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THE RELATIVE CANONICAL IDEAL OF THE KUMMER–ARTIN SCHREIER–WITT FAMILY OF CURVES

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Dedicated to Prof. Jannis A. Antoniadis on the occasion of his 70th birthday.

Abstract. — We study the canonical model of the Kummer–Artin Schreier–Witt flat family of curves over a ring of mixed characteristic. We first prove the relative version of a classical theorem by Petri, then use the model proposed by Bertin–Mézard to construct an explicit generating set for the relative canonical ideal. As a byproduct, we obtain a combinatorial criterion for a set to generate the canonical ideal, applicable to any curve satisfying the assumptions of Petri's theorem, except for plane quintics and trigonal curves.

Résumé. — Nous étudions le modèle canonique de la famille de courbes plate de Kummer–Artin Schreier–Witt sur un anneau de caractéristique mixte. Nous prouvons d'abord la version relative d'un théorème classique de Petri, puis utilisons le modèle proposé par Bertin–Mézard afin de construire un ensemble de générateurs explicite de l'idéal canonique relatif. De plus, nous obtenons un critère combinatoire pour qu'un ensemble engendre l'idéal canonique, applicable à toute courbe satisfaisante les hypothèses du théorème de Petri, á l'exception des planes quintiques et des courbes trigonales.

1. Introduction

1.1. The canonical ideal

Let X be a complete, non-singular, non-hyperelliptic curve of genus $g \geqslant 3$ over an algebraically closed field F of arbitrary characteristic. Let $\Omega_{X/F}$ denote the sheaf of holomorphic differentials on X and, for $n \geqslant 0$, let $\Omega_{X/F}^{\otimes n}$

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denote the n-th tensor power of $\Omega_{X/F}$. The following classical result is usually referred to in the bibliography as *Petri's Theorem*, even though it is due to Max Noether, Enriques and Babbage as well:

Theorem 1.1.

(1) *The canonical map*

$$\phi : \mathrm{Sym}(H^0(X, \Omega_{X/F})) \to \bigoplus_{n\geqslant 0} H^0(X, \Omega_{X/F}^{\otimes n})$$

is surjective.

(2) *The kernel I_X of ϕ is generated by elements of degree 2 and 3.*

(3) *I_X is generated by elements of degree 2 except in the following cases:*

  (a) *X is a non-singular plane quintic (in this case $g = 6$).*

  (b) *X is trigonal, i.e. a triple covering of $\mathbb{P}^1_F$.*

The standard terminology for the algebro-geometric objects relevant to Petri's Theorem uses the adjective *canonical*: the sheaf $\Omega_{X/F}$ is the *canonical bundle*, the ring $\bigoplus_{n\geqslant 0} H^0(X, \Omega_{X/F}^{\otimes n})$ is the *canonical ring*, the map ϕ is the *canonical map* and the kernel $I_X = \ker \phi$ is the *canonical ideal*. More details on the canonical map will be given in section 2; for a modern treatment over a field of arbitrary characteristic we refer to the article of B. Saint-Donat [20].

The problem of determining explicit generators for the canonical ideal has attracted interest by researchers over the years. A non-exhaustive list of techniques employed includes the use of Weierstrass semigroups [17], the theory of Gröbner bases [2], minimal free resolutions and syzygies [1]. The latter are also central to Green's conjecture, solved by Voisin in [28]. The purpose of this paper is to study Petri's Theorem in the context of *lifts of curves* as discussed below.

1.2. Lifts of curves

Let k be a field of prime characteristic $p > 0$. *A lift of k to characteristic 0* is the field of fractions L of any integral extension of the ring of Witt vectors $W(k)$, a classical construction by Witt [29] that generalizes the p-adic integers $\mathbb{Z}_p = W(\mathbb{F}_p)$. In what follows the field k will be assumed to be algebraically closed. Note that integral extensions of $W(k)$ are discrete valuation rings of mixed characteristic, with residue field k.

Consider a projective, non-singular curve $\mathcal{X}_0$ over k and let R be an integral extension of $W(k)$. *A lift of $\mathcal{X}_0/k$ to characteristic* 0, is a curve $\mathcal{X}_\eta$ over $L = \mathrm{Quot}R$, obtained as the generic fibre of a flat family of curves $\mathcal{X}/R$ whose special fibre is $\mathcal{X}_0/k$. Such lifts have been extensively used by arithmetic geometers to reduce characteristic p problems to the, much better understood, characteristic 0 case. One of the earliest uses of the idea of lifting is the approach of J.P. Serre [24] in an early attempt to define an appropriate cohomology theory which could solve the Weil conjectures. The lifting of an algebraic variety to characteristic zero is unfortunately not always possible and Serre was able to give such an example, see [25]. The progress made in deformation theory by Schlessinger [21] identified the lifting obstruction as an element in $H^2(X, T_X)$, see [23, 1.2.12], [11, 5.7 p.41].

1.3. Lifts of curves with automorphisms

Let $\mathcal{X}_0/k$ be a projective, non-singular curve as in the previous section. Such a curve can always lifted in characteristic zero, since the obstruction lives in the second cohomology which is always zero for curves. However, one might ask if it is possible to deform the curve together with its automorphism group, see [5]. This is not always possible, since Hurwitz's bound for the order of automorphism groups in characteristic 0 ensures that the answer for a general group G is negative, see [8, 15]. In the same spirit, J. Bertin in [3] provided an obstruction for the lifting based on the Artin representation which vanishes for cyclic groups. Note that, even in positive characteristic, the order of cyclic automorphism groups is bounded by the classical Hurwitz bound, see [14]. The existence of such a lift for cyclic p-groups was conjectured by Oort in [18] and was laid to rest three decades later by Obus–Wewers [16] and Pop [19].

In the meantime, the case for $G = \mathbb{Z}/p\mathbb{Z}$ was studied by Oort himself and Sekiguchi–Suwa [22, 27], who unified the theory of cyclic extensions of the projective line in characteristic p (*Artin–Schreier extensions*) and that of cyclic extensions of the projective line in characteristic 0 (*Kummer extensions*). The unified theory is usually referred to as *Kummer–Artin Schreier–Witt* theory or *Oort–Sekiguchi-Suwa* (OSS) theory. Using these results, Bertin–Mézard in [5] provided an explicit description of the affine model for the Kummer curve in terms of the affine model for the Artin–Schreier curve. Following this construction, Karanikolopoulos and the third author in [13] proposed the study of the Galois module structure of the relative curve $\mathcal{X}/R$. As a byproduct, they found an explicit basis of the

R-module of relative holomorphic differentials $H^0(\mathcal{X}, \Omega_{\mathcal{X}})$, using Boseck's work [6] on holomorphic differentials.

The main result of this paper is the determination of an explicit generating set for the relative canonical ideal of the unified Kummer–Artin Schreier–Witt theory, using the Bertin–Mézard model and the relative basis of [13] for 1-differentials. We conclude the introduction by giving an outline of our arguments and techniques.

1.4. Outline

In Section 2 we give details on the canonical map and we prove a combinatorial criterion for a subset of the canonical ideal to be a generating set. The main result of this section is Proposition 2.2 where we prove that to check if a set G of homogeneous polynomials of degree 2 generates the canonical ideal, it suffices to check whether $\dim_F (S/\langle \mathrm{in}_\prec(G)\rangle)_2 \leqslant 3(g-1)$. The above criterion reduces the problem of finding a generating set for the canonical ideal to counting initial terms; we note that the criterion is applicable to any curve satisfying the assumptions of Petri's theorem, with the exception of plane quintics and trigonal curves.

In Section 3 we formalize the lifting problem for the canonical ideal of the relative curve. First, we review the results of Bertin–Mézard on the explicit construction of the relative curve $\mathcal{X}/R$. Then, in Theorem 3.1, we define the relative canonical map and prove an analogue of Petri's Theorem for the relative curve $\mathcal{X}/R$, by constructing a diagram

$$\begin{tikzcd}
0 \arrow[r] & I_{\mathcal{X}_\eta} \arrow[r, hook] & S_L := L[\omega_1, \dots, \omega_g] \arrow[r, two heads, "\phi_\eta"] & \bigoplus_{n=0}^{\infty} H^0(\mathcal{X}_\eta, \Omega^{\otimes n}_{\mathcal{X}_\eta/L}) \arrow[r] & 0 \\
0 \arrow[r] & I_{\mathcal{X}} \arrow[r, hook] \arrow[u, hook, "\otimes_R L"] \arrow[d, two heads, "\otimes_R R/\mathfrak{m}"] & S_R := R[W_1, \dots, W_g] \arrow[r, two heads, "\phi"] \arrow[u, hook, "\otimes_R L"] \arrow[d, two heads, "\otimes_R R/\mathfrak{m}"] & \bigoplus_{n=0}^{\infty} H^0(\mathcal{X}, \Omega^{\otimes n}_{\mathcal{X}/R}) \arrow[r] \arrow[u, hook, "\otimes_R L"] \arrow[d, two heads, "\otimes_R R/\mathfrak{m}"] & 0 \\
0 \arrow[r] & I_{\mathcal{X}_0} \arrow[r, hook] & S_k := k[w_1, \dots, w_g] \arrow[r, two heads, "\phi_0"] & \bigoplus_{n=0}^{\infty} H^0(\mathcal{X}_0, \Omega^{\otimes n}_{\mathcal{X}_0/k}) \arrow[r] & 0
\end{tikzcd}$$

whose rows are exact and where each square is commutative. In Theorem 3.2, we give a Nakayama-type criterion that reduces the problem of finding a generating set for the relative canonical ideal $I_{\mathcal{X}}$ to finding compatible generating sets for the canonical ideals on the two fibres. In short, we prove that if G is a set of homogeneous polynomials in $I_{\mathcal{X}}$ such that $G \otimes_R L$ generates $I_{\mathcal{X}_\eta}$ and $G \otimes_R k$ generates $I_{\mathcal{X}_0}$ then G generates $I_{\mathcal{X}}$.

In Section 4 we state and prove results on the generators of the canonical ideal which are common for the two fibres. To facilitate the counting, we set a correspondence between the variables of the polynomial ring in Petri's Theorem and a discrete set of points $A \subseteq \mathbb{Z}^2$. In Proposition 4.2 we find a binomial ideal contained in the canonical ideal, leading us to build the generating sets for the two fibres on sets of binomials. Further, in Proposition 4.6, we extend the correspondence between the variables and the set A to a correspondence between the binomials and the Minkowski sum $A+A$, see [30, p. 28]. The cardinality of the Minkowski sum turns out to be too big, and we thus devote Section 4.3 to identify and study subsets of $A+A$ whose cardinalities are bounded by $3(g-1)$.

It turns out that these subsets of the Minkowski sum match exactly to the missing generators for the canonical ideals of the two fibers. These are studied in Section 5, which contains the main result of this paper, Theorem 5.5: The generators of the canonical ideal of the relative curve are either binomials of the form

$$W_{N_1,\mu_1} W_{N_1',\mu_1'} - W_{N_2,\mu_2} W_{N_2',\mu_2'}$$

or polynomials of the form

$$W_{N,\mu} W_{N',\mu'} - W_{N'',\mu} W_{N''',\mu'''} + \sum_{i=1}^{p-1} \sum_{j=j_{\min}(i)}^{(p-i)q} \lambda^{i-p} \binom{p}{i} c_{j,p-i} W_{N_j,\mu_i} W_{N_j',\mu_i'}.$$

The reader will have to refer to Section 5 for the details on the indices of the variables and the coefficients. For the proof of Theorem 5.5 we make essential use of our Nakayama-type Theorem 3.2 and Theorem 3.1, our analogue to Petri's Theorem, as reduction and thickening – à la Faltings [7] – are checked on the category of vector spaces, instead of the category of rings. To demonstrate our results, we use as a running example a genus 12 Kummer curve, see Examples 4.10, 4.13 and 5.4 .

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2. A criterion for generators of the canonical ideal

Throughout this section, X is a complete, non-singular, non-hyperelliptic curve of genus $g \geqslant 3$ over an algebraically closed field F of arbitrary characteristic, which is neither a plane quintic nor trigonal. As in the introduction, let $\Omega_{X/F}$ denote the sheaf of holomorphic differentials on X and, for $n \geqslant 1$, let $\Omega_{X/F}^{\otimes n}$ be the n-th tensor power of $\Omega_{X/F}$; its global sections $H^0(X, \Omega_{X/F}^{\otimes n})$ form an F-vector space of dimension $d_{n,g}$ where

$$d_{n,g} = \begin{cases} g, & \text{if } n = 1 \\ (2n-1)(g-1), & \text{if } n > 1. \end{cases} \tag{2.1}$$

The direct sum of the F-vector spaces $H^0(X, \Omega_{X/F}^{\otimes n})$ is equipped with the structure of a graded ring: multiplication in $\bigoplus_{n \geqslant 0} H^0(X, \Omega_{X/F}^{\otimes n})$ is defined via

$$\begin{aligned} H^0(X, \Omega_{X/F}^{\otimes n}) \times H^0(X, \Omega_{X/F}^{\otimes m}) &\to H^0(X, \Omega_{X/F}^{\otimes (n+m)}) \\ f dx^{\otimes n} \cdot g dx^{\otimes m} &\mapsto f g dx^{\otimes (n+m)}. \end{aligned}$$

Choosing coordinates $\omega_1, \dots, \omega_g$ for $\mathbb{P}_F^{g-1}$ one can identify the symmetric algebra $\mathrm{Sym}(H^0(X, \Omega_{X/F}))$ of Petri's Theorem with the graded polynomial ring $S := F[\omega_1, \dots, \omega_g]$ and we have that

$$S = \bigoplus_{n \geqslant 0} S_n \text{ where } S_n = \{f \in S : \deg f = n\}. \tag{2.2}$$

Choosing a basis $\mathbf{v} = \{f_1 dx, \dots, f_g dx\}$ for $H^0(X, \Omega_{X/F})$ allows us to extend the assignment $\omega_i \mapsto f_i dx$ and define a homogeneous map of graded rings

$$\begin{aligned} \phi : F[\omega_1, \dots, \omega_g] &\to \bigoplus_{n \geqslant 0} H^0(X, \Omega_{X/F}^{\otimes n}) \\ \omega_1^{a_1} \cdots \omega_g^{a_g} &\mapsto f_1^{a_1} \cdots f_g^{a_g} dx^{\otimes (a_1 + \cdots a_g)}. \end{aligned}$$

Note that when an emphasis on the basis $\mathbf{v}$ is desired, the map ϕ will be denoted by $\phi_{\mathbf{v}}$. The kernel of ϕ, denoted by I_X, is a graded ideal, so that in analogy to eq. (2.2) we may write

$$I_X = \bigoplus_{n \geqslant 0} (I_X)_n \text{ where } (I_X)_n = \{f \in I_X : \deg f = n\}.$$

In the context we are working, Petri's Theorem can be rewritten as follows:

Theorem 2.1. — *The canonical map ϕ is surjective and $I_X = \langle (I_X)_2 \rangle$.*

We fix a term order $\prec$ and note that each $f \in S$ has a unique leading term with respect to $\prec$, denoted by $\mathrm{in}_\prec(f)$. We define the initial ideal of I_X as $\mathrm{in}_\prec(I_X) = \langle \mathrm{in}_\prec(f) : f \in I_X \rangle$. If S_n, $(I_X)_n$ and $\mathrm{in}_\prec(I_X)_n$ are the $n-th$ graded pieces of S, I_X and $\mathrm{in}_\prec(I_X)$ respectively, then both $(I_X)_n$ and $\mathrm{in}_\prec(I_X)_n$ are F-subspaces of S_n and, since quotients commute with direct sums, we have that

$$(S/I)_n \cong S_n/I_n \text{ and } (S/\mathrm{in}_\prec(I))_n \cong S_n/\mathrm{in}_\prec(I)_n.$$

The proposition below gives a criterion for a subset of the canonical ideal to be a generating set:

Proposition 2.2. — *Let $G \subseteq I_X$ be a set of homogeneous polynomials of degree 2 in I_X. If*

$$\dim_F (S/\langle \mathrm{in}_\prec(G)\rangle)_2 \leqslant 3(g-1),$$

then $I_X = \langle G \rangle$.

Proof. — We note that since $G \subseteq I_X$, $\langle \mathrm{in}_\prec(G)\rangle_2$ is a subspace of $\mathrm{in}_\prec(I_X)_2$. Therefore

$$\begin{aligned}(2.3)\quad \dim_F (S/\mathrm{in}_\prec(I_X))_2 &= \dim_F S_2/\mathrm{in}_\prec(I_X)_2 \\ &\leqslant \dim_F S_2/\langle \mathrm{in}_\prec(G)\rangle_2 = \dim_F (S/\langle \mathrm{in}_\prec(G)\rangle)_2 .\end{aligned}$$

Moreover, by [26, Prop. 1.1]

$$\begin{aligned}(2.4)\qquad \dim_F (S/\mathrm{in}_\prec(I_X))_2 &= \dim_F (S/I_X)_2 \text{ and} \\ \dim_F (S/\langle \mathrm{in}_\prec(G)\rangle)_2 &= \dim_F (S/\langle G \rangle)_2 .\end{aligned}$$

By Petri's Theorem and eq. (2.1), we have that

$$(2.5)\qquad \dim_F (S/I_X)_2 = \dim_F H^0(X, \Omega_{X/F}^{\otimes 2}) = 3(g-1).$$

Combining eq. (2.3), (2.4), (2.5), and the hypothesis $\dim_F (S/\langle \mathrm{in}_\prec(G)\rangle)_2 \leqslant 3(g-1)$ gives

$$\dim_F \left(S/I_X\right)_2 = \dim_F \left(S/\langle G\rangle\right)_2 \Rightarrow (I_X)_2 = \langle G\rangle_2 \Rightarrow I_X = \langle (I_X)_2\rangle = \langle G\rangle$$

completing the proof. □

3. The canonical ideal of relative curves

Let k be an algebraically closed field of prime characteristic $\mathrm{char}(k) = p > 0$. Denote by $W(k)[\zeta]$ the ring of Witt vectors over k extended by a p-th root of unity ζ and let $\lambda = \zeta - 1$. By [12, Sec. 8.10] $W(k)[\zeta]$ is a discrete valuation ring with maximal ideal $\mathfrak{m}$ and residue field isomorphic to k. Let

$m \geqslant 1$ be a natural number not divisible by p; for any $1 \leqslant \ell \leqslant p-1$ we write $m = pq - \ell$ and consider, as in [13, Sec. 3], the local ring

$$R = \begin{cases} W(k)[\zeta][\![x_1, \dots, x_q]\!] & \text{if } \ell = 1 \\ W(k)[\zeta][\![x_1, \dots, x_{q-1}]\!] & \text{if } \ell \neq 1 \end{cases}$$

with maximal ideal $\mathfrak{m}_R = \langle \mathfrak{m}, \{x_i\} \rangle$. We write

$$K = \mathrm{Quot}\,(R/\mathfrak{m}) = \begin{cases} \mathrm{Quot}\,(k[\![x_1, \dots, x_q]\!]) & \text{if } \ell = 1 \\ \mathrm{Quot}\,(k[\![x_1, \dots, x_{q-1}]\!]) & \text{if } \ell \neq 1 \end{cases}$$

and consider the extension of the rational function field $K(x)$ given by $\mathcal{X}_0 : X^p - X = \frac{x^\ell}{a(x)^p}$, where

$$a(x) = \begin{cases} x^q + x_1 x^{q-1} + \dots + + x_{q-1} x + x_q & \text{if } \ell = 1 \\ x^q + x_1 x^{q-1} + \dots + x_{q-1} x & \text{if } \ell \neq 1. \end{cases} \tag{3.1}$$

Bertin–Mézard proved in [4, Sec. 4.3] that the curve above lifts to a curve over $L = \mathrm{Quot}(R)$ given by $\mathcal{X}_\eta : y^p = \lambda^p x^\ell + a(x)^p$ for $y = a(x)(\lambda X + 1)$, which is the normalization of $R[x]$ in $L(y)$. This gives rise to a family $\mathcal{X} \to \mathrm{Spec}(R)$, with special fibre $\mathcal{X}_0$ and generic fibre $\mathcal{X}_\eta$:

$$\begin{tikzcd} \mathrm{Spec}(k) \times_{\mathrm{Spec}(R)} \mathcal{X} = \mathcal{X}_0 \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d] & \mathcal{X}_\eta = \mathrm{Spec}(L) \times_{\mathrm{Spec}(R)} \mathcal{X} \arrow[l, hook'] \arrow[d] \\ \mathrm{Spec}(k) \arrow[r, hook] & \mathrm{Spec}(R) & \mathrm{Spec}(L) \arrow[l, hook'] \end{tikzcd} \tag{3.2}$$

For $n \geqslant 1$, we write $\Omega_{\mathcal{X}/R}^{\otimes n}$ for the sheaf of holomorphic polydifferentials on $\mathcal{X}$. We remark that since $H^0(\mathcal{X}, \Omega_{\mathcal{X}/R}^{\otimes n}) \otimes_R k \cong H^0(\mathcal{X}_0, \Omega_{\mathcal{X}_0/k}^{\otimes n})$ and $H^0(\mathcal{X}, \Omega_{\mathcal{X}/R}^{\otimes n}) \otimes_R L \cong H^0(\mathcal{X}_\eta, \Omega_{\mathcal{X}_\eta/L}^{\otimes n})$, by [10, Lemma II.8.9] the R-modules $H^0(\mathcal{X}, \Omega_{\mathcal{X}/R}^{\otimes n})$ are free of rank $d_{n,g}$ for all $n \geqslant 1$, with $d_{n,g}$ given by eq. (2.1). We select generators $W_1, \dots, W_g$ for the symmetric algebra $\mathrm{Sym}(H^0(\mathcal{X}, \Omega_{\mathcal{X}/R}))$ and identify it with the polynomial ring $R[W_1, \dots, W_g]$. Similarly, we identify the symmetric algebras $\mathrm{Sym}(H^0(\mathcal{X}_\eta, \Omega_{\mathcal{X}_\eta/L}))$ and $\mathrm{Sym}(H^0(\mathcal{X}_0, \Omega_{\mathcal{X}_0/k}))$ with the polynomial rings $L[\omega_1, \dots, \omega_g]$, $k[w_1, \dots, w_g]$ respectively. Our next result concerns the canonical embedding of the Bertin–Mézard family:

Theorem 3.1. — *Diagram* (3.2) *induces a deformation-theoretic diagram of canonical embeddings*

(3.3)

$$\begin{tikzcd}
0 \arrow[r] & I_{\mathcal{X}_\eta} \arrow[r, hook] & S_L := L[\omega_1,\dots,\omega_g] \arrow[r, two heads, "\phi_\eta"] & \bigoplus_{n=0}^{\infty} H^0(\mathcal{X}_\eta, \Omega_{\mathcal{X}_\eta/L}^{\otimes n}) \arrow[r] & 0 \\
0 \arrow[r] & I_{\mathcal{X}} \arrow[r, hook] \arrow[u, hook, "\otimes_R L"'] \arrow[d, two heads, "\otimes_R R/\mathfrak{m}"] & S_R := R[W_1,\dots,W_g] \arrow[r, two heads, "\phi"] \arrow[u, hook, "\otimes_R L"'] \arrow[d, two heads, "\otimes_R R/\mathfrak{m}"] & \bigoplus_{n=0}^{\infty} H^0(\mathcal{X}, \Omega_{\mathcal{X}/R}^{\otimes n}) \arrow[r] \arrow[u, hook, "\otimes_R L"'] \arrow[d, two heads, "\otimes_R R/\mathfrak{m}"] & 0 \\
0 \arrow[r] & I_{\mathcal{X}_0} \arrow[r, hook] & S_k := k[w_1,\dots,w_g] \arrow[r, two heads, "\phi_0"] & \bigoplus_{n=0}^{\infty} H^0(\mathcal{X}_0, \Omega_{\mathcal{X}_0/k}^{\otimes n}) \arrow[r] & 0
\end{tikzcd}$$

where $I_{\mathcal{X}_\eta} = \ker \phi_\eta,\ I_{\mathcal{X}} = \ker \phi,\ I_{\mathcal{X}_0} = \ker \phi_0$, *each row is exact and each square is commutative.*

Proof. — Exactness of the top and bottom row of diagram (3.3) are due to Theorem 2.1, the classical result of Enriques, Petri and M. Noether. To define the map ϕ of the middle row, we choose generators $f_1 dx, \dots, f_g dx$ for $H^0(\mathcal{X}, \Omega_{\mathcal{X}/R})$ such that $f_i dx \otimes_R 1_{R/\mathfrak{m}_R} = \phi_0(w_i) \in H^0(\mathcal{X}_0, \Omega_{\mathcal{X}_0/k})$ (this is possible by Nakayama's Lemma) for $i = 1, \cdots, g$ and note that the assignment $W_i \mapsto f_i dx$ gives rise to a homogeneous homomorphism of graded rings

$$\phi : R[W_1,\dots,W_g] \xrightarrow{\phi} \bigoplus_{n=0}^{\infty} H^0(\mathcal{X}, \Omega_{\mathcal{X}/R}^{\otimes n}).$$

To prove surjectivity of ϕ, we write $\phi = \bigoplus_{n=0}^{\infty} (\phi)_n$, where for each $n \in \mathbb{N}$

$$(\phi)_n : R[W_1,\dots,W_g]_n \xrightarrow{\phi_n} H^0(\mathcal{X}, \Omega_{\mathcal{X}/R}^{\otimes n})$$

is a homomorphism of finitely generated R-modules. By construction, $(\phi)_n \otimes 1_{R/\mathfrak{m}_R} = (\phi_0)_n$ is surjective for all n. Nakayama's Lemma then implies that $(\phi)_n$ is surjective for all n, and thus ϕ is surjective as well. □

We proceed with establishing a Nakayama-type criterion for a subset of the kernel $I_{\mathcal{X}}$ to generate the relative canonical ideal:

Theorem 3.2. — *Let G be a set of homogeneous polynomials in $I_{\mathcal{X}}$ such that $G \otimes_R L$ generates $I_{\mathcal{X}_\eta}$ and $G \otimes_R k$ generates $I_{\mathcal{X}_0}$. Then:*

(1) *For any $n \in \mathbb{N}$, the R-modules $(S_R/\langle G\rangle)_n$ are free of rank $d_{n,g}$.*
(2) $I_{\mathcal{X}} = \langle G\rangle$.

Proof.

For (1). — Let $n \in \mathbb{N}$. Since by assumption $G \otimes_R L$ and $G \otimes_R k$ generate $I_{\mathcal{X}_\eta}$ and $I_{\mathcal{X}_0}$ respectively, we have that

$$(S_R/\langle G\rangle)_n \otimes_R L \cong \left(S_L/I_{\mathcal{X}_\eta}\right)_n \text{ and } (S_R/\langle G\rangle)_n \otimes_R k \cong \left(S_k/I_{\mathcal{X}_0}\right)_n.$$

By Petri's Theorem 2.1 we get that

$$\left(S_L/I_{\mathcal{X}_\eta}\right)_n \cong H^0(\mathcal{X}_\eta, \Omega^{\otimes n}_{\mathcal{X}_\eta/L}) \text{ and } \left(S_k/I_{\mathcal{X}_0}\right)_n \cong H^0(\mathcal{X}_0, \Omega^{\otimes n}_{\mathcal{X}_0/k})$$

and by eq. (2.1)

$$\dim_L \left(S_L/I_{\mathcal{X}_\eta}\right)_n = \dim_k \left(S_k/I_{\mathcal{X}_0}\right)_n = d_{n,g}.$$

The result follows from [10, lemma II.8.9].

For (2). — let $s \in I_{\mathcal{X}}$ and assume for contradiction that $s \notin \langle G\rangle$. Since $s \otimes 1_L \in I_{\mathcal{X}_\eta}$ and $G \otimes_R L$ generates $I_{\mathcal{X}_\eta}$, there exist $g_i \in G$ and $s_i \in S_L$ such that $s \otimes 1_L = \sum s_i(g_i \otimes 1_L)$. Choosing $d \in R$ to be the gcd of the denominators of the coefficients of the s_i, we may clear denominators to obtain $ds \otimes 1_L = \sum ds_i(g_i \otimes 1_L)$, with $ds_i \in S_R$ or equivalently $ds = \sum ds_i g_i$ with $ds_i \in S_R$, implying that $ds \in \langle G\rangle$. If $s \notin \langle G\rangle$, then s is a torsion element of $S_R/\langle G\rangle$, with its homogeneous components being torsion elements of the R-modules $(S_R/\langle G\rangle)_n$ for some $n \in \mathbb{N}$. By (i), the latter are free R-modules, so we conclude that if $s \notin \langle G\rangle$ then s must be zero, completing the proof. ☐

Theorem 3.2 reduces the problem of determining the generating set of the relative canonical ideal to determining compatible generating sets for the canonical ideals of the two fibers. Thus, in the next section we study the canonical embeddings of the two fibers, while compatibility is studied in Section 5.

4. The Canonical Embedding of the Two Fibers

The family's generic fibre, given by $\mathcal{X}_\eta : y^p = \lambda^p x^\ell + a(x)^p$, for $y = a(x)(\lambda X + 1)$, is a cyclic ramified covering of the projective line and, by assumption, the order of the cyclic group is prime to the characteristic p. Boseck in [6] gives an explicit description of a basis for the global sections of holomorphic differentials of such covers. Following the notation of [13], Boseck's basis $\mathbf{b}$ for $H^0(\mathcal{X}_\eta, \Omega_{\mathcal{X}_\eta/L})$ is given by

$$\mathbf{b} = \left\{ x^N y^{-\mu} dx : \left\lfloor \frac{\mu \ell}{p} \right\rfloor \leqslant N \leqslant \mu q - 2,\ 1 \leqslant \mu \leqslant p-1 \right\}. \tag{4.1}$$

Using this analysis, the authors of [13] found an explicit basis for the global sections of holomorphic differentials on the special fibre, compatible to $\mathbf{b}$ in the sense of Theorem 3.2. The basis $\overline{\mathbf{c}}$ for $H^0(\mathcal{X}_0, \Omega_{\mathcal{X}_0/k})$ is given by (see [13, eq. (25), p. 2381]):

$$\overline{\mathbf{c}} = \left\{ x^N a(x)^{p-1-\mu} X^{p-1-\mu} dx : \left\lfloor \frac{\mu \ell}{p} \right\rfloor \leqslant N \leqslant \mu q - 2,\ 1 \leqslant \mu \leqslant p-1 \right\}. \tag{4.2}$$

The elements of $\mathbf{b}$ and $\overline{\mathbf{c}}$ are determined by the values of (N, μ), so we proceed with the study of the respective index set.

4.1. The index set A and the corresponding multidegrees

Let

$$A = \left\{ (N, \mu) : \left\lfloor \frac{\mu \ell}{p} \right\rfloor \leqslant N \leqslant \mu q - 2,\ 1 \leqslant \mu \leqslant p-1 \right\} \subseteq \mathbb{N}^2. \tag{4.3}$$

and note that by [6, eq. (34) p. 48]

$$|A| = \sum_{\mu=1}^{p-1} \left(\mu q - \left\lfloor \frac{\mu \ell}{p} \right\rfloor - 1 \right) = g. \tag{4.4}$$

Let $\{z_{N,\mu} : (N, \mu) \in A\}$ be a set of variables indexed by A. To each variable $z_{N,\mu}$ we assign the multidegree $\mathrm{mdeg}(z_{N,\mu}) = (1, N, \mu) \in \mathbb{N}^3$. Thus, if $S = F[\{z_{N,\mu}\}]$ is the polynomial ring over F, by assigning the multidegree $(0, 0, 0)$ to the elements of F, we get a multigrading on S via

$$\begin{aligned} \mathrm{mdeg}(z_{N_1,\mu_1} z_{N_2,\mu_2} \cdots z_{N_d,\mu_d}) \\ = (d, N_1 + N_2 + \cdots N_d, \mu_1 + \mu_2 + \cdots + \mu_d). \end{aligned} \tag{4.5}$$

We will refer to the first coordinate of the multidegree (4.5) as the *standard degree*.

Next, we consider the two polynomial rings $L[\{\omega_{N,\mu}\}]$ and $k[\{w_{N,\mu}\}]$ with variables indexed by the points $(N, \mu) \in A$. The results of this subsection apply to both fibers, so we introduce the following notation: We will write X to refer to either curve $\mathcal{X}_\eta$ or $\mathcal{X}_0$, F to refer to either field L or k, $\{z_{N,\mu}\}$ to refer to either set of variables $\{\omega_{N,\mu}\}$ or $\{w_{N,\mu}\}$, $S = F[\{z_{N,\mu}\}]$ to refer to either polynomial ring $L[\{\omega_{N,\mu}\}]$ or $k[\{w_{N,\mu}\}]$ and $f_{N,\mu} dx$ to refer to the basis elements of either $\mathbf{b}$ or $\overline{\mathbf{c}}$. Note that the multiplication in the canonical ring in particular implies that for any two 1-differentials $f_{N,\mu}dx$, $f_{N',\mu'}dx$ we have $f_{N,\mu}dx \cdot f_{N',\mu'}dx = f_{N+N',\mu+\mu'}dx^{\otimes 2}$.

Definition 4.1. — *Let $\prec_t$ be the lexicographic order on the variables $\{z_{N,\mu} : (N,\mu) \in A\}$. We define a new term order $\prec$ on the monomials of S as follows:*

$$z_{N_1,\mu_1} z_{N_2,\mu_2} \cdots z_{N_d,\mu_d} \prec z_{N'_1,\mu'_1} z_{N'_2,\mu'_2} \cdots z_{N'_s,\mu'_s} \text{ if and only if}$$

(1) *$d < s$ or*
(2) *$d = s$ and $\sum \mu_i > \sum \mu'_i$ or*
(3) *$d = s$ and $\sum \mu_i = \sum \mu'_i$ and $\sum N_i < \sum N'_i$*
(4) *$d = s$ and $\sum \mu_i = \sum \mu'_i$ and $\sum N_i = \sum N'_i$ and*

$$z_{N_1,\mu_1} z_{N_2,\mu_2} \cdots z_{N_d,\mu_d} \prec_t z_{N'_1,\mu'_1} z_{N'_2,\mu'_2} \cdots z_{N'_s,\mu'_s}.$$

4.2. The binomial part of the canonical ideal

For each $n \in \mathbb{N}$ we write $\mathbb{T}^n$ for the set of monomials of degree n in S and observe that the binomials below are contained in I_X.

Proposition 4.2. — *Let $z_{N_1,\mu_1} z_{N'_1,\mu'_1}$, $z_{N_2,\mu_2} z_{N'_2,\mu'_2} \in \mathbb{T}^2$ be such that* $\mathrm{mdeg}(z_{N_1,\mu_1} z_{N'_1,\mu'_1}) = \mathrm{mdeg}(z_{N_2,\mu_2} z_{N'_2,\mu'_2})$*. Then $z_{N_1,\mu_1} z_{N'_1,\mu'_1} - z_{N_2,\mu_2} z_{N'_2,\mu'_2} \in I_X$.*

Proof. — Since $\mathrm{mdeg}(z_{N_1,\mu_1} z_{N'_1,\mu'_1}) = \mathrm{mdeg}(z_{N_2,\mu_2} z_{N'_2,\mu'_2})$, we have that $N_1 + N'_1 = N_2 + N'_2$ and $\mu_1 + \mu'_1 = \mu_2 + \mu'_2$, so

$$\begin{aligned}\phi(&z_{N_1,\mu_1} z_{N'_1,\mu'_1} - z_{N_2,\mu_2} z_{N'_2,\mu'_2}) \\ &= f_{N_1+N'_1,\, \mu_1+\mu'_1} dx^{\otimes 2} - f_{N_2+N'_2,\, \mu_2+\mu'_2} dx^{\otimes 2} = 0. \qquad \square\end{aligned}$$

We collect the binomials of Proposition 4.2 in the set below.

Definition 4.3. — *Let*

$$G_1 = \left\{ z_{N_1,\mu_1} z_{N'_1,\mu'_1} - z_{N_2,\mu_2} z_{N'_2,\mu'_2} \in S : \begin{array}{r} z_{N_1,\mu_1} z_{N'_1,\mu'_1}, z_{N_2,\mu_2} z_{N'_2,\mu'_2} \in \mathbb{T}^2 \\ \text{and } \mathrm{mdeg}(z_{N_1,\mu_1} z_{N'_1,\mu'_1}) \\ = \mathrm{mdeg}(z_{N_2,\mu_2} z_{N'_2,\mu'_2}) \end{array} \right\}.$$

Next, we consider the Minkowski sum of A with itself, defined as

$$A + A = \{(N + N', \mu + \mu') \ : \ (N,\mu), (N',\mu') \in A\} \subseteq \mathbb{Z}^2$$

and note the following correspondence between points of $A + A$ and monomials in $\mathbb{T}^2$:

Corollary 4.4. — $(\rho,T)\in A+A \Leftrightarrow \exists\, z_{N,\mu}z_{N',\mu'}\in\mathbb{T}^2$ *such that*

$$\mathrm{mdeg}(z_{N,\mu}z_{N',\mu'})=(2,\rho,T).$$

Proof. — Follows directly from the definition of A given in eq. (4.3), since

$$(N,\mu)\in A \Leftrightarrow \exists\, z_{N,\mu}\in F[\{z_{N,\mu}\}] \text{ such that } \mathrm{mdeg}(z_{N,\mu})=(1,N,\mu). \quad \square$$

The correspondence of Corollary 4.4 is not one-to-one: for any $(\rho,T)\in A+A$, we set

$$B_{\rho,T}:=\{z_{N,\mu}z_{N',\mu'}\in\mathbb{T}^2\ :\ (\rho,T)=(N+N',\mu+\mu')\}$$

and observe that the differences of elements of $B_{\rho,T}$ are in G_1. Next, we define the map of sets:

Definition 4.5.

$$\begin{aligned}\sigma: A+A &\to \mathbb{T}^2\\ (\rho,T)&\mapsto \min_{\prec} B_{\rho,T}.\end{aligned}$$

We will use the map σ to show that $A+A$ is in bijection with a standard basis of $(S/\langle \mathrm{in}_{\prec}(G_1)\rangle)_2$:

Proposition 4.6. — $|A+A|=\dim_F\,(S/\langle \mathrm{in}_{\prec}(G_1)\rangle)_2$

Proof. — Let $(\rho,T)\in A+A$. By Corollary 4.4, $B_{\rho,T}$ is non-empty and, since $\prec$ is a total order, it has a unique minimal element. Hence, the map σ is well-defined, $1-1$ and it is immediate that $\sigma(A+A)=\mathbb{T}^2\setminus \mathrm{in}_{\prec}(G_1)$. Since $\langle \mathrm{in}_{\prec}(G_1)\rangle$ is a monomial ideal generated in degree 2 we remark that $\dim_F\,(S/\langle \mathrm{in}_{\prec}(G_1)\rangle)_2=|\mathbb{T}^2\setminus \mathrm{in}_{\prec}(G_1)|$, completing the proof. $\square$

4.3. A subset of $A+A$ of cardinality $3(g-1)$

By Proposition 2.2 and Proposition 4.6, the binomials of Definition 4.3 would generate I_X if $|A+A|\leqslant 3(g-1)$. It turns out that this is not the case in general. Thus we need to identify an appropriate subset of $A+A$ whose points are in bijection with the monomials that do not appear as leading terms of the generators of I_X. To do so, we introduce and study appropriate subsets $C(i)\subseteq A+A$ for $0\leqslant i\leqslant p$, which we will use in Section 5 to give the generators for the relative canonical ideal (see remark after Definition 5.2).

We proceed with a description of $A+A$ in terms of bounding inequalities. To this end, we fix the second coordinate of a point $(\rho,T)\in A+A$ and determine the bounds of the first coordinate.

Definition 4.7. — *Let $T \in \mathbb{Z}$ such that $(\rho, T) \in A + A$. We define*

$$b(T) = \min\left\{ \left\lfloor \frac{\mu\ell}{p} \right\rfloor + \left\lfloor \frac{\mu'\ell}{p} \right\rfloor : \text{ all } \mu, \mu' \text{ s.t. } T = \mu + \mu' \text{ and } 1 \leqslant \mu, \mu' \leqslant p-1 \right\}$$

Remark 4.8. — The properties of the floor function imply that $b(T)$ takes one of the following values:

$$b(T) = \begin{cases} \left\lfloor \frac{T\ell}{p} \right\rfloor, & \text{if } \forall\, 1 \leqslant \mu, \mu' \leqslant p-1 \text{ with } T = \mu + \mu' \\ & \text{we have } \left\lfloor \frac{\mu\ell}{p} \right\rfloor + \left\lfloor \frac{\mu'\ell}{p} \right\rfloor = \left\lfloor \frac{T\ell}{p} \right\rfloor \\ \left\lfloor \frac{T\ell}{p} \right\rfloor - 1, & \text{if } \exists\, 1 \leqslant \mu, \mu' \leqslant p-1 \text{ with } T = \mu + \mu' \\ & \text{and } \left\lfloor \frac{\mu\ell}{p} \right\rfloor + \left\lfloor \frac{\mu'\ell}{p} \right\rfloor = \left\lfloor \frac{T\ell}{p} \right\rfloor - 1. \end{cases}$$

For example, $b(p) = \ell - 1$, since $p = 1 + (p-1)$ and $\left\lfloor \frac{\ell}{p} \right\rfloor + \left\lfloor \frac{(p-1)\ell}{p} \right\rfloor = \ell - 1$. Similarly, $b(2p-2) = 2\ell - 2$, since $2p - 2 = (p-1) + (p-1)$ and $\left\lfloor \frac{(p-1)\ell}{p} \right\rfloor + \left\lfloor \frac{(p-1)\ell}{p} \right\rfloor = 2\ell - 2$.

Definition 4.7 allows us to give an alternative description of $A + A$ which follows directly from the description of A given in eq. (4.3).

Lemma 4.9.

$$A + A = \{(\rho, T) : 2 \leqslant T \leqslant 2(p-1),\ b(T) \leqslant \rho \leqslant Tq - 4\} \subseteq \mathbb{N}^2.$$

Example 4.10. — Consider the genus 12 Kummer curve with affine model $\mathcal{X}_\eta : y^5 = \lambda^5 x^3 + (x^2 + x_1 x)^5$. The Minkowski sum

$$A + A = \{(\rho, T) : 2 \leqslant T \leqslant 8,\ b(T) \leqslant \rho \leqslant 2T - 4\}$$

is depicted in Figure 4.1 below.

The following auxiliary lemma will also be useful.

Lemma 4.11. — *If $2 \leqslant T \leqslant p-1$ and $0 \leqslant \alpha \leqslant p-1$, then $b(T+\alpha) \leqslant b(T) + \alpha$.*

Proof. — If $\alpha = 0$, the result follows trivially. If $1 \leqslant \alpha \leqslant p-1$, then by Definition 4.7 we can choose a decomposition $T + \alpha - 1 = \mu + \mu'$ satisfying $1 \leqslant \mu, \mu \leqslant p-1$ and $b(T+\alpha-1) = \left\lfloor \frac{\mu\ell}{p} \right\rfloor + \left\lfloor \frac{\mu'\ell}{p} \right\rfloor$. Since $T + \alpha - 1 \leqslant 2p - 3$, we may assume without loss of generality that $\mu \leqslant p-2$ and thus $T + \alpha$ can be decomposed as $T + \alpha = (\mu + 1) + \mu'$. We then obtain that

$$\begin{aligned} b(T+\alpha) &\leqslant \left\lfloor \frac{(\mu+1)\ell}{p} \right\rfloor + \left\lfloor \frac{\mu'\ell}{p} \right\rfloor \\ &= \left\lfloor \frac{(\mu+1)\ell}{p} \right\rfloor - \left\lfloor \frac{\mu\ell}{p} \right\rfloor + \left\lfloor \frac{\mu\ell}{p} \right\rfloor + \left\lfloor \frac{\mu'\ell}{p} \right\rfloor \leqslant 1 + b(T+\alpha-1). \end{aligned}$$

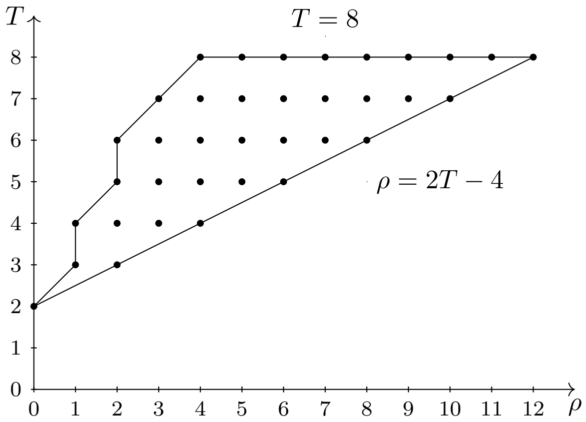

Figure 4.1. The set $A+A$ for $p=5,\ q=2,\ \ell=3$ corresponding to a genus 12 curve.

The result follows since

$$\begin{aligned} b(T+\alpha) &\leqslant b(T+\alpha-1)+1 \leqslant b(T+\alpha-2)+2 \\ &\leqslant \cdots \leqslant b(T+1)+\alpha-1 \leqslant b(T)+\alpha. \end{aligned}$$ □

We are ready to define the sets $C(i)$.

Definition 4.12. — *For $0 \leqslant i \leqslant p$ we let*

$$j_{\min}(i) = \begin{cases} 0, & \text{if } \ell = 1 \\ p-i, & \text{if } \ell \neq 1 \end{cases}$$

and define

$$C(i) = \left\{ (\rho,T) \in A+A : \begin{array}{l} (\rho+\ell, T+p) \text{ and } (\rho+j, T+p-i) \in A+A \\ \text{for } j_{\min}(i) \leqslant j \leqslant (p-i)q \end{array} \right\}.$$

Example 4.13. — For the genus 12 curve of Example 4.10, the red points in Figure 4.2 correspond to the set $C(0) = \{(0,2),(1,3),(2,3)\}$, which satisfies $|(A+A) \setminus C(0)| = 3(g-1) = 33$. In this particular case we have that $C(0) = C(1)$.

We note that a point $a = (\rho,T) \in C(i)$ determines the point $a_{-1} = (\rho+\ell, T+p) \in A+A$ as well as the collection of points $a_j = (\rho+j, T+p-i)$ of $A+A$ for $j_{\min}(i) \leqslant j \leqslant (p-i)q$.

Proposition 4.14. — $C(0) = \bigcap_{i=0}^{p} C(i)$.

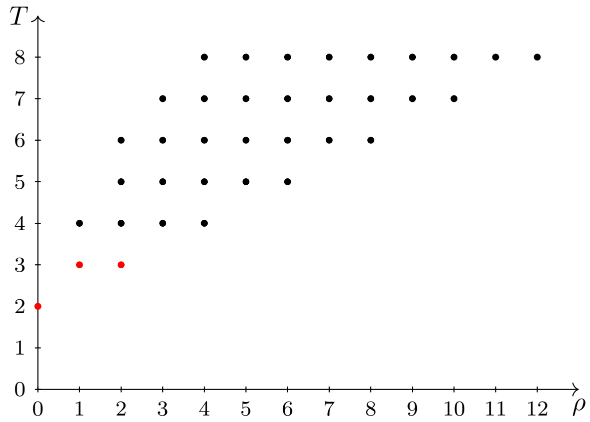

Figure 4.2. The red points correspond to $C(0) \subseteq A+A$ for $p=5,\ q=2,\ \ell=3$.

Proof. — It suffices to show that $C(0) \subseteq C(i)$ for $0 \leqslant i \leqslant p$. By Definition 4.12, this is equivalent to showing that if $(\rho, T) \in A+A$ and $(\rho+j, T+p) \in A+A$ for all $j_{\min}(0) \leqslant j \leqslant pq$ then $(\rho+j, T+p-i) \in A+A$ for all $j_{\min}(i) \leqslant j \leqslant (p-i)q$. First, we observe that

$$2 \leqslant T \leqslant T+p-i \leqslant p-2+p-i \leqslant 2(p-1)$$

and

$$\rho+j \leqslant \rho+(p-i)q \leqslant Tq-4+(p-i)q \leqslant (T+p-i)q-4.$$

For the lower bound of ρ, we distinguish the following cases:

- If $\ell=1$, then $j_{\min}(i)=0$. Since $(\rho,T) \in C(0)$, Definition 4.12 gives that $(\rho, T+p) \in C(0)$ and thus by Lemma 4.9 we get $b(T+p) \leqslant \rho$. We then have

$$b(T+p-i) \leqslant b(T+p) \leqslant \rho \leqslant \rho+j.$$

- If $\ell>1$ then $j_{\min}(i)=p-i$, and, by Lemma 4.11,

$$b(T+p-i) \leqslant b(T)+p-i \leqslant \rho+p-i \leqslant \rho+j.$$

We conclude that $2 \leqslant T \leqslant 2(p-1)$ and that $b(T+p-i) \leqslant \rho+j \leqslant (T+p-i)q-4$ for $j_{\min}(i) \leqslant j \leqslant (p-i)q$. Lemma 4.9 implies that $(\rho+j, T+p-i) \in A+A$, completing the proof. $\square$

We proceed with an auxiliary lemma to bound the cardinality of $(A+A) \setminus C(0)$.

Lemma 4.15. — *The cardinality of $C(0)$ satisfies*

$$|C(0)| > \sum_{T=2}^{p-2} (Tq - b(T) - 3) - \sum_{\substack{T=p-1 \\ b(T)=\left\lfloor \frac{T\ell}{p} \right\rfloor}}^{2p-2} 1.$$

Proof. — First we prove that the points of $C(0)$ satisfy the following bounding inequalities

$$C(0) = \{(\rho, T) \in A + A : M' \leqslant \rho \leqslant Tq - 4,\ 2 \leqslant T \leqslant p-2\}, \tag{4.6}$$

where

$$M' = \begin{cases} b(T), & \text{if } b(T+p) \leqslant b(T) + \ell \\ b(T)+1, & \text{if } b(T) + \ell < b(T+p). \end{cases}$$

Indeed, by definition, for all $j_{\min}(0) \leqslant j \leqslant pq$ we have that

$$\begin{aligned}(\rho, T) \in C(0) \Leftrightarrow &\ (\rho, T) \in A + A,\ (\rho + \ell, T + p) \in A + A \\ &\text{and } (\rho + j, T + p) \in A + A \\ \Leftrightarrow &\ 2 \leqslant T \leqslant p - 2 \text{ and } M \leqslant \rho \leqslant Tq - 4 \text{ (by Lemma 4.9)}\end{aligned}$$

where $M := \max\{b(T), b(T+p) - \ell, b(T+p) - j_{\min}(0)\}$. We will show that $M = M'$:

- If $\ell = 1$ then $j_{\min}(0) = 0$ and $b(T) = b(T+p) = 0$ since $\left\lfloor \frac{\mu\ell}{p} \right\rfloor = 0$ for all $1 \leqslant \mu \leqslant p - 1$. Hence $M = b(T) = 0$.
- If $\ell > 1$ then $j_{\min}(0) = p$, so $b(T+p) - j_{\min}(0) < b(T+p) - \ell$ and thus

  $$M = \max\{b(T), b(T+p) - \ell\}.$$

  If $b(T+p) \leqslant b(T)+\ell$, then $M = b(T)$, whereas if $b(T)+\ell < b(T+p)$, then it easily follows that

  $$b(T) = \left\lfloor \frac{T\ell}{p} \right\rfloor - 1 \text{ and } b(T+p) = \left\lfloor \frac{(T+p)\ell}{p} \right\rfloor = \left\lfloor \frac{T\ell}{p} \right\rfloor + \ell$$

  and so $M = b(T+p) - \ell = b(T) + 1$.

We thus have that $M = M'$, which completes the proof of eq. (4.6). We then have that

$$|C(0)| = \sum_{T=2}^{p-2} (Tq - M - 3) = \sum_{T=2}^{p-2} (Tq - b(T) - 3) - \sum_{\substack{T=2 \\ b(T)+\ell<b(T+p)}}^{p-2} 1.$$

When $2 \leqslant T \leqslant p-2$, the condition $b(T)+\ell < b(T+p)$ implies that

$$b(T+p) = \left\lfloor \frac{(T+p)\ell}{p} \right\rfloor .$$

Therefore

$$\sum_{\substack{T=2 \\ b(T)+\ell<b(T+p)}}^{p-2} 1 \leqslant \sum_{\substack{T=p+2 \\ b(T)=\lfloor \frac{T\ell}{p} \rfloor}}^{2p-2} 1 < \sum_{\substack{T=p-1 \\ b(T)=\lfloor \frac{T\ell}{p} \rfloor}}^{2p-2} 1$$

where the last inequality is strict by Remark 4.8. □

Finally, we show that the cardinality of $(A+A) \setminus C(0)$ is bounded by $3(g-1)$.

Lemma 4.16. — $|(A+A) \setminus C(0)| \leqslant 3(g-1)$.

Proof. — We successively have

$$\begin{aligned}
(4.7) \quad & |(A+A) \setminus C(0)| \\
&= \sum_{T=2}^{2(p-1)} (Tq - b(T) - 3) - |C(0)|, \text{ by Lemma 4.9} \\
&< \sum_{T=p-1}^{2(p-1)} (Tq - b(T) - 3) + \sum_{\substack{T=p-1 \\ b(T)=\lfloor \frac{T\ell}{p} \rfloor}}^{2(p-1)} 1, \text{ by Lemma 4.15} \\
&= \sum_{T=p-1}^{2(p-1)} \left(Tq - \left\lfloor \frac{T\ell}{p} \right\rfloor - 2\right) - \sum_{\substack{T=p-1 \\ b(T)=\lfloor \frac{T\ell}{p} \rfloor}}^{2(p-1)} 1 + \sum_{\substack{T=p-1 \\ b(T)=\lfloor \frac{T\ell}{p} \rfloor}}^{2(p-1)} 1, \text{ by Remark 4.8} \\
&= \sum_{T=p+1}^{2(p-1)} \left(Tq - \left\lfloor \frac{T\ell}{p} \right\rfloor - 2\right) + \left((p-1)q - \left\lfloor \frac{(p-1)\ell}{p} \right\rfloor - 2\right) \\
&\qquad + \left(pq - \left\lfloor \frac{p\ell}{p} \right\rfloor - 2\right).
\end{aligned}$$

By eq. (4.4), we have that

$$\sum_{T=1}^{p-1} \left(Tq - \left\lfloor \frac{T\ell}{p} \right\rfloor - 1\right) = g \qquad (4.8)$$

so we change the index in the sum of eq. (4.7) by setting $T' = T - p$:

$$\begin{aligned}(4.9)\quad \sum_{T=p+1}^{2(p-1)} \left(Tq - \left\lfloor \frac{T\ell}{p} \right\rfloor - 2\right) &= \sum_{T'=1}^{p-2} \left((T'+p)q - \left\lfloor \frac{(T'+p)\ell}{p} \right\rfloor - 2\right) \\ &= \sum_{T'=1}^{p-2} \left(T'q - \left\lfloor \frac{T'\ell}{p} \right\rfloor + m - 2\right), \text{ since } pq - \ell = m \\ &= \sum_{T'=1}^{p-2} \left(T'q - \left\lfloor \frac{T'\ell}{p} \right\rfloor - 1\right) + (m-1)(p-2).\end{aligned}$$

Next, we observe that

$$\begin{aligned}(4.10)\quad \sum_{T'=1}^{p-2} \left(T'q - \left\lfloor \frac{T'\ell}{p} \right\rfloor - 1\right) + \left((p-1)q - \left\lfloor \frac{(p-1)\ell}{p} \right\rfloor - 2\right) \\ = \sum_{T'=1}^{p-1} \left(T'q - \left\lfloor \frac{T'\ell}{p} \right\rfloor - 1\right) - 1.\end{aligned}$$

Combining relations (4.7), (4.8), (4.9) and (4.10) gives:

$$\begin{aligned}|(A+A) \setminus C(0)| &< \sum_{T'=1}^{p-1} \left(T'q - \left\lfloor \frac{T'\ell}{p} \right\rfloor - 1\right) - 1 + (m-1)(p-2) \\ &\qquad + \left(pq - \left\lfloor \frac{p\ell}{p} \right\rfloor - 2\right) \\ &= g - 1 + mp - 2m - p + 2 + m - 2 \\ &= g + (m-1)(p-1) - 2 \\ &= 3g - 2\end{aligned}$$

and changing $<$ to $\leqslant$ gives the desired

$$|(A+A) \setminus C(0)| \leqslant 3g - 3. \qquad \square$$

Combining Proposition 4.14 and Lemma 4.16 we directly get the following.

Corollary 4.17. — $|(A+A) \setminus \bigcap_{i=0}^{p} C(i)| \leqslant 3(g-1)$.

5. Thickening and reduction

In the notation of Section 3, let $\mathcal{X} \to \operatorname{Spec}(R)$ denote the family of curves with generic fiber

$$\mathcal{X}_\eta : y^p = \lambda^p x^\ell + a(x)^p \tag{5.1}$$

and special fiber

$$\mathcal{X}_0 : X^p - X = \frac{x^\ell}{a(x)^p} \tag{5.2}$$

where $y = a(x)(\lambda X + 1)$, and $a(x)$ is given by

$$a(x) = \begin{cases} x^q + x_1 x^{q-1} + \cdots + +x_{q-1}x + x_q, & \text{if } \ell = 1 \\ x^q + x_1 x^{q-1} + \cdots + x_{q-1}x, & \text{if } \ell \neq 1. \end{cases}$$

For each $0 \leqslant i \leqslant p$, we expand the $(p-i)$-th power of $a(x)$

$$a(x)^{p-i} = \sum_{j=j_{\min}(i)}^{(p-i)q} c_{j,p-i} x^j \tag{5.3}$$

where $j_{\min}(i)$ is 0 if $\ell = 1$ and $p-i$ if $\ell \neq 1$ as in Definition 4.12, and for $j_{\min}(i) \leqslant j \leqslant (p-i)q$, the coefficients $c_{j,p-i}$ are given by

$$c_{j,p-i} = \sum_{\substack{(t_0,\ldots,t_q)\in\mathbb{N}^q \\ t_1+2t_2\cdots+qt_q=j}} \binom{p-i}{t_0,\ldots,t_q} \prod_{s=0}^{q} x_s^{t_s}.$$

In [13] the authors prove that the free R-module $H^0(\mathcal{X}, \Omega_{\mathcal{X}/R})$ has basis

$$\mathbf{c} = \left\{ \frac{x^N a(x)^{p-1-\mu} X^{p-1-\mu}}{a(x)^{p-1}(\lambda X + 1)^{p-1}} dx : \left\lfloor \frac{\mu\ell}{p} \right\rfloor \leqslant N \leqslant \mu q - 2,\ 1 \leqslant \mu \leqslant p-1 \right\}.$$

Consider the canonical map

$$\phi_{\mathbf{c}} : S = R[\{W_{N,\mu}\}] \to \bigoplus_{n \geqslant 0} H^0(\mathcal{X}, \Omega_{\mathcal{X}/R}^{\otimes n}),$$

which maps a monomial $W_{N_1,\mu_1}^{a_1} \cdots W_{N_d,\mu_d}^{a_d}$ to the differential

$$\frac{x^{(a_1N_1+\cdots+a_dN_d)}(a(x)X)^{a_1(p-1-\mu_1)+\cdots+a_d(p-1-\mu_d)}}{a(x)^{(a_1+\ldots+a_d)(p-1)}(\lambda X+1)^{(a_1+\cdots+a_d)(p-1)}} dx^{\otimes(a_1+\cdots+a_d)}.$$

We write $I_{\mathcal{X}} = \ker \phi_{\mathbf{c}}$ for the canonical ideal and note that the following polynomials are in $I_{\mathcal{X}}$:

Proposition 5.1. — *Let $1 \leqslant i \leqslant p-1$. For $j_{\min}(i) \leqslant j \leqslant (p-i)q$, let $W_{N,\mu}W_{N',\mu'}$, $W_{N'',\mu''}W_{N''',\mu'''}$ and $W_{N_j,\mu_i}W_{N_j',\mu_i'}$ be any monomials of degree 2 in S satisfying*

$$\begin{aligned}\mathrm{mdeg}(W_{N'',\mu''}W_{N''',\mu'''}) &= \mathrm{mdeg}(W_{N,\mu}W_{N',\mu'}) + (0,\ell,p),\\ \mathrm{mdeg}(W_{N_j,\mu_i}W_{N_j',\mu_i'}) &= \mathrm{mdeg}(W_{N,\mu}W_{N',\mu'}) + (0,j,p-i).\end{aligned}$$

Then

$$W_{N,\mu}W_{N',\mu'} - W_{N'',\mu}W_{N''',\mu'''} + \sum_{i=1}^{p-1}\sum_{j=j_{\min}(i)}^{(p-i)q} \lambda^{i-p}\binom{p}{i} c_{j,p-i} W_{N_j,\mu_i}W_{N_j',\mu_i'} \in I_{\mathcal{X}}.$$

Proof. — Let

$$f := W_{N,\mu}W_{N',\mu'} - W_{N'',\mu}W_{N''',\mu'''} + \sum_{i=1}^{p-1}\sum_{j=j_{\min}(i)}^{(p-i)q} \lambda^{i-p}\binom{p}{i} c_{j,p-i} W_{N_j,\mu_i}W_{N_j',\mu_i'}$$

where

$$\begin{aligned} N''+N''' &= N+N'+\ell, \quad \mu''+\mu''' = \mu+\mu'+p\\ \text{and}\quad N_j+N_j' &= N+N'+j, \qquad \mu_i+\mu_i' = \mu+\mu'+p-i. \end{aligned} \tag{5.4}$$

We note that $f \in R[\{W_{N,\mu}\}]$, since by [5, Sec. 4.3]

$$p\cdot\lambda^s \equiv \begin{cases} 0 \mod \mathfrak{m}, & \text{for } -(p-1) < s < 0\\ -1 \mod \mathfrak{m}, & \text{for } s = -(p-1), \end{cases} \tag{5.5}$$

which implies that $\lambda^{i-p}\binom{p}{i} \in \mathfrak{m} \subseteq \mathfrak{m}_R \subseteq R$ for all $1 \leqslant i \leqslant p-1$. Applying the canonical map $\phi_{\mathbf{c}}$ to f gives

$$\begin{aligned} &\frac{x^{N+N'}\,(a(x)X)^{2(p-1)-(\mu+\mu')}}{(a(x)(\lambda X+1))^{2(p-1)}}dx^{\otimes 2}\\ &\qquad - \frac{x^{N''+N'''}\,(a(x)X)^{2(p-1)-(\mu''+\mu''')}}{(a(x)(\lambda X+1))^{2(p-1)}}dx^{\otimes 2}\\ &+\sum_{i=1}^{p-1}\sum_{j=j_{\min}(i)}^{(p-i)q}\lambda^{i-p}\binom{p}{i}c_{j,p-i}\frac{x^{N_j+N_j'}\,(a(x)X)^{2(p-1)-(\mu_i+\mu_i')}}{(a(x)(\lambda X+1))^{2(p-1)}}dx^{\otimes 2}, \end{aligned} \tag{5.6}$$

and using the relations of eq. (5.4) we may rewrite eq. (5.6) as

$$\phi_{\mathbf{c}}(f)=h\left(1-x^{\ell}\left(a(x)X\right)^{-p}+\sum_{i=1}^{p-1}\sum_{j=j_{\min}(i)}^{(p-i)q}\lambda^{i-p}\binom{p}{i}c_{j,p-i}x^{j}\left(a(x)X\right)^{i-p}\right),$$

where

$$h:=\frac{x^{N+N'}\left(a(x)X\right)^{2(p-1)-(\mu+\mu')}}{(a(x)(\lambda X+1))^{2(p-1)}}dx^{\otimes 2}.$$

Combining with the expansion of $a(x)^{p-i}$ in eq. (5.3) we get that

$$\phi_{\mathbf{c}}(f)=h\left(1-x^{\ell}\left(a(x)X\right)^{-p}+\sum_{i=1}^{p-1}\lambda^{i-p}\binom{p}{i}X^{i-p}\right)$$

and simplify the expression as follows:

$$\begin{aligned}\phi_{\mathbf{c}}(f)&=h\left(-x^{\ell}\left(a(x)X\right)^{-p}+\sum_{i=1}^{p}\lambda^{i-p}\binom{p}{i}X^{i-p}\right)\\&=h\left(-x^{\ell}\left(a(x)X\right)^{-p}-\lambda^{-p}X^{-p}+\sum_{i=0}^{p}\lambda^{i-p}\binom{p}{i}X^{i-p}\right)\\&=h\left(-x^{\ell}\left(a(x)X\right)^{-p}-\lambda^{-p}X^{-p}+\lambda^{-p}X^{-p}(\lambda X+1)^{p}\right).\end{aligned}\tag{5.7}$$

Finally, since $y=a(x)(\lambda X+1)$, eq. (5.7) is equivalent to eq. (5.1), so $\phi_{\mathbf{c}}(f)\otimes_R 1_L=0$ and thus $\phi_{\mathbf{c}}(f)=0$, completing the proof. ☐

We collect the polynomials of Proposition 5.1 in the set below:

Definition 5.2. — *Let*

$$\begin{aligned}G_2^{\mathbf{c}}=\Bigg\{&W_{N,\mu}W_{N',\mu'}-W_{N'',\mu''}W_{N''',\mu'''}\\&+\sum_{i=1}^{p-1}\sum_{j=j_{\min}(i)}^{(p-i)q}\lambda^{i-p}\binom{p}{i}c_{j,p-i}W_{N_j,\mu_i}W_{N'_j,\mu'_i}\in S\ :\\&\qquad\operatorname{mdeg}(W_{N'',\mu''}W_{N''',\mu'''})=\operatorname{mdeg}(W_{N,\mu}W_{N',\mu'})+(0,\ell,p),\\&\qquad\operatorname{mdeg}(W_{N_j,\mu_i}W_{N'_j,\mu'_i})=\operatorname{mdeg}(W_{N,\mu}W_{N',\mu'})+(0,j,p-i),\\&\qquad\qquad\textit{for } 0\leqslant i\leqslant p,\ j_{\min}(i)\leqslant j\leqslant(p-i)q\Bigg\}.\end{aligned}$$

Remark 5.3. — Let

$$g = W_{N,\mu}W_{N',\mu'} - W_{N'',\mu''}W_{N''',\mu'''} + \sum_{i=1}^{p-1}\sum_{j=j_{\min}(i)}^{(p-i)q} \lambda^{i-p}\binom{p}{i} c_{j,p-i} W_{N_j,\mu_i} W_{N'_j,\mu'_i}$$

be an element of $G_2^{\mathbf{c}}$ as above. By comparing the multidegree relations defining $G_2^{\mathbf{c}}$ with the description of $C(i)$ in Definition 4.12, we observe that the monomial $W_{N,\mu}W_{N',\mu'}$ corresponds to the point $(N+N', \mu+\mu')$ of $C(0)$. Moreover, the monomial $W_{N'',\mu''}W_{N''',\mu'''}$ of g corresponds to the point a_{-1} (see the comment preceding Proposition 4.14), while the monomials $W_{N_j,\mu_i}W_{N'_j,\mu'_i}$ of g in the double sum correspond to the points a_j.

Example 5.4. — In the context of Example 4.10 and Example 4.13, the monomial $W_{0,1}W_{0,1}$ corresponds to the point $(0,2) \in C(0)$. The blue points in the figure below correspond to the points a_{-1} and a_j as in the above discussion, and they define the element of $G_2^{\mathbf{c}}$ with initial term $W_{0,1}W_{0,1}$.

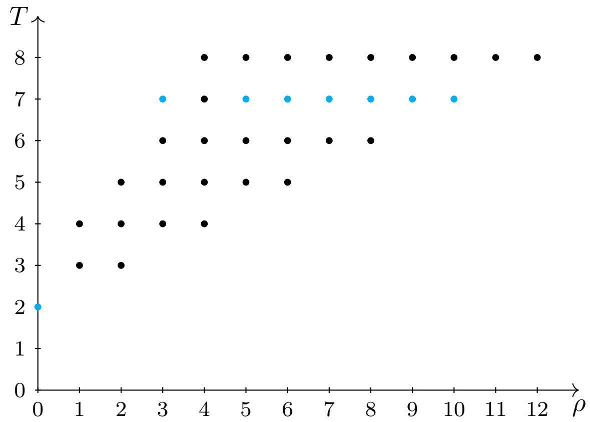

Figure 5.1. The blue points define the element of $G_2^{\mathbf{c}}$ with initial term $W_{0,1}W_{0,1}$.

We write $G_1^{\mathbf{c}}$ for the set of binomials in Definition 4.3. The main result of this section is the following:

Theorem 5.5. — $I_{\mathcal{X}} = \langle G_1^{\mathbf{c}} \cup G_2^{\mathbf{c}}\rangle$.

To prove Theorem 5.5, we will use the Nakayama-type criterion of Theorem 3.2 by showing that

$$\langle (G_1^{\mathbf{c}} \otimes_R k) \cup (G_2^{\mathbf{c}} \otimes_R k)\rangle = I_{\mathcal{X}_0} \quad \text{and} \quad \langle (G_1^{\mathbf{c}} \otimes_R L) \cup (G_2^{\mathbf{c}} \otimes_R L)\rangle \subseteq I_{\mathcal{X}_\eta}.$$

5.1. Compatibility with the special fiber

Rewrite the affine model for the family's special fiber given in eq. (5) as

$$\mathcal{X}_0 : 1 - x^{\ell} a(x)^{-p} X^{-p} - X^{-(p-1)} = 0. \tag{5.8}$$

Let $\overline{\mathbf{c}}$ be the basis for $H^0(\mathcal{X}_0, \Omega_{\mathcal{X}_0/k})$ as in eq. (4.2) consider the canonical map

$$\phi_{0,\overline{\mathbf{c}}} : S = k[\{w_{N,\mu}\}] \longrightarrow \bigoplus_{n \geqslant 0} H^0(\mathcal{X}_0, \Omega^{\otimes n}_{\mathcal{X}_0/k})$$

which maps a monomial $w^{a_1}_{N_1,\mu_1} \cdots w^{a_d}_{N_d,\mu_d}$ to the differential

$$x^{(a_1 N_1 + \cdots + a_d N_d)} \, (a(x)X)^{a_1(p-1-\mu_1)+\cdots+a_d(p-1-\mu_d)} \, dx^{\otimes(a_1+\cdots+a_d)}.$$

We write $I_{\mathcal{X}_0} = \ker \phi_{0,\overline{\mathbf{c}}}$ for the canonical ideal on the special fiber and note that the polynomials of Proposition 5.1 reduce to the following polynomials in $I_{\mathcal{X}_0}$:

Proposition 5.6. — *We have that $G_2^{\mathbf{c}} \otimes_R k = G_2^{\overline{\mathbf{c}}} \subseteq I_{\mathcal{X}_0}$, where*

$$G_2^{\overline{\mathbf{c}}} = \Bigg\{ w_{N,\mu} w_{N',\mu'} - w_{N'',\mu''} w_{N''',\mu'''} - \sum_{j=j_{\min}(1)}^{(p-1)q} c_{j,p-1} w_{N_j,\mu_j} w_{N'_j,\mu'_j} \in S \ : \\ \mathrm{mdeg}(w_{N'',\mu''} w_{N''',\mu'''}) = \mathrm{mdeg}(w_{N,\mu} w_{N',\mu'}) + (0,\ell,p), \\ \mathrm{mdeg}(w_{N_j,\mu_j} w_{N'_j,\mu'_j}) = \mathrm{mdeg}(w_{N,\mu} w_{N',\mu'}) + (0,j,p-1), \\ \text{for } j_{\min}(1) \leqslant j \leqslant (p-1)q \Bigg\}.$$

Proof. — Eq. (5.5) implies that in the expression

$$\sum_{i=1}^{p-1} \sum_{j=j_{\min}(i)}^{(p-i)q} \lambda^{i-p} \binom{p}{i} c_{j,p-i} W_{N_j,\mu_i} W_{N'_j,\mu'_i}$$

only the term for $i = 1$ survives reduction, giving that

$$\left(\sum_{i=1}^{p-1} \sum_{j=j_{\min}(i)}^{(p-i)q} \lambda^{i-p} \binom{p}{i} c_{j,p-i} W_{N_j,\mu_i} W_{N'_j,\mu'_i} \right) \otimes_R k \\ = - \sum_{j=j_{\min}(1)}^{(p-1)q} c_{j,p-1} w_{N_j,\mu_j} w_{N'_j,\mu'_j},$$

and equivalently

$$\Big(W_{N,\mu}W_{N',\mu'} - W_{N'',\mu}W_{N''',\mu'''} \\ + \sum_{i=1}^{p-1}\sum_{j=j_{\min}(i)}^{(p-i)q} \lambda^{i-p}\binom{p}{i} c_{j,p-i}W_{N_j,\mu_i}W_{N_j',\mu_i'}\Big) \otimes_R k \\ = w_{N,\mu}w_{N',\mu'} - w_{N'',\mu''}w_{N''',\mu'''} - \sum_{j=j_{\min}(1)}^{(p-1)q} c_{j,p-1}w_{N_j,\mu_j}w_{N_j',\mu_j'},$$

completing the proof. □

Remark 5.7. — The fact that $G_2^{\overline{\mathbf{c}}} \subseteq I_{\mathcal{X}_0}$ follows from the relative canonical embedding diagram of Theorem 3.1. However, the reader may also verify directly that $\phi_{0,\overline{\mathbf{c}}}\left(G_2^{\overline{\mathbf{c}}}\right) = 0$.

We write $G_1^{\overline{\mathbf{c}}}$ for the set of binomials in Definition 4.3 and remark that $G_1^{\mathbf{c}} \otimes_R k = G_1^{\overline{\mathbf{c}}} \subseteq I_{\mathcal{X}_0}$. To prove that $I_{\mathcal{X}_0} = \langle G_1^{\overline{\mathbf{c}}} \cup G_2^{\overline{\mathbf{c}}}\rangle$ we will use the dimension criterion of Proposition 2.2 and a series of lemmas. We consider the subset $C(1)$ of $A+A$ given by Definition 4.12

$$C(1) = \left\{ (\rho,T) \in A+A : \begin{array}{l} (\rho+\ell, T+p) \text{ and } (\rho+j, T+p-1) \in A+A \\ \text{for } j_{\min}(1) \leqslant j \leqslant (p-1)q \end{array} \right\},$$

and study its image under the map $\sigma : A+A \to \mathbb{T}^2$ given in Definition 4.5.

Lemma 5.8. — $\sigma(C(1)) \subseteq \mathrm{in}_\prec(G_2^{\overline{\mathbf{c}}})$.

Proof. — If $(\rho,T) \in C(1)$ then by definition $(\rho,T) \in A+A$, $(\rho+\ell, T+p) \in A+A$ and $(\rho+j, T+p-1) \in A+A$ for all $j_{\min}(1) \leqslant j \leqslant (p-1)q$. Hence the monomials $w_{N,\mu}w_{N',\mu'} := \sigma(\rho,T)$, $w_{N'',\mu''}w_{N''',\mu'''} := \sigma(\rho+\ell, T+p)$ and $w_{N_j,\mu_j}w_{N_j',\mu_j'} := \sigma(\rho+j, T+p-1)$ give rise to a polynomial

$$g = w_{N,\mu}w_{N',\mu'} - w_{N'',\mu''}w_{N''',\mu'''} - \sum_{j=j_{\min}(1)}^{(p-1)q} c_{j,p-1}w_{N_j,\mu_j}w_{N_j',\mu_j'},$$

which, by construction, satisfies $g \in G_2^{\overline{\mathbf{c}}}$ and $\mathrm{in}_\prec(g) = \sigma(\rho,T)$. □

Lemma 5.9. — $\dim_k \left(S/\langle \mathrm{in}_\prec\left(G_1^{\overline{\mathbf{c}}} \cup G_2^{\overline{\mathbf{c}}}\right)\rangle\right)_2 \leqslant |(A+A) \setminus C(1)|$.

Proof. — By Proposition 4.6 we have that $\sigma(A+A) = \mathbb{T}^2 \setminus \mathrm{in}_\prec(G_1^{\overline{\mathbf{c}}})$ and by Lemma 5.8 we have that $\sigma(C(1)) \subseteq \mathrm{in}_\prec(G_2^{\overline{\mathbf{c}}})$, so

$$\sigma\big((A+A) \setminus C(1)\big) \supseteq \mathbb{T}^2 \setminus \Big(\mathrm{in}_\prec(G_1^{\overline{\mathbf{c}}}) \cup \mathrm{in}_\prec(G_2^{\overline{\mathbf{c}}})\Big). \tag{5.9}$$

Since σ is one-to-one, eq. (5.9) gives

$$|(A+A)\setminus C(1)| = |\sigma\big((A+A)\setminus C(1)\big)| \geqslant \left|\mathbb{T}^2 \setminus \left(\mathrm{in}_\prec(G_1^{\bar{\mathbf{c}}}) \cup \mathrm{in}_\prec(G_2^{\bar{\mathbf{c}}})\right)\right|.$$

Finally, $\langle \mathrm{in}_\prec(G_1^{\bar{\mathbf{c}}}) \cup \mathrm{in}_\prec(G_2^{\bar{\mathbf{c}}})\rangle$ is a monomial ideal generated in degree 2 so

$$\dim_k \left(S/\langle \mathrm{in}_\prec(G_1^{\bar{\mathbf{c}}}) \cup \mathrm{in}_\prec(G_2^{\bar{\mathbf{c}}})\rangle\right)_2 = \left|\mathbb{T}^2 \setminus \left(\mathrm{in}_\prec(G_1^{\bar{\mathbf{c}}}) \cup \mathrm{in}_\prec(G_2^{\bar{\mathbf{c}}})\right)\right|,$$

completing the proof. □

Theorem 5.10. — $I_{\mathcal{X}_0} = \langle G_1^{\bar{\mathbf{c}}} \cup G_2^{\bar{\mathbf{c}}}\rangle$.

Proof. — By Proposition 4.2 and Proposition 5.6 we get that $\langle G_1^{\bar{\mathbf{c}}} \cup G_2^{\bar{\mathbf{c}}}\rangle \subseteq I_{\mathcal{X}_0}$. By Lemma 5.9 and Proposition 4.14 we get that

$$\dim_k \left(S/\langle \mathrm{in}_\prec(G_1^{\bar{\mathbf{c}}} \cup G_2^{\bar{\mathbf{c}}})\rangle\right)_2 \leqslant |(A+A)\setminus C(1)| \leqslant |(A+A)\setminus C(0)|$$

so Lemma 4.16 gives $\dim_k \left(S/\langle \mathrm{in}_\prec(G_1^{\bar{\mathbf{c}}} \cup G_2^{\bar{\mathbf{c}}})\rangle\right)_2 \leqslant 3(g-1)$. Proposition 2.2 implies that $I_{\mathcal{X}_0} = \langle G_1^{\bar{\mathbf{c}}} \cup G_2^{\bar{\mathbf{c}}}\rangle$, completing the proof. □

5.2. Compatibility with the generic fiber

Let $C(i)$ denote the subsets of $A+A$ given in Definition 4.12, where $0 \leqslant i \leqslant p$. By Proposition 4.14, $C(0) = \bigcap_{i=0}^{p} C(i)$. Thus, if $(\rho, T) \in C(0)$ then $(\rho, T) \in A+A$, $(\rho+\ell, T+p) \in A+A$ and $(\rho+j, T+p-i) \in A+A$ for all $j_{\min}(i) \leqslant j \leqslant (p-i)q$. Hence the monomials $W_{N,\mu}W_{N',\mu'} := \sigma(\rho,T)$, $W_{N'',\mu}W_{N''',\mu'''} := \sigma(\rho+\ell, T+p)$ and $W_{N_j,\mu_i}W_{N_j',\mu_i'} := \sigma(\rho+j, T+p-i)$ give rise to the polynomial

$$\begin{aligned} g = {} & W_{N,\mu}W_{N',\mu'} - W_{N'',\mu}W_{N''',\mu'''} \\ & + \sum_{i=1}^{p-1} \sum_{j=j_{\min}}^{(p-i)q} \lambda^{i-p}\binom{p}{i} c_{j,p-i} W_{N_j,\mu_i}W_{N_j',\mu_i'} \in G_2^{\mathbf{c}}. \end{aligned}$$

We comment that $\mathrm{in}_\prec(g) = \sigma(\rho, T)$.

Lemma 5.11. — $\dim_L \left(S/\langle \mathrm{in}_\prec\left(G_1^{\mathbf{c}} \cup G_2^{\mathbf{c}}\right)\rangle\right)_2 \otimes_R L \leqslant |(A+A)\setminus C(0)|$.

Proof. — By Proposition 4.6 we have that

$$\sigma(A+A) = \mathbb{T}^2 \setminus (\mathrm{in}_\prec(G_1^{\mathbf{c}}) \otimes_R L)$$

and by the preceding comment we have that $\sigma(C(0)) \subseteq \mathrm{in}_\prec(G_2^{\mathbf{c}}) \otimes_R L$, so

$$\sigma\big((A+A)\setminus C(0)\big) \supseteq \mathbb{T}^2 \setminus (\mathrm{in}_\prec(G_1^{\mathbf{c}}) \otimes_R L \cup \mathrm{in}_\prec(G_2^{\mathbf{c}}) \otimes_R L)\,. \tag{5.10}$$

Since σ is one-to-one, eq. (5.10) gives

$$\begin{aligned}|(A+A)\setminus C(0)| &= |\sigma\big((A+A)\setminus C(0)\big)| \\ &\geqslant |\mathbb{T}^2 \setminus (\mathrm{in}_\prec(G_1^{\mathbf{c}})\otimes_R L \cup \mathrm{in}_\prec(G_2^{\mathbf{c}})\otimes_R L)\,|.\end{aligned}$$

Finally, $\langle \mathrm{in}_\prec(G_1^{\mathbf{c}})\otimes_R L \cup \mathrm{in}_\prec(G_2^{\mathbf{c}})\otimes_R L)\rangle$ is a monomial ideal generated in degree 2 so

$$\dim_L (S/\langle \mathrm{in}_\prec (G_1^{\mathbf{c}}\cup G_2^{\mathbf{c}})\rangle)_2 \otimes_R L = |\mathbb{T}^2 \setminus (\mathrm{in}_\prec(G_1^{\mathbf{c}})\otimes_R L \cup \mathrm{in}_\prec(G_2^{\mathbf{c}})\otimes_R L)\,|,$$

completing the proof. □

5.3. Proof of Main Theorem

Proof of Theorem 5.5. — By Proposition 5.6 and Theorem 5.10 we get that $\langle (G_1^{\mathbf{c}}\otimes_R k)\cup(G_2^{\mathbf{c}}\otimes_R k)\rangle = I_{\mathcal{X}_0}$. By Proposition 4.2 and Proposition 5.1 we have that $\langle (G_1^{\mathbf{c}}\otimes_R L)\cup(G_2^{\mathbf{c}}\otimes_R L)\rangle \subseteq I_{\mathcal{X}_\eta}$. Lemma 5.11 and Proposition 4.16 imply that $\dim_L (S/\langle \mathrm{in}_\prec (G_1^{\mathbf{c}}\cup G_2^{\mathbf{c}})\rangle)_2\otimes_R L \leqslant |(A+A)\setminus C(0)| \leqslant 3(g-1)$, so by Proposition 2.2 we have that $\langle (G_1^{\mathbf{c}}\otimes_R L)\cup(G_2^{\mathbf{c}}\otimes_R L)\rangle = I_{\mathcal{X}_\eta}$. Hence, Theorem 3.2 gives that $I_{\mathcal{X}} = \langle G_1^{\mathbf{c}}\cup G_2^{\mathbf{c}}\rangle$. □

BIBLIOGRAPHY

[1] D. Bayer & D. Eisenbud, "Ribbons and their canonical embeddings", *Trans. Amer. Math. Soc.* **347** (1995), no. 3, p. 719-756.

[2] C. Berkesch & F.-O. Schreyer, "Syzygies, finite length modules, and random curves", in *Commutative algebra and noncommutative algebraic geometry. Vol. I*, Math. Sci. Res. Inst. Publ., vol. 67, Cambridge Univ. Press, New York, 2015, p. 25-52.

[3] J. Bertin, "Obstructions locales au relèvement de revêtements galoisiens de courbes lisses", *C. R. Acad. Sci. Paris Sér. I Math.* **326** (1998), no. 1, p. 55-58.

[4] J. Bertin & A. Mézard, "Déformations formelles des revêtements sauvagement ramifiés de courbes algébriques", *Invent. Math.* **141** (2000), no. 1, p. 195-238.

[5] ———, "Problem of formation of quotients and base change", *Manuscripta Math.* **115** (2004), no. 4, p. 467-487.

[6] H. Boseck, "Zur Theorie der Weierstrasspunkte", *Math. Nachr.* **19** (1958), p. 29-63.

[7] G. Faltings, "Calculus on arithmetic surfaces", *Ann. of Math. (2)* **119** (1984), no. 2, p. 387-424.

[8] B. Green & M. Matignon, "Liftings of Galois covers of smooth curves", *Compositio Math.* **113** (1998), no. 3, p. 237-272.

[9] P. Griffiths & J. Harris, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994, Reprint of the 1978 original, xiv+813 pages.

[10] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, New York-Heidelberg, 1977, xvi+496 pages.

[11] ———, *Deformation theory*, Graduate Texts in Mathematics, vol. 257, Springer, New York, 2010, viii+234 pages.

[12] N. Jacobson, *Basic algebra. II*, second ed., W. H. Freeman and Company, New York, 1989, xviii+686 pages.

[13] S. Karanikolopoulos & A. Kontogeorgis, "Integral representations of cyclic groups acting on relative holomorphic differentials of deformations of curves with automorphisms", *Proc. Amer. Math. Soc.* **142** (2014), no. 7, p. 2369-2383.

[14] S. Nakajima, "On abelian automorphism groups of algebraic curves", *J. London Math. Soc. (2)* **36** (1987), no. 1, p. 23-32.

[15] ———, "p-ranks and automorphism groups of algebraic curves", *Trans. Amer. Math. Soc.* **303** (1987), no. 2, p. 595-607.

[16] A. Obus & S. Wewers, "Cyclic extensions and the local lifting problem", *Ann. of Math. (2)* **180** (2014), no. 1, p. 233-284.

[17] G. Oliveira & K.-O. Stöhr, "Gorenstein curves with quasi-symmetric Weierstrass semigroups", *Geom. Dedicata* **67** (1997), no. 1, p. 45-63.

[18] F. Oort, "Lifting algebraic curves, abelian varieties, and their endomorphisms to characteristic zero", in *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, 1987, p. 165-195.

[19] F. Pop, "The Oort conjecture on lifting covers of curves", *Ann. of Math. (2)* **180** (2014), no. 1, p. 285-322.

[20] B. Saint-Donat, "On Petri's analysis of the linear system of quadrics through a canonical curve", *Math. Ann.* **206** (1973), p. 157-175.

[21] M. Schlessinger, "Functors of Artin rings", *Trans. Amer. Math. Soc.* **130** (1968), p. 208-222.

[22] T. Sekiguchi, F. Oort & N. Suwa, "On the deformation of Artin–Schreier to Kummer", *Ann. Sci. École Norm. Sup. (4)* **22** (1989), no. 3, p. 345-375.

[23] E. Sernesi, *Deformations of algebraic schemes*, Grundlehren der Mathematischen Wissenschaften, vol. 334, Springer-Verlag, Berlin, 2006, xii+339 pages.

[24] J.-P. Serre, "Sur la topologie des variétés algébriques en caractéristique p", in *Symposium internacional de topología algebraica International symposium on algebraic topology*, Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958, p. 24-53.

[25] ———, "Exemples de variétés projectives en caractéristique p non relevables en caractéristique zéro", *Proc. Nat. Acad. Sci. U.S.A.* **47** (1961), p. 108-109.

[26] B. Sturmfels, *Gröbner bases and convex polytopes*, University Lecture Series, vol. 8, American Mathematical Society, Providence, RI, 1996, xii+162 pages.

[27] N. Suwa & T. Sekiguchi, "Théorie de Kummer–Artin–Schreier et applications", *Journal de théorie des nombres de Bordeaux* **7** (1995), no. 1 (fr), p. 177-189.

[28] C. Voisin, "Green's canonical syzygy conjecture for generic curves of odd genus", *Compos. Math.* **141** (2005), no. 5, p. 1163-1190.

[29] E. Witt, "Zyklische Körper und Algebren der Charakteristik p vom Grad p^n. Struktur diskret bewerteter perfekter Körper mit vollkommenem Restklassenkörper der Charakteristik p", *J. Reine Angew. Math.* **176** (1937), p. 126-140.

[30] G. M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995, x+370 pages.

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