in the energy estimate (1.17), in the case $\|\rho_0\|_{L^1(\mathbb{R}^d)} = \|\Gamma\|_{L^1(\mathbb{R}^d)}$ we obtain:

$$\begin{aligned} \frac{1}{2}\left[\int_{\mathbb{R}^d} \rho(t,x)\left|u - \frac{\dot{\tau}}{\tau}x\right|^2 \mathrm{d}x + \int_{\mathbb{R}^d} \left|\nabla\sqrt{\rho}(t,x)\right|^2 \mathrm{d}x\right] \\ + \int_{\mathbb{R}^d} \rho(t,x)\ln(\rho(t,x))\mathrm{d}x + d\left(\ln(\tau(t)) + \frac{1}{\tau(t)^2}\right)\int_{\mathbb{R}} \rho(t,x)\mathrm{d}x \\ + \int_0^t \left[\int_{\mathbb{R}^d} \frac{\dot{\tau}}{\tau}\rho\left|u - \frac{\dot{\tau}}{\tau}x\right|^2 \mathrm{d}x + \nu\int_{\mathbb{R}^d} \rho\left|\mathbb{D}u - \frac{\dot{\tau}}{\tau}\right|^2\right]\mathrm{d}x\mathrm{d}s \leqslant C_0. \end{aligned}$$

Another point of view consists in recalling that in [11], the large time convergence of the second order momentum of R is established by using the a priori bounds provided by (1.17), *and* the information that the energy E defined in (1.2) is $o(\log t)$ as $t \to \infty$: even though this information is weaker than the expected boundedness of E (and even, decay), it seems to be needed in the proof, suggesting that either some tools are missing in the study of (R, U) to recover the energy inequality corresponding to (1.2) for the initial system, or that it is just not possible.

We complement the above results, valid for $\nu > 0$, with a global existence result in the case of the isothermal Korteweg equation ($\epsilon > 0$ and $\nu = 0$). The proof is fairly different from the case $\nu > 0$, since it is based on nonlinear Schrödinger equations, but is rather short. We choose to present this case so that the family of results in this paper is consistent. Mimicking [2, Definition 14], we set:

Definition 1.6. — *Let $d \geqslant 1$. Assume $\nu = 0$ and $\epsilon > 0$. Let $(\sqrt{\varrho_0}, \lambda_0) \in L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. We call global weak solution to* (1.1)*, associated to the initial data $(\sqrt{\varrho_0}, \lambda_0)$, any pair $(\sqrt{\varrho}, \sqrt{\varrho}u)$ such that if we define $\varrho := (\sqrt{\varrho})^2$, $j := \sqrt{\varrho} \times \sqrt{\varrho}u$, then we have:*

(i) *The following regularities:*

$$\sqrt{\varrho} \in L^\infty_{\mathrm{loc}}\left(0,\infty; H^1\left(\mathbb{R}^d\right)\right), \quad \sqrt{\varrho}u \in L^\infty_{\mathrm{loc}}\left(0,\infty; L^2\left(\mathbb{R}^d\right)\right),$$

with the compatibility condition

$$\sqrt{\varrho} \geqslant 0 \text{ a.e. on } (0,\infty)\times\mathbb{R}^d, \quad \sqrt{\varrho}u = 0 \text{ a.e. on } \{\varrho = 0\}.$$

(ii) *For every $T > 0$, for any test function $\varphi \in C_0^\infty([0,T[\times\mathbb{R}^d)$,*

$$\int_0^T\int_{\mathbb{R}^d}\left(\varrho\partial_t\varphi + j\cdot\nabla\varphi\right)\mathrm{d}t\mathrm{d}x + \int_{\mathbb{R}^d}\varrho_0\varphi(0)\mathrm{d}x = 0,$$

and for any test function $\eta \in C_0^\infty([0,T[\times\mathbb{R}^d;\mathbb{R}^d)$,

$$\int_0^T\int_{\mathbb{R}^d}\Big(j\cdot\partial_t\eta + (\sqrt{\varrho}u)\otimes(\sqrt{\varrho}u):\nabla\eta + \nabla\varrho\operatorname{div}\eta \\ + \epsilon^2\nabla\sqrt{\varrho}\otimes\nabla\sqrt{\varrho}:\nabla\eta - \frac{\epsilon^2}{4}\varrho\Delta\operatorname{div}\eta\Big)\mathrm{d}t\mathrm{d}x + \int_{\mathbb{R}^d}\lambda_0\cdot\eta(0)\mathrm{d}x = 0.$$

(iii) *(Generalized irrotationality condition) For almost every $t \geqslant 0$,*

$$\nabla\wedge j = 2\nabla\sqrt{\varrho}\wedge(\sqrt{\varrho}u)$$

holds in the sense of distributions.

Note that in the second point, the quantum pressure (right hand side of (1.1b)) has been recast in view of (1.12). Like before, whenever u is mentioned, it should be understood as

$$u = \frac{\sqrt{\varrho}u}{\sqrt{\varrho}}\mathbf{1}_{\sqrt{\varrho}>0}.$$

The generalized irrotationality condition, explained in [2, Remark 2], is the generalization of the property $\varrho\nabla\wedge u = 0$ of the smooth case $j = \varrho u$, to the notion of weak solution.

Also, Definition 1.6 is readily adapted to the case of (1.5) in the following statement. The first part of this result is the analogue of [2, Proposition 15] in the isothermal case.

Proposition 1.7. — *Let $d \geqslant 1$. Assume $\nu = 0$ and $\epsilon > 0$. Let $\psi_0 \in H^1\cap\mathcal{F}(H^\alpha)(\mathbb{R}^d)$ for some $0 < \alpha \leqslant 1$, and assume that the initial data for* (1.1) *are* well-prepared *in the sense that*

$$\varrho_0 = |\psi_0|^2, \quad j_0 = \epsilon\operatorname{Im}\left(\bar{\psi}_0\nabla\psi_0\right).$$

(1) *Then there exists a global weak solution to* (1.1)*. Furthermore, the energy $E(t)$ defined by* (1.2) *is conserved for all time $t \geqslant 0$.*
(2) *If $\psi_0 \in H^1\cap\mathcal{F}(H^1)(\mathbb{R}^d)$, then $(\sqrt{R},\sqrt{R}U)$ defined by*

$$\begin{aligned}\sqrt{\varrho}(t,x) &= \frac{1}{\tau(t)^{d/2}}\sqrt{R}\left(t,\frac{x}{\tau(t)}\right)\left(\frac{\|\varrho_0\|_{L^1}}{\|\Gamma\|_{L^1}}\right)^{1/2},\\ \sqrt{\varrho}u(t,x) &= \frac{1}{\tau(t)}\sqrt{R}U\left(t,\frac{x}{\tau(t)}\right)\left(\frac{\|\varrho_0\|_{L^1}}{\|\Gamma\|_{L^1}}\right)^{1/2} + \frac{\dot{\tau}(t)}{\tau(t)}x\sqrt{\varrho(t,x)},\end{aligned} \tag{1.24}$$

is a global weak solution to (1.5)*. The pseudo-energy* $\mathcal{E}$*, defined in* (1.6)*, solves* (1.7)*, where the dissipation is given by* (1.8)*. Equivalently, setting*

$$\mathcal{E} = \frac{1}{2\tau^2}\int_{\mathbb{R}^d}\left(\left|\sqrt{R}U\right|^2 + \epsilon^2\left|\nabla\sqrt{R}\right|^2\right) + \int_{\mathbb{R}^d}\left(R|y|^2 + R\log R\right),$$

$$\mathcal{D} = \frac{\dot\tau}{\tau^3}\int_{\mathbb{R}^d}\left|\sqrt{R}U\right|^2 + \epsilon^2\left|\nabla\sqrt{R}\right|^2,$$

we have

$$\mathcal{E}(t) + \int_0^t \mathcal{D}(s)\mathrm{d}s = \mathcal{E}(0), \quad \forall\, t \geqslant 0.$$

The proof of Proposition 1.7 relies on properties of the logarithmic Schrödinger equation, which is the natural candidate to provide solutions to (1.1), as opposed to the nonlinear Schrödinger equation with power-like nonlinearity in the polytropic case. The specificity of this nonlinearity explains the presence of a (fractional) momentum in the first part of the statement. We emphasize the fact that the special structure of the initial data (due to the use of Madelung transform) implies that the flow is irrotational (see also the last point of Definition 1.6 and [2, Remark 2] where it is discussed). In view of [11], we readily infer the following corollary, which is stronger than Corollary 1.4:

Corollary 1.8. — *In the second case of Proposition 1.7, every such global weak solution satisfies*

$$\int_{\mathbb{R}^d}\begin{pmatrix}1\\ y\\ |y|^2\end{pmatrix} R(t,y)\mathrm{d}y \to \int_{\mathbb{R}^d}\begin{pmatrix}1\\ y\\ |y|^2\end{pmatrix}\Gamma(y)\mathrm{d}y$$

and

$$R(t) \rightharpoonup \Gamma \quad \textit{in } L^1\left(\mathbb{R}^d\right), \textit{ as } t\to\infty.$$

Remark 1.9. — In view of the proof of Proposition 1.7, [13, Theorem 1.12] implies that Proposition 1.7 and its corollary (from [11]) remain valid in the case where the above pressure law $p(\varrho) = \varrho$ is replaced for instance by

$$p(\varrho) = c_0\varrho + \sum_{j=1}^N c_j\varrho^{\gamma_j}, \quad c_j > 0,\ 0 \leqslant j \leqslant N, \quad 1 < \gamma_j < \frac{d+2}{(d-2)_+}.$$

Remark 1.10. — Since our reformulation of (1.1) in terms of the unknowns (R,U) provides extra positivity properties, one may ask if the isothermal case can be obtained as the limit $\gamma\to 1$ in the barotropic case, where the pressure law is $p(\varrho) = \varrho^\gamma$, $\gamma > 1$. A first aspect is that such a limit might be possible only *locally in time*, for as proven in [11] (isothermal case) and [12] (barotropic case), ϱ enjoys dispersive properties with a rate that changes precisely for the value $\gamma = 1$. For bounded time, it is plausible that the limit $\gamma \to 1$ might be handled in terms of (R, U) (adapted to the case $\gamma > 1$) when $\epsilon > 0$ because of a further uniform bound $\sqrt{R} \in H^1(\mathbb{R}^d)$ due to the Korteweg term. On the other hand, having proven Theorem 1.3, one may ask if the solutions from Proposition 1.7 can be obtained through the inviscid limit $\nu \to 0$. Such a convergence has been proven in [8] for the barotropic case, and [17] for the (damped) isothermal case, both times in a periodic setting $x \in \mathbb{T}^d$. The damping in [17] can easily be removed, but in order to consider the case $x \in \mathbb{R}^d$, the order of the limits $\ell \to \infty$ and $\nu \to 0$ is certainly a delicate issue, which we leave out at this stage. Finally, both limits $\gamma \to 1$ and $\nu \to 0$ seem highly singular when $\epsilon = 0$ (or goes simultaneously to 0) even in terms of (R, U). Concerning the limit $\gamma \to 1$ for instance, the estimates established in [12] are then not uniform in γ.

Organization of the paper

Until the end of Section 4, we assume $\nu > 0$. In Section 2, we construct solutions to (1.20) on the torus $\mathbb{T}^d_\ell$ with strictly positive densities. In Section 3, we obtain solutions to (1.19) in the presence of drag forces, $r_0, r_1 > 0$, by passing to the limit $\delta_1, \delta_2, \eta_1, \eta_2 \to$ in (1.20). Theorem 1.3 is proved in Section 4, where we let $r_0, r_1 \to 0$ and $\ell \to \infty$ (with possibly $\epsilon \to 0$). Section 5 is devoted to the proof of Proposition 1.7 ($\nu = 0$, $\epsilon > 0$). In Appendix A, we give more details about the derivation of an identity appearing in Section 4.

2. Construction of solutions to the regularized system

We start this study by constructing weak solutions to the system (1.20) on the torus $\mathbb{T}^d_\ell$ with strictly positive densities and deriving further properties satisfied by these solutions. We recall that in system (1.20) the parameters $r_0, r_1, \epsilon > 0$ are positive, which will be hence assumed through this section.

System (1.20) is endowed with some estimates. We first note that, integrating (1.20a) we obtain the conservation of mass:

$$\int_{\mathbb{T}^d_\ell} R(t) = \int_{\mathbb{T}^d_\ell} R_0. \tag{2.1}$$

Then, by multiplying formally (1.20b) with U/τ^2 and combining with equation (1.20a), we obtain that reasonable solutions to (1.20) should satisfy the energy estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{\mathrm{reg}}(R,U)+\mathcal{D}_{\mathrm{reg}}(R,U)=\frac{2d\delta_1}{\tau^2}\int_{\mathbb{T}^d_\ell}R-\frac{\nu\dot\tau}{\tau^3}\int_{\mathbb{T}^d_\ell}R\,\mathrm{div}\,U, \tag{2.2}$$

where

$$\begin{aligned}\mathcal{E}_{\mathrm{reg}}(R,U)=\frac{1}{2\tau^2}&\int_{\mathbb{T}^d_\ell}\left(R|U|^2+\epsilon^2\left|\nabla\sqrt{R}\right|^2\right)\\
&+\int_{\mathbb{T}^d_\ell}\left(R|y|^2+R\log R+\frac{\eta_1}{\alpha+1}R^{-\alpha}\right)\\
&+\frac{\eta_2}{2\tau^2}\int_{\mathbb{T}^d_\ell}|\nabla\Delta^s R|^2,\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}&\mathcal{D}_{\mathrm{reg}}(R,U)\\
&\quad=\frac{\dot\tau}{\tau^3}\int_{\mathbb{T}^d_\ell}\left(R|U|^2+\epsilon^2\left|\nabla\sqrt{R}\right|^2+\eta_2\left|\nabla\Delta^s R\right|^2\right)+\frac{\nu}{\tau^4}\int_{\mathbb{T}^d_\ell}R|\mathbb{D}U|^2\\
&\qquad+\frac{\delta_2}{\tau^4}\int_{\mathbb{T}^d_\ell}|\Delta U|^2+\frac{\delta_1\eta_2}{\tau^4}\int_{\mathbb{T}^d_\ell}\left|\Delta^{s+1}R\right|^2+\frac{4\delta_1}{\tau^2}\int_{\mathbb{T}^d_\ell}\left|\nabla\sqrt{R_N}\right|^2\\
&\qquad+\frac{4\delta_1\eta_1}{\alpha\tau^2}\int_{\mathbb{T}^d_\ell}\left|\nabla R^{-\alpha/2}\right|^2+\frac{r_0}{\tau^4}\int_{\mathbb{T}^d_\ell}|U|^2+\frac{r_1}{\tau^4}\int_{\mathbb{T}^d_\ell}R|U|^4\\
&\qquad+\frac{\delta_1\epsilon^2}{2\tau^4}\int_{\mathbb{T}^d_\ell}R\left|\nabla^2\log R\right|^2.\end{aligned}$$

Note that the term appearing on the last line is obtained thanks to the exact formula:

$$\frac{1}{2}\int R\left|\nabla^2\log R\right|^2=\int\frac{\Delta\sqrt{R}}{\sqrt{R}}\Delta R.$$

On the other hand, multiplying formally (1.20a) by a smooth function Ψ and (1.20b) by a smooth vector field Φ yields respectively

$$\begin{aligned}\int_{\mathbb{T}^d_\ell}R_0\Psi(0)+\int_0^T\int_{\mathbb{T}^d_\ell}R\partial_t\Psi+\int_0^T\int_{\mathbb{T}^d_\ell}\frac{1}{\tau^2}RU\cdot\nabla\Psi&\\
+\delta_1\int_0^T\int_{\mathbb{T}^d_\ell}\frac{1}{\tau^2}R\Delta\Psi=0,&\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}(2.5)\quad &\int_{\mathbb{T}_\ell^d} R_0U_0\Phi(0) + \int_0^T\int_{\mathbb{T}_\ell^d} RU\cdot\partial_t\Phi + \int_0^T\int_{\mathbb{T}_\ell^d} \frac{1}{\tau^2}RU\otimes U : \nabla\Phi \\ &= \int_0^T\int_{\mathbb{T}_\ell^d} R\left(2y\cdot\Phi - \operatorname{div}\Phi\right) + r_0\int_0^T\int_{\mathbb{T}_\ell^d}\frac{1}{\tau^2}U\cdot\Phi \\ &+ r_1\int_0^T\int_{\mathbb{T}_\ell^d}\frac{1}{\tau^2}R|U|^2U\cdot\Phi \\ &+ \epsilon^2\int_0^T\int_{\mathbb{T}_\ell^d}\frac{1}{2\tau^2}\left[2\Delta\sqrt{R}\nabla\sqrt{R}\cdot\Phi + \Delta\sqrt{R}\sqrt{R}\operatorname{div}\Phi\right] \\ &+ \nu\int_0^T\int_{\mathbb{T}_\ell^d}\frac{1}{\tau^2}R\mathbb{D}U : \nabla\Phi + \nu\int_0^T\int_{\mathbb{T}_\ell^d}\frac{\dot\tau}{\tau}R\operatorname{div}\Phi \\ &+ \delta_1\int_0^T\int_{\mathbb{T}_\ell^d}\frac{1}{\tau^2}\nabla U : \nabla R\otimes\Phi + \delta_2\int_0^T\int_{\mathbb{T}_\ell^d}\frac{1}{\tau^2}\Delta U\cdot\Delta\Phi \\ &+ \eta_1\int_0^T\int_{\mathbb{T}_\ell^d} R^{-\alpha}\operatorname{div}\Phi \\ &+ \eta_2\int_0^T\int_{\mathbb{T}_\ell^d}\frac{1}{\tau^2}\Delta^{s+1}R\Delta^s\left[\nabla R\cdot\Phi + R\operatorname{div}\Phi\right].\end{aligned}$$

So, to define weak solutions to (1.20), we look for minimal regularity assumptions that are induced by energy estimate (2.2) and which make (2.4)-(2.5) meaningful for smooth test-functions. For this, we first recall the following lemma – which is reminiscent of [5, Lemma 2.1] with a slightly different statement – to estimate negative power of the density which naturally appear in the formulation (1.20):

Lemma 2.1. — *For $n\in\mathbb{N}^*$ and $\Omega=\mathbb{T}^d$ or $\Omega=\mathbb{R}^d$, there holds*

$$\begin{aligned}&\left\|\nabla^n\left(f^{-1}\right)\right\|_{L^2(\Omega)} \\ &\qquad\lesssim\left(1+\left\|f^{-1}\right\|_{L^4(\Omega)}+\left\|f^{-1}\right\|_{L^{2(n+1)}(\Omega)}\right)^{n+1}\left(1+\|f\|_{H^\sigma(\Omega)}\right)^n\end{aligned}$$

with $\sigma>n+d/2$.

Proof. — Recall the embedding $H^{d/2+0}(\Omega)\hookrightarrow L^\infty(\Omega)$. We compute

$$\left|\nabla^n\left(f^{-1}\right)\right|^2\lesssim\sum_{j=1}^n\sum_{i_1+\cdots+i_j=n}\frac{\left|\nabla^{i_1}f\right|^2\cdots\left|\nabla^{i_j}f\right|^2}{f^{2(j+1)}},$$

hence, for any $j \geqslant 1$, we have:

$$\begin{aligned}
&\int \frac{\left|\nabla^{i_1} f\right|^2 \cdots \left|\nabla^{i_j} f\right|^2}{f^{2(j+1)}} \,\mathrm{d}x \\
&\quad \lesssim \left\|\nabla^{i_1} f\right\|_{L^\infty(\Omega)}^2 \cdots \left\|\nabla^{i_j} f\right\|_{L^\infty(\Omega)}^2 \int f^{-2(j+1)} \,\mathrm{d}x \\
&\quad \lesssim \|f\|_{H^\sigma(\Omega}^{2j} \|f^{-1}\|_{L^{2(j+1)}}^{2(j+1)} \\
&\quad \lesssim \left(1 + \|f\|_{H^\sigma(\Omega)}\right)^{2n} \left(1 + \left\|f^{-1}\right\|_{L^{2(j+1)}(\Omega)}\right)^{2(n+1)} \\
&\quad \lesssim \left(1 + \|f\|_{H^\sigma(\Omega)}\right)^{2n} \left(1 + \|f^{-1}\|_{L^4(\Omega)} + \left\|f^{-1}\right\|_{L^{2(n+1)}(\Omega)}\right)^{2(n+1)},
\end{aligned}$$

which completes the proof of Lemma 2.1. □

Since $\mathcal{E}_{\mathrm{reg}}$ enables to control the H^{2s+1}-norm of R together with the mean of $R^{-\alpha}$, we may infer that, for $\alpha > 4$ and $s > d$, the energy estimate (2.2) implies that $1/R$ is continuous. We also recall that the Laplace equation on the torus enjoys classical elliptic estimate so that the dissipation $\mathcal{D}_{\mathrm{reg}}$ (note that $r_0, \delta_2 > 0$) yields $U \in L^2_{\mathrm{loc}}(\mathbb{R}^+; H^2(\mathbb{T}^d_\ell))$. Introducing the regularity expected for R and U into the continuity equation (1.20a) entails that $\partial_t R \in L^2_{\mathrm{loc}}(\mathbb{R}^+; H^1(\mathbb{T}^d_\ell))$. Then, our definition of weak solution to (1.20) reads as follows:

Definition 2.2. — *Given $(R_0, U_0) \in L^1(\mathbb{T}^d_\ell) \times L^2(\mathbb{T}^d_\ell)$, we say that (R, U) is a global weak solution to (1.20) associated to the initial data (R_0, U_0) if we have:*

(i) *(R, U) satisfies*

$$\begin{aligned}
&R \in H^1_{\mathrm{loc}}\left(\mathbb{R}^+; H^1\left(\mathbb{T}^d_\ell\right)\right) \cap C\left(\mathbb{R}^+; H^{2s}\left(\mathbb{T}^d_\ell\right)\right) \\
&\qquad\qquad\qquad \cap L^2_{\mathrm{loc}}\left(\mathbb{R}^+; H^{2s+2}\left(\mathbb{T}^d_\ell\right)\right), \\
&1/R \in C\left(\mathbb{R}^+ \times \mathbb{T}^d_\ell\right), \\
&U \in L^\infty_{\mathrm{loc}}\left(\mathbb{R}^+; L^2\left(\mathbb{T}^d_\ell\right)\right) \cap L^2_{\mathrm{loc}}\left(\mathbb{R}^+; H^2\left(\mathbb{T}^d_\ell\right)\right).
\end{aligned} \tag{2.6}$$

(ii) *Equation (2.4) holds true for any $\Psi \in \mathcal{D}([0,\infty) \times \mathbb{T}^d_\ell)$.*

(iii) *Equation (2.5) holds true for any $\Phi \in \mathcal{D}([0,\infty) \times \mathbb{T}^d_\ell)^d$.*

Remark 2.3. — Thanks to the above remarks, the regularity statement (i) is sufficient to obtain that all the terms in (2.4)-(2.5) are well-defined.

In this section, we restrict to initial data with smooth and strictly positive density. This means that we shall assume that (R_0, U_0) satisfy:

$$R_0 \in \mathcal{D}\left(\mathbb{T}^d_\ell\right), \quad U_0 \in L^2\left(\mathbb{T}^d_\ell\right), \quad \inf_{y \in \mathbb{T}^d_\ell} R_0(y) \geqslant \theta > 0. \tag{2.7}$$

The first main result of this section is the following proposition:

Proposition 2.4. — *Given initial data* (R_0, U_0) *satisfying* (2.7), *there exists a global solution* (R, U) *to* (1.20) *associated to* (R_0, U_0) *on the torus* $\mathbb{T}_\ell^d$, *which satisfies moreover the conservation of mass* (2.1) *and the energy estimate, for a.e.* $T \geqslant 0$,

$$\mathcal{E}_{\mathrm{reg}}(R,U)(T) + \int_0^T \mathcal{D}_{\mathrm{reg}}(R,U)(s)\mathrm{d}s \leqslant C_0\left(\mathcal{E}_{\mathrm{reg}}\left(R_0,U_0\right)\right), \tag{2.8}$$

for some constant $C_0 > 0$ *depending on* $\mathcal{E}_{\mathrm{reg}}(R_0, U_0)$.

Remark 2.5. — We note that the energy estimate (2.8) together with (2.1) entail that the solution we construct enjoys the following regularity properties, with norms corresponding to these spaces bounded with respect to $\mathcal{E}_{\mathrm{reg}}(R_0, U_0)$ only:

$$\begin{aligned}
R\left(1+|y|^2+|\log R|\right) &\in L^\infty_{\mathrm{loc}}\left(\mathbb{R}^+; L^1\left(\mathbb{T}_\ell^d\right)\right),\\
\sqrt{R}U &\in L^\infty_{\mathrm{loc}}\left(\mathbb{R}^+; L^2\left(\mathbb{T}_\ell^d\right)\right),\\
\sqrt{\nu}\,\sqrt{R}\mathbb{D}U &\in L^2_{\mathrm{loc}}\left(\mathbb{R}^+; L^2\left(\mathbb{T}_\ell^d\right)\right),\quad \epsilon\nabla\sqrt{R}\in L^\infty_{\mathrm{loc}}\left(\mathbb{R}^+; L^2\left(\mathbb{T}_\ell^d\right)\right),\\
\sqrt{r_0}\,U &\in L^2_{\mathrm{loc}}\left(\mathbb{R}^+; L^2\left(\mathbb{T}_\ell^d\right)\right),\ \sqrt{r_1}\,R^{\frac14}U\in L^4_{\mathrm{loc}}\left(\mathbb{R}^+; L^4\left(\mathbb{T}_\ell^d\right)\right),\\
\sqrt{\delta_2}\,\Delta U &\in L^2_{\mathrm{loc}}\left(\mathbb{R}^+; L^2\left(\mathbb{T}_\ell^d\right)\right),\ \sqrt{\eta_2}\,R\in L^\infty_{\mathrm{loc}}\left(\mathbb{R}^+; H^{2s+1}\left(\mathbb{T}_\ell^d\right)\right),\\
\eta_1^{\frac{1}{\alpha}}\,R^{-1} &\in L^\infty_{\mathrm{loc}}\left(\mathbb{R}^+; L^\alpha\left(\mathbb{T}_\ell^d\right)\right),\\
\sqrt{\delta_1\eta_1}\,\nabla R^{-\frac{\alpha}{2}} &\in L^2_{\mathrm{loc}}\left(\mathbb{R}^+; L^2\left(\mathbb{T}_\ell^d\right)\right),\\
\sqrt{\nu\epsilon^2}\nabla^2\sqrt{R} &\in L^2_{\mathrm{loc}}\left(\mathbb{R}^+; L^2\left(\mathbb{T}_\ell^d\right)\right),\ \left(\nu\epsilon^2\right)^{\frac14}\nabla R^{\frac14}\in L^4_{\mathrm{loc}}\left(\mathbb{R}^+; L^4\left(\mathbb{T}_\ell^d\right)\right),\\
\sqrt{\delta_1\eta_2}\,\Delta^{s+1}R &\in L^2_{\mathrm{loc}}\left(\mathbb{R}^+; L^2\left(\mathbb{T}_\ell^d\right)\right).
\end{aligned}$$

We refer to (1.13) for the regularity claim on the before-last line. Also, combining these bounds with Lemma 2.1, we obtain that, for arbitrary $T > 0$, there exists a $C(\mathcal{E}_{\mathrm{reg}}(R_0, U_0), \eta_1, \eta_2, \theta, T) > 0$ so that

$$\|1/R\|_{L^\infty\left((0,T)\times\mathbb{T}_\ell^d\right)} \leqslant C\left(\mathcal{E}_{\mathrm{reg}}(R_0,U_0),\eta_1,\eta_2,\theta,T\right). \tag{2.9}$$

The proof of Proposition 2.4 is the content of the next subsection. Then in the last subsection, we focus on a further estimate satisfied by the weak solutions that we construct.

2.1. Proof of Proposition 2.4.

The plan of the proof follows closely the method of [29]. In the whole section (R_0, U_0) is a fixed initial data satisfying (2.7).

2.1.1. Faedo-Galerkin approximation

Let $X_N = \operatorname{span}\{e_1, \dots, e_N\}$ be the finite-dimensional space corresponding to the projection in $L^2(\mathbb{T}^d_\ell)$ onto the first N Fourier modes. We consider the system whose unknowns are

$$(R_N, U_N) \in C\left(\mathbb{R}^+; H^{2s+1}\left(\mathbb{T}^d_\ell\right)\right) \times C\left(\mathbb{R}^+; X_N\right),$$

and composed by (1.20a) and the following weak formulation of (1.20b): for any $t \in (0,T)$ and any vector field $\phi \in (X_N)^d$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d_\ell} R_N U_N \cdot \phi &- \frac{1}{\tau^2}\int_{\mathbb{T}^d_\ell} R_N U_N \otimes U_N : \nabla\phi \\
&+ \int_{\mathbb{T}^d_\ell} R_N \left(2y\cdot\phi - \operatorname{div}\phi\right) \\
&+ \frac{r_0}{\tau^2}\int_{\mathbb{T}^d_\ell} U_N \cdot \phi \, dy + \frac{r_1}{\tau^2}\int_{\mathbb{T}^d_\ell} R_N |U_N|^2 U_N \cdot \phi \\
&+ \frac{\delta_1}{\tau^2}\int_{\mathbb{T}^d_\ell} \left(\left[\nabla R_N \cdot \nabla\right] U_N\right)\cdot\phi \\
&+ \frac{\epsilon^2}{2\tau^2}\int_{\mathbb{T}^d_\ell} \left[2\Delta\sqrt{R}_N \nabla\sqrt{R}_N \phi + \Delta\sqrt{R}_N\sqrt{R}_N \operatorname{div}\phi\right] \\
&+ \frac{\nu}{\tau^2}\int_{\mathbb{T}^d_\ell} R_N \mathbb{D}U_N : \nabla\phi \\
&+ \frac{\nu\dot{\tau}}{\tau}\int_{\mathbb{T}^d_\ell} R_N \operatorname{div}\phi + \frac{\delta_2}{\tau^2}\int_{\mathbb{T}^d_\ell} \Delta U_N \cdot \Delta\phi + \eta_1 \int_{\mathbb{T}^d_\ell} R_N^{-\alpha}\operatorname{div}\phi \\
&- \frac{\eta_2}{\tau^2}\int_{\mathbb{T}^d_\ell} R_N \nabla\Delta^{2s+1} R_N \cdot \phi = 0,
\end{aligned} \tag{2.10}$$

where we recall that $r_0, r_1, \epsilon > 0$. We complement the system with initial conditions:

$$\begin{gathered}
R_N|_{t=0} = R_0, \\
\left[\int_{\mathbb{T}^d_\ell} R_N U_N \cdot \phi\right]\Bigg|_{t=0} = \int_{\mathbb{T}^d_\ell} R_0 U_0 \cdot \phi, \qquad \forall\, \phi \in (X_N)^d.
\end{gathered} \tag{2.11}$$

We have the following existence result for this approximate system:

Proposition 2.6. — *Given $N \in \mathbb{N}^*$, there exists a global solution (R_N, U_N) to* (1.20a)-(2.10)-(2.11) *that satisfies the conservation of mass* (2.1) *and the energy inequality*

$$\sup_{t \in (0,T)} \mathcal{E}_{\mathrm{reg}}\left(R_N, U_N\right) + \int_0^T \mathcal{D}_{\mathrm{reg}}\left(R_N, U_N\right) \mathrm{d}t \leqslant C\left(\mathcal{E}_{\mathrm{reg}}\left(R_N, U_N\right)|_{t=0}\right), \tag{2.12}$$

for come constant $C > 0$ *depending on* $\mathcal{E}_{\mathrm{reg}}(R_N, U_N)|_{t=0}$.

Proof. — The local existence is obtained following [29] (see also [21]). The novelties with respect to this previous study are: the linearity of the pressure term, the time factors $\tau, \dot{\tau}$ and the new terms

$$\int_{\mathbb{T}^d_\ell} R\left(2y \cdot \phi - \operatorname{div} \phi\right), \quad \frac{r_0}{\tau^2} \int_{\mathbb{T}^d_\ell} U \cdot \phi, \quad \frac{\nu \dot{\tau}}{\tau} \int_{\mathbb{T}^d_\ell} R \operatorname{div} \phi.$$

However, these terms are harmless in the fixed-point approach of [29, Section 2], for instance.

The global existence is then a consequence of the energy estimate that we obtain as follows. Conservation of mass follows by integrating (1.20a). We may then take $\phi = U_N(t)/\tau^2(t)$ in (2.10) since it corresponds to writing the N equations obtained by setting $\phi = e_j$, $j = 1, \ldots, N$, and combining them with the coefficients defining U_N in this basis. This yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}_{\mathrm{reg}}(R_N, U_N) + \mathcal{D}_{\mathrm{reg}}(R_N, U_N) = \frac{2d\delta_1}{\tau^2} \int_{\mathbb{T}^d_\ell} R_N - \frac{\nu \dot{\tau}}{\tau^3} \int_{\mathbb{T}^d_\ell} R_N \operatorname{div} U_N. \tag{2.13}$$

We deduce the energy inequality by remarking that the right-hand side of (2.13) can be bounded by

$$\left(\frac{2d\delta_1}{\tau^2} + C \frac{\nu \dot{\tau}^2}{\tau^2}\right) \int_{\mathbb{T}^d_\ell} R_N + \frac{\nu}{2\tau^4} \int_{\mathbb{T}^d_\ell} R_N \left|\mathbb{D} U_N\right|^2 \leqslant C \frac{\left(1 + \dot{\tau}^2\right)}{\tau^2} \int_{\mathbb{T}^d_\ell} R_N + \frac{1}{2} \mathcal{D}_{\mathrm{reg}}\left(R_N, U_N\right),$$

using the conservation of mass together with

$$\int_0^\infty \frac{1 + \dot{\tau}^2(t)}{\tau^2(t)} \, \mathrm{d}t < \infty,$$

and recalling that $\mathcal{E}_{\mathrm{reg}}$ is nonnegative. □

2.1.2. Convergence of the approximate solutions

We split the proof into three steps: defining limits to the sequence of approximate solutions (R_N, U_N), improving the sense in which this sequence converges, passing to the limit in the weak formulation (2.10). In all the convergences mentioned in the proof, we have to extract subsequences that we do not relabel for conciseness.

Proof of Proposition 2.4. — So, let $\{(R_N, U_N)\}_N$ be the sequence of approximate solutions to (1.20a)-(2.10)-(2.11) given by Proposition 2.6. We note that we have initially $R_N(0,\cdot) = R_0$ and $R_N U_N(0,\cdot) = \mathbb{P}_N[R_0 U_0]$ where $\mathbb{P}_N$ stands for the $(L^2(\mathbb{T}^d_\ell))$-projection onto X_N. In particular, since by assumption $R_0 U_0 \in L^2(\mathbb{T}^d_\ell)$, we have

$$\mathcal{E}_{\mathrm{reg}}(R_N, U_N)|_{t=0} \leqslant \mathcal{E}_{\mathrm{reg}}(R_0, U_0). \tag{2.14}$$

Step 1. — From (2.14) and the energy inequality derived in Proposition 2.6, we infer that

$$\sup_{t\geqslant 0} \mathcal{E}_{\mathrm{reg}}(R_N, U_N) + \int_0^\infty \mathcal{D}_{\mathrm{reg}}(R_N, U_N) \leqslant C(\mathcal{E}_{\mathrm{reg}}(R_0, U_0)), \quad \forall\ N.$$

We obtain then uniform bounds on (R_N, U_N) in a series of spaces similar to the ones in Remark 2.5. We first extract from this list that we have uniform bounds with respect to N for:

$$\frac{1}{\tau}\sqrt{\eta_2} R_N \text{ in } L^\infty\left(\mathbb{R}^+; H^{2s+1}\left(\mathbb{T}^d_\ell\right)\right),$$
$$\left(\frac{\eta_1}{\alpha+1}\right)^{\frac{1}{\alpha}} \frac{1}{R_N} \text{ in } L^\infty\left(\mathbb{R}^+; L^\alpha\left(\mathbb{T}^d_\ell\right)\right),$$
$$\frac{1}{\tau}\sqrt{R_N} U_N \text{ in } L^\infty\left(\mathbb{R}^+; L^2\left(\mathbb{T}^d_\ell\right)\right).$$

Using the first bound, we can extract a subsequence so that R_N/τ converges to some R/τ in this same space (for the weak-$*$ topology). From the last bound, we obtain that (up to the extraction of a subsequence) $\sqrt{R_N} U_N/\tau$ converges to some V/τ in $L^\infty(\mathbb{R}^+; L^2(\mathbb{T}^d_\ell)) - w*$. Restricting to any time interval $(0,T)$ with $T < \infty$, the second bound with the first one and Lemma 2.1 imply that R_N is uniformly bounded from below on $(0,T)$ by a constant $C(\mathcal{E}_{\mathrm{reg}}(R_0, U_0), \eta_1, \eta_2, \theta, T)$. Hence, we have also

$$R \geqslant C\left(\mathcal{E}_{\mathrm{reg}}(R_0, U_0), \eta_1, \eta_2, \theta, T\right) \quad \text{in} \quad (0,T), \tag{2.15}$$

and we may set $U = V/\sqrt{R}$. We focus now on the restriction of these limits on $(0,T)$.

Step 2. — On $(0,T)$, we establish convergences of R_N and U_N in a stronger sense.

To this end, we now extract from the list given by Remark 2.5 uniform bounds for

$$\begin{aligned} R_N &\text{ in } L^\infty\left(0,T;H^{2s+1}\left(\mathbb{T}^d_\ell\right)\right)\cap L^2\left(0,T;H^{2s+2}\left(\mathbb{T}^d_\ell\right)\right), \\ 1/R_N &\text{ in } L^\infty\left((0,T)\times\mathbb{T}^d_\ell\right), \\ U_N &\text{ in } L^2\left(0,T;H^2\left(\mathbb{T}^d_\ell\right)\right). \end{aligned}$$

The continuity equation (1.20a) satisfied by R_N implies then that $\partial_t R_N$ is bounded in $L^2(0,T;H^1(\mathbb{T}^d_\ell))$. Combining classical weak-convergence results and Ascoli-Arzelà type arguments entails that:

$$\begin{aligned} &R_N \to R \text{ in } C\left([0,T];H^{2s}\left(\mathbb{T}^d_\ell\right)\right), \\ &R_N \rightharpoonup R \text{ in } L^2\left(0,T;H^{2s+2}\left(\mathbb{T}^d_\ell\right)\right)-w, \\ &R_N \rightharpoonup R \text{ in } H^1\left(0,T;H^1\left(\mathbb{T}^d_\ell\right)\right)-w. \end{aligned} \tag{2.16}$$

Given the bound by below on R_N (2.15), we also have that $1/R_N$ converges to $1/R$ in $C([0,T]\times\mathbb{T}^d_\ell)$.

Next, given the uniform bounds for U_N and R_N, and since $(e_k)_{k\in\mathbb{N}}$ is orthogonal for the H^2-scalar product, we have that R_NU_N and $\mathbb{P}_N[R_NU_N]$ are uniformly bounded in $L^2(0,T;H^2(\mathbb{T}^d_\ell))$ too. On the other hand, the weak formulation satisfied by the approximation (R_N,U_N) reads:

$$\begin{aligned} &\partial_t\left(\mathbb{P}_N\left[R_NU_N\right]\right) \\ &= \mathbb{P}_N\Bigg[-\frac{1}{\tau^2}\operatorname{div}\left(R_NU_N\otimes U_N\right) - 2yR_N - \nabla R_N + \eta_1\nabla R_N^{-\alpha} + \frac{r_0}{\tau^2}U_N \\ &\quad + \frac{r_1}{\tau^2}R_N|U_N|^2U_N + \frac{\delta_1}{\tau^2}\left(\nabla R_N\cdot\nabla\right)U_N + \frac{\epsilon^2}{2\tau^2}R_N\nabla\left(\frac{\Delta\sqrt{R_N}}{\sqrt{R_N}}\right) \\ &\quad + \frac{\nu}{\tau^2}\operatorname{div}\left(R_N\mathbb{D}U_N\right) + \frac{\nu\dot\tau}{\tau}\nabla R_N + \frac{\delta_2}{\tau^2}\Delta^2U_N + \frac{\eta_2}{\tau^2}R_N\nabla\Delta^{2s+1}R_N\Bigg] \\ &=: \mathbb{P}_N\left[F_N\right] \end{aligned}$$

Again we note here that $\mathbb{P}_N$ is orthogonal with respect to the H^s-scalar product, so that

$$\|\mathbb{P}_NF_N\|_{H^{-s}\left(\mathbb{T}^d_\ell\right)} \leqslant \|F_N\|_{H^{-s}\left(\mathbb{T}^d_\ell\right)}, \quad \forall\ s\in\mathbb{N}.$$

For s sufficiently large, we may then combine the various uniform estimates satisfied by (R_N,U_N) on $(0,T)$ to infer that $\partial_t(\mathbb{P}_N[R_NU_N])$ is uniformly bounded in $L^2(0,T;H^{-(2s+2)}(\mathbb{T}^d_\ell))$. To prove this, the main terms to be discussed are $\operatorname{div}(R_N U_N \otimes U_N)$ and $R_N|U_N|^2 U_N$ which can be handled (since $d \leqslant 3$) via the embedding $H^2(\mathbb{T}_\ell^d) \subset L^\infty(\mathbb{T}_\ell^d)$. To summarize, we know that $\mathbb{P}_N[R_N U_N]$ is bounded in $L^2(0,T;H^2(\mathbb{T}_\ell^d))$ and $\partial_t(\mathbb{P}_N[R_N U_N])$ is bounded in $L^2(0,T;H^{-(2s+2)}(\mathbb{T}_\ell^d))$. Aubin–Lions like arguments imply then that $\mathbb{P}_N[R_N U_N]$ converges in $L^2(0,T;H^1(\mathbb{T}_\ell^d))$. Due to the compactness of the embedding $H^2(\mathbb{T}_\ell^d) \subset H^1(\mathbb{T}_\ell^d)$ again, there exists a sequence $(\varepsilon_N)_N$ converging to 0 so that

$$\left\|\mathbb{P}_N\left[R_N U_N\right] - R_N U_N\right\|_{L^2\left(0,T;H^1\left(\mathbb{T}_\ell^d\right)\right)} \leqslant \varepsilon_N \left\|R_N U_N\right\|_{L^2\left(0,T;H^2\left(\mathbb{T}_\ell^d\right)\right)}.$$

Consequently, $(\mathbb{P}_N[R_N U_N])_N$ and $(R_N U_N)_N$ both converge to the vector-field RU in $L^2(0,T;H^1(\mathbb{T}_\ell^d))$. Moreover, since $(1/R_N)_{N\in\mathbb{N}}$ is uniformly bounded and R_N converges to R in a sufficiently regular space, this also implies that

$$U_N \to U \text{ in } L^2\left(0,T;H^1\left(\mathbb{T}_\ell^d\right)\right). \tag{2.17}$$

To end up this part on the convergence of U_N, we note that the uniform estimates satisfied by (R_N, U_N) also entail that U_N is bounded in $L^\infty(0,T;L^2(\mathbb{T}_\ell^d)) \cap L^2(0,T;H^2(\mathbb{T}_\ell^d))$ so that the limit U lies in these spaces.

Step 3. — Given the time-regularity of approximate solutions, R_N and $R_N U_N$ satisfy (2.4) for arbitrary $\Psi \in \mathcal{D}([0,\infty) \times \mathbb{T}_\ell^d)$, and (2.5) for arbitrary $\Phi \in \mathcal{D}([0,\infty);X_N)$, respectively. The two sets of convergence results (2.16) and (2.17) are then sufficient to pass to the limit in these weak formulations. Again, the main difficulty might be here to pass to the limit in $R_N|U_N|^2 U_N$. However, we note that R_N converges in the set of continuous functions while U_N is bounded in $L^\infty_{\mathrm{loc}}((0,\infty);L^2(\mathbb{T}_\ell^d))$ and converges in $L^2_{\mathrm{loc}}((0,\infty);H^1(\mathbb{T}_\ell^d))$ so that, by interpolation, it converges in $L^4_{\mathrm{loc}}((0,\infty);L^3(\mathbb{T}_\ell^d))$. At this point, (R,U) satisfies (2.4) for arbitrary $\Psi \in \mathcal{D}([0,\infty) \times \mathbb{T}_\ell^d)$ and (2.5) for arbitrary $\Phi \in \mathcal{D}([0,\infty);\bigcup_N X_N)$. We note then that for arbitrary $\Phi \in \mathcal{D}([0,\infty) \times \mathbb{T}_\ell^d)$, $\partial_t \mathbb{P}_N[\Phi]$ and $\mathbb{P}_N[\Phi]$ converge to $\partial_t \Phi$ in $C([0,\infty);L^2(\mathbb{T}_\ell^d))$ and Φ in $L^2(0,\infty;H^{2s+2}(\mathbb{T}_\ell^d))$, respectively. This is sufficient to extend (2.5) to arbitrary $\Phi \in \mathcal{D}([0,\infty) \times \mathbb{T}_\ell^d)$.

As for energy estimate, we note that (R_N, U_N) satisfies (2.12) for arbitrary N and the initial data verifies (2.14). Since $\mathcal{E}_{\mathrm{reg}}(R_N, U_N)$ is continuous with respect to topologies for which R_N, U_N converge strongly, while $\mathcal{D}_{\mathrm{reg}}(R_N, U_N)$ is continuous with respect to topologies for which R_N, U_N converge weakly, we obtain that (R,U) satisfies (2.8) in the limit $N \to \infty$. This concludes the proof of Proposition 2.4. □

Remark 2.7. — With arguments similar to the ones in Step 3 of the above proof, we can extend the weak form (2.5) of the momentum equation to any test-function $\Phi \in (L^2(0,T;H^{2s+1}(\mathbb{T}^d_\ell))^d$ having compact support and such that $\partial_t \Phi \in (L^2(0,T;L^2(\mathbb{T}^d_\ell)))^d$.

2.2. Further properties of weak solutions to the regularized problem

Along with the energy estimate (2.8), we only showed that we had a list of regularity properties satisfied by our weak solutions (R,U). Nevertheless, most of these estimates rely on the regularization parameters η_1, η_2, r_0, r_1, etc. In order to let these parameters vanish, we need other estimates on these solutions. This is the motivation of the following lemma:

LEMMA 2.8 (BD-entropy). — *Assume the initial data satisfies* (2.7). *Then there exist constants C_1, C_2, C_3 with dependencies mentioned in parentheses, such that, for arbitrary $T>0$, the global solution (R,U) to* (1.20) *constructed in Proposition 2.4 satisfies*

$$\begin{aligned}(2.18)\qquad \sup_{t\in(0,T)} \mathcal{E}^+_{\mathrm{BD,\,reg}}(R,U)(t) + \int_0^T \mathcal{D}_{\mathrm{BD,\,reg}}(R,U)(t)\,\mathrm{d}t \\ \leqslant C_1\left(\mathcal{E}_{\mathrm{reg}\,|\,t=0}, \mathcal{E}^+_{\mathrm{BD,\,reg}|t=0}\right) \\ + (\delta_1+\delta_2)\,C_2\left(r_0, r_1, \eta_1, \eta_2, \mathcal{E}_{\mathrm{reg}\,|\,t=0}, T\right) + C_3(r_0),\end{aligned}$$

where $\mathcal{E}^+_{\mathrm{BD,\,reg}}$ is the positive part of the BD-entropy defined by

$$\begin{aligned}&\mathcal{E}^+_{\mathrm{BD,\,reg}}(R,U) = \\ &\quad \frac{1}{2\tau^2}\int_{\mathbb{T}^d_\ell}\left(R\,|U+\nu\nabla\log R|^2 + \epsilon^2\left|\nabla\sqrt{R}\right|^2 - 2r_0(\log R)\mathbf{1}_{R\leqslant 1}\right) \\ &\qquad + \int_{\mathbb{T}^d_\ell}\left(R|y|^2 + R\log R + \frac{\eta_1}{\alpha+1}R^{-\alpha}\right) + \frac{\eta_2}{2\tau^2}\int_{\mathbb{T}^d_\ell}|\nabla\Delta^s R|^2,\end{aligned}$$

and its associated nonnegative dissipation is given by

$$
\begin{aligned}
&\mathcal{D}_{\mathrm{BD,\,reg}}(R,U)\\
&= \frac{\dot\tau}{\tau^3}\int_{\mathbb{T}^d_\ell}\!\left(R|U|^2+\epsilon^2\left|\nabla\sqrt{R}\right|^2+\eta_2\left|\nabla\Delta^s R_N\right|^2\right)\!+\frac{2r_0\nu\dot\tau}{\tau^3}\int_{\mathbb{T}^d_\ell}|\log R|\,\mathbf{1}_{R<1}\\
&\quad+\left(\frac{\delta_1\nu^2}{\tau^4}+\frac{\nu\epsilon^2}{\tau^4}+\frac{\delta_1\epsilon^2}{2\tau^4}\right)\int_{\mathbb{T}^d_\ell}R\left|\nabla^2\log R\right|^2+\left(\frac{4\nu}{\tau^2}+\frac{4\delta_1}{\tau^2}\right)\int_{\mathbb{T}^d_\ell}\left|\nabla\sqrt{R}\right|^2\\
&\quad+\left(\frac{\eta_1\nu\alpha}{4\tau^2}+\frac{4\delta_1\eta_1}{10\tau^2}\right)\int_{\mathbb{T}^d_\ell}\left|\nabla R^{-\frac{\alpha}{2}}\right|^2+\frac{\nu}{\tau^4}\int_{\mathbb{T}^d_\ell}R|\mathbb{A}U|^2\\
&\quad+\frac{(\eta_2\nu+\delta_1\eta_2)}{\tau^4}\int_{\mathbb{T}^d_\ell}\left|\Delta^{s+1}R\right|^2+\frac{\delta_2}{\tau^4}\int_{\mathbb{T}^d_\ell}|\Delta U|^2\\
&\quad+\frac{r_0}{\tau^4}\int_{\mathbb{T}^d_\ell}|U|^2+\frac{r_1}{\tau^4}\int_{\mathbb{T}^d_\ell}R|U|^4.
\end{aligned}
$$

Remark 2.9. — Below, we see the positive BD-entropy as the positive part of the complete BD-entropy:

$$
\begin{aligned}
&\mathcal{E}_{\mathrm{BD,\,reg}}(R,U)\\
&\qquad=\frac{1}{2\tau^2}\int_{\mathbb{T}^d_\ell}\left(R\left|U+\nu\nabla\log R\right|^2+\epsilon^2\left|\nabla\sqrt{R}\right|^2-2r_0\log R\right)\\
&\qquad\quad+\int_{\mathbb{T}^d_\ell}\left(R|y|^2+R\log R+\frac{\eta_1}{\alpha+1}R^{-\alpha}\right)+\frac{\eta_2}{2\tau^2}\int_{\mathbb{T}^d_\ell}|\nabla\Delta^s R|^2,
\end{aligned}
$$

and we note that we have then

$$
\mathcal{E}^+_{\mathrm{BD,\,reg}}=\mathcal{E}_{\mathrm{BD,\,reg}}-\mathcal{E}^-_{\mathrm{BD,\,reg}},\quad \mathcal{E}^-_{\mathrm{BD,\,reg}}=-\frac{r_0}{\tau^2}\int_{\mathbb{T}^d_\ell}\log R\,\mathbf{1}_{R>1}.
$$

Proof. — We consider in this proof (R,U) a weak solution to (1.20) constructed in Proposition 2.4. We have

$$
\begin{aligned}
\nabla R\in H^1_{\mathrm{loc}}\left(\mathbb{R}^+;L^2\left(\mathbb{T}^d_\ell\right)\right)&\cap L^\infty_{\mathrm{loc}}\left(\mathbb{R}^+;H^{2s-1}\left(\mathbb{T}^d_\ell\right)\right)\\
&\cap L^2_{\mathrm{loc}}\left(\mathbb{R}^+;H^{2s+1}\left(\mathbb{T}^d_\ell\right)\right),\\
1/R\in H^1_{\mathrm{loc}}\left(\mathbb{R}^+;L^2\left(\mathbb{T}^d_\ell\right)\right)&\cap L^\infty_{\mathrm{loc}}\left(\mathbb{R}^+;H^{2s}\left(\mathbb{T}^d_\ell\right)\right)\\
&\cap L^2_{\mathrm{loc}}\left(\mathbb{R}^+;H^{2s+2}\left(\mathbb{T}^d_\ell\right)\right).
\end{aligned}
$$

For s sufficiently large, we obtain that $\Phi=(\nu\nabla\log R)/\tau^2$ satisfies:

$$
\Phi\in\left(L^2_{\mathrm{loc}}\left(\mathbb{R}^+;H^{2s+1}\left(\mathbb{T}^d_\ell\right)\right)^d,\quad \partial_t\Phi\in L^2_{\mathrm{loc}}\left(\mathbb{R}^+;L^2\left(\mathbb{T}^d_\ell\right)\right)\right)^d.
$$

Hence, for arbitrary $\chi\in\mathcal{D}(0,\infty)$, we can take $\Phi=(\nu\nabla\log R)\chi/\tau^2$ as a test function in the weak formulation of the momentum equation (2.5).

Combining with a standard regularity estimate for (1.20a), we obtain that, in $\mathcal{D}'((0,T))$, there holds:

$$\begin{aligned}(2.19)\quad &\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\nu}{\tau^2}\int_{\mathbb{T}^d_\ell} RU\cdot\nabla\log R\right]+\frac{2\nu\dot{\tau}}{\tau^3}\int_{\mathbb{T}^d_\ell} RU\cdot\nabla\log R\\ &+\frac{\epsilon^2\nu}{\tau^4}\int_{\mathbb{T}^d_\ell} R\left|\nabla^2\log(R)\right|^2+\left(\frac{\nu}{\tau^2}-\frac{\nu^2\dot{\tau}}{\tau^3}\right)\int_{\mathbb{T}^d_\ell} 4\left|\nabla\sqrt{R}\right|^2\\ &+\frac{4\eta_1\nu}{\alpha}\int_{\mathbb{T}^d_\ell}\left|\nabla\sqrt{R^{-\alpha}}\right|^2+\frac{\eta_2\nu}{\tau^4}\int_{\mathbb{T}^d_\ell}\left|\Delta^{s+1}R\right|^2\\ &=\frac{2d\nu}{\tau^2}\int_{\mathbb{T}^d_\ell} R-\frac{r_0\nu}{\tau^4}\int_{\mathbb{T}^d_\ell} U\cdot\nabla\log R-\frac{r_1\nu}{\tau^4}\int_{\mathbb{T}^d_\ell}|U|^2U\cdot\nabla R\\ &-\frac{\nu^2}{\tau^4}\int_{\mathbb{T}^d_\ell} R\mathbb{D}U:\nabla^2\log R\\ &-\frac{\delta_1\nu}{\tau^4}\int_{\mathbb{T}^d_\ell}\nabla U:\nabla R\otimes\nabla\log R-\frac{\delta_2\nu}{\tau^4}\int_{\mathbb{T}^d_\ell}\Delta U\cdot\nabla\Delta\log R\\ &-\frac{\delta_1\nu}{\tau^4}\int_{\mathbb{T}^d_\ell}\frac{\Delta R}{R}\operatorname{div}(RU)+\frac{\nu}{\tau^4}\int_{\mathbb{T}^d_\ell} R\nabla U:\nabla^\top U.\end{aligned}$$

The proof of this identity is mostly technical. More details are provided in Appendix A. On the other hand, differentiating the continuity equation (1.20a) we obtain:

$$\partial_t\left(R\nabla\log R\right)+\frac{1}{\tau^2}\operatorname{div}\left(R\nabla\log R\otimes U\right)+\frac{1}{\tau^2}\operatorname{div}\left(R\nabla^\top U\right)=\frac{\delta_1}{\tau^2}\Delta\nabla R.$$

This identity holds in $L^2_{\mathrm{loc}}(\mathbb{R}^+;L^2(\mathbb{T}^d_\ell))$ so, we can multiply it with a truncation of $\nabla\log R/\tau^2$. This leads to the energy estimate:

$$\begin{aligned}(2.20)\quad &\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2\tau^2}R\left|\nabla\log R\right|^2\right]\\ &\qquad+\frac{\dot{\tau}}{\tau^3}\int_{\mathbb{T}^d_\ell} R\left|\nabla\log R\right|^2+\frac{\delta_1}{2\tau^4}\int_{\mathbb{T}^d_\ell}\Delta R\left|\nabla\log R\right|^2\\ &\qquad\qquad=\frac{1}{\tau^4}\int_{\mathbb{T}^d_\ell} R\nabla U:\nabla^2\log R+\frac{\delta_1}{\tau^4}\int_{\mathbb{T}^d_\ell}\Delta\nabla R\cdot\nabla\log R.\end{aligned}$$

In this last identity, we note that:

$$\begin{aligned}\int_{\mathbb{T}_\ell^d} \Delta\nabla R\cdot\nabla\log R &= -\int_{\mathbb{T}_\ell^d} \nabla^2 R : \nabla^2\log R\\ &= -\int_{\mathbb{T}_\ell^d} \nabla\left(R\nabla\log R\right) : \nabla\log R\\ &= -\frac{1}{2}\int_{\mathbb{T}_\ell^d} \nabla R\cdot\nabla\left|\nabla\log R\right|^2 - \int_{\mathbb{T}_\ell^d} R\left|\nabla^2\log R\right|^2\\ &= \frac{1}{2}\int_{\mathbb{T}_\ell^d} \Delta R\left|\nabla\log R\right|^2 - \int_{\mathbb{T}_\ell^d} R\left|\nabla^2\log R\right|^2.\end{aligned}$$

Consequently, we rewrite the previous energy identity (2.20) as:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2\tau^2}R\left|\nabla\log R\right|^2\right] + \frac{\dot\tau}{\tau^3}\int_{\mathbb{T}_\ell^d} 4\left|\nabla\sqrt{R}\right|^2 + \frac{\delta_1}{\tau^4}\int_{\mathbb{T}_\ell^d} R\left|\nabla^2\log R\right|^2&\\ = \frac{1}{\tau^4}\int_{\mathbb{T}_\ell^d} R\nabla U : \nabla^2\log R.&\end{aligned}\tag{2.21}$$

At this point, we combine (2.19)+ν^2(2.21), which yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\left\{\frac{1}{\tau^2}\int_{\mathbb{T}_\ell^d}\left(\nu RU\cdot\nabla\log R + \frac{\nu^2}{2}R\left|\nabla\log R\right|^2\right)\right\}\\ &+\frac{2\nu\dot\tau}{\tau^3}\int_{\mathbb{T}_\ell^d} RU\cdot\nabla\log R\\ &+\frac{4\nu}{\tau^2}\int_{\mathbb{T}_\ell^d}\left|\nabla\sqrt{R}\right|^2 + \left(\frac{\delta_1\nu^2}{\tau^4}+\frac{\epsilon^2\nu}{\tau^4}\right)\int_{\mathbb{T}_\ell^d} R\left|\nabla^2\log R\right|^2\\ &+\frac{4\eta_1\nu}{\alpha\tau^2}\int_{\mathbb{T}_\ell^d}\left|\nabla R^{-\frac{\alpha}{2}}\right|^2 + \frac{\eta_2\nu}{\tau^4}\int_{\mathbb{T}_\ell^d}\left|\Delta^{s+1}R\right|^2\\ &=\frac{2d\nu}{\tau^2}\int_{\mathbb{T}_\ell^d} R - \frac{r_0\nu}{\tau^4}\int_{\mathbb{T}_\ell^d} U\cdot\nabla\log R - \frac{r_1\nu}{\tau^4}\int_{\mathbb{T}_\ell^d}|U|^2U\cdot\nabla R\\ &\quad-\frac{\nu^2}{\tau^4}\int_{\mathbb{T}_\ell^d} R\mathbb{D}U : \nabla^2\log R + \frac{\nu^2}{\tau^4}\int_{\mathbb{T}_\ell^d} R\nabla U\cdot\nabla^2\log R\\ &\quad-\frac{\delta_1\nu}{\tau^4}\int_{\mathbb{T}_\ell^d}\nabla U : \nabla R\otimes\nabla\log R - \frac{\delta_2\nu}{\tau^4}\int_{\mathbb{T}_\ell^d}\Delta U\cdot\nabla\Delta\log R\\ &\quad-\frac{\delta_1\nu}{\tau^4}\int_{\mathbb{T}_\ell^d}\frac{\Delta R}{R}\operatorname{div}(RU) + \frac{\nu}{\tau^4}\int_{\mathbb{T}_\ell^d} R\nabla U : \nabla^\top U.\end{aligned}$$

Introducing $\mathbb{A}U = \frac{1}{2}(\nabla U - \nabla^\top U)$ the skew-symmetric part of ∇U, the second line of the right-hand side also reads

$$-\frac{\nu^2}{\tau^4}\int_{\mathbb{T}^d_\ell} R\mathbb{D}U : \nabla^2 \log R + \frac{\nu^2}{\tau^4}\int_{\mathbb{T}^d_\ell} R\nabla U \cdot \nabla^2 \log R \\ = \frac{\nu^2}{\tau^4}\int_{\mathbb{T}^d_\ell} R\mathbb{A}U : \nabla^2 \log R = 0,$$

since skew-symmetric and symmetric matrices are orthogonal for the matrix contraction. Remark also that from the continuity equation (1.20a) we get

$$\partial_t(\log R) + \frac{1}{\tau^2}\nabla \log R \cdot U + \frac{1}{\tau^2}\operatorname{div} U = \frac{\delta_1}{\tau^2}\frac{\Delta R}{R},$$

whence

$$-\frac{r_0\nu}{\tau^4}\int_{\mathbb{T}^d_\ell} U \cdot \nabla \log R \\ = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{r_0\nu}{\tau^2}\int_{\mathbb{T}^d_\ell} \log R\right] + \frac{2r_0\nu\dot{\tau}}{\tau^3}\int_{\mathbb{T}^d_\ell} \log R - \frac{r_0\nu\delta_1}{\tau^4}\int_{\mathbb{T}^d_\ell} \frac{\Delta R}{R}.$$

We finally obtain the identity:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{\tau^2}\int_{\mathbb{T}^d_\ell}\left(\nu RU \cdot \nabla \log R + \frac{\nu^2}{2}R|\nabla \log R|^2 - 2r_0\nu \log R\right)\right\} \\ &\quad + \frac{2\nu\dot{\tau}}{\tau^3}\int_{\mathbb{T}^d_\ell}(RU\cdot\nabla\log R - r_0\log R) \\ &\quad + \frac{4\nu}{\tau^2}\int_{\mathbb{T}^d_\ell}\left|\nabla\sqrt{R}\right|^2 + \left(\frac{\delta_1\nu^2}{\tau^4} + \frac{\epsilon^2\nu}{\tau^4}\right)\int_{\mathbb{T}^d_\ell} R\left|\nabla^2\log R\right|^2 \\ &\quad + \frac{4\eta_1\nu}{\alpha\tau^2}\int_{\mathbb{T}^d_\ell}\left|\nabla R^{-\frac{\alpha}{2}}\right|^2 + \frac{\eta_2\nu}{\tau^4}\int_{\mathbb{T}^d_\ell}\left|\Delta^{s+1}R\right|^2 \\ &\quad = \frac{2d\nu}{\tau^2}\int_{\mathbb{T}^d_\ell} R - \frac{r_0\nu\delta_1}{\tau^4}\int_{\mathbb{T}^d_\ell}\frac{\Delta R}{R} - \frac{r_1\nu}{\tau^4}\int_{\mathbb{T}^d_\ell}|U|^2U\cdot\nabla R \\ &\qquad - \frac{\delta_1\nu}{\tau^4}\int_{\mathbb{T}^d_\ell}\nabla U : \nabla R\otimes\nabla\log R - \frac{\delta_2\nu}{\tau^4}\int_{\mathbb{T}^d_\ell}\Delta U\cdot\nabla\Delta\log R \\ &\qquad - \frac{\delta_1\nu}{\tau^4}\int_{\mathbb{T}^d_\ell}\frac{\Delta R}{R}\operatorname{div}(RU) + \frac{\nu}{\tau^4}\int_{\mathbb{T}^d_\ell} R\nabla U : \nabla^\top U. \end{aligned} \tag{2.22}$$

We now integrate this identity with respect to time and combine with (2.2), observing that

$$\int_{\mathbb{T}^d_\ell} R|\mathbb{D}U|^2 - \int_{\mathbb{T}^d_\ell} R\nabla U : \nabla^\top U = \int_{\mathbb{T}^d_\ell} R|\mathbb{A}U|^2.$$

Thus, we obtain (with the notations of Remark 2.9) that, for almost all $T \geqslant 0$,

$$
\begin{aligned}
&\mathcal{E}_{\mathrm{BD,\,reg}}(R,U)(T) + \int_0^T \frac{\dot\tau}{\tau^3}\int_{\mathbb{T}^d_\ell}\left(R|U|^2 + \epsilon^2\left|\nabla\sqrt{R}\right|^2 + \eta_2\left|\nabla\Delta^s R\right|^2\right)\\
&\qquad + 2r_0\nu\int_0^T \frac{\dot\tau}{\tau^3}\int_{\mathbb{T}^d_\ell} |\log R|\,\mathbf{1}_{R<1}\\
&\qquad + \left(\delta_1\nu^2 + \nu\epsilon^2 + \frac{\delta_1\epsilon^2}{2}\right)\int_0^T \frac{1}{\tau^4}\int_{\mathbb{T}^d_\ell} R\left|\nabla^2\log R\right|^2\\
&\qquad + (\nu+\delta_1)\int_0^T \frac{4}{\tau^2}\int_{\mathbb{T}^d_\ell}\left|\nabla\sqrt{R}\right|^2 + (\nu+\delta_1)\int_0^T \frac{4\eta_1}{\tau^2\alpha}\int_{\mathbb{T}^d_\ell}\left|\nabla R^{-\frac{\alpha}{2}}\right|^2\\
&\qquad + \int_0^T \frac{\nu}{\tau^4}\int_{\mathbb{T}^d_\ell} R|\mathbb{A}U|^2 + \int_0^T \frac{(\eta_2\nu+\delta_1\eta_2)}{\tau^4}\int_{\mathbb{T}^d_\ell}\left|\Delta^{s+1}R\right|^2\\
&\qquad + \int_0^T \frac{\delta_2}{\tau^4}\int_{\mathbb{T}^d_\ell}|\Delta U|^2 + \int_0^T \frac{r_0}{\tau^4}\int_{\mathbb{T}^d_\ell}|U|^2 + \int_0^T \frac{r_1}{\tau^4}\int_{\mathbb{T}^d_\ell} R|U|^4\\
&\quad\leqslant -r_1\nu\int_0^T \frac{1}{\tau^4}\int_{\mathbb{T}^d_\ell}|U|^2U\cdot\nabla R - r_0\nu\delta_1\int_0^T \frac{1}{\tau^4}\int_{\mathbb{T}^d_\ell}\frac{\Delta R}{R}\\
&\qquad + 2r_0\nu\int_0^T \frac{\dot\tau}{\tau^3}\int_{\mathbb{T}^d_\ell}\log R\,\mathbf{1}_{R\geqslant 1}\\
&\qquad - \delta_1\nu\int_0^T \frac{1}{\tau^4}\int_{\mathbb{T}^d_\ell}\nabla U : \nabla R\otimes\nabla\log R\\
&\qquad - \delta_1\nu\int_0^T \frac{1}{\tau^4}\int_{\mathbb{T}^d_\ell}\frac{\Delta R}{R}\operatorname{div}(RU)\\
&\qquad - \delta_2\nu\int_0^T \frac{1}{\tau^4}\int_{\mathbb{T}^d_\ell}\Delta U\cdot\nabla\Delta\log R + 2d(\delta_1+\nu)\int_0^T \frac{1}{\tau^2}\int_{\mathbb{T}^d_\ell} R\\
&\qquad + \nu\int_0^T \frac{\dot\tau}{\tau^3}\int_{\mathbb{T}^d_\ell} R\operatorname{div}U + \mathcal{E}_{\mathrm{BD,\,reg}}(R_0,U_0).
\end{aligned}
$$

We denote by $I_1,\dots,I_8$ the integrals on the right-hand side of this inequality so that we have

$$
\mathcal{E}_{\mathrm{BD,\,reg}}(R,U)(T) + \int_0^T \mathcal{D}_{\mathrm{BD,\,reg}}(R,U)(t)\,\mathrm{d}t \leqslant \mathcal{E}_{\mathrm{BD,\,reg}}(R_0,U_0) + \sum_{k=1}^{8} I_k,
$$

and we estimate each of them separately. In the sequel, we denote by K and C constants (that may change from line to line). The constant K depends only on the parameters of the target system (namely ν,ε) and the initial energy $\mathcal{E}_{\mathrm{reg}}(R_0, U_0)$, while the constant C may depend also on T, the parameters $\epsilon, \nu, r_0, r_1, \eta_1, \eta_2$, and the initial energy $\mathcal{E}_{\mathrm{reg}}(R_0, U_0)$. But none of them depends on (δ_1, δ_2). We remark that the functions $\frac{1}{\tau^2}$, $\frac{\dot\tau^2}{\tau^2}$, $\frac{1}{\tau^3}$ and $\frac{\dot\tau}{\tau^3}$ are integrable in time over $\mathbb{R}_+$, which we shall use below.

For the term I_1, integrating by parts, applying Young inequality – and referring again to (2.8) – yields:

$$\begin{aligned}
|I_1| &\leqslant r_1\nu \int_0^T \frac{1}{\tau^4} \int_{\mathbb{T}^d_\ell} R|U|^2|\nabla U|,\\
&\leqslant K\left[\int_0^T \frac{r_1}{\tau^4}\int_{\mathbb{T}^d_\ell} R|U|^4 + \int_0^T \frac{\nu}{\tau^4}\int_{\mathbb{T}^d_\ell} R|\mathbb{D}U|^2\right] + \frac{1}{2}\int_0^T \frac{\nu}{\tau^4}\int_{\mathbb{T}^d_\ell} R|\mathbb{A}U|^2,\\
&\leqslant \frac{1}{2}\int_0^T \frac{\nu}{\tau^4}\int_{\mathbb{T}^d_\ell} R|\mathbb{A}U|^2 + K,
\end{aligned}$$

and we observe that the first term can be absorbed by the dissipation $\mathcal{D}_{\mathrm{BD,\,reg}}$.

For the term I_2, since $\alpha > 2$ and $s > 2$, there holds thanks to (2.8):

$$\begin{aligned}
|I_2| &\leqslant r_0\nu\delta_1 \int_0^T \frac{1}{\tau^4}\|\Delta R\|_{L^2}\ \left\|R^{-1}\right\|_{L^2}\\
&\leqslant \delta_1 K \sup_{(0,\,T)} \|\Delta R/\tau\|_{L^2} \sup_{(0,\,T)} \left\|R^{-\alpha}\right\|_{L^1}^{1/2} \int_0^T \frac{1}{\tau^3}\\
&\leqslant \delta_1 C.
\end{aligned}$$

For the term I_3, we have:

$$I_3 \leqslant 2r_0\nu \int_0^T \frac{\dot\tau}{\tau^3}\int_{\mathbb{T}^d_\ell} \log R\mathbf{1}_{R\geqslant 1} \leqslant r_0 K \int_0^T \frac{\dot\tau}{\tau^3}\int_{\mathbb{T}^d_\ell} R \leqslant r_0 K.$$

For the term I_4, Hölder inequality in space and Cauchy–Schwarz inequality in time yield

$$\begin{aligned}
|I_4| &= \delta_1\nu\left|\int_0^T \frac{1}{\tau^4}\int_{\mathbb{T}^d_\ell} \sqrt{R}\mathbb{D}U : \frac{\nabla R\otimes\nabla R}{R^{3/2}}\right|\\
&\leqslant \delta_1\sqrt{\nu T}\left[\int_0^T \frac{\nu}{\tau^4}\int_{\mathbb{T}^d_\ell} R|\mathbb{D}U|^2\right]^{\frac12} \sup_{(0,\,T)} \|\nabla R/\tau\|^2_{L^\infty} \sup_{(0,\,T)} \left[\int_{\mathbb{T}^d_\ell} \frac{1}{R^3}\right]^{\frac12}.
\end{aligned}$$

Using Sobolev embedding and (2.8), we obtain that, since $s > d/2$:

$$\sup_{(0,\,T)} \|\nabla R/\tau\|^2_{L^\infty} \leqslant K \sup_{(0,\,T)} \|\nabla\Delta^s R/\tau\|^2_{L^2} \leqslant C,$$

and then $|I_4| \leqslant \delta_1 C$.

For the term I_5, we split $I_5 = I_5^a + I_5^b$ where:

$$I_5^a = \delta_1 \nu \int_0^T \frac{1}{\tau^4} \int_{\mathbb{T}_\ell^d} \frac{\Delta R}{\sqrt{R}} \sqrt{R} \operatorname{div} U,$$

$$I_5^b = 2\delta_1 \nu \int_0^T \frac{1}{\tau^4} \int_{\mathbb{T}_\ell^d} \sqrt{R} U \cdot \nabla \sqrt{R}\, \Delta R.$$

As previously, we note in these inequalities that thanks to Sobolev embeddings and (2.8), there holds:

$$\sup_{(0,T)} \|\Delta R/\tau\|_{L^\infty} + \sup_{(0,T)} \|\nabla R/\tau\|_{L^\infty} + \sup_{(0,T)} \int_{\mathbb{T}_\ell^d} \frac{1}{R} \leqslant C.$$

Consequently, we have the following controls

$$\begin{aligned}|I_5^a| &\leqslant \delta_1 \left(\int_0^T \frac{\nu}{\tau^4} \int_{\mathbb{T}_\ell^d} R |\mathbb{D} U|^2 \right)^{\frac{1}{2}} \left(\int_0^T \frac{\nu}{\tau^2} \right)^{\frac{1}{2}} \\ &\quad \times \sup_{(0,T)} \|\Delta R/\tau\|_{L^\infty} \sup_{(0,T)} \left(\int_{\mathbb{T}_\ell^d} \frac{1}{R} \right)^{\frac{1}{2}} \\ &\leqslant \delta_1 C,\end{aligned}$$

and

$$\begin{aligned}\left|I_5^b\right| &\leqslant \delta_1 \left[\int_0^T \frac{\nu}{\tau^2} \right] \sup_{(0,T)} \left\| \sqrt{R} U/\tau \right\|_{L^2} \sup_{(0,T)} \left\| \nabla \sqrt{R}/\tau \right\|_{L^2} \sup_{(0,T)} \|\Delta R/\tau\|_{L^\infty} \\ &\leqslant \delta_1 C.\end{aligned}$$

For the term I_6 we have:

$$|I_6| \leqslant \delta_2 \int_0^T \frac{1}{2\tau^4} \int_{\mathbb{T}_\ell^d} |\Delta U|^2 + \delta_2 \nu^2 \int_0^T \frac{1}{2\tau^4} \int_{\mathbb{T}_\ell^d} |\nabla \Delta \log R|^2,$$

and we remark that

$$\nabla \Delta \log R = \frac{\nabla \Delta R}{R} - \frac{\Delta R \nabla R}{R^2} - 2 \frac{\nabla^2 R \nabla R}{R^2} + 2 \frac{|\nabla R|^2 \nabla R}{R^3},$$

so that, using Sobolev embedding and (2.8) we obtain:

$$\begin{aligned}\sup_{(0,T)} \|\nabla \Delta \log R\|_{L^2} &\leqslant K \sup_{(0,T)} \left(1 + \|\nabla \Delta^s R\|_{L^2}\right)^3 \sup_{(0,T)} \left(1 + \int_{\mathbb{T}_\ell^d} \frac{1}{R^3} \right) \\ &\leqslant C,\end{aligned}$$

which implies

$$|I_6| \leqslant \int_0^T \frac{\delta_2}{2\tau^4} \int_{\mathbb{T}_\ell^d} |\Delta U|^2 + \delta_2 C,$$

and we observe that the first term can be absorbed by the dissipation $\mathcal{D}_{\mathrm{BD,\,reg}}$.

For the last two terms, we have:

$$I_7 + I_8 \leqslant (2d(1+\nu)+\nu)\int_0^\infty \frac{1+\dot{\tau}^2}{\tau^2}\int_{\mathbb{T}_\ell^d} R + \int_0^\infty \frac{\nu}{\tau^4}\int_{\mathbb{T}_\ell^d} R|\mathbb{D}U|^2,$$

where we have used Cauchy–Schwarz and Young inequalities for I_8. Then, thanks to (2.8), we get

$$I_7 + I_8 \leqslant K.$$

Gathering the previous estimates yields

$$\begin{aligned}\mathcal{E}_{\mathrm{BD,\,reg}}(R,U)(T) + \frac{1}{2}\int_0^T \mathcal{D}_{\mathrm{BD,\,reg}}(R,U)\,\mathrm{d}t \\ \leqslant K + r_0 K + (\delta_1+\delta_2)C + \mathcal{E}^+_{\mathrm{BD,\,reg}}(R_0,U_0).\end{aligned}$$

To conclude, we only need to control the negative part of the BD-entropy, which is done by

$$\mathcal{E}^-_{\mathrm{BD}}(R,U)(T) := \frac{r_0}{\tau^2(T)}\int |\log R(T)|\,\mathbf{1}_{R(T)\geqslant 1} \leqslant K r_0 \int_{\mathbb{T}_\ell^d} R \leqslant r_0 K.$$

This concludes the proof of Lemma 2.8. □

3. Global weak solutions to isothermal fluids with drag forces

In this section we construct global weak solutions to the isothermal fluid system with drag forces, that is system (1.19) with $r_0, r_1 > 0$. We consider solutions on the torus $\mathbb{T}_\ell^d$ by passing to the limit in the regularizing parameters $\delta_1, \delta_2, \eta_1, \eta_2 \to 0$ from solutions to the regularized system (1.20). Let $r_0, r_1 > 0$, we define the energy and its corresponding dissipation for the system (1.19):

$$\begin{aligned}\mathcal{E}_{\mathrm{drag}}(R,U) &= \frac{1}{2\tau^2}\int_{\mathbb{T}_\ell^d}\left(R|U|^2 + \epsilon^2\left|\nabla\sqrt{R}\right|^2\right) + \int_{\mathbb{T}_\ell^d}\left(R|y|^2 + R\log R\right),\\ \mathcal{D}_{\mathrm{drag}}(R,U) &= \frac{\dot{\tau}}{\tau^3}\int_{\mathbb{T}_\ell^d}\left(R|U|^2 + \epsilon^2\left|\nabla\sqrt{R}\right|^2\right) + \frac{\nu}{\tau^4}\int_{\mathbb{T}_\ell^d} R|\mathbb{D}U|^2\\ &\quad + \frac{r_0}{\tau^4}\int_{\mathbb{T}_\ell^d}|U|^2 + \frac{r_1}{\tau^4}\int_{\mathbb{T}_\ell^d} R|U|^4,\end{aligned}$$

as well as the BD-entropy and its corresponding flux

$$\begin{aligned}\mathcal{E}^+_{\mathrm{BD,\,drag}}(R,U) = \\ &\frac{1}{2\tau^2}\int_{\mathbb{T}^d_\ell}\left(R\left|U+\nu\nabla\log R\right|^2+\epsilon^2\left|\nabla\sqrt{R}\right|^2-2r_0\log R\mathbf{1}_{R<1}\right)\\ &+\int_{\mathbb{T}^d_\ell}\left(R|y|^2+R\log R\right),\end{aligned}$$

$$\begin{aligned}\mathcal{D}_{\mathrm{BD,\,drag}}(R,U) = \\ &\frac{\dot\tau}{\tau^3}\int_{\mathbb{T}^d_\ell}\left(R|U|^2+\epsilon^2\left|\nabla\sqrt{R}\right|^2\right)+\frac{2r_0\nu\dot\tau}{\tau^3}\int_{\mathbb{T}^d_\ell}|\log R|\,\mathbf{1}_{R<1}\\ &+\frac{\nu\epsilon^2}{\tau^4}\int_{\mathbb{T}^d_\ell}R\left|\nabla^2\log R\right|^2+\frac{4\nu}{\tau^2}\int_{\mathbb{T}^d_\ell}\left|\nabla\sqrt{R}\right|^2+\frac{\nu}{\tau^4}\int_{\mathbb{T}^d_\ell}R|\mathbb{A}U|^2\\ &+\frac{r_0}{\tau^4}\int_{\mathbb{T}^d_\ell}|U|^2+\frac{r_1}{\tau^4}\int_{\mathbb{T}^d_\ell}R|U|^4.\end{aligned}$$

We note that these quantities correspond to what remains of the energy and entropy defined in Section 2 when the regularizing parameters δ_1,δ_2 and η_1,η_2 are sent to 0.

It is then natural to build-up a definition of global solution to the isothermal system with drag forces (1.19) with $r_0,r_1>0$ based on the only information that $\mathcal{E}_{\mathrm{drag}}$ and $\mathcal{E}^+_{\mathrm{BD,\,drag}}$ are $L^\infty(\mathbb{R}^+)$ while $\mathcal{D}_{\mathrm{drag}}$ and $\mathcal{D}_{\mathrm{BD,\,drag}}$ are $L^1(\mathbb{R}^+)$. For this, it turns out that it is more suitable to interpret the density R as the square of $\sqrt{R}$. Indeed, combining $\mathcal{E}_{\mathrm{drag}}$ and $\mathcal{E}^+_{\mathrm{BD,\,drag}}$ yields a bound on $R|\nabla\log(R)|^2=4|\nabla\sqrt{R}|^2$. Correspondingly, we write (1.19a) in terms of $\sqrt{R}$:

$$\partial_t\sqrt{R}+\frac{1}{\tau^2}\operatorname{div}\left(\sqrt{R}U\right)=\frac{1}{\tau^2}\sqrt{R}\operatorname{div}U, \tag{3.1}$$

while in (1.19b) we only rewrite the Korteweg term applying the identity (see [22]):

$$R\nabla\left(\frac{\Delta\sqrt{R}}{\sqrt{R}}\right)=\operatorname{div}\left(\sqrt{R}\nabla^2\sqrt{R}-\nabla\sqrt{R}\otimes\nabla\sqrt{R}\right),$$

so that we obtain:

$$\begin{aligned}(3.2)\quad &\partial_t(RU)\\ &+\frac{1}{\tau^2}\operatorname{div}\left(\sqrt{R}U\otimes\sqrt{R}U\right)+2yR+\nabla R+\frac{r_0}{\tau^2}U+\frac{r_1}{\tau^2}R|U|^2U\\ &\qquad=\frac{\epsilon^2}{2\tau^2}\operatorname{div}\left(\sqrt{R}\nabla^2\sqrt{R}-\nabla\sqrt{R}\otimes\nabla\sqrt{R}\right)+\frac{\nu}{\tau^2}\operatorname{div}(R\mathbb{D}U)\\ &\qquad+\frac{\nu\dot{\tau}}{\tau}\nabla R.\end{aligned}$$

This remark motivates the following definition.

Definition 3.1. — *Given positive parameters $r_0, r_1 > 0$ and initial data $(\sqrt{R_0}, \Lambda_0 = (\sqrt{R}U)_0) \in L^2(\mathbb{T}^d_\ell) \times L^2(\mathbb{T}^d_\ell)$, we call global weak solution to the isothermal system with drag forces* (1.19) *in $\mathbb{T}^d_\ell$ any pair*

$$\left(\sqrt{R}, U\right) \in C\left([0,\infty); H^1\left(\mathbb{T}^d_\ell\right) - w\right) \times L^2_{\mathrm{loc}}\left(\mathbb{R}^+; L^2\left(\mathbb{T}^d_\ell\right)\right),$$

satisfying

(i) *Further regularity properties:*

$$\sqrt{R}U \in C\left([0,\infty); L^2\left(\mathbb{T}^d_\ell\right) - w\right), \quad \nabla^2\sqrt{R} \in L^2_{\mathrm{loc}}\left(0,\infty; L^2\left(\mathbb{T}^d_\ell\right)\right).$$

(ii) *Equations* (3.1) *and* (3.2) *in the sense of distributions.*

(iii) *Initial data $\sqrt{R}|_{t=0} = \sqrt{R_0}$ and $\sqrt{R}(\sqrt{R}U)|_{t=0} = \sqrt{R_0}\Lambda_0$.*

Remark 3.2. — We note that, since $\sqrt{R}$ and $\sqrt{R}U$ are continuous with respect to time, we may give sense to the initial conditions required in item (iii) of the above definition.

Remark 3.3. — We observe the difference between the definition of weak solutions for the system without and with drag forces. When the latter are present $(r_0, r_1 > 0)$, U is well defined as a function, ∇U as a distribution and $\sqrt{R}\mathbb{D}U$ is well defined. However, in the original system without drag forces, U is not well defined and $\sqrt{R}\mathbb{D}U$ has to be understood as $\mathbf{S}_N$.

Theorem 3.4. — *Assume $r_0, r_1, \nu, \epsilon > 0$. Let $(\sqrt{R_0}, \Lambda_0 = (\sqrt{R}U)_0)$ be an initial data satisfying* (2.7) *and such that $\mathcal{E}_{\mathrm{drag}\,|\,t=0}, \mathcal{E}_{\mathrm{BD,\,drag}\,|\,t=0} < +\infty$. Then there exists a global weak solution (R, U) to the isothermal fluid system with drag forces* (1.19) *in $\mathbb{T}^d_\ell$, in the sense of Definition 3.1, associated to the initial data $(\sqrt{R_0}, \Lambda_0)$. Furthermore, there exist constants C_1 and C_2 (whose dependencies are mentioned in parenthesis) such that this solution satisfies the energy inequality*

$$\sup_{t\geqslant 0}\mathcal{E}_{\mathrm{drag}}(R,U)+\int_0^\infty \mathcal{D}_{\mathrm{drag}}(R,U)\,\mathrm{d}t \leqslant C_1\left(\mathcal{E}_{\mathrm{drag}\,|\,t=0}\right),$$

and also the BD-entropy inequality

$$\sup_{t\geqslant 0}\mathcal{E}_{\mathrm{BD,\,drag}}(R,U)+\int_0^\infty \mathcal{D}_{\mathrm{BD,\,drag}}(R,U)\,\mathrm{d}t \leqslant C_2\left(\mathcal{E}_{\mathrm{drag}\,|\,t=0},\ \mathcal{E}_{\mathrm{BD,\,drag}\,|\,t=0}\right).$$

Proof of Theorem 3.4. — The proof consists of three parts: starting with the regularized system (1.20), in the first one we pass to the limit in the parameters $\delta_1,\delta_2\to 0$, which shall give us the existence of global weak solutions to an intermediate system given by (1.20) with $\delta_1=\delta_2=0$; then we pass to the limit $\eta_1,\eta_2\to 0$ to obtain a weak solution to (1.19) on the torus. In the whole proof $(\sqrt{R_0},\Lambda_0=(\sqrt{R}U)_0)$ is a fixed initial data satisfying (2.7) and the drag parameters $(r_0,r_1)\in(0,\infty)^2$ are fixed.

Step 1. Limits $\delta_1,\delta_2\to 0$. — In this part, we fix $\eta_1>0$ and $\eta_2>0$ and we consider sequence of parameters δ_1,δ_2 converging to 0. To simplify notations we shall denote $\delta=(\delta_1,\delta_2)$ and drop the η_1,η_2 dependencies. We consider the sequence of global weak solutions $\{(R_\delta,U_\delta)\}_\delta$ to the regularized problem (1.20) associated to (R_0,U_0), as constructed in Proposition 2.4. First, we construct limits R and U of this sequence as in Step 1 of Section 2.1.2.

We proceed with improving the sense of the convergence of $\{(R_\delta,U_\delta)\}_\delta$ to these limits. For this, we fix an arbitrary finite $T>0$. Thanks to the energy and BD-entropy inequalities, this sequence verifies uniform estimates in the following spaces:

$$\begin{aligned}
R_\delta\left(1+|y|^2+|\log R_\delta|\right)&\text{ in } L^\infty\left(0,T;L^1\left(\mathbb{T}^d_\ell\right)\right),\\
\nabla\sqrt{R_\delta}&\text{ in } L^\infty\left(0,T;L^2\left(\mathbb{T}^d_\ell\right)\right),\\
\sqrt{\eta_2}R_\delta&\text{ in } L^\infty\left(0,T;H^{2s+1}\left(\mathbb{T}^d_\ell\right)\right),\\
\sqrt{R_\delta}U_\delta&\text{ in } L^\infty\left(0,T;L^2\left(\mathbb{T}^d_\ell\right)\right),\\
\sqrt{\nu}\sqrt{R_\delta}\nabla U_\delta&\text{ in } L^2\left(0,T;L^2\left(\mathbb{T}^d_\ell\right)\right).
\end{aligned}\tag{3.3}$$

Recalling (2.9), this entails that $\{R_\delta\}_\delta$ is bounded in $L^\infty(0,T;H^1(\mathbb{T}^d_\ell))$. Writing the weak form (2.4) with a test function $\Psi\in\mathcal{D}((0,T)\times\mathbb{T}^d_\ell)$, we obtain that:

$$\partial_t R_\delta=-\sqrt{R_\delta}\sqrt{R_\delta}\,\mathrm{div}(U_\delta)-2\sqrt{R_\delta}U_\delta\cdot\nabla\sqrt{R_\delta}+\frac{\delta_1}{\tau^2}\Delta R \quad \text{in } \mathcal{D}'\left((0,T)\times\mathbb{T}^d_\ell\right).$$

This implies that $\{\partial_t R_\delta\}_\delta$ is also bounded in $L^2(0,T;L^1(\mathbb{T}^d_\ell))$. Applying again Ascoli-Arzelà arguments yields $R_\delta \to R$ in $C([0,T];H^{2s}(\mathbb{T}^d_\ell))$ and, moreover, with the uniform bound from below on R_δ in (2.9), we get

$$R_\delta^{-1} \to R^{-1} \quad \text{in } C\left([0,T]\times\mathbb{T}^d_\ell\right).$$

On the other hand, we note that the above bound (3.3) also entails that $\{R_\delta U_\delta\}_\delta$ is bounded in $L^2(0,T;H^1(\mathbb{T}^d_\ell))$. Taking then $\Phi \in \mathcal{D}((0,T)\times\mathbb{T}^d_\ell)$ in (2.5), and recalling (1.12) which is satisfied by $R_\delta > 0$, we obtain (in $\mathcal{D}'((0,T)\times\mathbb{T}^d_\ell)$):

$$\begin{aligned}\partial_t\left(R_\delta U_\delta\right) = &-\frac{1}{\tau^2}\operatorname{div}\left(\sqrt{R_\delta}U_\delta\otimes\sqrt{R}_\delta U_\delta\right) - 2yR_\delta - \nabla R_\delta + \eta_1\nabla R_\delta^{-\alpha}\\ &-\frac{r_0}{\tau^2}U_\delta - \frac{r_1}{\tau^2}R_\delta\left|U_\delta\right|^2U_\delta - \frac{\delta_1}{\tau^2}\left(\nabla R_\delta\cdot\nabla\right)U_\delta\\ &+\frac{\epsilon^2}{2\tau^2}\left(\sqrt{R_\delta}\nabla^2\sqrt{R_\delta} - \nabla\sqrt{R_\delta}\otimes\nabla\sqrt{R_\delta}\right) + \frac{\nu}{\tau^2}\operatorname{div}\left(R_\delta\mathbb{D}U_\delta\right)\\ &+\frac{\nu\dot\tau}{\tau}\nabla R_\delta + \frac{\delta_2}{\tau^2}\Delta^2U_\delta + \frac{\eta_2}{\tau^2}R_\delta\nabla\Delta^{2s+1}R_\delta.\end{aligned}$$

Consequently, combining the uniform bounds in (3.3) with the uniform bounds in the following spaces (again due to the energy and BD-entropy inequalities):

$$\begin{aligned}\sqrt{r_0}\,U_\delta &\text{ in } L^2\left(0,T;L^2\left(\mathbb{T}^d_\ell\right)\right),\\ \sqrt{r_1}\,R_\delta^{\frac14}U_\delta &\text{ in } L^4\left(0,T;L^4\left(\mathbb{T}^d_\ell\right)\right),\\ \sqrt{\delta_2}\,\Delta U_\delta &\text{ in } L^2\left(0,T;L^2\left(\mathbb{T}^d_\ell\right)\right),\\ R_\delta &\text{ in } L^2\left(0,T;H^{2s+2}\left(\mathbb{T}^d_\ell\right)\right),\\ \eta_1^{\frac{1}{\alpha}}\,R_\delta^{-1} &\text{ in } L^\infty\left(0,T;L^\alpha\left(\mathbb{T}^d_\ell\right)\right),\\ \sqrt{\nu\epsilon^2}\,\nabla^2\sqrt{R_\delta} &\text{ in } L^2\left(0,T;L^2\left(\mathbb{T}^d_\ell\right)\right),\end{aligned} \tag{3.4}$$

we conclude that $\{\partial_t(R_\delta U_\delta)\}_\delta$ is bounded in $L^2(0,T;H^{-(2s+1)}(\mathbb{T}^d_\ell))$. This entails that $R_\delta U_\delta \to RU$ in $L^2(0,T;L^2(\mathbb{T}^d_\ell))$.

Thanks to the previous estimates and Aubin-Lions/Ascoli-Arzelà arguments, we obtain the following convergences:

$$
\begin{aligned}
R_\delta &\to R \text{ in } L^2\left(0,T;H^{2s+2}\left(\mathbb{T}^d_\ell\right)-w\right),\\
R_\delta &\to R \text{ in } C\left([0,T];H^{2s}\left(\mathbb{T}^d_\ell\right)\right),\\
R_\delta U_\delta &\to RU \text{ in } L^2\left(0,T;L^p\left(\mathbb{T}^d_\ell\right)\right), \quad \forall\ p<6\\
U_\delta &\to U \text{ in } L^2\left(0,T;L^2\left(\mathbb{T}^d_\ell\right)\right),\\
\sqrt{R_\delta}U_\delta &\to \sqrt{R}U \text{ in } L^p\left(0,T;L^2\left(\mathbb{T}^d_\ell\right)\right), \quad \forall\ p<\infty,\\
\sqrt{R_\delta}U_\delta &\to \sqrt{R}U \text{ in } C\left([0,T];L^2\left(\mathbb{T}^d_\ell\right)-w\right),\\
R_\delta^{\frac14}U_\delta &\to R^{\frac14}U \text{ in } L^p\left(0,T;L^p\left(\mathbb{T}^d_\ell\right)\right), \quad \forall\ p<4.
\end{aligned}
\tag{3.5}
$$

The above list of convergences shows that we can pass to the limit in the initial condition. It also readily implies that:

$$R_\delta U_\delta\otimes U_\delta \to RU\otimes U \text{ in } L^1\left(0,T;L^1\left(\mathbb{T}^d_\ell\right)\right), \tag{3.6}$$

$$R_\delta |U_\delta|^2 U_\delta \to R|U|^2U \text{ in } L^1\left(0,T;L^1\left(\mathbb{T}^d_\ell\right)\right), \tag{3.7}$$

$$\sqrt{R_\delta}U_\delta \to \sqrt{R}U \text{ in } L^2\left(0,T;L^2\left(\mathbb{T}^d_\ell\right)\right). \tag{3.8}$$

We can now pass to the limit in the equations (2.4)-(2.5) when $\delta\to 0$, by remarking that, using the above estimates, we have

$$
\begin{aligned}
&\delta_1\int_0^T\!\!\int \frac{1}{\tau^2}R_\delta\Delta\Psi \to 0,\\
&\delta_1\int_0^T\!\!\int \frac{1}{\tau^2}\nabla U_\delta:\nabla R_\delta\otimes\Phi \to 0,\\
&\delta_2\int_0^T\!\!\int \frac{1}{\tau^2}\Delta U_\delta\Delta\Phi \to 0,
\end{aligned}
$$

where Ψ and Φ are smooth test functions with compact support in $(0,T)\times\mathbb{T}^d_\ell$. We have hence constructed (R,U) which is a global weak solution to the intermediate system corresponding to (1.20) with $\delta_1=\delta_2=0$, and, passing to the limit $\delta\to 0$ in the energy (2.8) and BD-entropy (2.18) inequalities, the solution (R,U) satisfies moreover the energy inequality (2.8) with $\delta_1=\delta_2=0$ as well as the BD-entropy inequality (2.18) with $\delta_1=\delta_2=0$.

Before going further, we remark that the continuity equation (1.20a) holds almost everywhere. Since $R>0$ on any compact interval of time, this entails that $\sqrt{R}$ satisfies (3.1) in $\mathcal{D}'((0,\infty)\times\mathbb{T}^d_\ell)$.

Step 2. Limits $\eta_1,\eta_2\to 0$. — With similar conventions as in the previous step, we introduce now $\eta=(\eta_1,\eta_2)$ and we consider $\{(R_\eta,U_\eta)\}_\eta$ the sequence of global weak solutions associated with initial data $(\sqrt{R_0},\Lambda_0)$

constructed in the Step 1. Thanks to the energy and BD-entropy inequalities, we obtain again the following uniform bounds:

$$\begin{aligned} R_\eta\left(1+|y|^2+|\log R_\eta|\right) &\text{ in } L^\infty\left(0,T;L^1\left(\mathbb{T}^d_\ell\right)\right),\\ \nabla\sqrt{R_\eta} &\text{ in } L^\infty\left(0,T;L^2\left(\mathbb{T}^d_\ell\right)\right),\\ \sqrt{R_\eta}U_\eta &\text{ in } L^\infty\left(0,T;L^2\left(\mathbb{T}^d_\ell\right)\right),\\ \sqrt{R_\eta}\nabla U_\eta &\text{ in } L^2\left(0,T;L^2\left(\mathbb{T}^d_\ell\right)\right). \end{aligned} \tag{3.9}$$

Introducing this bound in (3.1) – so that we prove $\partial_t\sqrt{R_\eta}$ is bounded in $L^2(0,T;H^{-1}(\mathbb{T}^d_\ell))$ – and remarking that $\sqrt{R_\eta}$ is bounded in the space $L^\infty(0,T;H^1(\mathbb{T}^d_\ell))$, Aubin-Lions argument entails that

$$\sqrt{R_\eta}\to\sqrt{R} \text{ in } C\left([0,T];L^2\left(\mathbb{T}^d_\ell\right)\right) \text{ and } L^2\left(0,T;L^2\left(\mathbb{T}^d_\ell\right)\right).$$

Furthermore, thanks to the energy and BD-entropy inequalities, we have the uniform bounds:

$$\begin{aligned} \sqrt{r_0}\,U_\eta &\text{ in } L^2\left(0,T;L^2\left(\mathbb{T}^d_\ell\right)\right),\\ \sqrt{r_1}\,R_\eta^{\frac14}U_\eta &\text{ in } L^4\left(0,T;L^4\left(\mathbb{T}^d_\ell\right)\right),\\ r_0\log\left(\frac{1}{R_\eta}\right)_+ &\text{ in } L^\infty\left(0,T;L^1\left(\mathbb{T}^d_\ell\right)\right),\\ \epsilon\,\nabla^2\sqrt{R_\eta} &\text{ in } L^2\left(0,T;L^2\left(\mathbb{T}^d_\ell\right)\right),\\ \sqrt{\epsilon}\,\nabla R_\eta^{\frac14} &\text{ in } L^4\left(0,T;L^4\left(\mathbb{T}^d_\ell\right)\right). \end{aligned} \tag{3.10}$$

From these bounds, and arguing similarly as in Step 1, we get the convergences

$$\begin{aligned} U_\eta\to U &\text{ in } L^2\left(0,T;L^2\left(\mathbb{T}^d_\ell\right)\right)-w,\\ \sqrt{R_\eta}U_\eta\to\sqrt{R}U &\text{ in } C\left([0,T];L^2\left(\mathbb{T}^d_\ell\right)-w\right),\\ R_\eta^{\frac14}U_\eta\to R^{\frac14}U &\text{ in } L^4\left(0,T;L^4\left(\mathbb{T}^d_\ell\right)\right)-w,\\ R_\eta U_\eta\to RU &\text{ in } L^2\left(0,T;L^2\left(\mathbb{T}^d_\ell\right)\right). \end{aligned} \tag{3.11}$$

Furthermore, we remark that we have

$$R_\eta\left|U_\eta\right|^2U_\eta\to R|U|^2U \text{ a.e.}$$

so that we can apply the uniform bound on $\{R_\eta^{1/4}U_\eta\}_\eta$ to reproduce the arguments of [29, Lemma 2.3] to yield:

$$R_\eta U_\eta\otimes U_\eta\to RU\otimes U \text{ in } L^1\left(0,T;L^1\left(\mathbb{T}^d_\ell\right)\right).$$

With these convergences at-hand, we can already pass to the limit in the weak formulation of the continuity equation (2.4). For the weak formulation (2.5), we only need to prove the convergence to zero of the cold pressure term $\eta_1 \nabla R_\eta^{-\alpha}$ and the regularization term $\frac{\eta_2}{\tau^2} R_\eta \nabla \Delta^{2s+1} R_\eta$, since the other terms can be treated with the above convergences.

We recall that we have the estimates

$$
\begin{aligned}
\sqrt{\eta_2}\, R_\eta &\in L^\infty\left(0,T; H^{2s+1}\left(\mathbb{T}_\ell^d\right)\right),\\
\sqrt{\eta_2}\, \Delta^{s+1} R_\eta &\in L^2\left(0,T; L^2\left(\mathbb{T}_\ell^d\right)\right),\\
\eta_1^{\frac{1}{\alpha}}\, R_\eta^{-1} &\in L^\infty\left(0,T; L^\alpha\left(\mathbb{T}_\ell^d\right)\right),\\
\sqrt{\eta_1}\, \nabla R_\eta^{-\frac{\alpha}{2}} &\in L^2\left(0,T; L^2\left(\mathbb{T}_\ell^d\right)\right).
\end{aligned} \tag{3.12}
$$

On the one hand, from (3.12) and Fatou's lemma we obtain

$$
\int \log\left(\frac{1}{R}\right)_+ \mathrm{d}y = \int \liminf_{\eta \to 0} \log\left(\frac{1}{R_\eta}\right)_+ \mathrm{d}y < +\infty,
$$

which implies that $\operatorname{meas}(\{y \in \mathbb{T}_\ell^d \mid R(t,y) = 0\}) = 0$ for a.e. $t \in (0,T)$. Since we already know that $R_\eta \to R$ a.e. in (t,y), we deduce

$$
\eta_1 R_\eta^{-\alpha} \to 0 \text{ a.e. in } (t,y) \text{ when } \eta_1 \to 0.
$$

We now claim that the uniform estimate $\eta_1 R_\eta^{-\alpha} \in L^{\frac{5}{3}}((0,T) \times \mathbb{T}_\ell^d)$ holds, from which we deduce the convergence

$$
\eta_1 R_\eta^{-\alpha} \to 0 \text{ in } L^1\left(0,T; L^1\left(\mathbb{T}_\ell^d\right)\right) \text{ when } \eta_1 \to 0.
$$

Let us prove this claim: since $\sqrt{\eta_1}\, \nabla R_\eta^{-\frac{\alpha}{2}} \in L^2(0,T; L^2(\mathbb{T}_\ell^d))$ and $\sqrt{\eta_1}\, R_\eta^{-\frac{\alpha}{2}} \in L^\infty(0,T; L^2(\mathbb{T}_\ell^d))$, we get

$$
\sqrt{\eta_1}\, R_\eta^{-\frac{\alpha}{2}} \in L^2\left(0,T; H^1\left(\mathbb{T}_\ell^d\right)\right) \hookrightarrow L^2\left(0,T; L^6\left(\mathbb{T}_\ell^d\right)\right),
$$

whence $\eta_1 R_\eta^{-\alpha} \in L^1(0,T; L^3(\mathbb{T}_\ell^d))$. We finally obtain the claim by using the interpolation inequality

$$
\|f\|_{L^{\frac{5}{3}}\left((0,T)\times \mathbb{T}_\ell^d\right)} \leqslant \|f\|^{\frac{2}{5}}_{L^\infty\left(0,T\,;\,L^1\left(\mathbb{T}_\ell^d\right)\right)} \|f\|^{\frac{3}{5}}_{L^1\left(0,T; L^3\left(\mathbb{T}_\ell^d\right)\right)}.
$$

On the other hand, we now want to show that, for any test function $\Phi \in \mathcal{D}([0,T) \times \mathbb{T}_\ell^d)^d$,

$$
\eta_2 \int_0^T\!\!\int \frac{1}{\tau^2} \Delta^{s+1} R_\eta \Delta^s \left[\nabla R_\eta \cdot \Phi + R_\eta \operatorname{div} \Phi\right] \to 0 \text{ as } \eta_2 \to 0, \tag{3.13}
$$

and we only concentrate in the sequel on the most difficult term, that is corresponding to the $\Delta^s(\nabla R_\eta) \cdot \Phi$ term, the other ones being treated similarly. Recall that $R_\eta \in L^\infty(0,T;L^1\cap L^3(\mathbb{T}^d_\ell))$ uniformly in η thanks to (3.9), and also the interpolation inequality

$$\|f\|_{\dot H^{2s+1}(\mathbb{T}^d_\ell)} \lesssim \|f\|^{\frac{2s+1}{2s+2}}_{\dot H^{2s+2}(\mathbb{T}^d_\ell)} \|f\|^{\frac{1}{2s+2}}_{L^2(\mathbb{T}^d_\ell)}.$$

Therefore, denoting $0<a=\frac{2s+1}{2s+2}<1$, we have

$$\begin{aligned}
&\left|\eta_2\int_0^T\!\!\int \frac{1}{\tau^2}\Delta^{s+1}R_\eta\Delta^s(\nabla R_\eta)\cdot\Phi\right| \\
&\qquad\leqslant C_\Phi\,\eta_2\left\|\nabla^{2s+2}R_\eta\right\|_{L^2\left(0,T\,;\,L^2\left(\mathbb{T}^d_\ell\right)\right)}\left\|\nabla^{2s+1}R_\eta\right\|_{L^2\left(0,\,T\,;\,L^2\left(\mathbb{T}^d_\ell\right)\right)} \\
&\qquad\leqslant C_\Phi\,\eta_2^{\frac12-\frac a2}\left(\sqrt{\eta_2}\left\|\nabla^{2s+2}R_\eta\right\|_{L^2\left(0,\,T\,;\,L^2\left(\mathbb{T}^d_\ell\right)\right)}\right)^{1+\frac{2s+1}{(2s+2)}} \\
&\qquad\qquad\qquad\qquad\left\|\nabla^{2s+1}R_\eta\right\|^{\frac{1}{2s+2}}_{L^2\left(0,\,T\,;\,L^2\left(\mathbb{T}^d_\ell\right)\right)} \\
&\qquad\to 0 \text{ as } \eta_2\to 0.
\end{aligned}$$

This ends the proof that $(\sqrt{R},U)$ satisfies (3.2).

□

At this stage we have constructed a global weak solution $(\sqrt{R},U)$ to the isothermal fluid system (1.19) with drag forces $(r_0,r_1>0)$ on the torus $\mathbb{T}^d_\ell$, in the sense of Definition 3.1, for smooth initial data satisfying (2.7). Furthermore this solution verifies the energy and BD-entropy inequalities of the statement of the theorem, which are obtained straightforwardly in the limit $\eta\to 0$ from the associated inequalities for (R_η,U_η).

4. Global weak solutions in the whole space $\mathbb{R}^d$

The next steps consist in passing to the limit $r_0,r_1\to 0,\ell\to\infty$, and possibly $\epsilon\to 0$. To do so, we adapt the approach of [22], based on a suitable notion of renormalized solution. We emphasize the main steps of the proof and the technical modifications, and refer to [22] for other details.

4.1. Outline of the proof

The method introduced in [22] is based on the introduction of a new family of solutions to the Navier-Stokes system: the renormalized weak solutions. In our framework these solutions are defined as follows:

Definition 4.1 (Renormalized weak solution). — *Let* $\Omega = \mathbb{T}^d_\ell$ *or* $\Omega = \mathbb{R}^d$. *Let* $r_0, r_1 \geqslant 0$, $\epsilon \geqslant 0$ *and* $\nu > 0$. *Let* $(\sqrt{R_0}, \Lambda_0 = (\sqrt{R}U)_0) \in H^1 \cap \mathcal{F}(H^1)(\Omega) \times L^2(\Omega)$ *verify*

$$\sqrt{R_0} \geqslant 0 \text{ a.e. on } \Omega, \quad \left(\sqrt{R}U\right)_0 = 0 \text{ a.e. on } \left\{\sqrt{R_0} = 0\right\}.$$

We say that (R, U) *is a global renormalized weak solution to* (1.19) *in* Ω, *and associated to the initial data* $(\sqrt{R_0}, \Lambda_0)$, *if there exists a collection* $(\sqrt{R}, \sqrt{R}U, \mathbf{S}_K, \mathbf{T}_N)$ *satisfying*

(i) *The following regularities:*

$$\begin{aligned} (\langle y\rangle + |U|)\sqrt{R} &\in L^\infty_{\mathrm{loc}}\left(0,\infty; L^2(\Omega)\right), & \nabla\sqrt{R} &\in L^\infty_{\mathrm{loc}}\left(0,\infty; L^2(\Omega)\right), \\ \epsilon\nabla^2\sqrt{R} &\in L^2_{\mathrm{loc}}\left(0,\infty; L^2(\Omega)\right), & \mathbf{T}_N &\in L^2_{\mathrm{loc}}\left(0,\infty; L^2(\Omega)\right), \\ \sqrt{\epsilon}\nabla R^{1/4} &\in L^4_{\mathrm{loc}}\left(0,\infty; L^4(\Omega)\right), & r_1^{1/4}R^{1/4}U &\in L^4_{\mathrm{loc}}\left(0,\infty; L^4(\Omega)\right), \\ r_0^{1/2}U &\in L^2_{\mathrm{loc}}\left(0,\infty; L^2(\Omega)\right), & r_0 \log R &\in L^\infty_{\mathrm{loc}}\left(0,\infty; L^1(\Omega)\right), \end{aligned}$$

with the compatibility conditions

$$\sqrt{R} \geqslant 0 \text{ a.e. on } (0,\infty)\times\Omega, \quad \sqrt{R}U = 0 \text{ a.e. on } \left\{\sqrt{R} = 0\right\}.$$

(ii) *For any function* $\varphi \in W^{2,\infty}(\mathbb{R}^d)$, *there exist two measures* $f_\varphi, g_\varphi \in \mathcal{M}((0,\infty)\times\Omega)$ *with*

$$\|f_\varphi\|_{\mathcal{M}((0,\infty)\times\Omega)} + \|g_\varphi\|_{\mathcal{M}((0,\infty)\times\Omega)} \leqslant C\left\|\nabla^2\varphi\right\|_{L^\infty(\mathbb{R}^d)},$$

where the constant C *depends only on the solution* $(\sqrt{R}, \sqrt{R}U)$, *such that in* $\mathcal{D}'((0,\infty)\times\mathbb{R}^d)$,

$$\left\{\begin{aligned} &\partial_t\sqrt{R} + \frac{1}{\tau^2}\operatorname{div}\left(\sqrt{R}U\right) = \frac{1}{2\tau^2}\operatorname{Trace}(\mathbf{T}_N), && \text{(4.1a)}\\ &\partial_t\left(R\varphi(U)\right) + \frac{1}{\tau^2}\operatorname{div}\left(R\varphi(U)\otimes U\right) && \text{(4.1b)}\\ &\quad + 2yR\varphi'(U) + \varphi'(U)\nabla R + \frac{r_0}{\tau^2}U\varphi'(U) + \frac{r_1}{\tau^2}R|U|^2U\varphi'(U) \\ &\quad = \operatorname{div}\left(\frac{\nu}{\tau^2}\sqrt{R}\varphi'(U)\mathbf{S}_N + \frac{\epsilon^2}{2\tau^2}\varphi'(U)\mathbf{S}_K\right) \\ &\qquad + \frac{\nu\dot\tau}{\tau}\varphi'(U)\nabla R + f_\varphi, \end{aligned}\right.$$

with $\mathbf{S}_N$ *the symmetric part of* $\mathbf{T}_N$ *and the compatibility conditions:*

$$\sqrt{R}\varphi_i'(U)\left[\mathbf{T}_N\right]_{jk} = \partial_j\left(R\varphi_i'(U)U_k\right) - 2\sqrt{R}U_k\partial_j\sqrt{R} + g_\varphi,$$

$$\mathbf{S}_K = \sqrt{R}\nabla^2\sqrt{R} - \nabla\sqrt{R}\otimes\nabla\sqrt{R},$$

for any $i, j, k \in \{1, \cdots, d\}$.

(iii) *For any* $\psi \in C^\infty(\Omega)$,

$$\lim_{t \to 0} \int_\Omega \sqrt{R}(t,y)\psi(y)\,\mathrm{d}y = \int_\Omega \sqrt{R_0}(y)\psi(y)\,\mathrm{d}y,$$

$$\lim_{t \to 0} \int_\Omega \sqrt{R}(t,y)\left(\sqrt{R}U\right)(t,y)\psi(y)\,\mathrm{d}y = \int_\Omega \Lambda_0(y)\psi(y)\,\mathrm{d}y.$$

Recall the definition of global weak solutions for (1.19) on the torus in Definition 3.1 for the case $r_0, r_1 > 0$, or in Definition 1.1 for solutions in $\mathbb{R}^d$ with $r_0 = r_1 = 0$. The main interest of the notion of renormalized solutions lies in the fact that it is easier to construct solutions to (4.1). More precisely, it is easier to prove the weak stability of renormalized solutions, and to prove the following properties:

- For $r_0, r_1 \geqslant 0$, any renormalized weak solution is also a weak solution,
- In the case $r_0, r_1, \epsilon > 0$, the two notions are equivalent: any weak solution is a renormalized solution.

The proof of existence of weak solution to the quantum Navier Stokes system then reduces to three steps:

- Proving that the weak solutions with drag forces that we constructed previously are indeed renormalized solutions.
- Proving compactness of renormalized solutions in terms of the parameters r_0, r_1, ϵ and ℓ.
- Proving that renormalized solutions in the whole space provide weak solutions in $\mathbb{R}^d$.

4.2. Proof of the main theorem

Consider initial data $(\sqrt{R_0}, \Lambda_0 = (\sqrt{R}U)_0) \in H^1 \cap \mathcal{F}(H^1)(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ as in the assumption of Theorem 1.3. We first construct a sequence of initial data

$$\sqrt{R_{0,\ell}}, \Lambda_{0,\ell} \in H^1\left(\mathbb{T}^d_\ell\right) \times L^2\left(\mathbb{T}^d_\ell\right), \quad \forall\ \ell \in \mathbb{N}^*,$$

which enter the framework of Theorem 3.4. This shall yield an associated sequence $\{(\sqrt{R_\ell}, U_\ell)\}_{\ell \in \mathbb{N}^*}$ of weak solutions to the isothermal system (1.19) with drag forces $(r_0, r_1 > 0)$ on the torus $\mathbb{T}^d_\ell$. We design our sequence of truncated initial data so that, for well-chosen drag parameters, the energy and BD-entropy estimates of Theorem 3.4 yield uniform bounds for these solutions.

So, we consider a plateau function $\chi \in C_c^\infty(\mathbb{R}^d)$ and smoothing kernel $\zeta \in C_c^\infty(\mathbb{R}^d)$ such that

$$\mathbf{1}_{|y|\leqslant 1/2} \leqslant \chi \leqslant \mathbf{1}_{|y|<1}, \quad \operatorname{supp}(\zeta) \subset B(0,1), \quad \int_{\mathbb{R}^d} \zeta(y)\mathrm{d}y = 1,$$

and, for $\ell, \iota > 0$, we set

$$\chi_\ell(y) = \chi\left(\frac{y}{\ell}\right), \quad \zeta_\iota(y) = \frac{1}{\iota^d}\zeta\left(\frac{y}{\iota}\right).$$

Given $\ell \in \mathbb{N}^*$, $\iota > 0$ and $\theta > 0$ we define now $S^0_{\ell,\theta,\iota}$ and $\Lambda_{0,\ell}$ as

$$S^0_{\ell,\theta,\iota}(y) = \left(\sqrt{R_0}(y)\chi_\ell(y) + \theta\right) * \zeta_\iota, \quad \Lambda_{0,\ell}(y) = \Lambda_0(y), \quad \text{for} \quad y \in [-\ell,\ell]^d.$$

Since χ_ℓ is zero on the boundary of the box, the above formula for $S^0_{\ell,\theta,\iota}$ defines an initial data that is smooth, strictly positive, and periodic. The above candidate $(S^0_{\ell,\theta,\iota}, \Lambda_{0,\ell})$ satisfies then the assumptions of Theorem 3.4 whichever the value of $\theta, \iota > 0$. The main property of this construction is the following proposition.

Proposition 4.2. — *There exist sequences $(\theta_\ell)_{\ell\in\mathbb{N}^*}$ and $(\iota_\ell)_{\ell\in\mathbb{N}^*}$ such that, denoting*

$$\sqrt{R_{0,\ell}} := S^0_{\ell,\theta_\ell,\iota_\ell}, \quad \forall\ \ell \in \mathbb{N}^*,$$

we have:

$$\limsup_{\ell\to\infty} \int_{\mathbb{T}^d_\ell} R_{0,\ell}(x)\mathrm{d}x \leqslant \int_{\mathbb{R}^d} R_0,$$

$$\limsup_{\ell\to\infty} \int_{\mathbb{T}^d_\ell} \left|\nabla\sqrt{R}_{0,\ell}\right|^2 \leqslant \int_{\mathbb{R}^d} \left|\nabla\sqrt{R_0}\right|^2,$$

$$\limsup_{\ell\to\infty} \int_{\mathbb{T}^d_\ell} R_{0,\ell}|y|^2 \leqslant \int_{\mathbb{R}^d} R_0|y|^2.$$

Proof. — We note that

$$S^0_{\ell,\theta,\iota} \xrightarrow[\theta\to 0]{} \left(\sqrt{R_0}\chi_\ell\right) * \zeta_\iota =: S^0_{\ell,\iota} \text{ in } C^1(\mathbb{T}^d_\ell).$$

Since all the integrals involved in our proposition are continuous in $S^0_{\ell,\theta,\iota}$ for the C^1-topology, we may only prove the claimed inequalities by replacing $S^0_{\ell,\theta,\iota}$ with $S^0_{\ell,\iota}$.

Standard arguments with the convolution – combined with explicit computations of the truncation – entail that, for arbitrary $\iota > 0$:

$$\limsup_{\ell\to\infty} \int_{\mathbb{T}^d_\ell} \left|S^0_{\ell,\iota}\right|^2 \mathrm{d}x \leqslant \int_{\mathbb{R}^d} R_0,$$

$$\limsup_{\ell\to\infty} \int_{\mathbb{T}^d_\ell} \left|\nabla S^0_{\ell,\iota}\right|^2 \leqslant \int_{\mathbb{R}^d} \left|\nabla\sqrt{R_0}\right|^2.$$

Then, by a convexity argument and duality formulas for the convolution, we obtain that

$$\int_{\mathbb{T}_\ell^d} S^0_{\ell,\iota}|y|^2 = \int_{\mathbb{T}_\ell^d} \left|\left[\sqrt{R_0}\chi_\ell\right] * \zeta_\iota\right|^2 |y|^2 \leqslant \int_{\mathbb{T}_\ell^d} \left[\left|\sqrt{R_0}\chi_\ell\right|^2 * \zeta_\iota\right] |y|^2$$
$$\leqslant \int_{\mathbb{T}_\ell^d} \left|\sqrt{R_0}\chi_\ell\right|^2 \left((1+\iota)|y|^2 + C\iota\right),$$

for an absolute constant C. Consequently, we obtain again that, for arbitrary $\iota > 0$,

$$\limsup_{\ell\to\infty} \int_{\mathbb{T}_\ell^d} \left|S^0_{\ell,\iota}\right|^2 |y|^2 \leqslant \int_{\mathbb{R}^d} R_0(1+\iota)|y|^2 + C\iota^2 \int_{\mathbb{R}^d} R_0.$$

It thus suffices to consider a sequence $\iota_\ell \to 0$. □

Note that applying Lemma 1.2 to

$$\mathbf{1}_{[-\ell,\ell]^d}\sqrt{R_{0,\ell}},$$

viewed as a function on $\mathbb{R}^d$, we infer from the above proposition that $\int_{\mathbb{T}_\ell^d} R_{0,\ell}|\log R_{0,\ell}|$ is bounded uniformly in ℓ.

In what follows, we consider that $(\sqrt{R_{0,\ell}}, \Lambda_{0,\ell})_{\ell\in\mathbb{N}^*}$ is the sequence of initial data constructed in the previous proposition. Invoking Theorem 3.4 with these data for arbitrary $\ell \in \mathbb{N}^*$, we obtain a sequence $(\sqrt{R_\ell}, U_\ell)_{\ell\in\mathbb{N}^*}$ such that for arbitrary $\ell \in \mathbb{N}^*$, the pair $(\sqrt{R_\ell}, U_\ell)$ is a global weak solution to (1.19) on the torus $\mathbb{T}_\ell^d$. We denote also

$$r_{0,\ell} := \frac{1}{\ell + \left(\int_{\mathbb{T}_\ell^d} \log(R_{0,\ell})\mathbf{1}_{R_{0,\ell}<1}\right)^2}, \quad r_{1,\ell} := \frac{1}{\ell}, \quad \epsilon_\ell = \epsilon + \frac{1}{\ell},$$

and of course, these values affect the above mentioned sequence of solutions $(\sqrt{R_\ell}, U_\ell)_\ell$. These choices ensure that the associated sequence of initial energies $\mathcal{E}_{\mathrm{drag}}$ (resp. entropies $\mathcal{E}_{\mathrm{BD,\,drag}}$) converge to the energy $\mathcal{E}$ (resp. entropy $\mathcal{E}_{\mathrm{BD}}$) of $(\sqrt{R_0}, \Lambda_0)$. As a matter of fact, the somehow intricate choice for $r_{0,\ell}$ is motivated by this property, to obtain

$$r_{0,\ell} \int_{\mathbb{T}_\ell^d} \log\left(R_{0,\ell}\right) \mathbf{1}_{R_{0,\ell}<1} \underset{\ell\to\infty}{\longrightarrow} 0.$$

4.2.1. Weak solutions with drag forces are renormalized solutions

Given $\ell \in \mathbb{N}^*$, we first obtain that the weak solution we constructed in the previous step is a renormalized solution as stated in Definition 4.1. To start with, we note that, in the case with drag and when Ω is a torus, item (i) in

Definition 4.1 gathers all the regularity properties inherited from the energy and entropy estimates in Theorem 3.4. The only point that deserves more details is the construction of the tensor $\mathbf{T}_{N,\ell}$. We set:

$$\mathbf{T}_{N,\ell} = \sqrt{R_\ell}\nabla U_\ell.$$

This tensor is well defined (at least in $\mathcal{D}'((0,\infty)\times\mathbb{T}_\ell^d)$) since, thanks to the energy/entropy estimates, we have $U_\ell \in L^2_{\mathrm{loc}}((0,\infty)\times\mathbb{T}_\ell^d)$ and $\sqrt{R_\ell}\in L^2_{\mathrm{loc}}((0,\infty);H^1(\mathbb{T}_\ell^d))$. Furthermore, we control the symmetric part (resp. the skew-symmetric part) of $\mathbf{T}_{N,\ell}$ with the energy dissipation (resp. the BD-entropy dissipation) so that we obtain the expected $L^2_{\mathrm{loc}}((0,\infty);L^2(\mathbb{T}_\ell^d))$ regularity.

We proceed with item (ii) of the Definition 4.1, the last one being an obvious corollary to the time regularity of $(\sqrt{R_\ell},\, U_\ell)$ as stated in Definition 3.1. By definition, the pair $(\sqrt{R_\ell}, \sqrt{R_\ell}U_\ell)$ solves the continuity equation (4.1a), identifying the right-hand side of (3.1) as $\operatorname{div}\mathbf{T}_{N,\ell}$. The compatibility conditions for the tensor $\mathbf{S}_{K,\ell}$ can be seen as a definition.

The main point of the construction is to obtain the momentum equation in terms of renormalized solution (4.1b). We give here only the main ideas of the computation and refer the reader to [22, Section 3] for more details. In order to multiply the equation with $\varphi'(U_\ell)$, the first step is to regularize the momentum equation by truncating large and small values of $\sqrt{R_\ell}$ in order to take advantage of the good integrability properties of $R_\ell^{1/4}U_\ell$. To this end, we first remark that the continuity equation reads:

$$\partial_t\sqrt{R_\ell} + \frac{2}{\tau^2}R_\ell^{1/4}U_\ell\cdot\nabla R_\ell^{1/4} + \frac{1}{2\tau^2}\sqrt{R_\ell}\operatorname{div}U_\ell = 0.$$

Applying the bounds on $\nabla R_\ell^{1/4}$ stemming from (1.13) we obtain $\partial_t\sqrt{R_\ell}\in L^2_{\mathrm{loc}}((0,\infty)\times\mathbb{T}_\ell^d)$. Moreover, we also know that $\nabla\sqrt{R_\ell}\in L^\infty_{\mathrm{loc}}((0,\infty);L^2(\mathbb{T}_\ell^d))$. Consequently, for arbitrary $\phi\in C^1_c(0,\infty)$, $\phi(R_\ell)=\phi(\sqrt{R_\ell}^{\,2})$ enjoys the same time and space integrability. On the other hand, we remark that the momentum equation satisfied by $R_\ell U_\ell$ reads:

$$\partial_t\left(R_\ell U_\ell\right) + \frac{1}{\tau^2}\operatorname{div}\left(R_\ell U_\ell\otimes U_\ell\right) = \operatorname{div}\left(\sqrt{R_\ell}\mathbf{S}_\ell\right) - F_\ell,$$

where

$$\begin{aligned}\mathbf{S}_\ell = {}& \frac{\nu}{\tau^2}\sqrt{R_\ell}\mathbb{D}(U_\ell) + \frac{\epsilon_\ell}{2\tau^2}\left(\nabla^2\sqrt{R_\ell} - 4\nabla R_\ell^{1/4}\otimes\nabla R_\ell^{1/4}\right)\\ &+\left(\frac{\nu\dot\tau}{\tau}-1\right)\sqrt{R_\ell}\mathbf{I}_d,\end{aligned}$$

and

$$F_\ell = \frac{r_{0,\ell}}{\tau^2}U + \frac{r_{1,\ell}}{\tau^2}R_\ell\left|U_\ell\right|^2U_\ell + 2yR_\ell.$$

Here we denoted by $\mathbf{I}_d$ the identity matrix. Since

$$\sqrt{R}_\ell \in L^2_{\mathrm{loc}}\left((0,\infty);H^2\left(\mathbb{T}^d_\ell\right)\right) \subset L^2_{\mathrm{loc}}\left((0,\infty);L^\infty\left(\mathbb{T}^d_\ell\right)\right) \quad (d \leqslant 3),$$

we have $F_\ell \in L^{4/3}_{\mathrm{loc}}((0,\infty)\times\mathbb{T}^d_\ell)$ and $\sqrt{R_\ell}\mathbf{S}_\ell \in L^1_{\mathrm{loc}}((0,\infty)\times\mathbb{T}^d_\ell)$. On the left-hand side of the equation, we have:

$$R_\ell U_\ell = \sqrt{R}_\ell\left(\sqrt{R_\ell}U_\ell\right) \in L^2_{\mathrm{loc}}\left((0,\infty)\times\mathbb{T}^d_\ell\right),$$

$$R_\ell U_\ell \otimes U_\ell = \sqrt{R}_\ell\, R_\ell^{1/4}U_\ell \otimes R_\ell^{1/4}U_\ell \in L^1_{\mathrm{loc}}\left((0,\infty);L^2\left(\mathbb{T}^d_\ell\right)\right).$$

We thus have sufficient regularity to multiply the momentum equation with $\phi(R_\ell)$. We obtain:

$$\begin{aligned}\partial_t\left(\phi(R_\ell)R_\ell U_\ell\right) &+ \frac{1}{\tau^2}\mathrm{div}\left(R_\ell U_\ell\otimes\phi(R_\ell)U_\ell\right)\\ &= \mathrm{div}\left(\phi(R_\ell)\sqrt{R_\ell}\mathbf{S}_\ell\right) + \phi(R_\ell)F_\ell - \sqrt{R_\ell}\mathbf{S}_\ell\cdot\nabla\phi(R_\ell)\\ &\quad + \left(\partial_t\phi(R_\ell) + U_\ell\cdot\nabla\phi(R_\ell)\right)R_\ell U_\ell.\end{aligned}$$

At this point, we remark that we may also multiply the continuity equation (4.1a) with a suitable function of $\sqrt{R}_\ell$ in order to replace it with

$$\partial_t\phi(R_\ell) + U_\ell\cdot\nabla\phi(R_\ell) = -\frac{1}{\tau^2}\phi'(R_\ell)\sqrt{R_\ell}\,\mathrm{Trace}\,\mathbf{T}_{N,\ell}.$$

Introducing $V_\ell = \phi(R_\ell)U_\ell$, we have finally,

$$\begin{aligned}\partial_t\left(R_\ell V_\ell\right) &+ \frac{1}{\tau^2}\mathrm{div}\left(R_\ell U_\ell\otimes V_\ell\right)\\ &= \mathrm{div}\left(\phi\left(R_\ell\right)\sqrt{R_\ell}\mathbf{S}_\ell\right) + \phi\left(R_\ell\right)F_\ell - \sqrt{R_\ell}\mathbf{S}_\ell\cdot\nabla\phi\left(R_\ell\right)\\ &\quad - \frac{1}{\tau^2}R_\ell U_\ell\phi'\left(R_\ell\right)\sqrt{R_\ell}\,\mathrm{Trace}\,\mathbf{T}_{N,\ell}.\end{aligned}$$

Since ϕ truncates the small and large values of R_ℓ we may rewrite

$$V_\ell = R_\ell^{1/4}U_\ell\frac{\phi\left(R_\ell\right)}{R_\ell^{1/4}} \in L^4_{\mathrm{loc}}\left((0,\infty)\times\mathbb{T}^d_\ell\right).$$

We are then in position to multiply the i^{th} equation of the momentum equation by $\varphi'(V_\ell)$. With the help of Friedrich's lemma we obtain, on the left-hand side

$$\begin{aligned}&\left(\partial_t\left(R_\ell V_\ell\right) + \frac{1}{\tau^2}\mathrm{div}\left(R_\ell U_\ell\otimes V_\ell\right)\right)\cdot\varphi'(V_\ell)\\ &\qquad = \partial_t\left(\phi\left(R_\ell\right)R_\ell\varphi\left(V_\ell\right)\right) + \frac{1}{\tau^2}\mathrm{div}\left(R_\ell U_\ell\otimes\varphi(V_\ell)\right)),\end{aligned}$$

and, on the right-hand side:

$$\Big(\operatorname{div}\Big(\phi(R_\ell)\sqrt{R_\ell}\mathbf{S}_\ell\Big) + \phi(R_\ell)F_\ell - \sqrt{R_\ell}\mathbf{S}_\ell\cdot\nabla\phi(R_\ell) \\ - \frac{1}{\tau^2}R_\ell U_\ell \phi'(R_\ell)\sqrt{R_\ell}\,\mathrm{Trace}\,\mathbf{T}_{N,\,\ell}\Big)\cdot\varphi'(V_\ell) \\ = \operatorname{div}\Big(\phi(R_\ell)\sqrt{R_\ell}\mathbf{S}_\ell\cdot\varphi'(V_\ell)\Big) + \phi(R_\ell)\varphi'(V_\ell)\cdot F_\ell - [\mathbf{S}_\ell\cdot\nabla\phi(R_\ell)]\cdot\varphi'(V_\ell) \\ - \frac{1}{\tau^2}\phi'(R_\ell)\sqrt{R_\ell}R_\ell U_\ell\cdot\varphi'(V_\ell)\,\mathrm{Trace}\,\mathbf{T}_{N,\,\ell} - \phi(R_\ell)\sqrt{R_\ell}\mathbf{S}_\ell : \varphi''(V_\ell)\nabla V_\ell.$$

To obtain (4.1b), it remains to approximate the constant 1 with a suitable sequence of functions $(\phi_m)_{m\in\mathbb{N}}$. This construction is performed in [22] and [29]. We emphasize that, in this case with drag forces:

$$f_\varphi = \varphi''(U_\ell)\mathbf{S}_\ell : \sqrt{R_\ell}\nabla U_\ell \in L^1_{\mathrm{loc}}\left((0,\infty)\times\mathbb{T}^d_\ell\right),$$
$$\|f_\varphi\|_{L^1_{\mathrm{loc}}\left((0,\,\infty)\,\times\,\mathbb{T}^d_\ell\right)} \leqslant \|\varphi''\|_{L^\infty([0,\,\infty))}\left(\mathcal{E}_{\mathrm{drag}}\left(R^0_\ell, U^0_\ell\right) + \mathcal{E}_{\mathrm{BD,\,drag}}\left(R^0_\ell, U^0_\ell\right)\right).$$

Finally, the compatibility condition concerning $\mathbf{T}_{N,\,\ell}$ is obtained by noting that for arbitrary $\varphi \in W^{2,\,\infty}(\mathbb{R}^d)$ and $j,k\in\{1,\,\ldots,\,d\}$, we have:

$$\varphi'(U_\ell)R_\ell\partial_j U_{\ell,\,k} \\ = \partial_j\left(R_\ell\varphi'(U_\ell)U_{\ell,\,k}\right) - 2\sqrt{R_\ell}U_{\ell,\,k}\varphi'(U_\ell)\partial_j\sqrt{R_\ell} - R_\ell U_{\ell,\,k}\varphi''(U_\ell)\partial_j U_\ell,$$

which is obtained standardly by first regularizing $\sqrt{R_\ell}$ and U_ℓ. So, we have:

$$\sqrt{R_\ell}\varphi'(U_\ell)\mathbf{T}_{N,\,\ell,\,j,\,k} = \partial_j\left(R_\ell U_\ell\varphi'(U_\ell)U_{\ell,\,k}\right) - 2\sqrt{R_\ell}U_\ell\partial_j\sqrt{R_\ell} + g_{j,\,k,\,\varphi},$$

with $g_{j,\,k,\,\varphi}\in L^2_{\mathrm{loc}}((0,\infty);L^1(\mathbb{T}^d_\ell))$ satisfying

$$\|g_{j,\,k,\,\varphi}\|_{L^1_{\mathrm{loc}}\left((0,\infty)\times\mathbb{T}^d_\ell\right)} \\ \leqslant \|\varphi''\|_{L^\infty([0,\,\infty))}\left(\mathcal{E}_{\mathrm{drag}}\left(R^0_\ell,\,U^0_\ell\right) + \mathcal{E}_{\mathrm{BD,\,drag}}\left(R^0_\ell,\,U^0_\ell\right)\right).$$

4.2.2. Compactness of renormalized solutions and conclusion

We are now able to prove our main result Theorem 1.3. Since, in any case (i.e. with or without drag) renormalized solutions to (1.5) are weak solutions as defined in Definition 1.1 (see [22, Section 4]), we only show that, when we let the parameter $\ell\to\infty$, we can extract a subsequence from $(\sqrt{R_\ell},\sqrt{R_\ell}U_\ell)_{\ell\in\mathbb{N}^*}$ that converges to a renormalized solution to (1.5) on the whole space $\mathbb{R}^d$.

Proof of Theorem 1.3. — First, thanks to the energy and entropy estimates on the one hand, and the choice of initial data on the other hand, the sequences of renormalized solutions $\{(\sqrt{R_\ell}, \sqrt{R_\ell}U_\ell, \mathbf{T}_{N,\ell})\}_\ell$ are uniformly bounded in the following spaces, respectively:

$$\begin{aligned}\sqrt{R_\ell} \text{ in } L^\infty_{\mathrm{loc}}\left(0,\infty; H^1_{\mathrm{loc}}\left(\mathbb{R}^d\right)\right),\\ \sqrt{R_\ell}U_\ell \text{ in } L^\infty_{\mathrm{loc}}\left(0,\infty; L^2_{\mathrm{loc}}\left(\mathbb{R}^d\right)\right),\\ \mathbf{T}_{N,\ell} \text{ in } L^2_{\mathrm{loc}}\left(0,\infty; L^2_{\mathrm{loc}}\left(\mathbb{R}^d\right)\right).\end{aligned}$$

Furthermore, by the choice of our initial data, we have:

$$\limsup_{\ell\to\infty}\left(\left\|\sqrt{R_\ell}\right\|_{L^\infty_{\mathrm{loc}}\left(0,\infty\,;\,H^1\left(\mathbb{T}^d_\ell\right)\right)} + \left\|\sqrt{R_\ell}U_\ell\right\|_{L^\infty_{\mathrm{loc}}\left(0,\infty\,;\,L^2\left(\mathbb{T}^d_\ell\right)\right)} \right.$$
$$\left. + \left\|\mathbf{T}_{N,\ell}\right\|_{L^2_{\mathrm{loc}}\left(0,\infty\,;\,L^2\left(\mathbb{T}^d_\ell\right)\right)}\right) \leqslant C\left(\sqrt{R_0},\Lambda_0\right).$$

Consequently, by a standard Cantor extraction argument, we can construct

$$\begin{aligned}\sqrt{R} \text{ in } L^\infty_{\mathrm{loc}}\left(0,\infty; H^1\left(\mathbb{R}^d\right)\right),\\ \sqrt{R}U \text{ in } L^\infty_{\mathrm{loc}}\left(0,\infty; L^2\left(\mathbb{R}^d\right)\right),\\ \mathbf{T}_N \text{ in } L^2_{\mathrm{loc}}\left(0,\infty; L^2\left(\mathbb{R}^d\right)\right),\end{aligned}$$

so that, without relabelling the subsequences:

$$\begin{aligned}\sqrt{R_\ell} \rightharpoonup \sqrt{R} \text{ in } L^\infty_{\mathrm{loc}}\left(0,\infty; H^1_{\mathrm{loc}}\left(\mathbb{R}^d\right)\right) - w*,\\ \sqrt{R_\ell}U_\ell \rightharpoonup \sqrt{R}U \text{ in } L^\infty_{\mathrm{loc}}\left(0,\infty; L^2_{\mathrm{loc}}\left(\mathbb{R}^d\right)\right) - w*,\\ \mathbf{T}_{N,\ell} \rightharpoonup \mathbf{T}_N \text{ in } L^2_{\mathrm{loc}}\left(0,\infty; L^2_{\mathrm{loc}}\left(\mathbb{R}^d\right)\right) - w.\end{aligned}$$

In addition, we have also momentum and (if $\epsilon > 0$) second order bounds for $\sqrt{R_\ell}$ uniformly in ℓ so that $\sqrt{R}$ enjoys the further estimates:

$$\begin{aligned}\epsilon\nabla^2\sqrt{R} \in L^2_{\mathrm{loc}}\left(0,\infty; L^2\left(\mathbb{R}^d\right)\right),\\ \sqrt{\epsilon}\nabla R^{1/4} \in L^4_{\mathrm{loc}}\left(0,\infty; L^4\left(\mathbb{R}^d\right)\right),\\ \langle y\rangle\sqrt{R} \in L^\infty_{\mathrm{loc}}\left(0,\infty; L^2\left(\mathbb{R}^d\right)\right).\end{aligned}$$

We have now a candidate satisfying item (i) of the Definition 1.1 of renormalized solutions without drag forces on the torus. Furthermore, we can pass to the weak limit in the energy and entropy estimates on the torus so that these solutions satisfy (1.17) and (1.18).

We note that the above weak convergences of $\sqrt{R_\ell}, \sqrt{R_\ell}U_\ell$ and $\mathbf{T}_{N,\ell}$ are sufficient to pass to the limit in the continuity equation (4.1a). Reproducing the arguments for the limits $\eta_1, \eta_2 \to 0$ in the previous section (see also the proof of [22, Lemma 5.1]), we obtain that

$$\sqrt{R_\ell} \to \sqrt{R} \text{ in } C\left([0,\infty); L^2_{\mathrm{loc}}\left(\mathbb{R}^d\right)\right).$$

We note that, since we control the second momentum of $\sqrt{R_\ell}$, the convergence actually holds in $C([0,T];L^2(\mathbb{R}^d))$. When $\epsilon > 0$, by interpolation, we have also that

$$\sqrt{R_\ell} \to \sqrt{R} \text{ in } L^4_{\mathrm{loc}}\left((0,\infty);H^1_{\mathrm{loc}}\left(\mathbb{R}^d\right)\right).$$

We can then combine the strong convergence of $\sqrt{R_\ell}$ and the weak convergence of $\nabla^2\sqrt{R_\ell}$ to pass to the limit in the compatibility condition for $\mathbf{S}_K$.

It remains to pass to the limit in the renormalized momentum equation and the compatibility condition for $\mathbf{T}_N$. For this, we can again reproduce the arguments of [22] with the only integrability of $\sqrt{R_\ell}$. We obtain that $R_\ell U_\ell \to RU$ in $L^2_{\mathrm{loc}}((0,\infty);L^p_{\mathrm{loc}}(\mathbb{R}^d))$ for arbitrary $p < 3/2$. Introducing $U = RU/R\mathbf{1}_{R>0}$, we conclude that $R_\ell \to R$ and $U_\ell \to U$ a.e., and consequently that $R_\ell^\alpha \phi(U_\ell) \to R^\alpha\phi(U)$ in $L^p_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^d)$ for any bounded $\phi : \mathbb{R}^d \to \mathbb{R}^d$, $\alpha < 6$ and $p < 6/\alpha$. Given $\varphi \in W^{2,\infty}(\mathbb{R}^d)$, we remark that the remainder $f_{\ell,\varphi}$ is a bounded sequence of measures, so that we can extract a weakly converging sequence. The above convergences are then sufficient to pass to the limit in the renormalized momentum equations with φ satisfied by $(\sqrt{R_\ell}, U_\ell)$ and obtain (4.1b). We proceed similarly to pass to the limit in the renormalized compatibility condition for $\mathbf{T}_{N,\ell}$ and obtain the renormalized compatibility condition for $\mathbf{T}_N$. This ends the proof of Theorem 1.3. □

5. Global weak solutions to isothermal Korteweg equation

In this section, we explain how to prove Proposition 1.7. The idea is the same as in [2, Proposition 15] in the barotropic case, and we present the specificities of the isothermal case.

Formally, Proposition 1.7 stems from Madelung transform: consider the solution $\psi \in L^\infty_{\mathrm{loc}}(\mathbb{R};H^1(\mathbb{R}^d))$ to the logarithmic Schrödinger equation

$$i\epsilon\partial_t\psi + \frac{\epsilon^2}{2}\Delta\psi = \psi\log|\psi|^2; \quad \psi_{|t=0} = \psi_0. \tag{5.1}$$

Then $(\varrho, j) = \left(|\psi|^2, \epsilon\,\mathrm{Im}(\bar\psi\nabla\psi)\right)$ is a natural candidate for the conclusions of Proposition 1.7. Indeed, we compute

$$\partial_t\varrho = 2\,\mathrm{Re}\,\bar\psi\partial_t\psi = -\epsilon\,\mathrm{Im}\left(\bar\psi\Delta\psi\right) = -\operatorname{div}\left(\epsilon\,\mathrm{Im}\left(\bar\psi\nabla\psi\right)\right),$$

and, in view of the identity

$$\begin{aligned}\partial_t\nabla\psi &= \frac{i\epsilon}{2}\Delta\nabla\psi - \frac{i}{\epsilon}\nabla\left(\psi\log|\psi|^2\right),\\ \partial_t j &= \epsilon\,\mathrm{Im}\left(\nabla\psi\left(-\frac{i\epsilon}{2}\Delta\bar{\psi}+\frac{i}{\epsilon}\bar{\psi}\log|\psi|^2\right)\right)\\ &\quad+\epsilon\,\mathrm{Im}\left(\bar{\psi}\left(\frac{i\epsilon}{2}\Delta\nabla\bar{\psi}-\frac{i}{\epsilon}\nabla\left(\psi\log|\psi|^2\right)\right)\right)\\ &= \frac{\epsilon^2}{4}\nabla\Delta|\psi|^2-\epsilon^2\,\mathrm{div}\left(\mathrm{Re}\left(\nabla\bar{\psi}\otimes\nabla\psi\right)\right)-\nabla|\psi|^2.\end{aligned}$$

The above identities are true in the sense of distributions, provided (at least) that $\psi\in H^1(\mathbb{R}^d)$. Therefore, to show that (ϱ,j) is a solution to (1.1) with $\nu=0$, we have to rewrite the term $\mathrm{div}(\mathrm{Re}(\nabla\bar{\psi}\otimes\nabla\psi))$. In view of [2, Lemma 3], for $\psi\in H^1(\mathbb{R}^d)$, there exists $\phi\in L^\infty(\mathbb{R}^d)$ such that $\psi=\sqrt{\varrho}\phi$ a.e. in $\mathbb{R}^d$, $\sqrt{\varrho}\in H^1(\mathbb{R}^d)$, $\nabla\sqrt{\varrho}=\mathrm{Re}(\bar{\phi}\nabla\psi)$, so that if we set $\sqrt{\varrho}u:=\epsilon\,\mathrm{Im}(\bar{\phi}\nabla\psi)$, then $\sqrt{\varrho}u\in L^2(\mathbb{R}^d)$, $j=\sqrt{\varrho}\times\sqrt{\varrho}u$ and

$$\epsilon^2\,\mathrm{Re}\left(\nabla\bar{\psi}\otimes\nabla\psi\right)=\epsilon^2\nabla\sqrt{\varrho}\otimes\nabla\sqrt{\varrho}+(\sqrt{\varrho}u)\otimes(\sqrt{\varrho}u)\,.$$

In this case,

$$\phi(x)=\begin{cases}\frac{\psi(x)}{|\psi(x)|} & \text{if } \psi(x)\neq 0,\\ 0 & \text{if } \psi(x)=0,\end{cases}$$

so the compatibility condition $\sqrt{\varrho}u=0$ a.e. on $\{\sqrt{\varrho}=0\}$ is satisfied. Finally, by the definition of j,

$$\nabla\wedge j=\epsilon\,\mathrm{Im}\left(\nabla\bar{\psi}\wedge\nabla\psi\right),$$

and [2, Corollary 13] yields, for $\psi\in H^1(\mathbb{R}^d)$,

$$\nabla\wedge j=2\nabla\sqrt{\varrho}\wedge(\sqrt{\varrho}u)\,.$$

Note that in the barotropic case considered in [2, Proposition 15], $p(\varrho)=\varrho^\gamma$, $\gamma>1$, instead of the logarithmic Schrödinger equation (5.1), one faces the more standard nonlinear Schrödinger equation with a power-like nonlinearity,

$$i\epsilon\partial_t\psi+\frac{\epsilon^2}{2}\Delta\psi=c_\gamma|\psi|^{\gamma-1}\psi\,;\quad \psi_{|t=0}=\psi_0, \tag{5.2}$$

for some constant $c_\gamma>0$ whose exact value is irrelevant for the present discussion.

The Cauchy problem for (5.1) was solved initially in [15] locally in time for $\psi_0 \in L^2(\mathbb{R}^d)$, using the theory of monotone operators. To obtain a solution with an H^1 regularity, as well as the uniqueness of this solution, in [15, 16] (see also [14]) the authors have to change the sign in front of the nonlinearity in (5.1), so the Hamiltonian structure of the equation directly provides a priori estimates. In the case of (5.1), the formally conserved energy

$$E_{\mathrm{logNLS}} = \frac{\epsilon^2}{2}\int_{\mathbb{R}^d} |\nabla\psi|^2 + \int_{\mathbb{R}^d} |\psi|^2 \log|\psi|^2, \tag{5.3}$$

is not helpful because the region $\{|\psi| < 1\}$ yields a negative contribution, and cannot be controlled in terms of the H^1-norm. This is why in the present case, working in H^1 is not enough, and a (fractional) momentum is considered to, $\psi_0 \in \mathcal{F}(H^\alpha)$, that is,

$$\int_{\mathbb{R}^d} \langle x\rangle^{2\alpha} |\psi_0(x)|^2 \,\mathrm{d}x < \infty,$$

for some $0 < \alpha \leqslant 1$. Then (5.1) has a unique, global solution $\psi \in L^\infty_{\mathrm{loc}}(\mathbb{R}; H^1 \cap \mathcal{F}(H^\alpha))$. We refer to [13] for details. The first part of Proposition 1.7 follows.

To conclude and prove the second point of Proposition 1.7, introduce Ψ given by

$$\psi(t,x) = \frac{1}{\tau(t)^{d/2}}\Psi\left(t, \frac{x}{\tau(t)}\right)\left(\frac{\|\varrho_0\|_{L^1(\mathbb{R}^d)}}{\|\Gamma\|_{L^1(\mathbb{R}^d)}}\right)^{1/2} \exp\left(i\frac{\dot\tau(t)}{\tau(t)}\frac{|x|^2}{2\epsilon} - i\frac{\theta(t)}{\epsilon}\right),$$

where

$$\theta(t) = d\int_0^t \log\tau(s)\mathrm{d}s - t\log\left(\frac{\|\varrho_0\|_{L^1(\mathbb{R}^d)}}{\|\Gamma\|_{L^1(\mathbb{R}^d)}}\right).$$

It solves (see [13])

$$\begin{aligned} i\epsilon\partial_t\Psi + \frac{\epsilon^2}{2\tau(t)^2}\Delta\Psi &= \Psi\log|\Psi|^2 + |y|^2\Psi, \\ \Psi(0,y) &= \psi_0(y)\left(\frac{\|\varrho_0\|_{L^1(\mathbb{R}^d)}}{\|\Gamma\|_{L^1(\mathbb{R}^d)}}\right)^{1/2}. \end{aligned} \tag{5.4}$$

We check

$$\varrho(t,x) = |\psi(t,x)|^2 = \frac{1}{\tau(t)^d}\left|\Psi\left(t,\frac{x}{\tau(t)}\right)\right|^2 \frac{\|\varrho_0\|_{L^1(\mathbb{R}^d)}}{\|\Gamma\|_{L^1(\mathbb{R}^d)}},$$

so in view of (1.24), $R = |\Psi|^2$, and

$$\sqrt{\varrho}u(t,x) = \frac{j(t,x)}{\sqrt{\varrho(t,x)}} = \frac{\epsilon\,\mathrm{Im}\left(\bar{\psi}\nabla\psi\right)(t,x)}{|\psi(t,x)|}$$
$$= \frac{\epsilon}{\tau(t)^{1+d/2}}\,\mathrm{Im}\left(\frac{\Psi}{|\Psi|}\nabla\Psi\right)\left(t,\frac{x}{\tau(t)}\right)\left(\frac{\|\varrho_0\|_{L^1(\mathbb{R}^d)}}{\|\Gamma\|_{L^1(\mathbb{R}^d)}}\right)^{1/2}$$
$$+\frac{\dot{\tau}(t)}{\tau(t)}\frac{x}{\tau(t)^{d/2}}\left|\Psi\left(t,\frac{x}{\tau(t)}\right)\right|\left(\frac{\|\varrho_0\|_{L^1(\mathbb{R}^d)}}{\|\Gamma\|_{L^1(\mathbb{R}^d)}}\right)^{1/2}$$
$$= \frac{\epsilon}{\tau(t)^{1+d/2}}\,\mathrm{Im}\left(\frac{\Psi}{|\Psi|}\nabla\Psi\right)\left(t,\frac{x}{\tau(t)}\right)\left(\frac{\|\varrho_0\|_{L^1(\mathbb{R}^d)}}{\|\Gamma\|_{L^1(\mathbb{R}^d)}}\right)^{1/2}+\frac{\dot{\tau}(t)}{\tau(t)}x\sqrt{\varrho(t,x)},$$

hence, in view of (1.24),

$$\sqrt{R}U = \epsilon\,\mathrm{Im}\left(\frac{\bar{\Psi}}{|\Psi|}\nabla\Psi\right).$$

In view of [13], for $\psi_0 \in H^1 \cap \mathcal{F}(H^1)$, (5.4) has a global solution $\Psi \in L^\infty_{\mathrm{loc}}(\mathbb{R}; H^1 \cap \mathcal{F}(H^1))$, which satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\epsilon^2}{2\tau(t)^2}\|\nabla\Psi(t)\|_{L^2}^2 + \int_{\mathbb{R}^d}|\Psi(t,y)|^2\log|\Psi(t,y)|^2\mathrm{d}y\right.$$
$$\left.+\int_{\mathbb{R}^d}|y|^2\,|\Psi(t,y)|^2\,\mathrm{d}y\right) = -\frac{\epsilon^2\dot{\tau}(t)}{\tau(t)^3}\|\nabla\Psi(t)\|_{L^2}^2.$$

Integrating in time and rewriting the quantities involved in this relation in terms of $(\sqrt{R}, \sqrt{R}U)$, we recover (1.7).

Appendix A. Proof of identity (2.19)

Proof. — We recall that, the first step in the computation of (2.19) is to set $\Phi = \chi\nu\nabla\log R/\tau^2$ in (2.5). This yields:

$$\begin{aligned}
&\int_0^\infty\!\!\int_{\mathbb{T}^d_\ell} RU\cdot\partial_t\Phi + \int_0^\infty\!\!\int_{\mathbb{T}^d_\ell} \frac{1}{\tau^2} RU\otimes U : \nabla\Phi \\
&= \int_0^\infty\!\!\int_{\mathbb{T}^d_\ell} R\left(2y\cdot\Phi - \operatorname{div}\Phi\right) + r_0\int_0^\infty\!\!\int_{\mathbb{T}^d_\ell} \frac{1}{\tau^2} U\cdot\Phi \\
&\quad + r_1\int_0^\infty\!\!\int_{\mathbb{T}^d_\ell} \frac{1}{\tau^2} R|U|^2U\cdot\Phi \\
&\quad + \epsilon^2\int_0^\infty\!\!\int_{\mathbb{T}^d_\ell} \frac{1}{\tau^2}\left[\frac{\Delta\sqrt{R}}{\sqrt{R}}\operatorname{div}(R\Phi)\right] \\
&\quad + \nu\int_0^\infty\!\!\int_{\mathbb{T}^d_\ell} \frac{1}{\tau^2} R\mathbb{D}U : \nabla\Phi + \nu\int_0^\infty\!\!\int_{\mathbb{T}^d_\ell} \frac{\dot\tau}{\tau} R\operatorname{div}\Phi \\
&\quad + \delta_1\int_0^\infty\!\!\int_{\mathbb{T}^d_\ell} \frac{1}{\tau^2}\nabla U : \nabla R\otimes\Phi + \delta_2\int_0^\infty\!\!\int_{\mathbb{T}^d_\ell} \frac{1}{\tau^2}\Delta U\cdot\Delta\Phi \\
&\quad + \eta_1\int_0^\infty\!\!\int_{\mathbb{T}^d_\ell} R^{-\alpha}\operatorname{div}\Phi + \eta_2\int_0^\infty\!\!\int_{\mathbb{T}^d_\ell} \frac{1}{\tau^2}\Delta^{s+1}R\Delta^s\left[\nabla R\cdot\Phi + R\operatorname{div}\Phi\right].
\end{aligned}$$

We number the integrals on the right-hand side I_1 to I_{10} successively:

$$\begin{aligned}
&I_1 = \int_0^\infty \frac{\chi\nu}{\tau^2}\int_{\mathbb{T}^d_\ell}\Big(4|\nabla\sqrt{R}|^2 - 2dR\Big), \quad I_2 = r_0\int_0^\infty\frac{\chi\nu}{\tau^4}\int_{\mathbb{T}^d_\ell} U\cdot\nabla\log R,\\
&I_3 = r_1\int_0^\infty\frac{\chi\nu}{\tau^4}\int_{\mathbb{T}^d_\ell}|U|^2U\cdot\nabla R, \qquad I_4 = \epsilon^2\int_0^\infty\frac{\chi\nu}{\tau^4}\int_{\mathbb{T}^d_\ell} R\left|\nabla^2\log R\right|^2,\\
&I_5 = \int_0^\infty\frac{\chi\nu^2}{\tau^4}\int_{\mathbb{T}^d_\ell} R\mathbb{D}U : \nabla^2\log R, \qquad I_6 = -\int_0^\infty\frac{\chi\nu^2\dot\tau}{\tau^3}\int_{\mathbb{T}^d_\ell} 4|\nabla\sqrt{R}|^2,\\
&I_7 = \delta_1\int_0^\infty\frac{\chi\nu}{\tau^4}\int_{\mathbb{T}^d_\ell}\nabla U : \nabla R\otimes\nabla\log R,\\
&\qquad\qquad\qquad\qquad\qquad\qquad I_8 = \delta_2\int_0^\infty\frac{\chi\nu}{\tau^4}\int_{\mathbb{T}^d_\ell}\Delta U\cdot\nabla\Delta\log R,\\
&I_9 = \eta_1\int_0^\infty\frac{4\chi\nu}{\alpha\tau^2}\int_{\mathbb{T}^d_\ell}\left|\nabla\sqrt{R^{-\alpha}}\right|^2, \qquad I_{10} = \eta_2\int_0^\infty\frac{\chi\nu}{\tau^4}\int_{\mathbb{T}^d_\ell}\left|\Delta^{s+1}R\right|^2.
\end{aligned}$$

While, we rewrite the left-hand side:

$$\begin{aligned}
LHS = &-\left\langle\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\nu}{\tau^2}\int_{\mathbb{T}^d_\ell} RU\cdot\nabla\log R\right],\chi\right\rangle - \int_0^\infty\frac{2\chi\nu\dot\tau}{\tau^3}\int_{\mathbb{T}^d_\ell} RU\cdot\nabla\log R\\
&+\int_0^\infty\frac{\chi\nu}{\tau^2}\int_{\mathbb{T}^d_\ell} RU\cdot\nabla\partial_t\log R + \int_0^\infty\frac{\chi\nu}{\tau^4}\int_{\mathbb{T}^d_\ell} RU\otimes U : \nabla^2\log R,
\end{aligned}$$

where we denote with brackets the duality in the sense of distributions. We proceed by computing the third term (denoted L_1) in the right-hand side of this identity. For this, we remark that differentiating the continuity equation (1.20a), we obtain (in $L^2_{\mathrm{loc}}(\mathbb{R}^+; L^2(\mathbb{T}^d_\ell))$):

$$\partial_t \left(R\nabla \log R\right) = -\frac{1}{\tau^2}\operatorname{div}\left(R\nabla \log R \otimes U\right) - \frac{1}{\tau^2}\operatorname{div}\left(R\nabla U\right) + \frac{\delta_1}{\tau^2}\Delta\nabla R,$$

splitting the left-hand side of this identity and calling again the continuity equation, we conclude that:

$$\begin{aligned} R\partial_t\nabla\log R &= \frac{1}{\tau^2}\operatorname{div}(RU)\nabla\log R - \frac{\delta_1}{\tau^2}\Delta R\nabla\log R \\ &\quad - \frac{1}{\tau^2}\operatorname{div}(R\nabla\log R\otimes U) - \frac{1}{\tau^2}\operatorname{div}\left(R\nabla^\top U\right) + \frac{\delta_1}{\tau^2}\Delta\nabla R. \end{aligned}$$

We infer then that, a.e. (in $(0,\infty)$), we have:

$$\begin{aligned} \int_{\mathbb{T}^d_\ell} RU\cdot\partial_t\nabla\log R &= -\frac{1}{\tau^2}\int_{\mathbb{T}^d_\ell} RU\otimes U : \nabla^2\log R + \frac{1}{\tau^2}\int_{\mathbb{T}^d_\ell} R\nabla U^\top : \nabla U \\ &\quad - \frac{\delta_1}{\tau^2}\int_{\mathbb{T}^d_\ell}\frac{\Delta R}{R}\operatorname{div}(RU). \end{aligned}$$

Plugging this identity into LHS, we obtain:

$$\begin{aligned} LHS &= -\left\langle \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\nu}{\tau^2}\int_{\mathbb{T}^d_\ell} RU\cdot\nabla\log R\right], \chi\right\rangle - \int_0^\infty \frac{2\chi\nu\dot{\tau}}{\tau^3}\int_{\mathbb{T}^d_\ell} RU\cdot\nabla\log R \\ &\quad + \int_0^\infty \frac{\chi\nu}{\tau^4}\int_{\mathbb{T}^d_\ell} R\nabla U : \nabla^\top U - \delta_1\int_0^\infty\frac{\chi\nu}{\tau^4}\int_{\mathbb{T}^d_\ell}\frac{\Delta R}{R}\operatorname{div}(RU). \end{aligned}$$

Finally, combining the computations of the right-hand side and left-hand side, we re-interpret our identity as:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\nu}{\tau^2}\int_{\mathbb{T}^d_\ell} RU\cdot\nabla\log R\right] &+ \frac{2\nu\dot\tau}{\tau^3}\int_{\mathbb{T}^d_\ell} RU\cdot\nabla\log R\\
&+\frac{\epsilon^2\nu}{\tau^4}\int_{\mathbb{T}^d_\ell} R\left|\nabla^2\log(R)\right|^2+\left(\frac{\nu}{\tau^2}-\frac{\nu^2\dot\tau}{\tau^3}\right)\int_{\mathbb{T}^d_\ell}4|\nabla\sqrt{R}|^2\\
&+\frac{4\eta_1\nu}{\alpha}\int_{\mathbb{T}^d_\ell}\left|\nabla\sqrt{R^{-\alpha}}\right|^2+\frac{\eta_2\nu}{\tau^4}\int_{\mathbb{T}^d_\ell}\left|\Delta^{s+1}R\right|^2\\
&=\frac{2d\nu}{\tau^2}\int_{\mathbb{T}^d_\ell}R-\frac{r_0\nu}{\tau^4}\int_{\mathbb{T}^d_\ell}U\cdot\nabla\log R-\frac{r_1\nu}{\tau^4}\int_{\mathbb{T}^d_\ell}|U|^2U\cdot\nabla R\\
&\quad-\frac{\nu^2}{\tau^4}\int_{\mathbb{T}^d_\ell}R\mathbb{D}U:\nabla^2\log R\\
&\quad-\frac{\delta_1\nu}{\tau^4}\int_{\mathbb{T}^d_\ell}\nabla U:\nabla R\otimes\nabla\log R-\frac{\delta_2\nu}{\tau^4}\int_{\mathbb{T}^d_\ell}\Delta U\cdot\nabla\Delta\log R\\
&\quad-\frac{\delta_1\nu}{\tau^4}\int_{\mathbb{T}^d_\ell}\frac{\Delta R}{R}\operatorname{div}(RU)+\frac{\nu}{\tau^4}\int_{\mathbb{T}^d_\ell}\nabla U:\nabla^\top U.\end{aligned}$$

This completes the proof of identity (2.19). □

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