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METRISED AMPLE LINE BUNDLES IN NON-ARCHIMEDEAN GEOMETRY

by Yanbo FANG (*)

Abstract. — We study metrised ample line bundles on projective varieties over non-Archimedean fields from the point of view of commutative Banach algebras and analytic functions of several variables. Line bundle metrics induce sup norms on the graded algebra of sections; the global metric positivity is interpreted as the holomorphic convexity of the spectrum of the normed section algebra. A normed extension property is established using spectral and functional methods: restricted sections on a closed subvariety can be extended to sections on the ambient variety, with a sub-exponential asymptotic distorsion of sup norms.

Résumé. — On étudie fibrés en droites amples métrisés sur variétés projectives définies sur un corps non-archimédien, d'un point de vue d'algèbres de Banach commutatives et fonctions analytiques à plusieurs variables. Une métrique sur un fibré en droites induit une norme sup en l'algèbre graduée de sections; la positivité au sens global de cette métrique est interprétée comme la convexité holomorphe du spectre analytique de cette algèbre normée de sections. Une propriété d'extension normée est établie par techniques spectrale et fonctionnel: on peut étendre une section restreinte sur une sous-variété en une section sur la variété ambiante, avec un contrôle sous-exponentiel de la distorsion asymptotique des normes sup.

1. Introduction

1.1. Overview

Non-Archimedean analytic geometry studies analytic varieties defined over a non-Archimedean field such as $\mathbb{Q}_p$, $\mathbb{F}_p((t))$ or $\mathbb{C}((t))$. Like its prototype complex analytic geometry, it provides a ground for investigating algebraic varieties from an analytic point of view.

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In this article, we encode positively metrised ample line bundles on a projective variety by certain Banach algebras of sections, and establish quantitative properties of global sections with regard to the restriction map to closed subvarieties. Such analytic objects and quantitative properties appear naturally in Arakelov theory over local fields (see for example [3, 7, 10, 11, 13, 20, 30, 31]). We shall use the theory of Berkovich [2] for non-archimedean analytic spaces.

The set-up is as follows: let $(k, |\cdot|)$ be a complete ultrametric non-trivially valued field, X be a projective variety defined over k, and L be an ample line bundle on X, we shall study line bundle metrics ϕ on the analytification L^{an}.

1.2. Notions of metric positivity and the normed extension property

In the complex analytic setting, one can use differentio-geometric tools to study bundle metrics (see for example [17]). The positivity of a line bundle metric can be defined locally by the definite positivity of its curvature form (or current) $\mathrm{dd^c}\,\phi$. This metric positivity is best manifested by the so-called *normed extension property* for restricted sections. Given a closed subvariety Y of X, if L is ample, its algebro-geometric positivity implies that the restriction maps

$$H^0(X, L^{\otimes n}) \xrightarrow{|_Y} H^0(Y, L|_Y^{\otimes n}), \quad s_n \mapsto t_n$$

are surjective for all large $n \in \mathbb{N}$, thanks to Serre's vanishing theorem. Thus restricted sections on Y can be extended to sections on X. The metric positivity of ϕ improves this to a "quantitative surjectivity", giving an upper bound controlling the distorsion of sup norms in the asymptotic sense:

$$\inf_{\substack{s_n \in H^0(X, L^{\otimes n}) \\ s_n|_Y = t_n \neq 0}} \frac{\|s_n\|_{n\phi}}{\|t_n\|_{n\phi|_Y}} \leqslant C(n), \quad n \to \infty. \tag{1.1}$$

The following classical result gives a bound using the L^2-method for $\bar{\partial}$-equations.

THEOREM 1.1 ([6, 28]; [1, 24, 30]). — *Let ϕ be a metric of strictly positive curvature. Assume that X is smooth, then there exists a polynomial bound for* (1.1). *Namely, there are $C \in \mathbb{R}_{>0}$ and $n_Y \in \mathbb{N}$ that depend on*

(X, Y, L, ϕ), such that for any $n \geqslant n_Y$ and any $t_n \in H^0(Y, L|_Y^{\otimes n})$, one can find $s_n \in H^0(X, L^{\otimes n})$ such that $s_n|_Y = t_n$ and

$$\|s_n\|_{n\phi} \leqslant Cn^d \cdot \|t_n\|_{n\phi|_Y}.$$

As a consequence, take subvarieties of dimension zero, and let n goes to infinity, one deduces that any line bundle metric with a positive curvature can be approximated by a sequence of *Fubini–Study metrics* of various level n, each one induced by some norm on the space of global sections of $L^{\otimes n}$. In general, a sub-exponential bound as $\mathrm{e}^{\epsilon n}$ in (1.1), though weaker than the polynomial one n^d, is sufficient for this approximation. Thus the metric positivity in terms of local curvatures can equivalently be described using global norms.

In the non-Archimedean analytic setting, despite recent fast developments of a differentio-geometric structure [13, 21] underlying X_k^{an}, parallel tools are not yet fully available. Traditionally, a continuous line bundle metric is considered to be positive if it can be uniformly approximated by Fubini–Study metrics, instead of meeting local curvature requirements. This leads to the notion of *semipositive metrics* ([15, 30]) whose positivity is formulated in terms of global norms. This replacement of definition raises the question *whether the normed extension property still holds for closed subvarieties of general dimensions.* Such a metric positivity notion is accompanied by the technique of integral models for (X, L): one views Fubini–Study metrics as model metrics, and translates the concerned metric properties into algebro-geometric properties of these models, see for example [7, 8, 10, 11, 13, 20, 22, 30]. In this vein, a normed extension result is obtained in [15, 31], which is not *uniform* in the sense that the asymptotic range n_Y also depends on the choice of the restricted section (see Theorem 4.1).

1.3. Banach algebra techniques

We approach the normed extension property from a point of view of *Banach algebras and analytic functions of several variables.* This strategy originates from [5, 18, 26], where new proof of (1.1) is obtained with a weaker sub-exponential bound; besides, the varieties are no longer supposed to be smooth.

The strategy is sketched as follows. On the algebro-geometric level, by the tautological construction, the study of sections of L on X is equivalent to the study of functions on the total space of the dual line bundle $\mathbf{V}(L)$ (or equivalently functions on its affine cone $\mathbf{C}(L)$ after contracting the zero section). The central idea is, on the normed level, the global norm metric positivity for the metrised line bundle (L,ϕ) can be translated to the holomorphic convexity of its dual unit disc bundle $\mathbf{D}^\vee(L,\phi)$, which is a pre-compact subset in $\mathbf{V}(L)^{\mathrm{an}}$. Consequently, the desired extended section with a norm control is obtained from Cartan's vanishing theorem for coherent sheaves on this domain and the open mapping theorem in functional analysis. Besides, the local curvature positivity is translated to the pseudo-convexity of the dual unit disc bundle, and its equivalence with the global norm positivity is clearly shown by the classical Levi problem (Oka's theorem) in several complex variables.

We implement parts of these constructions concerning the global norm metric positivity in the non-Archimedean setting, leaving aside the issue of its equivalence with the local curvature positivity.

First, we give a *normed* tautological construction for a metrised line bundle (L,ϕ), where L is ample and ϕ is not necessarily positive in any sense. We consider the sup norm induced by $n\phi$ on $H^0(X,L^{\otimes n})$, gather all these spaces into the graded algebra of sections R of L, and assemble all these sup norms into a graded submultiplicative norm $|||\cdot|||_\phi$ on this algebra. We take the completion of the normed algebra $(R(L),|||\cdot|||_\phi)$ to a Banach algebra $\mathcal{R}(L,\phi)$, and call it the normed section algebra for (L,ϕ). The following identification relates the Fubini–Study envelope metric $\mathcal{P}(\phi)$ of ϕ and the Berkovich spectrum $\mathfrak{M}$ of the normed section algebra, thus one can view the later as a "normed affine cone" of the metrised pair $(L,\mathcal{P}(\phi))$. It allows us to translate the global norm metric positivity of $\mathcal{P}(\phi)$ to the holomorphic convexity of this spectrum.

Theorem 1.2 (3.16). — *The following diagram of maps between topological spaces, induced by the canonical inclusions and the contractions of zero-sections p^{an}, is Cartesian*

$$\begin{array}{ccc} \overline{\mathbf{D}}^\vee(L,\mathcal{P}(\phi)) & \xrightarrow{p^{\mathrm{an}}} & \mathfrak{M}(\mathcal{R}(L,\phi)) \\ \big\downarrow & & \big\downarrow \\ \mathbf{V}(L)^{\mathrm{an}} & \xrightarrow{p^{\mathrm{an}}} & \mathbf{C}(L)^{\mathrm{an}} \end{array}$$

Second, once ϕ is supposed to be positive in the global sense, we use the above geometric construction to get a normed extension property for restricted sections from any closed subvariety Y, with two different approaches.

The spectral method exploits the holomorphic convexity of $\mathfrak{M}(\mathcal{R}(L,\phi))$ via the so-called *holomorphic functional calculus* in the theory of Banach algebras. This improves the earlier result of [15, 30] by making the choice of n_Y uniform as in (1.1).

Theorem 1.3 (4.2). — *Let ϕ be an asymptotic Fubini–Study metric on L. Then there is a sub-exponential bound for* (1.1). *Namely, for any $\epsilon > 0$, there exists $n_Y \in \mathbb{N}$ such that for any $n \geqslant n_Y$ and any $t_n \in H^0(Y, L|_Y^{\otimes n})$, one can find $s_n \in H^0(X, L^{\otimes n})$ such that $s_n|_Y = t_n$ and*

$$\|s_n\|_{n\phi} \leqslant \mathrm{e}^{n\epsilon} \cdot \|t_n\|_{n\phi|_Y}.$$

There is a technical modification in its proof compared to the constructions in the $\mathbb{C}$-analytic setting: one replaces the open disc bundle and Frechét norms on the section algebra by the closed disc bundle and Banach norms. One reason for this change is that the vanishing properties for (open) Stein domains in non-Archimedean geometry were not established when this research was conducted (see however [23] for a recent advance), instead we use the technique of holomorphic functional calculus on spectra of Banach algebras to perform the normed extension. Besides, Banach norms are more compatible with the norm reduction technique in non-Archimedean analysis, which is related to the method of integral models and shall be used in a further work.

The functional method makes explicit calculation of $\mathfrak{M}(\mathcal{R}(L, \phi))$ in the diagonalizable Fubini–Study metric case and determines the normed section algebra to be an affinoid algebra, whose finiteness property provides the desired uniform bound in Theorem 1.3.

1.4. Scopes

We hope that our several analytic variables approach to metric positivity problems could be helpful for connecting the algebro-geometric approach via intersection theory on integral models, and the differentio-geometric approach via differential forms on Berkovich spaces.

The normed extension property has various applications, notably to the study of restricted volumes. Combination with the calculation of certain determinant norms leads to an “arithmetic Hilbert–Samuel formula” in Arakelov geometry over local fields, via the induction-by-dimension strategy in [1]. This will be treated in a forthcoming work (note that in [7, 10] this formula is established using intersection theory on integral models and Deligne pairing techniques). The normed extension property has also been applied in the proof of Nakai–Moishezon criterion of arithmetic ampleness, cf. [30], see also [20].

Our article is organized as follows: in Section 2 we recall some basic notions and properties about the analysis and the geometry over non-Archimedean fields; in Section 3 we study the metric positivity of line bundle metrics defined by global norms, and interpret it by the holomorphic convexity of their dual unit disc bundles; in Section 4 we obtain the normed extension property with sub-exponential bounds for restricted sections from any closed subvariety, using two independent methods.

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2. Preliminaries on analysis over non-Archimedean fields

In this section, one recalls basics of functional analysis of normed vector spaces and normed algebras over a complete ultrametric valued field, following [2, 4]. Results in Section 2.2.3, Section 2.3.2 and Section 2.4.2 are particularly pertinent to our study.

Throughout the section, one denotes by k a field and by $|\cdot|$ a non-trivial complete ultrametric absolute value. Unless specified, all k-algebras are supposed to be commutative and unitary (with $0 \neq 1$), and by convention all homomorphism of k-algebras are supposed to preserve the units.

2.1. Normed vector spaces

2.1.1. Basic notions

Let V be a vector space over k. A seminorm $\|\cdot\|$ on V is *ultrametric* if it satisfies the ultrametric triangle inequality $\|u+v\| \leqslant \max\{\|u\|, \|v\|\}$. The couple $(V, \|\cdot\|)$ is called a *seminormed vector space* over k. The seminorm restricts to a subseminorm on any subspace S by restriction and projects to a quotient-seminorm on any quotient space Q as follows

$$S \xrightarrow{\iota} V \xrightarrow{\pi} Q,\ \forall s \in S, \|s\|_{\mathrm{sub}} \coloneqq \|\iota(s)\|,\ \forall q \in Q, \|q\|_{\mathrm{quot}} \coloneqq \inf_{\pi(q')=q} \|q'\|.$$

The seminorm induces a topology on V and one can take the completion of V with respect to it. A linear map between two normed vector spaces is continuous if and only if it is bounded in operator norm. A complete normed k-vector space is a *Banach* space over k. Classical fundamental results such as the open mapping theorem, the closed graph theorem, the equivalent norm theorem, are valid for Banach spaces over k ([9, §1.3.3]). A continuous linear map is *admissible* if on its image, the induced quotient norm and the subspace norm are equivalent.

The space of norms on V is equipped with a pseudo-distance $\mathrm{d}(\|\cdot\|, \|\cdot\|') \coloneqq \sup_{v\in V\setminus\{0\}}|\log(\|v\|/\|v\|')|$ (it is a distance if V is of finite dimension). Two norms are equivalent $\|\cdot\| \sim \|\cdot\|'$ if there is $C \in \mathbb{R}_+$ such that $\mathrm{d}(\|\cdot\|, \|\cdot\|') \leqslant C$.

In the rest of this article, all (semi)norms are supposed to be ultrametric.

2.1.2. Orthogonal basis

Let $(V, \|\cdot\|)$ be a normed vector space over k. A set of vectors $\{v_i\}$ is *orthogonal* ([4, Definition 2.4.1.1]) if the ultrametric equality holds for any finite sum

$$\forall(\lambda_1, \dots, \lambda_n) \in k^n, \quad \left\| \sum_{i\in\{1,\dots,n\}} \lambda_i e_i \right\| = \max_{i\in\{1,\dots,n\}} \|\lambda_i e_i\|.$$

A norm $\|\cdot\|$ is *diagonalizable* ([7, Definition 1.10]) if there exists an orthogonal basis.

Proposition 2.1. — *Let V be a k-vector space of finite dimension. In the space of norms on V, diagonalizable norms are dense.* ([15, *Proposition 2.2*][7, *Proposition 1.19*])

2.2. Banach algebras

2.2.1. Basic constructions

Let A be a k-algebra (with the unit $\mathbf{1}$) and $\|\cdot\|$ be a seminorm on A (viewed as a vector space over k). It is *sub-multiplicative* if

$$\forall(a, b) \in A \times A, \quad \|ab\| \leqslant \|a\| \cdot \|b\|.$$

It is moreover *power-multiplicative* if the equality holds in the case $a = b$ and is *multiplicative* if the equality holds for general a, b. We call it an *algebra seminorm* on A if it is sub-multiplicative and $\|\mathbf{1}\| = 1$, and denote it by $|||\cdot|||$ in the following. The couple $(A, |||\cdot|||)$ is a *seminormed algebra*. An algebra seminorm induces algebra seminorms on subalgebras or quotient algebras because the submultiplicativity is preserved under restriction or taking quotient. A k-algebra homomorphism between two normed algebras is a *homomorphism of normed algebras* if it is continuous (or equivalently if it is bounded in operator norm).

Let $(A, |||\cdot|||)$ be a normed algebra. It is a *Banach algebra* if the induced topology on A is complete, and is denoted in calligraphic style by $\mathcal{A}$. If the norm is not complete, one can take the completion $(A, |||\cdot|||) \rightsquigarrow \mathcal{A}$.

2.2.2. Spectrum

Let $\mathcal{A}$ be a Banach algebra over k. Its *(Berkovich) spectrum* $\mathfrak{M}(\mathcal{A})$ is the topological space consisting of points z as multiplicative algebra seminorms $|\cdot|_z$ that are bounded from above by $|||\cdot|||$, equipped with the weak topology that makes applications $|f| : z \in \mathfrak{M} \mapsto |f|_z$ continuous for all $f \in \mathcal{A}$. For any subset V of $\mathfrak{M}(\mathcal{A})$, we denote by $\mathrm{Int}^t(V)$ the *topological interior* of V.

Points in the spectrum are in bijective correspondence to pairs $(\mathfrak{p}_z, |\cdot|_z)$ consisting of a closed ideal of $\mathcal{A}$ (the null-space of $|\cdot|_z$) and an absolute value on the residue algebra (the projection of $|\cdot|_z$) for which the canonical quotient homomorphism is continuous. The residue field at $\mathfrak{p}_z$ is denoted by $\kappa(z)$ and its completion with respect to $|\cdot|_z$ is denoted by $\mathcal{H}(z)$.

Let $\mathcal{A}$ be a k-Banach algebra. Then $\mathfrak{M}(\mathcal{A})$ is a non-empty compact Hausdorff topological space ([2, Theorem 1.2.1]). A homomorphism of Banach algebras $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ induces a map $\phi^\star : \mathfrak{M}(\mathcal{A}_2) \to \mathfrak{M}(\mathcal{A}_1)$ sending any element $z \in \mathfrak{M}(\mathcal{A}_2)$ to the point corresponding to the seminorm $|\phi(\cdot)|_z$, which is continuous with respect to the canonical topologies; if ϕ has dense image, then ϕ^* is injective on spectra. ([2, Remark 1.2.2 (iii)]).

2.2.3. Spectral seminorm

Let $(A, |||\cdot|||)$ be a seminormed k-algebra. The *spectral algebra seminorm* is given by

$$\forall f \in A, \quad |||f|||_{\mathrm{sp}} := \lim_{n\to\infty} |||f^n|||^{\frac{1}{n}}.$$

In general, $|||\cdot|||_{\mathrm{sp}}$ is only a seminorm even if $|||\cdot|||$ is a norm, and one has $|||\cdot|||_{\mathrm{sp}} \leqslant |||\cdot|||$. It is ultrametric if the original algebra seminorm is so. Its null-space is the *radical* rad of the seminormed algebra, and if it is $\{0\}$, the seminormed algebra is *semisimple*. Let $\mathcal{A}$ is a Banach algebra, though $|||\cdot|||$ is complete on A, the projection of its spectral $|||\cdot|||_{\mathrm{sp}}$ to $A/\operatorname{rad}(A)$ *may not be complete*, the completion of the normed algebra $(A/\operatorname{rad}(A), |||\cdot|||_{\mathrm{sp}})$ is the *spectral completion* of $\mathcal{A}$ and is denoted by $\mathcal{A}_{\mathrm{sp}}$. One has a canonical homomorphism $\mathcal{A} \xrightarrow{\mathrm{sp}} \mathcal{A}_{\mathrm{sp}}$.

The fundamental result of spectral seminorms due to Gelfand still holds true.

Proposition 2.2. — *Let $\mathcal{A}$ be a k-Banach algebra. Then* sp *induces an isomorphism on spectra $\mathfrak{M}(\mathcal{A}) \simeq \mathfrak{M}(\mathcal{A}_{\mathrm{sp}})$. For any $f \in A$, one has*

$$|||f|||_{\mathrm{sp}} = \max_{z \in \mathfrak{M}(A)} |f|_z.$$

([2, *Theorem 1.3.1*, *Corollaries 1.3.3*, *1.3.4*]).

2.3. Affinoid algebras

Affinoid algebras are a special kind of k-Banach algebras possessing nice finiteness properties.

2.3.1. Basic constructions

Affinoid algebras are k-Banach algebras that are quotient Banach algebras of Tate algebras. Consider the multi-variate polynomial algebra $k[\boldsymbol{T}]$ with $\boldsymbol{T} \coloneqq (T_1, \ldots, T_n)$ for a multi-radius $\boldsymbol{r} = (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$, the *Gauss norm of multi-radius* $\boldsymbol{r}$ on this algebra is defined by

$$\left|\left|\left| \sum_{J \in \mathbb{N}^n} a_J \cdot \boldsymbol{T}^J \right|\right|\right|_{\mathcal{T}_n(\boldsymbol{r})} = \max_J \left\{ |a_J| \cdot \boldsymbol{r}^J \right\}.$$

Note that the Gauss norm is multiplicative. The Banach algebra completion into convergent power series is the *Tate algebra* over k with multi-radius $\boldsymbol{r}$ and is denoted by $\mathcal{T}_n(\boldsymbol{r})$. It is *strict* if $\boldsymbol{r} = (1, \ldots, 1)$, in which case it is simply denoted by $\mathcal{T}_n$.

Thus a k-Banach algebra $\mathcal{A}$ is affinoid if there is a surjective homomorphism of Banach algebras $\mathcal{T}_n(\boldsymbol{r}) \to \mathcal{A}$ for some n and $\boldsymbol{r}$. It is strict if the Tate algebra can be taken to be strict, and in general, one can enlarge the base field to make an affinoid algebra strict. The Banach algebra norm on an affinoid algebra $\mathcal{A}$ is called an *affinoid algebra norm.*

Proposition 2.3. — *Let* $(A, \|\cdot\|)$ *be a normed* k*-algebra (with completion* $\mathcal{A}$*) where* A *is finitely generated. Then there exists an (in fact, many) algebra norm on* A*, denoted by* $\|\cdot\|^{\Diamond}$*, such that* $\|\cdot\| \leqslant \|\cdot\|^{\Diamond}$ *and the completion* $\mathcal{A}^{\Diamond}$ *of* $(A, \|\cdot\|^{\Diamond})$ *is affinoid with a homomorphism of Banach algebras* $\mathcal{A}^{\Diamond} \to \mathcal{A}$*.*

Proof. — Choose a set of generators of A and one can present A as a quotient of $k[\boldsymbol{T}]$. Equip the later with a Gauss–Tate algebra norm with multi-radius $\boldsymbol{r}$, and take the quotient algebra norm on A to be $\|\cdot\|^{\Diamond}$. If $\boldsymbol{r}$ is sufficiently large then one has $\|\cdot\| \leqslant \|\cdot\|^{\Diamond}$ by the ultrametricity and the submultiplicativity of algebra norms. The completion $\mathcal{A}$ of $(A, \|\cdot\|^{\Diamond})$ is the corresponding quotient of the Tate algebra $\mathcal{T}(\boldsymbol{r})$, hence is affinoid. The inequality implies that the identity homomorphism on A extends to a homomorphism of Banach algebras $\mathcal{A}^{\Diamond} \to \mathcal{A}$. □

2.3.2. Finiteness properties

Affinoid algebras enjoy nice finiteness properties thanks to the validity of Weierstrass division theorem ([4, Theorem 5.2.1.2]). As a consequence affinoid algebras are all Noetherian and their ideals are all closed ([2, Proposition 2.1.3]). In particular, Noether normalization still holds true for strict affinoid algebras.

Proposition 2.4. — *For a strict affinoid algebra* $\mathcal{A}$*, there exists an injective and admissible Banach algebra homomorphism* $\mathcal{T}_d \to \mathcal{A}$ *for some* $d \in \mathbb{N}$*, which endows* $\mathcal{A}$ *with a structure of* finite *Banach algebra over* $\mathcal{T}_d$*. Moreover,* d *equals the Krull dimension of* $\mathcal{A}$ ([4, *Corollary 6.1.2.2*]).

As a consequence, the spectral operation on an affinoid algebra seminorm enjoys a finiteness property in the following sense.

Proposition 2.5. — *Let* $\mathcal{A}$ *be an affinoid algebra. Then the spectral seminorm* $\|\cdot\|_{\mathcal{A},sp}$ *is a complete norm on* $\mathcal{A}/\operatorname{rad}\mathcal{A}$*, and the later is* $\mathcal{A}_{\mathrm{sp}}$*. If* $\mathcal{A}$ *is reduced, the spectral norm is equivalent to the original Banach norm, that is to say, there exists* $C \in \mathbb{R}_+$ *such that*

$$\|\cdot\|_{\mathcal{A}} \leqslant C \cdot \|\cdot\|_{\mathcal{A},\mathrm{sp}}.$$

([4, *Theorem 6.2.4.1*], [2, *Proposition 2.1.4.ii*]).

Another consequence is an automatic continuity property of homomorphisms of Banach $\mathcal{A}$-modules. In particular, one has

Proposition 2.6. — *Let $\mathcal{A}$ be an affinoid algebra. Then the category of finite Banach $\mathcal{A}$-algebras is equivalent to the category of finite $\mathcal{A}$-algebras. Concretely, let $\mathcal{B}$ be a Banach algebra and $\gamma : \mathcal{A}^{\oplus e} \to \mathcal{B}$ be a finite homomorphism of $\mathcal{A}$-modules for some $e \in \mathbb{N}$, then the quotient module norm $\gamma(|||\cdot|||_{\mathcal{A}}^{\oplus e})$ is equivalent to the Banach algebra norm $|||\cdot|||_{\mathcal{B}}$ ([2, Proposition 2.1.12]). In particular, $|||\cdot|||_{\mathcal{B}}$ is an affinoid algebra norm.*

Proof. — To show that $|||\cdot|||_{\mathcal{B}}$ is an affinoid algebra norm, look at the homomorphism of $\mathcal{A}$-algebras $\gamma' : \mathcal{A}\{r_1^{-1}T_1, \dots, r_e^{-1}T_e\} \to \mathcal{B}$ defined by $T_i \mapsto \gamma(1_i)$ where $1_i \in \mathcal{A}$ is the unit in the i-th component of $\mathcal{A}^{\oplus e}$ and $r_i = |||\gamma(1_i)|||_{\mathcal{B}}$. By construction γ' is both surjective and continuous. By the open mapping theorem of Banach spaces, $|||\cdot|||_{\mathcal{B}}$ is equivalent to the quotient of the algebra norm on $\mathcal{A}\{r_1^{-1}T_1, \dots, r_e^{-1}T_e\}$ which is affinoid, so $|||\cdot|||_{\mathcal{B}}$ is also affinoid. □

Corollary 2.7. — *Let A and B be reduced k-algebras and $\gamma : A \to B$ be an injective finite homomorphism of algebras. Assume that any element of B has a power in A. Let $|||\cdot|||_2$ be a power-multiplicative algebra norm on B and $|||\cdot|||_1$ be its restriction on A. If the completion $\mathcal{A}$ of $(A, |||\cdot|||_1)$ is an affinoid algebra, so is the completion $\mathcal{B}$ of $(B, |||\cdot|||_2)$.*

Proof. — The k-algebra $\mathcal{B}' := \mathcal{A} \otimes_A B$ is a finite $\mathcal{A}$-algebra, denote by γ the corresponding $\mathcal{A}$-module homomorphism. By the above Proposition, there is an affinoid algebra norm $|||\cdot|||'$ on $\mathcal{B}'$ with $|||\cdot|||' \sim \gamma(|||\cdot|||_1^{\oplus e})$. Since $\mathcal{B}'$ is affinoid, so is its spectral completion $\mathcal{B}'_{\mathrm{sp}}$ by Proposition 2.5, the later is equipped with the spectral norm $|||\cdot|||'_{\mathrm{sp}}$. It suffices to show that $|||\cdot|||'_{\mathrm{sp}} = |||\cdot|||_2$. Taking restriction on $\mathcal{A}$, one gets $[\gamma(|||\cdot|||_1^{\oplus e})] \restriction_{\mathcal{A}} \sim [|||\cdot|||'] \restriction_{\mathcal{A}}$. Since $[\gamma(|||\cdot|||_1^{\oplus e})] \restriction_{\mathcal{A}} \sim |||\cdot|||_1$, one gets $[|||\cdot|||'] \restriction_{\mathcal{A}} \sim |||\cdot|||_1$. Taking the spectral seminorm one gets $[|||\cdot|||'_{\mathrm{sp}}] \restriction_{\mathcal{A}} = |||\cdot|||_1$. By the assumption that every element in B has a power in A, one deduces that $|||\cdot|||'_{\mathrm{sp}} = |||\cdot|||_2$ as algebra norms on B, because they are both power-multiplicative. □

2.3.3. Locally ringed space structure

It is possible to endow the spectrum of an affinoid algebra with a locally ringed space structure. Let $\mathcal{A}$ be an affinoid algebra with spectrum Z. An *affinoid domain* is a closed subset V of Z homeomorphic to $(\iota_V)^\star(\mathfrak{M}(\mathcal{A}_V))$ for some affinoid algebra $\mathcal{A}_V$ and Banach algebra homomorphism $\iota_V : \mathcal{A} \to \mathcal{A}_V$ which satisfies the *universal mapping property* (the existence of a factorization) for any other affinoid algebra homomorphism $\phi : \mathcal{A} \to \mathcal{C}$ with $\phi^\star(\mathfrak{M}(\mathcal{C}))$ contained in V.

A finite intersection of affinoid domain is an affinoid domain, and affinoid domain of V is also an affinoid domain of Z. A *G-open* set is a finite union of affinoid domains in Z, and the *G-topology* on Z is the one with G-open sets as admissible opens and finite coverings as admissible coverings. One denotes by Z_G the space equipped with this G-topology. Note that any point $z \in Z$ has a fundamental system of (closed) neighbourhoods consisting of affinoid domains ([2, Proposition 2.2.3]).

Let $W = \bigcup_{i\in I} V_i$ be a G-open set in $\mathfrak{M}(\mathcal{A})$ where I is a finite set and $\{\mathcal{A}_i\}$ are associated affinoid algebras, one associate to W the Banach k-subalgebra

$$\mathcal{A}_W := \ker\left(\prod_{i\in I} \mathcal{A}_{V_i} \to \prod_{i,j\in I} \mathcal{A}_{V_i\cap V_j}\right),$$

Denote by $\|\!|\cdot|\!\|_W$ the sup seminorm on W, and the spectral completion of $\mathcal{A}_W$ with respect to this seminorm $\mathcal{A}_{W,\mathrm{sp}}$. One obtains the *structural pre-sheaf of affinoid algebras* on $\mathfrak{M}(\mathcal{A})_G$ and it is a sheaf thanks to a deep acyclicity theorem of Tate ([4, Proposition 8.2.2.5]) ([2, Proposition 2.2.5]). Passing to the canonical topology by assigning an open set U of Z the limit $\varprojlim_{W\subseteq U} \mathcal{A}_W$ where W is a G-open set, one obtains the *structural sheaf* $\mathscr{O}_Z$ of Z, which is a sheaf of local rings ([2, Section 2.3]).

The algebra of *analytic functions around a closed subset* Σ in an affinoid space Z is the limit $\varinjlim \Gamma(U, \mathscr{O}_Z)$ where U runs through open neighbourhoods of Σ, denoted by $\Gamma(\Sigma, \mathscr{O}_Z)$. It is equipped with the sup seminorm $f \mapsto \sup_{z\in\Sigma} |f|_z$ which is denoted by $\|\!|\cdot|\!\|_\Sigma$ ([2, Section 2.6]).

2.4. Spectral calculus

Gelfand–Shilov theory allows one to do multi-variable spectral calculus for (commutative) Banach algebras over $\mathbb{C}$. In particular, one can localize a homomorphism of Banach algebras onto a neighbourhood of its spectrum. Its non-Archimedean analogue is developed in [2, Chapter 7] following some studies in [29].

2.4.1. Holomorphic convex envelope

Definition 2.8. — *Let $\mathcal{A}$ be a Banach k-algebra. Let K be a compact subset of $\mathfrak{M}(\mathcal{A})$. The* holomorphic convex envelope *of K in $\mathfrak{M}(\mathcal{A})$ is defined as the subset*

$$\widehat{K} := \left\{ z \in \mathfrak{M}(\mathcal{A}) \mid \forall f \in \mathcal{A}, |f|_z \leqslant \sup_{z'\in K} |f|_{z'} \right\}$$

The subset K is said to be holomorphically convex *in $\mathfrak{M}(\mathcal{A})$ if $\widehat{K} = K$.*

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach k-algebras, and $\phi : \mathcal{A} \to \mathcal{B}$ be a homomorphism of Banach algebras. The *spectrum of the homomorphism* ϕ is defined as the image of $\mathfrak{M}(\mathcal{B})$ in $\mathfrak{M}(\mathcal{A})$ under $\phi^\star$, and is denoted by Σ_ϕ. The holomorphic convexity of Σ_ϕ depends on the density of the image of ϕ.

Proposition 2.9. — *Let $\mathcal{A}$ be a k-affinoid algebra, $\mathcal{B}$ be a Banach k-algebra. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a homomorphism of Banach k-algebras. Let $\mathcal{B}'$ be the closed subalgebra generated by the image of ϕ of $\mathcal{A}$ in $\mathcal{B}$ and let $\phi' : \mathcal{A} \to \mathcal{B}'$ be the restricted homomorphism. Then $\widehat{\Sigma_\phi} = \Sigma_{\phi'}$. If ϕ has dense image, then Σ_ϕ is holomorphically convex.* ([2, *Proposition 7.3.1*])

In case where the spectrum of a homomorphism is not holomorphically convex, one can add variables to the source algebra so that spectrum of the extended homomorphism is holomorphically convex. The following is an analogue of Arens–Calderón theorem.

Proposition 2.10. — *Let $\mathcal{A}$ be a k-affinoid algebra, $\mathcal{B}$ be a Banach k-algebra. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a homomorphism of Banach k-algebras. Then for any open neighbourhood U in $\mathfrak{M}(\mathcal{A})$ of the spectrum Σ_ϕ, there exists a homomorphism of Banach algebras extending ϕ*

$$\phi^\uparrow : \mathcal{A}^\uparrow \coloneqq \mathcal{A}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to \mathcal{B}$$

such that $\mathrm{pr}(\widehat{\Sigma_\phi}) \subseteq U$, *where* $\mathrm{pr} : \mathfrak{M}(\mathcal{A}^\uparrow) \to \mathfrak{M}(\mathcal{A})$ *is the canonical projection.* ([2, *Proposition 7.3.3*])

2.4.2. Holomorphic functional calculus

It is easy to localize the homomorphism of Banach algebras to any neighbourhood of its spectrum if the spectrum is holomorphically convex; otherwise, one uses Proposition 2.10 to perform such a localization.

Theorem 2.11. — *Let $\phi : \mathcal{A} \to \mathcal{B}$ be a homomorphism of Banach algebras from an affinoid algebra to a Banach algebra. Let $W \subseteq \mathfrak{M}(\mathcal{A})$ be a G-open set that is a neighbourhood of Σ_ϕ. Then there exists a Banach algebra homomorphism*

$$\theta_\phi : \mathcal{A}_{W,\mathrm{sp}} \to \mathcal{B}$$

satisfying $\phi = \theta_\phi \circ \iota_W$, where $\iota_W : \mathcal{A} \to \mathcal{A}_W \to \mathcal{A}_{W,\mathrm{sp}}$ is the Banach algebra homomorphism induced by the inclusion $W \subseteq \mathfrak{M}(\mathcal{A})$. ([2, *Theorem 7.3.2*])

Remark 2.12. — The original proof shows the existence of a Banach algebra homomorphism $\mathcal{A}_W \to \mathcal{B}$ that factorizes through the normed algebra $\Gamma(\Sigma_\phi, \mathscr{O})$ which is equipped with sup norm. The restriction homomorphism $\mathcal{A}_{W,\mathrm{sp}} \to \Gamma(\Sigma_\phi, \mathscr{O})$ (both equipped with sup norms) induced by

$\Sigma_\phi \hookrightarrow W$ is bounded, and its composition with the bounded homomorphism $\Gamma(\Sigma_\phi, \mathscr{O}) \to \mathcal{B}$ is our θ_ϕ.

2.5. Analytification of schemes of finite type

2.5.1. Analytification

Schemes of finite type over $\operatorname{Spec} k$ can be analytified to Berkovich analytic spaces. Locally, for a finitely generated k-algebra A_Z with Zariski spectrum Z, one first obtains a topological space Z^{an} consisting of (equivalent classes of) multiplicative seminorms equipped with the weak (canonical) topology. A locally ringed space structure is obtained by considering the sheaf $\mathscr{O}_{Z^{\mathrm{an}}}$ of *analytic functions*, which to any open set $U \subseteq Z^{an}$ associates functions $h : U \to \coprod_{z \in U} \mathcal{H}(z)$ that are local uniform limits of rational functions. A morphism between affine schemes induces a morphism between their analytifications. Globally, the analytification X^{an} of a finite type scheme X over $\operatorname{Spec} k$ is obtained via a process of glueing [2, §3].

2.5.2. Topological properties on the analytification

We list some topological properties that will be useful. The next one follows directly from the construction of Berkovich spectrum and its Hausdorff compactness.

Proposition 2.13. — *Let $(A, \|\!|\cdot|\!\|)$ be a normed algebra where A is a finitely generated k-algebra, and denote by $\mathcal{A}$ its k-Banach algebra completion. Then the canonical homomorphism of k-algebras from A to $\mathcal{A}$ induces a continuous map which embeds the Berkovich spectrum $\mathfrak{M}(\mathcal{A})$ into $(\operatorname{Spec} A)^{\mathrm{an}}$ as a compact closed subspace, and the canonical topology on $\mathfrak{M}(\mathcal{A})$ coincides with the induced topology from the canonical topology on $(\operatorname{Spec} A)^{\mathrm{an}}$.* ([2, *Remark 3.4.2*])

Below are more fundamental ones concerning scheme-theoretic properties.

Proposition 2.14. — *Let $\phi : X \to Y$ be a morphism of schemes of locally finite type over $\operatorname{Spec} k$. Then it induces a morphism of locally ringed spaces $\phi^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$. In particular, the space map ϕ^{an} is continuous, and it is (1) separated, (2) injective, (3) surjective, (4) an open immersion and (5) an isomorphism if and only if ϕ^{an} has the same property.* ([2, *Proposition 3.4.6*])

Theorem 2.15. — *If X is proper, then X^{an} is Hausdorff and compact.* ([2, *Theorem 3.4.8*])

3. Normed section algebra and its spectrum

In this section, we lay the framework for our study of a metrised line bundle (L,ϕ) by globally defined objects. We gather the normed vector spaces of global sections of $L^{\otimes n}$ equipped with the sup norm induced by $n\phi$, to form a normed algebra of sections $\mathcal{R}(L,\phi)$. We relate its Berkovich spectrum to the dual unit disc bundle of $(L,\mathcal{P}(\phi))$, and interpret geometrically the global norm metric positivity of $\mathcal{P}(\phi)$ as the holomorphic convexity of this spectrum in the analytic affine cone.

3.1. Section algebra and norms on it

We review the classical affine cone construction from the section algebra, and the ways to put norms on them.

3.1.1. Graded algebras

We only consider $\mathbb{N}$-graded algebras. The inclusion $S \to R$ of a graded k-subalgebra is *(TN)-isomorphic* if the inclusion $S_n \to R_n$ is an isomorphism of k-linear space for n large enough. The Proj construction gives k-scheme $\mathrm{Proj}(R)$ and invertible sheaf $\mathscr{O}(1)_R$. Denote the affine cone and its cone vertex, the total space of $\mathscr{O}(1)_R^{-1}$ and its zero section by

$$\mathbf{C}(R) \coloneqq \mathrm{Spec}_k R \hookleftarrow \mathbf{0}_R, \quad \mathbf{V}(R) \coloneqq \mathrm{Spec}_{\mathrm{Proj}(R)} \mathrm{Sym}\, \mathscr{O}_R(1) \hookleftarrow \mathbf{O}_R,$$

and the canonical contraction by

$$\mathbf{V}(R) \xrightarrow{p} \mathbf{C}(R).$$

If $S \to R$ is (TN)-isomorphic, it induces isomorphisms

$$\mathrm{Proj}(R) \to \mathrm{Proj}(S), \quad \mathbf{V}(R) \to \mathbf{V}(S), \quad \mathbf{C}(R) \setminus \mathbf{0}_R \to \mathbf{C}(S) \setminus \mathbf{0}_S.$$

3.1.2. Section algebras

Let k be a field. Let $\pi : X \to \mathrm{Spec}\, k$ be a scheme of finite type over $\mathrm{Spec}\, k$. For any invertible $\mathscr{O}_X$-module L and any $n \in \mathbb{N}_{\geqslant 1}$, denote the k-vector space of sections and the k-graded algebra of sections by

$$R_n(L) \coloneqq H^0(X, L^{\otimes n}), \quad R(L) \coloneqq \bigoplus_{n\in\mathbb{N}} R_n(L)$$

The corresponding geometric objects of $R(L)$ are denoted by

$$\mathbf{C}(L) := \operatorname{Spec} R(L) \hookleftarrow \mathbf{0}_L, \quad \mathbf{V}(L) := \operatorname{Spec}_{\mathscr{O}_X}(\operatorname{Sym} L) \hookleftarrow \mathbf{O}_L,$$
$$\mathbf{V}(L) \xrightarrow{p_L} \mathbf{C}(L).$$

These schemes are related as follows.

Proposition 3.1. — *Assume that X is separated and quasi-compact, and that L is ample. Then the following diagram is commutative.*

$$\begin{array}{ccccc} X & \xrightarrow{\mathbb{O}_L} & \mathbf{V}(L) & \xrightarrow{\pi_L} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p_L} & & \downarrow{\scriptstyle \pi} \\ \operatorname{Spec} k & \xrightarrow[\mathbf{0}_L]{} & \mathbf{C}(L) & \xrightarrow[\varpi_L]{} & \operatorname{Spec} k \end{array}$$

Moreover, the restriction of p_L to $\mathbf{V}(L) \setminus \mathbb{O}_L(X)$ is an open immersion, whose image is contained in $\mathbf{C}(L) \setminus \mathbf{0}_L$. If X is proper, then p_L restricted to $\mathbf{V}(L)\setminus\mathbb{O}_L(X)$ defines an isomorphism between $\mathbf{V}(L)\setminus\mathbb{O}_L(X)$ and $\mathbf{C}(L)\setminus\mathbf{0}_L$. ([19, *Proposition 8.8.2, Remark 8.8.3*])

For any $x \in X$ with residue field $\kappa(x)$, the base change induces:

$$L(x) := L \otimes_{\mathscr{O}_X} \kappa(x), \quad \mathbf{V}(L)(x) := \mathbf{V}(L) \otimes_{\mathscr{O}_X} \kappa(x) = \operatorname{Spec}_{\kappa(x)} \operatorname{Sym}_{\kappa(x)} L(x)$$
$$R(L)(x) := R(L) \otimes_k \kappa(x), \quad \gamma_L(x) : R(L)(x) \longrightarrow \operatorname{Sym}_{\kappa(x)} L(x)$$

where $\gamma_L(x)$ is a (TN)-isomorphic homomorphism of graded $\kappa(x)$-algebras.

Let Y be a closed subscheme of X. The restriction Y gives an $\mathscr{O}_Y$ invertible sheaf $L|_Y$ and induces homomorphisms of k-vector space and k-graded algebras

$$R_n(L) \twoheadrightarrow R_n(L_{X|Y}) \hookrightarrow R_n(L|_Y), \quad R(L) \twoheadrightarrow R(L_{X|Y}) \hookrightarrow R(L|_Y),$$

where the middle terms are images of the composition restriction homomorphisms.

3.1.3. Analytifications

One may consider Berkovich analytifications for schemes and morphisms in the previous part, to get X^{an} and an $\mathscr{O}_{X^{\mathrm{an}}}$-invertible sheaf L^{an}, together with $\mathbf{C}(L)^{\mathrm{an}}$, $\mathbf{V}(L)^{\mathrm{an}}$ and analytified morphisms such as p_L^{an}.

For any $x \in X^{\mathrm{an}}$, denotes by $\mathcal{H}(x)$ the completed residual field at x, the analytic base change induces:

$$L(x) := L \otimes_{\mathscr{O}_X} \mathcal{H}(x),$$
$$\mathbf{V}(L)^{\mathrm{an}}(x) := \mathbf{V}(L)^{\mathrm{an}} \otimes_{\mathcal{O}_X} \mathcal{H}(x) = (\operatorname{Spec}_{\mathcal{H}(x)} \operatorname{Sym} L(x))^{\mathrm{an}}$$

Tautologically, any $w \in \mathbf{V}(L)^{\mathrm{an}}$ corresponds to a pair (x, w_x) where $x \in X^{\mathrm{an}}$, $w_x \in \mathbf{V}(L)^{\mathrm{an}}(x)$ are induced by the valued field extensions $\mathcal{H}(w)/\mathcal{H}(x)/k$.

3.1.4. Global graded algebra norms

Let R be a graded k-algebra. For any $n \in \mathbb{N}$, let $\|\cdot\|_n$ be a norm on R_n. The family of norms $(\|\cdot\|_n)_{n\in\mathbb{N}}$ is said to be *sub-multiplicative* if

$$\forall (m,n) \in \mathbb{N}^2, s_m \in R_m,\ s_n \in R_n, \quad \|s_m \cdot s_n\|_{m+n} \leqslant \|s_m\|_m \cdot \|s_n\|_n, \tag{3.1}$$

power-multiplicative if moreover equality holds in (3.1) for $s_m = s_n$ and *multiplicative* if equality holds for products of general homogeneous elements. The *orthogonal sum* $\bigoplus_{n\in\mathbb{N}} \|\cdot\|$ is the norm on R defined by $(s_n)_{n\in\mathbb{N}} \mapsto \sup_{n\in\mathbb{N}} \|s_n\|_n$.

Let $|||\cdot|||$ be a norm on R. By restriction to homogeneous components $\{R_n\}$, one gets a family of norms denoted by $\{|||\cdot|||_{\downarrow n}\}_{n\in\mathbb{N}}$, and the orthogonal sum of this family gives back a norm on R denoted by $|||\cdot|||^+$ (explicitly, $|||s|||^+ = \max_{n\in\mathbb{N}} |||s_n|||$). By construction one has $|||\cdot||| \leqslant |||\cdot|||^+$, and if the equality holds, one says that $|||\cdot|||$ is *graded* (or *orthogonal*) for $\{R_n\}$. If $|||\cdot|||$ is an algebra norm, namely if it is submultiplicative with $|||\mathbf{1}||| = 1$, then $|||\cdot|||^+$ is also submultiplicative, hence an algebra norm: if $r = s \cdot t$, then

$$\forall m \in \mathbb{N}, \tag{3.2}$$

$$\begin{aligned} |||r_m||| = |||\sum_{n\in\mathbb{N}} s_n \cdot t_{m-n}||| &\leqslant \sup_{n\in\mathbb{N}}\{|||s_n||| \cdot |||t_{m-n}|||\} \\ &\leqslant \sup_{n\in\mathbb{N}}\{|||s_n|||\} \cdot \sup_{n\in\mathbb{N}}\{|||t_n|||\} = |||(s_n)_{n\in\mathbb{N}}|||^+ \cdot |||(t_n)_{n\in\mathbb{N}}|||^+. \end{aligned}$$

Proposition 3.2. — *There is a bijective map between the set of* graded *submultiplicative (resp. power-multiplicative, resp. multiplicative) norms $|||\cdot|||$ on R and the set of submultiplicative (resp. power-multiplicative, resp. multiplicative) families of norms $(\|\cdot\|_n)_{n\in\mathbb{N}}$ on $\bigoplus_{n\in\mathbb{N}} R_n$, given respectively by the restrictions and the orthogonal sum*

$$\forall n \in \mathbb{N}, \|\cdot\|_n := |||\cdot|||_{\downarrow n}; \quad |||\cdot||| := \bigoplus_{n\in\mathbb{N}} \|\cdot\|_n.$$

Proof. — It suffices to check that the inverse map (the orthogonal sum) preserves desired properties for multiplications. For submultiplicativity this follows from (3.2). For multiplicativity, let n_1 (resp. n_2) be the smallest homogeneous degree maximizing $\{|||s_n|||\}_{n\in\mathbb{N}}$ ($\{|||t_n|||\}_{n\in\mathbb{N}}$), then as $|||\cdot|||$ is multiplicative, (3.2) is an equality for $m = n_1 + n_2$, because the term $s_{n_1} \cdot t_{n_2}$

is the unique one of maximal norm among all products contributing to r_m, hence the orthogonal sum is also multiplicative. For power-multiplicativity the argument is similar. □

Corollary 3.3. — *Let $|||\cdot|||$ be a seminorm on R. If it is sub-multiplicative (resp. power-multiplicative, resp. multiplicative), so is $|||\cdot|||^+$.*

3.1.5. Spectral seminorm

Consider graded algebra R with a graded algebra norm $|||\cdot|||$. Denote by $\mathcal{R}(|||\cdot|||)$ the Banach algebra completion and by $\mathfrak{M}$ its Berkovich spectrum. By (3.2) the radical is a homogeneous ideal of R. The spectral seminorm $|||\cdot|||_{\mathrm{sp}}$ is given by $a \mapsto \lim_{n\to\infty} |||a^n|||^{\frac{1}{n}}$.

Lemma 3.4. — *Let R be a graded k-algebra and $|||\cdot|||$ be a graded algebra seminorm on R. If $|\cdot|_z$ is a multiplicative seminorm on R, so is $|\cdot|_z^+$. If $|\cdot|_z$ is bounded from above by $|||\cdot|||$ (hence corresponds to a point in $\mathfrak{M}$), so is $|\cdot|_z^+$.*

Proof. — The first assertion follows from Corollary 3.3. For the second assertion, by the assumption there exists $C > 0$ such that

$$|s|_z^+ = \max_{n\in\mathbb{N}} |s_n|_z \leqslant C \cdot \max_{n\in\mathbb{N}} |||s|||_{\downarrow n} = C \cdot |||s|||.$$

So $|\cdot|_z^+$ is also bounded from above by $|||\cdot|||$ and corresponds to a point in $\mathfrak{M}$. □

Proposition 3.5. — *Let R be a graded k-algebra and $|||\cdot|||$ be a graded algebra seminorm, then its spectral seminorm $|||\cdot|||_{\mathrm{sp}}$ is also graded.*

Proof. — By Corollary 3.3, the seminorm $|||\cdot|||_{\mathrm{sp}}^+$ is power-multiplicative since $|||\cdot|||_{\mathrm{sp}}$ is so, and one has $|||\cdot|||_{\mathrm{sp}} \leqslant |||\cdot|||_{\mathrm{sp}}^+$. It suffices to show the reverse inequality. For $s \in R$, let $n_0 \in \mathbb{N}$ be a homogeneous degree maximizing $\{|||s_n|||_{\mathrm{sp}}\}_{n\in\mathbb{N}}$. Since $\mathfrak{M}$ is compact and the function $|s_{n_0}|_{(\cdot)}$ is continuous on it, one can find a point z_0 such that $\sup_{z\in\mathfrak{M}} |s_{n_0}|_z$ is achieved at z_0. Then since $|||\cdot|||$ is graded, Lemma 3.4 implies that the point corresponding to the *graded* multiplicative seminorm $|\cdot|_{z_0}^+$ also belongs to $\mathfrak{M}$. Thus with Theorem 2.2, one has

$$|||s|||_{\mathrm{sp}} = \sup_{z\in\mathfrak{M}} |s|_z \geqslant |s|_{z_0}^+ = \max_{n\in\mathbb{N}} |s_n|_{z_0} \geqslant |s_{n_0}|_{z_0} = |||s_{n_0}|||_{\mathrm{sp}} = |||s|||_{\mathrm{sp}}^+.$$

Hence $|||\cdot|||_{\mathrm{sp}}^+$ is equal to $|||\cdot|||_{\mathrm{sp}}$ and is graded. □

Consider the one-variable symmetric algebra with the natural grading. Graded algebra norms on them are rather rigid.

Proposition 3.6. — *Let $S \subseteq k[T]$ be a (TN)-isomorphic graded subalgebra, then any graded power-multiplicative seminorm $|||\cdot|||$ on S is the restriction of the Gauss–Tate algebra norm of radius $r := |||T^n|||^{\frac{1}{n}}$ (for any large n). Let $|\cdot|_z$ be a multiplicative algebra seminorm on $k[T]$, then it is bounded from above by $|||\cdot|||$ if and only if $|T|_z \leqslant r$.*

Proof. — For any $\sum \lambda_n \cdot T^n \in S$, since the norm is graded and is power-mutliplicative,

$$\left|\left|\left|\sum \lambda_n \cdot T^n\right|\right|\right| = \max_{n\in\mathbb{N}} \left\{|||\lambda_n \cdot T^n|||\right\} = \max_{n\in\mathbb{N}} \left\{|\lambda_n|\cdot|||T^n|||\right\} = \max_{n\in\mathbb{N}} \left\{|\lambda_n|\cdot|||T|||^n\right\},$$

thus $|||\cdot|||$ identifies with the restriction of a Gauss–Tate norm. For the second assertion, if $|\cdot| \leqslant C \cdot |||\cdot|||$, as $|\cdot|$ is mulitplicative then

$$|T|_z = \lim_{n\to\infty} |T^n|^{\frac{1}{n}} \leqslant \lim_{n\to\infty} (C \cdot |||T^n|||)^{\frac{1}{n}} = r;$$

conversly if $|T|_z \leqslant r$, then $|\sum \lambda_n \cdot T^n|_z \leqslant |||\sum \lambda_n \cdot T^n|||$ for any $\sum \lambda_n \cdot T^n$ thanks to the ultrametricity of $|\cdot|_z$ and the gradedness of $|||\cdot|||$. ☐

Remark 3.7. — The above assertion is not true if one replaces $k[T]$ by $k[T_1, \ldots, T_d]$ for $d \geqslant 2$. In fact, the Gauss–Tate norm is multiplicative, so it is equal to its spectral norm. They are orthogonal with respect to the grading by total degree, and even better, orthogonal with respect to the multi-grading by the multi-degree of $k[T_1, \ldots, T_n]$. The particular feature for $n = 1$ is that the two gradings coincide.

3.1.6. Metrics and dual disc bundle

A *metric* on L^{an} is given as the assignment to any $x \in X^{\mathrm{an}}$ a norm $|(\cdot)|_\phi(x)$ on the $\mathcal{H}(x)$-vector space $L(x)$. It is upper (resp. lower) semicontinuous if the function $x \mapsto |s(x)|_\phi$ is so for any local section s over any Zariski open set. One writes additively the product of metrics, in particular the dilation of ϕ by a factor e^ϵ (given by $\{|\cdot|_\phi(x) \cdot \mathrm{e}^\epsilon\}_{x\in X^{\mathrm{an}}}$) for some $\epsilon \in \mathbb{R}$ is denoted by $\phi(\epsilon)$. The restriction of ϕ gives a metric $\phi|_Y$ on $L|_Y$.

Proposition 3.8. — *For any $x \in X^{\mathrm{an}}$, any metric ϕ determines a family of norms $\{|\cdot|_{n\phi}(x)\}_{n\in\mathbb{N}}$ on the graded pieces $\{L^{\otimes n}(x)\}_{n\in\mathbb{N}}$ of the graded coordinate algebra $\mathrm{Sym}_{\mathcal{H}(x)} L(x)$ of the fiber $\mathbf{V}(L)^{\mathrm{an}}(x)$. Its orthogonal sum $|||\cdot|||_\phi(x)$ is a graded multiplicative seminorm on $\mathrm{Sym}_{\mathcal{H}(x)} L(x)$ of Gauss–Tate type. The Banach algebra of completion of $(\mathrm{Sym}\, L(x), |||\cdot|||_\phi(x))$ is a Tate algebra over $\mathcal{H}(x)$, whose spectrum is an affinoid disc in $\mathbf{V}(L)^{\mathrm{an}}(x)$.*

Proof. — The family is clearly multiplicative, the conclusions follow from Proposition 3.2 and Proposition 3.6. □

The *dual closed (resp. open) unit disc bundle* of the metrised pair (L, ϕ) are

$$\overline{\mathbf{D}}^{\vee}(L,\phi) := \left\{ w \in \mathbf{V}(L)^{\mathrm{an}} : \forall e(x) \in L(x), |e(x)|_{w_x} \leqslant |e(x)|_{\phi} \right\},$$
$$\mathbf{D}^{\vee}(L,\phi) := \left\{ w \in \mathbf{V}(L)^{\mathrm{an}} : \forall e(x) \in L(x), |e(x)|_{w_x} < |e(x)|_{\phi} \right\}.$$

they are subspaces of $\mathbf{V}(L)^{\mathrm{an}}$ equipped with subspace topology, where w corresponds to the pair (x, w_x). By Proposition 3.6, one knows that $w \in \overline{\mathbf{D}}^{\vee}(L,\phi)$ if and only if $w_x \in \overline{\mathbf{D}}^{\vee}(L,\phi)(x)$.

3.1.7. Fubini–Study metrics

Let L be an invertible $\mathscr{O}_X$-sheaf. The evaluation maps $R_1(L)\otimes_k \mathcal{H}(x) \longrightarrow L(x)$ induces quotient seminorms $\|\cdot\|_{1,X|x}$ (scalar extension of $\|\cdot\|_1$ from k to $\mathcal{H}(x)$) on stalks $L^{\mathrm{an}}(x)$, and they are norms if the maps are surjective. These norms give rise to a *Fubini–Study* metric on L denoted by $\mathrm{FS}(\|\cdot\|_1)$ ([15, §3.1]). Further, it is *diagonalizable* if the norm $\|\cdot\|_1$ is diagonalizable on $R_1(L)$ (note that [7, Definition 5.2, Corollary 7.18] includes this property in their definition of F.-S. metrics).

Let $|\!|\!|\cdot|\!|\!|$ be an algebra norm on $R(L)$, and $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ be its restrictions to homogeneous components. The evaluation map induces a quotient algebra seminorm $|\!|\!|\cdot|\!|\!|_{X|x}$ on $\mathrm{Sym}_{\mathcal{H}(x)} L(x)$ and a sub-multiplicative family of quotient seminorms $\|\cdot\|_{n,X|x}$ on homogeneous components.

The *Fubini–Study envelope* metric $\mathcal{P}(|\!|\!|\cdot|\!|\!|)$ on L is given by the spectral limit

$$s(x) \mapsto \lim_{n\to\infty} \|s^{\otimes n}(x)\|_{n,X|x}^{\frac{1}{n}}, \quad |\cdot|_{\mathcal{P}(|\!|\!|\cdot|\!|\!|)}(x) = (|\!|\!|\cdot|\!|\!|_{X|x;\mathrm{sp}})_{\downharpoonright 1}. \tag{3.3}$$

In fact, such an envelope metric can be defined if $|\!|\!|\cdot|\!|\!|$ is an algebra norm on a (TN)-isomorphic subalgebra $S(L)$ of $R(L)$.

A metric ϕ on L is said to be *semipositive* if it is a Fubini–Study envelope metric (constructed from some family of norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$) and is continuous [15, §3.2]. The continuity requirement is equivalent to the uniformity in x of the pointwise convergence of metrics, thanks to the compactness of X^{an} by Theorem 2.15. We refer to [8, §5.4] and [7, §6.1] for a clear discussion of other various notions of semipositivity that have been proposed and studied historically in [8, 11, 14, 15, 20, 25, 30], etc.

3.1.8. Global norms induced by a metric

Let $|\cdot|_\phi$ be an upper semicontinuous metric on L. If X is proper over k, then X^{an} is compact. The *sup norms* on $\{R_n(L)\}_{n\in\mathbb{N}}$

$$\|\cdot\|_{n\phi} : s_n \mapsto \sup_{x\in X^{\mathrm{an}}} |s_n(x)|_\phi, \quad |||\cdot|||_\phi \coloneqq \bigoplus \|\cdot\|_{n\phi}$$

give the *sup algebra norm* on $R(L)$. For a reduced closed subscheme Y in X, the restriction map $R(L) \to R(L|_Y)$ induce quotient norms on $R_n(L_{X|Y})$ and $R(L_{X|Y})$

$$\|\cdot\|_{n\phi,X|Y} \coloneqq (\|\cdot\|_{n\phi})_{\mathrm{quot}}, \quad |||\cdot|||_{\phi,X|Y} \coloneqq (|||\cdot|||_\phi)_{\mathrm{quot}}.$$

Proposition 3.9. — *The family of norms $\{\|\cdot\|_{n\phi}\}_{n\in\mathbb{N}}$ is sub-multiplicative and power-multiplicative, so is $|||\cdot|||_\phi$. The family of quotient norms $\{\|\cdot\|_{n\phi,X|Y}\}_{n\in\mathbb{N}}$ is submultiplicative, so is the quotient norm $|||\cdot|||_{\phi,X|Y}$. Both are algebra norms.*

Proof. — Multiplication properties for the sup norm family is verified as follows: for $s_n \in R_n(L)$ and $s_m \in R_m(L)$, one has

$$\begin{aligned}\|s_m \cdot s_n\|_{n\phi} &= \sup_{x\in X^{\mathrm{an}}} |s_m(x)|_{m\phi} \cdot |s_n(x)|_{n\phi} \\ &\leqslant \sup_{x\in X^{\mathrm{an}}} |s_m(x)|_{m\phi} \cdot \sup_{x\in X^{\mathrm{an}}} |s_n(x)|_{n\phi} = \|s_m\|_{m\phi} \cdot \|s_n\|_{n\phi}\end{aligned}$$

and the equality is attained if $s_m = s_n$. By Proposition 3.2, the graded norm $|||\cdot|||_\phi$ is also sub-multiplicative and power-multiplicative. Further one has $|||\mathbf{1}|||_{0\phi} = 1$ as 0ϕ is the trivial metric on $\mathscr{O}_X$ with unit section $\mathbf{1}$. The submultiplicative property of quotient norms follows from that of sup norms. □

The completions ($\rightsquigarrow$) of normed k-algebras give Banach algebras

$$(R(L), |||\cdot|||_\phi) \rightsquigarrow \mathcal{R}(L,\phi), \quad (R(L_{X|Y}), |||\cdot|||_{\phi_{X|Y}}) \rightsquigarrow \mathcal{R}(L_{X|Y}, \phi_{X|Y}).$$

As X and Y are reduced and ϕ is a metric, both Banach algebras have radicals as $\{0\}$, because $|||\cdot|||_\phi$ is a power-multiplicative norm and $|||\cdot|||_{\phi_{X|Y}}$ is bounded from below by the power-multiplicative norm $|||\cdot|||_{\phi|_Y}$.

3.1.9. Overall assumptions

In the rest of the article, we make assumptions that X is an integral projective scheme over k and Y is a closed reduced subscheme; that L is an ample line bundle on X and $|\cdot|_\phi$ is an upper-semicontinuous metric on it.

3.2. Fubini–Study metrics

We list some basic properties for operations of passage between norm and metric, which were obtained in [15] or in [7, Section 6], some arguments are included.

3.2.1. Explicit expression for diagonalizable Fubini–Study metrics

Diagonalizable Fubini–Study metrics have explicit expressions. Below we write scalar n-tuple of k (with $|\cdot|$) and vector n-tuple of some V (with $\|\cdot\|$) and the operations as

$$\begin{gathered}\boldsymbol{a} := (a_1, \dots, a_n) \in k^n, \quad \boldsymbol{b} := (b_1, \dots, b_n) \in V^n,\\ \boldsymbol{a}\cdot\boldsymbol{b} := \sum a_i b_i \in V,\\ \boldsymbol{ab} := (a_1b_1, \dots, a_nb_n) \in V^n,\\ \boldsymbol{b}/\boldsymbol{a} := (b_1/a_1, \dots, b_n/a_n) \in V^n\\ |\boldsymbol{a}| := (|a_1|, \dots, |a_n|) \in \mathbb{R}^n, \quad \|\boldsymbol{b}\| := (\|b_1\|, \dots, \|b_n\|) \in \mathbb{R}^n\end{gathered}$$

and $\max\{\boldsymbol{r}\}$ (resp. $\min\{\boldsymbol{r}\}$) the number $\max\{r_i\}$ (resp. $\min\{r_i\}$) for an n-tuple $\boldsymbol{r}$ of $\mathbb{R}$.

Lemma 3.10. — *Let $(k', |\cdot|')$ be a valued field extension of $(k, |\cdot|)$. For any $\boldsymbol{r} \in \mathbb{R}_+^n$, and $\boldsymbol{\lambda}$ among k'^n, one has*

$$\inf_{\boldsymbol{\lambda}\cdot\mathbf{1}=1} \max\left\{|\boldsymbol{\lambda}|'\cdot\boldsymbol{r}\right\} = \min\{\boldsymbol{r}\}.$$

Proof. — On the one hand, let $l \in \{1, \dots, n\}$ be an index at which $\min\{\boldsymbol{r}\}$ is attained. By taking $\lambda_j = 0$ for $j \neq l$ and $\lambda_l = 1$, one sees that

$$\inf_{\boldsymbol{\lambda}\cdot\mathbf{1}=1} \max\left\{|\boldsymbol{\lambda}|'\cdot\boldsymbol{r}\right\} \leqslant r_l = \min\{\boldsymbol{r}\}.$$

On the other hand, since $|\cdot|'$ is ultrametric, if the sum $\boldsymbol{\lambda}\cdot\mathbf{1}$ is 1, then there exists at least one $m \in \{1, \dots, n\}$ such that $|\lambda_m|' \geqslant 1$, thus

$$\max\{|\boldsymbol{\lambda}|'\cdot\boldsymbol{r}\} \geqslant |\lambda_m|'\cdot r_m \geqslant r_m, \quad \inf_{\boldsymbol{\lambda}\cdot\mathbf{1}=1} \max\left\{|\boldsymbol{\lambda}|'\cdot\boldsymbol{r}\right\} \geqslant \min\{\boldsymbol{r}\}.$$

Hence the two sides are equal. □

Proposition 3.11. — *Let $\|\cdot\|_1$ be a diagonalizable norm on $R_1(L)$ with an orthogonal basis $\{T_j\} =: \boldsymbol{T}$. Then for any $x \in X^{\mathrm{an}}$ and $e(x) \in L(x) \setminus 0$,*

$$|e(x)^{-1}|_{\mathrm{FS}(\|\cdot\|_1)^\vee} = \max\left\{|\kappa_j|_{\mathcal{H}(x)}\cdot\|T_j\|_1^{-1}\right\}, \quad \kappa_j := T_j(x)/e(x) \in \mathcal{H}(x)$$

with the convention that $0^{-1} = +\infty$. In particular, $|e(x)|^{-1}_{\mathrm{FS}(\|\cdot\|_1)} \leqslant 1$ (resp. $<$) if and only if $|\kappa_j|_{\mathcal{H}(x)} \leqslant \|T_j\|_1$ (resp. $<$).

Proof. — By definition, for $\boldsymbol{\lambda}$ among n-tuples of $\mathcal{H}(x)$, one gets

$$|e(x)|_{\mathrm{FS}(\|\cdot\|_1)} = \inf_{(\boldsymbol{\lambda}\cdot\boldsymbol{T})(x)=e(x)} \|\boldsymbol{\lambda}\cdot\boldsymbol{T}\|_1 = \inf_{\boldsymbol{\lambda}\cdot\boldsymbol{\kappa}=1} \|\boldsymbol{\lambda}\cdot\boldsymbol{T}\|_1$$
$$= \inf_{\boldsymbol{\lambda}\boldsymbol{\kappa}\cdot\mathbf{1}=1} \max\{|\boldsymbol{\lambda}\boldsymbol{\kappa}|_{\mathcal{H}(x)}\cdot\|\boldsymbol{T}/\boldsymbol{\kappa}\|_1\}$$

which is equal to $\min\{\|\boldsymbol{T}/\boldsymbol{\kappa}\|_1\}$. Take the inverse one gets the desired equality. □

Corollary 3.12. — *Let* $\|\cdot\|_1$ *be a norm on* $R_1(L)$*, then* $\mathrm{FS}(\|\cdot\|_1)$ *is a continuous metric on* L ([15, *Proposition 3.1*]).

3.2.2. Passages between norms and metrics

Recall that we have operations passing between space of norms and space of metrics as follows

$$\{\|\cdot\|_n \text{ on } R_n(L)\} \underset{\|\cdot\|_{\phi_n}}{\overset{\mathrm{FS}(\|\cdot\|_n)}{\rightleftarrows}} \{\phi_n \text{ on } L^{\otimes n}\}$$

Both these spaces are equipped with distances (hence with induced topologies)

$$\mathrm{d}(\|\cdot\|_{n,1},\|\cdot\|_{n,2}) := \sup_{s\in R_n\setminus\{0\}} \left|\log\frac{\|s\|_{n,1}}{\|s\|_{n,2}}\right|;$$
$$\mathrm{d}(\phi_{n,1},\phi_{n,2}) := \sup_{x\in X^{\mathrm{an}}} \left|\log\frac{\|\cdot\|_{\phi_{n,1}}}{\|\cdot\|_{\phi_{n,2}}}\right|.$$

Proposition 3.13. — *Let* $|\cdot|_\phi$ *and* $|\cdot|_{\phi'}$ *be two upper-semicontinuous metrics on* L*, and* $\|\cdot\|$ *and* $\|\cdot\|'$ *be two norms on* $R_1(L)$*, then*

$$\mathrm{d}(\|\cdot\|_\phi,\|\cdot\|_{\phi'}) \leqslant \mathrm{d}(\phi,\phi'), \quad \mathrm{d}(\mathrm{FS}(\|\cdot\|),\mathrm{FS}(\|\cdot\|')) \leqslant \mathrm{d}(\|\cdot\|,\|\cdot\|').$$

Consequently, in the space of Fubini–Study metrics on L *equipped with the topology induced by* d*, diagonalizable ones are dense.*

Proof. — Both inequalities follows directly from the definitions of these distances. The density follows from Proposition 2.1. □

Proposition 3.14. — *If* ϕ *is Fubini–Study (for example* $\phi = \mathrm{FS}(\|\cdot\|_1)$ *for some norm* $\|\cdot\|_1$ *on* $R_1(L)$*), then for any* $n\in\mathbb{N}$*,* $\mathrm{FS}(\|\cdot\|_{n\phi}) = n\phi$ ([15, *Proposition 3.6*]). *If* ϕ *is semipositive (for example* $\phi = \mathcal{P}(|\!|\!|\cdot|\!|\!|)$ *for some algebra norm* $|\!|\!|\cdot|\!|\!|_1$ *on* $R(L)$*), then* $\mathcal{P}(|\!|\!|\cdot|\!|\!|_\phi) = \phi$.

Proof. — The second statement follows from the first: if the sequence $\mathrm{d}(\phi,\frac{1}{n}\mathrm{FS}(\|\cdot\|_n))$ tends to 0, so does $\mathrm{d}(\frac{1}{n}\mathrm{FS}(\|\cdot\|_{n\phi}),\frac{1}{n}\mathrm{FS}(\|\cdot\|_{\mathrm{FS}(\|\cdot\|_n)}))$, whose limit is $\mathrm{d}(\mathcal{P}(|\!|\!|\cdot|\!|\!|_\phi),\phi))$ by the first statement and the construction. □

3.3. Spectrum and envelope

We describe the Berkovich spectrum of the normed algebra of sections $\mathcal{R}(L,\phi)$, a compact set in the analytification of the Zariski spectrum $\mathbf{C}(L)^{\mathrm{an}}$, through the lens of the continous map induced by the zero-section contraction morphism

$$p^{\mathrm{an}}:\mathbf{V}(L)^{\mathrm{an}}\to\mathbf{C}(L)^{\mathrm{an}}.$$

Recall that for $w\in\mathbf{V}(L)^{\mathrm{an}}$ one denotes by $x\in X^{\mathrm{an}}$ and $w_x\in L(x)$ the corresponding points on the base and the fiber and by $z\in\mathbf{C}(L)^{\mathrm{an}}$ its image under p^{an}.

3.3.1. Spectrum of normed section algebra in terms of envelope metric

Proposition 3.15. — *Let $|||\cdot|||$ be a graded algebra norm on $R(L)$. Then $z\in\mathfrak{M}(\mathcal{R}(L,|||\cdot|||))$ if and only if one of the following criteria holds*

(1) *there exists $C(z)>0$ such that*

$$\forall s\in R(L),\quad |s|_z\leqslant C(z)\cdot|||s|||.$$

(2) *there exists $C(z)>0$ such that*

$$\forall s(x)\in R(L)(x),\quad |s(x)|_{w_x}\leqslant C(z)\cdot|||s(x)|||_{X|x}.$$

(3) *it holds that*

$$\forall s(x)\in R(L)(x),\quad |s(x)|_{w_x}\leqslant|||s(x)|||_{X|x;\mathrm{sp}},$$

where $|||\cdot|||_{X|x;\mathrm{sp}}$ is the spectral algebra seminorm of $|||\cdot|||_{X|x}$.

(4) *it holds that*

$$\forall s(x)\in\mathrm{Sym}_{\kappa(x)}L(x),\quad |s(x)|_{w_x}\leqslant|||s(x)|||'_{X|x;\mathrm{sp}},$$

where $|||\cdot|||'_{X|x;\mathrm{sp}}$ is the unique graded power multiplicative algebra seminorm on $\mathrm{Sym}_{\mathcal{H}(x)}L(x)$ whose restriction on $R(L)_{X|x}$ is $|||\cdot|||_{X|x;\mathrm{sp}}$.

(5) *it holds that*

$$\forall e(x)\in L(x),\ |e(x)|_{w_x}\leqslant|||e(x)|||'_{X|x;\mathrm{sp}},$$

(6) *it holds that*

$$\forall e(x)\in L(x),\ |e(x)|_{w_x}\leqslant|e(x)|_{\mathcal{P}(|||\cdot|||)}.$$

Consequently, $z\in\mathfrak{M}(\mathcal{R}(L,|||\cdot|||))$ if and only if $w\in\overline{\mathbf{D}}^{\vee}(L,\mathcal{P}(|||\cdot|||))$.

Proof. — Start with the evaluation map $(R(L), |||\cdot|||) \xrightarrow{ev_z} (\mathcal{H}(z), |\cdot|_z)$.

(1) unfolds the definition of the Berkovich spectrum.

($1 \Leftrightarrow 2$) as ev_z factorizes through $(R(L), |||\cdot|||) \xrightarrow{ev_x} (R(L)(x), |||\cdot|||_{X|x})$, and $|||\cdot|||_{X|x}$ is just the quotient of the scalar extension from $|||\cdot|||$.

($2 \Rightarrow 3$) by passing to the spectral seminorm using the power-multiplicativity of $|\cdot|_{w_x}$; $\Leftarrow$ is obvious.

($3 \Leftrightarrow 4$) thanks to Corollary 3.6 applied to the (TN)-isomorphism $R(L)(x) \to \mathrm{Sym}_{\mathcal{H}(x)} L(x)$ and $|||\cdot|||_{X|x,\mathrm{sp}}$ on the former.

($4 \Leftrightarrow 5$) by applying Proposition 3.6 to the one-variable polynomial algebra $\mathrm{Sym}_{\mathcal{H}(x)} L(x)$ and $|||\cdot|||'_{X|x,\mathrm{sp}}$.

($5 \Leftrightarrow 6$) is due to the relation (3.3) expressing the envelope metric as the spectral seminorm. □

For abbreviations, denote the contracted closed dual unit disc bundles by

$$\mathfrak{M}(\phi) := p^{\mathrm{an}}(\overline{\mathbf{D}}^\vee(L, \mathcal{P}(\phi))), \quad \mathfrak{M}^-(\phi) := p^{\mathrm{an}}(\overline{\mathbf{D}}^\vee(L, \phi)).$$

Note that the first is the spectrum of the Banach algebra $\mathcal{R}(L, \phi)$, while the second is *a priori* **not** a spectrum. They are both compact sets in $\mathbf{C}(L)^{\mathrm{an}}$. The contractions of open disc bundles are denoted by $\mathfrak{M}_o(\phi)$ and $\mathfrak{M}_o^-(\phi)$.

COROLLARY 3.16. — *Let $|||\cdot|||$ be a graded algebra norm on $R(L)$. The restriction of the map p_L^{an} on the dual unit disc bundle is surjective*

$$\overline{\mathbf{D}}^\vee(L, \mathcal{P}(|||\cdot|||)) \to \mathfrak{M}(\mathcal{R}(L, |||\cdot|||)).$$

Consequently, for any $s \in R(L)$, one has

$$\sup_{z \in \mathfrak{M}(\phi)} |s|_z = |||s|||_\phi.$$

PROPOSITION 3.17. — *Let ϕ be a continuous metric on L. Then $\mathfrak{M}_o^-(\phi)$ is an open subset of $\mathbf{C}(L)^{\mathrm{an}}$. In particular, for $\epsilon > 0$, $\mathfrak{M}^-(\phi)$ is contained in the topological interior of $\mathfrak{M}^-(\phi(\epsilon))$.*

Proof. — As ϕ is continuous, $\mathbf{D}^\vee(L, \phi) \setminus \mathbf{O}_L^{\mathrm{an}}(X^{\mathrm{an}})$ is an open set in $\mathbf{V}(L)^{\mathrm{an}}$, hence its image under p^{an} is an open set of $\mathbf{C}(L)^{\mathrm{an}} \setminus \mathbf{0}$, as the restriction of p^{an} is open by Proposition 3.1 combined with Proposition 2.14. Thus this partial image is an open set in $\mathbf{C}(L)^{\mathrm{an}}$, it is $\mathfrak{M}_o^-(\phi)$ punctured at the vertex $\mathbf{0}$. The vertex $\mathbf{0}$ is not on the topological boundary of $\mathfrak{M}_o(\phi)$, otherwise there would be a net of points $\{z_\alpha\}_{\alpha \in A}$ in $\mathbf{C}(L)^{\mathrm{an}}$ outside of $\mathfrak{M}_o^-(\phi)$ converging to $\mathbf{0}$, then for a finite set of global sections $\{s_i\}_{i \in I}$ that is base point free, all nets $\{|s_i(z_\alpha)|_\phi\}_{\alpha \in A}$ in $\mathbb{R}$ would converge to 0, which is impossible as $\max_{i \in I}\{|s_i(z)|_\phi\} \geqslant c$ for some $c > 0$ and any z outside. So $\mathfrak{M}_o^-(\phi)$ is open.

The second assertion follows immediately as $\mathfrak{M}_o^-(\phi(\epsilon))$ is an open subset of $\mathfrak{M}^-(\phi(\epsilon))$ containing $\mathfrak{M}^-(\phi)$. □

3.3.2. Spectrum of normed section algebra as geometric envelope

Proposition 3.18. — *For any $n \in \mathbb{N}$ and $s_n \in R_n(L)$, one has*

$$\sup_{z\in\mathfrak{M}^-(\phi)} |s_n|_z = \sup_{w\in\overline{\mathbf{D}}^\vee(L,\phi)} |s_n|_w = \sup_{x\in X^{\mathrm{an}}} |s_n(x)|_{n\phi} = \|s_n\|_{n\phi},$$

and for any $s \in R(L)$, one has

$$\sup_{z\in\mathfrak{M}^-(\phi)} |s|_z = \max_{n\in\mathbb{N}} \|s_n\|_{n\phi} = |||s|||_\phi.$$

Proof. — The equalities in the first assertion follows immediately from the construction. For the second assertion, consider the sup norm over the dual unit disc bundle on $R(L)$, thanks to the decomposition of w along p^{an}, it can be written as

$$\sup_{w\in\overline{\mathbf{D}}^\vee(L,\phi)} |\cdot|_w = \sup_{x\in X^{\mathrm{an}}} \sup_{w_x\in\overline{\mathbf{D}}^\vee(L,\phi)(x)} |\cdot|_{w_x} \tag{3.4}$$

As $\overline{\mathbf{D}}^\vee(L,\phi)(x)$ is a $\mathcal{H}(x)$-affinoid disc in $\mathbf{V}(L)^{\mathrm{an}}(x)$, the sup norm over $\overline{\mathbf{D}}^\vee(L,\phi)(x)$ is a Gauss–Tate norm hence is graded on the graded $\mathcal{H}(x)$-algebra $\mathrm{Sym}_{\mathcal{H}(x)} L(x)$, thus it is a graded norm when restricted to the graded $\mathcal{H}(x)$-algebra $R(L)(x)$, and is also graded on the graded k-algebra $R(L)$ as the base change preserves their gradings. Thus the above sup norm over $\overline{\mathbf{D}}^\vee(L,\phi)$ on $R(L)$ is graded because $\sup_{x\in X^{\mathrm{an}}}$ does not concern the grading. The gradedness of this sup norm on the unit disc bundle, together with the first assertion, imply that

$$\sup_{w\in\overline{\mathbf{D}}^\vee(L,\phi)} |s|_w = \max_{n\in\mathbb{N}} \left\{ \sup_{w\in\overline{\mathbf{D}}^\vee(L,\phi)} |s_n|_w \right\} = \max_{n\in\mathbb{N}} \left\{ \|s_n\|_{n\phi} \right\} = |||s|||_\phi. \quad \square$$

Recall that for any compact set K in an affinoid domain $\mathfrak{M}(\mathcal{A})$, one can construct its holomorphic envelope in $\mathfrak{M}(\mathcal{A})$ as the following compact set

$$\widehat{K} := \left\{ z \in \mathfrak{M}(\mathcal{A}),\ \forall f \in \mathcal{A}, |f|_z \leqslant \sup_{z'\in K} |f|_{z'} \right\}.$$

If $\mathcal{A}$ has a dense k-subalgebra A, then one can also use elements in A to define a *polynomial envelope* in $(\mathrm{Spec}\, A)^{\mathrm{an}}$. The two envelopes are identified upon the canonical inclusion map $\mathfrak{M}(\mathcal{A}) \to (\mathrm{Spec}\, A)^{\mathrm{an}}$.

Proposition 3.19. — *Let ϕ be an upper semicontinuous metric on L. Then*

$$\mathfrak{M}(\phi) = \widehat{\mathfrak{M}^-(\phi)},$$

where the envelope is taken in $\mathbf{C}(L)^{\mathrm{an}}$.

Proof. — One first shows the inclusion $\supseteq$. Since $\mathfrak{M}(\phi)$ is the spectrum of $\mathcal{R}(L,\phi)$, it is holomorphic convex in Z by Proposition 2.9. Since it contains $\mathfrak{M}^-(\phi)$, it also contains the holomorphic envelope of $\mathfrak{M}^-(\phi)$.

Conversely, for the inclusion $\subseteq$, observe that for any $s \in R(L)$, one has

$$\sup_{z\in\mathfrak{M}(\phi)} |s|_z = |||s|||_\phi = \sup_{z'\in\mathfrak{M}^-(\phi)} |s|_{z'},$$

where the first equality follows from Corollary 3.16 and the definition of $\mathfrak{M}(\phi)$, and the second one from Proposition 3.18. By the construction of envelope, $\mathfrak{M}(\phi)$ is contained in the envelope of $\mathfrak{M}^-(\phi)$. □

All results in Section 3.3 are valid if one replaces $R(L)$ by a (TN)-isomorphic subalgebra $S(L)$, thanks to Proposition 3.6.

3.4. Restriction to subvarieties

We examine the behavior of norms under the restriction to a subvariety Y. Note that $R(L_{X|Y})$ is a (TN)-isomorphic subalgebra of $R(L|_Y)$ as L is ample.

Proposition 3.20. — *Let $|\cdot|_\phi$ be an upper-semicontinuous metric on L. Then*

$$\mathcal{P}(|||\cdot|||_\phi)|_Y = \mathcal{P}(|||\cdot|||_{\phi,X|Y}).$$

Proof. — By definition, for any $y \in Y^{\mathrm{an}}$, one has

$$\mathcal{P}(|||\cdot|||_\phi)(y) = \lim_{n\to\infty} \frac{1}{n}\,\mathrm{FS}(\|\cdot\|_{n\phi})(y),$$

$$\mathcal{P}(|||\cdot|||_{\phi,X|Y})(y) = \lim_{n\to\infty} \frac{1}{n}\,\mathrm{FS}(\|\cdot\|_{n\phi,X|Y})(y).$$

For all large $n \in \mathbb{N}$, since the k-linear map $R_n(L) \to R_n(L|_Y)$ is surjective, and $\|\cdot\|_{n\phi,X|Y}$ is the quotient norm of $\|\cdot\|_{n\phi}$, one has

$$\mathrm{FS}(\|\cdot\|_{n\phi})(y) = \mathrm{FS}(\|\cdot\|_{n\phi,X|Y})(y).$$

Hence the two envelope metrics are equal. □

Lemma 3.21. — *If ϕ is a Fubini–Study envelope (resp. semipositive) metric on L, so is $\phi|_Y$ on $L|_Y$.*

Proof. — If ϕ is the pointwise limit on X^{an} of $\{\frac{1}{n}\,\mathrm{FS}(\|\cdot\|_n)\}_{n\in\mathbb{N}}$, where $\|\cdot\|_n$ is a norm on $R_n(L)$, then $\phi|_Y$ is the pointwise limit of $\{\frac{1}{n}\,\mathrm{FS}(\|\cdot\|_{n,X|Y})\}$ on Y^{an}. Additionally, the uniformity of convergence in $x \in X^{\mathrm{an}}$ passes to that for $y \in Y^{\mathrm{an}}$ by restriction. □

Proposition 3.22. — *Let ϕ be a semipositive metric on L. Consider two algebra norms $|||\cdot|||_{\phi|_Y}$ and $|||\cdot|||_{\phi,X|Y}$ on $R(L_{X|Y})$. Then the following three metrics are equal*

$$\mathcal{P}(|||\cdot|||_{\phi,X|Y}) = \mathcal{P}(|||\cdot|||_{\phi|_Y}) = \phi|_Y.$$

Proof. — By the assumption and Lemma 3.21, both ϕ and $\phi|_Y$ are semipositive, so Proposition 3.14 implies that both $\mathcal{P}(|||\cdot|||_\phi)|_Y$ and $\mathcal{P}(|||\cdot|||_{\phi|_Y})$ are equal to $\phi|_Y$. The former is further equal to $\mathcal{P}(|||\cdot|||_{\phi,X|Y})$ by Proposition 3.20. □

Corollary 3.23. — *Let ϕ be a semipositive metric on L. Then on $R(L_{X|Y})$, the spectral algebra seminorm of $|||\cdot|||_{\phi,X|Y}$ is equal to $|||\cdot|||_{\phi|_Y}$. The map induced by the canonical homomorphism is an isomorphism of spectra*

$$\mathfrak{M}(\mathcal{R}(L_{X|Y}, \phi_{X|Y})) \simeq \mathfrak{M}(\mathcal{R}(L_{X|Y}, \phi|_Y)).$$

Proof. — By Proposition 2.2, one has a homeomorphism

$$\mathfrak{M}(\mathcal{R}(L_{X|Y}, |||\cdot|||_{\phi,X|Y;\mathrm{sp}})) \simeq \mathfrak{M}(\mathcal{R}(L_{X|Y}, |||\cdot|||_{\phi,X|Y})) = \mathfrak{M}(\mathcal{R}(L_{X|Y}, \phi_{X|Y})),$$

by Proposition 3.22, one has

$$\mathfrak{M}(\mathcal{R}(L_{X|Y}, |||\cdot|||_{\phi,X|Y;\mathrm{sp}})) \simeq \mathfrak{M}(\mathcal{R}(L_{X|Y}, |||\cdot|||_{\phi|_Y})) = \mathfrak{M}(\mathcal{R}(L_{X|Y}, \phi|_Y)).$$

By Proposition 2.2, the two power-multiplicative algebra seminorms $|||\cdot|||_{\phi|_Y}$ and $|||\cdot|||_{\phi,X|Y;\mathrm{sp}}$ on $R(L_{X|Y})$ are equal, since they are both sup norms on the same spectrum. □

4. Normed extension for restricted sections

In this section, given a closed subvariety Y of X and a positively metrised ample line bundle (L, ϕ), we consider the problem of extension of sections of $L|_Y$ to sections of L, with a control of their sup norms induced by ϕ. We interpret the norm control as an upper bound of the quotient algebra norm $|||\cdot|||_{\phi,X|Y}$ by its spectral algebra norm $|||\cdot|||_{\phi_Y}$. We propose two methods to establish this upper bound.

4.1. Review and Outline of strategy

Recall that the restriction maps on spaces of global sections induces homomorphisms $R_n(L) \twoheadrightarrow R_n(L_{X|Y}) \hookrightarrow R_n(L|_Y)$ where the second one is a (TN)-isomorphism (i.e. induces isomorphisms on graded pieces of level above some fixed $n_0 \in \mathbb{N}$). A continuous metric ϕ induces sup norms $\|\cdot\|_{n\phi}$ and $\|\cdot\|_{n\phi|_Y}$. The goal is to find an appropriate asymptotic upper bound for the ratio

$$\inf_{\substack{s_n \in R_n(L) \\ s_n|_Y = t_n}} \frac{\|s_n\|_{n\phi}}{\|t_n\|_{n\phi|_Y}} \leqslant C(n). \tag{4.1}$$

Previously, a normed extension property with a sub-exponential bound of the following form is established in [15, 30]. It is not uniform as the asymptotic range n_Y depends also on the choice of restricted sections.

THEOREM 4.1. — *Let ϕ be a semipositive metric on L, then for any $\epsilon > 0$, any $t_1 \in H^0(Y, L|_Y)$, there is $n_Y \in \mathbb{N}$ such that for any $n \geqslant n_Y$, there exists $s_n \in H^0(X, L^{\otimes n})$ with $s_n|_Y = t_1^{\otimes n}$ and*

$$\|s_n\|_{n\phi} \leqslant \mathrm{e}^{n\epsilon} \cdot (\|t_1\|_{\phi|_Y})^n.$$

The proof is based on lattice norms induced by integral models and the study of the limit metric.

Our new perspective for this normed extension problem is to investigate normed *algebras* of sections instead of normed *vector spaces* of sections. From this point of view, the goal is to compare two algebra norms $|||\cdot|||_{\phi,X|Y}$ and $|||\cdot|||_{\phi_Y}$ on the graded algebra of restricted sections $R(L_{X|Y})$. More precisely, the difficult point is to bound the quotient norm *from above* by the sup norm. To get a uniform version as in [5], a key is to use various finiteness properties of affinoid algebras, implicitly via holomorphic functional calculus (Section 2.4.2) or explicitly via the spectral norm comparison (Section 2.3.2). The advantage of passing from vector spaces to an algebra is to work in a global affine geometric setting accompanied by techniques of Banach algebras.

Below in Section 4.2 we reformulate and simplify the proof of Theorem 4.1. In Section 4.3 and Section 4.4, we prove the uniform version of the normed extension property, where n_Y does not depend on t_n.

THEOREM 4.2. — *Let ϕ be a semipositive metric on L. Then for any $\epsilon > 0$, one may find $C_\epsilon \in \mathbb{R}_+$ such that on $R(L_{X|Y})$, one has*

$$|||\cdot|||_{\phi,X|Y} \leqslant C_\epsilon \cdot |||\cdot|||_{\phi(\epsilon)|_Y}.$$

Consequently, there exists $n_Y \in \mathbb{N}$ *such that for any* $n \geqslant n_Y$

$$\|\cdot\|_{\phi,X|Y} \leqslant \mathrm{e}^{n\epsilon} \cdot \|\cdot\|_{n\phi|_Y},$$

namely for any $t_n \in H^0(Y, L|_Y^{\otimes n})$*, one can find* $s_n \in H^0(X, L^{\otimes n})$ *with* $s_n|_Y = t_n$ *and*

$$\|s_n\|_{n\phi} \leqslant \mathrm{e}^{n\epsilon} \cdot \|t_n\|_{n\phi|_Y}.$$

4.2. Non-uniform version

To compare the two algebra norms, start with the spectral relation between them and the isomorphism of spectra (Corollary 3.23)

$$\text{(4.2)} \quad |||\cdot|||_{\phi,X|Y;\mathrm{sp}} = |||\cdot|||_{\phi|_Y}, \quad \mathfrak{M}(\mathcal{R}(L_{X|Y}, \phi_{X|Y})) \simeq \mathfrak{M}(\mathcal{R}(L_{X|Y}, \phi|_Y)).$$

Proof of the theorem 4.1. — For some large $m \in \mathbb{N}$, one has $t_1^{\otimes m} \in R(L_{X|Y})$, thus by Corollary 3.23 one has

$$\lim_{n\to+\infty} \|t_1^{\otimes mn}\|_{mn\phi,X|Y}^{\frac{1}{n}} = \|t_1^{\otimes m}\|_{m\phi|_Y}.$$

Note that $|||\cdot|||_{\phi,X|Y}$ is submultiplicative and $|||\cdot|||_{\phi|_Y}$ is power-multiplicative, so the sequence $\{\|t_1^{\otimes n}\|_{n\phi,X|Y}^{\frac{1}{n}}\}_{n\geqslant n_0}$ has a limit and one deduces that

$$\lim_{n\to+\infty} \|t_1^{\otimes n}\|_{n\phi,X|Y}^{\frac{1}{n}} = \lim_{n\to+\infty} \|t_1^{\otimes nm}\|_{nm\phi,X|Y}^{\frac{1}{nm}} = \|t_1^{\otimes m}\|_{m\phi|_Y}^{\frac{1}{m}} = \|t_1\|_{\phi|_Y}.$$

The conclusion unfolds this limit of quotient norms for powers of t_1. □

Remark 4.3. — The nature of the spectral norm construction makes n_Y depending on t_1, hence non-uniform for the normed extension.

4.3. Spectral approximation

In this subsection, we prove the uniform version using a spectral method. Instead of the isomorphism (4.2), we examine the strict inclusion of spectra by considering the dilated metric $\phi(\epsilon)$ for $\epsilon \in \mathbb{R}_+$

$$\mathfrak{M}(\mathcal{R}(L_{X|Y}, \phi_{X|Y})) \hookrightarrow W_\epsilon \hookrightarrow \mathfrak{M}(\mathcal{R}(L_{X|Y}, \phi|_Y(\epsilon))),$$

here W_ϵ is an auxiliary G-open set contained in between. The strategy is to translate the above inclusions of spectra to inequalities of their corresponding algebra norms:

$$|||\cdot|||_{\phi,X|Y} \lesssim |||\cdot|||_{W_\epsilon} \lesssim |||\cdot|||_{\phi|_Y(\epsilon)},$$

where $\lesssim$ means $\leqslant$ but omitting geometric constants that depends only on X, Y, L and ϕ. The first inequality, bounding from above a quotient norm $|||\cdot|||_{\phi,X|Y}$ by a sup norm $|||\cdot|||_{W_\epsilon}$, is non-trivial, we obtain it by constructing a Banach algebra "localization" homomorphism from the structural algebra of W using holomorphic functional calculus

$$\mathcal{W}_\epsilon \to \mathcal{R}(L_{X|Y}, \phi_{X|Y}).$$

The second inequality concerning sup norms follows directly from the inclusion.

First we localize the spectrum by affinoid domain covering.

Proposition 4.4. — *Let ϕ be a semipositive metric on L. For any $\epsilon > 0$, there exists a G-open set W_ϵ such that*

$$\mathfrak{M}(\mathcal{R}(L_{X|Y}, \phi_{X|Y})) \simeq \mathfrak{M}(\mathcal{R}(L_{X|Y}, \phi|_Y)) \subseteq W_\epsilon \subseteq \mathfrak{M}(\mathcal{R}(L_{X|Y}, \phi|_Y(\epsilon))).$$

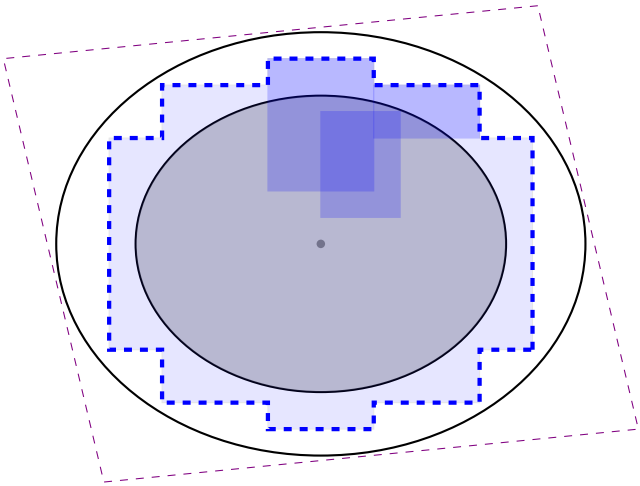
As a consequence one has $|||\cdot|||_{W_\epsilon} \leqslant |||\cdot|||_{\phi|_Y(\epsilon)}$.

Proof. — The first homeomorphism is from 3.23. To construct W_ϵ, note that every point in $\mathbf{C}(L_{X|Y})^{\mathrm{an}}$ has a neighbourhood system consisting of affinoid domains. Hence for any $z \in \mathfrak{M}(\mathcal{R}(L_{X|Y}, \phi|_Y))$, there is an affinoid domain neighbourhood $V(z)$. It can be chosen to be contained in $\mathrm{Int}^t(\mathfrak{M}(\mathcal{R}(L_{X|Y}, \phi|_Y(\epsilon))))$ as this open set contains $\mathfrak{M}(\mathcal{R}(L_{X|Y}, \phi|_Y))$ by Proposition 3.17.

One obtains a covering of $\mathfrak{M}(\mathcal{R}(L_{X|Y}, \phi|_Y))$ by open sets $\{\mathrm{Int}^t V(z)\}$ (for all z in this spectrum). Since this spectrum is compact, there exist finitely many points $\{z_1, \ldots, z_m\}$ such that $\{\mathrm{Int}^t V(z_i)\}_{i\in\{1,\ldots,m\}}$ form a covering. Let W_ϵ be the union of affinoid domains $\{V(z_i)\}_{i\in\{1,\ldots,m\}}$, then it is a G-open set satisfying the desired inclusion relations. (see the picture below)

The comparison of two sup algebra norms follows directly from this inclusion of spectrum and Theorem 2.2. □

Remark 4.5. — One denotes by $\mathcal{W}_\epsilon$ the spectral completion of the structural Banach k-algebra of W_ϵ, so it is equipped with sup norm $|||\cdot|||_{W_\epsilon}$ (see Section 2.3.3).



Spectral approximation

$$\mathfrak{M}(\mathcal{R}(L_{X|Y},\phi_{X|Y})) \subseteq W_\epsilon \subseteq \mathfrak{M}(\mathcal{R}(L_{X|Y},\phi|_Y(\epsilon))) \subseteq \mathfrak{M}(\mathcal{R}(L_{X|Y},\Diamond_\epsilon)$$

Then we localize a Banach algebra homomorphism.

Proposition 4.6. — *Let ϕ be a semipositive metric on L. For any $\epsilon > 0$, there exists a homomorphism of Banach k-algebras*

$$\theta(\epsilon) : \mathcal{W}_\epsilon \to \mathcal{R}(L_{X|Y},\phi_{X|Y})$$

whose restriction on $R(L_{X|Y})$ is the identity homomorphism. Consequently one has $|||\cdot|||_{\phi_{X|Y}} \leqslant C \cdot |||\cdot|||_{W_\epsilon}$ for some $C \in \mathbb{R}_{>0}$.

Proof. — By Proposition 2.3, one may construct an algebra norm $|||\cdot|||^{\Diamond_\epsilon}$ on $R(L_{X|Y})$ which dominates $|||\cdot|||_{\phi(\epsilon),X|Y}$, and gets an affinoid algebra (by completion) denoted by $\mathcal{R}(L_{X|Y},\Diamond_\epsilon)$ together with a homomorphism of Banach algebras $\sigma(\epsilon)$. Denote its composition with the canonical inclusion homomorphism by

$$\tau(\epsilon) : \mathcal{R}(L_{X|Y},\Diamond_\epsilon) \xrightarrow{\sigma(\epsilon)} \mathcal{R}(L_{X|Y},\phi(\epsilon)_{X|Y}) \hookrightarrow \mathcal{R}(L_{X|Y},\phi_{X|Y}),$$

this is a continuous homomorphism from an affinoid algebra to a Banach algebra.

This homomorphism is the identity on $R(L_{X|Y})$, it has dense image, so the induced map $\tau(\epsilon)^\star$ on spectra is injective hence with closed image (as the spectra are Hausdorff compact). The spectrum $\Sigma_{\tau(\epsilon)}$ can be identified with $\mathfrak{M}(\mathcal{R}(L_{X|Y},\phi_{X|Y}))$, which contained in W_ϵ.

One can perform spectral calculus for the homomorphism $\tau(\epsilon)$ and the G-open set W_ϵ which contains $\Sigma_{\tau(\epsilon)}$. By Theorem 2.11, there exists a homomorphism $\theta(\epsilon)$ of Banach algebras (which is the identity homomorphism on $R(L_{X|Y})$) that makes the following diagram commutative

$$\begin{array}{ccc} \mathcal{R}(L_{X|Y}, \Diamond_\epsilon) & \xrightarrow{\tau(\epsilon)} & \mathcal{R}(L_{X|Y}, \phi_{X|Y}) \; . \\ \Big\downarrow i_{W_\epsilon} & \nearrow_{\theta(\epsilon)} & \\ \mathcal{W}_\epsilon & & \end{array}$$

The comparison of algebra norms unfolds the boundedness of $\theta(\epsilon)$. □

Theorem 4.2 follows from this construction of Banach algebra homomorphism:

Proof of the theorem 4.2. — The comparison of algebra norms is obtained as a combination of inequalities in Propositions 4.4 and 4.6. One uses its version for some $\epsilon' < \epsilon$, and deduces that

$$\|t_n\|_{n\phi,X|Y} = |||t_n|||_{\phi,X|Y} \leqslant C_{\epsilon'} \cdot |||t_n|||_{\phi|_Y(\epsilon')} = C_{\epsilon'} \cdot \mathrm{e}^{n\epsilon'} \cdot \|t_n\|_{n\phi|_Y}.$$

It then suffices to take n_Y as the integer $\max\{\lceil \log(C_{\epsilon'})/(\epsilon - \epsilon')\rceil, n_0\}$. □

Remark 4.7. — The continuity of ϕ is necessary in this approach: it is needed in Propositions 4.4 to guarantee that there is enough space between $\mathfrak{M}(\mathcal{R}(L_{X|Y}, \phi_{X|Y}))$ and $\mathfrak{M}(\mathcal{R}(L_{X|Y}, \phi|_Y(\epsilon)))$, so that a G-open set W_ϵ can be inserted.

4.4. Functional approximation

In this subsection we give another proof. We first show that with a diagonalizable Fubini–Study metric, the normed section algebra is an affinoid algebra, and its finiteness property provides the desired norm control. The general case is obtained by a global approximation of algebra norms from this special case, whence the exponential factor is introduced.

PROPOSITION 4.8. — *Let ϕ be a diagonalizable Fubini–Study metric on $\mathscr{O}(1)$ over $\mathbb{P}^d$ induced by $\|\cdot\|_1$ on $R_1(\mathscr{O}(1))$. Then $\mathfrak{M}(\mathcal{R}(\mathscr{O}(1), \phi))$ is a polydisc in $\mathbf{C}(\mathscr{O}(1))^{\mathrm{an}}$. In particular, $\mathcal{R}(\mathscr{O}(1), \phi)$ is isomorphic to a Tate algebra.*

Proof. — By assumption there is an orthogonal basis $\{T_i\}$ of $(R_1(\mathscr{O}(1), \|\cdot\|_1)$. This basis induces an isomorphism $\mathbf{C}(\mathscr{O}(1))^{\mathrm{an}} \simeq (\mathbb{A}^{d+1})^{\mathrm{an}}$. Since ϕ is Fubini–Study, by Proposition 3.14 $\mathcal{P}(\phi) = \phi$. A point $z \in$

$\mathbf{C}(\mathscr{O}(1))^{\mathrm{an}} \setminus \mathbf{0}$ corresponds to a point $w \in \mathbf{V}(\mathscr{O}(1))^{\mathrm{an}} \setminus \mathbf{O}^{\mathrm{an}}$ hence to a pair (x, w_x), and by Proposition 3.15 z is in the spectrum if and only if $|w_x|_{\phi^\vee} \leqslant 1$. By Proposition 3.11 (taking $e(x) = w_x^{-1}$) this is equivalent to requiring that $\kappa_i := T_i(w_x) \in \mathcal{H}(x)$ satisfies $|\kappa_i|_{\mathcal{H}(x)} \leqslant \|T_i\|_1$ for all i. Since $|T_i(w_x)|_{\mathcal{H}(x)} = |T_i|_z$, if $\boldsymbol{r}$ is the multi-radius given by $\|\boldsymbol{T}\|_1$, then the condition can be reformulated as z belonging to the polydisc of radius $\boldsymbol{r}$ of center $\mathbf{0}$ in $(\mathbb{A}^{d+1})^{\mathrm{an}}$.

As the Banach algebra norm $|\!|\!|\cdot|\!|\!|_\phi$ is the sup norm on its spectrum, it is the Gauss–Tate norm of radius $\boldsymbol{r}$, so the normed section algebra is isomorphic to the Tate algebra $\mathcal{T}(\boldsymbol{r})$. □

Proposition 4.9. — *Let ϕ be a metric on L and let $m \in \mathbb{N}$ be an integer such that $L^{\otimes m}$ is very ample and that $m\phi$ is a diagonalizable Fubini–Study metric, then $\mathcal{R}(L, \phi)$, $\mathcal{R}(L_{X|Y}, \phi_{X|Y})$ and $\mathcal{R}(L|_Y, \phi|_Y)$ are affinoid algebras. Moreover, for any $l \in \mathbb{N}$, the three Veronese subalgebras $\mathcal{R}^{(l)}$ are affinoid algebras.*

Proof. — One first assume that L is very ample, with the induced projective embedding $X \xrightarrow{\iota} \mathbb{P}^{N_1}$. The Fubini–Study assumption implies that there is a norm $\|\cdot\|_1$ on $R_1(\mathscr{O}(1))$ such that the Fubini–Study metric $\psi := \mathrm{FS}(\|\cdot\|_1)$ on $\mathscr{O}(1)$ restricts to X to the Fubini–Study metric $\phi := \mathrm{FS}(\|\cdot\|_1)$ on L. By Proposition 4.8, the Banach algebra $\mathcal{R}(\mathscr{O}(1), \psi)$ is an affinoid algebra. Thus its quotient Banach algebra $\mathcal{R}(L, \psi_{\mathbb{P}^{N_1}|X})$ is also affinoid. As $|\!|\!|\cdot|\!|\!|_\phi$ is the spectral norm of $|\!|\!|\cdot|\!|\!|_{\psi,\mathbb{P}^{N_1}|X}$ by Proposition 3.23, and affinoid norms are preserved by passing to spectral seminorms, the spectral version $\mathcal{R}(L, \phi)$ is an affinoid algebra.

In general L is just ample, there exists $m \in \mathbb{N}$ such that $L^{\otimes m}$ is very ample, and that the canonical Veronese inclusion of section algebras $R(L^{\otimes m}) \to R(L)$ is finite. The restriction of $|\!|\!|\cdot|\!|\!|_\phi$ to the Veronese subalgebra is $|\!|\!|\cdot|\!|\!|_{m\phi}$. Since $\mathcal{R}(L^{\otimes m}, m\phi)$ is affinoid, so is $\mathcal{R}(L, \phi)$ by Corollary 2.7. Thus the quotient $\mathcal{R}(L_{X|Y}, \phi_{X|Y})$ and its spectral version $\mathcal{R}(L|_Y, \phi|_Y)$ are affinoid algebras.

The argument is similar for other Veronese subalgebras of level $l \in \mathbb{N}$: from the above one knows that $\mathcal{R}^{(m)}(L, \phi)$ can be presented as the *graded* quotient of some Tate algebra $\mathcal{T}_{N_m}$ (with equivalent norms) hence is affinoid; taking Veronese subalgebras one sees that $\mathcal{R}^{(ml)}(L, \phi)$ is also affinoid as it can be presented as the quotient of $(\mathcal{T}_{N_m})^{(l)}$ which is affinoid. Then consider the inclusion $\mathcal{R}^{(ml)}(L, \phi) \to \mathcal{R}^{(l)}(L, \phi)$ and use Corollary 2.7 to conclude that $\mathcal{R}^{(l)}(L, \phi)$ is affinoid. □

Theorem 4.2 follows from this affinoid property of normed section algebra and an approximation:

Proof of the theorem 4.2. — By the semipositivity assumption and Proposition 3.13, for any $\epsilon > 0$, there exists $m \in \mathbb{N}$ and a metric ϕ_m on L, such that $m\phi_m$ is a diagonalizable Fubini–Study metric on very ample line bundle $L^{\otimes m}$, and that $\mathrm{d}(\phi_m, \phi) \leqslant \frac{1}{3}\epsilon$, thus for any $n \in \mathbb{N}$,

$$\mathrm{d}(\|\cdot\|_{n\phi_m}, \|\cdot\|_{n\phi}) \leqslant \frac{1}{3}n\epsilon, \quad \mathrm{d}(\|\cdot\|_{n\phi_m,X|Y}, \|\cdot\|_{n\phi,X|Y}) \leqslant \frac{1}{3}n\epsilon.$$

By Proposition 4.9, the Banach algebra $\mathcal{R}(L_{X|Y}, (\phi_m)_{X|Y})$ is an affinoid algebra, so there exists $C_m \in \mathbb{R}_+$ such that

$$|||\cdot|||_{\phi_m,X|Y} \leqslant C_m \cdot |||\cdot|||_{\phi_m,Y}.$$

Combining this comparison with the above approximations, take n_Y to be

$\max\{\lceil \ln(C_m)/(\epsilon/3)\rceil, n_0\}$, then for any $n \geqslant n_Y$, one has

$$\begin{aligned}\|t_n\|_{n\phi,X|Y} &\leqslant \mathrm{e}^{\frac{1}{3}n\epsilon} \cdot \|t_n\|_{n\phi_m,X|Y} \leqslant \mathrm{e}^{\frac{1}{3}n\epsilon} \cdot C_m \cdot \|t_n\|_{n\phi_m|_Y} \\ &\leqslant \mathrm{e}^{\frac{2}{3}n\epsilon} \cdot C_m \cdot \|t_n\|_{n\phi|_Y} \leqslant \mathrm{e}^{n\epsilon} \cdot \|t_n\|_{n\phi|_Y}.\end{aligned}$$

Hence there exist $s_n \in R_n(L)$ with $\|s_n\|_{n\phi} \leqslant \mathrm{e}^{n\epsilon} \cdot \|t_n\|_{n\phi|_Y}$. □

Remark 4.10. — The continuity assumption on ϕ is necessary in this approach, for deducing a *uniform* convergence to ϕ in the Fubini–Study approximation by $\{\phi_m\}$.

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