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BASE DIVISORS OF BIG AND NEF LINE BUNDLES ON IRREDUCIBLE SYMPLECTIC VARIETIES

by Ulrike RIESS

ABSTRACT. — Under some conditions on the deformation type, which we expect to be satisfied for arbitrary irreducible symplectic varieties, we describe which big and nef line bundles on irreducible symplectic varieties have base divisors. In particular, we show that such base divisors are always irreducible and reduced. This is applied to understand the behaviour of divisorial base components of big and nef line bundles under deformations and for $K3^{[n]}$ -type and Kum^n -type.

RÉSUMÉ. — Nous décrivons quels fibrés en droites gros et nef, sur des variétés symplectiques irréductibles, ont des diviseurs de base, sous certaines conditions relatives au type de déformation dont nous nous attendons à ce qu'elles soient vraies pour toutes les variétés symplectiques irréductibles. En particulier, nous montrons que de tels diviseurs de base sont toujours réduits et irréductibles. Nous appliquons ces résultats pour comprendre le comportement après déformation des diviseurs de base des fibrés en droites gros et nef. Nous terminons en donnant une description très explicite pour les variétés de types $K3^{[n]}$ et Kum^n .

Introduction

Irreducible (holomorphic) symplectic varieties are a class of varieties that appears naturally in the classification of algebraic varieties with trivial first Chern class. The Beauville–Bogomolov decomposition theorem ([4, Theorem 1]) states that up to a finite étale cover every such variety can be decomposed into a product of three types of varieties: abelian varieties, irreducible symplectic varieties and (strict) Calabi–Yau varieties. Since this is known, there has been intense research on irreducible symplectic varieties.

Two-dimensional irreducible symplectic varieties are exactly the famous $K3$ surfaces. These surfaces have a rich geometry and are very well-studied.

Keywords: Irreducible symplectic varieties, hyperkähler manifolds, base divisors, Fujita’s conjecture.

2020 Mathematics Subject Classification: 14M99.

It turns out, that many results on K3 surfaces stem from more general phenomena for irreducible symplectic varieties.

In this article we study base divisors of big and nef line bundles on irreducible symplectic varieties.

Starting point are the known results on K3 surfaces:

PROPOSITION ([21]). — *Let X be a K3 surface with a big and nef line bundle $H \in \text{Pic}(X)$. Then H has base points if and only if $H = mE + C$, where $m \geq 2$, E is a smooth elliptic curve, and C is a smooth rational curve, such that $(E, C) = 1$. In this case, the base locus of H is exactly C .*

Although this article only deals with the complex setting, let us mention that over arbitrary algebraically closed fields of characteristic $\neq 2$ a similar result for ample line bundles was proved by Saint-Donat (see [28, Proposition 8.1]).

Mayer's proposition implies that for every ample line bundle H on a K3 surface, $2H$ is base point free. Remarkably, also for abelian varieties A , it is known that $2H$ is base point free for every ample line bundle $H \in \text{Pic}(A)$ (see [23]).

These results can be seen as part of a general conjecture of Fujita which predicts that for smooth, projective varieties $\omega_X + (\dim X + 1)H$ is base point free for any ample line bundle H (see e.g. [16, Conjecture 10.4.1] for the full statement of Fujita's conjecture). Note that for abelian varieties and for K3 surfaces, the bounds are even better than predicted by Fujita's conjecture. This suggests, that it might be particularly interesting to study questions related to base points of ample line bundles for irreducible symplectic varieties.

Let us mention that in general it is known, that for an ample line bundle H on a smooth projective variety X of dimension n , the line bundle $\omega_X + mH$ is globally generated for $m \geq \binom{n+1}{2}$ (see [3]).

In this article, we investigate the divisorial part of the base locus of big and nef line bundles on irreducible symplectic varieties. Before stating the main theorem in full generality, let me point out that we obtain a complete characterization of the base divisors for the two most prominent series of deformation types of irreducible symplectic varieties.

PROPOSITION (see Proposition 8.2). — *Let X be an irreducible symplectic variety of K3^[n]-type and $H \in \text{Pic}(X)$ a line bundle that is big and nef. Then H has a fixed divisor if and only if H is of the form $H = mL + F$, where $m \geq 2$, L is movable with $q(L) = 0$, and F is an irreducible reduced divisor of negative square with $(L, F)_q = 1$. In this case F is the fixed divisor of H .*

PROPOSITION (see Proposition 9.2). — *Let X be an irreducible symplectic variety of Kumⁿ-type, and H a big and nef line bundle. Then H is movable, i.e. it does not have a fixed divisor.*

These two results are consequences of the main theorem of this article. For the full statement, we first need the following notation:

Let X be an irreducible symplectic variety of dimension $2n$ and denote by $\overline{\mathcal{BK}}_X$ the closure of its birational Kähler cone (compare Section 2). A famous conjecture is

CONJECTURE (see Conjecture 4.4). — *Let $0 \neq L \in \text{Pic}(X) \cap \overline{\mathcal{BK}}_X$ be a primitive line bundle with $q(L) = 0$. Then $\dim h^0(X, L) = n + 1$ and $|L|$ induces a birational Lagrangian fibration to \mathbb{P}^n .*

On the other hand, the Riemann–Roch theorem for irreducible symplectic varieties (Theorem 1.3) implies that there exist constants $b_i \in \mathbb{Q}$ such that for each $L \in \text{Pic}(X)$ the Euler characteristic is given by $\chi(X, L) = \sum_{i=0}^n b_i q(L)^i$. Let us denote the appearing polynomial appearing by $RR_X(x) := \sum_{i=0}^n b_i x^i$.

The main theorem of this article is:

THEOREM (compare Theorem 4.7). — *Fix an irreducible symplectic variety X of dimension $2n$ for which $RR_X|_{\mathbb{Z}_{\geq 0}}$ is strictly monotonic and such that Conjecture 4.4 holds for X . Consider a big and nef line bundle $H \in \text{Pic}(X)$. Then H has a non-trivial base divisor if and only if there exists an irreducible reduced divisor F of negative square such that H is of the form $H = mL + F$, where $m \geq 2$, L is a primitive movable line bundle with $q(L) = 0$ and $(L, F)_q > 0$, such that $RR_X(q(H)) = \binom{m+n}{n}$. In this case F is exactly the fixed divisor of H .*

We expect that the conditions in the theorem are satisfied for arbitrary irreducible symplectic varieties. A discussion of the assumptions can be found in Section 6.

Note that in particular the base divisor is irreducible and reduced of negative square, whenever the theorem applies, which is a remarkable observation in itself.

One obtains the following corollary which fits in the framework of Fujita’s conjecture.

COROLLARY (see Corollary 4.9). — *Fix an irreducible symplectic variety X which satisfies the assumptions of the theorem. Pick a line bundle $H \in \text{Pic}(X)$ which is big and nef. Then the base locus of $2H$ does not contain a divisor.*

As another application of the main theorem one understands the behaviour of divisorial base loci under deformations:

THEOREM (see Theorem 7.1). — *Let X be an irreducible symplectic variety such that its deformation type satisfies the conditions in the main theorem. Then in a (semi-)polarized family the locus where the big and nef line bundle acquires a divisorial base locus is a disjoint union of certain Noether–Lefschetz divisors.*

In this article, we only discuss the phenomenon of divisorial base loci. For K3 surfaces, Mayer’s proposition shows that only divisorial base loci can occur for big and nef line bundles. However, for higher dimensional irreducible symplectic varieties, it turns out that the situation is more complicated. Therefore, we restrict ourselves to divisorial base loci in this article and present results towards base loci of higher codimension in [27].

In Sections 1 to 3, we state properties of irreducible symplectic varieties, including results on the birational Kähler cone and reflections in prime exceptional divisors.

The main theorem of this article (Theorem 4.7) is stated in its most general form in Section 4 and proved in Section 5. A discussion of the non-standard assumptions in the theorem can be found in Section 6.

Section 7 contains the application of the main theorem to the behaviour under deformations. The explicit description of base divisors for $K3^{[n]}$ -type and Kum^n -type can be found in Sections 8 and 9.

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1. Basic facts on irreducible symplectic varieties

This section contains a collection of basic facts on irreducible symplectic varieties, which we need later.

Let us start by giving the definition:

DEFINITION 1.1. — *An irreducible symplectic variety is a simply connected, smooth, projective complex variety X such that $H^0(X, \Omega_X^2)$ is generated by a nowhere degenerate two-form.*

Irreducible symplectic varieties are also known as “projective hyperkähler manifolds” and as “irreducible holomorphic symplectic varieties”. For an overview on irreducible symplectic varieties, we refer to [9, Part III], [26], and [11].

All known irreducible symplectic varieties are deformation equivalent to one of the following:

- the $2n$ -dimensional Hilbert scheme $\text{Hilb}^n(S)$ of n points on a K3 surface S ,
- the $2n$ -dimensional generalized Kummer variety $\text{Kum}^n(A)$ associated to an abelian surface A , which is constructed as a fibre of the summation map $\text{Hilb}^{n+1}(A) \rightarrow A$,
- two examples discovered by O’Grady (a ten-dimensional one [24] and a six-dimensional one [25]).

Irreducible symplectic varieties that are deformation equivalent to one of the first two series of examples are called $\text{K3}^{[n]}$ -type respectively Kum^n -type.

We will usually denote the even dimension of an irreducible symplectic variety X by $2n$, and we will not distinguish between $\text{Pic}(X)$ and $\text{NS}(X)$ which can be naturally identified.

We denote by q the Beauville–Bogomolov–Fujiki quadratic form on the second integral cohomology of an irreducible symplectic variety X , which satisfies the Fujiki relation $\int_X \alpha^{2n} = C_X \cdot q(\alpha)^n$ for every $\alpha \in H^2(X, \mathbb{Z})$ with a constant $C_X \in \mathbb{Q}_{>0}$ depending on X ([7]).

The signature of q is $(3, b_2 - 3)$, where $b_2 = \text{rk } H^2(X, \mathbb{Z})$. Restricted to $H^{1,1}(X, \mathbb{R})$ the signature of q is $(1, b_2 - 3)$.

Therefore one can define the *positive cone* $\mathcal{C}_X \subseteq H^{1,1}(X, \mathbb{R})$ as the connected component of $\{\alpha \in H^{1,1}(X, \mathbb{R}) \mid q(\alpha) > 0\}$ containing an ample class. Note that the positive cone contains all ample classes and that a nef class is big if and only if $q(H) > 0$.

Another important property of q is the following:

LEMMA 1.2 ([5, Proposition 4.2.(ii)]). — *Let X be an irreducible symplectic variety, and $E, F \in \text{Pic}(X)$ be effective divisors with no common component, then $q(E, F) \geq 0$.*

Using the Beauville–Bogomolov–Fujiki form, the Hirzebruch–Riemann–Roch formula takes a special form for irreducible symplectic varieties:

THEOREM 1.3 (Riemann–Roch for irreducible symplectic varieties, see [11, Section 1.11]). — *For an irreducible symplectic variety X there exist $b_i \in \mathbb{Q}$ such that for each $L \in \text{Pic}(X)$*

$$\chi(X, L) = \sum_{i=0}^n b_i q(L)^i.$$

Let us further mention the following result:

PROPOSITION 1.4 ([9, Theorem 5.9]). — *Let X be an irreducible symplectic variety of dimension $2n$. Then $\chi(\mathcal{O}_X) = n + 1$.*

We will frequently use the following basic observation.

LEMMA 1.5 ([10, Section 2.2]). — *Let X and X' be birational irreducible symplectic varieties. Then there exist open subsets $U \subseteq X$ and $U' \subseteq X'$ with $U \cong U'$, such that $X \setminus U$ and $X' \setminus U'$ have codimension at least two.*

2. The birational Kähler cone

In this section, we collect some useful facts about the birational Kähler cone of an irreducible symplectic variety.

DEFINITION 2.1. — *Let X be an irreducible symplectic variety of dimension $2n$ as before. Denote its Kähler cone by $\mathcal{K}_X \subseteq H^{1,1}(X, \mathbb{R})$. Define its birational Kähler cone as*

$$\mathcal{BK}_X := \bigcup_f f^*(\mathcal{K}_{X'}) \subseteq H^{1,1}(X, \mathbb{R}),$$

where the union is taken over all birational maps $f: X \dashrightarrow X'$ from X to another irreducible symplectic variety X' . Denote its closure by $\overline{\mathcal{BK}}_X \subseteq H^{1,1}(X, \mathbb{R})$.

Note that the pullback along $f: X \dashrightarrow X'$ is well-defined, since the indeterminacy locus is of codimension at least two by Lemma 1.5.

THEOREM 2.2 ([12, Proposition 4.2]). — *Let X be an irreducible symplectic variety. Then the closure of the birational Kähler cone of X can be described in the following way*

$$\overline{\mathcal{BK}}_X = \left\{ \alpha \in \overline{\mathcal{C}}_X \subseteq H^{1,1}(X, \mathbb{R}) \mid (\alpha, D)_q \geq 0, \forall D \subseteq X \text{ uniruled divisor} \right\}.$$

COROLLARY 2.3. — *Let X be an irreducible symplectic variety, $E \in \text{Pic}(X)$ an effective divisor and $H \in \overline{\mathcal{BK}}_X$. Then $(H, E)_q \geq 0$. In particular this applies for all nef line bundles $H \in \text{Pic}(X)$.*

Proof. — This was actually part of the proof of [12, Proposition 4.2]. \square

Remark 2.4. — In Theorem 2.2, we can restrict ourselves to considering irreducible effective divisors D such that $(D, \beta)_q < 0$ for some $\beta \in \overline{\mathcal{C}}_X$. Since D satisfies $(\alpha, D)_q > 0$ for all $\alpha \in \overline{\mathcal{BK}}_X \subseteq \overline{\mathcal{C}}_X$ (by Corollary 2.3), this implies that D is orthogonal to some element in \mathcal{C}_X in this case. Therefore the divisor D satisfies $q(D) < 0$. In this situation it was shown in [5], that D is always uniruled:

PROPOSITION 2.5 ([5, Proposition 4.7]). — *Every irreducible effective divisor $D \in \text{Pic}(X)$ with $q(D) < 0$ is uniruled.*

One can deduce the following standard fact:

COROLLARY 2.6. — *Fix an irreducible symplectic variety X . Let $\overline{\text{Mov}}(X)$ be the closure of the movable cone in $\text{Pic}(X)_{\mathbb{R}}$, i.e. the closure of the set*

$$\{r \cdot M \in \text{Pic}(X)_{\mathbb{R}} \mid r \in \mathbb{R}, M \in \text{Pic}(X) \text{ movable line bundle}\}.$$

Then $\overline{\text{Mov}}(X) = \overline{\mathcal{BK}}_X \cap \text{Pic}(X)_{\mathbb{R}}$.

Proof. — By Lemma 1.2 and Theorem 2.2 every movable line bundle $M \in \text{Pic}(X)$ is contained in $\overline{\mathcal{BK}}_X$. Conversely, every element $\alpha \in \overline{\mathcal{BK}}_X \cap \text{Pic}(X)$ is the pullback of an ample class on a birational model and therefore movable. \square

In general, the union in the definition of $\overline{\mathcal{BK}}_X$ can be infinite. Therefore it is important to observe the following:

PROPOSITION 2.7. — *Let X be an irreducible symplectic variety and fix $L \in \text{Pic}(X)$. Suppose $L \in \overline{\mathcal{BK}}_X \cap \mathcal{C}_X$. Then there exists a birational irreducible symplectic variety X' such that the associated line bundle $L' \in \text{Pic}(X')$ is nef. If $b_2(X) \neq 4$, the same holds for $L \in \overline{\mathcal{BK}}_X$ with $q(L) = 0$.*

The proof uses the theory of wall divisors (also known as MBM-classes), which can be used to describe the interior structure of $\overline{\mathcal{BK}}_X$.

Let W' be the set of divisors $W \in \text{Pic}(X)$ such that W^\perp contains a face of one of the Kähler chambers in $\overline{\mathcal{BK}}_X$, i.e. there is a birational map $f: X \dashrightarrow X'$ from X to another irreducible symplectic variety X' such that $W^\perp \cap f^*(\mathcal{K}_{X'})$ contains an open subset of W^\perp .

DEFINITION 2.8 (Wall-divisors/MBM-classes; compare [22, Def. 1.2], [1, Def. 1.13]). — *The set \mathcal{W} of wall divisors on X (also known as MBM-classes) is the union of $\varphi(\mathcal{W}')$ for all $\varphi \in \text{Mon}_{\text{Hdg}}^2(X)$, where $\text{Mon}_{\text{Hdg}}^2(X)$ denotes the monodromy operators on $H^2(X, \mathbb{Z})$ which preserve the Hodge structure.*

An important property of wall divisors is

PROPOSITION 2.9 ([1, Theorem 1.19]). — *Let X be an irreducible symplectic variety. The Kähler cone \mathcal{K}_X is a connected component of $\mathcal{C}_X \setminus \bigcup_{W \in \mathcal{W}} W^\perp$. Moreover, for every element $\alpha \in \mathcal{C}_X \setminus \bigcup_{W \in \mathcal{W}} W^\perp$ there exists $\varphi \in \text{Mon}_{\text{Hdg}}^2(X)$, such that $\varphi(\alpha) \in f^*(\mathcal{K}_{X'})$ for a birational irreducible symplectic variety X' as above.*

In order to deal with the case $q(L) = 0$ of Proposition 2.7, we further need the following:

THEOREM 2.10. — *Let X be an irreducible symplectic variety with $b_2(X) \neq 4$. Then the set*

$$\{q(W) \mid W \in h^{1,1}(X) \text{ (primitive) wall divisor on } X\}$$

is bounded.

Proof. — This is due to Amerik and Verbitsky. Their theorem [2, Theorem 3.17] states that for any irreducible symplectic variety X with $b_2(X) \geq 5$ the monodromy group acts on the set of primitive wall divisors with finitely many orbits. Indeed they show this even for the larger set of such classes in $H^2(X, \mathbb{Z})$. Since monodromy operators are isometries for q , this implies the statement. □

The other central ingredient is the following basic proposition:

PROPOSITION 2.11 ([19, Proposition 3.4]). — *Let X be an irreducible symplectic variety, $\Pi \subseteq \overline{\mathcal{C}_x}$ a (closed) rational polyhedral cone, and $N \in \mathbb{N}$ a fixed natural number. Then the set*

$$\{w \in H^{1,1}(X, \mathbb{Z}) \mid -N < q(w) < 0 \text{ and } w^\perp \cap \mathcal{C}_X \cap \Pi \neq \emptyset\}$$

is finite.

Using these results, we can prove Proposition 2.7:

Proof of Proposition 2.7. — From Proposition 2.9, one can deduce that

$$\overline{\mathcal{BK}}_X \cap \mathcal{C}_X = \bigcup_f f^*(\overline{\mathcal{K}_{X'}} \cap \mathcal{C}_X),$$

where the union is again taken over all $f: X \rightarrow X'$, where X' is a birational irreducible symplectic variety. This was not a priori clear, since the union could be infinite. Therefore, a line bundle in the intersection $L \in \text{Pic}(X) \cap \overline{\mathcal{BK}}_X \cap \mathcal{C}_X$ lies in one of the $f^*(\overline{\mathcal{K}}_{X'})$. Thus $L' := f^{-1*}(L) \in \overline{\mathcal{K}}_{X'}$ is nef.

It only remains to deal with the case $q(L) = 0$ under the assumption that $b_2(X) \neq 4$. Fix $L \in \overline{\mathcal{BK}}_X \cap \text{Pic}(X) \cap \partial \mathcal{C}_X$. We need to show that the associated line bundle L' is nef on some birational model.

Pick any (closed) rational polyhedral cone $\Pi \subseteq \overline{\mathcal{BK}}_X \cap \text{Pic}(X)_{\mathbb{R}}$ of dimension $\rho(X) := h^{1,1}(X, \mathbb{Z})$ such that $L \in \Pi$ (e.g. one can construct this by picking any rational polyhedral cone Π' in $\text{Nef}(X) = \overline{\text{Amp}(X)}$ of dimension $\rho(X)$, and choosing $\Pi := \Pi' + \mathbb{R}_{\geq 0}L$).

By Theorem 2.10, the square of primitive wall divisors is bounded, and therefore we can apply Proposition 2.11 to deduce that

$$\{w \in H^{1,1}(X, \mathbb{Z}) \mid w \text{ is wall divisor and } w^\perp \cap \mathcal{C}_X \cap \Pi \neq \emptyset\}$$

is a finite set.

In particular Π decomposes into a finite union of chambers of the form $\Pi \cap f^*(\text{Nef}(X'))$, where $f: X \dashrightarrow X'$ are birational irreducible symplectic varieties. Therefore, L lies in $f^*(\text{Nef}(X'))$ for one such X' . Thus the associated line bundle $L' := f^{-1*}(L)$ lies in $\text{Nef}(X')$ as claimed. \square

Remark 2.12. — The same arguments show: If $b_2(X) \neq 4$ and $\overline{\mathcal{BK}}_X$ is rationally polyhedral, then there are only finitely many chambers in $\overline{\mathcal{BK}}_X$.

The following useful property of line bundles in $A \in \text{Pic}(X) \cap \overline{\mathcal{BK}}_X$ with $q(A) \neq 0$ follows from Proposition 2.7:

LEMMA 2.13. — *Let X be an irreducible symplectic variety and pick a line bundle $A \in \text{Pic}(X) \cap \overline{\mathcal{BK}}_X \cap \mathcal{C}_X$. Then $h^0(X, A) = \chi(X, A)$.*

Proof. — Since $A \in \text{Pic}(X) \cap \overline{\mathcal{BK}}_X \cap \mathcal{C}_X$, Proposition 2.7 implies that there exists a birational irreducible symplectic variety X' such that the corresponding line bundle $A' \in \text{Pic}(X')$ is nef. Then A' is indeed big and nef since $q(A') = q(A) > 0$. Consequently

$$h^0(X, A) = h^0(X', A') = \chi(X', A') = \chi(X, A),$$

where the first equality uses that X and X' are isomorphic away from codimension two by Lemma 1.5. The second equality exploits that A' is big and nef, and therefore satisfies Kodaira vanishing, and the last equality holds since (X, M) and (X', M') are deformation equivalent (see [11, Theorem 4.6]). \square

3. Reflections for prime exceptional divisors

In this section we use Huybrechts' description of $\overline{\mathcal{BK}}_X$ and Markman's results on reflections for prime exceptional divisors to prove Proposition 3.4, which shows that arbitrary elements $\alpha \in \text{NS}(X)_{\mathbb{Q}} \cap \partial \mathcal{C}_X$ can be moved into $\overline{\mathcal{BK}}_X$, by a series of such reflections.

Let us first introduce some notations.

DEFINITION 3.1.

- (1) A divisor $D \in \text{Div}(X)$ is called prime exceptional divisor if D is reduced and irreducible and satisfies $q(D) < 0$.
- (2) For a prime exceptional divisor D define the reflection $R_D \in O(H^2(X, \mathbb{Q}))$ as

$$R_D(\alpha) := \alpha - \frac{2(D, \alpha)_q}{q(D)} D.$$

Remark 3.2. — Several statements in the previous sections involved prime exceptional divisors. In particular Proposition 2.5.

For the proof of Proposition 3.4, we will need the following result of Markman:

PROPOSITION 3.3 ([17, Proposition 6.2]). — For any prime exceptional divisor D the reflection R_D restricts to an integral morphism. In fact, it is a monodromy operator preserving the Hodge structure.

PROPOSITION 3.4. — Let $0 \neq \alpha \in \text{NS}(X)_{\mathbb{Q}} \cap \overline{\mathcal{C}}_X$ then there exists a composition $R = R_{D_{k-1}} \circ \cdots \circ R_{D_0}$ of reflections R_{D_i} associated to prime exceptional divisors D_i , such that $R(\alpha) \in \overline{\mathcal{BK}}_X$.

Proof. — This proof is similar to the analogue for K3 surfaces, as presented in [13, Remark VIII.2.12].

By passing to a multiple of α , we may assume that $\alpha \in H^2(X, \mathbb{Z})$ is an integral element. Set $\alpha_0 := \alpha$. Fix an ample class $h \in H^2(X, \mathbb{Z})$. For any element $\alpha_i \in H^2(X, \mathbb{Z}) \cap \overline{\mathcal{C}}_X$, the Beauville–Bogomolov pairing $(\alpha_i, h)_q$ is a positive integer. If $\alpha_i \notin \overline{\mathcal{BK}}_X$, then by Theorem 2.2 and Remark 2.4 there exists a prime exceptional divisor D_i with $(\alpha_i, D_i)_q < 0$. Note that $(D_i, h)_q > 0$ and $(\alpha_i, h)_q > 0$, since h is ample. Set $\alpha_{i+1} := R_{D_i}(\alpha_i)$, which is an element in the integral lattice $H^2(X, \mathbb{Z})$, since R_{D_i} is an integral

morphism by Proposition 3.3. Observe that

$$\begin{aligned} (\alpha_{i+1}, h)_q &= (R_{D_i}(\alpha_i), h)_q = (\alpha_i - \frac{2(D_i, \alpha_i)_q}{q(D_i)} D_i, h)_q \\ &= (\alpha_i, h)_q - \underbrace{\frac{2(D_i, \alpha_i)_q}{q(D_i)} \cdot (D_i, h)_q}_{>0} < (\alpha_i, h)_q. \end{aligned}$$

If $\alpha_{i+1} \notin \overline{\mathcal{BK}}_X$, repeat the above for α_{i+1} . Since $(\alpha_0, h) > (\alpha_1, h) > (\alpha_2, h) > \dots$ is a descending sequence of positive integers, this procedure needs to stop for some $k \in \mathbb{N}$, which implies that $\alpha_k \in \overline{\mathcal{BK}}_X$. Set $R := R_{D_{k-1}} \circ \dots \circ R_{D_0}$. This concludes the proof, since $R(\alpha) = \alpha_k \in \overline{\mathcal{BK}}_X$. \square

Furthermore, we will need the following inequality for exceptional prime divisors:

LEMMA 3.5 (follows from [18, Lemma 3.7]). — *Let X be an irreducible symplectic variety. Suppose $D \in \text{Pic}(X)$ is an irreducible (and reduced) divisor with $q(D) < 0$. Recall that $\text{div}(D) := \text{gcd}\{(\alpha, D)_q \mid \alpha \in H^2(X, \mathbb{Z})\}$. Then $q(D) \mid 2 \text{div}(D)$. In particular*

$$-\frac{1}{2}q(D) \leq \text{div}(D).$$

4. Main Theorem

In this section, we state the main theorem of this article and deduce some immediate consequences.

For the assumptions of the theorem in its most general form, we need to introduce some notation first.

Let X be an irreducible symplectic variety. Recall that by the Riemann–Roch theorem for irreducible symplectic varieties (Theorem 1.3) there exist $b_i \in \mathbb{Q}$ such that for each $L \in \text{Pic}(X)$

$$\chi(X, L) = \sum_{i=0}^n b_i q(L)^i.$$

DEFINITION 4.1. — *For a given irreducible symplectic variety X , denote the polynomial appearing in the Riemann–Roch formula by $RR_X(x) := \sum_{i=0}^n b_i x^i$, where the b_i are as above.*

The polynomial RR_X only depends on the deformation type of X .

Remark 4.2. — For the main theorem (Theorem 4.7), we will need to assume that RR_X is strictly monotonic. Note that without any assumptions, it is not even clear that for an arbitrary irreducible symplectic variety X and an ample line bundle $H \in \text{Pic}(X)$ there are non-trivial global sections in $H^0(X, H)$. Our assumptions on RR_X will exclude this problem.

Another statement which we will frequently need for the main theorem is that the following conjecture holds for the variety X which we consider:

DEFINITION 4.3. — *A line bundle $L \in \text{Pic}(X)$ is said to induce a rational Lagrangian fibration to \mathbb{P}^n if there exists a birational map $f: X \dashrightarrow X'$ to an irreducible symplectic variety X' , and a fibration $\varphi: X' \rightarrow \mathbb{P}^n$, such that the birational transform L' of L on X' is the pullback of an ample line bundle.*

CONJECTURE 4.4. — *Let X be an irreducible symplectic variety of dimension $2n$ and $0 \neq L \in \text{Pic}(X) \cap \overline{\mathcal{BK}}_X$ be a primitive line bundle with $q(L) = 0$. Then $\dim h^0(X, L) = n + 1$ and $|L|$ induces a birational Lagrangian fibration to \mathbb{P}^n .*

This is a classical conjecture on irreducible symplectic varieties. There is a weaker version which does not make any predictions on the base space of the Lagrangian fibration. For its history we refer to [29, p. 3]. Note however, that the base space of a Lagrangian fibration is automatically isomorphic to \mathbb{P}^n if it is smooth (see [8] and [14]). Smoothness of the base is not known in general, but also conjectured. Furthermore, the presented statement of the conjecture implies that the pullback $\varphi^*\mathcal{O}(1)$ is primitive. This property is conjectured to hold in general and has been subject of [15]. The version of the conjecture stated here also implies the existence of the birational model X' on which L' is nef. By Proposition 2.7 this is automatic whenever $b_2(X) \neq 4$.

Conjecture 4.4 is known to hold in many cases:

THEOREM 4.5 ([20, Corollary 1.1]). — *Conjecture 4.4 holds for all irreducible symplectic varieties of $\text{K3}^{[n]}$ -type and of Kum^n -type.*

Remark 4.6. — Trivially, if X satisfies $\text{Pic}(X) \cap \partial \mathcal{C}_X = 0$, Conjecture 4.4 automatically holds for X (since there is no such line bundle L).

We can now formulate the main theorem in its most general form:

THEOREM 4.7. — *Let X be a $2n$ -dimensional irreducible symplectic variety for which $RR_X|_{\mathbb{Z}_{\geq 0}}$ is strictly monotonic and such that Conjecture 4.4 holds for X . Consider a big and nef line bundle $H \in \text{Pic}(X)$. Then H has*

a non-trivial base divisor if and only if there exists an irreducible reduced divisor F of negative square such that H is of the form $H = mL + F$, where $m \geq 2$, L is a primitive movable line bundle with $q(L) = 0$ and $(L, F)_q > 0$, such that $RR_X(q(H)) = \binom{m+n}{n}$. In this case F is exactly the fixed divisor of H .

Let us first note some immediate consequences, before we give the proof in Section 5.

COROLLARY 4.8. — *If X is an irreducible symplectic variety such that $RR_X|_{\mathbb{Z}_{\geq 0}}$ is strictly monotonic and with $\text{Pic}(X) \cap \partial\mathcal{C}_X = 0$, then no big and nef line bundle on X has fixed components.*

COROLLARY 4.9. — *Fix an irreducible symplectic variety X such that $RR_X|_{\mathbb{Z}_{\geq 0}}$ is strictly monotonic and Conjecture 4.4 holds for X . Let $H \in \text{Pic}(X)$ be a big and nef line bundle. Then $2H$ does not have a divisorial base component.*

Proof. — For a big and nef line bundle $H \in \text{Pic}(X)$ Theorem 4.7 shows that if H has a divisorial base component, H is of the form $H = mL + F$, where $m \geq 2$, L is movable with $q(L) = 0$, and F is the fixed part of H , which is an irreducible reduced divisor of negative square, and $(L, F)_q > 0$.

Therefore $2H = 2mL + 2F$. Assume for contradiction that $2H$ has a divisorial base component F' . Then Theorem 4.7 shows that $2H = m'L' + F'$, where F' is the base locus of $2H$, and $q(L') = 0$. However, the base locus of $2H$ is contained in the base locus of H , which implies $F' = F$. Therefore $m'L' = 2H - F = 2mL + F$. For the last term use $(L, F)_q > 0$ and Lemma 3.5 to show that $q(m'L') = q(2mL + F) = 4m(L, F)_q + q(F) > 0$ which gives the desired contradiction to $q(m'L') = 0$. □

5. Proof of the main theorem (Theorem 4.7)

In this section we prove the main theorem of this article. The general structure of this proof was inspired by the case of K3 surfaces (compare [13, Proof of Corollary 3.15]).

A crucial observation in the study of base components for irreducible symplectic varieties is the following:

LEMMA 5.1. — *Let X be an irreducible symplectic variety and $0 \neq F \in \text{Pic}(X)$ be a fixed divisor (i.e. $h^0(X, F) = 1$). If $RR_X|_{\mathbb{Z}_{\geq 0}}$ is monotonic and if X satisfies Conjecture 4.4 then $q(F) < 0$.*

Proof. — Suppose for contradiction that $F \in \overline{\mathcal{C}}_X$. Apply Proposition 3.4 (and its proof) to see that, after a finite number of reflections associated to prime exceptional divisors, $A := R_{D_{k-1}} \circ \dots \circ R_{D_0}(F) \in \overline{\mathcal{BK}}_X$ lies in the closure of the birational Kähler cone. Recall that in the proof of Proposition 3.4, we set $F_0 := F$. As long as $F_i \notin \overline{\mathcal{BK}}_X$, we successively pick effective divisors D_i with $q(D_i) < 0$ and $(D_i, F_i)_q < 0$, and define $F_{i+1} := R_{D_i}(F_i) = F_i - \frac{2(D_i, F_i)_q}{q(D_i)} D_i$. By the choice of the D_i , each of these reflections subtracts a positive multiple of the effective divisor D_i .

Consequently

$$F = A + \sum_{i=0}^{k-1} a_i D_i \quad \text{with } a_i = \frac{2(D_i, F_i)_q}{q(D_i)} > 0.$$

Note that $a_i \in \mathbb{Z}$ is integral by Lemma 3.5. Therefore, it suffices to show that no element $A \in \overline{\mathcal{BK}}_X$ is a fixed divisor.

If $q(A) > 0$, then by assumption $RR_X(q(A)) \geq RR_X(0) = n + 1 > 1$, which together with Lemma 2.13 implies

$$h^0(X, A) = \chi(X, A) = RR_X(q(A)) > 1.$$

In particular A is not a fixed divisor.

If $q(A) = 0$ Conjecture 4.4 implies that A is movable. In both cases this shows that F could not be a fixed divisor. □

Using Lemma 5.1 we can show:

LEMMA 5.2. — *Let X be an irreducible symplectic variety which satisfies Conjecture 4.4 and such that the map $RR_X|_{\mathbb{Z}_{\geq 0}}$ is strictly monotonic. Fix a big and nef bundle $H \in \text{Pic}(X)$ and consider the decomposition $H = M + F$ into the movable and fixed part. If in this situation $q(M) > 0$, then $F = 0$.*

Proof. — By assumption, M is movable. Therefore $M \in \text{Pic}(X) \cap \overline{\mathcal{BK}}_X$ by Corollary 2.6. Since $q(M) > 0$, Lemma 2.13 applies and yields

$$\chi(X, H) = h^0(X, H) = h^0(X, M) = \chi(X, M),$$

where the first equality exploits that H is big and nef, and therefore satisfies Kodaira vanishing, the second equality holds since M is the movable part of H , and the third equality is the statement of Lemma 2.13.

This implies $q(H) = q(M)$, since by assumption $RR_X|_{\mathbb{Z}_{\geq 0}}$ is strictly monotonic, and in particular injective. Then

$$\begin{aligned}
 q(M) = q(H) = q(M + F) &= q(M) + 2(M, F)_q + q(F) \quad \text{implies} \\
 0 = 2(M, F)_q + q(F) &= \underbrace{(M, F)_q}_{\geq 0} + \underbrace{(H, F)_q}_{\geq 0}.
 \end{aligned}$$

The fact that both terms on the right hand side are at least zero, follows from Lemma 1.2. This equality implies that $(M, F)_q = 0 = (H, F)_q$. Therefore

$$0 = (H, F)_q = (M + F, F)_q = \underbrace{(M, F)_q}_{=0} + q(F).$$

By Lemma 5.1, we know that $q(F) < 0$ whenever F is not trivial. It follows that $F = 0$ as claimed. □

Using these results, we can complete the proof of the main theorem

Proof of Theorem 4.7. — Let X be an irreducible symplectic variety satisfying the conditions of Theorem 4.7 and $H = M + F \in \text{Pic}(X)$ be a big and nef line bundle with its decomposition into movable and fixed part. Suppose that $H \neq M$ or equivalently that the fixed divisor $F \neq 0$ is not trivial. It follows from Lemma 5.2 that $q(M) = 0$. Using that $RR_X|_{\mathbb{Z}_{\geq 0}}$ is strictly monotonic one obtains

$$h^0(X, M) = h^0(X, H) = \chi(X, H) = RR_X(q(H)) > RR_X(0) = n + 1,$$

since M is the movable part of H , by Kodaira vanishing, Theorem 1.3, and Proposition 1.4. Denote $M = m \cdot L$, where L is the primitive line bundle in the same ray and $m \geq 0$. By Conjecture 4.4, $|L|$ induces a rational Lagrangian fibration to \mathbb{P}^n and therefore $h^0(X, L) = n + 1$. Thus $m \geq 2$.

We need to observe that F is irreducible. Let $F = \sum a_i F_i$ be the decomposition of F into its irreducible components. Lemma 5.1 shows that $q(F) < 0$. Since $0 < (H, F)_q = (M, F)_q + q(F) = \sum_i a_i (M, F_i)_q + q(F)$, there exists at least one i_0 such that the irreducible component F_{i_0} satisfies $(M, F_{i_0})_q > 0$.

Define $H' := M + F_{i_0} = mL + F_{i_0}$. We will show that $H' \in \overline{\mathcal{BK}}_X \cap \mathcal{C}_X$. By choice of i_0 one has $(L, F_{i_0})_q > 0$, and thus $(L, F_{i_0})_q \geq \text{div}(F_{i_0}) > 0$. On the other hand, since F_{i_0} is a fixed divisor, Lemma 5.1 shows that $q(F_{i_0}) < 0$, and thus Lemma 3.5 implies that

$$(5.1) \quad -q(F_{i_0}) \leq 2 \text{div}(F_{i_0}) \leq 2(L, F_{i_0})_q.$$

Consequently

$$\begin{aligned}
 q(H') &= q(mL + F_{i_0}) \\
 &= \underbrace{m^2 q(L)}_{=0} + 2(m-1)(L, F_{i_0})_q + \underbrace{2(L, F_{i_0})_q + q(F_{i_0})}_{\geq 0} > 0.
 \end{aligned}$$

For any irreducible effective divisor $D \neq F_{i_0} \in \text{Pic}(X)$ (in particular for uniruled divisors) use Lemma 1.2 to see

$$(H', D)_q = (mL + F_{i_0}, D)_q = \underbrace{m(L, D)_q}_{\geq 0} + \underbrace{(F_{i_0}, D)_q}_{\geq 0} \geq 0.$$

Finally observe that

$$(H', F_{i_0})_q = (mL + F_{i_0}, F_{i_0})_q = m(L, F_{i_0})_q + q(F_{i_0}) \geq 0,$$

where the last inequality follows again from (5.1). Thus $H' \in \overline{\mathcal{BK}}_X \cap \mathcal{C}_X$ by Theorem 2.2.

Similar to Lemma 5.2 we show that $F - F_{i_0} = 0$: Since M is the movable part of H , observe that $h^0(X, H) = h^0(X, H') = h^0(X, M)$. Using Lemma 2.13 twice, thus gives $\chi(X, H) = h^0(X, H) = h^0(X, H') = \chi(X, H')$, and thus $q(H) = q(H')$ by the injectivity of $RR_X|_{\mathbb{Z}_{\geq 0}}$. Therefore

$$\begin{aligned}
 0 &= q(H) - q(H') = (H, H' + F - F_{i_0})_q - q(H') \\
 &= (H' + F - F_{i_0}, H')_q + (H, F - F_{i_0})_q - q(H') \\
 &= \underbrace{(F - F_{i_0}, H')_q}_{\geq 0} + \underbrace{(H, F - F_{i_0})_q}_{\geq 0},
 \end{aligned}$$

where the last inequalities follow from Corollary 2.3. Consequently $(H, F - F_{i_0})_q = 0 = (H', F - F_{i_0})_q$, and thus

$$q(F - F_{i_0}) = (H - H', F - F_{i_0})_q = (H, F - F_{i_0})_q - (H', F - F_{i_0})_q = 0.$$

Since $F - F_{i_0}$ is a fixed divisor, Lemma 5.1 implies that $F - F_{i_0} = 0$. Therefore $F = F_{i_0}$ is irreducible and reduced and in addition $(F, L)_q = (F_{i_0}, L)_q > 0$.

In order to establish the condition on $RR_X(q(H))$, begin with the following observation. For a primitive movable line bundle $L \in \text{Pic}(X) \cap \overline{\mathcal{BK}}_X$ with $q(L) = 0$, use Conjecture 4.4 for X to obtain a birational model X' , where the corresponding line bundle L' induces a Lagrangian fibration to \mathbb{P}^n . In particular

$$(5.2) \quad h^0(X, mL) = h^0(X', mL') = h^0(\mathbb{P}^n, \mathcal{O}(m)) = \binom{m+n}{n},$$

where the first equality follows from the fact that X and X' are isomorphic away from codimension two by Lemma 1.5.

Therefore the theorem follows from the above, since by Kodaira vanishing $h^0(X, H) = \chi(X, H) = RR_X(q(H))$, and because a line bundle of the form $H = mL + F$ has base locus along F if and only if $h^0(X, H) = h^0(X, mL)$. \square

6. Discussion of the assumptions in the statement

In this section we discuss the assumptions in the statement of Theorem 4.7.

First note that Theorem 4.7 applies to all varieties of $K3^{[n]}$ -type and of Kum^n -type: Conjecture 4.4 holds in these cases by Theorem 4.5. The condition that $RR_X|_{\mathbb{Z}_{\geq 0}}$ is strictly monotonic, can either be verified directly, or deduced from the following lemma:

Let U be the standard hyperbolic lattice of rank two, i.e. the lattice with associated matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

LEMMA 6.1. — *Fix an irreducible symplectic variety X such that*

- (1) *Conjecture 4.4 is satisfied for all deformations of X ,*
- (2) *$H^2(X, \mathbb{Z})$ is an even lattice with respect to the Beauville–Bogomolov–Fujiki form, and*
- (3) *$H^2(X, \mathbb{Z})$ contains a copy of U .*

Then the restriction $RR_X|_{\mathbb{Z}_{\geq 0}} : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}$ is strictly monotonic and in particular injective.

Remark 6.2. — This lemma should serve as an indication that it is reasonable to ask whether $RR_X|_{\mathbb{Z}_{\geq 0}}$ is strictly monotonic for all irreducible symplectic varieties. Conjecture 4.4 is a classical conjecture on irreducible symplectic varieties. Furthermore all known examples of irreducible symplectic varieties satisfy 2 and 3. It is not known, whether these conditions hold for arbitrary irreducible symplectic varieties. I expect, that in general the question whether $RR_X|_{\mathbb{Z}_{\geq 0}}$ is strictly monotonic should be easier to answer than Conjecture 4.4.

Proof. — The polynomial RR_X only depends on the deformation type of X . Therefore by passing to a suitable deformation of X we may assume that $\text{Pic}(X) \cong U$. Let $E, F \in \text{Pic}(X)$ be the generators of U that span the boundary of the positive cone. For $k > 0$ define the element $B_k := F + kE \in$

U . Then $q(B_k) = 2k$ and therefore

$$RR_X(2k) = RR_X(q(B_k)) = \chi(X, B_k).$$

The closure of the birational Kähler cone $\overline{\mathcal{BK}}_X$ is cut out by uniruled irreducible (and reduced) divisors D with $q(D) < 0$ (compare Theorem 2.2). By Lemma 3.5 every such divisor D satisfies $\frac{1}{2}|q(D)| \leq \text{div}(D)$. Express D as $aE + bF$ for some $a, b \in \mathbb{Z}$. Then the equality gives

$$\frac{1}{2}|2ab| \leq \text{gcd}(a, b),$$

which has solutions for $|a| = |b| = 1$ or $a = 0$ or $b = 0$. Thus the only solution with $q(D) < 0$ is $\pm(E - F)$.

Therefore there are two possible cases:

Case 1: Both elements $E - F$ and $F - E$ are not effective, and thus $\overline{\mathcal{BK}}_X = \mathcal{C}_X$. — By Conjecture 4.4 both E and F are effective with $h^0(X, E) = h^0(X, F) = n + 1 > 1$, where $n = \dim(X)/2$ (both are giving a rational Lagrangian fibration to \mathbb{P}^n). In particular observe that

$$\begin{aligned} h^0(X, B_1) &= h^0(X, E + F) > h^0(X, F) = n + 1, \quad \text{and} \\ h^0(X, B_{k+1}) &= h^0(X, B_k + E) > h^0(X, B_k) \quad \forall k > 0. \end{aligned}$$

Case 2: $\overline{\mathcal{BK}}_X \neq \mathcal{C}_X$. — In this case there exists a unique prime exceptional divisor: either $E - F$ or $F - E$. Without loss of generality we may assume $D := F - E$ is effective, and thus $E \in \overline{\mathcal{BK}}_X$ induces a rational Lagrangian fibration to \mathbb{P}^n by Conjecture 4.4. Therefore $h^0(X, E) = n + 1 > 1$, and $F = E + D$ implies $h^0(X, F) \geq h^0(X, E) = n + 1 > 1$. Thus

$$h^0(X, B_1) = h^0(X, F + E) = h^0(X, 2E + D) \geq h^0(X, 2E) > n + 1$$

and

$$h^0(X, B_{k+1}) = h^0(X, B_k + E) > h^0(X, B_k) \quad \forall k > 0.$$

In both cases the elements B_k are contained in $\overline{\mathcal{BK}}_X \cap \mathcal{C}_X$ (for Case 2 note that $(B_k, D)_q = (F + kE, F - E)_q = k - 1 \geq 0$). Therefore by Lemma 2.13 $\chi(X, B_k) = h^0(X, B_k)$.

Together, this yields that in both cases:

$$\begin{aligned} RR_X(2) &= \chi(X, B_1) = h^0(X, B_1) > n + 1 = RR_X(0), \quad \text{and} \\ RR_X(2k + 2) &= \chi(X, B_{k+1}) = h^0(X, B_{k+1}) \\ &> h^0(X, B_k) = \chi(X, B_k) = RR_X(2k) \quad \forall k > 0, \end{aligned}$$

which concludes the proof. □

For arbitrary deformation types, the condition that RR_X is strictly monotonic is known to hold up to dimension 6:

PROPOSITION 6.3. — *Let X be an irreducible symplectic variety of dimension ≤ 6 . Then in the Riemann–Roch polynomial $RR_X(x) := \sum_{i=0}^n b_i x^i$ all b_i are non-negative. Since $b_n > 0$ this implies that RR_X is strictly monotonic.*

Proof. — For dimension 6 this is shown in [6]. In fact the proof of [6, Corollary 3.4] gives an explicit description for the b_i in this case and shows that they are non-negative.

The same arguments work for dimension < 6 .

Finally, let us explain, why $b_n > 0$. From the proof of [9, Corollary 23.18] one sees that $\int_X \alpha^{2n} = (2i)!b_n q(\alpha)^i$ for every $\alpha \in H^2(X, \mathbb{Z})$. Therefore b_n differs by a positive factor from the Fujiki constant, which is known to be positive. □

7. Behaviour under deformation

In this section we discuss the behaviour of divisorial base loci of big and nef line bundles on irreducible symplectic varieties under deformations.

For a given irreducible symplectic variety X , fix a marking $\eta : H^2(X, \mathbb{Z}) \xrightarrow{\cong} \Lambda$ for some lattice Λ . Denote by \mathcal{M}_Λ^X the moduli space of marked irreducible symplectic varieties deformation equivalent to (X, η) . Pick an element $h \in \Lambda$, such that $H := \eta^{-1}(h)$ is a big and nef line bundle on X . Denote by $\mathcal{M}_{\Lambda, h}^X$ the moduli space of semi-polarized marked irreducible symplectic varieties deformation equivalent to (X, η) , i.e. the subspace of the moduli space \mathcal{M}_Λ^X , where the class h still corresponds to a big and nef line bundle. To fix notation, $t \in \mathcal{M}_{\Lambda, h}^X$ parametrizes an irreducible symplectic variety X_t with marking $\eta_t : H^2(X_t, \mathbb{Z}) \cong \Lambda$ such that $H_t := \eta_t^{-1}(h)$ is a big and nef line bundle.

THEOREM 7.1. — *Let X be an irreducible symplectic variety such that its deformation type satisfies the conditions of the main theorem (Theorem 4.7). Fix a marking η and a class $h \in \Lambda$ as above. Then the locus in the semi-polarized moduli space $\mathcal{M}_{\Lambda, h}^X$, where the semi-polarization has a non-trivial base divisor, is a union of disjoint Noether–Lefschetz divisors (the Noether–Lefschetz divisors corresponding to the possible base divisors).*

Proof. — First note that the condition that RR_{X_t} is strictly monotonic ensures that for each $t \in \mathcal{M}_{\Lambda, h}^X$ the semi-polarization H_t is effective (since H_t is big and nef, it satisfies Kodaira vanishing and thus $h^0(X_t, H_t) = \chi(X_t, H_t) = RR_{X_t}(q(H_t)) > 0$ by assumption). Thus it makes sense to study the base divisor.

Consider a small open subset $T \subseteq \mathcal{M}_{\Lambda, h}^X$ such that there exists a universal family $\pi: \mathcal{X} \rightarrow T$ with semi-polarization $\mathcal{H} \in \text{Pic}(\mathcal{X})$ (this is possible by [11, 1.12]). Let $X_t := \pi^{-1}t$ and $H_t := \mathcal{H}|_{X_t}$. Then again by Kodaira vanishing $H^k(X_t, H_t) = 0$ for all $t \in T$ and $k > 0$. Therefore, by base change $\pi_*(\mathcal{H})$ is a vector bundle with $H^0(X_t, H_t) = \pi_*\mathcal{H}|_t$. Thus after shrinking T , one can assume that all global sections of H_t are restrictions of global sections of \mathcal{H} . This implies that the base locus of \mathcal{H} , which is a closed set, is equal to the union of the base loci of the H_t . In particular, the locus in T where H_t has a base divisor is closed. It thus suffices to prove the statement locally in $\mathcal{M}_{\Lambda, h}^X$, and we can assume that there exists a universal family.

If there exists t_0 such that H_{t_0} has non-trivial divisorial base locus, then one can apply the main theorem (Theorem 4.7). One sees that H_{t_0} is of the form $H_{t_0} = mL_{t_0} + F_{t_0}$, where $m \geq 2$, L_{t_0} is a primitive movable line bundle with $q(L_{t_0}) = 0$ and $(L_{t_0}, F_{t_0})_q > 0$, such that $h^0(X_{t_0}, H_{t_0}) = RR_{X_{t_0}}(q(H_{t_0})) = \binom{m+n}{n}$, F_{t_0} is a prime exceptional divisor with $q(F_{t_0}) < 0$.

Consider $T' \subset T$ the Noether–Lefschetz locus, where the class of F_{t_0} stays $(1, 1)$, and thus there is a deformation $\mathcal{F} \in \text{Pic}(\mathcal{X}_{T'})$. We will show, that H_t has non-trivial base divisor for all $t \in T'$, and that the base divisor is exactly $F_t := \mathcal{F}|_{X_t}$.

As observed above, Kodaira vanishing implies that $h^0(X_t, H_t)$ is constant and thus

$$(7.1) \quad h^0(X_t, H_t) = h^0(X_{t_0}, H_{t_0}) = \binom{m+n}{n}.$$

Since F_{t_0} is a prime exceptional divisor, it stays effective under deformation by [18, Proposition 5.2], i.e. for every $t \in T'$ the divisor F_t is effective.

Set $L_t := \frac{1}{m}(H_t - F_t)$, which is a primitive, integral class. Since both H_t and F_t are of $(1, 1)$ type along T' , the same holds for L_t .

CLAIM 7.2. — *For every $t \in T'$ the line bundle L_t is effective with $h^0(X_t, mL_t) \geq h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = \binom{m+n}{n}$ for all m .*

Proof of claim. — Since $L_{t_0} \in \overline{\mathcal{BK}}_{X_{t_0}}$, also for very general $t \in T'$ the line bundle $L_t \in \overline{\mathcal{BK}}_{X_t}$ (use Theorem 2.2 to see this is satisfied for every t which does not admit an additional uniruled divisor in $\text{Pic}(X_t)$). Further, note that $q(L_t) = q(L_{t_0}) = 0$. Consequently, for general $t \in T'$ Conjecture 4.4 implies that L_t induces a rational Lagrangian fibration to \mathbb{P}^n , and thus $h^0(X_t, mL_t) \geq h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = \binom{m+n}{n}$. The claim follows by semicontinuity. □

To conclude the proof of Theorem 7.1 use that by construction $H_t = mL_t + F_t$ for every $t \in T'$, where we saw above that all involved line bundles are effective. We therefore have

$$\binom{m+n}{n} \stackrel{(7.1)}{=} h^0(X_t, H_t) \geq h^0(X_t, mL_t) \stackrel{7.2}{\geq} \binom{m+n}{n}.$$

Thus all involved inequalities must be equalities, in particular $h^0(X_t, H_t) = h^0(X_t, mL_t)$, which implies that F_t is contained in the base locus of H_t and then Theorem 4.7 implies that the base divisor of H_t is irreducible and reduced, and therefore coincides with F_t . □

8. Base divisors for $K3^{[n]}$ -type

In this section we improve the main theorem (Theorem 4.7) for irreducible symplectic varieties of $K3^{[n]}$ -type. We need the following Riemann–Roch formula.

PROPOSITION 8.1 (Riemann–Roch for $K3^{[n]}$ -type, [9, Example 23.19]). *Let X be of $K3^{[n]}$ -type, and $L \in \text{Pic}(X)$ be a line bundle. Then*

$$\chi(X, L) = \binom{\frac{1}{2}q(L) + n + 1}{n}.$$

With this in mind, we find the following characterization for base divisors of big and nef line bundles on irreducible symplectic varieties of $K3^{[n]}$ -type.

PROPOSITION 8.2. — *Let X be an irreducible symplectic variety of $K3^{[n]}$ -type and $H \in \text{Pic}(X)$ a line bundle that is big and nef. Then H has a fixed divisor if and only if H is of the form $H = mL + F$, where $m \geq 2$, L is movable with $q(L) = 0$, and F is an irreducible reduced divisor of negative square with $(L, F)_q = 1$. In this case F is the fixed divisor of H .*

Proof. — Let $H \in \text{Pic}(X)$ be a big and nef line bundle on X . Recall that Theorem 4.7 can be applied for $K3^{[n]}$ -type by Section 6. This shows that H has non-trivial base divisor if and only if $H = mL + F$ where $m \geq 2$, L is movable with $q(L) = 0$, and F is an irreducible reduced divisor of negative square, and $(L, F)_q > 0$, such that $h^0(X, H) = \binom{m+n}{n}$. Therefore, we only need to observe that the additional condition $(L, F)_q = 1$ is equivalent to $h^0(X, H) = \binom{m+n}{n}$.

Since H is big and nef by assumption, Kodaira vanishing and the Riemann–Roch for $K3^{[n]}$ -type (see Proposition 8.1) imply that

$$h^0(X, H) = \chi(X, H) = \binom{\frac{1}{2}q(H) + n + 1}{n}.$$

In this case $H = mL + F$ shows that

$$(8.1) \quad q(H) = q(mL + F) = 2m(L, F)_q + q(F) \\ = 2(m - 1)(L, F)_q + \underbrace{2(L, F)_q + q(F)}_{\geq 0},$$

where the inequality of the last term follows from Lemma 3.5 since $(L, F)_q > 0$.

Therefore

$$(8.2) \quad h^0(X, H) = \binom{\frac{1}{2}q(H) + n + 1}{n} \geq \binom{(m - 1)(L, F)_q + n + 1}{n} \\ \stackrel{(*)}{\geq} \binom{(m - 1) + n + 1}{n} = \binom{m + n}{n},$$

where equality in $(*)$ holds if only if $(L, F)_q = 1$.

To see the equivalence $h^0(X, H) = \binom{m+n}{n}$ if and only if $(L, F)_q = 1$, first suppose $h^0(X, H) = \binom{m+n}{n}$. In this case equality holds in all places in (8.2). In particular, there is equality in $(*)$, whence $(L, F)_q = 1$.

For the other implication, suppose that $(L, F)_q = 1$. Since $q(F) < 0$, Lemma 1.2 implies that F is a fixed divisor, $(L, F)_q = 1$ ensures that $\text{div}(F) = 1$. Note that the $\text{K3}^{[n]}$ -lattice $\Lambda_{\text{K3}^{[n]}}$ is an even lattice. Therefore use Lemma 3.5 to see that $-q(F) \leq 2 \text{div}(F) = 2$, and thus $q(F) = -2$. In particular, $2(L, F)_q + q(L) = 2 \cdot 1 - 2 = 0$, which shows that both inequalities in (8.2) are equalities in this case. Consequently, $h^0(X, H) = \binom{m+n}{n}$. \square

9. Base divisors for Kumⁿ-type

Similar as in the previous section, we need the following

PROPOSITION 9.1 (Riemann–Roch for Kumⁿ-type, [9, Example 23.20]). *Let X be an irreducible symplectic variety of Kumⁿ-type, and $L \in \text{Pic}(X)$ be a line bundle. Then*

$$\chi(X, L) = (n + 1) \binom{\frac{1}{2}q(L) + n}{n}.$$

This can be used to exclude divisorial base components for big and nef line bundles on varieties of Kumⁿ-type:

PROPOSITION 9.2. — *Let X be an irreducible symplectic variety of Kumⁿ-type, and H a big and nef line bundle. Then H is movable, i.e. it does not have a fixed divisor.*

Proof. — Suppose for contradiction that H has non-trivial base divisor. Note that Theorem 4.7 can be applied to Kum ^{n} -type by Section 6. Thus there exists an irreducible reduced divisor F of negative square such that H is of the form $H = mL + F$, where $m \geq 2$, L is a primitive movable line bundle with $q(L) = 0$ and $(L, F)_q > 0$, such that $RR_X(q(H)) = \binom{m+n}{n}$. We only need to see that this cannot happen for X of Kum ^{n} -type.

As in (8.1)

$$q(H) = 2(m - 1)(L, F)_q + \underbrace{2(L, F)_q + q(F)}_{\geq 0}.$$

We will use this to show, that there are no positive integral values for m and n such that

$$(9.1) \quad RR_X(q(H)) = \binom{m+n}{n} = \frac{(m+n)!}{n! \cdot m!}.$$

Distinguish between two cases:

Case 1: If $(L, F)_q = 1$. — Since the Kum ^{n} -lattice is even and $q(F) < 0$, in this case $0 \leq 2(L, F)_q + q(F) = 2 + q(F) \leq 0$, and thus in fact there is equality everywhere, $q(F) = -2$ and

$$(9.2) \quad q(H) = 2(m - 1)(L, F)_q = 2(m - 1).$$

With this in mind use Riemann–Roch for Kum ^{n} -type (Proposition 9.1) to see that the left hand side of (9.1) is

$$\begin{aligned} RR_X(q(H)) &= \chi(X, H) = (n + 1) \binom{\frac{1}{2}q(H) + n}{n} \\ &\stackrel{(9.2)}{=} (n + 1) \binom{m - 1 + n}{n} = (n + 1) \frac{(m + n - 1)!}{n! \cdot (m - 1)!} \end{aligned}$$

Therefore, (9.1) becomes

$$\begin{aligned} (n + 1) \frac{(m + n - 1)!}{n! \cdot (m - 1)!} &= \frac{(m + n)!}{n! \cdot m!} \\ \iff (n + 1) \cdot 1 &= \frac{(m + n)}{m} \\ \iff nm + m &= m + n \\ \iff 1 &= m. \end{aligned}$$

which is a contradiction to $m \geq 2$, which would hold if H were a big and nef line bundle with a fixed divisor.

Case 2: If $(L, F)_q \geq 2$. — Again, we want to verify, that there are no positive integral solutions m, n of (9.1). Note that in this case

$$q(H) = 2(m-1)(L, F)_q + \underbrace{2(L, F)_q + q(F)}_{\geq 0} \geq 4(m-1).$$

In addition, $m \geq 2$ immediately implies that $2(m-1) \geq m$. Since RR_X is monotonic for Kumⁿ-type this implies that

$$\begin{aligned} RR_X(q(H)) &= n \binom{\frac{1}{2}q(H) + n - 1}{n-1} \geq (n+1) \binom{2(m-1) + n}{n} \\ &\geq (n+1) \binom{m+n}{n} > \binom{m+n}{n}. \end{aligned}$$

This is the desired contradiction to the equality in (9.1). \square

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