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HOROSPHERES IN TEICHMÜLLER SPACE AND MAPPING CLASS GROUP

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ABSTRACT. — We study the geometry of horospheres in Teichmüller space of Riemann surfaces of genus g with n punctures, where $3g-3+n \geqslant 2$. We show that every C^1-diffeomorphism of Teichmüller space to itself that preserves horospheres is an element of the extended mapping class group. Using the relation between horospheres and metric balls, we obtain a new proof of Royden's Theorem that the isometry group of the Teichmüller metric is the extended mapping class group.

RÉSUMÉ. — Nous étudions la géométrie des horosphéres dans l'espace de Teichmüller des surfaces de Riemann de genre g avec n perforations, oú $3g - 3 + n \geqslant 2$. Nous montrons que chaque C^1-difféomorphisme de l'espace de Teichmüller dans lui-même qui préserve les horosphéres est un élément du groupe mapping class étendu. En utilisant la relation entre les horosphéres et les boules métriques, nous obtenons une nouvelle preuve du théoréme de Royden qui affirme que le groupe d'isométrie de la métrique de Teichmüller est le groupe mapping class étendu.

1. Introduction

In this paper, we study the geometry of horospheres in Teichmüller space. As an application, we give a new proof of Royden's Theorem that every isometry of Teichmüller space with respect to the Teichmüller metric is induced by an element of the mapping class group. Our results rely heavily on the theory of measured foliations as found and developed in [4, 12, 16, 24].

1.1. Background

Let $S = S_{g,n}$ be a Riemann surface of genus g with n punctures, and let $\mathcal{T}_{g,n}$ be the Teichmüller space of S. We endow $\mathcal{T}_{g,n}$ with the Teichmüller metric. Throughout this paper, we assume that $3g - 3 + n \geqslant 2$.

Much of the study of Teichmüller space is inspired by analogies with negatively curved spaces. The Teichmüller metric is a complete Finsler metric, with very rich geometry involving extremal lengths of measured foliations. The Teichmüller geodesic flow and horocycle flow are ergodic on the moduli space, with respect to the Masur–Veech measure.

Let $\mathcal{MF} = \mathcal{MF}(S)$ be the space of measured foliations on S. Denote the space of projective classes in $\mathcal{MF}$ by $\mathcal{PMF}$. Topologically, $\mathcal{PMF}$ is a sphere of dimension $6g - 7 + 2n$. Thurston [24] showed that $\mathcal{T}_{g,n}$ admits a natural compactification, whose boundary can be identified with $\mathcal{PMF}$. A generic pair of transverse measured foliations $\mathcal{F}, \mathcal{G} \in \mathcal{MF}$ determines a unique Teichmüller geodesic, which has the projective classes of $\mathcal{F}$ and $\mathcal{G}$ as its "limits" on $\mathcal{PMF}$.

1.2. Main theorems

Level sets of extremal length functions in Teichmüller space, associated with measured foliations, are called *horospheres*. The notion is motivated by the fact that extremal length functions on $\mathcal{T}_{g,n}$ are Hamiltonian functions of the Teichmüller horocycle flow [19].

DEFINITION 1.1. — *We say that a diffeomorphism $f : \mathcal{T}_{g,n} \to \mathcal{T}_{g,n}$ preserves horospheres if the image of any horosphere under f is a horosphere.*

Remark 1.2. — In this paper, we require that f is a C^1-diffeomorphism. The smoothness is just used to show that the inverse f^{-1} also preserves horospheres (see Lemma 4.1).

Our main result is the following:

THEOREM 1.3. — *Let $f : \mathcal{T}_{g,n} \to \mathcal{T}_{g,n}$ be a diffeomorphism that preserves horospheres. Then f is induced by an element of the extended mapping class group.*

Remark 1.4. — We exclude the case that $(g, n) = (1, 0), (1, 1)$ or $(0, 4)$, when $\mathcal{T}_{g,n}$ is isometric to the hyperbolic plane $\mathbb{H}^2$. In this two-dimensional case, level sets of extremal length functions are horocycles in $\mathbb{H}^2$. For instance, on the Teichmüller space of flat tori, any point $\tau \in \mathbb{H}^2$ corresponds

to a marked Riemann surface defined as the quotient space of $\mathbb{C}$ by a lattice generated by $\langle z \mapsto z + 1, z \mapsto z + \tau \rangle$; and the extremal length of the closed curve corresponding to $(1, 0)$ is equal to $1/\operatorname{Im} \tau$. It is not hard to check that if $f : \mathbb{H}^2 \to \mathbb{H}^2$ is a diffeomorphism that preserves horocycles, then f also preserves geodesics. Any bijection between hyperbolic space that preserves geodesics is an isometry [13]. Thus $f \in \operatorname{PSL}(2, \mathbb{R})$. However, the mapping class group of the torus is $\operatorname{PSL}(2, \mathbb{Z})$.

The proof of Theorem 1.3 is inspired by Ivanov's geometric proof of Royden's Theorem [12]. The vague idea is that, the action of f on horospheres should induce an action on the space of (projective) measured foliations. In fact, there is a subset of $\mathcal{MF}$ with full measure, on which the action induced by f is an isomorphism.

Let us explain more details. A measured foliation is *indecomposable* if it is equivalent either to a simple closed curve or to some minimal component with an ergodic measure (see Section 2.2 for the precise definition). Denote by $\mathcal{MF}_{ind}$ the set of indecomposable measured foliations. It is well known that $\mathcal{MF}_{ind}$ is a subset of $\mathcal{MF}$ with full measure. For $\mathcal{F} \in \mathcal{MF}$ and $X \in \mathcal{T}_{g,n}$, we denote by $\operatorname{HS}(\mathcal{F}, X)$ the horosphere associated with $\mathcal{F}$ and passing through X.

With the above terminologies, we prove:

PROPOSITION 1.5. — *Let $f : \mathcal{T}_{g,n} \to \mathcal{T}_{g,n}$ be a diffeomorphism that preserves horospheres. Assume that $\mathcal{F} \in \mathcal{MF}_{ind}$ and $f[\operatorname{HS}(\mathcal{F}, X)] = \operatorname{HS}(\mathcal{G}, Y)$. Then*

(1) $f[\operatorname{HS}(\mathcal{F}, Z)] = \operatorname{HS}(\mathcal{G}, f(Z))$ *for all $Z \in \mathcal{T}_{g,n}$;*
(2) $\mathcal{G} \in \mathcal{MF}_{ind}$.

Thus f induces a natural action on $\mathcal{MF}_{ind}$, denoted by f_*. We further show that f_* preserves the relation of zero intersection (see Proposition 4.6). There is a characterization of measured foliations corresponding to simple closed curves in terms of the dimension of zero intersection subspace. As a result, we can show that f_* induces an automorphism of the complex of curves of S. It follows from a result of Ivanov that f_* is given by an element of the extended mapping class group (see Theorem 2.6).

To prove Theorem 1.3, we can reduce to the case that $f[\operatorname{HS}(\mathcal{F}, X)] = \operatorname{HS}(\mathcal{F}, Y)$, for all $\mathcal{F}$ corresponding to simple closed curves. We study the condition when two or three such horospheres are tangent to each other (see Lemma 3.15 and Lemma 3.16), and use this to show that f is equal to the identify map on a dense subset of $\mathcal{T}_{g,n}$. Then the continuity of f implies that $f = \operatorname{id}$.

Using analytic nature of the Teichmüller metric, Royden [22] (and extended by Earle and Kra [3]) proved that

THEOREM 1.6 (Royden). — *If $3g - 3 + n \geqslant 2$, then every isometry of $\mathcal{T}_{g,n}$ with respect to the Teichmüller metric is induced by an element of the extended mapping class group.*

Ivanov [12] gave an alternative proof of Royden's Theorem, by investigation of the asymptotic geometry of Teichmüller geodesic rays. In Section 5, we observe that there is a direct relation between horospheres and level sets of Busemann functions, when the measured foliations defining the horospheres are indecomposable. Consider any isometry f of $\mathcal{T}_{g,n}$, we show that f preserves horospheres associated to indecomposable measured foliations. Again, f induces an isomorphism of $\mathcal{MF}_{ind}$. The proof of Theorem 1.3 can be adapted to show that f is induced by an element of the extended mapping class group. Thus we obtain a new proof of Royden's Theorem.

1.3. Organization of the article

In Section 2 we give the preliminaries on Teichmüller theory and measured foliations. The geometry of horospheres is investigated in Section 3. We prove Proposition 1.5 and Theorem 1.3 in Section 4. Theorem 1.6 is proved in Section 6.

2. Preliminaries

In this section, we briefly recall the background material on Teichmüller theory of Riemann surfaces and measured foliations.

2.1. Teichmüller space

Let S be a Riemann surface of genus g with n punctures, with $3g - 3 + n \geqslant 2$. The *Teichmüller space* $\mathcal{T}_{g,n}$ is the space of equivalence classes of pairs (X, f), where $f : S \to X$ is an orientation-preserving diffeomorphism (known as a marking). The equivalence relation is given by $(X, f) \sim (Y, g)$ if there is a conformal mapping $\phi : X \to Y$ so that $g^{-1} \circ \phi \circ f$ is isotopic to the identity map of S.

The Teichmüller space $\mathcal{T}_{g,n}$ has a complete distance, called the *Teichmüller distance* $d_{\mathcal{T}}(\cdot, \cdot)$. For any two points $[(X, f)], [(Y, g)] \in \mathcal{T}_{g,n}$ the distance is defined by

$$d_{\mathcal{T}}([(X, f)], [(Y, g)]) = \frac{1}{2} \inf_{h} \log K(h),$$

where h ranges over all quasiconformal mappings $h : X \to Y$ such that $h \circ f$ is homotopic to g, and $K(h)$ is the maximal quasiconformal dilatation of h.

For simplicity, we shall denote a point in $\mathcal{T}_{g,n}$ by a Riemann surface X, without explicit reference to the marking or to the equivalence relation.

2.2. Measured foliations

A *measured foliation* $\mathcal{F}$ on S is a foliation (with a finite number of singularities) with a transverse invariant measure. This means that if the local coordinates send the regular leaves of $\mathcal{F}$ to horizontal arcs in $\mathbb{R}^2$, then the transition functions on $\mathbb{R}^2$ are of the form $(f(x, y), \pm y + c)$ where c is a constant, and the measure is given by $|\mathrm{d}y|$. The allowed singularities of $\mathcal{F}$ are topologically the same as those that occur at $z = 0$ in the line field of $z^{p-2}\mathrm{d}z^2$, $p \geqslant 3$ (we allow $p = 1$ at the puncture of S). A leaf of $\mathcal{F}$ is called *critical* if it contains a singularity of $\mathcal{F}$. The union of compact critical leaves is called the *critical graph*.

Let $\mathcal{S}$ be the set of free homotopy classes of non-trivial, non-peripheral simple closed curves on S. The *intersection number* $i(\gamma, \mathcal{F})$ of a simple closed curve γ with a measured foliation $\mathcal{F}$ endowed with transverse measure μ is defined by

$$i(\gamma, \mathcal{F}) = \inf_{\gamma'} \int_{\gamma'} \mathrm{d}\mu,$$

where the infimum is taken over all simple closed curves γ' in the isotopy class of γ.

Two measured foliations $\mathcal{F}$ and $\mathcal{F}'$ are *measure equivalent* if, for all $\gamma \in \mathcal{S}$, $i(\gamma, \mathcal{F}) = i(\gamma, \mathcal{F}')$. Denote by $\mathcal{MF} = \mathcal{MF}(S)$ the space of equivalence classes of measured foliations on S.

Two measured foliations $\mathcal{F}$ and $\mathcal{F}'$ are *projectively equivalent* if there is a constant $b > 0$ such that $\mathcal{F} = b \cdot \mathcal{F}'$, i.e. $i(\gamma, \mathcal{F}) = b \cdot i(\gamma, \mathcal{F}')$ for all $\gamma \in \mathcal{S}$. The space of projective equivalence classes of foliations is denoted by $\mathcal{PMF}$.

Thurston showed that $\mathcal{MF}$ is homeomorphic to a $6g-6+2n$ dimensional ball and $\mathcal{PMF}$ is homeomorphic to a $6g-7+2n$ dimensional sphere. The set $\mathcal{S}$ is dense in $\mathcal{PMF}$. For more details on measured foliations, see [4].

We will use the ergodic decomposition of a measured foliation later in this paper. By removing the critical graph, a measured foliation $\mathcal{F}$ is decomposed into a finite number of connected components, each of which is either a cylinder foliated by closed leaves or a minimal component on which every leaf is dense. Furthermore, the transverse measure on a minimal component D can be represented as a finite sum of projectively distinct ergodic measures:

$$\mu|_D = \sum_k \mu_{D,k}.$$

We refer to [10, 16] for more details.

A measured foliation $\mathcal{F}'$ is called an *indecomposable component* of $\mathcal{F}$ if it is either one of the cylindrical components of $\mathcal{F}$, or it is measure equivalent to some minimal component of $\mathcal{F}$ endowed with one of the ergodic measures. A measured foliation $\mathcal{F}$ is *indecomposable* if it has a unique indecomposable component. We denote the set of indecomposable measured foliations on S by $\mathcal{MF}_{ind}$.

Thus an indecomposable measured foliation is equivalent to either a weighted simple close curve or a minimal component on a subsurface with an ergodic measure. In particular, uniquely ergodic measured foliations are indecomposable. We recall that a measured foliation $\mathcal{F}$ is *uniquely ergodic* if it is minimal and any topologically equivalent measured foliation is measure equivalent to a multiple of $\mathcal{F}$.

Usually, we will represent a measured foliation $\mathcal{F}$ as a finite sum

$$\mathcal{F} = \sum_{i=1}^{k} \mathcal{F}_i$$

of mutually disjoint $(i(\mathcal{F}_i, \mathcal{F}_j) = 0)$ and distinct indecomposable measured foliations. In the literature, such a (unique) decomposition is called the *ergodic decomposition* of $\mathcal{F}$ [16].

The next lemma will be used later.

LEMMA 2.1 ([25]). — *Let $\{\mathcal{F}_i\}_{i=0}^{k}$ be a set of projectively distinct, indecomposable elements of $\mathcal{MF}$ such that $i(\mathcal{F}_i, \mathcal{F}_j) = 0$ for all i and j. Then for any $\varepsilon > 0$, there exists a simple closed curve $\beta \in \mathcal{S}$ such that*

$$i(\mathcal{F}_i, \beta) < i(\mathcal{F}_0, \beta)\ \varepsilon,\ \forall\ i \neq 0.$$

2.3. Quadratic differentials

A holomorphic quadratic differential q on $X \in \mathcal{T}_{g,n}$ is a tensor which is locally represented by $q = q(z)\mathrm{d}z^2$, where $q(z)$ is a holomorphic function on the local conformal coordinate z of X. We allow holomorphic quadratic differentials to have at most simple poles at the punctures of X. Denote the vector space of holomorphic quadratic differentials on X by $Q(X)$.

The cotangent space of $\mathcal{T}_{g,n}$ at X can be naturally identified with $Q(X)$. We define the L^1-norm on $\mathcal{Q}(X)$ by

$$\|q\| = \int_X |q|.$$

Denote by $\mathcal{QT}_{g,n}$ the cotangent bundle of $\mathcal{T}_{g,n}$, and let $\mathcal{Q}^1\mathcal{T}_{g,n}$ be the unit cotangent bundle of $\mathcal{T}_{g,n}$.

A pair of measured foliations $\{\mathcal{F}, \mathcal{G}\}$ is *transverse* if $i(\mathcal{F}, \gamma) + i(\mathcal{G}, \gamma) > 0$ for all $\gamma \in \mathcal{MF}$. Any $q \in Q(X)$ gives rise to a pair of transverse measured foliations $\mathcal{F}_v(q)$ and $\mathcal{F}_h(q)$ on X, called the *vertical* and *horizontal measured foliations* of q, respectively. The vertical foliation $\mathcal{F}_v(q)$ (resp. horizontal foliation $\mathcal{F}_h(q)$) is defined by the foliation of the direction field $q(z)\mathrm{d}z^2 < 0$ (resp. $q(z)\mathrm{d}z^2 > 0$) with the transverse measure $|\mathrm{Re}\,\sqrt{q}|$ (resp. $|\mathrm{Im}\,\sqrt{q}|$).

On the other hand, according to a fundamental result of Hubbard and Masur [9], for any measured foliation $\mathcal{F} \in \mathcal{MF}$, there is a unique holomorphic quadratic differential $q \in Q(X)$ such that $\mathcal{F}_v(q)$ is measure equivalent to $\mathcal{F}$. The quadratic differential q is called the *Hubbard–Masur differential* of $\mathcal{F}$.

Let $X = (X, f) \in \mathcal{T}_{g,n}$ and $q \in Q(X)$. For any $t \in \mathbb{R}$, consider the normalized solution f_t of the Beltrami equation

$$\frac{\partial f}{\partial \overline{z}} = \tanh(t)\frac{|q|}{q}\frac{\partial f}{\partial z}$$

on X. We obtain a geodesic in the Teichmüller space:

$$\mathbf{G}_q : \mathbb{R} \to \mathcal{T}_{g,n},$$
$$t \mapsto (X_t, f_t \circ f).$$

where $X_t = f_t(X)$. We call $\mathbf{G}_q$ the *Teichmüller geodesic associated to q*.

We say a Teichmüller geodesic is determined by a pair of transverse measured foliations $\{\mathcal{F}, \mathcal{G}\}$, if it is defined by a holomorphic quadratic differential whose vertical and horizontal foliations are in the projective classes of $\mathcal{F}$ and $\mathcal{G}$. Any pair of transverse measured foliations determines a unique Teichmüller geodesic [7].

2.4. Extremal length

Extremal length is an important tool in the study of the Teichmüller metric. The notion is due to Ahlfors and Beurling.

Let $X = (X, f) \in \mathcal{T}_{g,n}$ and $\alpha \in \mathcal{S}$. The extremal length $\mathrm{Ext}_X(\alpha)$ is defined by

$$\mathrm{Ext}_X(\alpha) = \sup_\rho \frac{\ell_\rho(f(\alpha))^2}{\mathrm{Area}(X, \rho)},$$

where the supremum is taken over all conformal metrics ρ on X and $\ell_\rho(f(\alpha))$ denotes the geodesic length of $f(\alpha)$ in the metric ρ. Kerckhoff [14] proved that the definition of extremal length extends continuously to $\mathcal{MF}$. One can show that the extremal length of a measured foliation $\mathcal{F}$ satisfies

$$\mathrm{Ext}_X(\mathcal{F}) = \|q\|,$$

where q is the Hubbard–Masur differential of $\mathcal{F}$.

The following formula of Kerckhoff [14] is very useful to understand the geometry of Teichmüller distance.

THEOREM 2.2. — *For any $X, Y \in \mathcal{T}_{g,n}$, the Teichmüller distance between X and Y is given by*

$$d_\mathcal{T}(X, Y) = \frac{1}{2} \log \sup_{\alpha \in \mathcal{S}} \frac{\mathrm{Ext}_X(\alpha)}{\mathrm{Ext}_Y(\alpha)}.$$

The following inequality is due to Minsky [21], see also Gardiner–Masur [7].

THEOREM 2.3 (Minsky). — *Let $\{\mathcal{F}, \mathcal{G}\} \in \mathcal{MF}$ be a pair of transverse measured foliations. Then for any $X \in \mathcal{T}_{g,n}$, we have*

$$i(\mathcal{F}, \mathcal{G})^2 \leqslant \mathrm{Ext}_X(\mathcal{F}) \, \mathrm{Ext}_X(\mathcal{G}).$$

Moreover, the equality is obtained if and only if X belongs to the unique Teichmüller geodesic determined by $\mathcal{F}$ and $\mathcal{G}$ (i.e., the horizontal and vertical foliations are in the projective classes of $\mathcal{F}$ and $\mathcal{G}$).

COROLLARY 2.4. — *For any $X \in \mathcal{T}_{g,n}$ and $\mathcal{F} \in \mathcal{MF}$, we have*

$$\mathrm{Ext}_X(\mathcal{F}) = \sup_{\gamma \in \mathcal{S}} \frac{i(\mathcal{F}, \gamma)^2}{\mathrm{Ext}_X(\gamma)}.$$

Proof. — By Minsky's inequality,

$$\mathrm{Ext}_X(\mathcal{F}) \geqslant \sup_{\gamma \in \mathcal{S}} \frac{i(\mathcal{F}, \gamma)^2}{\mathrm{Ext}_X(\gamma)}.$$

On the other hand, let q be the Hubbard–Masur differential of $\mathcal{F}$, and let $\mathcal{G}$ be the horizontal measured foliation of q. Then

$$\mathrm{Ext}_X(\mathcal{F}) = \frac{i(\mathcal{F}, \mathcal{G})^2}{\mathrm{Ext}_X(\mathcal{G})}.$$

By the density of weighted simple closed curves in $\mathcal{MF}$, we are done. □

The following first variational formula of extremal length is called Gardiner's formula.

THEOREM 2.5 ([5, 6]). — Let $\mu = \mu(z)\frac{\mathrm{d}\bar{z}}{\mathrm{d}z}$ be a Beltrami differential on X that represents a tangent vector of $\mathcal{T}(S)$ at X. Then for any measured foliation $\mathcal{F} \in \mathcal{MF}$,

$$\mathrm{d}\,\mathrm{Ext}_X(\mathcal{F})[\mu] = 2\,\mathrm{Re} \iint_X \mu(z)q(z)\,\mathrm{d}x\mathrm{d}y,$$

where q is the Hubbard–Masur differential of $\mathcal{F}$.

2.5. Complex of curves and mapping class group

The *complex of curves* was introduced into the study of Teichmüller spaces by Harvey [8], as an analogue of the Tits building of a symmetric space. The vertex set of the complex of curves $\mathcal{C}(S)$ is given by $\mathcal{S}$. Two vertices $\alpha, \beta \in \mathcal{S}$ are connected by an edge if they have disjoint representations. For any two vertices α, β, we define the distance $d_{\mathcal{S}}(\alpha, \beta)$ to be the minimal number of edges connecting α and β.

The *mapping class group* $\mathrm{Mod}(S)$ is the group of homotopy classes of orientation-preserving diffeomorphisms $\sigma : S \to S$. Every mapping class $[\sigma]$ acts on $\mathcal{T}_{g,n}$ by changing the markings:

$$[(X, f)] \to [(X, f \circ \sigma^{-1})].$$

Denote by $\mathrm{Mod}^{\pm}(S)$ the extended mapping class group, which contains $\mathrm{Mod}(S)$ as a subgroup of index two.

It is clear that $\mathrm{Mod}^{\pm}(S)$ acts on $\mathcal{C}(S)$ as a group of automorphisms.

THEOREM 2.6. — If S is not a sphere with $\leqslant 4$ punctures, nor a torus with $\leqslant 2$ punctures, then every automorphism of $\mathcal{C}(S)$ is given by an element of $\mathrm{Mod}^{\pm}(S)$.

The above theorem was proved by Ivanov [11] for surfaces of genus $g \geqslant 2$ and by Korkmaz [15] for $g \geqslant 1$. See also Luo [17]. We remark that, for the torus with two punctures, $\mathcal{C}(S_{1,2})$ is isomorphic to $\mathcal{C}(S_{0,5})$. Instead of $\mathrm{Mod}^{\pm}(S_{1,2})$, the automorphism group of $\mathcal{C}(S_{1,2})$ is $\mathrm{Mod}^{\pm}(S_{0,5})$,

3. The geometry of horospheres

In this section, we study the geometry of horospheres in $\mathcal{T}_{g,n}$. Although the Teichmüller metric is neither non-positively curved nor δ-hyperbolic, we will show that the asymptotic geometry of horospheres, for those associated with indecomposable measured foliations, have similar properties to those in the hyperbolic space.

3.1. Horoballs and horospheres

Given $\mathcal{F} \in \mathcal{MF}$, the extremal length function $X \mapsto \mathrm{Ext}_X(\mathcal{F})$ is a C^1-function on $\mathcal{T}_{g,n}$.

DEFINITION 3.1. — *Let $\mathcal{F} \in \mathcal{MF}$ and $s \in \mathbb{R}_+$. The open horoball associated to $\mathcal{F}$ is defined by*

$$\mathrm{HB}(\mathcal{F}, s) = \{X \in \mathcal{T}_{g,n} \,|\, \mathrm{Ext}_X(\mathcal{F}) < s\}.$$

The associated closed horoball is defined as

$$\overline{\mathrm{HB}}(\mathcal{F}, s) = \{X \in \mathcal{T}_{g,n} \,|\, \mathrm{Ext}_X(\mathcal{F}) \leqslant s\}.$$

The associated horosphere is defined as

$$\mathrm{HS}(\mathcal{F}, s) = \{X \in \mathcal{T}_{g,n} \,|\, \mathrm{Ext}_X(\mathcal{F}) = s\}.$$

Remark 3.2. — Let $\mathcal{F} \in \mathcal{MF}$ and $k, s \in \mathbb{R}_+$. Since $\mathrm{Ext}_X(k \cdot \mathcal{F}) = k^2 \mathrm{Ext}_X(\mathcal{F})$, we have

$$\mathrm{HS}(k \cdot \mathcal{F}, k^2 s) = \mathrm{HS}(\mathcal{F}, s).$$

LEMMA 3.3. — *For any $X, Y \in \mathcal{T}_{g,n}$, there exists a measured foliation $\mathcal{F} \in \mathcal{MF}$ and $t \in \mathbb{R}_+$ such that $X, Y \in \mathrm{HS}(\mathcal{F}, t)$.*

Proof. — It suffices to prove that there is a measured foliation $\mathcal{F} \in \mathcal{MF}$ such that $\mathrm{Ext}_X(\mathcal{F}) = \mathrm{Ext}_Y(\mathcal{F})$. Let $\Phi : \mathcal{MF} \to \mathbb{R}$ defined by

$$\Phi(\mathcal{F}) = \mathrm{Ext}_X(\mathcal{F}) - \mathrm{Ext}_Y(\mathcal{F}).$$

It is obvious that Φ is continuous. Let $f : X \to Y$ be the Teichmüller map from X to Y, and $q \in Q(X)$ be the holomorphic quadratic differential associated to f. Denote by $\mathcal{F}_h(q)$ (resp. $\mathcal{F}_v(q)$) the horizontal foliation (resp. vertical foliation) of q. Then we have $\Phi(\mathcal{F}_h(q)) < 0$ and $\Phi(\mathcal{F}_v(q)) > 0$. Since $\mathcal{MF} \setminus \{0\}$ is connected, the mean value theorem of continuous function implies that there is a measured foliation $\mathcal{F}^{'}$ such that $\Phi(\mathcal{F}^{'}) = 0$. $\qquad\square$

Remark 3.4. — In general, the horosphere containing $X, Y \in \mathcal{T}_{g,n}$ is not unique.

Let $\mathcal{F} \in \mathcal{MF}$ and $X \in \mathcal{T}_{g,n}$, we denote by

$$\mathrm{HS}(\mathcal{F}, X) = \{Y \in \mathcal{T}_{g,n} \mid \mathrm{Ext}_Y(\mathcal{F}) = \mathrm{Ext}_X(\mathcal{F})\}.$$

Let $T_X = T_X \mathcal{T}_{g,n}$ denote the tangent space of $\mathcal{T}_{g,n}$ at X, and let $\mathrm{Gr}(T_X)$ denote the set of linear subspaces of T_X of dimension $6g - 7 + 2n$.

With the above notation, we define a map $T : \mathcal{MF} \to \mathrm{Gr}(T_X)$ by

$$T(\mathcal{F}) = T_X \mathrm{HS}(\mathcal{F}, X),$$

where $T_X \mathrm{HS}(\mathcal{F}, X)$ denotes the tangent space of $\mathrm{HS}(\mathcal{F}, X)$ at X.

LEMMA 3.5. — *Let $\mathcal{F}, \mathcal{G} \in \mathcal{MF}$. Then $T(\mathcal{F}) = T(\mathcal{G})$ if and only if $\mathcal{G}$ is projectively equivalent to $\mathcal{F}_h(q)$ or $\mathcal{F}_v(q)$, where $q \in Q(X)$ is the Hubbard–Masur differential of $\mathcal{F}$.*

Proof. — Assume that $T(\mathcal{F}) = T(\mathcal{G})$. According to the definition of tangent space and the C^1-property of extremal length function $\mathrm{Ext}_X(\mathcal{F})$, we have

$$\mathrm{d}\,\mathrm{Ext}_X(\mathcal{F})[\mu] = \mathrm{d}\,\mathrm{Ext}_X(\mathcal{G})[\mu] = 0, \quad \forall\, \mu \in T_X \mathrm{HS}(\mathcal{F}, X).$$

Moreover, since the subspace of T_X tangent to the horosphere has codimension one, there is a non-zero constant $k \in \mathbb{R}$ such that

$$\mathrm{d}\,\mathrm{Ext}_X(\mathcal{F})[\nu] = k \cdot \mathrm{d}\,\mathrm{Ext}_X(\mathcal{G})[\nu] \neq 0, \quad \forall\, \nu \in T_X \setminus T_X \mathrm{HS}(\mathcal{F}, X).$$

It follows that $\mathrm{d}\,\mathrm{Ext}_X(\mathcal{F})$ is a real multiple of $\mathrm{d}\,\mathrm{Ext}_X(\mathcal{G})$.

Let q_1 and q_2 be the holomorphic quadratic differentials on X which realize the measured foliations $\mathcal{F}$ and $\mathcal{G}$, respectively. Using Gardiner's formula (Theorem 2.5), we have

$$\mathrm{d}\,\mathrm{Ext}_X(\mathcal{F})[\mu] = 2\,\mathrm{Re} \iint_X \mu(z) q_1(z) \mathrm{d}x \mathrm{d}y$$

$$= k \cdot \left(2\,\mathrm{Re} \iint_X \mu(z) q_2(z) \mathrm{d}x \mathrm{d}y \right), \quad \forall\, \mu \in T_X.$$

This implies that $q_1 = k q_2$. As a result, if $k > 0$, then $\mathcal{G}$ is projectively equivalent to $\mathcal{F}$; if $k < 0$, then $\mathcal{G}$ is projectively equivalent to $\mathcal{F}_h(q_1)$.

To prove $T(\mathcal{F}) = T(\mathcal{G})$ under the assumption that $\mathcal{G}$ is projectively equivalent to $\mathcal{F}_h(q)$ or $\mathcal{F}_v(q)$, we can apply the above argument in the converse direction. $\qquad \square$

COROLLARY 3.6. — *If $\mathcal{G}$ is not projectively equivalent to $\mathcal{F}$, then $\mathrm{HS}(\mathcal{G}, X) \neq \mathrm{HS}(\mathcal{F}, X)$.*

Let $V \in \mathrm{Gr}(T_X)$. Suppose that $\langle \mu_1, \ldots, \mu_{6g-7+2n} \rangle = V$. Each μ_i induces a linear function

$$\widehat{\mu}_i : Q(X) \to \mathbb{R}$$

by

$$\widehat{\mu}_i(q) = \langle \mu_i, q \rangle = 2 \, \mathrm{Re} \int_X \mu_i(z) q(z) \mathrm{d}x \mathrm{d}y.$$

Let $V_i = \mathrm{Ker}(\widehat{\mu}_i)$. It is clear that V_i is a linear subspace of $Q(X)$ and $\dim(V_i) = 6g - 7 + 2n$. Let

$$V^* = \bigcap_{i=1}^{6g-7+2n} V_i.$$

It follows from linear algebra that V^* is a linear subspace with $\dim(V^*) \geqslant 1$. This implies that there is a $q \in V^*$ and $q \neq 0$ such that

$$\langle \mu_i, q \rangle = 0, i = 1, \ldots, 6g - 7 + 2n.$$

Hence $T(\mathcal{F}_h(q)) = \langle \mu_1, \ldots, \mu_{6g-7+2n} \rangle$. This shows that

LEMMA 3.7. — *The map T defined above is surjective. Thus for any linear subspace $V \in \mathrm{Gr}(T_X)$, there is a measured foliation $\mathcal{F}$ such that the tangent space of horosphere $\mathrm{HS}(\mathcal{F}, X)$ at X is V.*

Gardiner and Masur [7] proved that

LEMMA 3.8. — *Every horosphere in $\mathcal{T}_{g,n}$ is a hypersurface homeomorphic to the Euclidean space $\mathbb{R}^{6g-7+2n}$.*

3.2. Relation between horospheres

Let $X \in \mathcal{T}_{g,n}$ and let $\mathbf{A}$ be a subset of $\mathcal{T}_{g,n}$, we define

$$d_{\mathcal{T}}(X, \mathbf{A}) = \inf_{Y \in \mathbf{A}} d_{\mathcal{T}}(X, Y).$$

If there is a point $Y \in \mathbf{A}$ such that $d_{\mathcal{T}}(X, Y) = d_{\mathcal{T}}(X, \mathbf{A})$, then Y is called a *foot* of X on $\mathbf{A}$.

LEMMA 3.9. — *Let $0 < s < t$ and $\mathcal{F} \in \mathcal{MF}$. Then horospheres $\mathrm{HS}(\mathcal{F}, s)$ and $\mathrm{HS}(\mathcal{F}, t)$ are equidistant, i.e. for any $X \in \mathrm{HS}(\mathcal{F}, s)$, we have*

$$d_{\mathcal{T}}(X, \mathrm{HS}(\mathcal{F}, t)) = \frac{1}{2} \log \frac{t}{s}.$$

Moreover, any $X \in \mathrm{HS}(\mathcal{F}, s)$ has a unique foot on $\mathrm{HS}(\mathcal{F}, t)$.

Proof. — According to Kerckhoff's formula, we have

$$d_{\mathcal{T}}(X,Y) \geqslant \frac{1}{2} \log \frac{\mathrm{Ext}_Y(\mathcal{F})}{\mathrm{Ext}_X(\mathcal{F})} = \frac{1}{2} \log \frac{t}{s}$$

for any $X \in \mathrm{HS}(\mathcal{F}, s), Y \in \mathrm{HS}(\mathcal{F}, t)$. Thus

$$d_{\mathcal{T}}(X, \mathrm{HS}(\mathcal{F}, t)) \geqslant \frac{1}{2} \log \frac{t}{s}.$$

If we choose $Y \in \mathrm{HS}(\mathcal{F}, t)$ as the Teichmüller deformation of X in the direction $q \in Q(X)$ with $\mathcal{F}_v(q) = \mathcal{F}$, then

$$d_{\mathcal{T}}(X, Y) = \frac{1}{2} \log \frac{t}{s}.$$

As a result,

$$d_{\mathcal{T}}(X, \mathrm{HS}(\mathcal{F}, t)) = \frac{1}{2} \log \frac{t}{s}.$$

Note that $Y \in \mathrm{HS}(\mathcal{F}, t)$ is a foot of X if and only if

$$d_{\mathcal{T}}(X, Y) = \frac{1}{2} \log \frac{\mathrm{Ext}_Y(\mathcal{F})}{\mathrm{Ext}_X(\mathcal{F})}.$$

By the uniqueness of Teichmüller map, the above equality holds if and only if Y is the Teichmüller deformation of X in the direction $q \in Q(X)$ with $\mathcal{F}_v(q) = \mathcal{F}$. This implies that the foot Y is unique. □

We call the Teichmüller geodesic passing through $X \in \mathrm{HS}(\mathcal{F}, s)$ and $Y \in \mathrm{HS}(\mathcal{F}, t)$ such that Y is the foot of X on $\mathrm{HS}(\mathcal{F}, t)$ a geodesic *perpendicular* to the family of horospheres $\mathrm{HS}(\mathcal{F}, s), s \in \mathbb{R}_+$.

To obtain further results, we first consider the asymptotic estimates of extremal length functions on a given horosphere.

Fix a horosphere $\mathrm{HS}(\mathcal{F}, t)$. Consider the function $\mathrm{Ext}_X(\mathcal{G})$ for any $\mathcal{G} \in \mathcal{MF}$, where X runs over all points belong to $\mathrm{HS}(\mathcal{F}, t)$. We will write $\mathcal{F} = \sum_i \mathcal{F}_i$ as the ergodic decomposition of $\mathcal{F}$. If each indecomposable component of $\mathcal{G}$ is projectively equivalent to one of the indecomposable components of $\mathcal{F}$, we denote by $\mathcal{G} \prec \mathcal{F}$; otherwise, $\mathcal{G} \nprec \mathcal{F}$.

LEMMA 3.10. — *Fix a horosphere* $\mathrm{HS}(\mathcal{F}, t)$. *For any* $\mathcal{G} \in \mathcal{MF}$, *we have*

(1) *If* $i(\mathcal{F}, \mathcal{G}) \neq 0$, *then*

$$\inf_{X \in \mathrm{HS}(\mathcal{F},t)} \mathrm{Ext}_X(\mathcal{G}) > 0, \quad \sup_{X \in \mathrm{HS}(\mathcal{F},t)} \mathrm{Ext}_X(\mathcal{G}) = \infty.$$

(2) *If* $i(\mathcal{F}, \mathcal{G}) = 0$ *and* $\mathcal{G} \prec \mathcal{F}$, *then*

$$\sup_{X \in \mathrm{HS}(\mathcal{F},t)} \mathrm{Ext}_X(\mathcal{G}) < \infty.$$

(3) *If $i(\mathcal{F}, \mathcal{G}) = 0$ and $\mathcal{G} \not\prec \mathcal{F}$, then*

$$\sup_{X \in \mathrm{HS}(\mathcal{F}, t)} \mathrm{Ext}_X(\mathcal{G}) = \infty.$$

Proof.

(1). — According to Minsky's inequality (Theorem 2.3), we have

$$\mathrm{Ext}_X(\mathcal{F})\, \mathrm{Ext}_X(\mathcal{G}) \geqslant i(\mathcal{F}, \mathcal{G})^2 > 0.$$

Then

$$\inf_{X \in \mathrm{HS}(\mathcal{F}, t)} \mathrm{Ext}_X(\mathcal{G}) \geqslant \frac{i(\mathcal{F}, \mathcal{G})^2}{t}.$$

For the supremum, we use the action of horocycle flow on $\mathcal{T}_{g,n}$. Choose any $X \in \mathrm{HS}(\mathcal{F}, t)$. Denote by q the Hubbard–Masur differential of $\mathcal{F}$ on X and $\mathcal{G}'$ the horizontal measured foliations of q. In local coordinates $z = x+iy$ on which $q = \mathrm{d}z^2$, the horocycle flow $h^s : \mathcal{QT}_{g,n} \to \mathcal{QT}_{g,n}$ acts on q by

$$\begin{pmatrix} \mathrm{d}y \\ \mathrm{d}x \end{pmatrix} \mapsto \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{d}y \\ \mathrm{d}x \end{pmatrix} = \begin{pmatrix} \mathrm{d}y + s\mathrm{d}x \\ \mathrm{d}x \end{pmatrix}.$$

The horocycle flow acts on the horosphere $\mathrm{HS}(\mathcal{F}, t)$. For any closed curve $\gamma \in \mathcal{S}$, its length under the flat metric $|h^s(q)|$ has an explicit lower bound:

$$\int_\gamma |s\mathrm{d}x + \mathrm{d}y| \geqslant |s| i(\mathcal{F}, \gamma) - i(\mathcal{G}'\gamma).$$

The area of $h^s(q)$ is equal to $\|q\|$. Denote by X_s the projection of $h^s(q)$ on $\mathcal{T}_{g,n}$. Then $X_s \in \mathrm{HS}(\mathcal{F}, t)$. By definition of extremal length, we have

$$\mathrm{Ext}_{X_s}(\gamma) \geqslant \frac{s^2 i(\mathcal{F}, \gamma)^2 - 2s\, i(\mathcal{F}, \gamma) i(\mathcal{G}', \gamma) + i(\mathcal{G}', \gamma)^2}{\|q\|}$$

$$\geqslant \frac{s^2 i(\mathcal{F}, \gamma)^2}{2\|q\|}$$

when s is sufficiently large. By continuity, the above inequality applies to general measured foliations. In particular, when $i(\mathcal{F}, \mathcal{G}) \neq 0$, we have

$$\sup_{X \in \mathrm{HS}(\mathcal{F}, t)} \mathrm{Ext}_X(\mathcal{G}) = \infty.$$

(2). — Decompose $\mathcal{F}$ into its indecomposable components

$$\mathcal{F} = \sum_{i=1}^{k} \mathcal{F}_i.$$

According to Lenzhen–Masur [16, Theorem C], there exists a sequence of multi-curves

$$\sum_{i=1}^{k} s_n^i \gamma_n^i \to \mathcal{F}, \quad s_n^i \gamma_n^i \to \mathcal{F}_i, \quad n \to \infty.$$

Note that each γ_n^i may itself be a multi-curve. By continuity, we have $\mathrm{Ext}_X(\sum_{i=1}^{k} s_n^i \gamma_n^i) \to \mathrm{Ext}_X(\mathcal{F})$ as $n \to \infty$. By definition of extremal length, we have

$$\mathrm{Ext}_X(s_n^i \gamma_n^i) \leqslant \mathrm{Ext}_X\left(\sum_{j=1}^{k} s_n^j \gamma_n^j\right).$$

Then we have

$$\mathrm{Ext}_X(\mathcal{F}_i) \leqslant \mathrm{Ext}_X(\mathcal{F}) = t.$$

Since $\mathcal{G} \prec \mathcal{F}$, we can write $\mathcal{G}$ as $\mathcal{G} = \sum_{i=1}^{k} a_i \mathcal{F}_i$, $a_i \geqslant 0$. Use the definition of extremal length and the above result of Lenzhen–Masur again, we have

$$\mathrm{Ext}_X\left(\sum_{i=1}^{k} a_i \mathcal{F}_i\right) \leqslant \left(\sum_{i=1}^{k} a_i \sqrt{\mathrm{Ext}_X(\mathcal{F}_i)}\right)^2$$

$$\leqslant \left(\sum_{i=1}^{k} a_i\right)^2 t.$$

(3). — Without loss of generality, we may assume that $\mathcal{G}$ is indecomposable and $\mathcal{G}$ is disjoint from $\mathcal{F}$. By Lemma 2.1, for any $M \in \mathbb{R}_+$ there is $\beta \in \mathcal{S}$ such that

$$\frac{i(\mathcal{G}, \beta)}{i(\mathcal{F}, \beta)} \geqslant M.$$

Since uniquely ergodic measured foliations are dense in $\mathcal{MF}$, there is a uniquely ergodic measured foliation $\mathcal{F}'$ such that

$$\frac{i(\mathcal{G}, \mathcal{F}')}{i(\mathcal{F}, \mathcal{F}')} \geqslant \frac{M}{2}.$$

It is clear that $\{\mathcal{F}, \mathcal{F}'\}$ is a pair of transverse measured foliations. By Theorem 2.3, there is a point $X \in \mathrm{HS}(\mathcal{F}, t)$ satisfying

$$\mathrm{Ext}_X(\mathcal{F}) = \frac{i(\mathcal{F}, \mathcal{F}')^2}{\mathrm{Ext}_X(\mathcal{F}')}.$$

In fact, X is the intersection point of $\mathrm{HS}(\mathcal{F}, t)$ with the Teichmüller geodesic determined by $\mathcal{F}$ and $\mathcal{F}'$. It follows that

$$\mathrm{Ext}_X(\mathcal{G}) \geqslant \frac{i(\mathcal{G}, \mathcal{F}')^2}{\mathrm{Ext}_X(\mathcal{F}')} \geqslant \left(\frac{M}{2}\right)^2 \mathrm{Ext}_X(\mathcal{F}).$$

This implies that

$$\sup_{X \in \mathrm{HS}(\mathcal{F},t)} \mathrm{Ext}_X(\mathcal{G}) = +\infty.$$

The proof is complete.   □

Remark 3.11. — If $\mathrm{HS}(\mathcal{F}, t)$ is a horosphere and $\mathcal{G} \in \mathcal{MF}$ satisfies

$$\sup_{X \in \mathrm{HS}(\mathcal{F},t)} \mathrm{Ext}_X(\mathcal{G}) \leqslant M$$

for some $M \in \mathbb{R}_+$, then

$$\mathrm{HB}(\mathcal{F}, t) \subset \mathrm{HB}(\mathcal{G}, s)$$

when $s \geqslant M$. We recall that $\mathrm{HB}(\mathcal{F}, t)$ is the horoball whose boundary is $\mathrm{HS}(\mathcal{F}, t)$, see Definition 3.1.

The following corollary is immediate:

COROLLARY 3.12. — *Given any horosphere* $\mathrm{HS}(\mathcal{F}, t)$, *we have*

(1) *If* $i(\mathcal{F}, \mathcal{G}) \neq 0$, *then there exists* $s_0 = s_0(\mathcal{G}, t) > 0$ *such that*

$$\mathrm{HB}(\mathcal{F}, t) \cap \mathrm{HB}(\mathcal{G}, s) = \emptyset, \quad s < s_0.$$

(2) *If* $i(\mathcal{F}, \mathcal{G}) = 0$ *and* $\mathcal{G} \prec \mathcal{F}$, *then there exists* $s_0 = s_0(\mathcal{G}, t) > 0$ *such that*

$$\mathrm{HB}(\mathcal{F}, t) \subset \mathrm{HB}(\mathcal{G}, s), \quad s > s_0.$$

Recall that $\mathcal{MF}_{ind}$ is defined as the set of indecomposable measured foliations on S.

PROPOSITION 3.13. — *Let* $\mathcal{F} \in \mathcal{MF}_{ind}$ *and* $s \in \mathbb{R}_+$. *If there exist* $\mathcal{G} \in \mathcal{MF}$ *and* $t \in \mathbb{R}_+$ *such that*

$$\mathrm{HB}(\mathcal{F}, s) \subset \mathrm{HB}(\mathcal{G}, t),$$

then $\mathcal{G} = k\mathcal{F}$, *for some* $k \in \mathbb{R}_+$.

Proof. — By definition,

$$\sup_{X \in \mathrm{HB}(\mathcal{F},s)} \mathrm{Ext}_X(\mathcal{G}) \leqslant t.$$

It follows from Lemma 3.10 that $\mathcal{G} \prec \mathcal{F}$. Since $\mathcal{F}$ is indecomposable, $\mathcal{G}$ must be a multiple of $\mathcal{F}$.   □

3.3. Horospheres tangent to each other

DEFINITION 3.14. — *Two horospheres* $\mathrm{HS}(\mathcal{F}, s)$ *and* $\mathrm{HS}(\mathcal{G}, t)$ *are tangent to each other if they satisfy the following conditions:*

(1) $\mathrm{HB}(\mathcal{F}, s) \cap \mathrm{HB}(\mathcal{G}, t) = \emptyset$,
(2) $\mathrm{HS}(\mathcal{F}, s) \cap \mathrm{HS}(\mathcal{G}, t) \neq \emptyset$.

If $\mathrm{HS}(\mathcal{F}, s)$ and $\mathrm{HS}(\mathcal{G}, t)$ are tangent to each other, then it is necessary that $\{\mathcal{F}, \mathcal{G}\}$ is a pair of transverse measured foliations. This follows from Lemma 3.5. In fact, at any $X \in \mathrm{HS}(\mathcal{F}, s) \cap \mathrm{HS}(\mathcal{G}, t)$, we have $T(\mathcal{F}) = T(\mathcal{G})$ (otherwise, one can check that the hypothesis $\mathrm{HB}(\mathcal{F}, s) \cap \mathrm{HB}(\mathcal{G}, t) = \emptyset$ is not satisfied). And then $\mathcal{F}$ and $\mathcal{G}$ should be equivalent to the vertical and horizontal measured foliations of some quadratic differential.

The following lemma will show that, if two horospheres $\mathrm{HS}(\mathcal{F}, s)$ and $\mathrm{HS}(\mathcal{G}, t)$ are tangent to each other, then they have a unique intersection point. This is not obvious from the above definition.

LEMMA 3.15. — *Let* $\{\mathcal{F}, \mathcal{G}\}$ *be a pair of transverse measured foliations. Then the horospheres* $\mathrm{HS}(\mathcal{F}, s)$ *and* $\mathrm{HS}(\mathcal{G}, t)$ *are tangent to each other if and only if*

$$(3.1) \qquad\qquad s \cdot t = i(\mathcal{F}, \mathcal{G})^2.$$

When the condition holds, $\mathrm{HS}(\mathcal{F}, s)$ *and* $\mathrm{HS}(\mathcal{G}, t)$ *have a unique intersection point.*

Proof.

($\Longleftarrow$). — Assume that $s \cdot t = i(\mathcal{F}, \mathcal{G})^2$. Let $\mathbf{G}_{\mathcal{F}, \mathcal{G}}$ be the Teichmüller geodesic determined by $\mathcal{F}$ and $\mathcal{G}$. Since the extremal length of $\mathcal{F}$ is a strictly monotonic function along $\mathbf{G}_{\mathcal{F}, \mathcal{G}}$, there exists a unique Riemann surface $X_s \in \mathbf{G}_{\mathcal{F}, \mathcal{G}}$ such that $\mathrm{Ext}_{X_s}(\mathcal{F}) = s$. Since $\mathrm{Ext}_{X_s}(\mathcal{F}) \, \mathrm{Ext}_{X_s}(\mathcal{G}) = i(\mathcal{F}, \mathcal{G})^2$, $\mathrm{Ext}_{X_s}(\mathcal{G})$ must be equal to t.

By our construction, $X_s \in \mathrm{HS}(\mathcal{F}, s) \cap \mathrm{HS}(\mathcal{G}, t)$.

It remains to show that $\mathrm{HB}(\mathcal{F}, s) \cap \mathrm{HB}(\mathcal{G}, t) = \emptyset$. For any point $X \in \mathrm{HB}(\mathcal{F}, s)$, we have $\mathrm{Ext}_X(\mathcal{F}) < s$. Using Minsky's inequality (Theorem 2.3), we obtain

$$\mathrm{Ext}_X(\mathcal{G}) \geqslant \frac{i(\mathcal{F}, \mathcal{G})^2}{\mathrm{Ext}_X(\mathcal{F})} > \frac{i(\mathcal{F}, \mathcal{G})^2}{s} = t.$$

This implies that $X \notin \mathrm{HB}(\mathcal{G}, t)$. Similarly, for any point $Y \in \mathrm{HB}(\mathcal{G}, t)$, we have $Y \notin \mathrm{HB}(\mathcal{F}, s)$.

Thus $\mathrm{HS}(\mathcal{F}, s)$ is tangent to $\mathrm{HS}(\mathcal{G}, t)$ at X_s.

($\Longrightarrow$). — Conversely, assume that $\mathrm{HS}(\mathcal{F}, s)$ and $\mathrm{HS}(\mathcal{G}, t)$ are tangent to each other. Set $r = \frac{i(\mathcal{F}, \mathcal{G})^2}{t}$. From the above argument, the horospheres $\mathrm{HS}(\mathcal{F}, r)$ and $\mathrm{HS}(\mathcal{G}, t)$ tangent to each other at X_r. We claim that $r = s$.

In fact, if $s < r$, then the closed horoball $\overline{\mathrm{HB}}(\mathcal{F}, s)$ is contained in $\mathrm{HB}(\mathcal{F}, r)$. This implies that $\overline{\mathrm{HB}}(\mathcal{F}, s) \cap \overline{\mathrm{HB}}(\mathcal{G}, t) = \emptyset$, which contradicts our assumption.

If $s > r$, then $X_r \in \mathrm{HB}(\mathcal{G}, s)$, which contradicts with the fact that $X_r \in \mathrm{HS}(\mathcal{F}, r) \cap \mathrm{HS}(\mathcal{G}, t)$.

We have proved the equivalence. The above proof also shows that any $X \in \mathrm{HS}(\mathcal{F}, s) \cap \mathrm{HS}(\mathcal{G}, t)$ lies on the geodesic $\mathbf{G}_{\mathcal{F}, \mathcal{G}}$. Thus $X = X_s$ is unique. $\qquad\square$

The next lemma studies the question when a triple of horospheres are tangent to each other. For simplicity, we only consider measured foliations corresponding to simple closed curves. This is sufficient for application in the proof of Theorem 1.3.

A pair of simple closed curves $(\alpha, \beta) \in \mathcal{S} \times \mathcal{S}$ is *filling* if $i(\alpha, \gamma) + i(\beta, \gamma) > 0$ for all $\gamma \in \mathcal{S}$. If (α, β) is filling, there is a unique Teichmüller geodesic determined by α and β. We shall denote such a geodesic by $\mathbf{G}_{\alpha, \beta}$.

LEMMA 3.16. — *Suppose that all the pairs $(\alpha, \beta), (\alpha, \gamma), (\beta, \gamma)$ are filling. Then there exist unique $s, t, r \in \mathbb{R}_+$ such that the horospheres $\mathrm{HS}(\alpha, r)$, $\mathrm{HS}(\beta, s)$ and $\mathrm{HS}(\gamma, t)$ are tangent to each other.*

Proof. — By Theorem 2.3, any $X \in \mathbf{G}_{\alpha, \beta}$ must satisfy

$$\mathrm{Ext}_X(\alpha) \, \mathrm{Ext}_X(\beta) = i(\alpha, \beta)^2.$$

Let

$$k = \frac{i(\alpha, \gamma)^2}{i(\beta, \gamma)^2}.$$

There is a unique $X_0 \in \mathbf{G}_{\alpha, \beta}$ such that

$$\frac{\mathrm{Ext}_{X_0}(\alpha)}{\mathrm{Ext}_{X_0}(\beta)} = k.$$

Let $r = \mathrm{Ext}_{X_0}(\alpha), s = \mathrm{Ext}_{X_0}(\beta)$ and

$$t = \frac{i(\alpha, \gamma)^2}{r} = \frac{i(\beta, \gamma)^2}{s}.$$

According to Lemma 3.15, the horospheres $\mathrm{HS}(\alpha, r), \mathrm{HS}(\beta, s)$ and $\mathrm{HS}(\gamma, t)$ are tangent to each other. Moreover, the solution (s, t, r) is unique. $\qquad\square$

Triples of horospheres tangent to each other are flexible in Teichmüller space:

LEMMA 3.17. — *Let $(\alpha, \beta) \in \mathcal{S} \times \mathcal{S}$ be filling. Given any $t \in \mathbb{R}_+$ and $\epsilon > 0$, there exists a simple closed curve $\gamma \in \mathcal{S}$ such that both (α, γ) and (β, γ) are filling and*

$$\left| \frac{i(\alpha, \gamma)}{i(\beta, \gamma)} - t \right| < \epsilon.$$

Proof. — It is not hard to show that the map

$$\frac{i(\alpha, \cdot)}{i(\beta, \cdot)} : \mathcal{PMF} \to \mathbb{R}_{\geqslant 0} \cup \{+\infty\}$$

is continuous. Since $\mathcal{PMF}$ is path-connected and

$$\frac{i(\alpha, \alpha)}{i(\beta, \alpha)} = 0, \frac{i(\alpha, \beta)}{i(\beta, \beta)} = +\infty,$$

we have for any $t \in \mathbb{R}_+$, there is a measured foliation $\mathcal{F}$ such that

$$\frac{i(\alpha, \mathcal{F})}{i(\beta, \mathcal{F})} = t.$$

By the density of uniquely ergodic measured foliations, we may assume that $\mathcal{F}$ is uniquely ergodic and

$$\left| \frac{i(\alpha, \mathcal{F})}{i(\beta, \mathcal{F})} - t \right| < \frac{\epsilon}{2}.$$

It is obvious that $(\alpha, \mathcal{F})$ (and also $(\beta, \mathcal{F})$) are transverse. If we choose $\gamma \in \mathcal{S}$ sufficiently close to $[\mathcal{F}]$ in $\mathcal{PMF}$, then both $(\alpha, \gamma), (\beta, \gamma)$ are filling and

$$\left| \frac{i(\alpha, \gamma)}{i(\beta, \gamma)} - t \right| < \epsilon. \qquad \square$$

4. Horosphere-preserving diffeomorphisms

This section contains the proof of Proposition 1.5 and Theorem 1.3, as stated in Section 1. Throughout this section, $f : \mathcal{T}_{g,n} \to \mathcal{T}_{g,n}$ will denote a diffeomorphism that preserves horospheres.

4.1. Proof of Proposition 1.5

We first prove:

LEMMA 4.1. — *The inverse map f^{-1} also preserves horospheres.*

Proof. — Assume that $f(X) = Y$. Let $W = \langle \mu_1, \ldots, \mu_{6g-7+2n} \rangle$ be the tangent space of $\mathrm{HS}(\mathcal{F}, Y)$ at Y. Then the pull-back of W by f

$$V = \langle f^*(\mu_1), \ldots, f^*(\mu_{6g-7+2n}) \rangle$$

is a linear subspace of $T_X \mathcal{T}_{g,n}$ of dimension $6g - 7 + 2n$.

There is a quadratic differential $q \in Q(X)$ such that V is the tangent space of $\mathrm{HS}(\mathcal{F}_v(q), X)$ and $\mathrm{HS}(\mathcal{F}_h(q), X)$ at X (see Lemma 3.7). Since f preserves horospheres, it maps $\mathrm{HS}(\mathcal{F}_v(q), X)$ and $\mathrm{HS}(\mathcal{F}_h(q), X)$ to two horospheres, denoted by $\mathrm{HS}(\mathcal{F}', Y)$ and $\mathrm{HS}(\mathcal{G}', Y)$.

Note that $\mathrm{HS}(\mathcal{F}', Y)$ and $\mathrm{HS}(\mathcal{G}', Y)$ are tangent to each other at Y, and their tangent space at Y is equal to W. According to Lemma 3.5, we have $\mathcal{F} = k\mathcal{F}'$ or $\mathcal{F} = k\mathcal{G}'$ for some $k \in \mathbb{R}_+$. Thus (see Remark 3.2) $\mathrm{HS}(\mathcal{F}, Y) = f[\mathrm{HS}(\mathcal{F}_v(q), X)]$ or $f[\mathrm{HS}(\mathcal{F}_h(q), X)]$. The proof is complete. □

Remark 4.2. — The C^1-smoothness of f is used in the proof of Lemma 4.1. If we assume that $f : \mathcal{T}_{g,n} \to \mathcal{T}_{g,n}$ is a homeomorphism and both f, f^{-1} preserve horospheres, then the results in this section are still valid.

COROLLARY 4.3. — *The map f preserves horoballs.*

Proof. — Let $\mathrm{HS}(\mathcal{F}, s)$ be a horosphere. Assume that $\mathrm{HS}(\mathcal{G}, t) = f[\mathrm{HS}(\mathcal{F}, s)]$. Note that $\mathcal{T}_{g,n}$ is separated by $\mathrm{HS}(\mathcal{G}, t)$ into $\mathrm{HB}(\mathcal{G}, t)$ and $\mathcal{T}_{g,n}/\overline{\mathrm{HB}}(\mathcal{G}, t)$. We claim that

$$f[\mathrm{HB}(\mathcal{F}, s)] = \mathrm{HB}(\mathcal{G}, t), f[\mathcal{T}_{g,n}/\mathrm{HB}(\mathcal{F}, s)] = \mathcal{T}_{g,n}/\mathrm{HB}(\mathcal{G}, t).$$

Choose any $X \in \mathrm{HS}(\mathcal{F}, s)$. There is a unique quadratic differential $q \in Q(X)$ such that $\mathcal{F}_h(q) = \mathcal{F}$. According to Lemma 3.14, we know that $\mathrm{HS}(\mathcal{F}, X)$ is tangent to $\mathrm{HS}(\mathcal{F}_v(q), X)$ at X. This implies that $f[\mathrm{HS}(\mathcal{F}, X)]$ is tangent to $f[\mathrm{HS}(\mathcal{F}_v(q), X)]$ at $f(X)$. In particular,

$$f[\mathrm{HS}(\mathcal{F}_v(q), X)] \subset \mathcal{T}_{g,n}/\mathrm{HB}(\mathcal{G}, t).$$

Since $\mathrm{HS}(\mathcal{F}_v(q), X) \subset \mathcal{T}_{g,n}/\mathrm{HB}(\mathcal{F}, s)$, we have

$$f[\mathcal{T}_{g,n}/\mathrm{HB}(\mathcal{F}, s)] = \mathcal{T}_{g,n}/\mathrm{HB}(\mathcal{G}, t).$$

The proof is complete. □

Now we prove Proposition 1.5:

PROPOSITION 1.5. — *Let $f : \mathcal{T}_{g,n} \to \mathcal{T}_{g,n}$ be a diffeomorphism that preserves horospheres. Assume that $\mathcal{F} \in \mathcal{MF}_{ind}$ and $f[\mathrm{HS}(\mathcal{F}, X)] = \mathrm{HS}(\mathcal{G}, Y)$. Then*

(1) *$f[\mathrm{HS}(\mathcal{F}, Z)] = \mathrm{HS}(\mathcal{G}, f(Z))$ for all $Z \in \mathcal{T}_{g,n}$;*
(2) *$\mathcal{G} \in \mathcal{MF}_{ind}$.*

Proof of Proposition 1.5.

(1). — Let $Z \in \mathcal{T}_{g,n}$. By Lemma 4.1, we can assume that

$$f^{-1}[\mathrm{HS}(\mathcal{G}, f(Z))] = \mathrm{HS}(\mathcal{F}', Z).$$

Consider the case that $\mathrm{Ext}_Y(\mathcal{G}) \leqslant \mathrm{Ext}_{f(Z)}(\mathcal{G})$. By Corollary 4.3, we have

$$f^{-1}[\mathrm{HB}(\mathcal{G}, Y)] = \mathrm{HB}(\mathcal{F}, X), f^{-1}[\mathrm{HB}(\mathcal{G}, f(Z))] = \mathrm{HB}(\mathcal{F}', Z).$$

Since $\mathrm{HB}(\mathcal{G}, Y) \subset \mathrm{HB}(\mathcal{G}, f(Z))$, we have $\mathrm{HB}(\mathcal{F}, X) \subset \mathrm{HB}(\mathcal{F}', Z)$. It follows from Proposition 3.13 that $\mathcal{F}' = k\mathcal{F}$ for some $k \in \mathbb{R}_+$. It follows that

$$\mathrm{HS}(\mathcal{F}, Z) = \mathrm{HS}(\mathcal{F}', Z) \quad \text{and} \quad f[\mathrm{HS}(\mathcal{F}, Z)] = \mathrm{HS}(\mathcal{G}, f(Z)).$$

The case that $\mathrm{Ext}_Y(\mathcal{G}) \geqslant \mathrm{Ext}_{f(Z)}(\mathcal{G})$ can be proved in the same way.

(2). — Suppose not, $\mathcal{G}$ has more than one indecomposable component. Let $\mathcal{G}_0$ be an indecomposable component of $\mathcal{G}$. According to Corollary 3.12, there exists $Y_0 \in \mathcal{T}_{g,n}$ such that

$$\mathrm{HB}(\mathcal{G}, Y) \subset \mathrm{HB}(\mathcal{G}_0, Y_0).$$

Assume that $f^{-1}[\mathrm{HB}(\mathcal{G}_0, Y_0)] = \mathrm{HB}(\mathcal{F}_0, X_0)$, where $X_0 = f^{-1}(Y_0)$. Then we have

$$\mathrm{HB}(\mathcal{F}, X) \subset \mathrm{HB}(\mathcal{F}_0, X_0).$$

Applying Proposition 3.13 to $\mathcal{F}$, which is assumed to be indecomposable, we have $\mathcal{F}_0 = k \cdot \mathcal{F}$ for some $k \in \mathbb{R}_+$. This implies that

$$\mathrm{HS}(\mathcal{G}, Y_0) = f[\mathrm{HS}(\mathcal{F}_0, X_0)] = \mathrm{HS}(\mathcal{G}_0, Y_0).$$

This leads to a contradiction, since $\mathcal{G}_0 \neq k\mathcal{G}$ for any $k \in \mathbb{R}_+$. $\qquad\square$

COROLLARY 4.4. — *Let $\mathcal{F} \in \mathcal{MF}_{ind}$ and $s \in \mathbb{R}_+$. Then there exists $\mathcal{G} \in \mathcal{MF}_{ind}$ such that $f[\mathrm{HS}(\mathcal{F}, s)] = \mathrm{HS}(\mathcal{G}, t(s))$, where $t(s)$ is a strictly monotonically increasing function of s.*

4.2. The map f induces an automorphism of $\mathcal{C}(\mathcal{S})$

As above, let

$$f : \mathcal{T}_{g,n} \to \mathcal{T}_{g,n}$$

be a diffeomorphism that preserves horospheres. It follows from Proposition 1.5 and Remark 3.2 that f induces a bijection

$$f_* : \mathcal{MF}_{ind} \to \mathcal{MF}_{ind}.$$

Moreover, f_* maps projective equivalence classes to projective equivalence classes.

Denote by $\mathcal{UMF}$ be the set of uniquely ergodic measured foliations. It is clear that $\mathcal{S}$ and $\mathcal{UMF}$ are contained in $\mathcal{MF}_{ind}$.

For $\mathcal{F} \in \mathcal{MF}_{ind}$, we denote

$$\mathcal{N}(\mathcal{F}) = \{\mathcal{G} \in \mathcal{MF}_{ind} \,|\, i(\mathcal{F}, \mathcal{G}) = 0\}.$$

Two measured foliations $\mathcal{F}$ and $\mathcal{G}$ are *topologically equivalent* if they (considered without their transverse measures) are isotopic up to Whitehead moves.

LEMMA 4.5. — *Let $\mathcal{F}, \mathcal{G} \in \mathcal{MF}_{ind}$. Then $\mathcal{N}(\mathcal{F}) = \mathcal{N}(G)$ if and only if $\mathcal{F}$ and $\mathcal{G}$ are topologically equivalent. We have $\mathcal{N}(\mathcal{F}) = \{k \cdot \mathcal{F} \,|\, k \in \mathbb{R}_+\}$ if and only if $\mathcal{F} \in \mathcal{UMF}$.*

Lemma 4.5 was proved in [12, Theorem 4.1].

PROPOSITION 4.6. — *The map $f_* : \mathcal{MF}_{ind} \to \mathcal{MF}_{ind}$ satisfies:*

$$i(f_*(\mathcal{F}), f_*(\mathcal{G})) = 0 \iff i(\mathcal{F}, \mathcal{G}) = 0.$$

Proof. — Let $\mathcal{F}, \mathcal{G} \in \mathcal{MF}_{ind}$ with $i(\mathcal{F}, \mathcal{G}) = 0$. We define $\mathcal{F} + \mathcal{G}$ be the measured foliation equivalent to $i(\mathcal{F}, \cdot) + i(\mathcal{G}, \cdot)$ (when $\mathcal{F}$ and $\mathcal{G}$ are multi-curves, $\mathcal{F} + \mathcal{G}$ is the union of the curves).

Take a sequence of $X_k \in \mathcal{T}_{g,n}$ such that $\text{Ext}_{X_k}(\mathcal{F} + \mathcal{G}) \to 0$ as $k \to \infty$. By the monotonicity of extremal length, we have

$$\text{Ext}_{X_k}(\mathcal{F}) \leqslant \text{Ext}_{X_k}(\mathcal{F} + \mathcal{G}), \text{Ext}_{X_k}(\mathcal{G}) \leqslant \text{Ext}_{X_k}(\mathcal{F} + \mathcal{G}).$$

Then

$$\text{Ext}_{X_k}(\mathcal{F}) \to 0, \text{Ext}_{X_k}(\mathcal{G}) \to 0$$

as $k \to \infty$. Equivalently, we have

$$X_k \in \text{HS}(\mathcal{F}, s_k) \cap \text{HS}(\mathcal{G}, t_k)$$

with $s_k, t_k \to 0$ as $k \to \infty$. By Corollary 4.4, $Y_k := f(X_k)$ belongs to

$$f\left(\text{HS}(\mathcal{F}, s_k)\right) \cap f\left((\text{HS}(\mathcal{G}, t_k)\right) := \text{HS}(\mathcal{F}', s_k') \cap \text{HS}(\mathcal{G}', t_k'),$$

with $s_k', t_k' \to \infty$ as $k \to \infty$. This implies that $i(\mathcal{F}', \mathcal{G}') = 0$. Otherwise, the product $\text{Ext}_{Y_k}(\mathcal{F}') \text{Ext}_{Y_k}(\mathcal{G}')$ is bounded below by $i(\mathcal{F}', \mathcal{G}')^2$, which is impossible. □

Combining Lemma 4.5 with Proposition 4.6, we obtain:

COROLLARY 4.7. — *The map f_* satisfies $f_*(\mathcal{UMF}) = \mathcal{UMF}$.*

PROPOSITION 4.8. — *The map f gives rise to an automorphism of the complex of curves $f_* : \mathcal{C}(S) \to \mathcal{C}(S)$.*

Proof. — It suffices to prove that $f_*(\gamma) \in \mathcal{S}$ when $\gamma \in \mathcal{S}$. According to Proposition 4.6, we can assume that $\mathcal{G} = f_*(\gamma) \in \mathcal{MF}_{ind}$. Denote by $\widetilde{\mathcal{G}}$ the unmeasured foliation obtained from $\mathcal{G}$ by forgetting the measure.

First, we observe that the dimension of the space of transverse measures on $\widetilde{\mathcal{G}}$ is one. If not, there exists some other $\mathcal{G}' \in \mathcal{MF}_{ind}$ which is topologically equivalent to $\mathcal{G}$, but not projectively equivalent. Denote $\mathcal{F} = f_*^{-1}(\mathcal{G}') \in \mathcal{MF}_{ind}$. By Proposition 4.6, we have

$$\mathcal{N}(\gamma) = \mathcal{N}(f_*^{-1}(\mathcal{G})) = \mathcal{N}(f_*^{-1}(\mathcal{G}')) = \mathcal{N}(\mathcal{F}),$$

since $\mathcal{N}(\mathcal{G}) = \mathcal{N}(\mathcal{G}')$. Applying Lemma 4.5 we conclude that $\mathcal{F}$ and γ are projectively equivalent. This is a contradiction to the assumption that $\mathcal{G}$ and $\mathcal{G}'$ are not projective equivalent.

There are three possibilities:

(i). — $\mathcal{G} \in \mathcal{S}$. This is what we want to prove.

(ii). — $\mathcal{G}$ is a uniquely ergodic measured foliation on S. This can not happen because $\gamma = f_*^{-1}(\mathcal{G}) \notin \mathcal{UMF}$.

(iii). — The remaining case is that $\mathcal{G}$ is uniquely ergodic on X_0, which is a proper subsurface of S. Let β be a boundary component of X_0. Denote $\mathcal{F} = f_*^{-1}(\beta)$. Then $\mathcal{F} \in \mathcal{MF}_{ind}$ is either a simple closed curve or a minimal ergodic component. In both cases, there always exists $\alpha \in \mathcal{S}$ such that $i(\mathcal{F}, \alpha) \neq 0$ and $i(\gamma, \alpha) = 0$. Then we have

$$i(\beta, f_*(\alpha)) \neq 0, i(\mathcal{G}, f_*(\alpha)) = 0.$$

Due to our construction, any measured foliation that intersects with the boundary component β must also intersects with $\mathcal{G}$ (since the measured foliation must cross the collar neighborhood of β and $\mathcal{G}$ is filling on the subsurface X_0). This leads to a contraction. □

4.3. Proof of Theorem 1.3

The proof is motivated by [12, §5].

Proof of Theorem 1.3. — Let $f : \mathcal{T}_{g,n} \to \mathcal{T}_{g,n}$ be a diffeomorphism that preserves horospheres. We show that f is induced by an element of the extended mapping class group.

By Theorem 4.8, f induces an automorphism f_* of the complex of curve $\mathcal{C}(S)$. Then the theorem of Ivanov (Theorem 2.6) implies that f_* acts on $\mathcal{C}(S)$ as an element ϕ of the extended mapping class group. Replacing f by $\phi^{-1} \circ f$, we can assume that $f_* = \text{id} : \mathcal{S} \to \mathcal{S}$. It remains to prove that $f = \text{id}$ on $\mathcal{T}_{g,n}$.

Let (α, β) be a pair of filling simple closed curves. Denote by $\mathbf{G}_{\alpha,\beta}$ the Teichmüller geodesic determined by α and β. We first claim that

$$f(\mathbf{G}_{\alpha,\beta}) = \mathbf{G}_{\alpha,\beta}.$$

In fact, for any $X \in \mathbf{G}_{\alpha,\beta}$, $\mathrm{HS}(\alpha, X)$ is tangent to $\mathrm{HS}(\beta, X)$ at X (see Lemma 3.15). Since $f_*(\alpha) = \alpha, f_*(\beta) = \beta$ and f preserves horospheres, $\mathrm{HS}(\alpha, f(X))$ is tangent to $\mathrm{HS}(\beta, f(X))$ at $f(X)$. Thus $f(X) \in \mathbf{G}_{\alpha,\beta}$. This shows that f preserves the Teichmüller geodesic $\mathbf{G}_{\alpha,\beta}$.

We next show that f is identity on $\mathbf{G}_{\alpha,\beta}$. We parameterize $\mathbf{G}_{\alpha,\beta}$ by $t \to \mathbf{G}_{\alpha,\beta}(t)$ such that the pair of measured foliation $(\mathcal{F}, \mathcal{G})$ is changed to $(e^t\alpha, e^{-t}\beta)$. Without loss of generality, we may also assume that

$$\frac{\mathrm{Ext}_{\mathbf{G}_{\alpha,\beta}(0)}(\alpha)}{\mathrm{Ext}_{\mathbf{G}_{\alpha,\beta}(0)}(\beta)} = 1.$$

According to Lemma 3.17, for each t, there exists a simple closed curve γ such that both (α, γ) and (β, γ) are filling, and

$$k = \frac{i(\alpha, \gamma)^2}{i(\beta, \gamma)^2} \approx e^{2t}.$$

There is a unique $X \in \mathbf{G}_{\alpha,\beta}$ such that

$$\frac{\mathrm{Ext}_X(\alpha)}{\mathrm{Ext}_X(\beta)} = k.$$

Let $r = \mathrm{Ext}_X(\alpha), s = \mathrm{Ext}_X(\beta)$. Then the proof of Lemma 3.16 shows that the horospheres $\mathrm{HS}(\alpha, r), \mathrm{HS}(\beta, s)$ are tangent to the horosphere $\mathrm{HS}(\gamma, \frac{i(\alpha,\gamma)^2}{r})$. Note that X is close to $\mathbf{G}_{\alpha,\beta}(t)$.

As a result, there is a dense subset $\mathbf{G} \subset \mathbf{G}_{\alpha,\beta}$ such that for every $X \in \mathbf{G}$, there exists a simple closed curve γ and $r, s, t \in \mathbb{R}_+$ such that the horospheres $\mathrm{HS}(\gamma, r), \mathrm{HS}(\alpha, s), \mathrm{HS}(\beta, t)$ are tangent to each other, and X is the tangent point of $\mathrm{HS}(\alpha, s)$ and $\mathrm{HS}(\beta, t)$. The images of the above triple of horospheres under f are also horospheres tangent to each other (still associated with α, β, γ). As we have observed in Lemma 3.16, X is the unique solution of the tangent problem. Thus $f(X) = X$.

Since the set of Teichmüller geodesics determined by filling pairs of simple closed geodesics is dense in $\mathcal{T}_{g,n}$ [20], it follows from continuity that $f = \mathrm{id}$. $\qquad\square$

5. Metric balls and Busemann functions

For $X \in \mathcal{T}_{g,n}$ and $r \in \mathbb{R}_+$, we denote by

$$\mathrm{B}(X, r) = \{Y \in \mathcal{T}_{g,n} \mid d_{\mathcal{T}}(X, Y) < r\}$$

the open metric ball of radius r centered at X. The closure of $\mathrm{B}(X, r)$ is called a closed metric ball. It is not hard to show that (see [1, Lemma 3.2])

LEMMA 5.1. — *Every closed metric ball in $\mathcal{T}_{g,n}$ is a countable intersection of closed horoballs.*

In this section, we prove the following converse result. Recall that $\mathcal{MF}_{ind}$ is the set of indecomposable measured foliations on S.

LEMMA 5.2. — *Let $\mathcal{F} \in \mathcal{MF}_{ind}$. Then every open horoball in $\mathcal{T}_{g,n}$ associated to $\mathcal{F}$ is a nested union of open metric balls.*

The about lemma was proved in special cases, by Bourque and Rafi [1] in the case that $\mathcal{F}$ is a simple closed curve, and by Masur [18] in the case that $\mathcal{F}$ is uniquely ergodic.

Our proof is to show that horospheres associated with indecomposable measured foliations are level sets of Busemann functions.

5.1. Busemann functions

Let $\mathcal{F} \in \mathcal{MF}$. Denote by $\mathbf{G}(\,\cdot\,) : [0, \infty) \to \mathcal{T}_{g,n}$ the Teichmüller geodesic ray determined by $\mathcal{F}$ and $X_0 = \mathbf{G}(0)$.

DEFINITION 5.3 (Busemann function). — *With above notation, the Busemann function associated to $\mathbf{G}(\,\cdot\,)$ is the map $B_{\mathbf{G}} : \mathcal{T}_{g,n} \to \mathbb{R}$ defined by*

$$B_{\mathbf{G}}(\,\cdot\,) = \lim_{t \to \infty} \left(d_{\mathcal{T}}(\,\cdot\,, \mathbf{G}(t)) - d_{\mathcal{T}}(\mathbf{G}(0), \mathbf{G}(t)) \right).$$

To see the convergence, let $X \in \mathcal{T}_{g,n}$ and denote

$$D(t) = d_{\mathcal{T}}(X, \mathbf{G}(t)) - d_{\mathcal{T}}(\mathbf{G}(0), \mathbf{G}(t))$$
$$= d_{\mathcal{T}}(X, \mathbf{G}(t)) - t.$$

We observe that $D(t)$ is a bounded non-increasing function. In fact, $D(t) \geqslant -d(\mathbf{G}(0), X)$. For any $0 \leqslant t_1 \leqslant t_2$, we have

$$D(t_2) = d_{\mathcal{T}}(X, \mathbf{G}(t_2)) - t_2$$
$$= d_{\mathcal{T}}(X, \mathbf{G}(t_2)) - d_{\mathcal{T}}(\mathbf{G}(t_2), \mathbf{G}(t_1)) - t_1$$
$$\leqslant d_{\mathcal{T}}(X, \mathbf{G}(t_1)) - t_1$$
$$= D(t_1).$$

For simplicity, we denote $B = B_{\mathbf{G}}$. Let

$$\mathrm{L}(B, s) = \{X \in \mathcal{T}_{g,n} \,|\, B(X) = s\}$$

and

$$\mathrm{SL}(B, s) = \{X \in \mathcal{T}_{g,n} \mid B(X) < s\},$$

denote the level set and sub-level set of B, respectively. Note that $X_0 \in \mathrm{L}(B, 0)$. Let

$$\mathrm{S}(X, \varepsilon) = \{Y \in \mathcal{T}_{g,n} \mid d_{\mathcal{T}}(X, Y) = \varepsilon\}$$

and

$$\mathrm{B}(X, \varepsilon) = \{Y \in \mathcal{T}_{g,n} \mid d_{\mathcal{T}}(X, Y) < \varepsilon\}$$

be the metric sphere and the metric ball with center $X \in \mathcal{T}_{g,n}$ and radius $\varepsilon \in \mathbb{R}_+$, respectively.

DEFINITION 5.4. — *Let $\{\mathbf{M}_n\}$ be a sequence of non-empty subsets of $\mathcal{T}_{g,n}$. We define the upper and lower limits of the sequence as follows:*

(1) *The upper limit $\overline{\lim}\,\mathbf{M}_n$ consists of all accumulation points of any sequences $\{X_n\}$ with $X_n \in \mathbf{M}_n$. Thus $X \in \overline{\lim}\,\mathbf{M}_n$ if and only if each $\mathrm{S}(X, \varepsilon)$, $\varepsilon > 0$, intersects with infinitely many $\mathbf{M}_n$.*

(2) *The lower limit $\underline{\lim}\,\mathbf{M}_n$ consists of all points X such that each $\mathrm{S}(X, \varepsilon)$, $\varepsilon > 0$, intersects with all but a finite number of $\mathbf{M}_n$.*

By definition, $\underline{\lim}\,\mathbf{M}_n \subset \overline{\lim}\,\mathbf{M}_n$. If $\underline{\lim}\,\mathbf{M}_n = \overline{\lim}\,\mathbf{M}_n$, we denote by $\lim\mathbf{M}_n$.

The next lemma describes the relationship between metric spheres and level sets of Busemann functions. The proof is obtained in [2], which applies to a general geodesic metric spaces. We give the proof here for convenience of the readers.

LEMMA 5.5. — *Let $\mathrm{S}(\mathbf{G}(t), t)$ and $\mathrm{B}(\mathbf{G}(t), t)$, respectively, be the metric sphere and the metric ball with center at $\mathbf{G}(t)$ and passing through X_0. Then*

$$\lim_{t \to \infty} \mathrm{S}(\mathbf{G}(t), t) = \mathrm{L}(B, 0) \quad and \quad \lim_{t \to \infty} \mathrm{B}(\mathbf{G}(t), t) = \mathrm{SL}(B, 0).$$

Proof. — We first show that $\overline{\lim}\,\mathrm{S}(\mathbf{G}(t), t) \subset \mathrm{L}(B, 0)$.

Let Y be an arbitrary accumulation point, that is, there is a sequence $X_n \in \mathrm{S}(\mathbf{G}(t_n), t_n)$ such that $X_n \to Y$ as $t_n \to \infty$. It suffices to show that $B(Y) = 0$. In fact, by definition,

$$\begin{aligned}
B(Y) &= \lim_{n \to \infty} \{d_{\mathcal{T}}(Y, \mathbf{G}(t_n)) - t_n\} \\
&= \lim_{n \to \infty} \{d_{\mathcal{T}}(X_n, \mathbf{G}(t_n)) - t_n\} \\
&= \lim_{n \to \infty} \{d_{\mathcal{T}}(X_0, \mathbf{G}(t_n)) - t_n\} \\
&= 0.
\end{aligned}$$

It remains to prove that $\mathrm{L}(B,0) \subset \underline{\lim}\, \mathrm{S}(\mathbf{G}(t), t)$.

Choose any $Y \in \mathrm{L}(B,0)$. As we have noted above, the Busemann function $B(\,\cdot\,)$ is the limit of a non-increasing sequence of functions $d_{\mathcal{T}}(\,\cdot\,, \mathbf{G}(t)) - t$. It follows that

$$d_{\mathcal{T}}(Y, \mathbf{G}(t)) - t \geqslant B(Y) = 0.$$

Thus the distance between Y and $\mathbf{G}(t)$ is greater than t. Consider the geodesic segment $\overline{Y\mathbf{G}(t)}$ connecting Y and $\mathbf{G}(t)$. It intersects with the metric sphere $\mathrm{S}(\mathbf{G}(t), X)$ at some point X_t. Then

$$
\begin{aligned}
d_{\mathcal{T}}(Y, X_t) &= d_{\mathcal{T}}(Y, \mathbf{G}(t)) - d_{\mathcal{T}}(X_t, \mathbf{G}(t)) \\
&= d_{\mathcal{T}}(Y, \mathbf{G}(t)) - d_{\mathcal{T}}(X_0, \mathbf{G}(t)) \\
&\to B(Y) - B(X_0) \\
&= 0
\end{aligned}
$$

This implies that when t large enough, we have

$$B(Y, \varepsilon) \cap \mathrm{S}(\mathbf{G}(t), t) \neq \emptyset,$$

and then

$$Y \in \underline{\lim}\, \mathrm{S}(\mathbf{G}(t), t).$$

The proof of the sub-level set is similar. $\qquad\square$

5.2. Formula of Busemann functions

There is an explicit formula of $B_{\mathbf{G}}$, due to Walsh [26]. As before, we denote by $\mathbf{G}(\,\cdot\,)$ the Teichmüller geodesic ray determined by $\mathcal{F}$ and $X_0 = \mathbf{G}(0)$. Denote by $\mathcal{G}$ the horizontal foliation of the Hubbard–Masur differential of $\mathcal{F}$ on X_0.

THEOREM 5.6 ([26, Corollary 1.2]). — Let $\mathcal{F} = \sum_j \mathcal{F}_j$ be the ergodic decomposition of $\mathcal{F}$. Then the Busemann function $B_{\mathbf{G}}(\,\cdot\,)$ is given by

$$B_{\mathbf{G}}(X) = \frac{1}{2} \log \sup_{\gamma \in \mathcal{S}} \frac{E_{\mathcal{F}}(\gamma)}{\mathrm{Ext}_X(\gamma)} - \frac{1}{2} \log \sup_{\gamma \in \mathcal{S}} \frac{E_{\mathcal{F}}(\gamma)}{\mathrm{Ext}_{X_0}(\gamma)},$$

where $E_{\mathcal{F}}(\gamma)$ is defined by

$$E_{\mathcal{F}}(\gamma) = \sum_j \frac{i(\mathcal{F}_j, \gamma)^2}{i(\mathcal{F}_j, \mathcal{G})}.$$

By Walsh's formula, if the measured foliation $\mathcal{F} \in \mathcal{MF}_{ind}$, then

$$B_{\mathbf{G}}(X) = \frac{1}{2} \log \sup_{\gamma \in \mathcal{S}} \frac{i(\mathcal{F}, \gamma)^2}{\operatorname{Ext}_X(\gamma)} - \frac{1}{2} \log \sup_{\gamma \in \mathcal{S}} \frac{i(\mathcal{F}, \gamma)^2}{\operatorname{Ext}_{X_0}(\gamma)}$$

$$= \frac{1}{2} \log \operatorname{Ext}_X(\mathcal{F}) - \frac{1}{2} \log \operatorname{Ext}_{X_0}(\mathcal{F}).$$

The last equality holds because of Corollary 2.4. Hence we obtain the following:

PROPOSITION 5.7. — *Horospheres in $\mathcal{T}_{g,n}$ associated with $\mathcal{F} \in \mathcal{MF}_{ind}$ are level sets of the corresponding Busemann functions.*

Remark 5.8. — We can prove that a horosphere associated with a measured foliation $\mathcal{F}$ is the level of a Busemann function if and only if $\mathcal{F}$ is indecomposable [23].

5.3. Proof of Lemma 5.2

It is a direct corollary of Lemma 5.5 and Proposition 5.7.

6. A proof of Royden's Theorem

Denote by $\mathcal{B}$ the set of Busemann functions on $\mathcal{T}_{g,n}$. For $X \in \mathcal{T}_{g,n}$ and $\mathcal{F} \in \mathcal{MF}$, we denote by $B(X, \mathcal{F})$ the Busemann function of the Teichmüller geodesic ray $\mathbf{G}(t)$ determined by $\mathcal{F}$ and $X = \mathbf{G}(0)$. The level set $\{Z \in \mathcal{T}_{g,n} \,|\, B(X, \mathcal{F})(Z) = 0\}$ will be denoted by $\mathrm{L}(X, \mathcal{F})$.

Let $f : \mathcal{T}_{g,n} \to \mathcal{T}_{g,n}$ be an isometry of the Teichmüller metric. Since f maps Teichmüller geodesic rays to Teichmüller geodesic rays, f defines a transformation $f_* : \mathcal{B} \to \mathcal{B}$. Given $X \in \mathcal{T}_{g,n}$ and $\mathcal{F} \in \mathcal{MF}$, we denote

$$f_*(B(X, \mathcal{F})) = B(Y, \mathcal{G}),$$

where $Y = f(X) \in \mathcal{T}_{g,n}$ and $\mathcal{G} \in \mathcal{MF}$. It projects to map from $\mathcal{MF}$ to itself, which is still denoted by f_*.

Proof of Royden's Theorem. — Using the proof of Theorem 1.3, it suffices to show that f preserves horospheres determined by indecomposable measured foliations.

Step 1: f preserves level sets of Busemann functions. — In fact, f maps Teichmüller geodesic rays to Teichmüller geodesic rays, and f maps metric spheres to metric spheres. Thus by Lemma 5.5, we have

$$f(\mathrm{L}(X, \mathcal{F})) = \mathrm{L}(Y, \mathcal{G}).$$

Note that f also preserves sub-level sets of Busemann functions.

Step 2: If $\mathcal{F} \in \mathcal{MF}_{ind}$, then $\mathcal{G} \in \mathcal{MF}_{ind}$. — If not, $\mathcal{G} \notin \mathcal{MF}_{ind}$. Let $\mathcal{G} = \sum \mathcal{G}_j$ be its ergodic decomposition. We claim:

LEMMA 6.1. — *For all $\mathcal{G}_j$ in the ergodic decomposition of $\mathcal{G}$, The sub-level set $\mathrm{SL}(Y, \mathcal{G}) := \{Z \in \mathcal{T}_{g,n} \,|\, B(Y, \mathcal{G})(Z) < 0\}$ is contained in the horoball $\mathrm{HB}(\mathcal{G}_j, s)$ for some $s > 0$.*

Proof. — To prove the claim, we use Walsh's formula for the Busemann function. Up to an additive constant, $B(Y, \mathcal{G})(\cdot)$ is of the form

$$B(Y, \mathcal{G})(Z) = \frac{1}{2} \log \sup_{\gamma \in \mathcal{S}} \frac{\sum_k c_k i(\mathcal{G}_k, \gamma)^2}{\mathrm{Ext}_Z(\gamma)}.$$

Thus, up to an additive constant,

$$B(Y, \mathcal{G})(Z) \geqslant \frac{1}{2} \log \sup_{\gamma \in \mathcal{S}} \frac{i(\mathcal{G}_j, \gamma)^2}{\mathrm{Ext}_Z(\gamma)} - \log \sqrt{c_j}.$$

As we have observed in Section 5.2, by Corollary 2.4,

$$\frac{1}{2} \log \sup_{\gamma \in \mathcal{S}} \frac{i(\mathcal{G}_j, \gamma)^2}{\mathrm{Ext}_Z(\gamma)} = \frac{1}{2} \log \mathrm{Ext}_Z(\mathcal{G}_j).$$

Thus $\mathrm{SL}(Y, \mathcal{G})$ is contained in some horoball of $\mathcal{G}_j$. $\qquad\square$

Since $\mathcal{F}, \mathcal{G}_j \in \mathcal{MF}_{ind}$, for any $Z \in \mathcal{T}_{g,n}$, we have

$$\mathrm{L}(Z, \mathcal{F}) = \mathrm{HS}(\mathcal{F}, Z)$$

and

$$\mathrm{L}(Z, \mathcal{G}_j) = \mathrm{HS}(\mathcal{G}_j, Z), \quad \mathrm{SL}(Z, \mathcal{G}_j) = \mathrm{HB}(\mathcal{G}_j, Z).$$

By the above lemma, there is some $Z_0 \in \mathcal{T}_{g,n}$ such that

$$\mathrm{HB}(\mathcal{F}, X) = f^{-1}(\mathrm{SL}(Y, \mathcal{G})) \subset f^{-1}(\mathrm{HB}(\mathcal{G}_j, Z_0)).$$

Let $\mathcal{F}_j$ be a measured foliation such that

$$\mathrm{SL}(f^{-1}(Z_0), \mathcal{F}_j) = f^{-1}(\mathrm{HB}(\mathcal{G}_j, Z_0)).$$

Then we have

$$\mathrm{HB}(\mathcal{F}, X) \subset \mathrm{SL}(f^{-1}(Z_0), \mathcal{F}_j).$$

Apply Lemma 6.1 again, for each $\mathcal{F}_j$, $\mathrm{SL}(f^{-1}(Z_0), \mathcal{F}_j)$ must contained in some horoballs associated to each ergodic component of $\mathcal{F}_j$. It follows from Proposition 3.13 that any such ergodic component of $\mathcal{F}_j$ is projectively equivalent to $\mathcal{F}$. It turns out that each $\mathcal{F}_j$ is projectively equivalent to $\mathcal{F}$. And then all the $\mathcal{G}_j$ are projectively equivalent to each other. This leads to a contradiction with the assumption that $\mathcal{G} \notin \mathcal{MF}_{ind}$.

By Step 2, we have shown that f preserves horospheres of indecomposable measured foliations. Using the proof of Theorem 1.3, we conclude that f is an element of the extended mapping class group. The proof is complete. □

Remark 6.2. — We can apply the decomposition of measured foliations and Walsh's formula to study the action of an isometry on Teichmüller geodesics, and then give another proof of Royden's Theorem [23].

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