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Pentagon representations and complex projective structures on closed surfaces

PENTAGON REPRESENTATIONS AND COMPLEX PROJECTIVE STRUCTURES ON CLOSED SURFACES

by Thomas LE FILS

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ABSTRACT. — We define a class of representations of the fundamental group of a closed surface of genus 2 to $\mathrm{PSL}_2(\mathbb{C})$: *the pentagon representations*. We show that they are exactly the non-elementary $\mathrm{PSL}_2(\mathbb{C})$-representations of surface groups that do not admit a Schottky decomposition, i.e. a pants decomposition such that the restriction of the representation to each pair of pants is an isomorphism onto a Schottky group. In doing so, we exhibit a gap in the proof of Gallo, Kapovich and Marden that every non-elementary representation of a surface group to $\mathrm{PSL}_2(\mathbb{C})$ is the holonomy of a projective structure, possibly with one branched point of order 2. We show that pentagon representations arise as such holonomies and repair their proof.

RÉSUMÉ. — Nous définissons une classe de représentations du groupe fondamental d'une surface fermée de genre 2 dans $\mathrm{PSL}_2(\mathbb{C})$: les représentations pentagones. Nous montrons que ce sont exactement les représentations d'un groupe de surface dans $\mathrm{PSL}_2(\mathbb{C})$ qui n'admettent pas de décomposition de Schottky, *i.e.* de décomposition en pantalons telle que la restriction de la représentation à chaque pantalon est un isomorphisme sur un groupe de Schottky. Ce faisant, nous exhibons une lacune dans la preuve de Gallo, Kapovich et Marden du fait que toute représentation non-élémentaire d'un groupe de surface dans $\mathrm{PSL}_2(\mathbb{C})$ est l'holonomie d'une structure projective, avec éventuellement un point de branchement d'ordre 2. Nous montrons que les représentations pentagones sont de telles holonomies et réparons leur preuve.

1. Introduction

Denote by $\Sigma_{g,n}$ an oriented compact surface of genus g with n boundary components, and by $\Gamma_{g,n}$ a fundamental group of $\Sigma_{g,n}$ for all $g, n \geqslant 0$. For simplicity, we denote $\Sigma_{g,0}$ by Σ_g and $\Gamma_{g,0}$ by Γ_g.

A complex projective structure on Σ_g is a (G, X) structure with $G = \mathrm{PSL}_2(\mathbb{C})$ and $X = \mathbb{CP}^1$, that is the datum of an atlas of charts with values

in $\mathbb{CP}^1$, whose transition maps are restrictions of Möbius transformations. We can also allow branched points in the definition and get the notion of branched projective structure, see [7, Section 1.4] for a concise definition. We denote by $\mathcal{P}(\Sigma_g)$, resp. $\mathcal{P}_b(\Sigma_g)$, the set of unbranched complex projective structures, resp. the set of projective structures with a single branched point, of order 2. The datum of such a structure on Σ_g gives rise to a holonomy map, well-defined up to conjugacy (see [19, Chapter 3] for more information on (G, X)-structures). Hence we have a map:

$$\mathrm{hol} : \mathcal{P}(\Sigma_g) \sqcup \mathcal{P}_b(\Sigma_g) \to \mathrm{Hom}(\Gamma_g, \mathrm{PSL}_2(\mathbb{C}))/\mathrm{PSL}_2(\mathbb{C}).$$

This map establishes a relationship between $\mathrm{PSL}_2(\mathbb{C})$-representations of surface groups and projective structures. The study of this relationship has a long history. There is a natural complex structure on $\mathcal{P}(S)$, induced by its identification with the quadratic forms (see [9] for example). Hejhal, Earl and Hubbard [5, 11, 12] showed that the map $\mathrm{hol}_{|\mathcal{P}(\Sigma_g)}$ is a local biholomorphism. However, it is known that $\mathrm{hol}_{|\mathcal{P}(\Sigma_g)}$ is neither injective nor a covering map. We refer to [4] for more information about projective structures.

The question of finding which representations arise as the holonomy of a complex projective structure has been open for a long time. Poincaré himself asked it in the case where Σ is a punctured sphere, see [15, Paragraph 4]. Very recently, Gupta announced an answer for every punctured surface [10]. In [7], Gallo, Kapovich and Marden provided a complete answer for closed surfaces. They showed that the image of hol is the set of non-elementary representations. The main part of the theorem is the proof that every non-elementary representation is in the image of hol.

The strategy of [7] consists in first proving that every non-elementary representation admits a *Schottky decomposition* in the following sense.

DEFINITION 1.1. — *A Schottky decomposition for a representation* ρ : $\Gamma_g \to \mathrm{PSL}_2(\mathbb{C})$ *is a pants decomposition of* $\Sigma_g = \cup P_i$, *such that for all* i, *the restriction* $\rho_{|P_i} : \pi_1(P_i) \to \mathrm{PSL}_2(\mathbb{C})$ *is an isomorphism onto a Schottky group.*

Once such a decomposition is found, the authors put a projective structure on each pair of pants P_i, whose holonomy is given by the restriction $\rho_{|P_i}$. Then they glue the pants together with cylinders. It might be required to add a branched point of order 2 in one of the pairs of pants in order to make all the gluings possible.

Perhaps one main contribution of the present article is to exhibit a gap in the proof by Gallo, Kapovich and Marden of the existence of a Schottky

decomposition for every non-elementary representation. We will establish later (see Theorem 1.4 below), essentially along their proof, that such a decomposition exists provided $g \geqslant 3$. However, in genus 2, the hyperelliptic involution yields counterexamples that we now introduce.

Let us recall that the mapping class group $\mathrm{Mod}(\Sigma_2)$ of Σ_2 has its center generated by the hyperelliptic involution. Let $\varphi \in \mathrm{Homeo}^+(\Sigma_2)$ be one of its representatives. The orbifold fundamental group Γ of Σ_2/φ has the following presentation:

$$\Gamma = \langle q_1, q_2, q_3, q_4, q_5, q_6 \mid q_i^2 = 1, q_1 q_2 \ldots q_6 = 1 \rangle.$$

The group Γ_2 is naturally an index two subgroup of Γ (see Section 3).

DEFINITION 1.2. — *A* pentagon representation *is a non-elementary representation that is the restriction of a representation* $\rho : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ *such that* $\rho(q_i) = \mathrm{id}$ *for exactly one* $1 \leqslant i \leqslant 6$.

This definition is reminiscent of the hourglass representations considered in [14]. We will see that the action of $\mathrm{Mod}(\Sigma_2) \simeq \mathrm{Out}^+(\Gamma_2)$ on the conjugacy classes of representations $\Gamma_2 \to \mathrm{PSL}_2(\mathbb{C})$ preserves the pentagon representations. Since there are only two pants decompositions of Σ_2 up to the action of the mapping class group, it suffices to check that these decompositions cannot be Schottky decompositions of a pentagon representation to prove the following.

PROPOSITION 1.3. — *A pentagon representation does not admit a Schottky decomposition.*

As a first step into reparing the proof of [7], we prove that there exists a Schottky decomposition for every other non-elementary ρ.

THEOREM 1.4. — *A representation* $\rho : \Gamma_g \to \mathrm{PSL}_2(\mathbb{C})$ *admits a Schottky decomposition if and only if* ρ *is non-elementary and is not a pentagon representation.*

In fact we find a Schottky decomposition with g pairs of pants glued to themselves in the non-pentagon case. This allows us to use the second part of [7], and hence fix the gap of the proof in this case.

As a corollary of this characterization of the pentagon representations, we will see that even if they do not admit a Schottky decomposition, they still have a *loxodromic decomposition*.

COROLLARY 1.5. — *If* $\rho : \Gamma_g \to \mathrm{PSL}_2(\mathbb{C})$ *is non-elementary, then there exists a pants decomposition of* Σ_g *whose boundary curves are taken by* ρ *to loxodromic isometries.*

We hence also repair the proof of this corollary which was used for example in [18, Proposition 1].

We show that the pentagon representations have odd Stiefel–Whitney class, thus they cannot be in $\mathrm{hol}(\mathcal{P}(\Sigma_2))$ (see [7, Corollary 11.2.3]). It remains to understand whether they are in $\mathrm{hol}(\mathcal{P}_b(\Sigma_2))$ or not. We answer positively:

THEOREM 1.6. — *A pentagon representation is the holonomy of a branched projective structure with exactly one branched point, which is of order* 2.

Theorem 1.6 somehow complements a recent theorem of Baba [1], which states that every unbranched complex projective structure is obtained by gluing Schottky pants as in [7]. Indeed the analogous theorem for $\mathcal{P}_b(\Sigma_2)$ cannot hold for the pentagon representations have no Schottky decomposition and yet are in $\mathrm{hol}(\mathcal{P}_b(\Sigma_2))$.

In view of possible generalizations, we provide some alternative (and maybe simpler) proofs of some intermediate results of [6], that we use to prove Theorem 1.4. In particular, we give a new proof of the existence of a special handle in $\Sigma_{g,n}$: a subsurface which is a once punctured torus, whose fundamental group is generated by two elements a and b such that $\rho(a)$ and $\rho(b)$ are loxodromic without a common fixed point, where $\rho : \Gamma_{g,n} \to \mathrm{PSL}_2(\mathbb{C})$ is non-elementary and $g \geqslant 1$. Our approach begins by finding a handle on which the restriction of ρ is non-elementary. This leads us to study the non-elementary representations of the punctured torus, up to the action of the mapping class group. This study is reminiscent of the work of Goldman in [8]. In particular we prove the following.

THEOREM 1.7. — *If $\rho : \Gamma_{1,1} \to \mathrm{PSL}_2(\mathbb{C})$ is non-elementary, then there exists simple loops a and b generating $\Gamma_{1,1}$ such that $\rho(a)$ and $\rho(b)$ are loxodromic.*

It follows that:

COROLLARY 1.8. — *If $g \geqslant 1$, and $\rho : \Gamma_{g,n} \to \mathrm{PSL}_2(\mathbb{C})$ is non-elementary, then there exists a simple curve $\gamma \in \Gamma_{g,n}$ such that $\rho(\gamma)$ is loxodromic.*

Note that this theorem does not hold for $g = 0$. Indeed, Baba observed that some $\mathrm{PSL}_2(\mathbb{R})$-representations of $\Gamma_{0,4}$ studied in [2] are non-elementary and send every simple curve to an elliptic isometry. Later in [3], Deroin and Tholozan exhibited a class of non-elementary representations of $\Gamma_{0,n}$ into $\mathrm{PSL}_2(\mathbb{R})$ that send every simple closed curve to a non-hyperbolic isometry for every $n \geqslant 4$.

Let us now describe the organization of the paper. In Section 2, we recall general facts about curves on surfaces before proving the existence of a special handle given a non-elementary ρ. We proceed as described above and prove Theorem 1.7. Then we study pentagon represensions in Section 3. Section 4 is devoted to the proof of Theorem 1.4. Finally, we show that the pentagon representations are in $\mathrm{hol}(\mathcal{P}_b(\Sigma_2))$ in Section 5.

Acknowledgements

2. Special handle

2.1. Reminder of curves on surfaces

If γ and δ are elements of $\Gamma_{g,n}$, the loop $\gamma\delta$ is the path following δ first and then γ. We also define the commutator of γ and δ to be $[\gamma, \delta] = \delta^{-1}\gamma^{-1}\delta\gamma$.

We fix a system of *standard generators* $a_1, b_1, \ldots a_g, b_g, c_1, \ldots c_n$ of $\Gamma_{g,n}$, in the same way as Figure 2.1 for $(g, n) = (2, 1)$.
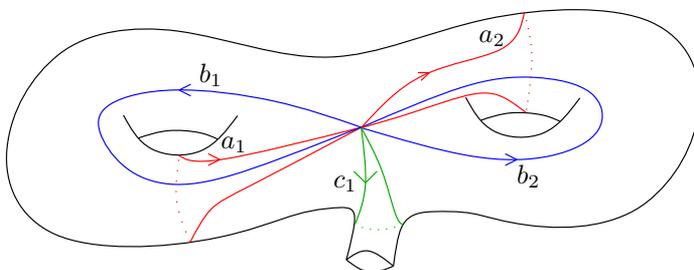

Figure 2.1. Standard generators for $\Gamma_{2,1}$.

Denote by $\mathcal{S}_{\mathrm{ns}}$ the set of $\gamma \in \Gamma_{g,n}$ such that γ is freely homotopic to an essential non-separating simple closed curve. If $f \in \mathrm{Homeo}^+(\Sigma_g, *)$, i.e. f fixes the base point $*$ of Γ_g, we denote by f_* the automorphism of Γ_g induced by f.

LEMMA 2.1. — *Given γ and δ in $\mathcal{S}_{\mathrm{ns}}$, there exists $f \in \mathrm{Homeo}^+(\Sigma_{g,n}, *)$ such that $f_*(\gamma) \in \{\delta, \delta^{-1}\}$.*

Proof. — This is a consequence of classification of compact surfaces. See [6, Section 1.3]. Note that if $g \geqslant 1$, and $* \notin \partial\Sigma_{g,n}$, we can even choose f such that $f_*(\gamma) = \delta$. $\qquad\square$

Now assume that $g \geqslant 1$, and that $* \notin \partial\Sigma_{g,n}$.

LEMMA 2.2. — *Let $\gamma, \delta \in \mathcal{S}_{\mathrm{ns}}$ be such that they have representatives that cross only at $*$ as in Figure 2.2.*
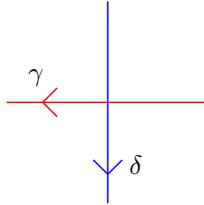

Figure 2.2. Handle.

There exists $f \in \mathrm{Homeo}^+(\Sigma_{g,n}, *)$ such that $f_*(\gamma) = a_1$ and $f_*(\delta) = b_1$.

Proof. — This is also a consequence of the classification of surfaces, see [6, Section 1.3.3]. $\qquad\square$

2.2. Existence of a non-elementary handle

Let us recall that $\mathrm{PSL}_2(\mathbb{C})$ is the group of isometries of $\mathbb{H}^3$. Its action on $\mathbb{H}^3$ extends to the Gromov boundary $\partial\mathbb{H}^3 \simeq \mathbb{CP}^1$ where it acts by Möbius transformations. We say that a homomorphism $\rho : G \to \mathrm{PSL}_2(\mathbb{C})$ is elementary if there exists $z \in \mathbb{H}^3 \cup \partial\mathbb{H}^3$ such that $\rho(G) \cdot z$ is finite. We have, see for example [17, Chapter 5], the following characterization.

PROPOSITION 2.3 ([17]). — *A homomorphism $\rho : G \to \mathrm{PSL}_2(\mathbb{C})$ is elementary if and only if there exists a set $S \subset \mathbb{H}^3 \cup \partial\mathbb{H}^3$ containing 1 or 2 points such that $\rho(G)$ stabilizes S.*

Suppose $g \geqslant 1$ and $(g, n) \neq (1, 0)$. Fix a non-elementary $\rho : \Gamma_{g,n} \to \mathrm{PSL}_2(\mathbb{C})$. We also assume that the base point of $\Gamma_{g,n}$ is not on the boundary $\partial\Sigma_{g,n}$.

DEFINITION 2.4. — *A non-elementary handle is a subsurface of* $S \subset \Sigma_{g,n}$ *which is a once punctured torus, such that the restriction* $\rho_{|S} : \Gamma_{1,1} \to \mathrm{PSL}_2(\mathbb{C})$ *is non-elementary.*

Let us recall three lemmas from [7]. We will assume that the reader is familiar with three dimensional hyperbolic geometry, in particular with the notions of loxodromic, elliptic, parabolic isometries and their axes, as introduced for example in [13, Section 3.5].

LEMMA 2.5. — *Suppose* $\alpha, \beta \in \mathrm{PSL}_2(\mathbb{C})$.

- *If* α *is loxodromic, and* β *does not send its attractive fixed point to its repulsive one (resp. its repulsive one to its attractive one), then there exists* $K > 0$ *such that* $\alpha^k \beta$ *is loxodromic for all* $k \geqslant K$ *(resp. for all* $k \leqslant -K$*). Moreover, the trace of* $\alpha^k \beta$ *can be made arbitrarily large.*
- *If* α *is parabolic and* β *does not fix its fixed point, then* $\alpha^k \beta$ *is loxodromic for* $|k|$ *large enough.*

Proof. — This is a trace computation, see [7, Lemma 2.2.1]. □

LEMMA 2.6. — *If there exists* $\gamma \in \mathcal{S}_{\mathrm{ns}}$ *such that* $\rho(\gamma)$ *is loxodromic or parabolic, then there is a handle in* $\Sigma_{g,n}$ *on which the restriction of* ρ *is non-elementary.*

Proof. — We can assume that $\gamma = a_1$. If $\rho(b_1)$ stabilizes the set of fixed points of $\rho(\gamma)$, take $c \in \{a_2^{-1}, b_2, \ldots, b_g, c_1, \ldots, c_n\}$ such that $\rho(c)$ does not, and apply a homeomorphism so that (a_1, cb_1) becomes (a_1, b_1), which exists by Lemma 2.2.

If $\rho(a_1)$ is parabolic, then the group generated by $\rho(a_1)$ and $\rho(b_1)$ is non-elementary. This is also the case if $\rho(a_1)$ is loxodromic and $\rho(b_1)$ does not share a fixed point with $\rho(a_1)$. Otherwise denote by p this common fixed point. Choose $c \in \{a_2^{-1}, b_2, \ldots, b_g, c_1, \ldots, c_n\}$ such that $\rho(c)$ does not fix p. Then $(ca_1^{-1}, a_1^k b_1)$ is such a handle for some k. Indeed there exists $K > 0$ such that $\rho(a_1^k b_1)$ is loxodromic for $k \geqslant K$ or $k \leqslant -K$. Moreover the fixed point of $\rho(a_1^k b_1)$ which is not p is different for any two k in that range: if $\rho(a_1^k b_1)(q) = \rho(a_1^{k+m} b_1)(q) = q$, then $\rho(a_1)(q) = \rho(b_1)(q) = q$ and $q = p$. Thus we can take a k in that range that does not share a fixed point with $\rho(ca_1^{-1})$. □

LEMMA 2.7. — *If* α *and* β *are elliptic with different axes and* $\alpha\beta$ *is elliptic, then:*

(1) *The axes of* α *and of* β *lie in a plane* P.
(2) *If they are disjoint, they are orthogonal to a plane.*

(3) *The axis of $\alpha\beta$ is not contained in P.*

Proof. — Decompose α and β as: $\alpha = s_{\ell_2}s_{\ell_1}$ and $\beta = s_{\ell_3}s_{\ell_2}$ where s_{ℓ_i} is the elliptic involution with axis ℓ_i. See [7, Lemma 3.4.1, 3.4.3]. $\qquad\square$

PROPOSITION 2.8. — *Let $g \geqslant 1$ and $n \geqslant 0$ be such that $n \neq 0$ if $g = 1$. Let $\rho : \Gamma_{g,n} \to \mathrm{PSL}_2(\mathbb{C})$ be a non-elementary representation. There exists a non-elementary handle on $\Sigma_{g,n}$.*

Proof. — We may assume $g \geqslant 2$ or $n \geqslant 2$ because the proposition is obvious otherwise. By contradiction suppose that the restriction of ρ to any handle on $\Sigma_{g,n}$ is elementary. For every $\gamma \in \mathcal{S}_{\mathrm{ns}}$, $\rho(\gamma)$ is elliptic or the identity. As before, we can assume that $\rho(a_1)$ is not the identity, that $\rho(b_1)$ is not the identity and does not have the same axis as $\rho(a_1)$. Their axes cross since a_1 and b_1 bound a handle on which ρ is elementary. Pick $c' \in \{a_2, b_2^{-1}, \ldots, b_g^{-1}, c_1^{-1}, \ldots, c_n^{-1}\}$ so that $\rho(c')$ does not fix the common fixed point of $\rho(a_1)$ and $\rho(b_1)$. Let $c = c'b_1^{-1}a_1^{-1}$, so that any two out of a_1, b_1, c form a handle.

The axes of $\rho(a_1), \rho(b_1)$ and $\rho(c)$ form a triangle T.

Since $\rho(ca_1b_1) = \rho(c')$ is elliptic, the axis of $\rho(c)$ is coplanar with the axis of $\rho(a_1b_1)$. The only plane that contains both the axis of $\rho(c)$ and the common fixed point of $\rho(a_1)$ and $\rho(b_1)$ is the one spanned by T. This is a contradition by Lemma 2.7: the axis of $\rho(a_1b_1)$ is not coplanar with both the axis of $\rho(a_1)$ and the axis of $\rho(b_1)$. $\qquad\square$

2.3. Non-elementary representations of the punctured torus

The aim of this subsection is to prove Theorem 1.7. Let $\rho : \Gamma_{1,1} \to \mathrm{PSL}_2(\mathbb{C})$ be a non-elementary representation. Our strategy is to precompose ρ by automorphisms of $\Gamma_{1,1}$ induced by Dehn twists along curves homotopic to the standard generators a and b of $\Gamma_{1,1}$. The automorphisms we consider are defined by $(\varphi_k(a), \varphi_k(b)) = (b^k a, b)$ and $(\psi_k(a), \psi_k(b)) = (a, a^k b)$ for $k \in \mathbb{Z}$.

LEMMA 2.9. — *If there exists $\gamma \in \mathcal{S}_{\mathrm{ns}}$ such that $\rho(\gamma)$ is parabolic or loxodromic, then Theorem 1.7 holds.*

Proof. — We may assume that $a \in \{\gamma, \gamma^{-1}\}$ by Lemma 2.1. If $\rho(a)$ is loxodromic, then so is $\rho(a^k b)$ for some k. Apply the Dehn twist that changes (a, b) to $(a, a^k b)$. If $\rho(a)$ is parabolic, then $\rho(a^k b)$ is loxodromic if $|k|$ is large enough. We change (a, b) to $(a, a^k b)$ with a Dehn twist, and return to the previous case. $\qquad\square$

2.3.1. Existence of a loxodromic

We now show that it is not possible for ρ to send every $\gamma \in \mathcal{S}_{\mathrm{ns}}$ to an elliptic element or to the identity. Assume that is does by contradiction.

LEMMA 2.10. — *The representation ρ has some conjugate into $\mathrm{PSL}_2(\mathbb{R})$: it preserves a plane.*

Proof. — Since ρ is non-elementary, the isometries $\rho(a)$ and $\rho(b)$ are not the identity, and their axes do not cross. Since $ab \in \mathcal{S}_{\mathrm{ns}}$, it follows from Lemma 2.7 that these axes are orthogonal to a plane. This plane and its boundary circle are preserved by both $\rho(a)$ and $\rho(b)$, hence they are preserved by $\rho(\Gamma_{1,1})$. $\qquad\square$

We will thus now assume that $\rho(\Gamma_{1,1}) \subset \mathrm{PSL}_2(\mathbb{R})$ and consider the elements of $\rho(\Gamma_{1,1})$ as isometries of the hyperbolic plane.

LEMMA 2.11. — *There exists $N \geqslant 1$ such that for all $\gamma \in \mathcal{S}_{\mathrm{ns}}$, $\rho(\gamma)^N = \mathrm{id}$.*

Here we denoted by id the element $\pm \mathrm{Id}$ of $\mathrm{PSL}_2(\mathbb{C})$.

Proof. — Let us first show that $\rho(\gamma)$ has finite order for every $\gamma \in \mathcal{S}_{\mathrm{ns}}$. Suppose for a contradiction that there exists $\gamma \in \mathcal{S}_{\mathrm{ns}}$ such that $\rho(\gamma)$ has infinite order. We may assume without loss of generality that $\gamma = a$. Let us write $\rho(a^n) = s_{\ell_n} s_{\ell'}$ and $\rho(b) = s_{\ell'} s_{\ell''}$ as products of reflections across geodesics, where ℓ' is the geodesic passing through both the fixed point of $\rho(a)$ and the fixed point of $\rho(b)$. Take $n \geqslant 1$ such that ℓ_n does not cross ℓ''. Then $a^n b \in \mathcal{S}_{\mathrm{ns}}$ and $\rho(a^n b)$ is not elliptic, which is a contradiction.
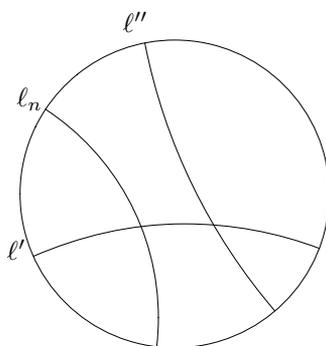

Figure 2.3. Product of reflections.

This can also be seen with a computation.

By Selberg's lemma, there is a torsion free subgroup Λ of $\rho(\Gamma_{1,1})$ of finite index n. Let $\gamma \in \mathcal{S}_{\mathrm{ns}}$. Two of the cosets $\Lambda, \rho(\gamma)\Lambda, \ldots \rho(\gamma)^n\Lambda$ are equal. Thus $\rho(\gamma)^j \in \Lambda$ for a $j \leqslant n$. This implies that $\rho(\gamma)^j = \mathrm{id}$ and it thus suffices to take $N = n!$. $\square$

We get a contradiction from this lemma.

LEMMA 2.12. — *We can increase the order of $\rho(a)$ or $\rho(b)$ by applying a Dehn twist.*

Proof. — Write $\rho(a) = s_{\ell_2}s_{\ell_1}$ and $\rho(b) = s_{\ell_3}s_{\ell_2}$ as products of reflections. Since $\rho(ba)$ is elliptic, the lines ℓ_1, ℓ_2 and ℓ_3 form a triangle, see Figure 2.4. Let us denote by θ_i, for $i \in \{1,3\}$, the angle of this triangle at the crossing of ℓ_i and ℓ_2.
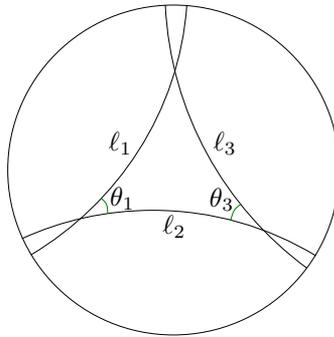

Figure 2.4. Triangle (ℓ_1, ℓ_2, ℓ_3).

Suppose $\theta_3 \leqslant \theta_1$. The fixed points of $\rho(b^{-1}a)$ and of $\rho(ba)$ are separated by the line ℓ_2. Indeed if not, one could construct a triangle with total angle greater than π. Let q be the intersection of ℓ_1 with its perpendicular that goes through the fixed point of $\rho(b)$. The function that maps $p \in \ell_1$ to the angle of the triangle formed by p and the two fixed points of $\rho(a)$ and $\rho(b)$ decreases when p moves away from q. If ℓ_3 intersects ℓ_1 in the part of $\mathbb{H}^2 \setminus \ell_2$ that does not contain q, then the order of $\rho(ba)$ is less than the order of $\rho(a)$. In this case change the handle to (ba, b). If not, change the handle to $(b^{-1}a, b)$.

The case where $\theta_1 \leqslant \theta_3$ is symmetric. $\square$

2.3.2. Special handle

DEFINITION 2.13. — *A special handle is a non-elementary handle with standard generators a, b such that $\rho(a)$ and $\rho(b)$ are loxodromic, and such that $\rho(a)$ does not send a fixed point of $\rho(b)$ to the other.*

Thanks to Theorem 1.7, we can suppose that the standard generators a and b are sent by ρ to loxodromic isometries. We now modify this handle to get a special one. Our proof relies on the following results of [7, Lemma 2.2.1, 2.2.2, 2.2.3], obtained by trace computations.

LEMMA 2.14 ([7]).

(1) *If ξ interchanges the fixed points of $\beta^k \alpha$ and of $\beta^{k+1} \alpha$ which are loxodromic, then $\xi \alpha \xi = \alpha^{-1}$ and $\xi \beta \xi = \alpha^{-1} \beta^{-1} \alpha$.*

(2) *Suppose α and β are loxodromic and α sends a fixed point of β to the other. Then β sends a fixed point of α to the other if and only if $\operatorname{Tr}^2(\alpha) = \operatorname{Tr}^2(\beta)$.*

(3) *If α and β are loxodromic, the fixed points of $\alpha^m \beta$ converge to p^* and $\beta(p_*))$ (resp. p_* and $\beta^{-1}(p_*))$ when $m \to \infty$ (resp. $m \to -\infty$) where p^* is the attractive fixed point of α and p_* its repulsive one.*

Later we will need a little bit more than the existence of a special handle, as proved in [7].

PROPOSITION 2.15. — *Let $\rho : \Gamma_{1,1} \to \operatorname{PSL}_2(\mathbb{C})$ be a non-elementary representation. There exists a homeomorphism f of $\Sigma_{1,1}$ fixing the boundary pointwise such that $(f(a), f(b))$ is a special handle. Moreover, given a finite set $A \subset \mathbb{CP}^1$ and $\xi \in \operatorname{PSL}_2(\mathbb{C})$ we can assume that no fixed point of $\rho \circ f_*(b)$ lies in A and that ξ does not interchange the fixed points of $\rho \circ f_*(a)$.*

Proof. — Since $\alpha = \rho(a)$ does not exchange the fixed points of $\beta = \rho(b)$, we have that $\beta^k \alpha$ is loxodromic if $k \geqslant K$ or $k \leqslant -K$ for some $K > 0$. If ξ does not interchange the fixed points of $\beta^k \alpha$ for such a k, apply the Dehn twist that changes (a, b) into $(b^k a, b)$.

If however ξ interchanges the fixed points of both $\alpha^k \beta$ and $\alpha^{k+1} \beta$, with k and $k+1$ in that range, then Lemma 2.14 shows that $\xi \alpha \xi = \alpha^{-1}$ and $\xi \beta \xi = \alpha^{-1} \beta^{-1} \alpha$. Note that we cannot have $\xi \beta^{-1} \xi = \beta$, otherwise α and β would have the same fixed points.

Thus if it is the case, we apply a sequence of Dehn twists that change the handle like this:

$$(a, b) \to (b^{-1} a, b) \to (b^{-1}, ab) \to (b^{-1}, b^{-1} ab).$$

And we return to the beginning. Therefore we may assume that ξ does not interchange the fixed points of α.

Since β does not interchange the fixed points of α, there exists $K > 0$ such that $\alpha^m\beta$ is loxodromic for $m \geqslant K$ or for $m \leqslant -K$. Now suppose there is an infinite sequence of m in that range such that α sends a fixed point of $\alpha^m\beta$ to the other one. Since the fixed points of $\alpha^m\beta$ tend to p and $\beta^{-1}(q)$ where p, q are the fixed points of α, we have $\alpha(p) = \beta^{-1}(q)$ or $\alpha\beta^{-1}(q) = p$. Thus $p = \beta^{-1}(q)$ and $\alpha^m\beta(p) = q$ for all m. But if we increase K, we can assume $|\operatorname{Tr}(\alpha^m\beta)| \neq |\operatorname{Tr}(\alpha)|$ for $m \geqslant K$ or $m \leqslant -K$. Since $\alpha^m\beta$ sends one fixed point of α to the other and thanks to Lemma 2.14, α does not send a fixed point of $\alpha^m\beta$ to the other, a contradiction.

The fixed points of $\alpha^m\beta$ are disjoint from those of $\alpha^n\beta$ if $n \neq m$. Otherwise, α and β would have a common fixed point. Hence we can also suppose that no fixed point of $\alpha^m\beta$ lies in A. $\qquad\square$

3. Pentagon representations

3.1. Definition

Let Γ be the group with the following presentation:

$$\Gamma = \langle q_1, q_2, q_3, q_4, q_5, q_6 \mid q_i^2 = 1, q_1q_2 \ldots q_6 = 1\rangle.$$

Define ι by $\iota(a_1) = q_2q_1$, $\iota(b_1) = q_2q_3$, $\iota(a_2) = q_5q_4$ $\iota(b_2) = q_5q_6$. We have $\iota([a_2, b_2][a_1, b_1]) = (q_6q_5q_4)^2(q_3q_2q_1)^2 = 1$, so $\iota : \Gamma_2 \to \Gamma$ is a well-defined map. The homomorphism ι is injective and identifies Γ_2 with an index two subgroup of Γ.

Let us recall from the introduction that a pentagon representation is the restriction of a representation $\rho : \Gamma \to \operatorname{PSL}_2(\mathbb{C})$ that kills (i.e. sends to the identity) exactly one q_i, and such that $\rho \circ \iota$ is non-elementary. We leave it as an elementary exercise to show that if two or more q_i are killed by ρ, then ρ is elementary. This property is invariant under conjugation, so it is a property of characters.

Before going on to the study of those representations, let us give an example that motivates the terminology. Consider a right-angled pentagon in the hyperbolic plane and denote its vertices by $x_1, \ldots, x_5$. Define $\rho : \Gamma \to \operatorname{PSL}_2(\mathbb{R})$ by $\rho(q_i) = s_{x_i}$ for $i \leqslant 5$ where s_{x_i} is the elliptic involution of the hyperbolic plane fixing x_i, and by $\rho(q_6) = \operatorname{id}$. This is well-defined because s_{x_i} is the product of the reflection across the lines $(x_{i-1}x_i)$ and

(x_i, x_{i+1}) (in cyclic notation) so $\rho(q_1, \ldots q_6) = \mathrm{id}$. The representation $\rho \circ \iota$ is non-elementary and is thus a pentagon representation.

We now show that pentagon representations have odd Stiefel–Whitney class.

PROPOSITION 3.1. — *A pentagon representation does not lift to $\mathrm{SL}_2(\mathbb{C})$.*

Proof. — Let $\rho : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ be such that $\rho \circ \iota$ is a pentagon representation. Take $\widetilde{q}_i \in \mathrm{SL}_2(\mathbb{C})$ such that $\pm\widetilde{q}_i = \rho(q_i)$ for $i \leqslant 6$.

Note that $\widetilde{q}_i^{-1} = -\widetilde{q}_i$ if $\rho(q_i) \neq \mathrm{id}$ because $\rho(q_i)$ is conjugate to $\pm \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Of course if $\rho(q_i) = \mathrm{id}$, then $\widetilde{q}_i^{\,2} = \mathrm{Id}$.

Now $\widetilde{q}_1\widetilde{q}_2 \ldots \widetilde{q}_6$ is a lift of $\rho(q_1 q_2 \ldots q_6) = \mathrm{id}$, so $\widetilde{q}_1\widetilde{q}_2 \ldots \widetilde{q}_6 = \epsilon\, \mathrm{Id}$, with $\epsilon = \pm 1$.

Let $\widetilde{a_1} = \widetilde{q}_2\widetilde{q}_1$, $\widetilde{b_1} = \widetilde{q}_2\widetilde{q}_3$, $\widetilde{a_2} = \widetilde{q}_5\widetilde{q}_4$ and $\widetilde{b_2} = \widetilde{q}_5\widetilde{q}_6$.

$$[\widetilde{a_2}, \widetilde{b_2}][\widetilde{a_1}, \widetilde{b_1}] = \widetilde{q}_6^{\,-1}\widetilde{q}_5^{\,-1}\widetilde{q}_4^{\,-1}\widetilde{q}_6\widetilde{q}_5\widetilde{q}_4\widetilde{q}_3^{\,-1}\widetilde{q}_2^{\,-1}\widetilde{q}_1^{\,-1}\widetilde{q}_3\widetilde{q}_2\widetilde{q}_1$$

$$= -(\widetilde{q}_6\widetilde{q}_5\widetilde{q}_4)^2(\widetilde{q}_3\widetilde{q}_2\widetilde{q}_1)^2$$

$$= -(\widetilde{q}_6\widetilde{q}_5\widetilde{q}_4)\epsilon\widetilde{q}_3\widetilde{q}_2\widetilde{q}_1$$

$$= -\epsilon^2\, \mathrm{Id} = -\,\mathrm{Id}\,.$$

Therefore $\rho \circ \iota$ does not lift to $\mathrm{SL}_2(\mathbb{C})$.   □

3.2. Action of the mapping class group

The mapping class group of Σ_2 acts naturally on $\mathrm{Hom}(\Gamma_2, \mathrm{PSL}_2(\mathbb{C}))/\mathrm{PSL}_2(\mathbb{C})$ as follows: $[f] \cdot [\rho] = [\rho \circ f_*^{-1}]$ where f_* is the outer automorphism of Γ_2 induced by f.

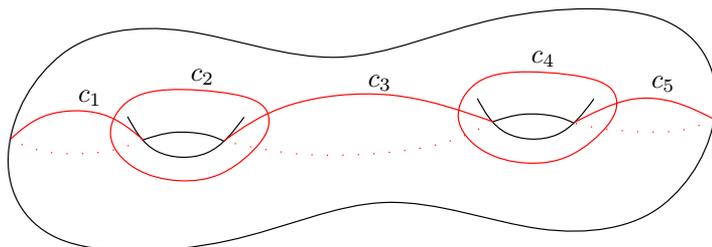PROPOSITION 3.2. — *This action preserves pentagon representations.*

Figure 3.1. Generators of $\mathrm{Mod}(\Sigma_2)$.

Proof. — For $1 \leqslant i \leqslant 5$, let σ_i be the automorphism of Γ defined by $\sigma_i(q_i) = q_{i+1}$, $\sigma_i(q_{i+1}) = q_{i+1}q_iq_{i+1}$ and $\sigma_i(q_j) = q_j$ for $j \neq i, i+1$. The mapping class group is generated by the Lickorish generators: the Dehn twists along the curves c_i drawn in Figure 3.1; see [6, Chapter 4].

The outer automorphism $[\varphi_i]$ of Γ_2 induced by a Dehn twist along the curve c_i has a representative $\varphi_i \in \mathrm{Aut}^+(\Gamma_2)$ such that the following diagram commutes:

$$\begin{array}{ccc} \Gamma_2 & \xrightarrow{\varphi_i} & \Gamma_2 \\ \iota \downarrow & & \downarrow \iota \\ \Gamma & \xrightarrow{\sigma_i} & \Gamma. \end{array}$$

Hence $[\rho \circ \iota \circ \varphi_i] = [\rho \circ \sigma_i \circ \iota]$ and $\rho \circ \sigma_i$ kills exactly one q_i if ρ does. □

4. Schottky decomposition

The aim of this section is to prove Theorem 1.4: the absence of a Schottky decomposition characterizes the pentagon representations among the non-elementary representations.

4.1. Pentagons are not Schottky

Let us first show that a pentagon representation does not admit a Schottky decomposition.

Proof. — Let $\rho : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ be such that $\rho \circ \iota$ is a pentagon representation. Let us consider the two pants decomposition of Σ_2 as shown in Figure 4.1. The first one is not a Schottky decomposition for ρ. Indeed since there is a q_i killed by ρ, its restriction to one of the two handles must be elementary. The second pants decomposition is not a Schottky decomposition for ρ either since one can check that the image of one of the boundary curves has order 2.

For every $g \geqslant 2$, the mapping class group $\mathrm{Mod}(\Sigma_g)$ acts on the set $\mathcal{P}_g$ of pants decompositions of Σ_g. Let us recall, see for example [16], that the quotient $\mathcal{P}_g / \mathrm{Mod}(\Sigma_g)$ is in bijection with the isomorphism classes of connected trivalent graphs with $2g-2$ vertices. In particular there are only two pants decompositions of a surface of genus 2 up to the action of $\mathrm{Mod}(\Sigma_2)$. Now if there is a Schottky decomposition P for ρ, there exists a positive homeomorphism f of Σ taking one of these two pants decompositions to P. The pentagon representation $[\rho \circ f_*]$ admits a Schottky decomposition with one of those two pants decompositions, which is a contradiction. □
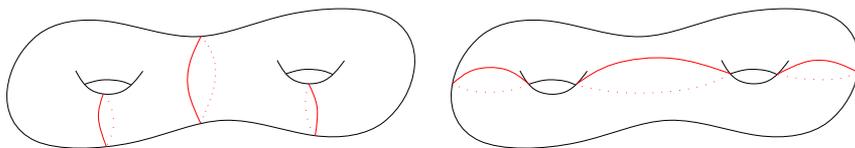
Figure 4.1. Pants decompositions of Σ_2.

4.2. Non-Schottky are pentagons

4.2.1. Tools to form a Schottky decomposition

The following proposition is a rephrasing of the paragraph 4.4 of [7]:

PROPOSITION 4.1. — Let $g \geqslant 2$, $n \geqslant 0$ and $\rho : \Gamma_{g,n} \to \mathrm{PSL}_2(\mathbb{C})$. Suppose that (a_1, b_1) is a special handle, and that $\rho(b_2 a_1^{-1})$ does not exchange the fixed points of $\rho(b_1)$. Suppose moreover that $\rho(b_2) \neq \mathrm{id}$ or that $\rho(a_2)$ does not interchange the fixed points of $\rho(a_1)$. A Dehn twist of order n, along a curve d_k, freely homotopic to $b_2 a_1^{-1} b_1^k$ transforms (a_1, b_1) in a non-elementary handle, and $\rho(a_2)$ (or $\rho(b_2 a_2)$) in a loxodromic isometry for some k, n.
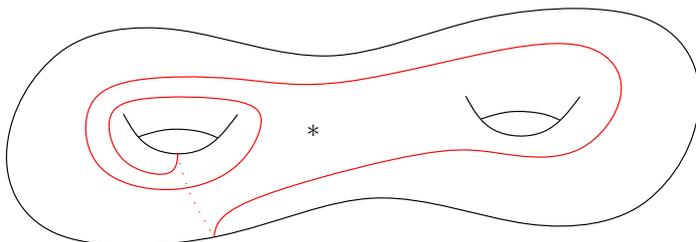

Figure 4.2. Dehn Twist.

The proof is very similar to the one of [7]; we slightly simplify its beginning and modify its end.

Proof. — There exists $K > 0$ such that $\rho(a_1^{-1} b_1^k)$ is loxodromic for $|k| \geqslant K$. The isometry $\delta_k = \rho(b_2 a_1^{-1} b_1^k)$ is also loxodromic for $k \geqslant K$ or $k \leqslant -K$, increasing K if necessary.

Fix such a k so that $k+1$ is also in that range, and let $\delta = \delta_k$. There is at most one n such that $\rho(b_1)\delta^n$ shares a given fixed point with $\rho(a_1^{-1} b_1^k)$. For if $\rho(b_1)\delta^n(p) = \rho(a_1^{-1} b_1^k)(p) = p = \rho(b_1)\delta^{n+m}(p)$, then $\delta(p) = p = \rho(b_1)(p) = \rho(a_1)(p)$, which is a contradiction since (a_1, b_1) is a special handle.

Thus there exists $N(k) \geqslant 0$ such that $\rho(b_1)\delta^n$ and $\rho(a_1^{-1}b_1^k)$ do not share a fixed point for $|n| \geqslant N(k)$: the image of the handle is non-elementary.

LEMMA 4.2. — *It is not possible for both $\rho(a_2)$ and $\rho(b_2 a_2)$ to interchange the fixed points of δ_k and of δ_{k+1}.*

Proof. — We proceed by contradiction.

- If $\rho(b_2) = \mathrm{id}$, then $\rho(a_2)$ interchanges the fixed points of $\rho(a_1^{-1}b_1^k)$ and of $\rho(a_1^{-1}b_1^{k+1})$. By Lemma 2.14, $\rho(a_2)$ interchanges the fixed points of $\rho(a_1)$.
- If $\rho(b_2) \neq \mathrm{id}$, then $\rho(b_2)$ fixes the fixed points of δ_k and of δ_{k+1}, and has at most two fixed points in $\partial\mathbb{H}^3$. Hence δ_k and δ_{k+1} share a fixed point p with $\rho(b_2)$. This implies that $\rho(b_2 a_1^{-1}b_1^k)(p) = \rho(a_1^{-1}b_1^k)(p) = p$ and $\rho(a_1^{-1}b_1^{k+1})(p) = p$. Hence both $\rho(b_1)$ and $\rho(a_1)$ fix p. This is a contradiction since (a_1, b_1) is a special handle. $\qquad\square$

It follows that $\rho(a_2)\delta^n$, or $\rho(b_2 a_2)\delta^n$ is loxodromic for some $n \geqslant N(k)$ or $n \leqslant -N(k)$, increasing $N(k)$ if necessary. Note that its trace can be made arbitrarily large. $\qquad\square$

Let us recall a tool from [7] to construct Schottky groups.

LEMMA 4.3. — *Suppose α, β and δ are loxodromic, and that neither α nor β shares a fixed point with δ.*

Then $\delta^n \alpha \delta^{-n}$ and β generate a Schottky group for $|n|$ large enough.

The following proposition explains how we can construct a Schottky decomposition, following [7].

PROPOSITION 4.4. — *Let $g \geqslant 2$ and $\rho : \Gamma_g \to \mathrm{PSL}_2(\mathbb{C})$. Suppose we can cut the surface along non-separating simple closed curves such that we get a surface of genus 1, with a special handle, and such that the boundary curves are loxodromic with pairwise different images. Then ρ admits a Schottky decomposition.*

Proof. — Observe that cutting the surface along a non-separating simple closed curve creates two boundary components, hence the number of boundaries is at least 2. Suppose we want to construct a Schottky pair of pants from the boundaries d_1 and d_2, such that $\rho(d_1) \neq \rho(d_2)$. By Proposition 2.15, we may assume that $\rho(b_1)$ does not fix the fixed points of $\rho(d_1 d_2^{-1})$. Then, it is not possible for both $\rho(d_1 a_1^{-1})$ and $\rho(d_2 a_1^{-1})$ to interchange the fixed points of $\rho(b_1)$.

We can now apply the arguments of 5.2 and 5.3 of [7] to form a Schottky pair of pants from those boundary components, and remove it from the

surface. We make sure the trace of the new boundary is larger than the others'. Following [7], we produce a Schottky pants decomposition.   □

The mistake of [7] lies in its paragraph 5.5, where a Schottky pair of pants is found, but the non-elementary handle is not always kept. We avoid using this part of their proof.

4.2.2. The genus 2 case

Let $\rho : \Gamma_2 \to \mathrm{PSL}_2(\mathbb{C})$ be a non-elementary representation that does not admit a Schottky decomposition.

The following proposition is an adaptation of [7, Paragraph 4.5].

PROPOSITION 4.5. — *We can change ρ by some ρ_1 such that $[\rho_1] = f \cdot [\rho]$ with $f \in \mathrm{Homeo}^+(\Sigma_2)$, so that (a_1, b_1) is a special handle, and $\rho(b_2)$ is loxodromic.*

Let us explain why this result, combined with Theorem 1.4, implies Corollary 1.5. By Theorem 1.4, we just have to consider the pentagon representations. If a pentagon representation ρ is of the form above, it admits an extension $\rho' : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ such that $\rho'(q_4) = \mathrm{id}$. But then $\rho(a_1)$, $\rho([a_2, b_2]) = \rho'((q_5 q_6)^2) = \rho(b_2)^2$ and $\rho(b_2)$ are loxodromic. We thus consider a pants decomposition defined by curves freely homotopic to a_1, $[a_2, b_2]$ and b_2.

Proof. — We can assume (a_1, b_1) is a special handle. Let us start by changing the handle (a_2, b_2) by $(a_2, a_2 b_2)$ in the case where $\rho(a_2) = \rho(b_2)$ and they are of order 2. Return to the notation (a_2, b_2).

If $\rho(b_2) = \mathrm{id}$, apply Proposition 2.15 to make sure that $\rho(a_2)$ does not interchange the fixed points of $\rho(a_1)$. Then we can apply Proposition 4.1 to turn $\rho(a_2)$ into a loxodromic isometry, because $\rho(b_2 a_1^{-1})$ is loxodromic and hence cannot interchange two points.

If however $\rho(b_2) \neq \mathrm{id}$, apply Proposition 2.15 to make sure that $\rho(b_2 a_2)$ does not interchange the fixed points of $\rho(a_1)$.

If $\rho(b_2 a_1^{-1})$ does not interchange the fixed points of $\rho(b_1)$, we can apply Proposition 4.1 to make $\rho(a_2)$ loxodromic.

If it does, then suppose that $\rho(a_2^{-1} a_1^{-1})$ does not interchange the fixed points of $\rho(b_1)$. Then we modify the handle (a_2, b_2) by a homeomorphism to $(b_2 a_2, a_2^{-1})$. It is a composition of Dehn twists in the handle that changes it as follows:

$$(a_2, b_2) \to (a_2, a_2^{-1} b_2) \to (b_2 a_2, a_2^{-1}).$$

We can apply Proposition 4.1 since we made sure that $\rho(b_2 a_2)$ does not interchange the fixed points of $\rho(a_1)$.

Finally if both $\rho(a_2^{-1}a_1^{-1})$ and $\rho(b_2 a_1^{-1})$ interchange the fixed points of $\rho(b_1)$, then $\rho(a_2^{\pm 1}b_2 a_1^{-1})$ does not, for it would imply that $\rho(a_2)$ fixes them and then $\rho(a_1)$ would interchange them. We can make sure that $\rho(a_2^{\pm 1}b_2) \neq$ id, because we are not in the case where $\rho(b_2) = \rho(a_2)$ is of order 2. We then apply a Dehn twist that does:

$$(a_2, b_2) \to (a_2, a_2^{\pm 1}b_2).$$

We can then use Proposition 4.1.

We have made $\rho(a_2)$ loxodromic. But again we can change the handle (a_2, b_2) as before to make sure $\rho(a_2^{-1})$ is loxodromic. The handle (a_1, b_1) is non-elementary and we can improve it to a special handle with Proposition 2.15. □

We now just have to consider pentagon representations in this special form.

PROPOSITION 4.6. — *A homomorphism* $\rho : \Gamma_2 \to \mathrm{PSL}_2(\mathbb{C})$ *that is non-elementary and does not admit a Schottky decomposition is a pentagon representation.*

Proof. — The axes of $\rho(a_1)$ and of $\rho(b_1)$ do not cross in $\mathbb{H}^3 \cup \partial\mathbb{H}^3$, so there exists a unique line ℓ orthogonal to both of them. Let $q_2 = s_\ell$ be the elliptic involution with axis ℓ. Then $q_1 = q_2\rho(a_1)$ and $q_3 = q_2\rho(b_1)$ are elliptic involutions.

We have $\rho(a_2)^{-1}\rho(b_2)^{-1}\rho(a_2) = \rho(b_2)$. Indeed, otherwise we could cut the surface along a curve freely homotopic to b_2, and use Proposition 4.4. The isometry $q_5 = \rho(a_2)$ interchanges the fixed points of $\rho(b_2)$, and is an elliptic involution with an axis orthogonal to the one of $\rho(b_2)$. Hence $q_6 = q_5\rho(b_2)$ has order 2.

We have id $= \rho([a_2 b_2][a_1, b_1]) = (q_6 q_5)^2(q_3 q_2 q_1)^2$. This implies that $q_1 q_2 q_3 = q_6 q_5 \circ r$ where r commutes with $q_6 q_5$ and is such that $r^2 = $ id because of the following lemma.

LEMMA 4.7. — *If f and g are loxodromic isometries such that $f^2 = g^2$, either $f = g$ or $f = g \circ r$ where r is an elliptic involution with the same axis as f and g.*

Proof. — They have the same axis so after conjugating we can write $f(z) = \lambda z$ and $g(z) = \mu z$. We have $\lambda^2 = \mu^2$; hence $\lambda = \mu$ or $\lambda = -\mu$. □

Thanks to Proposition 4.4, $\rho(b_2 a_1^{-1})$ must interchange the fixed points of $\rho(b_1)$. Indeed otherwise we could apply Proposition 4.1 to improve the situation and suppose that $\rho(a_2)$ is loxodromic. Then $\rho(a_2)^{-1}\rho(b_2)^{-1}\rho(a_2) = \rho(b_2)$ would be impossible.

We have $\rho(b_2 a_1^{-1}) = q_5 q_6 q_1 q_2 = r q_3$ and $\rho(b_1) = q_2 q_3$. Therefore $r q_3$ is of order 2 and $r q_3 = (r q_3)^{-1} = q_3 r$. Moreover, $(r q_3) q_2 q_3 (r q_3) = q_3 q_2$, thus $r q_3 q_2 r = q_3 q_2$. The centralizer of r contains $q_1 q_2 q_3$, q_3 and $q_3 q_2$, hence a non-elementary group. This implies that $r = \mathrm{id}$. $\qquad\square$

4.2.3. Genus $g \geqslant 3$

In this subsection we consider the case $g \geqslant 3$.

PROPOSITION 4.8. — *Let $g \geqslant 3$ and $\rho : \Gamma_g \to \mathrm{PSL}_2(\mathbb{C})$ be a non-elementary representation. There exists a Schottky decomposition for ρ.*

Proof. — Thanks to Proposition 4.4, it suffices to show that we can cut the surface along non-separating curves having different loxodromic images.

We can assume that (a_1, b_1) is a special handle. We can also suppose that each $\rho(b_i)$ is loxodromic, by applying successively the arguments of Proposition 4.5. We can moreover suppose that the traces of the $\rho(b_i)$ are pairwise distinct.

Use Proposition 2.15 to make sure that $\rho(b_2)$ does not fix the fixed points of $\rho(b_1)$. It is not possible for both $\rho(b_3 a_1^{-1})$ and $\rho(b_2 b_3 a_1^{-1})$ to interchange the fixed points of $\rho(b_1)$, because that would mean that $\rho(b_2)$ fixes them. We now apply the Proposition 4.1 in the handle (a_3, b_3) or $(a_3, b_2 b_3)$. Note that $\rho(b_3) \neq \mathrm{id}$ and $\rho(b_2 b_3) \neq \mathrm{id}$ for the trace of $\rho(b_2)$ is different from the trace of $\rho(b_3)$. This changes $\rho(a_3)$ into a loxodromic isometry and leaves $\rho(b_3)$ unchanged. Thus both $\rho(a_3)$ and $\rho(b_3)$ are loxodromic; we cut the surface along a curve freely homotopic to a_3. We cannot have $\rho(a_3) = \rho(b_3)^{-1} \rho(a_3)^{-1} \rho(b_3)$ for it would imply that $\rho(b_3)$ interchanges the fixed points of $\rho(a_3)$.

We can repeat the argument with the other handles while there are at least two handles to cut. Note that the images of the boundary components are modified by conjugation at each step.

We are left with a special handle, $2(g-2)$ boundary components, and a handle that we want to cut. We may assume that (a_1, b_1) is a special handle, and that the handle we wish to cut is (a_2, b_2), and that $\rho(b_2)$ is loxodromic. Make sure as before that $\rho(b_3)$ does not fix the fixed points of $\rho(b_1)$.

If $\rho(b_2 a_1^{-1})$ does not interchange the fixed points of $\rho(b_1)$, then we apply Proposition 4.1 in the handle (a_2, b_2). Otherwise, we apply the same proposition in the handle $(a_2, b_2 a_3^{-1})$. The boundary components $d_1 = b_3$ and $d_2 = b_3^{-1} a_3^{-1} b_3$ corresponding to the handle (a_3, b_3) are changed as follows: $(d_1, d_2) \to (d_1, \zeta^{-n} d_2 \zeta^n)$ where $\zeta = a_1^{-1} b_1^k b_2 b_3$, and k and n come from the Proposition 4.1. We cannot have $d_1 = \zeta^{-n} d_2 \zeta^n = \zeta^{-n-1} d_2 \zeta^{n+1}$

for it would imply $d_1 = d_2$. Thus we can assume that $d_1 \neq d_2$ after this Dehn twist is done.

We can thus cut again, to have a genus 1 surface with a special handle, and $2(g-1)$ boundary components, with any two of them having different images.   □

5. Projective structure

The goal of this section is to prove Theorem 1.6. As before, we can modify ρ by the action of the mapping class group since the property we are interested in is invariant under this action.

Even if the non-elementary handle is not kept, we still find a Schottky pair of pants following [7, Section 5.5].

PROPOSITION 5.1. — *There exists ρ_1 such that $[\rho_1] = f \cdot [\rho]$ for some $f \in \mathrm{Homeo}^+(\Sigma_2)$, such that $\rho_1(a_2)$ and $\rho_1(b_2^{-1} a_2^{-1} b_2)$ generate a Schottky group, and such that $\rho_1(b_1)$ is loxodromic and $\rho_1(a_1)$ interchanges the fixed points of $\rho_1(b_1)$.*

Proof. — We may assume that ρ is as in Proposition 4.5. Denote by ρ' an extension of ρ to Γ. Cut Σ_2 along a curve freely homotopic to b_2, so that we get a genus 1 surface with 2 boundary components having loxodromic images: $d_1 = b_2$ and $d_2 = a_2^{-1} b_2^{-1} a_2$. Their images are equal : $\rho'(q_4) = \mathrm{id}$ and $\rho'(q_5 q_6 q_5 q_5) = \rho'(q_5 q_6)$. The isometries $\rho((b_1^{-k} a_1) d_1 (a_1^{-1} b_1^k))$ and $\rho(d_2)$ generate a Schottky group for $|k|$ large enough. Indeed $\rho(d_2) = \rho'(q_5 q_6)$ cannot fix a fixed point of $\rho(b_1) = \rho'(q_2 q_3)$ for $\rho(a_1^{-1} b_2) = \rho'(q_1 q_2 q_5 q_6) = \rho'(q_3)$ interchanges those of $\rho(b_1)$ and thus $\rho(a_1)$ would send a fixed point of $\rho(b_1)$ to the other. Similarly, $\rho(a_1^{-1} a_1 d_1 a_1^{-1})$ interchanges the fixed points of $\rho(b_1)$, hence $\rho(a_1 d_1 a_1^{-1})$ does not fix any of them. We thus get a Schottky pair of pants by Lemma 4.3.

We may assume that the Schottky pair of pants comes from cutting the handle (a_2, b_2). One of q_1, q_2 and q_3 is killed by ρ'. We can assume that it is q_1, applying a homeomorphism of the handle (a_1, b_1) if necessary. Since $\rho(a_1)^{-1} \rho(b_1) \rho(a_1) = \rho(b_1)^{-1}$, the map $\rho(b_1)^2 = \rho([a_1, b_1])^{-1} = \rho([a_2, b_2])$ is loxodromic, and so is $\rho(b_1)$.   □

We now put a projective structure on $\Sigma_{1,1}$ whose holonomy is the non-Schottky part of the previous proposition. Namely, this holonomy is given by ρ which maps the standard generators a_1, b_1 of $\Gamma_{1,1}$ to $\rho(a_1)$, which is loxodromic, and to $\rho(b_1)$ which is an involution interchanging the fixed points of $\rho(a_1)$.

PROPOSITION 5.2. — *There exists a projective structure with a single branched point of order 2 on $\Sigma_{1,1}$ such that its holonomy is ρ, and such that the developing map embeds the boundary curve in $\mathbb{CP}^1$.*
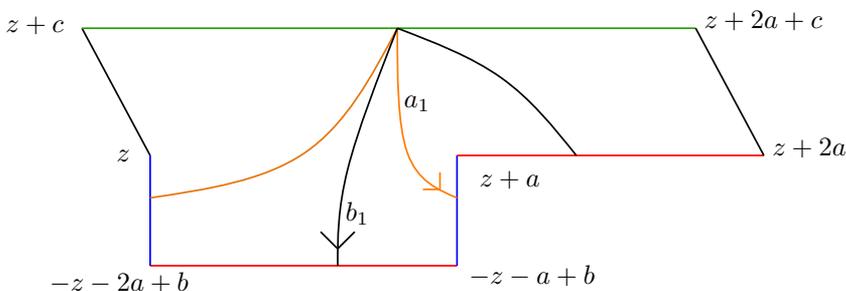

Figure 5.1. *Affine structure on $\Sigma_{1,1}$.*

Proof. — Up to conjugacy we can suppose that $\rho(a_1)(z) = \lambda^{-2}z$ with $|\lambda| > 1$. We can adapt the coordinate z to have $\rho(b_1)(z) = \mu z^{-1}$ with an arbitrary $\mu \in \mathbb{C} \setminus \{0\}$. The choice of μ will be given in the sequel. Let $a \in \mathbb{C}$ be such that $e^a = \lambda^2$. Choose z, b, c such that the points $-z - 2a + b$, $-z - a + b$, $z + a$, $z + 2a$, $z + 2a + c$, $z + c$ and z form an L-shaped polygon P as in Figure 5.1.

The identifications of the sides by $z \mapsto z + a$ for the blue ones, $z \mapsto -z + b$ for the red ones, and $z \mapsto z + 2a$ for the black ones give an affine structure on S, with a cone point of angle 4π. Taking the exponential of small enough charts defines a complex projective structure on $\Sigma_{1,1}$ with a single branched point of order 2.

The holonomy of this projective structure maps a_1 to $z \mapsto \lambda^{-2}z$. Indeed, if $z = e^\omega$, then $e^{\omega - a} = \lambda^{-2}z$. Similarly, it maps b_1 to $z \mapsto e^b z^{-1}$.

Since $\Re(a) > 0$, the developing map embeds the boundary curve in $\mathbb{CP}^1$.   □

We are now able to prove Theorem 1.6. We are reduced to the case where ρ is of the form of Proposition 5.1. Put the branched projective structure as above on the handle that is not Schottky. We can put a projective structure on the Schottky handle with the desired holonomy that is compatible (i.e. that we can glue to the other one), possibly with a branched point of order 2 (see [7, Paragraph 7,8,9]).

But it is not possible for ρ to be the holonomy of a branched projective structure with two branched points of order 2, since it would imply that it lifts to $\mathrm{SL}_2(\mathbb{C})$ (see [7, Corollary 11.2.3]), contradicting Proposition 3.1.

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