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SEPARATION PROFILE, ISOPERIMETRY, GROWTH AND COMPRESSION

by Antoine GOURNAY & Corentin LE COZ (*)

ABSTRACT. — We give lower and upper bounds for the separation profile (introduced by Benjamini, Schramm & Timár) for various graphs using isoperimetric profile, volume growth and Hilbertian compression. For graphs which have polynomial isoperimetry and growth, we show that the separation profile is bounded between n^a and n^b for some $a, b \in (0, 1)$. For many amenable groups, we prove a lower bound of $n/\log(n)^a$ for some $a > 1$, and for groups admitting “good” embeddings into an ℓ^p space we prove an upper bound of $n/\log(n)^b$ for some $b \in (0, 1)$. We show that solvable groups of exponential growth cannot have a separation profile bounded above by a sublinear power function. We also introduce a notion of local separation, with applications for percolation clusters of \mathbb{Z}^d and graphs which have polynomial isoperimetry and growth.

RÉSUMÉ. — Pour différents types de graphes, nous établissons des bornes inférieures et supérieures sur leur profil de séparation (introduit par Benjamini, Schramm & Timár), en utilisant le profil isopérimétrique, la fonction de croissance et la compression dans les espaces de Hilbert. Dans le cas des graphes de dimension isopérimétrique supérieure à 1 et à croissance polynomiale, nous montrons que le profil de séparation est compris entre deux fonctions puissance, avec des exposants compris strictement entre 0 et 1. Pour de nombreux groupes moyennables, nous montrons une borne inférieure de la forme $n/\log(n)^a$ avec $a > 1$ et, pour les groupes ayant des « bons » plongements vers un espace ℓ^p une borne supérieure de la forme $n/\log(n)^b$ avec b compris strictement entre 0 et 1. Nous prouvons que le profil de séparation d’un groupe résoluble à croissance exponentielle n’est jamais dominé par une fonction puissance sous-linéaire. Nous introduisons également une notion de séparation locale, avec des applications aux composantes de percolation de \mathbb{Z}^d et aux graphes de dimension isopérimétrique supérieure à 1 et à croissance polynomiale.

Keywords: Cheeger constants, Coarse geometry, Geometric group theory.

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1. Introduction

The separation profile was first defined by Benjamini, Schramm & Timár in [7]. The separation function at n is the supremum over all subgraphs of size n , of the number of vertices needed to be removed from the subgraph, in order to cut into connected pieces of size at most $n/2$. The introduction of this function was motivated by the study of regular maps between metric spaces, because the separation profile is monotonous under regular maps (up to constant factors). A map between two graphs of bounded degree is said to be regular if it is Lipschitz, and if the cardinality of the preimages of singletons is uniformly bounded (see Definition 6.2); for example, quasi-isometric and coarse embeddings between connected graphs are regular maps.

Hume’s work about linear separation profiles (see [22]) led to an equivalent definition of the separation profile using the Cheeger constant h . The definition of the separation profile that we use is the following:

$$\text{Sep}(n) = \sup_{F \subset V\Gamma, |F| \leq n} |F|h(F),$$

where $h(F)$ denotes the Cheeger constant of F (see Section 2 for details).

This was studied by Hume in [22] with the aim of finding large classes of expanders. His work was continued by Hume, Mackay & Tessera [24] who introduced L^p -variants of these profiles and recently Hume & Mackay [23] studied the case of groups with low separation profiles ($\prec \log(n)$). On the opposite side, the separation profiles of expanders is linear (along a subsequence).

The subject matter of the current paper is to estimate the separation profile for various types of graphs, using other known information such as growth, isoperimetry and compression. Our focus is mostly on Cayley graphs, because this is where most information is available on other properties of the graph (such as growth, isoperimetry and compression). However our methods do not really require the high level of regularity that Cayley graphs possess.

A first group of results regards graphs G which are “polynomial” in some sense. Here Λ_G denotes the isoperimetric profile of G , defined by $\Lambda_G(n) = \inf \left\{ \frac{|\partial A|}{|A|} : A \subset G, |A| \leq n \right\}$. Isoperimetry gives rise to a notion of dimension: if for some $K > 0$, $\Lambda_G(n) \geq Kn^{-1/d}$, one says the graph has d -dimensional isoperimetry. The growth will be measured by $b_n = \sup_x |B_n(x)|$ where $B_n(x)$ is the ball of radius n centred at x . Roughly

speaking, a (d_1, d_2) -polynomial graph is a graph of growth bounded by n^{d_2} and of isoperimetric dimension at least d_1 , see Definition 6.9 for details.

THEOREM 1.1.

- Let G be a graph such that $\frac{K_1}{n^{1/d}} \leq \Lambda_G(n) \leq \frac{K_2}{n^{1/d}}$ for some constants K_1 and K_2 , then, $\exists K_3 > 0$ such that for all n , $\frac{\text{Sep}(n)}{n} \geq \frac{K_3}{n^{1/d}}$.
- Let G be a graph such that $b_n \leq K_1 n^d$ for some constant K_1 , then, $\exists K_2 > 0$ such that for all n , $\frac{\text{Sep}(n)}{n} \leq \frac{K_2 \log n}{n^{1/d}}$.
- Let G be a $(1+\epsilon, d)$ -polynomial graph. Then, for any $\delta > d$ there are constants K_1 and K_2 such that, for all n , $K_1 n^{\epsilon(1+\epsilon)/\delta^2} \leq \text{Sep}(n) \leq K_2 n^{(d-1)/d} \log n$.

See Proposition 4.1, Proposition 4.3 and Corollary 5.4 for details.

This theorem can be used on Cayley graphs of nilpotent groups (for which a sharper upper bound was already given by Hume, Mackay & Tessera [24]), but our method applies also to other type of graphs, such as pre-fractal Sierpinski carpets (on this subject, see Gibson & Pivarski [16] for isoperimetry and Gladkova & Shum [17] for a study of the relationship between conformal dimension and separation profile in graphs of fractals).

Our methods also yield results on the infinite percolation components of \mathbb{Z}^d , and more generally on polynomial graphs. Since the percolation component always includes arbitrary large balls, it is more interesting to introduce a local variant of the separation profile in this context, namely the *local separation at v* , where v is a vertex of the graph (see Section 6):

$$\text{Sep}_G^v(n) := \sup_{F < B_G(v,r); |B_G(v,r)| \leq n} |F| \cdot h(F).$$

Local separation profiles are also monotone under regular maps, see Proposition 6.3. In that case, we show that $\frac{\text{Sep}_G^v(n)}{n}$ is bounded below by a function of the type $\frac{1}{n^\alpha}$, for every vertices in the polynomial case, see theorem below.

THEOREM 1.2. — Let G be a (d_1, d_2) -polynomial graph. Then for any $\eta \in (0, 1)$ there exists $c > 0$ such that for any vertex v and any integer n :

$$\text{Sep}_G^v(n) \geq cn^{(1-\eta) \frac{d_1^2(d_1-1)}{d_2^3}}.$$

(see Corollaries 6.11 and 6.21 for details), and for vertices that stay exponentially close to the origin in the \mathbb{Z}^d percolation case:

THEOREM 1.3. — Let \mathcal{C}_∞ be a supercritical phase percolation cluster of \mathbb{Z}^d . Then for any $\epsilon \in (0, 1)$ There exists almost surely $c > 0$ such that

for n large enough, if $\|x\|_\infty \leq \exp\left(n^{(1-\varepsilon)\frac{d}{d-1}}\right)$, then we have:

$$\text{Sep}_{\mathcal{C}_\infty}^x(n) \geq cn^{\frac{d-1}{d}}.$$

(see Corollary 6.13).

In this case, the inclusion in \mathbb{Z}^d shows that this lower bound is optimal. The same methods as those of Theorem 1.1 can also be applied to groups of intermediate growth:

THEOREM 1.4. — *Let G be a Cayley graph of a groups of intermediate growth with $K_1 e^{n^a} \leq b_n \leq K_2 e^{n^b}$ where $K_1, K_2 > 0$ and $a, b \in]0, 1[$. Then*

- $\exists K_3 > 0$ such that there are infinitely many n with $\frac{\text{Sep}(n)}{n} \geq \frac{K_3}{(\log n)^{1+\frac{1}{a}}}$.
- $\exists K_4 > 0$ such that for any n , $\frac{\text{Sep}(n)}{n} \leq \frac{K_4}{(\log n)^{\frac{1}{b}-1}}$.

See Proposition 4.7 and Corollary 5.3 for details. The upper bound is obtained here using the growth assumption. Relations between growth and separation is studied in subsection 5.1.

Inside the realms of graphs having a logarithmic isoperimetric profile, the lower bounds obtained are listed in the upcoming theorem. (The list is not exhaustive, one could also get a lower bound for any group whose isoperimetric profile is known, e.g. $\mathbb{Z} \wr \mathbb{Z}$.) Note that it is reasonable to compare $\frac{\text{Sep}(n)}{n}$ with $\Lambda_G(n)$ for two reasons. First, for nilpotent groups those two functions coincide up to some multiplicative constants (see Hume, Mackay & Tessera [24]). Second, the underlying mechanism which allow us to provide such bounds relies on the known estimates of $\Lambda_G(n)$.

THEOREM 1.5. — *Let G be a Cayley graph.*

<i>If, for some $a > 0$, $\Lambda_G(N)$ is ...</i>	<i>then, for infinitely many N's, $\frac{\text{Sep}(N)}{N}$ is ...</i>
$\asymp \frac{1}{\log(N)^a}$	$\asymp \frac{\Lambda_G(N)}{\log(N)}$
$\asymp \frac{1}{\log^a(\log(N))}$	$\asymp \frac{\Lambda_G(N)}{\log(N)^C}$ (for some C)
$\asymp \frac{1}{(\log \dots \log \log N)^a}$	$\asymp \frac{\Lambda_G(N)}{N^\varepsilon}$, where ε can be arbitrarily small

These estimates on the isoperimetric profile are known for polycyclic groups which are not nilpotent (first row of the table with $a = 1$), wreath products $F \wr N$ where F is finite and N is a nilpotent group whose growth is

polynomial of degree d (first row with $a = 1/d$), iterated wreath products $F \wr (F \wr N)$ where F is finite and N is a nilpotent group whose growth is polynomial of degree d (second row with $a = 1/d$), solvable groups in general (third row, see [37] and [38]). See Propositions 4.7 and 4.13 for details.

Using the last row of the table, we prove the following theorem:

THEOREM 1.6. — *Let G be a finitely generated solvable group. If there exists $\epsilon \in (0, 1)$ and $c > 0$ such that for any large enough integer n we have*

$$\text{Sep}_G(n) \leq cn^{1-\epsilon},$$

then G is virtually nilpotent.

See Theorem 4.18 for details. It is a very natural question to wonder whether $\text{Sep}(n) \preceq n^{1-\epsilon}$ for G amenable implies that G is virtually nilpotent, see Question 7.7. However, our proofs techniques break down completely in the general case, see Section 4.5. Here is a direct application of Theorem 4.18 and Theorem 1.4: (see Definition 6.2 for the definition of a regular map)

COROLLARY 1.7. — *Let G be a finitely generated group which is either solvable or of subexponential growth. If there exists a regular map from G to \mathbb{H}^d (the d -dimensional hyperbolic space), then G is virtually nilpotent.*

The case of G solvable was already addressed by Hume & Sisto [25, Corollary 1.3] in the case of coarse embeddings, with a completely different proof.

Remark 1.8. — In the context of the current paper, limited to graphs, it might not be clear what a regular map to \mathbb{H}^d is. Either replace \mathbb{H}^d by any uniform lattice or see Hume, Mackay & Tessera [24] for the generalisation of separation profiles and regular maps to Riemannian manifolds with bounded geometry.

Though our lower bounds apply to a vast array of groups, we show, using constructions of Erschler [14] and Brioussell & Zheng [9], that there are groups for which those methods fail to give a significant bound (see Section 4.5).

Our upper bounds on the separation profile come either from the growth of balls (for graphs which do not have exponential growth, see Theorems 5.3 and 5.4) or from the compression exponents:

THEOREM 1.9. — *Let G be a bounded degree graph which has compression exponent (in some L^p -space) equal to $\alpha > 0$. Then, for any $c < \frac{\alpha}{2-\alpha}$, there is a constant K so that, for any N , $\frac{\text{Sep}(N)}{N} \leq \frac{K}{(\log N)^c}$.*

See Corollary 5.6 for details.

In addition to the afore-mentioned examples, the previous theorem applies to products of hyperbolic groups (these have $\alpha = 1$ [39, Corollary 2]). This shows that the separation profile of a product of hyperbolic groups satisfies $\frac{\text{Sep}(n)}{n} \leq \frac{K_1}{(\log n)^{1-\varepsilon}}$ (for any $\varepsilon \in (0, 1)$). This is quite sharp, since such a group always have $\frac{\text{Sep}(n)}{n} \simeq \frac{1}{\log n}$, as soon as at least two of these hyperbolic groups are non-elementary. Indeed, in this case it contains up to quasi-isometry the product $T \times T$, where T is the infinite binary tree (e.g. from the Tits alternative for hyperbolic groups). This implies that its separation profile satisfies $\frac{\text{Sep}(n)}{n} \geq \frac{1}{\log n}$ [7, Lemma 1.3, Theorem 3.5]. On the other hand, any product of hyperbolic groups coarsely embeds in a product of the form $T \times \cdots \times T$ [10], which implies that its separation profile satisfies $\frac{\text{Sep}(n)}{n} \preceq \frac{1}{\log n}$.

It might be disappointing to see that the lower bounds of Theorem 1.5 are sometimes much smaller than the original isoperimetric profile (e.g. a power of $\log N$ is much larger than $\log \log N$). However, our upper bounds show that such a dramatic loss cannot be avoided (see Theorem 1.9, Corollary 5.6 and Remark 4.14).

In fact, for polycyclic groups, free metabelian groups, lamplighters on \mathbb{Z} with finite lamps, lamplighters on \mathbb{Z}^2 with finite lamps and some iterated wreath products such as $F \wr (F \wr \mathbb{Z})$, Theorem 1.5 gives that $\frac{\text{Sep}(n)}{n}$ is infinitely often $\geq K_1(\log n)^{-c_1}$ while it is (for all n) $\leq K_2(\log n)^{-c_2}$, from Theorem 1.9.

More precisely, $c_2 < 1$ can be arbitrarily close to 1 (but there is no control on the constant K_2 as far as we know) Furthermore, for polycyclic groups and lamplighters on \mathbb{Z} with finite lamps $c_1 = 2$. For lamplighters on \mathbb{Z}^2 with finite lamps $c_1 = \frac{3}{2}$. For free metabelian groups $c_1 > 1 + \frac{1}{r}$ is arbitrarily close to $1 + \frac{1}{r}$, where r is the rank of the group.

The case of $F \wr (F \wr \mathbb{Z})$ is of particular interest, since it shows that the appearance of the logarithmic factor in the lower bound is necessary (see Remark 5.7). This also shows that there are amenable groups for which $\frac{\text{Sep}(n)}{n}$ decays much faster than $\Lambda_G(n)$.

Organisation of the paper

Section 2 contains the basic definitions. In Section 3.1, we make the first step towards a lower bound by looking at sets which have a small boundary to content ratio. These optimal set turn out to have a high separation. This

estimate is then used in Section 3.2 to get lower bounds on the separation profile from the isoperimetric profile. Section 4 is devoted to concrete estimates in various Cayley graphs and self-similar graphs (Section 4.1.1). The proof of Theorem 1.6 (as well as some further lower bounds) is done in Section 3.3. Groups where the methods do not yield a lower bound are constructed in Section 4.5. Upper bounds on the separation profile are done in Section 5: via growth in Section 5.1 and via compression in Section 5.2. Local separation profiles are studied in Section 6, with applications to infinite percolation component in \mathbb{Z}^d (Section 6.2) and to polynomial graphs (Section 6.2 and 6.4). Questions are presented in Section 7.

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2. Definitions

The set of vertices of a graph G will be denoted VG , while the set of edges will be written EG . The set of edges consists of ordered pairs of vertices: $EG \subset VG \times VG$. Since the separation profile is monotone under quasi-isometric embeddings only for graphs of bounded degree, we will always work with this hypothesis.

DEFINITION 2.1. — *Let G be a graph. For any subset $A \subset VG$, its boundary is the set $\partial A = \{(a, b) \in EG, a \in A \Leftrightarrow b \in A^c\}$.*

DEFINITION 2.2. — *The isoperimetric profile of a graph G is the function $\Lambda_G : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ defined by*

$$\Lambda_G(n) = \inf_{A \subset VG, |A| \leq n} \frac{|\partial A|}{|A|}.$$

Note that, for an infinite connected graph, the isoperimetric profile never takes the value 0.

DEFINITION 2.3. — For any finite graph G , the Cheeger constant $h(G)$ of G is

$$h(G) = \min_{A \subset G, |A| \leq \frac{|G|}{2}} \frac{|\partial A|}{|A|}.$$

Let Γ be an infinite graph. For any finite subset $F \subset V\Gamma$, let \tilde{F} be the subgraph of Γ induced by F , defined more precisely by:

- $V\tilde{F} = F$
- $E\tilde{F} = \{(a, b) \in V\Gamma \mid a, b \in F\}$

By a small abuse of notation, the Cheeger constant of $F \subset V\Gamma$ is $h(F) = h(\tilde{F})$.

DEFINITION 2.4. — Let Γ be an infinite graph of bounded degree. The separation profile of Γ is the function $\text{Sep} : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ defined by

$$\text{Sep}(n) = \sup_{F \subset V\Gamma, |F| \leq n} |F| \cdot h(F).$$

As remarked by Hume, Mackay & Tessera in [24], it comes naturally from [22, Proposition 2.2. and Proposition 2.4.] that this definition is equivalent to the original one from Benjamini, Schramm & Timár [7].

One may notice that we use here the edge-boundaries, unlike Hume who uses vertex-boundaries. However, under the assumption that the graph has a bounded degree those two differ only by a constant factor.

Remark 2.5. — In any graph with at least one edge there is a trivial lower bound on $\text{Sep}(n)$: $\text{Sep}(n) \geq 1$ or $\frac{\text{Sep}(n)}{n} \geq \frac{1}{n}$. Recall that Benjamini, Schramm & Timár [7, Theorem 2.1] showed that a graph with bounded separation admits a regular map into a tree.

Lastly, the following convention will be used to compare functions: $f(n) \preceq g(n)$ if there is a constant $K > 0$ so that $f(n) \leq Kg(n)$. $f(n) \simeq g(n)$ if $f(n) \preceq g(n)$ and $g(n) \preceq f(n)$ (most of the time not with the same constant).

3. Lower bound on the separation from isoperimetry

3.1. Optimal sets and their Cheeger constant

This section is devoted to the following question: given an infinite graph that has a rather large isoperimetric profile, does the same holds for the Cheeger constants of its finite subgraphs?

There are two simple negative answers. First, in any infinite connected graph, there is an infinite subset L so that the graph induced to L is an infinite half-line. However, a finite subgraph of the half-line has the weakest Cheeger constant. Clearly, one needs to restrict a bit more the sets considered. It turns out the right thing to do is to restrict only to sets F which are “optimal” for the isoperimetric problem.

Second, consider the infinite regular tree; a graph with strong isoperimetric profile (i.e. the isoperimetric profile is bounded from below by a constant). Any graph induced by a finite subset has (again) the weakest possible Cheeger constant.

The aim of this section is to show that in graphs without strong isoperimetric profiles, the Cheeger constant on the optimal induced subsets is still fairly strong.

DEFINITION 3.1 (Optimal sets and integers). — *Let Γ be an infinite graph*

- *A subset F of $V\Gamma$ is called optimal if $\frac{|\partial F|}{|F|} = \Lambda_\Gamma(|F|)$, i.e.:*

$$\forall G \subset V\Gamma, \quad |G| \leq |F| \Rightarrow \frac{|\partial G|}{|G|} \geq \frac{|\partial F|}{|F|}.$$

- *An integer n will be called optimal if there exists an optimal subset of cardinality n .*

LEMMA 3.2. — *Assume F is optimal. Then:*

$$2h(F) \geq \Lambda_\Gamma\left(\frac{|F|}{2}\right) - \Lambda_\Gamma(|F|).$$

Proof. — The Cheeger constant for the [finite] graph induced on F is given by looking at subsets F_1 of VF of size at most equal to $\frac{|F|}{2}$, and trying to minimise $\frac{|\partial_F F_1|}{|F_1|}$, where ∂_F denotes the boundary in F .

Let F_1 be a subset of VF of size at most equal to $\frac{|F|}{2}$, let $F_2 = F \setminus F_1$. For any (disjoint) subsets A and B of $V\Gamma$, we denote by $E(A, B)$ the set of edges of Γ that have one endpoint in A , and the other in B . We have:

$$\begin{aligned} \partial F_1 &= E(F_1, V\Gamma \setminus F_1) \\ &= E(F_1, V\Gamma \setminus F) \sqcup E(F_1, F_2) \\ &= E(F_1, V\Gamma \setminus F) \sqcup \partial_F F_1. \end{aligned}$$

Similarly,

$$\begin{aligned} \partial F_2 &= E(F_2, V\Gamma \setminus F) \sqcup \partial_F F_2 \\ &= E(F_2, V\Gamma \setminus F) \sqcup \partial_F F_1. \end{aligned}$$

Then we have

$$2|\partial_F F_1| = |\partial F_1| + |\partial F_2| - |\partial F|.$$

Moreover, we have $\Lambda_\Gamma(|F_1|) \geq \Lambda_\Gamma(|F|/2)$, and as F is optimal, $\frac{|\partial F_2|}{|F_2|} \geq \frac{|\partial F|}{|F|}$ and $\Lambda_\Gamma(|F|) = \frac{|\partial F|}{|F|}$.

Using these facts, we can deduce that

$$\begin{aligned} 2|\partial_F F_1| &\geq |F_1| \Lambda_\Gamma(|F_1|) + |F_2| \cdot \frac{|\partial F|}{|F|} - |F| \cdot \frac{|\partial F|}{|F|} \\ &= |F_1| \Lambda_\Gamma(|F_1|) - |F_1| \cdot \frac{|\partial F|}{|F|} \\ &\geq |F_1| (\Lambda_\Gamma(|F|/2) - \Lambda_\Gamma(|F|)). \end{aligned}$$

Since this is true for any $F_1 \subset VF$ of size at most $|F|/2$, this concludes the proof. □

One already sees that these methods give basically nothing in graphs with a strong isoperimetric profile, since Λ_Γ is nearly constant. This is however to be expected since there can be no general reasonable bound in this family of graphs (the typical example would be infinite k -regular trees).

COROLLARY 3.3. — *If $n > 0$ is optimal, then:*

$$2 \frac{\text{Sep}(n)}{n} \geq \Lambda_\Gamma(n/2) - \Lambda_\Gamma(n).$$

Proof. — Assume F is optimal of cardinality n . Then $2 \frac{\text{Sep}(n)}{n} \geq 2h(F) \geq \Lambda_\Gamma(n/2) - \Lambda_\Gamma(n)$ □

3.2. A lower bound on the separation profile from isoperimetry

The following theorem is a consequence of the previous lemma and it applies to a large class of graphs, providing that the isoperimetric profile tends to 0 without reaching it (equivalently, the graph is amenable and has no finite connected components). It can be simplified in the case of graphs with “many symmetries”, see Theorem 3.8.

THEOREM 3.4. — *Let G be an infinite connected amenable graph of bounded degree. Assume there is an increasing function $p : \mathbb{N} \rightarrow \mathbb{N}$ so that for any n there is a $k \in (n, p(n)]$ such that k is optimal. Choose $\varepsilon \in (0, 1)$. Let $n \geq 1$ be an integer. Let $m \geq 1$ be such that $\Lambda_G(m) \leq (1 - \varepsilon)\Lambda_G(n)$.*

Then there exists an $N \in [n, p(m)]$ such that

$$\frac{\text{Sep}(N)}{N} \geq \varepsilon \frac{\Lambda_G(n)}{4 \log\left(\frac{p(m)}{n}\right) + 4}.$$

For example, it applies to graphs where the isoperimetric profile is bounded above and below: $\frac{C_1}{n^a} \leq \Lambda_G(n) \leq \frac{C_2}{n^a}$. Given an optimal integer n_o the next optimal integer n_p happens at the latest when $\frac{C_1}{n_o^a} \geq \frac{C_2}{n_p^a}$, or in other words $n_p \leq (\frac{C_2}{C_1})^{1/a} n_o$ (and the function p is linear). We study this specific case in Theorem 4.1

Proof. — We use $\Lambda = \Lambda_G$ throughout this proof. Let $n_0 = \min\{k \geq n \mid k \text{ is optimal}\}$. We define recursively $i \in \mathbb{N}$, $n_{i+1} = \max\{k \in (n_i, 2n_i] \mid k \text{ is optimal}\}$, if this set is non-empty, and, otherwise, $n_{i+1} = \min\{k \mid k \in [2n_i, p(n)] \text{ and } k \text{ optimal}\}$.

Notice that $\forall i, n_{i+2} \geq 2n_i$. Let i_{\max} be the first index i for which $n_i \geq m$. Then

$$n_{i_{\max}-1} \leq m \leq n_{i_{\max}} \leq p(n_{i_{\max}-1}) \leq p(m).$$

Using Lemma 3.2,

$$\forall i \in [0, i_{\max}], \quad 2 \frac{\text{Sep}(n_i)}{n_i} \geq \Lambda(n_i/2) - \Lambda(n_i).$$

Summing up all these inequalities, one gets

$$2 \sum_{i=0}^{i_{\max}} \frac{\text{Sep}(n_i)}{n_i} \geq \Lambda(n_0/2) - \Lambda(n_{i_{\max}}) + \sum_{i=0}^{i_{\max}-1} \Lambda(n_{i+1}/2) - \Lambda(n_i).$$

Either $\frac{1}{2}n_{i+1} \leq n_i$ (so that $\Lambda(n_{i+1}/2) \geq \Lambda(n_i)$ since Λ is decreasing) or $\Lambda(n_{i+1}/2) \geq \Lambda(n_i)$ because n_{i+1} is the next optimal integer after n_i . Either way, the sum on the right-hand side is positive, and consequently

$$\begin{aligned} 2 \sum_{i=0}^{i_{\max}} \frac{\text{Sep}(n_i)}{n_i} &\geq \Lambda(n_0/2) - \Lambda(n_{i_{\max}}) \\ &\geq \Lambda(n) - \Lambda(m) \\ &\geq \varepsilon \Lambda(n). \end{aligned}$$

To justify the second inequality, note that $\Lambda(m) \geq \Lambda(n_{i_{\max}})$ by monotonicity of Λ (and $m \leq n_{i_{\max}}$) To see that $\Lambda(n_0/2) \geq \Lambda(n)$ consider two cases:

- if $n_0 \leq 2n$, then (since Λ is non-increasing) $\Lambda(n_0/2) \geq \Lambda(n)$
- otherwise $n_0 \geq 2n$, therefore $\lfloor n_0/2 \rfloor$ is not optimal, so $\Lambda(n_0/2) = \Lambda(n)$.

From there, we can deduce that

$$\exists j \in [0, i_{\max}], \quad \frac{\text{Sep}(n_j)}{n_j} \geq \frac{\varepsilon}{2} \frac{\Lambda(n)}{i_{\max} + 1}.$$

But recall that $\forall i, n_{i+2} \geq 2n_i$. Consequently, $n_{i_{\max}} \geq 2^{\lfloor \frac{i_{\max}}{2} \rfloor} n_0$. Furthermore, $n_{i_{\max}} \leq p(m)$. This implies: $2^{\lfloor \frac{i_{\max}}{2} \rfloor} n_0 \leq p(m)$. Since $n_0 = n$,

$2^{\lfloor \frac{i_{\max}}{2} \rfloor} \leq \frac{p(m)}{n}$ Thus,

$$i_{\max} + 1 \leq 2^{\lfloor \frac{i_{\max}}{2} \rfloor} + 2 \leq 2 \log_2 \left(\frac{p(m)}{n} \right) + 2.$$

The claim follows if we choose $N := n_j$. □

DEFINITION 3.5. — We shall say that a graph G has partial self-isomorphisms, if, for every finite set $F \subset VG$, there exists another finite F' such that $F \cap F' = \emptyset$ and the graph induced on $F \cup \partial F$ is isomorphic (as a finite graph) to $F' \cup \partial F'$.

Note that having partial self-isomorphisms implies the graph is infinite. This property is satisfied by fairly natural classes of graphs such as Cayley graphs, graphs with vertex-transitive (or edge-transitive) automorphisms and self-similar graphs.

LEMMA 3.6. — Let G be a graph which has partial self-isomorphisms. Assume n is an optimal integer. The set $\{k \in (n, 2n] \mid k \text{ is optimal}\}$ is not empty.

Proof. — If there is an optimal subset F whose size is $(n, 2n]$, there is nothing to prove. Otherwise, let us construct such an optimal set of size $2n$.

Let F be an optimal set of size n . Using the property of partial self-isomorphisms, there is a set F' with $|F| = |F'|$, $|\partial F| = |\partial F'|$, and $F \cap F' = \emptyset$. Hence $|F \cup F'| = 2n$ and $\frac{|\partial(F \cup F')|}{|F \cup F'|} \leq \frac{|\partial F|}{|F|}$. Since we assumed there are no optimal sets whose size is in $(n, 2n]$, then for any $G \subset \Gamma$ such that $|G| \leq 2n$, we have $\frac{|\partial G|}{|G|} \geq \frac{|\partial F|}{|F|} \geq \frac{|\partial(F \cup F')|}{|F \cup F'|}$. Therefore $F \cup F'$ is optimal. □

Since there is always an optimal integer ($n = 1!$), any graph with partial self-isomorphisms always has infinitely many optimal n .

Remark 3.7. — Even without the assumption that the graph G has partial self-isomorphisms, it is still possible to get some information on optimal integers, using the bounds on the isoperimetric profile. This is what is done in Theorem 3.10.

This strategy is very well adapted to polynomial graphs, so we applied it also in the proof of Theorem 6.10, taking a function $p(n)$ satisfying the more restrictive condition $\Lambda_G(p(n)) \leq \Lambda_G(n)/2$. It simplifies the proof without any loss.

THEOREM 3.8. — Assume Γ is a connected amenable graph of bounded degree with partial self-isomorphisms. Let $n \geq 1$ and $\varepsilon \in (0, 1)$. Let $m \geq 1$ be such that $\Lambda_\Gamma(m) \leq (1 - \varepsilon)\Lambda_\Gamma(n)$.

Then there exists an $N \in [n, 2m]$ such that

$$\frac{\text{Sep}(N)}{N} \geq \varepsilon \frac{\Lambda_\Gamma(n)}{4 \log(m/n) + 8}.$$

Proof. — This result comes naturally from Theorem 3.4 and Lemma 3.6. □

3.3. A qualitative approach

In this subsection, we give an application of Theorem 3.4 using a single upper bound on the isoperimetric profile. In applications, this gives an improvement on the lower bounds obtained on the separation profile, but this comes to the cost of a weaker control on the frequency of the integers for which the bound holds. Inspired by the formulation of Theorem 3.4, we quantify the decreasing of real functions in the following way:

DEFINITION 3.9. — Let $f: \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ be a continuous non-increasing function such that f tends to 0 at infinity. For any $\delta \in (0, 1)$, we define the δ -geometric decay function of f as:

$$p_f^\delta(x) := f^{-1}(\delta f(x)) = \min \{x' \mid f(x') \leq \delta f(x)\}.$$

We define the δ -geometric decay function of a function from \mathbb{N}^* to $\mathbb{R}_{>0}$ as the δ -geometric decay function of either a natural extension, either a piecewise affine extension.

We can state the following theorem:

THEOREM 3.10. — Let G be an infinite connected amenable graph of bounded degree. Let g be a continuous non-increasing positive function such that

- $\lim_{n \rightarrow \infty} g(n) = 0$,
- for any large enough n , $\Lambda_G(n) \leq g(n)$.

Then, for infinitely many integers n there exists $N \in [n, p_g^{1/8}(n)]$ such that

$$\frac{\text{Sep}_G(N)}{N} \geq \frac{1}{8} \frac{\Lambda_G(n)}{\log\left(\frac{p_g^{1/8}(n)}{n}\right) + 1}.$$

Proof of Theorem 3.10. — This follows from the two lemmas below. □

LEMMA 3.11. — *Let G be an infinite connected amenable graph of bounded degree. Then for any $n \geq 1$ there exists an integer $N \in [n, p_{\Lambda_G}^{1/4}(n)]$ such that*

$$\frac{\text{Sep}_G(N)}{N} \geq \frac{1}{8} \frac{\Lambda_G(n)}{\log\left(\frac{p_{\Lambda_G}^{1/4}(n)}{n}\right) + 1}.$$

Proof. — This is straightforward using Theorem 3.4, taking $\varepsilon = \frac{1}{2}$, $m = \lfloor p_{\Lambda_G}^{1/2}(n) \rfloor \in [n, p_{\Lambda_G}^{1/4}(n)]$ and $p(m) = \lfloor p_{\Lambda_G}^{1/4}(n) \rfloor$. Note that in the degenerated cases where $m = p(m)$ the integer $p(m)$ is optimal since $\frac{\Lambda_G(p(m))}{\Lambda_G(p(m)-1)} \leq 1/2$. \square

LEMMA 3.12. — *Let $f, g: \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ two continuous non-increasing functions such that $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} g(x) = 0$.*

We assume that for any $x > 0$ we have $f(x) \leq g(x)$. Then there exists infinitely many positive integers $(n_i)_{i \geq 0}$ such that for any i :

$$p_f^{1/4}(n_i) \leq p_g^{1/8}(n_i).$$

Proof. — Suppose for a contradiction that there exists an integer N such that we have

$$(3.1) \quad p_f^{1/4}(n) > p_g^{1/8}(n) \quad \text{for any } n \geq N.$$

We claim that we have, for any $k \geq 0$,

$$(3.2) \quad p_f^{1/4 \circ k}(N) \geq p_g^{1/8 \circ k}(N).$$

We prove this by induction. If $k = 0$, then this is obvious. Now, if (3.2) is satisfied for some $k \geq 0$, we have:

$$\begin{aligned} p_f^{1/4 \circ (k+1)}(N) &= p_f^{1/4}\left(p_f^{1/4 \circ k}(N)\right) \\ &\geq p_f^{1/4}\left(p_g^{1/8 \circ k}(N)\right) && \text{by (3.2)} \\ &\geq p_g^{1/8}\left(p_g^{1/8 \circ k}(N)\right) && \text{by (3.1)} \\ &= p_g^{1/8 \circ (k+1)}(N). \end{aligned}$$

This proves (3.2) for every $k \geq 0$.

Let m be such that $2^m > \frac{g(N)}{f(N)}$. We compose m times the $1/4$ -geometric decay function of f :

$$\begin{aligned} f\left(p_f^{1/4 \circ m}(N)\right) &= \frac{1}{4^m} f(N) \\ &> \frac{1}{8^m} g(N) \quad \text{by definition of } m, \\ &= g\left(p_g^{1/8 \circ m}(N)\right) \\ &\geq g\left(p_f^{1/4 \circ m}(N)\right) \quad \text{by (3.2) and } g \text{ non-increasing,} \end{aligned}$$

which contradicts the initial assumption that $f \leq g$. □

4. Applications

In this section, we give applications of Theorems 3.4, 3.8, and 3.10. We use isoperimetric profiles that have already been computed in the literature. In the course of this investigation, there are three factors that come into play:

- The geometry / the symmetries of the graph: the function $p(n)$ of Theorem 3.4.
- The decay of the isoperimetric profile.
- Inaccurate knowledge of the isoperimetric profile: when we only have loose bounds on the isoperimetric profile.

Our goal on this section is not to give an exhaustive overview of possible applications, but only to apply Theorem 3.4 in situations that seemed interesting to us. In Section 4.4, we give an application of Theorem 3.10 to solvable groups. In Section 4.5 we look at limit cases, where those theorems give no information.

Bendikov, Pittet & Sauer [4, Table 1 on p. 52] contains many reference for the isoperimetric profile of groups. As noted in Erschler [13, §1] the isoperimetric profile Λ_G is connected to the Følner function F by the relation:

$$\Lambda_G(N) \simeq \frac{1}{F^{-1}(N)}.$$

4.1. Isoperimetric profile decaying as a power of N

Recall that virtually nilpotent groups (equivalently groups for which the cardinality of a ball of radius r is bounded by polynomials in r) are the

only groups where $\Lambda_G(n)$ is of the form $\frac{1}{n^{1/d}}$ (where d is the degree of the polynomial); see Pittet & Saloff-Coste [35, Theorem 7.1.5] or [34, Theorem 3.4].

A polynomial upper bound on the isoperimetric profile is given by Benjamini & Papasoglu [6, Theorem 2.1] for doubling planar graphs.

PROPOSITION 4.1. — *Let G be a graph of bounded degree such that if n is large enough, the following inequality holds:*

$$(4.1) \quad \frac{C_1}{n^\beta} \leq \Lambda_G(n) < \frac{C_2}{n^\beta},$$

for some constants $C_1, C_2, \beta > 0$.

Then if n is large enough:

$$\frac{\text{Sep}(n)}{n} \geq A \cdot \Lambda_G(n),$$

for some constant A .

One may notice that this proposition applies for very general graphs (although the hypotheses on G imply that it is amenable and has no finite connected component).

Proof. — We use $\Lambda = \Lambda_G$ throughout this proof. Let n_0 be an integer such that (4.1) holds for any $n \geq n_0$. For any such n ,

$$\Lambda \left(\left(\frac{C_2}{C_1} \right)^{1/\beta} \cdot n \right) \leq \frac{C_1}{n^\beta} \leq \Lambda(n).$$

Therefore with the notations of Theorem 3.4, we can take

$$p(n) = \left(\frac{C_2}{C_1} \right)^{1/\beta} n.$$

Let C be the smallest integer larger than $\left(\frac{C_2}{\frac{1}{2} \cdot C_1} \right)^{1/\beta}$. Let n be an integer such that $n \geq n_0$ and let m be the smallest integer such that $m \geq Cn$. Since $m^\beta \geq 2 \frac{C_2}{C_1} n^\beta$, it follows that $\Lambda(m) \leq \frac{C_2}{m^\beta} \leq \frac{\frac{1}{2} \cdot C_1}{n^\beta} \leq \frac{1}{2} \Lambda(n)$.

Then we can apply Theorem 3.4: there is a $N \in [n, 2m]$ such that

$$\frac{\text{Sep}(N)}{N} \geq \frac{\varepsilon}{4} \frac{\Lambda(n)}{\log \left(\frac{p(m)}{n} \right) + 1} = \frac{\varepsilon}{4} \frac{\Lambda(n)}{\log \left(\left(\frac{C_2}{C_1} \right)^{1/\beta} \cdot \frac{m}{n} \right) + 1}.$$

Since $m/n - 1 \leq C$, and $\log(m/n) \leq \log(m/n - 1) + 1 \leq \log(C) + 1$, finally we get:

$$\frac{\text{Sep}(N)}{N} \geq K\Lambda(n),$$

with $K = \varepsilon \left(4 \log \left(\left(\frac{C_2}{C_1} \right)^{1/\beta} \right) + \log(C) + 1 \right)^{-1}$.

The proof shows the lower bound for some (mysterious) infinite set of integers. To show it for any large enough integer: suppose that $k \geq 4C^2$, then (since k can be the upper bound of an interval with some integer N), we have:

$$\frac{\text{Sep}(k)}{k} \geq \frac{K}{2C + 1} \frac{1}{k^\beta}.$$

Indeed, assume $k \geq n_0$ and $k \geq 4C^2$. We know that there exists an integer $N \in [\lfloor \frac{k}{2C} \rfloor, k]$ such that $\frac{\text{Sep}(N)}{N} \geq K\Lambda(k)$. Then we have:

$$\frac{\text{Sep}(k)}{k} \geq \frac{\text{Sep}(N)}{(2C + 1)N} \geq \frac{K}{2C + 1} \cdot \frac{1}{N^\beta} \geq \frac{K}{2C + 1} \cdot \frac{1}{k^\beta} \geq \frac{K}{(2C + 1) \cdot C_2} \cdot \Lambda(k).$$

This concludes the proof. □

Note that in the case of virtually nilpotent groups, and more generally for vertex-transitive graphs with polynomial growth, Hume, Mackay & Tessera [24, Theorem 7] shows that the inequality of Proposition 4.1 is sharp.

Remark 4.2. — If we assume that the graph G has partial self-isomorphisms then we can take $p(n) = 2n$ according to Lemma 3.6. Therefore we may improve the constant K_1 of Proposition 4.1.

PROPOSITION 4.3. — *If one assumes that for a graph of bounded degree there exist some positive constants C_1, C_2, α , and β , with $1 > \alpha > \beta$, such that we have*

$$\frac{C_1}{n^\alpha} \leq \Lambda_G(n) \leq \frac{C_2}{n^\beta},$$

for any positive integer n , then there exists $A > 0$ such that for any $n > 0$ we have:

$$\text{Sep}(n) \geq A \cdot \frac{n^\gamma}{\log(n)},$$

with:

- $\gamma = \frac{\beta(1-\alpha)}{\alpha}$ if G has partial self-isomorphisms,
- $\gamma = \frac{\beta^2(1-\alpha)}{\alpha^2}$ otherwise.

Proof. — Without assuming partial self-isomorphisms, we can apply Theorem 3.4, with $p(n) \simeq n^{\frac{\alpha}{\beta}}$ and $m \simeq n^{\frac{\alpha}{\beta}}$. Then for any integer n we have an integer $N \in \left[n, Cn^{\left(\frac{\alpha^2}{\beta^2}\right)} \right]$ such that $\text{Sep}(N) \succeq \frac{N \cdot \Lambda_G(n)}{\log(n)}$. Now, let k be an positive integer. Let $n = \left(\frac{k}{C}\right)^{\frac{\beta^2}{\alpha^2}}$. Then there exists some $N \in [n, k]$ such that

$$\text{Sep}(N) \succeq \frac{N \cdot \Lambda_G(n)}{\log(n)} \succeq \frac{n}{n^\alpha \log(n)} = \frac{n^{1-\alpha}}{\log(n)} \simeq \frac{k^\gamma}{\log(k)},$$

with $\gamma = \frac{\beta^2(1-\alpha)}{\alpha^2}$. Since $\text{Sep}(k) \geq \text{Sep}(N)$, we get the announced lower bound.

If the graph has partial self-isomorphisms, then Lemma 3.6 shows that $p(n) = 2n$ is a valid choice and the rest of the proof is similar. □

4.1.1. Application to pre-fractal Sierpinski carpets

Gibson & Pivarski showed in [16] some results on isoperimetry in pre-fractal graphical Sierpinski carpets.

Pre-fractal Sierpinski carpets are built using an iterating process. We consider a squared fundamental domain F_1 which is a union of little squares, obtained by removing subsquares in an admissible way which is satisfied by standard Sierpinski carpets (see [16, Definition 2.1 and §2.2.2]). We can consider F_1 as a pattern. We make copies of F_1 in such a way that we reproduce this pattern at a larger scale. We get a bigger square that we can call F_2 . That is the first step of this process, and the pre-fractal Sierpinski carpet is the limit object that we get iterating the process indefinitely.

The carpet is then a subset of \mathbb{R}^2 , which is a union of little squares. The associated graph is obtained putting a vertex in the centre of each of these squares, and linking vertices with an edge if and only if their squares share a common face in the carpet.

We use the notations of [16]: F_1 is the fundamental domain of the pre-fractal, m_F is the number of sub-squares in F_1 and R is the number of columns of F_1 with one or more squares removed.

THEOREM 4.4 (Gibson & Pivarski [16]). — *Let X be an admissible two-dimensional pre-fractal graphical Sierpinski carpet. Then*

$$\Lambda_X(n) \asymp n^{\frac{\log(R)}{m_F} - 1}.$$

Proof. — The lower bound comes from [16, Theorem 4.4], together with [16, Corollary 3.2], and [16, Corollary 3.8] to convert the result of the Theorem for the so-called graphical isoperimetry.

The upper bound comes from the construction of explicit subsets [16, Lemma 4.1], together with [16, Lemma 3.3] (to get *graphical* subsets) and [16, Corollary 3.2]. \square

The construction of pre-fractal Sierpinski carpets can be generalised in higher dimensions. The following theorem holds for standard Sierpinski carpets of any dimension.

THEOREM 4.5 (see Gibson & Pivarski [16]). — *Let X be the d -dimensional pre-fractal graphical standard Sierpinski carpet. Then*

$$\Lambda_X(n) \asymp n^{-\frac{\log(3^d-1) - \log(3^{d-1}-1)}{\log(3^d-1)}}.$$

Proof. — The proof is very similar to the latest proof, using d -dimensional results: [16, Corollary 4.6] and [16, Corollary 4.2]. \square

In this context, Proposition 4.1 applies, so we can deduce the following corollary:

COROLLARY 4.6. — *Under the assumptions of Theorem 4.4 or of Theorem 4.5, there exists $n_0, K_1 > 0$ such that*

$$\forall n \geq n_0 \quad \frac{\text{Sep}_X(n)}{n} \geq K_1 \cdot \Lambda_X(n).$$

As pointed by the anonymous referee, this estimate is very close to a lower bound of Gladkova & Shum [17] (for the $S(d, 3, \{1\}, 1)$ graphical fractal in their terminology).

4.2. Isoperimetric profile with logarithmic decay

Before moving on to the next class of examples, let us recall that for polycyclic groups of exponential growth (as well as solvable groups with finite Prüfer rank and geometrically elementary solvable groups) the isoperimetric profile is known to be of the form $\frac{C_1}{\log(n)} \leq \Lambda_G(n) \leq \frac{C_2}{\log(n)}$; see Pittet & Saloff-Coste [35, Theorem 7.2.1], [34, Theorem 3.4] for polycyclic groups, and Bendikov, Pittet & Sauer [4, Table 1], Pittet & Saloff-Coste [33] and Tessera [40] for more general statements.

Also a group of intermediate growth (i.e. a group where the cardinality of balls are such that $e^{n^a} \asymp |B_n| \asymp e^{n^b}$) are known to have a bound $\frac{C_1}{(\log n)^{\frac{1}{a}}} \asymp \Lambda_G(n) \asymp \frac{C_2}{(\log n)^{\frac{1}{b}-1}}$. The lower bound comes from Coulhon & Saloff-Coste [12, Theorem 1], the upper bound can be deduced from Lemma 5.1, and using the monotonicity of the isoperimetric profile.

Lastly, wreath products $F \wr N$ where F is finite and N has polynomial growth of degree d have an isoperimetric profile of the form $\frac{C_1}{(\log n)^{\frac{1}{d}}} \preccurlyeq \Lambda_G(n) \preccurlyeq \frac{C_2}{(\log n)^{\frac{1}{d}}}$; see Pittet & Saloff-Coste [34, §4 and §7] or Erschler [13, Theorem 1].

PROPOSITION 4.7. — *Let $C_1, C_2, \alpha, \beta > 0$. Let G be an infinite connected amenable graph of bounded degree with partial self-isomorphisms such that $\frac{C_1}{\log^\alpha(n)} \leq \Lambda_G(n) \leq \frac{C_2}{\log^\beta(n)}$, for any large enough n .*

Then there exists a constant K_1 such that for infinitely many N 's, the following inequality holds:

$$\frac{\text{Sep}(N)}{N} \geq K_1 \frac{\Lambda_G(N)}{\log N}.$$

Remark 4.8. — For this class of examples, we chose to apply Theorem 3.10. We could also have used Theorem 3.8. This would give a less interesting bound (of the form $\frac{\Lambda_G(N)}{(\log N)^{\alpha/\beta}}$), but, on the other hand, it gives an estimation of the frequency of the integers N satisfying the inequality.

We will use the following fact.

FACT 4.9. — *Let $g(n) = \frac{c}{(\log n)^\beta}$. Then we have $p_g^{1/8}(n) = n^{(8^{1/\beta})}$, for any $n \geq 1$.*

Proof. — Let m, n be such that $m = n^{(8^{1/\beta})}$. Then, $g(m) = \frac{1}{8} \frac{c}{(\log n)^\beta} = \frac{1}{8}g(n)$, which is the required equality. □

Proof of Proposition 4.7. — Combining Theorem 3.10 with Fact 4.9, we obtain that there exists infinitely many integers $(n_i)_{i \geq 0}$ such that for any i there exists $N_i \in [n_i, p_g^{1/8}(n_i)]$ such that, if n_i is large enough,

$$\begin{aligned} \frac{\text{Sep}_G(N_i)}{N_i} &\geq \Lambda_G(n_i) \frac{1/8}{\log\left(\frac{p_g^{1/8}(n_i)}{n_i}\right)+1} \\ &\geq \Lambda_G(N_i) \frac{1/8}{\log(p_g^{1/8}(n_i))+1} \\ &= \Lambda_G(N_i) \frac{1/8}{8^{1/\beta} \log(n_i)+1} \\ &\geq \Lambda_G(N_i) \frac{1/16}{8^{1/\beta} \log(N_i)}. \end{aligned} \quad \square$$

We deduce the three following corollaries.

COROLLARY 4.10. — *Assume G is a polycyclic group of exponential growth (or, more generally, a solvable group with finite Prüfer rank or a*

geometrically elementary solvable group), then there exists a constant K so that for infinitely many N 's we have:

$$\frac{\text{Sep}(N)}{N} \geq K \frac{\Lambda_G(N)}{\log(N)} \simeq \frac{1}{\log(N)^2}.$$

COROLLARY 4.11. — Assume G is a wreath product $F \wr N$ where F is finite and N has polynomial growth of degree d . Then there exists a constant K so that for infinitely many N 's we have:

$$\frac{\text{Sep}(N)}{N} \geq K \frac{\Lambda_G(N)}{\log(N)} \simeq \frac{1}{\log(N)^{\frac{d+1}{d}}}.$$

COROLLARY 4.12. — Let G be a group of intermediate growth with $e^{n^a} \preccurlyeq |B_n| \preccurlyeq e^{n^b}$. Then there is a constant K so that for infinitely many N 's we have:

$$\frac{\text{Sep}(N)}{N} \geq K \frac{\Lambda_G(N)}{\log N} \asymp \frac{1}{(\log N)^{1+1/a}}.$$

4.3. Isoperimetric profile with iterated logarithmic decay

There are explicit groups where the isoperimetric profile decays with a power of iterated logarithms. Example of such groups are iterated wreath products $F \wr (F \wr N)$ where F is finite and N is a nilpotent group whose growth is polynomial of degree d . For such groups, one has $\Lambda_G(n) \simeq \frac{1}{(\log \log n)^{1/d}}$. Iterating further the wreath products (with finite groups) gives a profile with more iterated logarithms; see Erschler [13, Theorem 1] or Gromov [19, §8.1].

$$\text{Let } \underbrace{\log \log \cdots \log(x)}_{k \text{ times}} \text{ and } \underbrace{\exp \exp \cdots \exp(x)}_{j \text{ times}}.$$

PROPOSITION 4.13. — Let G be an infinite connected amenable graph with partial self-isomorphisms such that there exists an integer k and $C, \beta > 0$ such that $\Lambda_G(n) \leq \frac{C}{(\log^{(k)} n)^\beta}$, for any large enough n . Then there exists some positive constants K and C such that the following inequality holds for infinitely many N 's:

$$\frac{\text{Sep}(N)}{N} \geq K \frac{\Lambda_G(N)}{\exp^{(k-1)}(C \log^{(k)} N)}.$$

As in Section 4.2, we chose to apply Theorem 3.10. We could also have used Theorem 3.8, see Remark 4.8.

Remark 4.14. — When $k = 2$, the denominator of the right-hand-side is a power of $\log(n)$. In that case, the lower bound is significantly weaker than Λ_G , because a power of $\log(N)$ is significantly bigger than $\log \log(N)$. This tendency continues for isoperimetric profiles which are even closer to being constant. For example, if $\Lambda_G(n) \simeq 1/\log \log \log(n)$, then the bound on $\text{Sep}(N)/N$ is of the form $\Lambda_G(N)/(\log N)^{\log^\eta(\log N)}$, with $\eta = 1 - C$.

However, this phenomenon is not only an artefact of the proof. Indeed, the upper bounds obtained in Corollary 5.6 together with Theorem 1.1 of [28] shows that the separation profile of $\mathbb{Z}_2(\mathbb{Z}_2 \wr \mathbb{Z})$ cannot dominate $\frac{n}{(\log n)^c}$ for some $c > 0$, while its isoperimetric profile is equivalent to $\frac{1}{\log \log n}$. Beyond amenable groups, the gap may be even larger: virtually free groups have a bounded separation profile (so $\text{Sep}(N)/N \rightarrow 0$), while their isoperimetric profile is equivalent to a positive constant.

Remark 4.15. — Note that for any integer k and $C > 0$, we have for any $\varepsilon \in (0, 1)$, we have

$$\exp^{(k-1)}\left(C \log^{(k)} N\right) \leq N^\varepsilon.$$

Indeed, let $\varepsilon > 0$. For N large enough the following inequalities hold:

$$\log^{(k)}(N) \leq \frac{\log^{(k-1)}(N)}{2C} \leq \frac{1}{C} \log^{(k-1)}(N^\varepsilon).$$

Therefore we have:

$$C \log^{(k)}(N) \leq \log^{(k-1)}(N^\varepsilon),$$

i.e.:

$$\exp^{(k-1)}\left(C \log^{(k)}(N)\right) \leq N^\varepsilon.$$

To prove Proposition 4.13, we will use the following fact.

FACT 4.16. — Let $g(n) = \frac{c}{(\log^{(d-1)}(n))^\beta}$. Then for any large enough n we have

$$p_g^{1/8}(n) \leq \exp^{(k)}\left(8^{1/\beta} \log^{(k)}(n)\right).$$

Proof. — Let m, n be such that $m \geq \exp^{(k)}\left(8^{1/\beta} \log^{(k)}(n)\right)$. Let us write $m' = \log^{(k)}(m)$ and $n' = \log^{(k)}(n)$. We have $m' \geq 8^{1/\beta} n'$. Hence, we have $g(m) \leq \frac{1}{8} g(n)$, which is the required inequality. \square

Proof of Proposition 4.13. — Combining Theorem 3.10 with Fact 4.16, we get that there exists infinitely many integers $(n_i)_{i \geq 0}$ such that for any

i there exists $N_i \in \left[n_i, p_g^{1/8}(n_i) \right]$ such that, if n_i is large enough,

$$\begin{aligned} \frac{\text{Sep}_G(N_i)}{N_i} &\geq \Lambda_G(n_i) \frac{1/8}{\log\left(\frac{p_g^{1/8}(n_i)}{n_i}\right)+1} \\ &\geq \Lambda_G(N_i) \frac{1/8}{\log(p_g^{1/8}(n_i))+1} \\ &\geq \Lambda_G(N_i) \frac{1/16}{\exp^{(k-1)}(8^{1/\beta} \log^{(k)}(n_i))} \\ &\geq \Lambda_G(N_i) \frac{1/16}{\exp^{(k-1)}(8^{1/\beta} \log^{(k)}(N_i))}. \end{aligned} \quad \square$$

COROLLARY 4.17. — *Assume $G = F \wr (F \wr N)$ where F is a finite group and N is a nilpotent group whose growth is polynomial of degree d . Then there are constants $K, C > 1$ so that for infinitely many integers N 's, we have:*

$$\frac{\text{Sep}(N)}{N} \geq K \frac{\Lambda_G(N)}{\log(N)^C} \simeq \frac{1}{\log(\log(N))^{1/d} \cdot \log(N)^C}.$$

It is hard to tell if this lower bound is sharp. Note however, that some power of log need to be present as we exhibit an upper bound which also decays as a power of log, see Remark 5.7.

4.4. Solvable groups

In this subsection, we give an application of Proposition 4.13 to solvable groups. We prove the following theorem:

THEOREM 4.18. — *Let G be a finitely generated solvable group. If there exists $\epsilon \in (0, 1)$ and $c > 0$ such that for any large enough integer n we have*

$$\text{Sep}_G(n) \leq cn^{1-\epsilon},$$

then G is virtually nilpotent.

It is known that any nilpotent group of rank d has a separation profile equivalent to $n^{\frac{d-1}{d}}$ (see [24, Theorem 7]). We show here that, among solvable groups, the separation profile is able to reveal nilpotence. This is quite sharp since the separation profile of the classical lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ (as well as any polycyclic group) is bounded above by $\frac{n}{\log(n)}$ (since it has finite Assouad dimension, see Hume [22, Theorem 1.5]; our Corollary 5.6 gives a slightly weaker bound for these groups). Note that this result is definitively not true in general as non-amenable groups give counterexamples: for any $d \geq 3$, the d -dimensional hyperbolic space has a separation

profile equivalent to $n^{\frac{d-2}{d-1}}$ (see [7, Proposition 4.1.]) or, more spectacularly, free groups has a bounded separation profile. This theorem partially answers a question posed to us independently by David Hume and Jérémie Briussel:

QUESTION 4.19. — *Is there an exponential growth solvable group Γ such that $\text{Sep}_\Gamma(n) \not\asymp \frac{n}{\log(n)}$?*

See Question 7.4 for further discussions of this topic.

Proof of Theorem 4.18. — Recall that for any group Δ we define the derived series of Δ as the sequence of groups $(\Delta^{(i)})_{i \geq 0}$ defined inductively by $\Delta^{(0)} = \Delta$, $\Delta^{(i)} = [\Delta^{(i-1)}, \Delta^{(i-1)}]$. A group is solvable if and only if $\Delta^{(i)} = \{e\}$ for some i and the smallest such i is called the derived length of Δ .

Let r be the size of a finite generating set of G and d be the derived length of G . If G is an abelian group, then the conclusion of Theorem 4.18 is valid, then we can assume that d is at least equal to 2. Let \mathbf{F}_r be “the” free group on r generators, labelled by a generating set of G of size r , and let $\mathbf{S}_{d,r} := \mathbf{F}_r / \mathbf{F}_r^{(d)}$ be the free solvable on r generators of derived length d . G is a quotient of $\mathbf{S}_{d,r}$, considering the well-defined surjective group homomorphism

$$\pi_G : \mathbf{S}_{d,r} \rightarrow G.$$

From Tessera [40, Proposition 5.5], we have

$$\Lambda_G \leq \Lambda_{\mathbf{S}_{d,r}}.$$

Additionally, L. Saloff-Coste and T. Zheng explicited in the introduction of [38] the isoperimetric profile of the free solvable groups, namely:

$$\Lambda_{\mathbf{S}_{d,r}}(n) \simeq \left(\frac{\log^{(d)}(n)}{\log^{(d-1)}(n)} \right)^{1/r}.$$

Combining those two inequalities (and the fact that $\log^{(d)}(n) \leq \sqrt{\log^{(d-1)}(n)}$ for $n \geq 4^{d-1}$), we get that there exists some constants c, d , r such that for any large enough n , we have

$$(4.2) \quad \Lambda_G(n) \leq \frac{c}{\left(\log^{(d-1)}(n)\right)^{1/2r}}.$$

Let us assume that G has an exponential volume growth. From Theorem 1 of [12], there exists a positive constant c_l such that the following inequality

is true for any n :

$$(4.3) \quad \Lambda_G(n) \geq \frac{c_l}{\log(n)}.$$

Then combining Proposition 4.13 and (4.2), we get that there exists $K, C > 0$ such that for infinitely many integers N , we have:

$$\frac{\text{Sep}(N)}{N} \geq K \frac{\Lambda_G(N)}{\exp^{(k-1)}\left(C \log^{(k)}(N)\right)}.$$

Let $\varepsilon > 0$. From Remark 4.15, we then have, for any large enough such N ,

$$\begin{aligned} \frac{\text{Sep}_G(N)}{N} &\geq \Lambda_G(N) N^{-\varepsilon/2} \\ &\geq \frac{c_l}{\log N} N^{-\varepsilon/2} \quad \text{from (4.3)} \\ &\geq N^{-\varepsilon}. \end{aligned}$$

We have shown that if the group G has exponential volume growth, then its separation profile dominates along a subsequence every sublinear power function. By contraposition, if the assumptions of Theorem 4.18 are satisfied, that means that the group G does not have exponential growth, meaning that it is virtually nilpotent according to the usual dichotomy for solvable groups (see for example [20]). □

4.5. Limitations of Theorem 3.8

Now we will be interested in graphs where the conclusion of Theorem 3.8 becomes trivial.

A first source of loss of sharpness is the uncertainty on the isoperimetric profile. This happens for example when the profile is bounded by different numbers of iterated logarithms. However, we will consider this limitation due to lack of information as superficial. The approach of Section 3.3 supports this point of view.

A second source, which we believe to be much deeper, is the decay of the isoperimetric profile. As we noticed before, Theorem 3.8 gives nothing in the case of graphs with almost constant isoperimetric profiles, since the integer m doesn't exist for n large enough. We will see in this subsection an example of isoperimetric profile for which the conclusion of the Theorem 3.8 is trivial.

Erschler [14] and Briussel & Zheng [9] give an explicit construction of groups with up-to-constant prescribed⁽¹⁾ isoperimetric profile. We can deduce that the examples below has instances in Cayley graphs, which leaves little hope to generalise Theorem 4.18 to amenable groups.

In the thesis of the first author, explicit examples of isoperimetric profiles where Theorem 3.8 gives no information are given.

5. Upper bounds on the separation profile

5.1. From growth

The aim of this subsection is to obtain upper bound from on the separation profile using the growth of balls in the graphs. Let d denote the combinatorial distance in the graph, then $B_n(x) = \{v \in VG \mid d(x, v) \leq n\}$ is the ball of radius n with centre x .

In order to effectively apply this method, the upper bound on the size $B_n(x)$ should be independent of the choice of the ball's centre x .

LEMMA 5.1. — Assume G is a graph such that $\sup_{x \in VG} |B_n(x)| \leq e^{f(n)}$ and $\frac{f(n)}{n} \rightarrow 0$. For any subgraph G' let $\beta_n(x)$ be the cardinality of a ball [inside the subgraph] of radius n centred at x . Let n_0 be such that $\sup_{n \geq n_0} \frac{f(n)}{n} \leq 1$. Then for any $n \geq n_0$ and $x \in VG$ there is a $\ell \in [n, 2n]$ such that

$$\frac{\beta_{\ell+1}(x) - \beta_\ell(x)}{\beta_\ell(x)} \leq \frac{2f(n)}{n}.$$

Proof. — Let us alleviate notation by using $\beta_j := \beta_j(x)$ and $\sigma_n = \beta_n - \beta_{n-1}$. Let $C_{n,k} = \min_{i \in [n, n+k]} \frac{\beta_{i+1}}{\beta_i}$. Then $\beta_{n+k} \geq (C_{n,k})^k \beta_n$. However, since the growth of the extra k steps is bounded by $e^{f(k)}$, $\beta_{n+k} \leq \beta_{n-1} + \sigma_n e^{f(k)} = \beta_n + \sigma_n(e^{f(k)} - 1)$. Thus

$$(C_{n,k})^k \leq 1 + \frac{\sigma_n}{\beta_n} (e^{f(k)} - 1) \leq e^{f(k)}.$$

This implies that $C_{n,k} - 1 \leq e^{f(k)/k} - 1$. If $\frac{f(k)}{k} \leq 1$ then $C_{n,k} - 1 \leq e^{f(k)/k} - 1 \leq (e - 1) \frac{f(k)}{k} \leq 2 \frac{f(k)}{k}$. Taking $k = n$ yields the conclusion. \square

⁽¹⁾In those examples, isoperimetric profiles aren't prescribed exactly, but up to constants. However, using Theorem 3.10 we can have the same bounds (up to constants) on infinitely many N 's as using Theorem 3.4 if the isoperimetric profile was exactly prescribed.

For the next proposition f^{-1} denotes the generalised inverse (but one can also simply assume f is a continuous increasing function defined on the reals).

PROPOSITION 5.2. — *Let G be a graph so that $\sup_{x \in VG} |B_n(x)| \leq e^{f(n)}$ for a function f with $\lim_{n \rightarrow \infty} \frac{f(n)}{n} = 0$. Assume the degree of the vertices is bounded by d . Then there is a constant K (depending on f) such that for any integer $N > K$,*

$$\frac{\text{Sep}(N)}{N} \leq 4d \frac{f\left(\frac{f^{-1}(\ln \frac{N}{2})-1}{2}\right)}{f^{-1}(\ln \frac{N}{2}) - 1}.$$

Proof. — For any subset $F \subset VG$ of cardinality N , consider the balls $B'_n(x)$ of radius n in the subgraph induced by F . (It does not matter where the centre x of the ball is, if one could choose, it would probably be best to choose a point that realises the diameter of F). Note that $|B'_n(x)| \leq |B_n(x)| \leq e^{f(n)}$. Let n_0 be the largest integer such that $e^{f(n_0)} \leq N/2$. Applying Lemma 5.1 with $n = \lfloor n_0/2 \rfloor$ implies that for some $k \in [\frac{n_0-1}{2}, n_0]$,

$$\frac{|\partial B'_k|}{|B'_k|} \leq d \frac{|B'_{k+1}| - |B'_k|}{|B'_k|} \leq d \frac{2f(n)}{n}.$$

Consequently, the Cheeger constant of F is at most $2d \frac{f(n)}{n}$. This shows that $\frac{\text{Sep}(N)}{N} \leq 2d \frac{f(n)}{n}$, as long as $\frac{f(n)}{n} \leq 1$. The constant K is $e^{f(n_1)}$, where n_1 is the smallest integer so that $\frac{f(n)}{n} \leq 1$ for any $n \geq n_1$. □

COROLLARY 5.3. — *Let G be a graph so that $\sup_{x \in VG} |B_n(x)| \leq K_1 e^{K_2 n^\alpha}$ for some constants $K_1, K_2 > 0$ and $\alpha \in [0, 1)$. Then, there are constants $L_1, L_2 > 0$ so that, for any $N > L_1$ large enough,*

$$\frac{\text{Sep}(N)}{N} \leq \frac{L_2}{(\ln N)^{\frac{1}{\alpha}-1}}.$$

Proof. — Use Proposition 5.2 with $f(x) = K_2 x^\alpha + \log K_1$. □

COROLLARY 5.4. — *Let G be a graph so that $\sup_{x \in VG} |B_n(x)| \leq K n^d$ for some constants $K, d > 0$. Then, there are constants $L_1, L_2 > 0$ so that, for any $N > L_1$ large enough,*

$$\frac{\text{Sep}(N)}{N} \leq \frac{L_2 \log(N)}{N^{1/d}}.$$

Proof. — Use Proposition 5.2 with $f(x) \simeq d \log x + \log K$. □

5.2. From compression

Another upper bound on the separation profile can be obtained if the groups has a good embedding in L^p -spaces. Recall that for a finite graph X and a 1-Lipschitz embedding $F : X \hookrightarrow Y$ in a L^p -space, the distortion of F is

$$\text{dist}(F) = \max_{x,y \in X, x \neq y} \frac{d(x,y)}{\|F(x) - F(y)\|_Y}.$$

where d is the combinatorial distance on X . The L^p -distortion of X is $c_p(X) := \inf\{\text{dist}(F) \mid F : X \hookrightarrow L^p\}$.

P.-N. Jolissaint and Valette [27, Theorem 1.1] (combined with [27, Proposition 3.3] as well as an estimate on the p -spectral gap from Amghibeche [1]) give the following lower bound on the L^p distortion for a finite graph X with n vertices and Cheeger constant h :

$$(5.1) \quad c_p(X) \geq K \log(n)h,$$

where the constant K depends only on p and the maximal degree.

Distortion can also be studied for infinite graphs, but we will here rely on the notion of compression. If G is a [connected] infinite graph, then a compression function for a 1-Lipschitz embedding $\Phi : G \hookrightarrow L^p$ is a function ρ so that

$$(5.2) \quad \forall x, y \in G, \quad \rho(d(x,y)) \leq \|\Phi(x) - \Phi(y)\|_{L^p}.$$

The compression exponent of Φ is $\alpha(\Phi) = \liminf_{x \rightarrow \infty} \frac{\log \rho(x)}{\log x}$ and the compression exponent of the graph is $\alpha(G) = \sup_{\Phi} \alpha(\Phi)$.

PROPOSITION 5.5. — *Let G be a connected graph of bounded degree which admits an embedding in a L^p -space (for some p) with compression function $\rho(x) = k_1 + k_2x^a$. Then there is a constant K so that*

$$\frac{\text{Sep}(n)}{n} \leq \frac{K}{(\log n)^{a/(2-a)}}.$$

Proof. — Assume X is a finite subgraph of G of cardinality n with Cheeger constant h , maximal degree k and diameter δ .

First note that the diameter of X is bounded by the other quantities. Indeed, pick $x, y \in X$, that realise the diameter and let $\delta' = \lfloor \frac{\delta-1}{2} \rfloor$. Then the balls of radius δ' around x and y are disjoint and at least one of them does not cover more than half the vertices. The ratio $\frac{h}{k}$ then dictates a minimal growth: $\frac{n}{2} \geq (1 + \frac{h}{k})^{\delta'}$, that is $\delta' \leq \frac{\log \frac{n}{2}}{\log(1 + \frac{h}{k})}$. For $\delta \geq 6$, this gives

$$\frac{\delta}{3} \leq \frac{\log n}{\log(1 + \frac{h}{k})}.$$

Since G admits a 1-Lipschitz embedding Φ , this embedding restricts to X and (5.2) can be rewritten as

$$\forall x, y \in X, \quad \frac{d(x, y)}{\rho(d(x, y))} \geq \frac{d(x, y)}{\|\Phi(x) - \Phi(y)\|_{L^p}}.$$

The left-hand-side gets only bigger if one looks at $d(x, y) = \delta$. By taking the maximum on the right hand side, this leads to

$$\frac{\delta}{\rho(\delta)} \geq c_p(X).$$

Using the bound (5.1) of P.-N. Jolissaint and Valette [27], this gives

$$\frac{\delta}{\rho(\delta)} \geq Kh \log(n).$$

For n large enough, δ is also large ($\delta \geq \frac{\log n}{\log k}$) so that $\frac{\delta}{\rho(\delta)} \leq k_3 \delta^{1-a}$ for some constant k_3 . Next using the bound on δ above:

$$Kh \log(n) \leq k_3 \delta^{1-a} \leq 3^{1-a} k_3 \frac{(\log n)^{1-a}}{(\log(1 + \frac{h}{k}))^{1-a}}.$$

This can be rewritten as

$$Kh(\log(1 + \frac{h}{k}))^{1-a} \leq 3^{1-a} k_3 (\log n)^{-a}.$$

Since $h \leq k$, $\log(1 + \frac{h}{k}) \geq \frac{h}{k} \log 2$. With new constants, the inequality reads:

$$h^{2-a} \leq K'(\log n)^{-a}.$$

This means that any [connected] subset X of cardinality n inside G has a Cheeger constant of at most $K''(\log n)^{-a/(2-a)}$. From the definition of $\text{Sep}(n)$ it follows that $\text{Sep}(n) \preceq n(\log n)^{-a/(2-a)}$. □

COROLLARY 5.6. — *Assume G is a graph with bounded degree and compression exponent α (in some L_p -space). Then for any $c < \frac{\alpha}{2-\alpha}$ there is a constant K so that*

$$\frac{\text{Sep}(n)}{n} \leq \frac{K}{(\log n)^c}.$$

Here is a [non-exhaustive] list of Cayley graphs for which the compression exponent is known (references below). This table does not always use the case $p = 2$; in fact taking $p \rightarrow 1$ or $p \rightarrow \infty$ often gives better bounds, see Naor & Peres [30, Lemma 2.1].

$\alpha(G)$	$\frac{\alpha}{2-\alpha}$	Groups
1	1	polycyclic groups ^(a) , the lamplighter group over \mathbb{Z} with finite lamps ^(a) , hyperbolic groups ^(b) , Baumslag–Solitar groups ^(c) , 3-manifolds groups ^(d)
$\rightarrow 1$	1	lamplighter group over \mathbb{Z} with lamps in $\mathbb{Z}^{(e)}$, lamplighter over H of polynomial growth with finite lamps or lamps in $\mathbb{Z}^{(f)}$
$\frac{1}{2}$	$\frac{1}{3}$	lamplighter over \mathbb{Z}^2 with lamp group H having $\alpha_2(H) \geq \frac{1}{2}$ ^(g) , Thompson’s group $F^{(h)}$
$\frac{1}{2-2^{1-k}}$	$\frac{1}{3-2^{2-k}}$	iterated wreath products of the form: $\underbrace{(\dots((\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}) \dots)}_{k \text{ “}\mathbb{Z}\text{” terms}} \wr \mathbb{Z}^{(i)}$
$\geq \frac{1-\gamma}{1+\gamma}$	$\geq \frac{1-\gamma}{1+3\gamma}$	groups with return probability after n steps of a SRW $\leq K_2 e^{-K_1 n^\gamma}$ ^(j)
$\geq 1 - \nu$	$\geq \frac{1-\nu}{1+\nu}$	groups of intermediate growth with $b_n \leq e^{n^\nu}$ ^(k)
$\rightarrow \geq \frac{1}{d-1}$	$\geq \frac{1}{2d-3}$	free solvable groups $\mathbf{S}_{d,r}$ of derived length d when $d > 1$ ^(l)

Table’s references:

- (a) Tessera [39, Theorems 9 and 10]
- (b) Bonk & Schramm [8] and Buyalo & Schroeder [11]
- (c) Jolissaint & Pillon [26, Corollary 2]
- (d) Hume [21, Theorem 5.4]
- (e) Naor & Peres [30, Lemma 7.8] and [31, Theorem 6.1]; the bound is $\max\{\frac{p}{2p-1}, \frac{2}{3}\}$, take $p \rightarrow 1$
- (f) Naor & Peres [31, Theorem 3.1]; the bound is $\max\{\frac{1}{p}, \frac{1}{2}\}$, take $p \rightarrow 1$
- (g) Naor & Peres [30, Theorem 3.3]
- (h) Arzhantseva, Guba & Sapir [2, Theorem 1.3]
- (i) Naor & Peres [30, Corollary 1.3]
- (j) see [18, Theorem 1.1]
- (k) see either [18, Theorem 1.3(b)] or Tessera [39, Proposition 14]. In that case, the bound obtained on $\text{Sep}(n)$ by Proposition 5.2 is better.
- (l) see Sale [36, Corollary 4.2]; the bound on is $\frac{1}{p(d-1)}$ for $p \in [1, 2]$, so take $p \rightarrow 1$.

There are many other groups for which one can compute the compression (the above list does not exhaust the results in the references). For

example, $\alpha(G \times H) = \min(\alpha(G), \alpha(H))$. There are also further results: on HNN-extensions see Jollissaint & Pillon [26], on relatively hyperbolic groups see Hume [21], on wreath products see Li [28].

Note that Proposition 5.5 is fairly sharp (in this generality). Indeed, if one looks at the product of two trees, then the compression exponent is 1. This means Proposition 5.5 ensures, for every $c < 1$, the existence of K_c such that $\frac{\text{Sep}(n)}{n} \leq \frac{K_c}{(\log n)^c}$. On the other hand, it was shown by Benjamini, Schramm & Timár [7, Theorem 3.5] that the separation profile of such a space is $\frac{\text{Sep}(n)}{n} \simeq \frac{1}{\log n}$.

Remark 5.7. — The above corollary shows that there are amenable groups G for which $\frac{\text{Sep}(n)}{n}$ decreases much more quickly than $\Lambda_G(n)$. For example, $G = F \wr (F \wr \mathbb{Z})$ has $\Lambda_G(n) \simeq \frac{1}{\log \log n}$. On the other hand this group has an isometric embedding in a Cayley graph of $\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})$. In particular, its compression exponent is at least $\frac{4}{7}$. This implies that $\frac{\text{Sep}(n)}{n} \preceq \frac{1}{(\log n)^c}$ for any $c < \frac{2}{5}$.

Remark 5.8. — It is possible to show that if there is an embedding with $\rho(x) \geq K_1(\log^{(k)} n)^\alpha$ (where $\log^{(k)}$ denotes k iterated logarithms) then the conclusion of Proposition 5.5 is that

$$\frac{\text{Sep}(n)}{n} \leq \frac{K'}{(\log^{(k+1)} n)^{\alpha/2}}.$$

Compression function of this sort follow from the methods of [18, Paragraph before Remark 3.4]. It can be shown that any [amenable] group where $P^n(e) \leq K_1 \exp(n / \log^{(k)} n)$ has an embedding in some Hilbert space with $\rho(x) \geq (\log^{(k)} n)^{1/2}$. In fact (thanks to Kesten’s criterion for amenability), one can get for any amenable group an upper bound on the separation profile.

6. Local separation profiles

In this section, we will study a local variant of the separation profile. We found it relevant in two contexts. First, in \mathbb{Z}^d percolation clusters, where considering classical separation profile is trivial: since in almost every percolation configuration one can find arbitrary large balls, the profile is almost surely equal to the separation profile of \mathbb{Z}^d . Second, it can tackle the issue of the density of high separation subgraphs in non-vertex transitive graphs.

We will first define it, give a local version of Theorem 3.4 (Theorem 6.8), and some applications to percolation clusters in \mathbb{Z}^d , and to graphs of polynomial growth and of isoperimetric dimension larger than one, that we call *polynomial graphs*. Finally, we will give a theorem with a more abstract approach, that also applies to polynomial graphs, see Theorem 6.19.

DEFINITION 6.1. — *Let (G, v) be a rooted graph. Let $\rho : \mathbb{R}_{\geq 1} \rightarrow \mathbb{R}_{\geq 1}$ be a non-decreasing function. We define the (ρ, v) -local separation profile as:*

$$\text{Sep}_G^{\rho, v}(n) := \sup_{|F| \leq n \text{ and } F \subset B_G(v, \rho(n))} |F| \cdot h(F).$$

In comparison to the classical separation profile, which is defined as $\text{Sep}_G(n) = \sup_{|F| \leq n} |F| \cdot h(F)$, there is an extra condition restricting the subgraphs to lie in a given sequence of balls. One can think of it as searching for graph with big separation, but not too far from x ; the “not too far”-part is quantified by the function ρ .

As for the classical separation profile, this local variant gives obstructions for the existence or regular maps (see Lemma 1.3 of [7]). We remind the reader the definition of a regular map:

DEFINITION 6.2. — *Let X and Y be two graphs of uniformly bounded degrees. A map $f : X \rightarrow Y$ is said to be regular if there exists a constant $\kappa > 0$ such that the following two conditions are satisfied:*

- $\forall x_1, x_2 \in X \ d(f(x_1), f(x_2)) \leq \kappa d(x_1, x_2),$
- $\forall y \in Y \ |f^{-1}(\{y\})| \leq \kappa.$

The local separation profiles satisfies the following monotonicity:

PROPOSITION 6.3. — *Let (X, x_0) and (Y, y_0) be two rooted graphs of uniformly bounded degrees. Let $\rho : \mathbb{R}_{\geq 1} \rightarrow \mathbb{R}_{\geq 1}$ be a non-decreasing function. Let $f : X \rightarrow Y$ be a regular map such that $f(x_0) = y_0$. Then there exists a constant $K > 0$ such that for any n :*

$$\text{Sep}_X^{\rho, x_0}(n) \leq K \text{Sep}_Y^{\rho(K \cdot), y_0}(n).$$

Proof. — The same proof as the proof of Lemma 1.3 of [7] works. □

Remark 6.4. — Recall that the constant K appears both in factor of the separation profile, and in the argument of ρ (we define $\rho(K \cdot)$ by $n \mapsto \rho(Kn)$).

Note that if $\rho \succeq n$, the (v, ρ) -local separation profile coincides with the usual separation profile for vertex-transitive graphs.

The smallest (interesting) local profile is what we obtain choosing ρ to be the generalised inverse of the volume growth: $\rho(n) = \gamma_v^{-1}(n) = \sup\{x \geq$

$0 \mid \gamma_v(x) \leq n\}$, with $\gamma_v(n)$ denoting the size of the ball of radius n and centred at v . In that case we restrict the graphs investigated to lie in a ball of cardinality (almost) n .

In the situations we investigate, we get upper bounds in the case $\rho = \gamma_v^{-1}$. Then in what follows, we will restrict ourselves to this case. In this situation, the condition “ $|F| \leq n$ ” is redundant and we will drop the ρ from our notation:

$$\text{Sep}_G^v(n) := \text{Sep}_G^{\gamma_v^{-1},v}(n) = \sup_{F \subset B_G(v,r); |B_G(v,r)| \leq n} |F| \cdot h(F).$$

Therefore, we do not study this notion in its full generality, but we believe nevertheless that it can be relevant in some probabilistic contexts.

Local separation profile will be studied in two cases: first, \mathbb{Z}^d percolation clusters, then, graphs of isoperimetric dimension greater than one and of polynomial growth.

6.1. A local version of Theorem 3.4

6.1.1. Statement of the Theorem

Before stating the theorem, we will introduce some notations for local isoperimetry.

DEFINITION 6.5. — We say that $F \subset G$ is (n, v) -optimal if:

- $F \subset B(v, n)$,
- $\forall A \subset F \frac{|\partial A|}{|A|} \geq \frac{|\partial F|}{|F|}$.

As before, we will say that an integer r is (n, v) -optimal if there exists an (n, v) -optimal set of cardinality r .

To adapt the previous result to our context, we need to introduce a local version of the isoperimetric profile:

DEFINITION 6.6. — Let G be a graph. Let $v \in G$ and n be a positive integer. We define for any $r > 0$:

$$\Lambda_n^v(r) = \inf_{A \subset B_G(v,n), |A| \leq r} \frac{|\partial A|}{|A|}.$$

This is a mixed profile between the classical and the isoperimetric profile inside the balls introduced by Tessera in [39].

We can now state the local versions of Lemma 3.2 and Theorem 3.4. The proofs of the corresponding statements still work in this local context, we will not write them again:

LEMMA 6.7. — *Let F be a (n, v) -optimal subset of a graph G . Then:*

$$2h(F) \geq \Lambda_n^v \left(\frac{|F|}{2} \right) - \Lambda_n^v(|F|).$$

THEOREM 6.8. — *Let G be a connected infinite graph of bounded degree. Let $v \in G$, n be a positive integer, and k be an integer. Assume there is a non-decreasing function $p : [0, k] \rightarrow [0, |B(x, n)|]$ so that for any $r \in [0, k]$ there is an $r_{op} \in (r, p(r)]$ such that r_{op} is optimal. Choose $\varepsilon \in (0, 1)$. Let $r_1, r_2 \in [0, k]$ be such that $\Lambda_n^v(r_2) \leq (1 - \varepsilon)\Lambda_n^v(r_1)$.*

Then there exists an $r' \in [r_1, p(r_2)]$ such that

$$\frac{\text{Sep}^v(r')}{r'} \geq \varepsilon \frac{\Lambda_n^v(r_1)}{4 \log\left(\frac{p(r_2)}{r_1}\right) + 4}.$$

6.2. Application to polynomial graphs and \mathbb{Z}^d percolation clusters.

We will apply Theorem 6.10 in graphs of polynomial growth and of dimension greater than one. We will call such a graph a *polynomial graph*. We will show that around any point the separation is bounded below by some power of n . We start with the definition of a polynomial graph:

DEFINITION 6.9. — *Let G be a graph. Let d_1 and d_2 be two positive reals. We say that G is (d_1, d_2) -polynomial if there exist $b, g > 0$ such that:*

- *For any vertex v and any integer n $|B(v, n)| \leq bn^{d_2}$,*
- *For any $V \subset VG$, $|\partial V| \geq g|V|^{\frac{d_1-1}{d_1}}$.*

The upcoming theorem will apply both to polynomial graphs and to percolation clusters of \mathbb{Z}^d . Therefore the assumptions of this theorem are less restrictive, and polynomial graphs will be a particular case where they are satisfied. In particular, we do not require every subset of vertices to satisfy the isoperimetric inequality, but only some large enough subset.

THEOREM 6.10. — *Let G be a connected infinite graph of bounded degree. We assume that there exist $d_1, d_2 > 1$ such that G locally has a growth rate at most d_2 and a large scale isoperimetric dimension at least d_1 . Namely, we assume that there exist some functions $f, g, b > 0$ such that for any vertex v and any integer n :*

- (1) $|B(v, n)| \leq b(v) \cdot n^{d_2}$,
- (2) For any $A \subset B(v, n)$ such that $|A| \geq f(v, n)$, $|\partial A| \geq g(v) \cdot |A|^{1-1/d_1}$.

We make the additional assumption that for any vertex v there exists an integer n_ω such that for any integer $n \geq n_\omega$ we have $f(v, n) \leq |B(v, n)|$.

Then for any $\eta \in (0, 1)$, there exist $c(v), K(v), \beta > 0$ (with $\beta > d_1$), such that for any vertex v and any large enough integer n , when $f(v, n) \leq cg(v) \frac{\beta-d_1}{d_1-1} n^{\frac{d_1^2(1-\eta)^2}{d_2^2}}$, then we have:

$$\text{Sep}_G^v(n) \geq Kg(v)^\beta n^{\alpha(1-\eta)}, \text{ with } \alpha = \frac{d_1^2(d_1-1)}{d_2^3}.$$

Moreover, if $d_1 = d_2$ the conclusion is also true with $\eta = 0$. In this case, the constant K depends on the logarithm of g .

Before proving this theorem in subsection 6.3, we will state the corollaries we obtain in the two particular cases that interest us. First, to polynomial graphs:

COROLLARY 6.11. — *Let G be a (d_1, d_2) -polynomial graph. Then for any $\eta \in (0, 1)$ there exists $c > 0$ such that for any vertex v and any integer n :*

$$\text{Sep}^v(n) \geq cn^{(1-\eta)\frac{d_1^2(d_1-1)}{d_2^3}}.$$

Moreover, if $d_1 = d_2$ the conclusion is also true with $\eta = 0$.

Remark 6.12. — If d_1 equals d_2 , we get the expected exponent $\frac{d_1-1}{d_1}$; this is optimal in the case of vertex-transitive graphs, see Benjamini, Schramm & Timár [7].

As a second application, we study local separation in \mathbb{Z}^d percolation clusters. We obtain the following corollary:

COROLLARY 6.13. — *Let $p > p_c(\mathbb{Z}^d)$. Let ω be a percolation configuration of \mathbb{Z}^d of parameter p . Let \mathcal{C}_∞ be an (almost surely unique) infinite connected component of ω . Let $\varepsilon \in (0, 1)$. Then there exist $c(d, p) > 0$ and, for almost every ω , an integer l_ω such that for any $n \geq l_\omega$ and for any $x \in \mathcal{C}_\infty$ such that $\|x\|_\infty \leq \exp\left(n^{(1-\varepsilon)\frac{d}{d-1}}\right)$, we have:*

$$\text{Sep}_{\mathcal{C}_\infty}^x(n) \geq cn^{\frac{d-1}{d}}.$$

This theorem will be deduced from a result on isoperimetry by G. Pete [32]. The forebears of this result can be found in the work of Barlow [3] and Benjamini & Mossel [5] (see [32] for more details on the history).

THEOREM (Pete [32, Corollary 1.3]). — *For all $p > p_c(\mathbb{Z}^d)$ there exist $c_3(d, p) > 0, \alpha(d, p) > 0$ and (for almost all percolation configurations ω) an*

integer n_ω such that for all $n > n_\omega$, all connected subsets $S \subset \mathcal{C}_\infty \cap [-n, n]^d$ with size $|S| \geq c_3(\log n)^{\frac{d-1}{d}}$, we have $|\partial_{\mathcal{C}_\infty} S| \geq \alpha|S|^{1-1/d}$.

Proof of Corollary 6.13. — From this theorem, one can deduce that we can apply Theorem 6.10 to almost every percolation configuration with $d_1 = d_2 = d$, $f(v, n) = c_3(\log(\|v\|_\infty + n))^{\frac{d-1}{d}}$, and $g(v) = \alpha$. \square

6.3. Proof of Theorem 6.10

To show Theorem 6.10, we start with two lemmas. First, we can deduce from isoperimetry a lower bound on the growth of the graph:

LEMMA 6.14. — *Let G be a connected infinite graph of bounded degree satisfying the assumptions of Theorem 6.10. Let $v \in G$. Then there exists $b'(v) > 0$ such that for any large enough n , we have:*

$$|B(v, n)| \geq b'(v) \cdot g(v)^{d_1} \cdot n^{d_1}.$$

Proof. — We can substitute $n \mapsto |B(v, n)|$ with an piecewise affine function $B(t)$ that takes the same values on integer points. Then, for every $n > n_\omega$, we get:

$$\begin{aligned} B(n)^{1/d_1} - B(n_\omega)^{1/d_1} &= \frac{1}{d_1} \int_{n_\omega}^n \frac{B'(t)}{B(t)^{1-1/d_1}} dt \\ &\geq \frac{1}{d_1} \sum_{r=n_\omega}^{n-1} \frac{B(r+1) - B(r)}{B(r+1)^{1-1/d_1}} \\ &= \frac{1}{d_1} \sum_{r=n_\omega}^{n-1} \frac{B(r+1) - B(r)}{B(r)^{1-1/d_1}} \left(\frac{B(r)}{B(r+1)} \right)^{1-1/d_1} \\ &\geq \frac{1}{d_1} \sum_{r=n_\omega}^{n-1} \frac{B(r+1) - B(r)}{B(r)^{1-1/d_1}} \frac{1}{D^{1-1/d_1}} \\ &\geq \frac{1}{d_1} \sum_{r=n_\omega}^{n-1} \frac{|\partial B(v, r)|}{B(r)^{1-1/d_1}} \frac{1}{D^{2-1/d_1}} \\ &\geq \frac{g(v)}{d_1 D^{2-1/d_1}} \cdot (n - n_\omega), \end{aligned}$$

where D is a bound on the degrees of the vertices of G . This implies, for any large enough n ,

$$B(n)^{1/d_1} \geq \frac{g(v)}{2d_1 D^{2-1/d_1}} n. \quad \square$$

Second, we can deduce an upper bound on the isoperimetric ratio of balls using growth:

LEMMA 6.15. — *Let G be a connected infinite graph of bounded degree satisfying the assumptions of Theorem 6.10. Let $v \in G$ and $\eta \in (0, 1)$.*

Then there exists $a > 0$ such that for any large enough integer n , there exists an integer r between $n^{1-\eta}$ and $2n$ such that:

$$\frac{|\partial B(v, r)|}{|B(v, r)|} \leq \frac{a}{|B(v, r)|^{1/d_1}}.$$

Moreover, if $d_1 = d_2$ the conclusion is also true for $\eta = 0$.

To show this lemma, we will use the following facts, that we will prove later:

FACT 6.16. — Let G be a connected infinite graph of bounded degree satisfying the assumptions of Theorem 6.10. Let $v \in G$ and $\eta \in (0, 1)$.

Then there exists $A > 0$ such that for any non-negative integer n there exists $m \in [n^{1-\eta}, n]$ such that $|B(v, 2m)| \leq A|B(v, m)|$.

Moreover, if $d_1 = d_2$ the conclusion is also true for $\eta = 0$: there exists $A > 0$ such that for any non-negative integer n we have $|B(v, 2n)| \leq A|B(v, n)|$.

FACT 6.17. — Let G be a connected infinite graph of bounded degree satisfying the assumptions of Theorem 6.10.

Let $A > 0, v$ be a vertex of G and m be an integer such that $|B(v, 2m)| \leq A|B(v, m)|$. Then there exists an integer r between m and $2m$ such that:

$$\frac{|\partial B(v, r)|}{|B(v, r)|} \leq \frac{2 \log(A)}{r}.$$

Before proving those facts, we give a proof of Lemma 6.15:

Proof of Lemma 6.15. — According to the Facts 6.16 and 6.17, there exists $A > 0$ such that for any non-negative integer n there exists $r \in [n^{1-\eta}, 2n]$ such that

$$\frac{|\partial B(v, r)|}{|B(v, r)|} \leq \frac{\log(A)}{r}.$$

We assume that $n^{1-\eta}$ is large enough so that it satisfies the assumptions of Lemma 6.14. From this lemma, we have $r \leq \frac{|B(v, r)|^{1/d_1}}{b'(v)^{1/d_1} g(v)}$. Therefore $\frac{|\partial B(v, r)|}{|B(v, r)|} \leq \frac{a}{|B(v, r)|^{1/d_1}}$ with $a = g(v) \log(A) b'(v)^{1/d_1}$. □

We will now prove the facts.

Proof of Fact 6.16. — Let A be such that $\frac{\eta}{2} \log(A) \geq d_2 + \log(b + 1)$, and let n be a positive integer. Then:

- if $n \leq \exp\left(\frac{2}{\eta}\right)$, then up to taking a larger A , we can show that the conclusion of the fact holds, since G is of bounded degree.

- otherwise, we assume by contradiction that for any integer m in the interval $[n^{1-\eta}, n]$, we have $|B(x, 2m)| > A \times |B(x, m)|$. Then we have:

$$\begin{aligned} |B(x, n)| &\geq A^{\log(n^\eta)-1} |B(x, n^{1-\eta})| \\ &\geq A^{\log(n^\eta)-1} \\ &\geq A^{\log(n^\eta)/2} \quad \text{as } n \geq \exp\left(\frac{2}{\eta}\right) \\ &\geq \exp\left(\frac{\eta}{2} \log(n) \log(A)\right) \\ &\geq \exp(d_2 \log(n) + \log(b + 1)) \\ &= (b + 1)n^{d_2}. \end{aligned}$$

(Our logarithms and exponentials are in base 2.) This contradicts the assumption on the growth of the graph.

If $d_1 = d_2$, the assumption on the growth of G and the conclusion of Lemma 6.14 give the announced result with $A = \frac{b}{b'} 2^{d_2}$. □

Proof of Fact 6.17. — We assume by contradiction that for any r between n and $2n$ we have $\frac{|\partial B(v,r)|}{|B(v,r)|} > \frac{2 \log(A)}{r}$. That implies in particular the following inequality: $\frac{|B(v,r+1)| - |B(v,r)|}{|B(v,r)|} > \frac{2 \log(A)}{r}$. Summing-up these inequalities, we have:

$$\sum_{r=m}^{2m} \frac{|B(v, r + 1)| - |B(v, r)|}{|B(v, r)|} > 2 \log(A) \sum_{r=m}^{2m} \frac{1}{r}.$$

Then we consider an piecewise affine function $B(t)$ that coincides with $|B(v, t)|$ on integer points. We get:

$$\log\left(\frac{B(2m)}{B(m)}\right) = \int_m^{2m} \frac{B'(t)}{B(t)} dt > 2 \log(A) \int_m^{2m} \frac{1}{t} dt > \log(A).$$

Therefore $B(2m) > AB(m)$, which is a contradiction. □

We are now able to prove Theorem 6.10:

Proof of Theorem 6.10. — Let v be a vertex of G and $n \geq n_\omega$ be an integer large enough so that we can apply Lemmas 6.14 and 6.15. We will require n to be (a priori) even larger in the following, satisfying some conditions that will appear later. Let η be a real of the interval $(0, 1)$, that may be equal to zero if $d_1 = d_2$.

FACT 6.18. — *Let $r \in [g(v)^{d_1} f(v, n)^{d_1}, |B(v, n)|]$. Let F a subset of $B(v, n)$ of cardinality at most r . Then, we have $\frac{|\partial F|}{|F|} \geq g(v)r^{-1/d_1}$.*

Proof. — If $|F| \leq f(v, n)$, then since G is infinite and connected, we have $|\partial F| \geq 1$. From the lower bound on r we can deduce

$$\frac{|\partial F|}{|F|} \geq \frac{1}{|F|} \geq \frac{1}{f(v, n)} \geq g(v)r^{-1/d_1}.$$

If $|F| \geq f(v, n)$, then we have from assumption (2)

$$\frac{|\partial F|}{|F|} \geq g(v)|F|^{-1/d_1} \geq g(v)r^{-1/d_1}. \quad \square$$

It follows that we have:

$$(is1) \quad \forall r \in [g(v)^{d_1} f(v, n)^{d_1}, |B(v, n)|] \quad \Lambda_n^v(r) \geq g(v)r^{-1/d_1}.$$

Let r be an integer in $\left[\max \left(8^{d_2} b(v), 4^{d_2} b(v) n_\omega^{\frac{d_2}{1-\eta}} \right), |B(v, n)| \right]$. Let r' be the biggest integer such that $|B(v, 2r')| \leq r$. According to Lemma 6.15, there exists an integer r'' between $r'^{1-\eta}$ and $2r'$ such that $\frac{|\partial B(v, r'')|}{|B(v, r'')|} \leq \frac{a}{|B(v, r'')|^{1/d_1}}$.

Since $B(v, 2r' + 2) \geq r$, we get from the growth assumption 1 of Theorem 6.10 that $r' \geq \frac{1}{2} \left(\frac{r}{b(v)} \right)^{1/d_2} - 2 \geq \frac{1}{4} \left(\frac{r}{b(v)} \right)^{1/d_2}$. Then we have $r'' \geq \frac{r^{(1-\eta)/d_2}}{4^{(1-\eta)b(v)^{(1-\eta)/d_2}}} \geq n_\omega$. Therefore, by Lemma 6.14 we have: $|B(v, r'')| \geq b'(v) \cdot g(v)^{d_1} \cdot r''^{d_1} \geq \frac{b'(v)g(v)^{d_1}}{4^{(1-\eta)d_1} b(v)^{(1-\eta)\frac{d_1}{d_2}}} r^{(1-\eta)\frac{d_1}{d_2}}$. We can deduce the follow-

ing inequality, setting $g'(v) = \frac{a \cdot 4^{(1-\eta)b(v)\frac{d_1}{d_2}}}{b'(v)^{1/d_1}}$:

$$(is2) \quad \forall r \in \left[4^{d_2} b(v) n_\omega^{\frac{d_2}{1-\eta}}, |B(v, n)| \right] \quad \Lambda_n^v(r) \leq \frac{g'(v)}{g(v)} r^{-\frac{(1-\eta)}{d_2}}.$$

Note that if $d_1 = d_2$, then this is also true with $\eta = 0$.

Let us set $s = \left(\frac{2g'(v)}{g(v)^2} \right)^{\frac{d_2}{1-\eta}}$. From the inequalities (is1) and (is2), we can deduce that whenever r_1 and r_2 , respectively in the validity domains of (is1) and (is2), satisfy $r_2 \geq s \cdot r_1^{\frac{d_2}{d_1(1-\eta)}}$, we have $\Lambda(r_2) \leq \frac{1}{2} \Lambda(r_1)$. From this inequality we can deduce that $p: r \mapsto s \cdot r^{\frac{d_2}{d_1(1-\eta)}}$ is a suitable function to apply Theorem 6.8.

Let r_1 be the biggest integer such that $p(p(r_1)) \leq |B(v, n)|$. Then we have $p(p(r_1 + 1)) \geq |B(v, n)|$. Since n is at least equal to n_ω , we can use Lemma 6.14, which gives $|B(v, n)| \geq b' \cdot g(v)^{d_1} \cdot n^{d_1}$. This yields:

$$\left(\frac{2g'(v)}{g(v)^2} \right)^{\left(\frac{d_2}{1-\eta} + \frac{d_2^2}{d_1(1-\eta)^2} \right)} \cdot (r_1 + 1)^{\frac{d_2^2}{d_1^2(1-\eta)^2}} \geq b' \cdot g(v)^{d_1} \cdot n^{d_1}.$$

Therefore, if n is large enough,

$$(6.1) \quad r_1 \geq c^{d_1} \cdot g(v) \left(\frac{d_1^3(1-\eta)^2}{d_2^2} + 2 \frac{d_1^2(1-\eta)}{d_2} + 2d_1(1-\eta) \right) n^{\frac{d_1^3(1-\eta)^2}{d_2^2}},$$

with

$$c = \frac{b' \frac{d_1(1-\eta)^2}{d_2^2}}{(3g'(v))^{\frac{d_1(1-\eta)}{d_2} + \frac{1}{d_1}}}.$$

Then if n is large enough r_1 is in the validity domain of (is2). Moreover, if we set

$$\beta = \frac{d_1^2(d_1 - 1)(1 - \eta)^2}{d_2^2} + 2 \frac{d_1(d_1 - 1)(1 - \eta)}{d_2} + 2(d_1 - 1)(1 - \eta) + 1,$$

we get that if $f(v, n) \leq cg(v) \frac{\beta - d_1}{d_1 - 1} n^{\frac{d_1^2(1-\eta)^2}{d_2^2}}$, then r_1 is in the validity domain of (is1). We find out the condition on $f(v, n)$ that is made in the statement of Theorem 6.10. Under this assumption, we can apply Theorem 6.8 with $r_2 = p(r_1)$. This gives:

$$(6.2) \quad \begin{aligned} \text{Sep}^v(|B(v, n)|) &\geq \text{Sep}^v(p(r_2)) \\ &\geq r_1 \frac{\Lambda_n^v(r_1)}{8 \log\left(\frac{p(r_2)}{r_1}\right) + 8}. \end{aligned}$$

First, from (is1) and the lower bound (6.1) on r_1 , we have:

$$r_1 \Lambda_n^v(r_1) \geq g(v) r_1^{\frac{d_1 - 1}{d_1}} \geq \frac{b' \frac{d_1(d_1 - 1)(1 - \eta)^2}{d_2^2}}{(3g'(v))^{\frac{d_1(d_1 - 1)(1 - \eta)}{d_2} + \frac{d_1 - 1}{d_1}}} \cdot g(v)^\beta n^{\frac{d_1^2(d_1 - 1)(1 - \eta)^2}{d_2^2}}.$$

Second, we have $p(r_2) \leq |B(v, n)| \leq b(v) \cdot n^{d_2}$, from the assumption (1) of Theorem 6.10. We now set $N = |B(v, n)|$, and define D to be a bound on the degrees of the vertices of G .

Then, after calculation we deduce from (6.2), using the two previous inequalities, and the lower bound (6.1) on r_1 , that if N is large enough we have

$$\text{Sep}^v(N) \geq Kg(v)^\beta \frac{N^\alpha}{\log(N)},$$

with:

$$\begin{aligned}
 K &= \frac{b' \frac{d_1(d_1-1)(1-\eta)^2}{d_2^2}}{9 \frac{d_2^3 - d_1^3(1-\eta)^2}{d_2^2} (3g'(v)) \frac{d_1(d_1-1)(1-\eta) + \frac{d_1-1}{d_1^2} b \frac{d_1^2(d_1-1)(1-\eta)^2}{d_2^3}}{d_2}}, \\
 \alpha &= \frac{d_1^2(d_1-1)(1-\eta)^2}{d_2^3}, \\
 \beta &= \frac{d_1^2(d_1-1)(1-\eta)^2}{d_2^2} + 2 \frac{d_1(d_1-1)(1-\eta)}{d_2} + 2(d_1-1)(1-\eta) + 1.
 \end{aligned}$$

Up to considering a larger η , we can substitute $(1-\eta)^2$ with $(1-\eta)$ in the expression of α , and remove the $\log(N)$ term that appear in the lower bound on $\text{Sep}^v(N)$. This gives the desired inequality in Theorem 6.10.

If we make the additional assumption that $d_1 = d_2$, we can set $d = d_1$. Our previous inequalities were valid with $\eta = 0$, which we now assume to be true. Then our previous inequalities turn into

$$\frac{p(r_2)}{r_1} \leq bc^{-d} g(v)^d,$$

and

$$r_1 \Lambda_n^v(r_1) \geq \frac{1}{3} g'(v)^{-\frac{(d-1)(d^2+1)}{d^2}} g(v)^\beta N^{\frac{d-1}{d}}.$$

Thus, the inequality (6.2) yields, if N is large enough:

$$\text{Sep}^v(N) \geq K g(v)^\beta N^\alpha,$$

with:

$$\begin{aligned}
 K &= \frac{1}{24} \frac{g'(v)^{-\frac{(d-1)(d^2+1)}{d^2}}}{\log(bc^{-d} g(v)^d) + 1}, \\
 \alpha &= \frac{d-1}{d}, \\
 \beta &= 5d - 4.
 \end{aligned}$$

This ends the proof of Theorem 6.10. □

6.4. Another approach for polynomial graphs.

In this subsection, we study local separation in graphs of polynomial growth and of isoperimetric dimension greater than 1. Using a more abstract and simple approach, we show again that around any point the separation is bounded below by a power of n , that improves Theorem 6.10 in some cases. We will prove a statement in a slightly more general context than polynomial graphs, with a local flavour, which is very natural regarding to the proof. We will then formulate the theorem in the setting of polynomial graphs (Corollary 6.21). Here is our theorem:

THEOREM 6.19. — *Let G be an infinite graph of bounded degree such that there exists $d_2 \geq d_1 > 1$ and two positive functions $b(v)$ and $g(v, n)$ such that for any vertex v and any positive integer n :*

- $\gamma_v(n) := |B(v, n)| \leq b(v)n^{d_2}$,
- for any $V \subset B(v, n)$, $|\partial V| \geq g(v, \gamma_v(n))|V|^{\frac{d_1-1}{d_1}}$.

We assume moreover that $d_1^2 > d_2 - d_1$. Then for any $\eta > 0$ there exists $s > 0$ depending only on d_1, d_2, b and η such that for any positive integer n and any vertex v :

$$\text{Sep}_G^v(n) \geq s \cdot g(v, n)^\beta \cdot n^{(1-\eta)\alpha} \quad \text{with } \alpha = \frac{(d_1-1)(d_1^2-(d_2-d_1))}{d_1^2 d_2},$$

$$\text{and } \beta = \frac{d_1^2+d_1-1}{d_1}.$$

Moreover, if $d_1 = d_2$ the conclusion is also true for $\eta = 0$.

Remark 6.20. — The conclusion of the theorem implies in particular that the classical (or global) separation profile is bounded below: For any $\eta > 0$ there exists $s(v, \eta) > 0$ such that for any positive integer n :

$$\text{Sep}_G(n) \geq s \cdot g(v, n)^\beta \cdot n^{(1-\eta)\alpha}.$$

This corollary follows, using the terminology introduced in Definition 6.9:

COROLLARY 6.21. — *Let G be a (d_1, d_2) -polynomial graph such that $d_2 - d_1 < d_1^2$. Then for any $\eta \in (0, 1)$ there exists $c > 0$ such that for any vertex v and any integer n :*

$$\text{Sep}_G^v(n) \geq cn^{(1-\eta)\alpha}, \quad \text{with } \alpha = \frac{(d_1 - 1)(d_1^2 - (d_2 - d_1))}{d_1^2 d_2}.$$

Moreover, if $d_1 = d_2$ the conclusion is also true for $\eta = 0$.

Remark 6.22. — As in Corollary 6.11, in the case where d_1 equals d_2 we get the expected exponent $\frac{d_1-1}{d_1}$, optimal in the case of vertex-transitive graphs. If d_1 is smaller than d_2 one can notice that Corollaries 6.11 and 6.21 do not give the same exponents (the best can be given by one or the other, depending on the values of d_1 and d_2), which is an interesting demonstration of the fact that, despite the use of the same ingredients, the two approaches are essentially different. Dropping the $(1-\eta)$ factor, the exponents of Corollaries 6.11 and 6.21 are respectively $\frac{d_1^2(d_1-1)}{d_2^3}$ and $\frac{(d_1-1)(d_1^2-(d_2-d_1))}{d_1^2 d_2}$. For example, with $d_1 = 2$ and $d_2 = 3$, Corollary 6.21 gives $\frac{1}{4}$, which is bigger than $\frac{4}{27}$, given by Corollary 6.11. With $d_1 = 2$ and $d_2 = \frac{11}{2}$, Corollary 6.11 gives $\frac{32}{1331} \approx 0.024$, which is bigger than $\frac{1}{44} \approx 0.023$ given by Corollary 6.21. Moreover, Corollary 6.21 gives nothing when $d_2 \geq d_1^2 + d_1$, while Corollary 6.11 always gives a consistent result.

Let us explain the strategy of the proof of Theorem 6.19. We call *isoperimetric ratio* of a set the ratio between the size of its boundary and its size, $\frac{|\partial \cdot|}{|\cdot|}$. Our goal is to find, for any n , a subset X of $B(v, n)$ for which we can bound below its cardinality and its Cheeger constant in order to get a bound on $|X|h(X)$. Adapting slightly the proof of Lemma 3.2, we see that to bound its Cheeger constant, it suffices for X to verify two conditions: first that it has a lower (or equal) isoperimetric ratio than its subsets, and second that the isoperimetric ratio of its small (less than a half) subsets is bigger, by a controlled factor greater than 1. To get those properties, we proceed recursively: starting from a ball $B(v, n)$, we take smaller and smaller subsets that violates the second condition, and when there is no such small subset, we finally take a subset of the resulting set that minimises the isoperimetric ratio. Our hypothesis on the growth of the graph gives an upper bound on the isoperimetric ratio the size of the boundary of $B(v, n)$, and the hypothesis on the isoperimetric dimension ensures a lower bound on the cardinality of the final set and on its isoperimetric ratio, leading to a bound on its Cheeger constant.

In the proof of Theorem 6.19, we will use the following lemma. As mentioned above, this is a local version of Lemma 3.2.

LEMMA 6.23. — *Let X be a finite subset of an infinite graph, satisfying the following properties:*

- $\forall Y \subset X \quad \frac{|\partial Y|}{|Y|} \geq \frac{|\partial X|}{|X|}$,
- $\forall Y \subset X \quad \left(|Y| \leq \frac{|X|}{2} \Rightarrow \frac{|\partial Y|}{|Y|} \geq (1 + \varepsilon) \frac{|\partial X|}{|X|} \right)$.

Then,

$$2h(X) \geq \varepsilon \frac{|\partial X|}{|X|}.$$

Proof. — The proof is very similar to the proof of Lemma 3.2. Let F_1 be a subset of X such that $|F_1| \leq \frac{|X|}{2}$. We denote $F_2 = X \setminus F_1$. Then we have:

$$\begin{aligned} 2|\partial_X F_1| &= |\partial F_1| + |\partial F_2| - |\partial X| \\ &\geq (1 + \varepsilon) \frac{|\partial X|}{|X|} |F_1| + \frac{|\partial X|}{|X|} |F_2| - |\partial X| \\ &= \varepsilon \frac{|\partial X|}{|X|} |F_1| + \frac{|\partial X|}{|X|} (|F_1| + |F_2|) - |\partial X| \\ &= \varepsilon \frac{|\partial X|}{|X|} |F_1|. \end{aligned}$$

Then we have $2 \frac{|\partial_X F_1|}{|F_1|} \geq \varepsilon \frac{|\partial X|}{|X|}$. Since this is true for any subset F_1 of X containing at most half of its points, we have shown the announced inequality. □

Proof of Theorem 6.19. — Let v be a vertex of G . We start by giving a doubling property of the graph G . Let η be a real of the interval $(0, 1)$, that may be equal to zero if $d_1 = d_2$. Let n be an integer at least equal to 2. Let m, A , and r be given by Facts 6.16 and 6.17. Then we have:

- $n^{1-\eta} \leq r \leq 2n$
- $\frac{|\partial B(v,r)|}{|B(v,r)|} \leq \frac{\log(A)}{r}$

Let us write $F_1 = B(v, r)$ the ball of G centred at v of radius r . Let $g = g(v, |B(v, 2n)|)$ and ε be a positive real small enough so that $2^{\frac{1}{d_1+1}} \leq \frac{2^{1/d_1}}{1+\varepsilon}$.

We construct a finite decreasing sequence $(F_i)_i$ by induction, in the following way: let i be a positive integer. If F_i is defined, then:

- If there exists a subset of A of F_i such that $|A| \leq \frac{|F_i|}{2}$ and $\frac{|\partial A|}{|A|} \leq (1 + \varepsilon) \frac{|\partial F_i|}{|F_i|}$, then we take F_{i+1} being such a set.
- Otherwise, we stop the sequence.

Let k denote the number of terms of this sequence. From the isoperimetric dimension hypothesis we have: $|F_k|^{-1/d_1} \leq \frac{1}{g} \frac{|\partial F_k|}{|F_k|} \leq \frac{(1+\varepsilon)^k}{g} \frac{|\partial F_1|}{|F_1|} \leq \frac{(1+\varepsilon)^k}{g} \frac{\log(A)}{r}$, therefore we can deduce that $|F_k| \geq \frac{g^{d_1} r^{d_1}}{\log(A)^{d_1} (1+\varepsilon)^{k d_1}}$.

By construction, we have $|F_k| \leq 2^{-k} |F_1|$. Hence, we can deduce that

$$2^{k/d_1} |F_1|^{-1/d_1} \leq |F_k|^{-1/d_1} \leq \frac{(1 + \varepsilon)^k \log(A)}{g r},$$

which means that $2^{\frac{k}{d_1+1}} \leq \left(\frac{2^{1/d_1}}{1+\varepsilon}\right)^k \leq \frac{\log(A)}{g} \frac{|F_1|^{1/d_1}}{r} \leq \frac{\log(A) b^{1/d_1}}{g} \frac{r^{d_2/d_1}}{r} = \frac{\log(A) b^{1/d_1}}{g} r^{\frac{d_2-d_1}{d_1}}$.

Then, since $(1 + \varepsilon)^{k d_1} \leq 2^{\left(\frac{1}{d_1} - \frac{1}{d_1+1}\right) k d_1} = 2^{\frac{k}{d_1+1}}$, we can deduce that, with $c = \log(A)^{-(d_1+1)} b^{-1/d_1}$,

$$|F_k| \geq c \cdot g^{d_1+1} \cdot \frac{r^{d_1}}{r^{\frac{d_2-d_1}{d_1}}} = c \cdot g^{d_1+1} \cdot r^{\frac{d_1^2 - (d_2-d_1)}{d_1}}.$$

We can take a final set X minimising $\frac{|\partial|}{| \cdot |}$ among subsets of F_k .

Therefore, X satisfies the following properties:

- $\forall Y \subset X \quad \frac{|\partial Y|}{|Y|} \geq \frac{|\partial X|}{|X|}$
- $\forall Y \subset X \quad \left(|Y| \leq \frac{|X|}{2} \Rightarrow \frac{|\partial Y|}{|Y|} \geq (1 + \varepsilon) \frac{|\partial X|}{|X|}\right)$

Then, we can apply Lemma 6.23 to X . We get

$$2h(X) \geq \varepsilon \frac{|\partial X|}{|X|}.$$

By construction of F_k , we have $|X| \geq |F_k|/2$. We have:

$$\begin{aligned} |X|h(X) &\geq \frac{\varepsilon}{2} \cdot |\partial X| \\ &\geq \frac{\varepsilon}{2} g \cdot |X|^{\frac{d_1-1}{d_1}} \\ &\geq \frac{\varepsilon}{2} g 2^{\frac{1-d_1}{d_1}} \cdot |F_k|^{\frac{d_1-1}{d_1}} \\ &\geq \frac{\varepsilon}{2} g 2^{\frac{1-d_1}{d_1}} c^{\frac{d_1-1}{d_1}} g^{\frac{(d_1-1)(d_1+1)}{d_1}} \cdot r^{\frac{(d_1-1)(d_1^2-(d_2-d_1))}{d_1^2}} \\ &\geq c' \cdot g^{\frac{d_1^2+d_1-1}{d_1}} \cdot n^{d_2(1-\eta)\alpha}. \end{aligned}$$

With $c' = \frac{\varepsilon}{2} \cdot 2^{\frac{1-d_1}{d_1}} \cdot c^{\frac{d_1-1}{d_1}}$ and $\alpha = \frac{(d_1-1)(d_1^2-(d_2-d_1))}{d_1^2 d_2}$.

We have shown that there exists a positive constant c' such that for any integer $n \geq 2$ and any vertex v , we have:

$$\begin{aligned} \text{Sep}_G^v(|B(v, 2n)|) &\geq c' g(v, |B(v, 2n)|)^{\frac{d_1^2+d_1-1}{d_1}} \cdot n^{d_2(1-\eta)\alpha} \\ &\geq \frac{c'}{b^{(1-\eta)\alpha} 2^{d_2(1-\eta)\alpha}} g(v, |B(v, 2n)|)^{\frac{d_1^2+d_1-1}{d_1}} \cdot |B(v, 2n)|^{(1-\eta)\alpha}. \end{aligned}$$

The announced result follows. □

7. Questions

Although we showed that there are plenty of optimal integers, it turns out it's incredibly hard to describe optimal sets. In the case of \mathbb{Z}^d this can probably be achieved with the Loomis–Whitney inequality (see [29]).

QUESTION 7.1. — *Give an explicit description of the optimal sets in the discrete Heisenberg groups (or in any amenable group which is not virtually Abelian).*

For the “continuous” version of the Heisenberg group, this is an old open question. But perhaps the discrete case is easier.

More generally, one could ask whether it is possible to find the optimal sets in semi-direct products of “well-known cases”: assuming the optimal sets of the [finitely generated] groups G_1 and G_2 are known [for some generating sets S_1 and S_2], can the optimal sets of $G_1 \times G_2$ be of the form $F_1 \times F_2$ (where F_i is an optimal set for the Cayley graph of G_i w.r.t. S_i)?

Another interesting question on optimal sets would be the following:

QUESTION 7.2. — *If G is a graph whose isoperimetric profile is known up to a multiplicative constant, what can we say about the density of sets whose separation is good?*

Let us shortly describe two interpretations of this question. First, Proposition 4.3 only uses the fact that $p(n) \leq Kn^c$ for some $K > 0$ and $c > 1$. This gives a fairly low density of optimal integers, leaving open the possibility for much higher densities. For example, if $K = 1$ and $c = 2$, then the sequence of optimal integers could be as sparse as $2, 4, 16, 256, \dots$

Second (in the spirit of local separation), one could also fix some n , r and K and look at the density of vertices x for which a ball of radius r contains a set of size n which is up to a multiplicative factor of K as hard to cut as the best set for that given n .

Here are many inequalities between the separation and isoperimetric profile which seem natural (they might be easy, or hard, to prove or disprove):

QUESTION 7.3.

(1) *If G is the Cayley graph of a group, more generally a vertex-transitive graphs, $\frac{\text{Sep}(N)}{N} \stackrel{?}{\asymp} \Lambda_G(N)$.*

(2) *If G is the Cayley graph of an amenable group,*

$$\frac{\text{Sep}_G(N)}{N} \stackrel{?}{\asymp} \Lambda_G(N/2) - \Lambda_G(N).$$

(3) *If G is the Cayley graph of a polycyclic group, $\frac{\text{Sep}_G(N)}{N} \stackrel{?}{\asymp} \frac{1}{\log n}$. (For such groups $\Lambda_G(N) \simeq \frac{1}{\log n}$.)*

(4) *If G is the Cayley graph of a group, is $\frac{\text{Sep}(N)}{N} \stackrel{?}{\asymp} N(\Lambda_G(N-1) - \Lambda_G(N))$.*

The following associated question was also posed to us in connection with Question 4.19:

QUESTION 7.4. — *Does the classical lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ coarsely embed in any exponential growth solvable group?*

A positive answer to this question would give a (negative) answer to Question 4.19. In fact, regular maps from the lamplighter to solvable groups (of exponential growth) would be enough (and should be easier to produce). Note that one cannot replace the lamplighter with a polycyclic group (of exponential growth) in Question 7.4. Indeed, the asymptotic dimension increases under a regular map (see Benjamini, Schramm and Timàr [7, §6]) and the classical lamplighter has asymptotic dimension 1 while polycyclic

groups have dimension ≥ 2 (they are finitely presented; see Gentimis [15] for both results). Consequently, there are no regular maps from any polycyclic group to the classical lamplighter group (which is a solvable group).

It is very natural to ask Question 7.4 more generally for exponential growth amenable groups. However, in [23], Hume and Mackay gave examples of elementary hyperbolic groups with an arbitrary low profile, along a subsequence. Then, the lamplighter group cannot coarsely embed in those groups.

The following interesting question was suggested by the referee in light of Corollary 1.7:

QUESTION 7.5. — *Are there amenable groups of exponential growth which have a regular embedding in an hyperbolic space of dimension $d > 1$.*

QUESTION 7.6. — *Does there exists a vertex-transitive graph G such that for some/any vertex v we have $\text{Sep}_G^{\gamma_v^{-1}, v} \prec \text{Sep}_G$? With G amenable?*

This question is linked with the issue of controlling the diameter of high separation graphs. Indeed, we expect those graphs to have a small diameter but finding such a counter-example would be very interesting.

The following question is very natural, after Theorem 1.6 which answers positively when G is solvable and Theorem 1.4 which does it for groups of subexponential growth.

QUESTION 7.7. — *For G amenable, does $\text{Sep}(n) \preceq n^{1-\epsilon}$ implies that G is virtually nilpotent?*

BIBLIOGRAPHY

- [1] S. AMGHIBECH, “Eigenvalues of the discrete p -Laplacian for graphs”, *Ars Combin.* **67** (2003), p. 283-302.
- [2] G. N. ARZHANTSEVA, V. S. GUBA & M. V. SAPIR, “Metrics on diagram groups and uniform embeddings in a Hilbert space”, *Comment. Math. Helv.* **81** (2006), no. 4, p. 911-929.
- [3] M. T. BARLOW, “Random walks on supercritical percolation clusters”, *Ann. Probab.* **32** (2004), no. 4, p. 3024-3084.
- [4] A. BENDIKOV, CH. PITTET & R. SAUER, “Spectral distribution and L^2 -isoperimetric profile of Laplace operators on groups”, *Math. Ann.* **354** (2012), no. 1, p. 43-72.
- [5] I. BENJAMINI & E. MOSSEL, “On the mixing time of a simple random walk on the super critical percolation cluster”, *Probab. Theory Related Fields* **125** (2003), no. 3, p. 408-420.
- [6] I. BENJAMINI & P. PAPASOGLU, “Growth and isoperimetric profile of planar graphs”, *Proc. Amer. Math. Soc.* **139** (2011), no. 11, p. 4105-4111.
- [7] I. BENJAMINI, O. SCHRAMM & Á. TIMÁR, “On the separation profile of infinite graphs”, *Groups Geom. Dyn.* **6** (2012), no. 4, p. 639-658.

- [8] M. BONK & O. SCHRAMM, “Embeddings of Gromov hyperbolic spaces”, *Geom. Funct. Anal.* **10** (2000), no. 2, p. 266-306.
- [9] J. BRIEUSSEL & T. ZHENG, “Speed of random walks, isoperimetry and compression of finitely generated groups”, *Ann. of Math. (2)* **193** (2021), no. 1, p. 1-105.
- [10] S. BUYALO, A. DRANISHNIKOV & V. SCHROEDER, “Embedding of hyperbolic groups into products of binary trees”, *Invent. Math.* **169** (2007), no. 1, p. 153-192.
- [11] S. BUYALO & V. SCHROEDER, *Elements of asymptotic geometry*, EMS Monographs in Mathematics, European Mathematical Society (EMS), Zürich, 2007, xii+200 pages.
- [12] TH. COULHON & L. SALOFF-COSTE, “Isopérimétrie pour les groupes et les variétés”, *Rev. Mat. Iberoamericana* **9** (1993), no. 2, p. 293-314.
- [13] A. ERSCHLER, “On isoperimetric profiles of finitely generated groups”, *Geom. Dedicata* **100** (2003), p. 157-171.
- [14] ———, “Piecewise automatic groups”, *Duke Math. J.* **134** (2006), no. 3, p. 591-613.
- [15] T. GENTIMIS, “Asymptotic dimension of finitely presented groups”, *Proc. Amer. Math. Soc.* **136** (2008), no. 12, p. 4103-4110.
- [16] L. R. GIBSON & M. PIVARSKI, “Isoperimetric profiles on the pre-fractal Sierpinski carpet”, *Fractals* **18** (2010), no. 4, p. 433-449.
- [17] V. GLADKOVA & V. SHUM, “Separation profiles of graphs of fractals”, <https://arxiv.org/abs/1810.08792>, 2018.
- [18] A. GOURNAY, “The Liouville property and Hilbertian compression”, *Ann. Inst. Fourier (Grenoble)* **66** (2016), no. 6, p. 2435-2454.
- [19] M. GROMOV, “Entropy and isoperimetry for linear and non-linear group actions”, *Groups Geom. Dyn.* **2** (2008), no. 4, p. 499-593.
- [20] P. DE LA HARPE, *Topics in geometric group theory*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2000, vi+310 pages.
- [21] D. HUME, “Direct embeddings of relatively hyperbolic groups with optimal ℓ^p compression exponent”, *J. Reine Angew. Math.* **703** (2015), p. 147-172.
- [22] ———, “A continuum of expanders”, *Fund. Math.* **238** (2017), no. 2, p. 143-152.
- [23] D. HUME & J. M. MACKAY, “Poorly connected groups”, *Proc. Amer. Math. Soc.* **148** (2020), no. 11, p. 4653-4664.
- [24] D. HUME, J. M. MACKAY & R. TESSERA, “Poincaré profiles of groups and spaces”, *Rev. Mat. Iberoam.* **36** (2020), no. 6, p. 1835-1886.
- [25] D. HUME & A. SISTO, “Groups with no coarse embeddings into hyperbolic groups”, *New York J. Math.* **23** (2017), p. 1657-1670.
- [26] P.-N. JOLISSAINT & T. PILLON, “ L^p compression of some HNN extensions”, *J. Group Theory* **16** (2013), no. 6, p. 907-913.
- [27] P.-N. JOLISSAINT & A. VALETTE, “ L^p -distortion and p -spectral gap of finite graphs”, *Bull. Lond. Math. Soc.* **46** (2014), no. 2, p. 329-341.
- [28] S. LI, “Compression bounds for wreath products”, *Proc. Amer. Math. Soc.* **138** (2010), no. 8, p. 2701-2714.
- [29] L. H. LOOMIS & H. WHITNEY, “An inequality related to the isoperimetric inequality”, *Bull. Amer. Math. Soc.* **55** (1949), p. 961-962.
- [30] A. NAOR & Y. PERES, “Embeddings of discrete groups and the speed of random walks”, *Int. Math. Res. Not. IMRN* (2008), article no. 076 (34 pages).
- [31] ———, “ L_p compression, traveling salesmen, and stable walks”, *Duke Math. J.* **157** (2011), no. 1, p. 53-108.
- [32] G. PETE, “A note on percolation on \mathbb{Z}^d : isoperimetric profile via exponential cluster repulsion”, *Electron. Commun. Probab.* **13** (2008), p. 377-392.

- [33] CH. PITTET & L. SALOFF-COSTE, “Random walks on finite rank solvable groups”, *J. Eur. Math. Soc. (JEMS)* **5** (2003), no. 4, p. 313-342.
- [34] CH. PITTET & L. SALOFF-COSTE, “Amenable groups, isoperimetric profiles and random walks”, in *Geometric group theory down under (Canberra, 1996)*, de Gruyter, Berlin, 1999, p. 293-316.
- [35] ———, “A survey on the relationships between volume growth, isoperimetry, and the behavior of simple random walk on Cayley graphs, with examples”, <https://math.unice.fr/~indira/papers/surveyPS.pdf>, 2001.
- [36] A. W. SALE, “Metric behaviour of the Magnus embedding”, *Geom. Dedicata* **176** (2015), p. 305-313.
- [37] L. SALOFF-COSTE & T. ZHENG, “Random walks and isoperimetric profiles under moment conditions”, *Ann. Probab.* **44** (2016), no. 6, p. 4133-4183.
- [38] ———, “Isoperimetric profiles and random walks on some permutation wreath products”, *Rev. Mat. Iberoam.* **34** (2018), no. 2, p. 481-540.
- [39] R. TESSERA, “Asymptotic isoperimetry on groups and uniform embeddings into Banach spaces”, *Comment. Math. Helv.* **86** (2011), no. 3, p. 499-535.
- [40] ———, “Isoperimetric profile and random walks on locally compact solvable groups”, *Rev. Mat. Iberoam.* **29** (2013), no. 2, p. 715-737.

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