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Jan Dobrowolski

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SETS, GROUPS, AND FIELDS DEFINABLE IN VECTOR SPACES WITH A BILINEAR FORM

by Jan DOBROWOLSKI (*)

Abstract. — We study definable sets, groups, and fields in the theory of infinite-dimensional vector spaces over an algebraically closed field of any fixed characteristic different from 2 equipped with a nondegenerate symmetric (or alternating) bilinear form. First, we define a notion of dimension of a definable set, which enjoys many properties of Morley rank in strongly minimal theories. Then, using this dimension notion as the main tool, we prove that all definable groups are (algebraic-by-abelian)-by-algebraic. We conclude that every definable field is definably isomorphic to the field of scalars of the vector space. We derive some other consequences of good behaviour of the dimension, e.g. every generic type in any definable set is a definable type; every set is an extension base; every definable group has a definable connected component.

We also prove analogous results working over real closed fields.

Résumé. — Nous étudions des ensembles, des groupes et des corps définissables dans la théorie des espaces vectoriels de dimension infinie sur un corps algébriquement clos de caractéristique différente de 2 munis d'une forme bilinéaire symétrique (ou alternée) non dégénérée. Tout d'abord, nous définissons une notion de dimension d'un ensemble définissable, qui possède de nombreuses propriétés de rang de Morley dans les théories fortement minimales. Ensuite, en utilisant cette notion de dimension comme outil principal, nous prouvons que tous les groupes définissables sont (algébriques-par-abéliens)-par-algébriques. Nous concluons que tout corps définissable est définissablement isomorphe au corps des scalaires de l'espace vectoriel. Nous déduisons d'autres conséquences du bon comportement de la dimension, par exemple chaque type générique dans tout ensemble définissable est un type définissable ; chaque ensemble est une base d'extension ; chaque groupe définissable a une composante connexe définissable.

Nous démontrons également des résultats analogues en travaillant sur des corps réels clos.

1. Introduction

There are two kinds of motivation for the study undertaken in this paper.

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The first is improving our understanding of definable sets and other definable objects (such as groups and fields) in classical mathematical structures. There is a variety of this kind of results in numerous contexts; we mention few of them. In algebraically closed fields there is a very well-behaved notion of dimension on definable sets (given by the algebraic dimension of the Zariski closure of a set, which coincides with a more general notion of Morley rank) and the following well-known description of definable groups and fields follows from results by Weil, Hrushovski, and van den Dries (see [1, 6, 7, 22]).

Fact 1.1. — *Let K be an algebraically closed field. Then:*

(1) *The groups definable in K are precisely the algebraic groups over K.*
(2) *Every field definable in K is definably isomorphic to K.*

Variants of these statements for separably closed fields were proved in [17]. In the real closed fields and their *o*-minimal expansions, again, there is a very nice notion of dimension, and Pillay's Conjecture provides a link between definable groups and Lie groups. Moreover, the following was proved in [18].

Fact 1.2. — *Every infinite field definable in an o-minimal structure is either real closed or algebraically closed.*

There are many more results on groups definable in fields and in their expansions such as differential fields, fields with a generic automorphism, or valued fields. In a different flavour, it was proved in [2] that there are no infinite fields definable in free groups. Groups definable in ordered vector spaces over ordered division rings were studied in [10].

Our second motivation is understanding certain phenomena in NSOP_1 structures - a very broad class of "tame" structures studied intensively in recent years, with the vector spaces with a generic bilinear form being one of the main algebraic examples. This motivation is addressed most directly in Section 8, which, however, relies on our study of dimension in earlier sections.

A systematic study of vector spaces with a bilinear form was first undertaken in [11]. Several fundamental results concerning completeness, model completeness, and quantifier elimination were established there. As finite-dimensional vector spaces with a bilinear form are definable in the underlying field of scalars, only the infinite-dimensional case goes really beyond the (model-theoretic) study of the field. The main focus in [11] was on the theory T_∞ of infinite-dimensional vector spaces over an algebraically closed field of any fixed characteristic different from 2 with a nondegenerate symmetric (or alternating) bilinear form (this is a slight abuse of notation, as this means in fact considering a family of different theories, depending on whether the form is assumed to be symmetric or alternating, and also on the characteristic of the field of scalars, all of which are denoted by T_∞). A certain independence relation $\mathop{\smile\!\!\!\!\!\!\mid\;}^{\Gamma}$ on models on T_∞ was constructed there, and it was proved that it shares many nice properties with forking independence in stable theories (forking independence is a central notion in model theory generalising linear independence in vector spaces and algebraic independence in algebraically closed fields to abstract contexts). These results were later used in [4] to prove that T_∞ is NSOP_1. T_∞ was further studied in [14], where the canonical independence relation in NSOP_1 theories called Kim-independence (and denoted $\mathop{\smile\!\!\!\!\!\!\mid\;}^{K}$) was introduced, and described in particular in T_∞ (some corrections are needed in that description, see Proposition 8.12 and the discussion preceding it). It was then deduced in [14] that $\mathop{\smile\!\!\!\!\!\!\mid\;}^{\Gamma}$ is strictly stronger than $\mathop{\smile\!\!\!\!\!\!\mid\;}^{K}$.

In [3] it was proved that (the completions of) the theories of vector spaces with a nondegenerate bilinear form over an NIP (another tameness property studied extensively in model theory) field satisfy a generalisation of NIP called NIP_2; in particular, T_∞ and T_∞^{RCF} (see the paragraph below) are examples of NIP_2 theories which are not NIP.

In this paper, we study the theory (strictly speaking, the theories) T_∞ and the theory (two theories) T_∞^{RCF} of infinite-dimensional vector spaces over a real closed field equipped with a nondegenerate alternating bilinear form or a nondegenerate symmetric positive-definite bilinear form (RCF stands for the theory of real closed fields). In the final chapter of [11] (12.5) it was asked whether every group definable in T_∞ is finite Morley rank-by-abelian-by finite Morley rank, which we confirm in Section 7. We also prove that finite Morley rank quotients of groups definable in T_∞ by definable normal subgroups are definable in T_∞ and hence they are algebraic (by "algebraic" we mean definably isomorphic to an algebraic group over the field of scalars), thus obtaining that all groups definable in T_∞ are (algebraic-by-abelian)-by-algebraic. This conclusion is optimal in the sense that none of the three components in "(algebraic-by-abelian)-by-algebraic" can be omitted (see Remark 7.4). Using our theorem about groups, we deduce that every field definable in T_∞ is finite-dimensional, and hence either finite or definably isomorphic to the field of scalars. We also prove analogous results about groups and fields definable in T_∞^{RCF}. As our main tool, we develop a notion of dimension on sets definable in T_∞ and T_∞^{RCF}, whose good behaviour has several other consequences which may be of independent interest.

Most of the arguments in the paper are carried out simultaneously for T_∞, where we use Morley rank to define dimension, and for T_∞^{RCF}, where we use a topological dimension (called *o-minimal dimension*) for this purpose. Except Section 8 where we focus on model-theoretic properties of T_∞, the only significant difference between the two cases is that in T_∞ every definable set has finite multiplicity with respect to our dimension notion, which does not hold in T_∞^{RCF}. Because of this, we need separate arguments for T_∞ and T_∞^{RCF} in the proof of Corollary 6.4. Our proof of finiteness of multiplicity in T_∞ implies in particular that given a system of finitely many equations using the linear space operations and the bilinear form, the algebraic varieties obtained by intersecting the set of solutions of the system with finite-dimensional nondegenerate linear subspaces have uniformly bounded number of irreducible components of maximal dimension in the sense of algebraic geometry (cf. Theorem 6.3(1)).

The paper is organised as follows. In Section 2 we recall some basic facts about Morley rank and the *o*-minimal (topological) dimension, and about model theory of vector spaces with bilinear forms.

In Section 3 we review the notions of dimension and codimension of a definable subset of the vector sort V introduced in [11], filling a gap in the construction.

In Section 4 we extend the notion of dimension to arbitrary definable sets and types in T_∞ and T_∞^{RCF}, and we prove that it has properties similar to those of Morley rank in strongly minimal theories (Corollary 4.13).

In Section 5 we prove an analogue of Lascar's equality for T_∞ and T_∞^{RCF}, and we relate our notion of dimension to the linear dimension.

In Section 6 we define multiplicity of a definable set in analogy with Morley degree, and we prove that every set definable in T_∞ has finite multiplicity. Using this, we prove that a quotient of a group definable in T_∞ by a definable normal subgroup is algebraic provided that it has finite Morley rank (and, using some additional argument, an analogous result for T_∞^{RCF}). We also derive another consequence of finiteness of multiplicity in T_∞: in every definable set there are only finitely many complete generic types (over any fixed model), and each of them is a definable type.

In Section 7 we first observe that every group definable in T_∞ has a definable connected component, and then we prove the main results of this paper: every group definable in T_∞ is (algebraic-by-abelian)-by-algebraic, and every field definable in T_∞ has finite dimension, hence is either finite or definably isomorphic to the field of scalars K. Simultaneously, we prove the corresponding results for T_∞^{RCF}.

In Section 8 we prove that every set of parameters in T_∞ is an extension base (i.e. T_∞ satisfies the existence axiom for forking independence) and we give a description of Kim-independence in T_∞ over arbitrary sets, correcting in particular the description of Kim-independence over models in T_∞ given in [14]. Finally, we prove that in every group G definable in T_∞ the $\mathop{\smile\!\!\!\!\!\vert}^{\,\Gamma}$-generics are precisely the generics in the sense of dimension (in particular $\mathop{\smile\!\!\!\!\!\vert}^{\,\Gamma}$-generics exist in G), and that the additive group $(V,+)$ of the vector sort does not have any $\mathop{\smile\!\!\!\!\!\vert}^{\,K}$-generics over any set.

All sections except the last one (Section 8) require only a very basic understanding of first-order logic, and should be accessible to readers familiar with concepts such as a model, a complete theory, a type (i.e. a consistent set of formulas), and quantifier elimination.

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2. Preliminaries

2.1. Morley rank and the o-minimal dimension

Let T be a complete theory, and let $M \models T$.

Definition 2.1. — *The Morley rank of a formula ϕ over M defining a set S, denoted* $\mathrm{RM}(\phi)$ *or* $\mathrm{RM}(S)$, *is an ordinal or -1 or ∞, defined by first recursively defining what it means for a formula to have Morley rank at least α for some ordinal α:*

- $\mathrm{RM}(S) \geqslant 0$ *iff* $S \neq \emptyset$.
- *If $\alpha = \beta + 1$ is a successor, then* $\mathrm{RM}(S) \geqslant \alpha$ *iff for every $n \in \omega$ there are disjoint sets $(X_i)_{i\in\{1,\dots,n\}}$ definable in some elementary extension N of M such that* $\mathrm{RM}(X_i) \geqslant \beta$ *and $X_i \subseteq \phi(N)$ for each $i \in \{1,\dots,n\}$.*
- *If λ is a limit ordinal then* $\mathrm{RM}(S) \geqslant \lambda$ *iff* $\mathrm{RM}(S) \geqslant \alpha$ *for every $\alpha < \lambda$.*

Finally, $\mathrm{RM}(S) = \alpha$ *when* $\mathrm{RM}(S) \geqslant \alpha$ *and for no $\beta > \alpha$ one has* $\mathrm{RM}(S) \geqslant \beta$. *Also, we set* $\mathrm{RM}(S) = \infty$ *if* $\mathrm{RM}(S) \geqslant \alpha$ *for every* $\alpha \in \mathrm{Ord}$.

If $\mathrm{RM}(S) \in \mathrm{Ord}$*, then the Morley degree of* S*, denoted by* $\mathrm{DM}(S)$*, is the maximal number of definable sets of Morley rank* $\mathrm{RM}(S)$ *into which* S *can be partitioned.*

If $M = (F, +, \cdot, 0, 1)$ is an algebraically closed field (which is essentially the only case in which we consider Morley rank in this paper), passing to an elementary extension N of M in the above definition is not necessary - the sets X_i may be chosen to be definable in M.

A one-sorted structure M (or its theory $\mathrm{Th}(M)$) is called *strongly minimal* if $\mathrm{RM}(x = x) = \mathrm{DM}(x = x) = 1$ where x is a single variable of the only sort of M. Equivalently, every definable subset of any model $\mathfrak{C} \models \mathrm{Th}(M)$ is either finite of co-finite. Any algebraically closed field is strongly minimal. For p equal zero or a prime number, ACF_p denotes the (complete) theory of algebraically closed fields of characteristic p.

Fact 2.2 ([16]). — *An infinite field has finite Morley rank if and only if it is algebraically closed (if and only if it is strongly minimal).*

If K is an algebraically closed field and X is an algebraic subset of K^n for some $n \in \omega$, then $\mathrm{RM}(X)$ is the dimension of X in the sense of algebraic geometry, and $\mathrm{DM}(X)$ is the number of irreducible components of X of maximal dimension.

In real closed fields Morley rank of any infinite set is equal to ∞, but there is another useful notion of dimension (having various equivalent definitions).

Definition 2.3. — *Let* $(R, +, \cdot, \leqslant)$ *be a real closed field (or, more generally, an o-minimal structure). For a nonempty definable* $X \subseteq R^k$ *the (topological) dimension of* X*, denoted by* $\dim_t(X)$*, is the greatest number* n *such that a nonempty definable open (in the order topology) subset of* R^n *embeds definably into* X*. We also put* $\dim_t(\emptyset) = -1$*.*

Again, for algebraic subsets of R^n where $R \models \mathrm{RCF}$, $\dim_t$ coincides with the dimension in the sense of algebraic geometry.

Definition 2.4. — *We say an* S*-valued (where* S *is any set) rank* rk *on the collection of all sets definable in* T *is* definable *(over* $\emptyset$*) if for any formula* $\phi(x, y)$ *over* $\emptyset$*,* $n \in S$*, and* $\mathfrak{C} \models T$ *the set* $\{a \in \mathfrak{C} : \mathrm{rk}(\phi(x, a)) = n\}$ *is definable over* $\emptyset$*.*

For the following properties of Morley rank and the topological dimension consult e.g. [21, Section 6.2] and [8, Section 4.1]; for (6) see [13, Lemma 3].

Fact 2.5. — *Let* rk *be either Morley rank in a strongly minimal theory, or the topological dimension in a real closed field (or in any o-minimal theory). Suppose* X_1 *and* X_2 *are definable. Then:*

(0) $\mathrm{rk}(X_1) \in \omega \cup \{-1\}$ *and* $\mathrm{rk}(X_1) = 0$ *iff* X_1 *is finite and nonempty.*
(1) *If* $X_1 \subseteq X_2$*, then* $\mathrm{rk}(X_1) \leqslant \mathrm{rk}(X_2)$*.*
(2) $\mathrm{rk}(X_1 \cup X_2) = \max(\mathrm{rk}(X_1), \mathrm{rk}(X_2))$*.*
(3) *If there is a definable bijection between* X_1 *and* X_2*, then* $\mathrm{rk}(X_1) = \mathrm{rk}(X_2)$*.*
(4) *More generally, if* $f : X_1 \to X_2$ *is a definable surjection and there is* $d \in \omega$ *is such that* $\mathrm{rk}(f^{-1}(y)) = d$ *for each* $y \in X_2$*, then* $\mathrm{rk}(X_1) = \mathrm{rk}(X_2) + d$ *unless* X_2 *is empty.*

    In particular, if $\emptyset \neq Z \subseteq Y \times X$ *and there is* $d \in \omega$ *such that* $\mathrm{rk}(\{x \in X : (y, x) \in Z\}) = d$ *for every* $y \in Y$*, then* $\mathrm{rk}(Z) = \mathrm{rk}(Y) + d$*.*
(5) rk *is definable over* $\emptyset$*.*

    Additionally, in any algebraically closed fields RM *satisfies definable multiplicity property:*
(6) *For any* $n, d \in \omega$ *and a formula* $\phi(x, y)$ *the set*

$$\{a \in \mathfrak{C} : \mathrm{RM}(\phi(x, a)) = n, \mathrm{DM}(\phi(x, a)) = d\}$$

is definable over $\emptyset$*, and only for finitely many pairs* (n, d) *this set is nonempty.*

    In any strongly minimal theory we also have:
(7) *If* $\mathrm{RM}(X_1) < \mathrm{RM}(X_2)$*, then* $\mathrm{DM}(X_1 \cup X_2) = \mathrm{DM}(X_2)$*.*
(8) $\mathrm{DM}(X_1 \cup X_2) \leqslant \mathrm{DM}(X_1) + \mathrm{DM}(X_2)$
(9) *If* $f : X \to Y$ *is a definable surjection such that* $\mathrm{DM}(Y) = m \in \omega$ *and there are* $s, m' \in \omega$ *such that*

$$\mathrm{RM}(f^{-1}(y)) = s \text{ and } \mathrm{DM}(f^{-1}(y)) \leqslant m'$$

for every $y \in Y$*, then* $\mathrm{DM}(Y) \leqslant \mathrm{DM}(X) \leqslant mm'$*.*

(9) above can be proved in the same way as Proposition 6.2(4).

Finally, let us mention that if K is an algebraically closed field or a real closed field, then it admits (uniform) *elimination of imaginaries* (EI), that is, if E is any definable equivalence relation on K^n then the quotient K^n/E is in a definable bijection (in the structure K with the sort K^n/E added) with a definable subset of K^m for some m. However, the theories T_∞ and T_∞^{RCF} considered in this paper do not admit EI, hence we will need some extra care when dealing with quotients there.

2.2. Generic bilinear forms

We start by recalling some notation from [11]. Let L be the two-sorted language with sorts V (vectors), and K (scalars), containing constant symbols 0_V, 0_K, and 1_K, as well as binary function symbols: $+_V$, $+_K$, $\circ_K$, γ, $[\cdot,\cdot]$, which we shall interpret as: vector addition, field addition, field multiplication, scalar multiplication, and a bilinear form on the vector space.

We fix p to be 0 or a prime number different from 2, and we let $T_0 = \mathrm{ACF}_p$, the (complete) theory of algebraically closed fields of characteristic p. As the value of p does not play any role in the paper (and in the results of [11]), it is omitted in the notation below.

Definition 2.6. — *Let $m \in \omega \cup \{\infty\}$ and T_0 be either ACF_p or RCF. By ${}_ST_m^{T_0}$ [respectively, ${}_AT_m^{T_0}$] we denote the L-theory expressing that the sort K is a model of T_0, the sort V is an m-dimensional vector space over K, and that $[\cdot,\cdot]$ is a nondegenerate symmetric [respectively, alternating] K-bilinear form on V, and additionally ${}_ST_m^{\mathrm{RCF}}$ says that $[\cdot,\cdot]$ is positive-definite. We will write $T_m^{T_0}$ to mean either ${}_ST_m^{T_0}$ or ${}_AT_m^{T_0}$. We will also simply write T_m to mean $T_m^{\mathrm{ACF}_p}$, which is consistent with* [11, *Chapter 12*], *and T_m^* to mean either $T_m^{\mathrm{ACF}_p}$ or T_m^{RCF}.*

If $m \in \omega$ then ${}_ST_m^{T_0}$ is consistent only when m is even, so below we will always assume that $m = \infty$ or m is even in the symmetric case.

Definition 2.7. — *For any $n < \omega$ let $\theta_n(X_1, \dots, X_n)$ be the L-formula saying that the vectors $X_1, \dots, X_n$ are linearly independent. Let L_θ be the expansion of L obtained by adding to L a symbol θ_n for each n (which we shall interpret as the relation given by the formula θ_n).*

For any $n \in \{1, 2, \dots\}$ let $F_n : V^{n+1} \to K^n$ be a definable function sending any tuple $(v_1, \dots, v_{n+1})$ with $v_1, \dots, v_n$ linearly independent and $v_{n+1} \in \mathrm{Lin}_K(v_1, \dots, v_n)$ to the unique tuple $(a_1, \dots, a_n) \in K^n$ such that $v_{n+1} = a_1v_1 + \dots + a_nv_n$ (and any other tuple to $(0_K, \dots, 0_K)$, say). In [11, Corollary 9.2.3] Granger claimed that $T_m^{T_0}$ has quantifier elimination in the language $L_\theta \cup L_K$, where L_K is any language on K bidefinable with $(K, +, \cdot)$ in which K has quantifier elimination. D. MacPherson has later pointed out that there is a problem with this result, unless one adds function symbols for each F_n to the language: for example, working over an algebraically closed field, if α is a scalar transcendental over the prime field and $v \neq 0$ is a vector with $[v, v] = 0$, then the tuples $(v, \alpha v, \alpha^2 v)$ and $(v, \alpha v, \alpha^3 v)$ have the same $L_\theta \cup L_R$-quantifier-free type, where L_R is the language of rings, but they do not have the same type, as the formula $\phi(x, y, z) = \exists u(y = ux \wedge z = u^2x)$ is satisfied by $(v, \alpha v, \alpha^2 v)$ but not by $(v, \alpha v, \alpha^3 v)$.

A. Chernikov and N. Hempel have proved that indeed $T_m^{T_0}$ eliminates quantifiers in $L_\theta \cup L_K \cup \{F_n : n \in \omega\}$. Let us remark here that, in the symmetric positive-definite case over a real closed field, the functions F_n are equal to some terms in the language L_θ, hence adding the F_n's to the language is necessary only in the alternating case. For let $v_1, \ldots, v_n \in V$ be linearly independent, and $v_{n+1} = \sum_{i \leqslant n} a_i v_i$ for some $a_1, \ldots, a_n \in K$. Let A be the $n \times n$-matrix $([v_i, v_j])_{i,j \leqslant n}$ and note that $A(a_1, \ldots, a_n)^T = ([v_1, v_{n+1}], \ldots, [v_n, v_{n+1}])^T$. Note that if $b_1, \ldots, b_n$ is such that

$$A(b_1, \ldots, b_n)^T = ([v_1, v_{n+1}], \ldots, [v_n, v_{n+1}])^T$$

then $(b_1, \ldots, b_n) = (a_1, \ldots, a_n)$ as otherwise $\Sigma_{i \leqslant n}(a_i - b_i)v_i$ would be a non-zero vector orthogonal to $v_1, \ldots, v_n$, hence orthogonal to itself, which is a contradiction. So the equation

$$A(x_1, \ldots, x_n)^T = ([v_1, v_{n+1}], \ldots, [v_n, v_{n+1}])^T$$

has exactly one solution $(x_1, \ldots, x_n) = (a_1, \ldots, a_n)$, and so A is a non-singular matrix and

$$F(v_1, \ldots, v_{n+1}) = (a_1, \ldots, a_n) = A^{-1}([v_1, v_{n+1}], \ldots, [v_n, v_{n+1}])^T,$$

hence $F(v_1, \ldots, v_{n+1})$ is equal to a term in L_θ. Summarising, we have:

Fact 2.8. — *Put $L_\theta^F := L_\theta \cup \{F_n : n \in \omega\}$ and let T_0 be a completion of the theory of fields admitting quantifier elimination in a language L_K. Then, for every $m \in \omega$, the theories ${}_ST_m^{T_0}$ and ${}_AT_m^{T_0}$ have quantifier elimination in $L_\theta^F \cup L_K$.*

In particular, for every $m \in \omega \cup \{\omega\}$ (with m even in the alternating case) the theories ${}_ST_m$ and ${}_AT_m$ have quantifier elimination in L_θ^F, ${}_ST_m^{\mathrm{RCF}}$ has quantifier elimination in $L_\theta \cup \{\leqslant\}$ (by the discussion above) and ${}_AT_m^{\mathrm{RCF}}$ has quantifier elimination in $L_\theta^F \cup \{\leqslant\}$, where $\leqslant$ is a binary relation symbol interpreted as the unique field ordering on K in the real closed case.

The following fact follows from the proof of [11, Corollary 9.2.9]: although in the case $T_m^* = {}_ST_m^{\mathrm{RCF}}$ it does not formally follow from [11, Corollary 9.2.9] as in a real closed field not all elements have square roots, this condition is only used to transform a normal basis to an orthonormal basis (see the proof of [11, Proposition 9.1.5]), which clearly can be done over any real closed field if $[\cdot, \cdot]$ is positive definite.

Fact 2.9. — *For any $m \in \omega \cup \{\infty\}$ the L-theory T_m^* is complete.*

For a set or a tuple A, by $V(A)$ we mean the set of vectors belonging to A, and $\mathrm{acl}(A)$ [respectively, $\mathrm{dcl}(A)$] denotes the model-theoretic algebraic [definable] closure of A, that is, the set of elements whose type over

A has finitely many realisations [only one realisation]. The following fact easily follows from quantifier elimination (cf. [11, Proposition 9.5.1, Proposition 12.4.1]).

Fact 2.10. — *Let $M = (V, K) \models T^*_\infty$ and $A \subseteq M$. Then:*

(1) *For any $v \in V \setminus \mathrm{Lin}_K(A)$ the type $tp(v/A)$ is implied by $p_{v,A}(x) := \{[x,a] = [v,a] : a \in V(A)\} \cup \{[x,x] = [v,v]\} \cup \{\theta_n(a_1,\dots,a_n) \to \theta_{n+1}(a_1,\dots,a_n,x) : a_1,\dots,a_n \in V(A)\}$.*

(2) $\mathrm{acl}(A) \subseteq \mathrm{Lin}_K(V(A))$.

Proof.

(1). — Suppose $w \models p_{(v,A)}$. Then $v, w \notin \mathrm{Lin}_K(V(A))$, so there is a K-linear isomorphism $g : \mathrm{Lin}_K(V(A) \cup \{v\}) \to \mathrm{Lin}_K(V(A) \cup \{w\})$ fixing $\mathrm{Lin}_K(V(A))$ pointwise and sending v to w. Then g preserves $[\cdot,\cdot]$, so $g \cup \mathrm{id}_K$ is an elementary map by quantifier elimination. In particular, $\mathrm{tp}(w/A) = \mathrm{tp}(v/A)$.

(2). — By finite character of acl we may assume that A is finite. Now, if $v \notin \mathrm{Lin}_K(V(A))$ then for any $u \in V$ which is orthogonal to $V(A) \cup \{v\}$ we have by (1) that $\mathrm{tp}(v + u/A) = \mathrm{tp}(v/A)$, so in particular $\mathrm{tp}(v/A)$ has infinitely many realisations, i.e. $v \notin \mathrm{acl}(A)$. □

3. Dimension on V

This section is in a large part a review of the results from [11, Subsection 12.4], where the notions of dimension and codimension of a definable subset of the vector sort V in T_∞ were introduced. However, the definition of codimension there uses a false claim (see Remark 3.11 below), so we provide an argument fixing it.

In the rest of this paper, T^*_∞ means either T_∞, in which case we put $\mathrm{rk} = \mathrm{RM}$, or T^{RCF}_∞, in which case we put $\mathrm{rk} = \dim_t$ (see Definition 3.4 below). When we write $X \subseteq V$ we mean that X is a set of single elements of the sort V, but when we write $X \subseteq M$ where M is a model [or when we say that X is definable in M], we mean that X is a [definable] set of arbitrary finite compatible tuples in M. We will be working in a fixed $\aleph_0$-saturated model $\mathfrak{C} \models T^*_\infty$, which means every type in a single variable over a finite subset of M is realised in M. By Fact 2.10 it is easy to see that this is equivalent to saying that the field of scalars $K(\mathfrak{C})$ has infinite transcendence degree over its prime subfield (we will need $\aleph_0$-saturation only to choose generic elements in the proof of Theorem 7.3).

As in [11], *we deal with the case of a symmetric bilinear form unless stated otherwise*, and the alternating case can be treated analogously by replacing an orthonormal basis by a symplectic basis. We will occasionally point out the main differences between the symmetric and the alternating case. In fact, the alternating case tends to be easier, as the condition $[x,x] = [v,v]$ in the type $p_{v,A}(x)$ implying $\mathrm{tp}(v/A)$ (see Fact 2.10(1)) is trivially satisfied by any vector x, so it can be omitted.

The following definition was introduced (in a more general version) in [11, Section 12.1].

Definition 3.1. — *If $M = (K(M), V(M)) \models T^*_\infty$ and $V(M)$ is countably dimensional over $K(M)$, then an* approximating sequence *for M is a sequence $(N_r)_{r\in\omega}$ of substructures of M with $K(N_r) = K(M)$ such that $N_r \models T_r$, $M = \bigcup_{r\in\omega} N_r$, and $N_r \subseteq N'_r$ for all $r \leqslant r'$.*

In the alternating case, an approximating sequence is a sequence $(N_r)_{r\in\{2,4,\dots\}}$ satisfying analogous properties.

We will write $M = \bigcup^a_r N_r$ to mean that $(N_r)_r$ is an approximating sequence for M (so in particular, $M \models T^*_\infty$).

The following fact follows by, for example, the proofs of Theorem 1 and Corollary 1 in [12, Chapter II.2].

Fact 3.2. — *If $M \models {}_ST^*_\infty$ and $V(M)$ has dimension $\aleph_0$ over $K(M)$, then M has an approximating sequence $(N_r)_{r\in\omega}$, and for any such sequence we can find by the Gram–Schmidt process an orthonormal basis $(e_i)_{i\in\{1,2,\dots\}}$ for $V(M)$ over $K(M)$ such that $V(N_r) = \mathrm{Lin}_{K(M)}(e_1,\dots,e_r)$ for each $r \in \omega$. Similarly, in the alternating case, if $V(M)$ is countably dimensional over $K(M)$ then we can find an approximating sequence $(N_r)_{r\in\{2,4,\dots\}}$ for M and a symplectic basis $(e_i, f_i)_{i\in\omega}$ for $V(M)$ over $K(M)$ such that $T_{2r} = \mathrm{Lin}_{K(M)}(e_1, f_1,\dots,e_r,f_r)$ for every $r \in \omega$. In both cases, given an orthonormal [symplectic] basis B for some N_r with $r \in \omega$ [$r \in \{2l : l \in \omega\}$], we can find such an orthonormal [symplectic] basis for M (or for any $N_{r'}$ with $r' \geqslant r$) which extends B.*

Moreover, both in the symmetric and the alternating case, if $v_1,\dots,v_m \in V(M)$, then there is a $K(M)$-linear subspace V_0 of $V(M)$ such that $v_1,\dots,v_m \in V_0$ and $(K(M), V_0) \models T_{2m}$, and there is an approximating sequence $(N_r)_r$ for M with $N_{2m} = (K(M), V_0)$.

By (the proof of) [11, Lemma 10.1.3] and quantifier elimination we have:

Fact 3.3. — *Let $r \in \omega \cup \{\infty\}$ and $N = (V,K) \models T^*_r$.*

(1) *If $r \in \omega$, then the structure N is definable (over some parameters) in the pure field $(K,+,\cdot)$.*

(2) *For any $n \in \omega$, all definable [$\emptyset$-definable] in N subsets of K^n are definable [$\emptyset$-definable] in the pure field $(K, +, \cdot)$.*

Definition 3.4. — *Let $N = (V, K) \models T_m^{\mathrm{RCF}}$ for some $m \in \omega$. For any set X definable in N we put $\dim_t(X) := \dim_t^K(f[X])$ where f is any definable bijection between X and a subset of K^n for some n (note that $f[X]$ is definable in $(K, +, \cdot)$ by Fact 3.3(1)). This does not depend on the choice of f, because for any other definable bijection g between X and a subset of $K^{m'}$, the sets $f[X]$ and $g[X]$ are in a K-definable bijection by Fact 3.3(2).*

The following was stated in [11, Corollary 12.4.2] for definable subsets of V and $T_\infty^* = T_\infty$, but exactly the same proof works for definable subsets of any V^k and $T_\infty^* \in \{T_\infty, T_\infty^{\mathrm{RCF}}\}$ using quantifier elimination and definability of rk (Fact 2.5(5)).

Remark 3.5. — If $M = \bigcup_r^a N_r$, $M' = \bigcup_r^a N'_r$, $R \in \omega$ and X is a set definable over $N_R \cap N'_R$, then $\mathrm{rk}_{N_r}(X \cap N_r) = \mathrm{rk}_{N'_r}(X \cap N'_r)$ for all $r \geqslant R$.

Remark 3.6. — If X is a set definable in T_∞^* over a model $M = \bigcup_r^a N_r$ and $X(M) \subseteq N_R$ for some $R \in \omega$, then for any $r \geqslant R$ we have

$$\mathrm{rk}_{N_r}(X \cap N_r) = \mathrm{rk}_{N_R}(X \cap N_R).$$

If $* = \mathrm{ACF}_p$, then also $\mathrm{rk}_{N_R}(X \cap N_R) = \mathrm{RM}_M(X(M))$.

Proof. — If $* = \mathrm{ACF}_p$, then, as the definable subsets of $X(M) = X \cap N_r = X \cap N_R$ in the sense of N_R, N_r and M all coincide by Fact 3.3, we get $\mathrm{rk}_{N_r}(X \cap N_r) = \mathrm{rk}_{N_R}(X \cap N_R) = \mathrm{rk}_M(X(M))$.

If $* = \mathrm{RCF}$, then the equality $\mathrm{rk}_{N_R}(X \cap N_R) = \mathrm{rk}_{N_R}(X \cap N_r) = \mathrm{rk}_{N_r}(X \cap N_r)$ follows directly from Definition 3.4, as an N_R-definable bijection between $X \cap N_r$ and a set definable in K is in particular N_r-definable. □

For a tuple of parameters [tuple of single variables, respectively] a, by $l(a)$ we will mean the number of vectors [vector variables] in a.

By Fact 2.10(1) we have:

Fact 3.7. — *Let M be a countably dimensional model of ${}_ST_\infty^*$ with an orthonormal basis $(e_i)_{i \in \{1,2,\dots\}}$ and put $N_r = (K(M), \mathrm{Lin}_{K(M)}(e_1, \dots, e_r))$ for every $r \in \omega$. Suppose $R \in \omega$ and $a \in V(M) \backslash V(N_R)$. If we put $\beta_i = [a, e_i]$ for $i = 1, \dots, e_R$ and $\gamma = [a, a]$, then $tp(a/N_R)$ is isolated by the formula*

$$\phi_{a,R}(x) := \bigwedge_{i=1,\dots,R} [x, e_i] = \beta_i \wedge [x, x] = \gamma \wedge \theta_{R+1}(e_1, \dots, e_R, x)$$

(note that if $*=\mathrm{RCF}$ *then the formula* $\theta_{R+1}(e_1,\dots,e_R,x)$ *can be omitted here, as* $\gamma\neq\sum_{i=1,\dots,R}\beta_i^2$ *since* $a\notin N_R$*, so* $\bigwedge_{i=1,\dots,R}[x,e_i]=\beta_i\wedge[x,x]=\gamma$ *implies* $\theta_{R+1}(e_1,\dots,e_R,x)$*).*

Remark 3.8. — Suppose $n\in\omega$ and $M=(K,V_0)$ is a model of ${}_ST^*_n$ (i.e. a model of ${}_ST_n^{\mathrm{ACF}_p}$ or of ${}_ST_n^{\mathrm{RCF}}$). Let $c\in K$ and assume $c>0$ in the real closed case. Then

$$\mathrm{rk}_M(\{v\in V_0:[v,v]=c\})=n-1,$$

and if $M\models{}_ST_m^{\mathrm{ACF}_p}$ and $n\geqslant 3$, then we also have $\mathrm{DM}(\{v\in V_0:[v,v]=c\})=1$.

Proof. — Choose an orthonormal basis $(e_1,\dots,e_n)$ of V_0 over K. Then $\sum x_ie_i\mapsto(x_1,\dots,x_n)$ gives a definable bijection between $\{v\in V_0:[v,v]=c\}$ and $\{(x_1,\dots,x_n)\in K^n:x_1^2+\dots+x_n^2=c\}$, which has dimension $n-1$ when $*=\mathrm{ACF}_p$, and topological dimension $n-1$ when $*=\mathrm{RCF}$ (and $c>0$), hence $\mathrm{rk}(\{v\in V_0:[v,v]=c\})=n-1$. Assume now $M\models{}_ST_m^{\mathrm{ACF}_p}$. If $c\neq 0$ then the sphere $\{(x_1,\dots,x_n)\in K^n:x_1^2+\dots+x_n^2=c\}$ is known to be irreducible for any $n\geqslant 2$. Also $\{(x_1,\dots,x_n)\in K^n:x_1^2+\dots+x_n^2=c\}$ is known to be irreducible for any $n\geqslant 3$ (this can be deduced, for example, from the Eisenstein criterion). Hence, $\mathrm{DM}(\{v\in V_0:[v,v]=c\})=1$ for any $n\geqslant 3$ and any $c\in K$. □

Corollary 3.9. — *With the notation of Fact 3.7, for any* $r>R$ *with* $a\in N_r$ *we have* $\mathrm{rk}_{N_r}(\phi_{a,R}(N_r))=r-R-1$ *and, if* $*=$*ACF*$_p$ *and* $r\geqslant R+3$*, then* $\mathrm{DM}_{N_r}(\phi_{a,R}(N_r))=1$.

Proof. — Put $V_0:=\mathrm{Lin}_{K(M)}(e_{R+1},\dots,e_r)$. Then clearly $(K(M),V_0)\models{}_ST^*_{r-R}$. Let a_0 be the projection of a on $V(N_R)$. Then $w\mapsto w-a_0$ gives a definable bijection between $\phi_{a,R}(N_r)$ and $\{v\in V_0:[v,v]=[a,a]-[a_0,a_0]\}$. Hence the conclusion follows by Remark 3.8 (note that if $*=\mathrm{RCF}$ then $[a,a]-[a_0,a_0]\neq 0$ as $a\notin V(N_R)$). □

Proposition 3.10. — *Suppose* $M=\bigcup^a_{r\in\omega}N_r$ *and* $X\subseteq V$ *is a set definable by a formula* $\phi(x,a)$ *which is not contained in any finite-dimensional subspace of* V*. Let* $R\in\omega$ *be minimal such that* $R\geqslant 4l(a)+1$ *and* $a\subseteq N_R$*. Then there is* $d\leqslant 2l(a)+1$ *such that for any* $r\geqslant R$ *we have*

$$\mathrm{rk}_{N_r}(X\cap N_r)=r-d.$$

By Fact 3.5, d *does not depend on the choice of* M *and* $(N_r)_{r\in\omega}$*.*

Moreover, if $*=$*ACF*$_p$ *and* $r\geqslant R+2$ *for* R *as above, then* $\mathrm{DM}(X\cap N_r)=\mathrm{DM}(X\cap N_{R+2})$*.*

Proof. — By modifying N_r's for $r < R$ (using Fact 3.2), we may assume that $a \subseteq N_{2l(a)}$. Choose an orthonormal basis $(e_0, e_1, \dots)$ for M such that $N_r = \mathrm{Lin}_{K(M)}(e_1, \dots, e_r)$ for every $r \in \omega$.

Claim 1. — *There is $d \in \{0, 1, \dots, 2l(a)+1\}$ such that for any $r \geqslant 2l(a)+1$ we have $\mathrm{rk}_{N_r}((X \backslash N_{2l(a)}) \cap N_r) = r - d$.*

Proof of the claim. — For any $v \in (X \backslash N_{2l(a)}) \cap N_r$ there is at least one, and at most two vectors in $\phi_{v,N_{2l(a)}}(N_{2l(a)+1})$ (as defined in Fact 3.7). Namely, if $v = v_0 + v_1$ where $v_0 \in N_{2l(a)}$ and v_1 is orthogonal to $N_{2l(a)}$, then w must be of the form $v_0 + v_1'$ where $v_1' \in \mathrm{Lin}_{K(M)}(e_{2l(a)+1})$ and $[v_1', v_1'] = [v_1, v_1]$. Clearly there are two possibilities on such a v_1' in case $v_1 \neq 0$, and they are additive inverses of each other, and one such vector if $v_1 = 0$. Thus we have a definable surjection

$$f^r_{2l(a)+1} : (X \backslash N_{2l(a)}) \cap N_r \to (X \backslash N_{2l(a)}) \cap N_{2l(a)+1} / \sim$$

sending $v \in (X \backslash N_{2l(a)}) \cap N_r$ to the at most two-element set $\phi_{v,N_{2l(a)}}(N_{2l(a)+1})$, where $\sim$ is the relation identifying $v_0 + v_1$ with $v_0 - v_1$ for $v_0 \in N_{2l(a)}$ and $v_1 \in \mathrm{Lin}_{K(M)}(e_{2l(a)+1})$. Put

$$t \coloneqq \mathrm{rk}_{N_{2l(a)+1}}(\mathrm{im}(f^r_{2l(a)+1})) = \mathrm{rk}_{N_{2l(a)+1}}((X \backslash N_{2l(a)}) \cap N_{2l(a)+1} / \sim)$$

($\sim$ above actually does not change the rank by Fact 2.5(4), as all $\sim$-classes are finite, and hence of rank 0). Clearly $t \leqslant 2l(a)+1$. Now, for any $w \in (X \backslash N_{2l(a)}) \cap N_{2l(a)+1}$ we have that $(f^r_{2l(a)+1})^{-1}([w]_\sim) = \phi_{w,N_{2l(a)}}(N_r)$, which, by Corollary 3.9, has rank $r - 2l(a) - 1$, and degree 1 when $* = \mathrm{ACF}_p$ and $r \geqslant R+2$. Hence, by Fact 2.5(4) we get that

$$\mathrm{rk}_{N_r}((X \backslash N_{2l(a)}) \cap N_r) = r - 2l(a) - 1 + t = r - d$$

for $d \coloneqq 2l(a) + 1 - t$. As t did not depend on r, neither does d, so we are done. $\square$

As $X \cap N_r = ((X \backslash N_{2l(a)}) \cap N_r) \cup (X \cap N_{2l(a)})$ and $\mathrm{rk}(X \cap N_{2l(a)}) \leqslant 2l(a) \leqslant r - d$ for $r \geqslant 4l(a)+1$, we conclude by Fact 2.5(2) that $\mathrm{rk}_{N_r}(X \cap N_r) = r - d$ for every $r \geqslant 4l(a)+1$.

Now assume that $* = \mathrm{ACF}_p$ and $r \geqslant R+2$. We have $\mathrm{RM}((X \backslash N_{2l(a)}) \cap N_r) > \mathrm{RM}(X \cap N_{2l(a)})$, which, by Fact 2.5(7), implies that

$$\begin{aligned} \mathrm{DM}(X \cap N_r) &= \mathrm{DM}((X \backslash N_{2l(a)}) \cap N_r) = \mathrm{DM}((X \backslash N_{2l(a)}) \cap N_{2l(a)+1} / \sim) \\ &= \mathrm{DM}((X \backslash N_{2l(a)}) \cap N_{R+2}) = \mathrm{DM}(X \cap N_{R+2}) \end{aligned}$$

where the second and third equalities follow by Fact 2.5(9) applied to $f^r_{2l(a)+1}$ and to $f^{R+2}_{2l(a)+1}$, respectively. $\square$

Remark 3.11. — Proposition 12.4.1 from [11] uses the claim stated in the paragraph preceding it which says that for X and $(N_r)_{r\in\omega}$ as above (with $* = \mathrm{ACF}_p$), then one has $\mathrm{RM}_{N_r}(X \cap N_r) \leqslant \mathrm{RM}_{N_{r+1}}(X \cap N_{r+1}) + 1$ for every r. This is not true even if we assume that X is definable over N_r: for example, if $X = V \setminus \mathrm{Lin}_K(e_1, \dots, e_r)$, then $\mathrm{RM}_{N_r}(X \cap N_r) = \mathrm{RM}(\emptyset) = -1$, but $\mathrm{RM}_{N_{r+1}}(X \cap N_{r+1}) = r + 1$.

Remark 3.12. — If $[\cdot,\cdot]$ is alternating rather than symmetric, then in the setting of Proposition 3.10 we get that there is $d \leqslant 2l(a)$ such that for any $R \geqslant 2l(a)$ for which $a \subseteq N_{2R}$ we have $\mathrm{rk}_{N_{2R}}(X \cap N_{2R}) = 2R - d$ and if $* = \mathrm{ACF}_p$ then $\mathrm{DM}_{N_{2r}}(X \cap N_{2r}) = 1$ for any $r > R$. The argument is very similar to that in the symmetric case: First, by Fact 3.2 we can find a substructure $N \models T^*_{2l(a)}$ of N_{2R} containing a with $K(N) = K(M)$, so we may assume that $a \subseteq N_{2l(a)}$. Next, we choose $(e_i, f_i)_{i\in\omega}$ such that $(e_i, f_i)_{i\leqslant R}$ is a symplectic basis for N_{2R} for every R and let $\pi : N_{2R} \to N_{2l(a)}$ be the projection with respect to the basis $(e_i, f_i)_i$. Then for any $R > l(a)$ we have that $X \cap N_{2R} = (X \cap N_{2l(a)}) \cup ((\pi(X \cap N_{2R}) \oplus \mathrm{Lin}_{K(M)}((e_i, f_i)_{l(a)<i\leqslant R})) \setminus N_{2l(a)})$ has rank $2R - 2l(a) + \mathrm{rk}(\pi(X \cap N_{2R})) = 2R - 2l(a) + \mathrm{rk}(\pi(X \cap N_{2(l(a)+1)}))$, so we can put $d \coloneqq 2l(a) - \mathrm{rk}(\pi(X \cap N_{2(l(a)+1)}))$, and the second assertion follows as in the symmetric case.

Below we continue working with the symmetric case, the arguments in the alternating case being virtually the same.

Having Proposition 3.10, the rest of the arguments from [11, Subsection 12.4] go through unchanged.

FACT/DEFINITION 3.13 ([11, Proposition 12.4.1, Corollary 12.4.2, Definition 12.4.3]). — *Let $X \subseteq V$ be non-empty and definable in T_∞ over a finite tuple a. Then there exists $d \leqslant 2l(a)+1$ such that whenever $M = \bigcup_r^a N_r$, and $R \in \omega$ is such that $a \subseteq N_R$ and $R \geqslant 4l(a) + 1$, then:*

$$\mathrm{rk}_{N_n}(X \cap N_n) = d \text{ for all } n \geqslant R \text{ or } \mathrm{rk}_{N_n}(X \cap N_n) = n - d \text{ for all } n \geqslant R.$$

In the first case, we write $\mathrm{Dim}(X) = d$ and $\mathrm{Codim}(X) = \infty$, and in the second case we write $\mathrm{Dim}(X) = \infty$ and $\mathrm{Codim}(X) = d$. In the first case d can be chosen not greater than $2l(a)$.

FACT 3.14 ([11, Theorem 12.4.5]). — *Let X be a definable subset of the vector sort V. Then:*

(1) *Exactly one of $\mathrm{Dim}(X)$ and $\mathrm{Codim}(X)$ is finite.*
(2) *If $\phi(x, y)$ is a formula with x a single variable, then there are formulas without parameters $(\psi_n(y))_{n\in\omega}$ and $(\chi_n(y))_{n\in\omega}$ such that,*

for each $n \in \omega$*, one has* $\mathrm{Dim}(\phi(x,b)) = n \iff \models \psi_n(b)$ *and* $\mathrm{Codim}(\phi(x,b)) = n \iff \models \chi_n(b)$*.*

(3) $\mathrm{Dim}(X)$ *is finite iff* X *is contained in a finite-dimensional subspace of* V*, and in this case* $\mathrm{rk}(X) = \mathrm{Dim}(X)$*.*

Remark 3.15. — It is clear from the above result that there are formulas $\psi_{\mathrm{fin}}(y)$ and $\chi_{\mathrm{fin}}(y)$ such that $\mathrm{Dim}(\phi(x,b)) \in \omega \iff \models \psi_{\mathrm{fin}}(b)$ and $\mathrm{Codim}(\phi(x,b)) \in \omega \iff \models \chi_{\mathrm{fin}}(b)$.

4. Dimension on all definable sets

In this section, we define a notion of dimension of an arbitrary set definable in T^*_∞ and we study its properties. On definable subsets of V it is going to distinguish between infinite-dimensional sets of distinct codimensions, so formally it is not an extension of Dim. Thus we are going to denote it by dim rather than Dim to avoid confusion. We continue working in T^* with $* \in \{\mathrm{ACF}_p, \mathrm{RCF}\}$.

Let $I = \{f \in (\mathbb{Z},+)^\omega : f(n) = 0 \text{ for almost all } n \in \omega\} \leqslant (\mathbb{Z},+)^\omega$. Consider the quotient group:

$$S := (\mathbb{Z},+)^\omega / I.$$

For a function $f : \omega \to \mathbb{Z}$ we will write $[f]$ to mean f/I, and when f is a given by a linear function over $\mathbb{Z}$, i.e. there are $d_0, d_1 \in \mathbb{Z}$ such that $f(n) = d_0 + d_1 n$ for every $n \in \omega$, we shall identify f with the linear polynomial $d_0 + d_1 n$ in variable n. For example, $[n]$ denotes the class of the function $g : \omega \to \mathbb{Z}$ given by $g(n) = n$ for any n. Now put

$$S_{\mathrm{lin}} := \{[d_0 + d_1 n] : d_0, d_1 \in \mathbb{Z}\} \leqslant S.$$

We will write $[f] \leqslant [g]$ if $f(k) \leqslant g(k)$ for almost all $k \in \omega$. For a partial function $f : \omega \nrightarrow \mathbb{Z}$ with domain co-finite in ω, by $[f]$ we will mean $[\bar{f}]$ for any $\bar{f} : \omega \to \mathbb{Z}$ extending f.

Remark 4.1. — $(S_{\mathrm{lin}}, +, \leqslant)$ is an ordered abelian group isomorphic to $(\mathbb{Z} \times \mathbb{Z}, +, \leqslant_{lex})$.

We will write $[f] < [g]$ when $[f] \leqslant [g]$ but $[f] \neq [g]$.

Definition 4.2. — *Suppose* X *is a non-empty set definable in* T^*_∞ *over a model* $M = \bigcup_r^a N_r$*. Let* $f_{X,M,(N_r)_{r\in\omega}} : \omega \to \mathbb{Z}$ *be given by* $f_{X,M,(N_r)_{r\in\omega}} = \mathrm{rk}_{N_r}(X \cap N_r)$ *for each* r*. Put*

$$\dim(X) := [f_{X,M,(N_r)_{r\in\omega}}] \in S.$$

We also put $\dim(\emptyset) = -1$*.*

In the alternating case we define $\dim(X)$ to be the class of the function

$$f_{X,M,(N_r)_{r\in\{2,4,\dots\}}} : \{2,4,\dots\} \to \mathbb{Z}$$

with respect to being equal except finitely many points. However, we will see in Theorem 4.10 that the dimension of any definable set is given by a linear function (both in the symmetric and the alternating case), so, having Theorem 4.10, we can naturally identify $\dim(X)$ with an element of S_{lin} also in the alternating case.

Remark 4.3. — By Remark 3.5, if X is also definable over $M' = \bigcup_{r\in\omega}^{a} N'_r$, then $[f_{X,M,(N_r)_{r\in\omega}}] = [f_{X,M',(N'_r)_{r\in\omega}}]$, so the definition of $\dim(X)$ is independent of the choice of the model M and the approximating sequence $(N_r)_{r\in\omega}$.

We now aim to prove that the dimension of any set definable in T^*_∞ belongs to S_{lin} (so in particular the dimensions of the definable sets are linearly ordered). This will be proved first for definable subsets of V^k by induction on k simultaneously with some other statements. In particular, we define below a family of finite sets $D_{k,l} \subseteq S_{lin}$ which will turn out to contain the dimension of any subset of V^k definable over a set containing at most l vectors.

Definition 4.4. — *For any $k, l \in \omega$ put*

$$D_{k,l} := \left\{ \begin{matrix} [d_0 + d_1 n] : 0 \leqslant d_1 \leqslant k \\ \textit{and } -d_1(2l+1) - k(k-1) \leqslant d_0 \leqslant (k-d_1)2l + k(k-1) \end{matrix} \right\} \subseteq S_{\mathrm{lin}}.$$

The following property of the sets $D_{k,l}$ will be used in the inductive proof of Theorem 4.10.

Remark 4.5. — $D_{k,l} + D_{1,k+l} \subseteq D_{k+1,l}$ for any $l, k \in \omega$.

Proof. — Suppose $[d_0 + d_1 n] \in D_{k,l}$ and $[d'_0 + d'_1 n] \in D_{1,k+l}$. Then clearly $d_1 + d'_1 \leqslant k+1$ and

$$-d_1(2l+1) - k(k-1) \leqslant d_0 \leqslant (k-d_1)2l + k(k-1)$$

as well as

$$-d'_1(2k+2l+1) \leqslant d'_0 \leqslant (1-d'_1)(2k+2l)$$

so

$$\begin{aligned} &- d_1(2l+1) - k(k-1) - d'_1(2k+2l+1) \\ &\qquad \leqslant d_0 + d'_0 \leqslant (k-d_1)2l + k(k-1) + (1-d'_1)(2k+2l), \end{aligned}$$

which gives what we need, as

$$\begin{aligned}
&-d_1(2l+1)-k(k-1)-d_1'(2k+2l+1)\\
&\qquad\qquad = -(d_1+d_1')(2l+1)-2d_1'k-k(k-1)\\
&\qquad\qquad \geqslant -(d_1+d_1')(2l+1)-2k-k(k-1)\\
&\qquad\qquad = -(d_1+d_1')(2l+1)-k(k+1)
\end{aligned}$$

and, similarly, on the right-hand side:

$$\begin{aligned}
&(k-d_1)2l+k(k-1)+(1-d_1')(2k+2l)\\
&\qquad\qquad = (k+1-d_1-d_1')2l+k(k-1)+(1-d_1')2k\\
&\qquad\qquad \leqslant (k+1-d_1-d_1')2l+k(k+1).
\end{aligned}$$

Hence $[d_0+d_1n]+[d_0'+d_1'n]=[d_0+d_0'+(d_1+d_1')n]\in D_{k+1,l}$. ☐

Definition 4.6. — *Let $\alpha:\omega^2\to\omega$ be any function such that:*

- $\alpha(k,l)\geqslant 2kl+2l+2k^2+1$ *for any* $k,l\in\omega$.
- $\alpha(k+m,l)\geqslant\alpha(k,l+m)$ *and* $\alpha(k+m,l)\geqslant\alpha(k,l)$ *for any* $k,l,m\in\omega$.

Clearly, such a function can be constructed recursively on k. We will say that a definable set $X\subseteq V^k$ is nice, *if $X=\emptyset$ or for each a over which X is definable one has $\dim(X)=[d_0+d_1n]\in D_{k,l(a)}$, and whenever $M=\bigcup^a_{r\in\omega}N_r$, $R\geqslant\alpha(k,l(a))$, and $a\subseteq N_R$, then we have*

$$\mathrm{rk}_{N_R}(X\cap N_R)=d_0+d_1R.$$

In the above situation, we know by the definition of dim that if $\dim(X)=[d_0+d_1n]$ then the equality $\mathrm{rk}_{N_R}(X\cap N_R)=d_0+d_1R$ holds for sufficiently large R, but the niceness property, saying that it holds for any R with $R\geqslant\alpha(k,l(a))$ and $a\subseteq N_R$, allows us to choose R uniformly when we work with a uniformly definable family, which will be crucial in the proof of Lemma 4.9(3) below.

Note that by Fact 3.13 we have that any definable $X=\phi(\mathfrak{C},a)\subseteq V$ is nice: If $\mathrm{Dim}(X)=d_0\in\omega$, then $0\leqslant d_0\leqslant 2l(a)$ and $\dim(X)=[d_0]$, so the inequalities $-d_1(2l(a)+1)\leqslant d_0\leqslant(k-d_1)2l(a)$ are satisfied as $d_1=0$ and $k=1$. If $\mathrm{Codim}(X)\in\omega$, then $\dim(X)=[d_0+n]$ for $d_0=-\mathrm{Codim}(X)$, so $d_1=1$ and $-2l(a)-1\leqslant d_0\leqslant 0$, so again the required inequalities hold. In both cases the equality $\mathrm{rk}_{N_R}(X\cap N_R)=d_0+d_1R$ holds for any $R\geqslant 4l(a)+1$ with $a\subseteq N_R$, hence for any $R\geqslant\alpha(1,l(a))$, as $\alpha(1,l(a))\geqslant 4l(a)+1$.

We will eventually see in Theorem 4.10 that all sets definable in T^*_∞ are nice.

Lemma 4.7. — *If* $[d_0+d_1n], [d'_0+d'_1n] \in D_{k,l}$ *and* $[d_0+d_1n] > [d'_0+d'_1n]$, *then* $d_0 + d_1r > d'_0 + d'_1r$ *for any* $r \geqslant \alpha(k,l)$.

Proof. — If $d_1 = d'_1$ then $d_0 > d'_0$, and the inequality is obvious, so assume $d_1 > d'_1$. Then, by the inequalities in the definition of niceness we get:

$$\begin{aligned}
&d_0 + d_1r - (d'_0 + d'_1r)\\
&\quad = d_0 - d'_0 + (d_1 - d'_1)r\\
&\quad \geqslant -d_1(2l+1) - k(k-1) - ((k-d'_1)2l + k(k-1)) + (d_1 - d'_1)\alpha(k,l)\\
&\quad = (d'_1 - d_1)2l - 2kl - d_1 - 2k(k-1) + (d_1 - d'_1)\alpha(k,l)\\
&\quad = (d_1 - d'_1)(\alpha(k,l) - 2l) - 2kl - 2k(k-1) - d_1\\
&\quad \geqslant \alpha(k,l) - 2l - 2kl - 2k^2 > 0,
\end{aligned}$$

so $d_0 + d_1r > d'_0 + d'_1r$. □

Lemma 4.8. — *If* $M = \bigcup_r^a N_r$ *and* $\emptyset \neq X = \phi(M,a)$ *for some formula* $\phi(x;y)$, *then* $X \cap N_r \neq \emptyset$ *for any* $r \geqslant 2l(xy)$ *such that* $a \subseteq N_r$.

Proof. — This is similar to the proof of Fact 2.10(1). Let $c \in X$ and put $l \coloneqq l(a)$. We can find $e_1, \ldots, e_r, e'_{2l+1}, \ldots, e'_r$ such that $(e_1, \ldots, e_r)$ and $(e_1, \ldots, e_{2l}, e'_{2l+1}, \ldots, e'_r)$ are orthonormal sequences,

$$V(a) \subseteq \mathrm{Lin}_{K(M)}(e_1, \ldots, e_{2l}),$$
$$V(c) \subseteq \mathrm{Lin}_{K(M)}(e_1, \ldots, e_{2l}, e'_{2l+1}, \ldots, e'_r)$$

and

$$V(N_r) = \mathrm{Lin}_{K(M)}(e_1, \ldots, e_r).$$

Then letting $f = \mathrm{id}_{K(M)} \cup F$ where F is a $K(M)$-linear function sending $(e_1, \ldots, e_{2l}, e'_{2l+1}, \ldots, e'_r)$ to $(e_1, \ldots, e_r)$, we see by quantifier elimination that $\mathrm{tp}(f(c)/a) = \mathrm{tp}(c/a)$. In particular, $f(c) \in X \cap N_r$. □

Lemma 4.9.

(1) *If* $X \subseteq Y$ *then* $\dim(X) \leqslant \dim(Y)$
(2) *If* $X_1, X_2 \subseteq V^k$ *are nice then*

$$\dim(X_1 \cup X_2) = \max(\dim(X_1), \dim(X_2)).$$

If additionally X_1 *and* X_2 *are definable over every tuple of parameters over which* X *is definable, then* $X_1 \cup X_2$ *is also nice.*
(3) *Let* $\pi : V^{k+m} \to V^k$ *be the projection on the last* k *coordinates (where* $m \geqslant 1$*). Suppose* $X \subseteq V^{k+m}$ *is definable and non-empty, all sections* $X_y = \{x \in V^m : (x,y) \in X\}$ *with* $y \in \pi[X]$ *are*

nice and they all have same dimension s, and $\pi[X]$ is nice. Then $\dim(X) = s + \dim(\pi[X])$.

If additionally $m = 1$ then X is nice.

Proof.

(1). — Suppose $X = \phi(\mathfrak{C}, a)$, $Y = \psi(\mathfrak{C}, b)$, $M = \bigcup_{r\in\omega}^{a} N_r$ and $a, b \subseteq N_R$ for some $R \in \omega$. Then for any $r \geqslant R$ we have $X \cap N_r \subseteq Y \cap N_r$, so $\mathrm{rk}_{N_r}(X \cap N_r) \leqslant \mathrm{rk}_{N_r}(Y \cap N_r)$ by Fact 2.5(1). Hence $\dim(X) \leqslant \dim(Y)$.

(2). — Suppose $X_1 = \phi(\mathfrak{C}, a)$, $X_2 = \psi(\mathfrak{C}, b)$, $\dim(X_1)] = [d_0 + d_1 n]$, $\dim X_2 = [d_0' + d_1' n]$, $M = \bigcup_{r\in\omega}^{a} N_r$, and $R \geqslant \alpha(k, \max(l(a), l(b)))$ is such that $a, b \subseteq N_R$. We may assume $\dim(X_1) \geqslant \dim(X_2)$. For any $r \geqslant R$ we have by Fact 2.5(2) that

$$\begin{aligned}\mathrm{rk}_{N_r}((X_1 \cup X_2) \cap N_r) &= \mathrm{rk}_{N_r}((N_r \cap X_1) \cup (N_r \cap X_2)) \\ &= \max(\mathrm{rk}_{N_r}(X_1 \cap N_r), \mathrm{rk}_{N_r}(X_2 \cap N_r)),\end{aligned}$$

which equals $d_0 + d_1 r$ for almost all $r \in \omega$, and hence $\dim(X_1 \cup X_2) = [d_0 + d_1 n] = \max(\dim(X_1), \dim(X_2))$.

Suppose additionally that X_1 and X_2 are definable over any tuple of parameters over which $X_1 \cup X_2$ is definable, and consider any c such that $X_1 \cup X_2$ (so also X_1 and X_2) is definable over c. Then the above remains true for any $r \geqslant \alpha(k, l(c))$ with $c \subseteq N_r$. For any such r, we know by niceness of X_1 and X_2 that $\mathrm{rk}_{N_r}(X_1 \cap N_r) = d_0 + d_1 r$ and $\mathrm{rk}_{N_r}(X_2 \cap N_r) = d_0' + d_1' r$. By Lemma 4.7 we have $d_0 + d_1 r \geqslant d_0' + d_1' r$, so $\mathrm{rk}_{N_R}((X_1 \cup X_2) \cap N_r) = \max(d_0 + d_1 r, d_0' + d_1' r) = d_0 + d_1 r$, and hence $X_1 \cup X_2$ is nice. Assume $X = \phi(\mathfrak{C}, a)$ and put $l = l(a)$. Let $d_0, d_1 \in \omega$ be such that $s = [d_0 + d_1 n]$; as the sections of X are nice, we have that $[d_0 + d_1 n] \in D_{m,k+l}$. Consider any $M = \bigcup_{r\in\omega}^{a} N_r$, and $r \geqslant \alpha(k + m, l)$ with $a \subseteq N_r$. Put $Y = \pi[X]$.

For any $y \in Y \cap N_r$ we have $(X \cap N_r)_y = X_y \cap N_r$, so, as $r \geqslant \alpha(k+m, l) \geqslant \alpha(m, l + k) = \alpha(m, l(ay))$ and $X_y \subseteq V^m$ is a nice set definable over ay, we get

$$\mathrm{rk}_{N_r}((X \cap N_r)_y) = d_0 + d_1 r.$$

Note also that if $y \in Y = \pi[X]$ then X_y is a non-empty set definable over ay, so as, $r \geqslant \alpha(k + m, l) > 2(k + m + l)$, it must meet N_r by Lemma 4.8. Thus, $Y \cap N_r = \pi[X \cap N_r]$. Hence, by Fact 2.5(4), we have $\mathrm{rk}_{N_r}(X \cap N_r) = \mathrm{rk}_{N_r}(Y \cap N_r) + d_0 + d_1 r$. As Y is nice and $r \geqslant \alpha(k + m, l) \geqslant \alpha(k, l)$, we get that $\mathrm{rk}_{N_r}(Y \cap N_r) = d_0' + d_1' r$ for d_0', d_1' such that $\dim(Y) = [d_0' + d_1' n] \in D_{k,l}$. So $\mathrm{rk}_{N_r}(X \cap N_r) = d_0 + d_0' + (d_1 + d_1')r$. Hence $\dim(X) = [d_0 + d_0' + (d_1 + d_1')n] = s + \dim(Y)$.

If, additionally, $m = 1$, then $\dim(X) \in D_{k,l} + D_{1,k+l} \subseteq D_{k+1,l}$ by Remark 4.5, so X is nice. □

Theorem 4.10. — *We work in T^*_∞.*

(1) *For any $k \in \omega$, every non-empty definable subset of V^k is nice. In particular, $\dim(X) \in D_{k,l(a)}$ for any $X \subseteq V^k$ definable over a finite tuple a.*
(2) *Suppose $k \in \omega$, $x = (x_1, \dots, x_k)$ where each x_i is a variable of the sort V, and y is an arbitrary tuple of variables. Then for any formula $\phi(x; y)$ over $\emptyset$ and any $s \in D_{k,l(y)}$ the set*

$$D_{\phi(x;y),s} := \{a \in \mathfrak{C} : \dim(\phi(x; a)) = s\}$$

is $\emptyset$-definable.

Proof. — We will prove (1) and (2) simultaneously by induction on k.

When $k = 1$, we know that (1) and (2) both hold by Section 3.

Suppose now $k \geqslant 1$ and (1) and (2) are true for $1, 2, \dots, k$. Consider any formula $\phi(x; y)$ over $\emptyset$ with $x = (x_1, \dots, x_{k+1})$, where each x_i is a variable of the sort V.

Consider any $a \in \mathfrak{C}$ compatible with y, and write $X_a = \phi(\mathfrak{C}; a) \subseteq V^{k+1}$. For $b \in V^k$ put

$$X_{b,a} = \phi(\mathfrak{C}; b, a) \subseteq V.$$

For any $s \in D_{1,k+l(y)}$ let $\chi_s(x_2, \dots, x_{k+1}; y)$ be a formula over $\emptyset$ such that

$$\models \chi_s(v_2, \dots, v_{k+1}; w) \iff \dim(\phi(\mathfrak{C}; v_2, \dots, v_{k+1}, w)) = s \text{ for all } v_2, \dots, v_{k+1}, w \in \mathfrak{C},$$

(such a formula exists, as (2) holds for $k = 1$). Put $X_{s,a} = \{b \in V : \dim(X_{b,a}) = s\} = \chi(\mathfrak{C}; a)$.

Then letting $\pi : V^{k+1} \to V^k$ be the projection on the last k coordinates, we have for each $s \in D_{1,k+l(y)}$ and each $b \in X_{s,a}$ that $\dim((\pi|_{X_a})^{-1}(b)) = \dim(X_{b,a}) = s$ and $X_{b,a}$ is nice by the inductive hypothesis, as is $X_{s,a}$. Thus, by Lemma 4.9(3), we get that $(\pi|_{X_a})^{-1}[X_{s,a}]$ is nice. Now

$$X_a = \bigcup_{s \in D_{1,k+l(y)}} (\pi|_{X_a})^{-1}[X_{s,a}]$$

and $(\pi|_{X_a})^{-1}[X_{s,a}]$ is nice for each $s \in D_{1,k+l(y)}$, so by Lemma 4.9(2) we conclude that X_a is nice, which proves part 1 of the theorem for $k + 1$.

Lemma 4.9 gives us also that

$$\dim(X_a) = \max_{s \in D_{1,k+l(y)}} \dim((\pi|_{X_a})^{-1}[X_{s,a}])$$

and

$$\dim((\pi|_{X_a})^{-1}[X_{s,a}]) = s + \dim(X_{s,a}).$$

Hence, putting

$$I = D_{1,k+l(y)} \times D_{k,l(y)},$$

we get that for any $a \in \mathfrak{C}$ compatible with y we have

$$\dim(\phi(x_1, \dots, x_{k+1}; a)) \in \{s + t : (s,t) \in I\}.$$

So fix any $(s_0, t_0) \in I$, and put $I_{=s_0+t_0} = \{(s,t) \in I : s + t = s_0 + t_0\}$ and $I_{>s_0+t_0} = \{(s,t) \in I : s + t > s_0 + t_0\}$. Then

$$\begin{aligned} &\dim(\phi(x_1, \dots, x_{k+1}; a)) = s_0 + t_0 \\ &\iff \left(\bigvee_{(s,t) \in I_{=s_0+t_0}} \dim(\chi_s(x_2, \dots, x_{k+1}; a)) = t \right) \\ &\qquad \wedge \left(\bigwedge_{(s,t) \in I_{>s_0+t_0}} \neg \dim(\chi_s(x_2, \dots, x_{k+1}; a)) = t \right). \end{aligned}$$

By the inductive hypothesis, for any $(s,t) \in I$ the condition

$$\dim(\chi_s(x_2, \dots, x_{k+1}; a)) = t$$

is definable (in the variable a), so, by the above equivalence, the condition $\dim(\phi(x_1, \dots, x_{k+1}; a)) = s_0 + t_0$ is definable as well. This proves that (2) holds for $k + 1$. □

Remark 4.11. — If there is a definable bijection f between $X \subseteq V^k$ and $Y \subseteq V^{k'}$, then $\dim(X) = \dim(Y)$.

Proof. — If X, Y and f are all definable over N_R, where $M = \bigcup_r^a N_r$ and $R \in \omega$, then for any $r \geqslant R$ we have that $f[X \cap N_r] \subseteq Y \cap N_r$ as $\operatorname{dcl}(N_r) = N_r$ by Fact 2.10, so $X \cap N_r \subseteq f^{-1}[Y \cap N_r]$, and similarly $f^{-1}[Y \cap N_r] \subseteq X \cap N_r$. Hence

$$\operatorname{rk}_{N_r}(X \cap N_r) = \operatorname{rk}_{N_r}(f^{-1}[Y \cap N_r]) = \operatorname{rk}_{N_r}(Y \cap N_r)$$

by Fact 2.5(3), so $\dim(X) = \dim(Y)$. □

We now extend the definition of dimension to all sets definable in T_∞.

Definition 4.12. — *If X is any set definable in T^*_∞, so $X \subseteq V^k \times K^m$ for some $k, m \in \omega$, then we let $\dim(X) = \dim(X')$, where X' is any definable subset of $V^{k'}$ for some $k' \in \omega$ for which there is a definable (with parameters) bijection between X and X'. Such an X' always exists, as we have a definable injection $f_{k,(e_1,\dots,e_m)} : V^k \times K^m \to V^{k+1}$ given by*

$f_{k,(e_1,\dots,e_m)}(v, a_1, \dots, a_m) = (v, a_1e_1 + \cdots + a_me_m)$, *where* $(e_1, \dots, e_m)$ *is any fixed linearly independent tuple of vectors from* V.

Moreover, $\dim(X)$ *is well defined by Remark 4.11 above.*

Now we summarise the properties of dim following from what we have proved so far.

Corollary 4.13. — *We work in* T^*_∞

(1) $\dim$ *is* $\emptyset$ *definable.*
(2) *If* $X \subseteq Y$ *are definable then* $\dim(X) \leqslant \dim(Y)$.
(3) $\dim(X_1 \cup X_2) = \max(\dim(X_1), \dim(X_2))$ *for any definable* X_1 *and* X_2.
(4) *If* $f : X \to Y$ *is a definable surjection such that* $\dim(f^{-1}(y)) = s$ *for each* $y \in Y$, *then* $\dim(X) = \dim(Y) + s$ *unless* Y *is empty.*

Proof.

(1). — Consider any formula $\phi(x, y; z)$ where x is a variable of the sort V^k and y is a variable of the sort K^m, and any $s \in S_{\mathrm{lin}}$. Let x' be a variable of the sort V^{k+1} and put

$$\psi(x'; z, e_1, \dots, e_m) = (x' \in \mathrm{im}(f_{k,(e_1,\dots,e_m)}) \wedge \phi(f^{-1}_{k,(e_1,\dots,e_m)}(x'); z)),$$

where $e_1, \dots, e_m$ are some linearly independent vectors from V. By Theorem 4.10(2) there is a formula $\chi_s(z, e_1, \dots, e_m)$ such that, for any z,

$$\begin{aligned} \models \chi_s(z, e_1, \dots, e_m) &\iff \dim(\psi(x'; z, e_1, \dots, e_m)) = s \\ &\iff \dim(\phi(x, y; z)) = s. \end{aligned}$$

As this holds for any linearly independent vectors $e_1, \dots, e_m$, we may replace the formula $\chi_s(z, e_1, \dots, e_m)$ by the $L(\emptyset)$-formula

$$\exists_{v_1,\dots,v_m}(\theta_m(v_1, \dots, v_m) \wedge \chi_s(z, v_1, \dots, v_m))$$

(2). — Suppose $X \subseteq Y \subseteq V^k \times K^m$. Then

$$f_{k,(e_1,\dots,e_m)}[X] \subseteq f_{k,(e_1,\dots,e_m)}[Y],$$

so

$$\dim(X) = \dim(f_{k,(e_1,\dots,e_m)}[X]) \leqslant \dim(f_{k,(e_1,\dots,e_m)}[Y]) = \dim(Y)$$

by Lemma 4.9(1).

(3). — This follows by Lemma 4.9(2) using the injection $f_{k,(e_1,\dots,e_m)}$ again.

(4). — As any subset of $V^k \times K^m$ is in a definable bijection with a subset of $V^{k'} \times K^{m'}$ for any $k' \geqslant k, m' \geqslant m$, we may assume (by modifying X, Y and f) that there are k and m such that $X, Y \subseteq V^{k-1} \times K^m$. Then applying $f_{k-1,(e_1,\ldots,e_m)}$ we may assume $X, Y \subseteq V^k$. Put

$$Z \coloneqq \{(x,y) \in X \times Y : y = f(x)\},$$

and let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the projections. Note that $\dim(X) = \dim(Z)$ as $\pi_1|_Z : Z \to X$ is a definable bijection. Moreover, for any $y \in Y$ we have $X_y = f^{-1}(y)$ has dimension s. Thus, by Lemma 4.9(3) we have $\dim(X) = \dim(Z) = \dim(\pi_2[Z]) + s = \dim(Y) + s$. □

Note the above properties correspond to the main properties of Morley rank in strongly minimal theories (and of the topological dimension in RCF) listed in Fact 2.5. However, a major difference is that the set of values of dim is not well ordered. Nevertheless, if we work with a fixed finite tuple of variables and a fixed finite tuple of parameters, the set of possible values of dim is finite.

Remark 4.14. — If $X \subseteq V^k$ is definable in T^*_∞ then $\dim(X) \in \omega$ if and only if $X \subseteq (V_0)^k$ for some finite dimensional $K(\mathfrak{C})$-linear subspace V_0 of V. Moreover, if these equivalent conditions hold and $* =\mathrm{ACF}_p$ then $\dim(X) = [\mathrm{RM}(X)]$.

Proof. — The implication from right to left follows from Remark 3.6.

Assume $\dim(X) = [d] \in \omega$. Then, for each $i \in \{1, \ldots, k\}$, the projection $\pi_i(X)$ of X on the i-th coordinate must have finite dimension (bounded by d), so, by Fact 3.14(3), there is some finite dimensional $V_i \leqslant V$ such that $\pi(X) \subseteq V_i$. This means that $X \subseteq (\Sigma_i V_i)^k$ so we can put $V_0 \coloneqq \Sigma_i V_i$.

The "moreover" clause now follows by Remark 3.6 again. □

Finally, we define the dimension of a type. As the set of values of dim in T^*_∞ is not well ordered, in general we need to use its Dedekind completion $\overline{S_{\mathrm{lin}}}$.

Definition 4.15. — *Let $\pi(x)$ be a partial finitary type (i.e. x is a finite tuple of variables) in T^*_∞ over a set A. We put*

$$\dim(\pi(x)) \coloneqq \inf_{\pi(x)\vdash\phi(x)\in L(\mathfrak{C})} \dim(\phi(x)) = \inf_{\pi(x)\vdash\phi(x)\in L(A)} \dim(\phi(x)) \in \overline{S_{\mathrm{lin}}}$$

(the two infima are equal by Corollary 4.13(2), as any formula implied by $\pi(x)$ is implied by a conjuction of finitely many formulas from $\pi(x)$). Note that $\dim(\pi(x)) \in S_{\mathrm{lin}}$ if A contains only finitely many vectors, as in this case the dimension of any formula in x over A belongs to the finite set $D_{l(x),l(A)}$.

Proposition 4.16. — *Let $\pi(x)$ be a partial finitary type in T^*_∞ over A. Then there exists $p(x) \in S(A)$ with $\dim(p(x)) = \dim(\pi(x))$.*

Proof. — Put

$$p_0(x) := \pi(x) \cup \{\neg\phi(x) \in L(A) : \dim(\phi(x)) < \dim(\pi(x))\}.$$

We claim that $p_0(x)$ is consistent. For if not, then there is a finite $\pi_0(x) \subseteq \pi(x)$ and formulas $\phi_1(x), \dots, \phi_n(x)$ such that $\dim(\phi_i(x)) < \dim(\pi(x))$ for every i and $\bigwedge \pi_0(x) \wedge (\bigwedge_{1\leqslant i\leqslant n} \neg\phi_i(x))$ is inconsistent. Then $\models \bigwedge \pi_0(x) \rightarrow \bigvee_{1\leqslant i\leqslant n} \phi_i(x)$. But, by Lemma 4.9(b),

$$\begin{aligned}\dim(\pi_0) &= \dim\left(\left(\bigwedge \pi_0\right) \wedge \bigvee_{1\leqslant i\leqslant n} \phi_i(x)\right)\\ &= \max_{1\leqslant i\leqslant n} \dim\left(\pi_0 \wedge \phi_i(x)\right) < \dim\left(\pi(x)\right),\end{aligned}$$

which is a contradiction.

Hence p_0 is consistent, and we can take p to be any completion of p_0. □

Notation 4.17. — For $s, s' \in S_{lin}$ we will write:

- $s \sim s'$ if $s - s' \in \{[d] : d \in \mathbb{Z}\}$,
- $s \lesssim s'$ if $s \sim s'$ or $s \leqslant s'$,
- $s \ll s'$ if $s \leqslant s'$ and $\neg(s \sim s')$.

Definition 4.18.

(1) *We write* $\dim(a/b)$ *to mean* $\dim(tp(a/b))$*. By the discussion in Definition 4.15, if a and b are finite, then* $\dim(a/b) \in S_{lin}$*.*

(2) *If X is a set (type)-definable over a and $b \supseteq a$, then we say that an element $c \in X$ is* generic *[quasi-generic] in X over b if* $\dim(c/b) = \dim(X)$ *[*$\dim(c/b) \sim \dim(X)$*].*

By Proposition 4.16, for any X definable over a and any $b \supseteq a$ there exists a generic in X over b (in some model of T^*_∞ containing b). If b is finite, then such a generic can be found in $\mathfrak{C}$, as we are assuming that $\mathfrak{C}$ is $\aleph_0$-saturated.

5. Lascar's equality and the connection between $\dim$ and $\dim_{\mathrm{Lin}}$

The following additivity property is an analogue of Lascar's equality, which holds, for example, for Morley rank in strongly minimal theories (and more generally, for Lascar U-rank assuming the the ranks in the statement are finite).

Proposition 5.1 (Lascar's equality for dim). — *If* $a, b, c \in \mathfrak{C}$ *are finite tuples, then* $\dim(ab/c) = \dim(a/bc) + \dim(b/c)$.

Proof. — First, we will show that $\dim(ab/c) \geqslant \dim(a/bc) + \dim(b/c)$. Consider any formula $\phi(x, y) \in tp(ab/c)$. Then $\phi(x; b, c) \in tp(a/bc)$, so $s := \dim(\phi(x; b, c)) \geqslant \dim(a/bc)$. Now, by Corollary 4.13(1) there is a formula $\chi(y; c)$ over c such that

$$\models \chi(d; c) \iff \dim(\phi(x; d; c)) = s$$

for any d compatible with y. Then $\chi(y; c) \in tp(b/c)$, so $t := \dim(\chi(y; c)) \geqslant \dim(tp(b/c))$. Now, by Corollary 4.13(4) applied to $\phi(x, y; c) \wedge \chi(y; c)$ and the projection on the y-coordinate, we get that

$$\dim(\phi(x, y; c)) \geqslant \dim(\phi(x, y; c) \wedge \chi(y; c)) = s + t \geqslant \dim(a/bc) + \dim(b/c).$$

This shows that $\dim(ab/c) \geqslant \dim(a/bc) + \dim(b/c)$.

Now, choose a formula $\psi(x; b, c) \in tp(a/bc)$ such that

$$s' := \dim(\psi(x; b, c)) = \dim(a/bc).$$

Again by Corollary 4.13(1), there is a formula $\chi(y; c)$ over c such that

$$\models \chi(d; c) \iff \dim(\psi(x; d, c)) = s'$$

for any d compatible with y. Clearly $\chi(y; c) \in tp(b/c)$ so if we choose $\xi(y; c) \in tp(b/c)$ such that $t' := \dim(\xi(y; c)) = \dim(b/c)$, then we also have $\chi(y; c) \wedge \xi(y; c) \in tp(b/c)$, hence $\dim(\chi(y; c) \wedge \xi(x; c)) = t'$. Now applying Corollary 4.13(4) to the formula

$$\delta(x, y; c) := \psi(x, y; c) \wedge \chi(y; c) \wedge \xi(y; c)$$

and the projection on the y-coordinate, we get $\dim(\delta(x, y; c)) = s' + t'$. As $\delta(x, y; c) \in tp(ab/c)$, we conclude that $\dim(ab/c) \leqslant s' + t' = \dim(a/bc) + \dim(b/c)$. □

Proposition 5.2. — *If* a, b *are finite tuples and* $\dim(a/b) = [d_0 + d_1 n]$, *then* d_1 *is equal to the linear dimension* $\dim_{\mathrm{Lin}}(a/b)$ *of* $V(a)$ *over* $V(b)$, *that is, the size of a maximal subset of* $V(a)$ *which is* K*-linearly independent over* $\mathrm{Lin}_K(V(b))$.

Proof. — Put $l := \dim_{\mathrm{Lin}}(a/b)$ and let $(a_1, \ldots, a_k)$ be all vectors in a, and let $(c_1, \ldots, c_m)$ be all scalars in a. We may assume $(a_1, \ldots, a_l)$ is a maximal subtuple of a which is K-linearly independent over $\mathrm{Lin}_K(V(b))$. Write $V(b) = \{b_1, \ldots, b_p\}$. Let $\phi(x_1, \ldots, x_k)$ be a formula over b expressing that $x_{l+1}, \ldots, x_k \in \mathrm{Lin}_K(V(b), x_1, \ldots, x_l)$, and let $f : \phi(\mathfrak{C}) \times K^m \to V^l \times K^m \times K^{(k-l)(l+p)}$ be a map sending a tuple $(x_1, \ldots, x_k, y_1, \ldots, y_m)$ to $(x_1, \ldots, x_l, y_1, \ldots, y_m, A)$ where A is an $(l + p) \times (k - l)$-matrix such that

$A(x_1,\dots,x_l,b_1,\dots,b_p)^T=(x_{l+1},\dots,x_k)$. Then f is a b-definable injection of $\phi(\mathfrak{C})\times K^m$ into $V^l\times K^{m+(k-l)(l+p)}$. As $(a_1,\dots,a_k)\models\phi(x_1,\dots,x_k)$, we get

$$\begin{aligned}\dim(a/b)&\leqslant\dim(\phi(\mathfrak{C})\times K^m)\\&\leqslant\dim(V^l\times K^{m+(k-l)(l+p)})=[m+(k-l)(l+b)+ln].\end{aligned}$$

This shows that $d_1\leqslant l=\dim_{\mathrm{Lin}}(a/b)$.

It is left to prove that $d_1\geqslant\dim_{\mathrm{Lin}}(a/b)$, which we do by induction on $\dim_{\mathrm{Lin}}(a/b)$. If $\dim_{\mathrm{Lin}}(a/b)=1$ then $\dim(a/b)\geqslant\dim(a_1/b)$ and $a_1\notin\mathrm{Lin}_K(V(b))$ so $\dim(a_1/b)$ is infinite by Fact 3.14(3), i.e. $d_1\geqslant 1$. For the inductive step, assume $\dim_{\mathrm{Lin}}(a/b)=l+1$ and (without loss of generality) that $(a_1,\dots,a_{l+1})$ is K-linearly independent over $\mathrm{Lin}_K(V(b))$. By the inductive hypothesis, $\dim(a_1,\dots,a_l/b)\gtrsim[ln]$ and $\dim(a_{l+1}/ba_1,\dots,a_l)\gtrsim[n]$, so by Lascar's equality we get $\dim(a/b)\geqslant\dim(a_1,\dots,a_{l+1}/b)=\dim(a_1,\dots,a_l/b)+\dim(a_{l+1}/ba_1,\dots,a_l)\gtrsim[(l+1)n]$, as required. □

Corollary 5.3. — *For any finite tuples a,b,c we have $\dim(a/b)\sim\dim(a/bc)$ if and only if $\mathrm{Lin}_K(V(ab))\cap\mathrm{Lin}_K(V(bc))=\mathrm{Lin}_K(V(b))$.*

Proof. — If $\mathrm{Lin}_K(V(ab))\cap\mathrm{Lin}_K(V(bc))=\mathrm{Lin}_K(V(b))$ then any tuple $(a_1,\dots,a_d)$ of elements of $V(a)$ which is K-linearly independent over $\mathrm{Lin}_K(V(b))$ is also linearly independent over $\mathrm{Lin}_K(V(bc))$, so $\dim(a/b)\sim\dim(a/bc)$ by Proposition 5.2.

Conversely, assume $\dim(a/b)\sim\dim(a/bc)$ and let $(a_1,\dots,a_d)$ be a maximal tuple of elements of $V(a)$ which is K-linearly independent over $V(bc)$. By Proposition 5.2 and the assumption, $(a_1,\dots,a_d)$ is also maximal K-linearly independent over $V(b)$. Hence, any element of $\mathrm{Lin}_K(V(ab))\cap\mathrm{Lin}_K(V(bc))$ is of the form

$$\sum_{i\leqslant d}\alpha_ia_i+b_1=c_1$$

for some $\alpha_i\in K$, $b_1\in\mathrm{Lin}_K(V(b))$, and $c_1\in\mathrm{Lin}_K(V(bc))$, so $\sum_{i\leqslant d}\alpha_ia_i=c_1-b_1\in\mathrm{Lin}_K(V(bc))$. As $(a_1,\dots,a_d)$ is linearly independent over $\mathrm{Lin}_KV(bc)$, we get that $\sum_{i\leqslant d}\alpha_ia_i=0$ and $\Sigma_{i\leqslant d}\alpha_ia_i+b_1=b_1\in\mathrm{Lin}_K(V(b))$. Thus,

$$\mathrm{Lin}_K(V(ab))\cap\mathrm{Lin}_K(V(bc))=\mathrm{Lin}_K(V(b)).$$ □

6. Finiteness of multiplicity and its consequences

In this section we will define multiplicity of a set definable in T_∞ in analogy with Morley degree and we will prove that the multiplicity of any set definable in T_∞ is finite. We will deduce that any group interpretable in T_∞ which has finite Morley rank is definable in T_∞, and hence is an algebraic group over K (we will also prove an analogous result for T_∞^{RCF}), as well as some other consequences of finiteness of multiplicity, including definability of generic types in T_∞.

Definition 6.1. — *Let X be definable in T_∞. We let the multiplicity of X, written $\mathrm{Mlt}(X)$, be the maximal number $m \in \omega$ such that there are pairwise disjoint definable sets $X_1, \ldots, X_m$ with $X_i \subseteq X$ and $\dim(X_i) = \dim(X)$ for each $i \in \{1, \ldots, m\}$ if such a number m exists, and ∞ otherwise.*

Proposition 6.2. — *We work in T_∞.*

(1) *If $X \subseteq Y$ and $\dim(X) = \dim(Y)$ then $\mathrm{Mlt}(X) \leqslant \mathrm{Mlt}(Y)$.*
(2) *If $\dim(X_1) < \dim(X_2)$, then $\mathrm{Mlt}(X_1 \cup X_2) = \mathrm{Mlt}(X_2)$.*
(3) *If $\dim(X_1) = \dim(X_2) = s$ then $Mtl(X_1 \cup X_2) \leqslant \mathrm{Mlt}(X_1) + \mathrm{Mlt}(X_2)$, and equality holds when $\dim(X_1 \cap X_2) < s$.*
(4) *If $f : X \to Y$ is a definable function such that $\mathrm{Mlt}(Y) = m \in \omega$ and there are $s \in S_{\mathrm{lin}}$ and $m' \in \omega$ such that $\dim(f^{-1}(y)) = s$ and $\mathrm{Mlt}(f^{-1}(y)) \leqslant m'$ for every $y \in Y$, then $\mathrm{Mlt}(X) \leqslant mm'$.*
(5) *If $\dim(X) \in \omega$ then $\mathrm{Mlt}(X) = \mathrm{DM}(X)$.*

Proof. — (1), (2), and (3) follow easily from the definition of Mlt and the properties of dim (Corollary 4.13) and (5) follows from Proposition 4.14.

Let us prove (4). Let $Y_1, \ldots, Y_m$ be sets partitioning Y with $\dim(Y_i) = \dim(Y)$. By (3) applied to the sets $f^{-1}[Y_1], \ldots, f^{-1}[Y_m]$ we may assume that $m = 1$ and $Y_1 = Y$. Suppose for a contradiction that there are pairwise disjoint $X_1, \ldots, X_{m'+1} \subseteq X$ with $\dim(X_i) = \dim(X)$. For each $i \in \{1, \ldots, m' + 1\}$ put $Z_i := \{y \in Y : \dim(f^{-1}(y) \cap X_i) = s\} \subseteq Y$. Then each Z_i is definable by Corollary 4.13(1) and $\dim(Z_i) = \dim(Y)$ by Corollary 4.13(4) applied to f and to $f|_{X_i}$. As $\mathrm{Mlt}(Y) = 1$, using induction and (2) we easily get that $\dim(\bigcap_{i \in \{1,\ldots,m'+1\}} Z_i) = \dim(Y)$. In particular, there exists $y \in \bigcap_{i \in \{1,\ldots,m'+1\}} Z_i$ and we have that $(f^{-1}[y] \cap X_i)_{i \in \{1,\ldots,m'+1\}}$ are pairwise disjoint subsets of $f^{-1}(y)$ of dimension s, a contradiction to $\mathrm{Mlt}(f^{-1}(y)) = m'$. $\square$

Theorem 6.3. — *We work in T_∞.*

(1) *For every formula $\phi(x; y)$ over $\emptyset$ there exists $m_{\phi(x;y)} \in \omega$ such that for every $R \in \omega$ and every $N \models T_R$ containing a we have $\mathrm{DM}_N(\phi(\mathfrak{C}, a) \cap N) \leqslant m_{\phi(x;y)}$.*
(2) *Every formula in T_∞ has finite multiplicity.*

Proof. — Using the functions $f_{k,e_1,\dots,e_m}$ (see Definition 4.12), we may assume that x is a tuple of k vector variables for some $k \in \omega$. We will now prove the statement by induction on k.

For any fixed $R_0 \in \omega$, by quantifier elimination in T_{R_0} and Fact 2.5(6) we easily get a bound on $\mathrm{DM}_N(\phi(\mathfrak{C};a) \cap N)$ with $a \subseteq N \models T_{R_0}$ depending only on $\phi(x;y)$ and on R_0. Also, we know by the proof of Proposition 3.10 that if $\dim(\phi(\mathfrak{C};a)) \in \omega$ then $\phi(M,a) \subseteq N_{2l(a)}$ for some $M = \bigcup_r^a N_r$ with $a \subseteq N_{2l(a)}$. Hence we may restrict ourselves to considering only $R \geqslant \alpha(l(x), l(y))$ and a such that $\dim(\phi(\mathfrak{C};a)) \notin \omega$.

First, assume $k = 1$ so x is a single vector variable. If $R \geqslant \alpha(1, l(y))$ (so $R \geqslant 4l(a)+3$), $a \in N \models T_R$ and $\mathrm{Dim}(\phi(\mathfrak{C};a)) \notin \omega$, then by Proposition 3.10 (with R there equal to $4l(a)+1$ here), $\mathrm{DM}_N(\phi(\mathfrak{C};a) \cap N)$ is equal to $\mathrm{DM}_{N'}(\phi(\mathfrak{C};a) \cap N')$ for any $N' \models T_{4l(a)+3}$ containing a with $K(N) = K(M)$ and $N' \subseteq N$. This, in turn, is bounded independently from a and N by quantifier elimination in $T_{4l(a)+3}$ and Fact 2.5(6), which completes the proof when $k = 1$.

Now, assume that $k \geqslant 1$ and we have numbers $m_{\psi(x_1,\dots,x_i;y)}$ satisfying the assertion for each $\phi(x_1,\dots,x_i;y)$ with $x_1,\dots,x_i$ being single variables of the sort V and $i \leqslant k$. Consider any $\phi(x;y)$ with $x = (x_1,\dots,x_{k+1})$, where each x_i is a single variable of the sort V. As every definable set is nice, there are at most $D := |D_{1,k+l(y)}|$ possibilities $s_1,\dots,s_D \in D_{1,k+l(y)}$ on $\mathrm{rk}(\phi(x_1;v_2,\dots,v_{k+1},w))$ for $v_2,\dots,v_{k+1},w \in \mathfrak{C}$ with $\phi(x_1;v_2,\dots,v_{k+1},w) \neq \emptyset$. For each $i \leqslant D$ let $\chi_{s_i}(x_2,\dots,x_{k+1},y)$ be a formula over $\emptyset$ such that

$$\models \chi_{s_i}(v_2,\dots,v_{k+1},w) \iff \dim(\phi(x_1;v_2,\dots,v_{k+1},w)) = s_i.$$

Put

$$m_{\phi(x_1,\dots,x_{k+1};y)} := \sum_{i \leqslant D} (m_{\phi(x_1;x_2,\dots,x_{k+1},y)} m_{\chi_{s_i}(x_2,\dots,x_{k+1};y)})$$

(the numbers on the right-hand side are already defined by the inductive hypothesis). Consider any $R \geqslant \alpha(k+1, l(y))$, $N \models T_R$, and $a \subseteq N$ compatible with y. As $\alpha(k+1, l(y)) \geqslant \alpha(1, k+l(y))$, for every $i \leqslant D$ there is $t_i \in \omega$ such that for every $v_2,\dots,v_{k+1},w \in N$, if $\models \chi_{s_i}(v_2,\dots,v_{k+1},w)$ then $\mathrm{RM}_N(\phi(\mathfrak{C};v_2,\dots,v_{k+1},w) \cap N) = t_i$.

Let $\pi : V^{k+1} \to V^k$ be the projection on the last k coordinates. Put $X = \phi(\mathfrak{C};a)$ and $X_{s_i} := X \cap \pi^{-1}[\chi_{s_i}(\mathfrak{C};a)]$ for each $i \leqslant D$.

By Lemma 4.8 we get (as in the proof of Lemma 4.9(3)) that $\pi[X_{s_i} \cap N] = \pi[X_{s_i}] \cap N = \chi_{s_i}(\mathfrak{C};a) \cap N$. Also, for any $v_2,\dots,v_{k+1} \in \pi[(X_{s_i} \cap N)]$ we know that

$$\begin{aligned}\mathrm{DM}&((\pi|_{X_{s_i}\cap N})^{-1}(v_2,\dots,v_{k+1}))\\ &\qquad = \mathrm{DM}(N\cap \pi|_{X_{s_i}}{}^{-1}(v_2,\dots,v_{k+1}))\\ &\qquad = \mathrm{DM}(N\cap \phi(\mathfrak{C};v_2,\dots,v_{k+1},a)) \leqslant m_{\phi(x_1;x_2,\dots,x_{k+1},y)}.\end{aligned}$$

Thus, by Fact 2.5(8) and by Fact 2.5(9) applied to the functions $\pi|_{X_{s_i}\cap N}$ we have

$$\begin{aligned}\mathrm{DM}_N(X\cap N) &\leqslant \sum i \leqslant D\mathrm{DM}_N(X_{s_i}\cap N)\\ &\leqslant \sum_{i\leqslant D}(m_{\phi(x_1;x_2,\dots,x_{k+1},y)}\mathrm{DM}_N(\chi_{s_i}(\mathfrak{C};a)\cap N))\\ &\leqslant \sum_{i\leqslant D}(m_{\phi(x_1;x_2,\dots,x_{k+1},y)}m_{\chi_{s_i}(x_2,\dots,x_{k+1};y)}),\end{aligned}$$

as required. This completes the induction.
Let's now prove (2). Choose $M=\bigcup_r^a N_r$ containing a. Let $m := m_{\phi(x,y)}$ be the number given by (1). We claim that $\mathrm{Mlt}(\phi(x,a))\leqslant m$. If not, then there exist pairwise disjoint sets $X_1,\dots,X_{m+1}\subseteq \phi(M,a) =: X$ definable in M over some finite $b\subseteq M$. Let $R\geqslant \alpha(l(x),l(ab))$ be such that each X_i is definable over N_R and $ab\subseteq N_R$. Then, as X and all X_i's are nice, we have $\mathrm{RM}_{N_R}(X_i\cap N_R) = \mathrm{RM}_{N_R}(X\cap N_R)$ for every $i\leqslant m+1$, so $\mathrm{DM}_{N_R}(X\cap N_R) > m$, which contradicts the choice of m. ☐

Corollary 6.4. — *If G is a group definable in T_∞^* and $H \lhd G$ is a definable normal subgroup such that $\dim(G)-\dim(H)\in\omega$, then G/H is definably isomorphic to a group definable in T_∞^* of finite dimension.*

Proof. — Put $d := \dim(G)-\dim(H)\in\omega$. Let a be a finite tuple over which H and G are definable in a variable x, and choose $M=\bigcup_r^a N_r$ and $R\geqslant\alpha(l(x),l(a))$ such that $a\subseteq N_R$. Then, by niceness of H and G, we have $\mathrm{rk}_{N_r}(G\cap N_r)=\mathrm{rk}_{N_r}(H\cap N_r)+d$ so $\mathrm{rk}_{N_r}((G\cap N_r)/(H\cap N_r))=d$ for every $r\geqslant R$. Note that $H\cap N_r\lhd G\cap N_r$ for every $r\geqslant R$ as each N_r is dcl-closed.

Claim 1. — *There is $r_0\geqslant R$ such that for every $r\geqslant r_0$ the definable embedding of groups*

$$h_{r_0,r} : (G\cap N_{r_0})/(H\cap N_{r_0}) \to (G\cap N_r)/(H\cap N_r)$$

given by $g(H\cap N_{r_0})\mapsto g(H\cap N_r)$ is surjective.

Proof of the claim.

Case 1: $* = \mathrm{ACF}_p$. — By Theorem 6.3 we know there is m such that $\mathrm{DM}_{N_r}(G \cap N_r) \leqslant m$ for every $r \in \omega$, so also $\mathrm{DM}_{N_r}((G \cap N_r)/(H \cap N_r)) \leqslant m$ for any $r \geqslant R$ by Fact 2.5(9). If $h_{r,r+1}$ is not surjective for some $r \geqslant R$, then $h_{r,r+1}[(G \cap N_r)/(H \cap N_r)] = (G \cap N_r)/(H \cap N_{r+1})$ is a proper subgroup of the group $(G \cap N_{r+1})/(H \cap N_{r+1})$ of the same dimension d, so

$$\begin{aligned}\mathrm{DM}_{N_r}((G \cap N_r)/(H \cap N_r)) &= \mathrm{DM}_{N_{r+1}}((G \cap N_r)/(H \cap N_{r+1})) \\ &< \mathrm{DM}_{N_{r+1}}((G \cap N_{r+1})/(H \cap N_{r+1})),\end{aligned}$$

so, by boundedness of $\mathrm{DM}_{N_r}((G \cap N_r)/(H \cap N_r))$ (by m) there is $r_0 \geqslant R$ such that for every $r \geqslant r_0$ the embedding $h_{r,r+1}$ is surjective, and so is $h_{r_0,r} = h_{r-1,r}h_{r-2,r-1}\dots h_{r_0,r_0+1}$.

Case 2: $* = \mathrm{RCF}$. — We claim that $r_0 \coloneqq R$ works. For any $r \geqslant r_0$ we have that $\dim_t((G \cap N_r)/(H \cap N_r)) = d = \dim_t((G \cap N_R)/(H \cap N_R)) = \dim_t((G \cap N_R)/(H \cap N_r))$, so, by Fact 2.5(0) and (4), the index

$$[(G \cap N_r)/(H \cap N_r) : (G \cap N_R)/(H \cap N_R)] = [G \cap N_r : (G \cap N_R) \cdot (H \cap N_r)]$$

is finite. Note that the group $G \cap N_R$ normalises $H \cap N_r$ so $G_0 \coloneqq (G \cap N_R) \cdot (H \cap N_r) = \{x \cdot y : x \in G \cap N_R, y \in H \cap N_r\}$ is a definable subgroup of $G \cap N_r$. Now for any $g \in G \cap N_r$ the coset $gG_0 \in (G \cap N_r)/G_0$ is algebraic in N_r over N_R. As RCF eliminates imaginaries and algebraic closure coincides with definable closure in RCF, this implies that the coset $g \cdot G_0$ is definable over N_R, hence also over N_{r_0}. Also, $g \cdot G_0$ is definable over a, g, so, as $r_0 \geqslant 4l(x) + 2l(a)$, we get by Lemma 4.8 that $g \cdot G_0 \cap N_{r_0} \neq \emptyset$. This shows that $h_{r_0,r}$ is surjective, which completes the proof of the claim. □

By the claim, for every $g \in G(M)$ there is $g' \in G \cap N_{r_0}$ with $gH = g'H$. As $M \prec \mathfrak{C}$, we must also have that for every $g \in G(\mathfrak{C})$ there is $g' \in G \cap \mathrm{Lin}_{K(\mathfrak{C})}(N_{r_0})$ with $g/H = g'/H$, so

$$G/H = (G \cap \mathrm{Lin}_{K(\mathfrak{C})}(N_{r_0}))/(H \cap \mathrm{Lin}_{K(\mathfrak{C})}(N_{r_0}))$$

is definable in $K(\mathfrak{C})$ (by elimination of imaginaries in K), and hence it is definable in $\mathfrak{C}$ and has finite dimension. □

Remark 6.5. — If X is definable in T_∞ and E is a definable equivalence relation on X such that $\mathrm{RM}(X/E) < \omega$ (in $\mathfrak{C}$ expanded by the sort X/E and the quotient map $X \to X/E$), then for every $s \in S_{lin}$ for which there is $x \in X$ with $\dim(x_E) = s$ we have $\dim(X_s) - s \in \omega$, where $X_s = \{x \in X : \dim(x_E) = s\}$.

Proof. — Put $l \coloneqq \mathrm{RM}(X/E)$, and let $M = \bigcup_{r\in\omega}^{a} N_r$ with X and E definable over some finite $b \subseteq N_R$ for some $R \geqslant \alpha(\mathfrak{l}(x), l(x) + \mathfrak{l}(b))$, where x is a variable in which X is definable. If $\dim(X_s) = [d_0 + d_1 n]$ and $s =$

$[d'_0 + d'_1 n]$, then for every $r \geqslant R$ we have $\dim(x_E) \cap N_r = d'_0 + d'_1 r$ for each $x \in X \cap N_r$ by niceness of the xb-definable set x_E. But, as $\mathrm{RM}_{N_r}(X_s \cap N_r/E) \leqslant \mathrm{RM}(X/E) = l$, we get by Fact 2.5(4) applied to the quotient map $X_s \cap N_r \to (X_s \cap N_r)/E$ that

$$\mathrm{RM}_{N_r}(X_s \cap N_r) = d'_0 + d'_1 r + \mathrm{RM}_{N_r}((X_s \cap N_r)/E) \leqslant l + d'_0 + d_1 r,$$

(note $(X_s \cap N_r)/E$ is definable in N_r by elimination of imaginaries in ACF_p). This shows that $\dim(X_s) \leqslant [d'_0 + d'_1 n + l] = s + l$, so $\dim X_s - s \leqslant l$. □

Corollary 6.6. — *Let G be a group definable in T^*_∞ and let $H \triangleleft G$ be a definable normal subgroup. Then, the following are equivalent:*

(1) $\dim(G) - \dim(H) \in \omega$.
(2) *G/H is definably isomorphic to a definable in T^*_∞ group of finite dimension (hence to an algebraic group over K if $* =ACF_p$ and to a semialgebraic group over K if $* = \mathrm{RCF}$).*

If $* =ACF_p$, then these conditions are also equivalent to:

(3) $\mathrm{RM}(G/H) < \omega$.

Proof. — (1) implies (2) by Corollary 6.4. (2) implies (3) by Remark 4.14 and it implies (1) as well by Corollary 4.13(4) applied to the quotient map $G \to G/H$ (where we identify G/H with a definable group definably isomorphic to it). (3) implies (1) by Remark 6.5 applied to the equivalence relation E on G given by: $E(g, g') \iff gH = g'H$. □

From finiteness of multiplicity in T_∞, we also conclude definability of generic types.

Proposition 6.7.

(1) *Let X be definable in T_∞ over a model M. Put $m = \mathrm{Mlt}(X)$. Then there are exactly m complete generic types in X over M.*
(2) *Let $M \models T_\infty$ and let $p(x) \in S(M)$ be such that $\dim(p(x)) \in S_{lin}$. Then $p(x)$ is definable. Hence, each generic type in every definable set is definable.*

Proof.

(1). — Suppose first that there are $m + 1$ distinct generics $p_1, \ldots, p_{m+1} \in S(M)$ in X. Let $\phi(x)$ be a formula over M defining the set X. Choose pairwise inconsistent formulas $\phi_i(x) \in p_i$ for $i \leqslant m + 1$. Then, as $\phi_i(x) \wedge \phi(x) \in p_i$ for each i, we must have $\dim(\phi_i(x) \wedge \phi(x)) = \dim(X)$ as each p_i is generic in X. This shows that $\mathrm{Mlt}(X) \geqslant m$, a contradiction.

On the other hand, by definability of dim (Corollary 4.13(1)) we can find pairwise disjoint $X_1, \ldots, X_m \subseteq X$ definable over M with $\dim(X_i) = X$ for each i, and choose a generic $p_i \in X_i$ for each i. Then p_i's are pairwise distinct generics in X.

(2). — As $\dim(p(x)) \in S_{\mathrm{lin}}$, we can choose $\phi(x) \in p(x)$ such that

$$\dim(\phi(x)) = \dim(p(x)).$$

By definability of dim there are formulas $\phi_1(x), \dots, \phi_m(x)$ over M of dimension $\dim(\phi(x))$ which partition $\phi(x)$, and one of them must belong to $p(x)$. So we may assume $\mathrm{Mlt}(\phi(x)) = 1$. Now consider any formula $\psi(x; y)$. Then for any $a \subseteq M$ compatible with y we have that $\psi(x, a) \in p(x)$ iff $\dim(\psi(x; a) \wedge \phi(x)) = \dim(\phi(x))$: If $\psi(x, a) \in p(x)$ then $\psi(x; a) \wedge \phi(x) \in p(x)$ so $\dim(\psi(x; a) \wedge \phi(x)) = \dim(\phi(x))$; conversely, if the latter holds, then the generic type in $\psi(x; a) \wedge \phi(x)$ over M is also generic in $\phi(x)$, so is equal to $p(x)$ by (1), as $\mathrm{Mlt}(\phi(x)) = 1$. Thus $\psi(x; a) \in p(x)$.

As the condition $\dim(\psi(x; a) \wedge \phi(x)) = \dim(\phi(x))$ is definable by Corollary 4.13(1), we get that $p(x)$ is a definable type. □

7. Definable groups and fields

In this section we will prove our main results about groups and fields definable in T^*_∞. Let us start with some examples. Clearly, any algebraic group over the field of scalars K is definable in T_∞ and any semialgebraic group over K is definable in T^{RCF}_∞. Another class of examples is obtained from the natural actions of linear algebraic groups over K on Cartesian powers of the (infinite-dimensional) vector space V:

Example 7.1. — Let $M = (V, K)$ be a model of T^*_∞ and $k \in \omega$.

(1) Suppose $H \leqslant GL_k(K)$ is a linear algebraic group. Consider the semidirect product $G := V^k \rtimes H$, where the action of H on V^k is induced by scalar multiplication. Then G is definable in M in a natural way, with its universe being a definable subset of $V^k \times K^{k^2}$ consisting of pairs (v, A) with $v \in V^k$ and $A \in H$.

(2) Let $(G, \cdot)$ be the Heisenberg group of $[\cdot, \cdot]$, that is, $G = V \times V \times K$ and $(v, w, a) \cdot (v', w', a') = (v + v', w + w', a + a' + [v, w'])$ for $(v, w, a), (v', w', a') \in G$. Then $(G, \cdot)$ is definable in M (in an obvious way).

We say a definable group G is *connected* if it has no definable subgroup of finite index.

Remark 7.2. — Every group definable in T_∞ has a connected component, that is, a definable connected subgroup of finite index.

Proof. — Let M be a model over which G is definable. By Proposition 6.7 there are only finitely many generic types in G in $S(M)$. Let $p_1, \dots, p_m$ be all of them. Then for any $i \leqslant m$ and $g \in G(M)$ we have that $g \cdot p_i(x) := \{\phi(g^{-1} \cdot x) : \phi(x) \in p_i(x)\} \in S(M)$ is also a generic in G, so G acts naturally on $p_1, \dots, p_m$. Let G_0 be the kernel of this action. Now, if we choose pairwise inconsistent $\phi_i(x) \in p_i(x)$ of dimension $\dim(G)$ and multiplicity 1, then $G_0 = \{g \in G : \bigwedge_i \dim(\phi_i(x) \wedge \phi_i(g^{-1} \cdot x)) = \dim(G)\}$ (cf. the proof of Proposition 6.7), so G_0 is definable by Corollary 4.13. As $[G : G_0] < \omega$, we must have $\dim(G_0) = \dim(G)$. Now only one of the types $p_1, \dots, p_m$ contains the formula "$x \in G_0$", as otherwise we would have some $g_i \in G_0 \cap \phi_i(M)$ and $g_j \in G_0 \cap \phi_j(M)$ for $i \neq j$, so $g_i g_j^{-1} \cdot p_j = p_i$, a contradiction, as $g_i g_j^{-1} \in G_0$. Hence G_0 has only one generic type, and so $\mathrm{Mlt}(G_0) = 1$ by Proposition 6.7. This clearly implies that G_0 is connected. □

By a [semi] algebraic group in our context we mean a group interpretable in T^*_∞ which is definably isomorphic to a [semi] algebraic group over the field of scalars K. Thus, for example, although the group $(V, +)$ might be abstractly isomorphic to the group $(K, +)$ in a particular model $(K, V) \models T_\infty$, it is not an algebraic group in our sense, as there is no definable bijection between V and any set definable in K. Accordingly, we say that a definable group G is ([semi] algebraic-by-abelian)-by-[semi] algebraic, if there are definable $N \triangleleft G$ and $N_0 \triangleleft N$ such that N_0 and G/N are [semi] algebraic and N/N_0 is abelian.

Let $g, h \in G$ where G is a group. We will usually write the product of g and h as gh omitting the multiplication symbol. To avoid confusion with a pair, below we will use commas in tuples. By g^h we mean the conjugate hgh^{-1} of g by h, and by $[g, h]$ we mean the commutator $ghg^{-1}h^{-1}$ of g and h. By G' we denote the commutator subgroup of G, that is, the subgroup of G generated by the set $\{[x, y] : x, y \in G\}$.

Theorem 7.3. — *Let G be a group definable in T^*_∞. Then G is (algebraic-by-abelian)-by-algebraic when $* = ACF_p$ and (semialgebraic-by-abelian)-by-semialgebraic when $* = RCF$.*

Proof. — Let G be a group definable in T_∞ [or in T^{RCF}_∞] over some finite tuple a. We may assume that $G \subseteq V^k$ for some $k \in \omega$ and that a is a subtuple of any element of G.

Put

$$N := \{x \in G : \dim(C_G(x)) \sim \dim(G)\}.$$

Claim 1. — *$N \triangleleft G$ and N is a-definable.*

Proof of the claim. — First, we show that N is a subgroup of G. Take any $g_1, g_2 \in G$. Let $M = \bigcup_{r\in\omega}^{a} N_r$ and $R \in \omega$ be such that $g_1, g_2, a \subseteq N_R$. Consider any $r \geqslant R$. Note that $C_G(g_1) \cap N_r = C_{G\cap N_r}(g_1)$ and $C_G(g_2) \cap N_r = C_{G\cap N_r}(g_2)$ are both subgroups of the group $G \cap N_r \leqslant G$, as N_r is definably closed by Fact 2.10(2). Hence $(G \cap N_r)/(C_G(g_1) \cap C_G(g_2) \cap N_r)$ embeds N_r-definably into $((G\cap N_r)/(C_G(g_1)\cap N_r)) \times ((G\cap N_r)/(C_G(g_2)\cap N_r))$ by

$$g(C_G(g_1) \cap C_G(g_2) \cap N_r) \mapsto (g(C_G(g_1) \cap N_r), g(C_G(g_2) \cap N_r)).$$

So

$$\begin{aligned}&\mathrm{rk}_{N_r}((G \cap N_r)/(C_G(g_1) \cap C_G(g_2) \cap N_r))\\ &\qquad \leqslant \mathrm{rk}_{N_r}((G \cap N_r)/(C_G(g_1) \cap N_r)) + \mathrm{rk}_{N_r}((G \cap N_r)/(C_G(g_2) \cap N_r)).\end{aligned}$$

By Fact 2.5(4) applied to the corresponding quotient maps, this means that

$$\begin{aligned}&\mathrm{rk}_{N_r}(G \cap N_r) - \mathrm{rk}_{N_r}(C_G(g_1) \cap C_G(g_2) \cap N_r)\\ &\leqslant \mathrm{rk}_{N_r}(G\cap N_r) - \mathrm{rk}_{N_r}(C_G(g_1)\cap N_r) + \mathrm{rk}_{N_r}(G\cap N_r) - \mathrm{rk}_{N_r}(C_G(g_2)\cap N_r).\end{aligned}$$

As this holds for any $r \geqslant R$, we get that

$$\begin{aligned}&\dim(G) - \dim(C_G(g_1) \cap C_G(g_2))\\ &\qquad\qquad \leqslant \dim(G) - \dim(C_G(g_1)) + \dim(G) - \dim(C_G(g_2)) \in \omega.\end{aligned}$$

As $C_G(g_1 \cdot g_2) \supseteq C_G(g_1) \cap C_G(g_2)$, we conclude that $\dim(C_G(g_1 \cdot g_2)) \sim \dim(G)$, so $g_1 \cdot g_2 \in N$ and N is a subgroup of G. Also, for any $g \in G$ and $h \in N$ we have that $\dim(C_G(h^g)) = \dim((C_G(h))^g) = \dim(C_G(h))$, as $(C_G(h))^g$ and $C_G(h)$ are in a definable bijection. This shows that N is normal in G. Finally, N is a-definable by Corollary 4.13(1). □

Let h be a generic in G over a and let g be a generic in G over a, h. Write $h = (w_1, \ldots, w_k)$ and $g = (w_{k+1}, \ldots, w_{2k})$ (where $w_i \in V$ for each $i \leqslant 2k$). Let $j_1, \ldots, j_l \in \{1, \ldots, 2k\}$ be such that $w_{j_1}, \ldots, w_{j_l}$ is a basis of $W := \mathrm{Lin}_K(w_1, \ldots, w_{2k})$ over K.

For any $x = (v_1, \ldots, v_k) \in G$ there are $i_1, \ldots, i_m \in \{1, \ldots, k\}$ such that $(v_{i_1}, \ldots, v_{i_m}, w_{j_1}, \ldots, w_{j_l})$ is a basis of $\mathrm{Lin}(W, v_1, \ldots, v_k)$. As this is expressible by a formula $\phi(x)$ with parameters h, g and there are only finitely many possibilities on the tuple $(i_1, \ldots, i_m) \in \{1, \ldots, k\}^m$ (with $m \leqslant k$), by Corollary 4.13(3) there must be some such tuple for which the set

$$X := \left\{\begin{matrix} x = (v_1, \ldots, v_k) \in G : (v_{i_1}, \ldots, v_{i_m}, w_{j_1}, \ldots, w_{j_l}) \\ \text{is a basis of } \mathrm{Lin}_K(W, v_1, \ldots, v_k) \end{matrix}\right\}$$

is generic in G. We may assume $(i_1,\dots,i_m)=(1,\dots,m)$. Notice that for any $x=(v_1,\dots,v_k)\in X$ we have

$$gx^h \in \mathrm{dcl}(x,g,h) \subseteq \mathrm{Lin}_K(w_{i_1},\dots,w_{i_l},v_1,\dots,v_m),$$

so we can define a function let $f : X \to K^{k(l+m)}$ such that for every $x=(v_1,\dots,v_k)\in X$

$$\text{if } f(x)=Y=(Y_1,Y_2) \text{ with } Y_1\in M_{l\times k}(K), Y_2\in M_{m\times k}(K)$$
$$\text{then } gx^h = Y_1\cdot(w_{i_1},\dots,w_{i_l})^T + Y_2\cdot(v_1,\dots,v_m)^T.$$

As f is a definable function and $\dim(\mathrm{im}(f)) \leqslant [k(l+m)]$, by Corollary 4.13(4) there must be some $C=(C_1,C_2)\in K^{k(l+m)}$ such that

$$\dim(f^{-1}(C)) \sim \dim(G).$$

Then for $x=(v_1,\dots,v_k)\in f^{-1}(C)$ we have

$$gx^h = C_1\cdot(w_{i_1},\dots,w_{i_l}) + C_2\cdot(v_1,\dots,v_m).$$

By Lemma 4.16 we can choose $g_1\in f^{-1}(C)$ such that $\dim(g_1/C,h,g,a) = \dim(f^{-1}(C)) \sim \dim(G)$, and $g_2\in f^{-1}(C)$ such that $\dim(g_2/C,h,g,g_1,a) \sim \dim(G)$. Write $g_1=(v_1,\dots,v_k)$ and $g_2=(v'_1,\dots,v'_k)$. So

$$gg_1^h = C_1\cdot(w_{i_1},\dots,w_{i_l})^T + C_2\cdot(v_1,\dots,v_m)^T$$

and

$$gg_2^h = C_1\cdot(w_{i_1},\dots,w_{i_l})^T + C_2\cdot(v'_1,\dots,v'_m)^T.$$

So $t \coloneqq C_1\cdot(w_{i_1},\dots,w_{i_l})^T = gg_1^h - C_2\cdot(v_1,\dots,v_m)^T \in dcl(g_1,gg_1^h,C,a)$, hence

$$gg_2^h = t + C_2\cdot(v'_1,\dots,v'_m)^T \in dcl(g_1,gg_1^h,g_2,C,a).$$

Thus,

$$(g_1^{-1}g_2)^h = (gg_1^h)^{-1}gg_2^h \in dcl(g_1,gg_1^h,g_2,C,a).$$

So, choosing h_1 to be a generic in $\mathrm{tp}(h/g_1,gg_1^h,g_2,C,a)$ over h,g_1,gg_1^h,g_2,C,a, we get

$$(g_1^{-1}g_2)^h = (g_1^{-1}g_2)^{h_1},$$

hence

$$g_1^{-1}g_2 \in C_G(h_1^{-1}h).$$

Claim 2. — $\dim(h_1^{-1}h/g_1^{-1}g_2,a) \sim \dim(G)$.

Proof of the claim. — By Lascar's equality (Proposition 5.1) we have

$$\begin{aligned}(7.1)\qquad \dim(h/C,a) &= \dim(h,C/a) - \dim(C/a)\\ &\geqslant \dim(h/a) - \dim(C/a) \sim \dim(G)\end{aligned}$$

as h is generic in G over a and $\dim(C/a) \in \omega$. Also

$$(7.2)\qquad \dim(g_1/h,C,a) \sim \dim(G)$$

by the choice of g_1. Now, as g_1 is quasi-generic over g,h,a we have that

$$\dim(g_1,g,h/a) = \dim(g_1/g,h,a) + \dim(g/h,a) + \dim(h/a) \sim 3\dim(G)$$

by Lascar's equality, but also

$$\dim(g_1,g,h/a) = \dim(g/g_1,h,a) + \dim(g_1,h/a),$$

so $\dim(g/g_1,h,a) \sim \dim(G)$ as $\dim(g_1,h/a) \leqslant 2\dim(G)$. Hence

$$\begin{aligned}(7.3)\quad &\dim(gg_1^h/g_1,h,C,a)\\ &\qquad\sim \dim(gg_1^h/g_1,h,a) = \dim(g/g_1,h,a) \sim \dim(G),\end{aligned}$$

where the equality follows by invariance of dim under definable bijections. We also have

$$(7.4)\qquad \dim(g_2/gg_1^h,g_1,h,C,a) \sim \dim(G)$$

by the choice of g_2. Now, by (1), (2), (3),(4), and Lascar's equality we have $\dim(g_2,gg_1^h,g_1,h/C,a) \sim 4\dim(G)$. But

$$\dim(g_2,gg_1^h,g_1,h/C,a) = \dim(h/g_2,gg_1^h,g_1,C,a) + \dim(g_2,gg_1^h,g_1,C,a),$$

so

$$\dim(h/g_2,gg_1^h,g_1,C,a) \sim \dim(G)$$

as $\dim(g_2,gg_1^h,g_1,C,a) \lesssim 3\dim(G)$. As h_1 is generic in $\operatorname{tp}(h/g_1,gg_1^h,g_2,C,a)$ over (h,g_1,gg_1^h,g_2,C,a), it follows that

$$\dim(h_1^{-1}h/h,g_1,gg_1^h,g_2,C,a) = \dim(h_1/h,g_1,gg_1^h,g_2,C,a) \sim \dim(G),$$

so also $\dim(h_1^{-1}h/g_1^{-1}g_2,a) \sim \dim(G)$ which completes the proof of the claim. □

As $h_1^{-1}h \in \mathfrak{C}_G(g_1^{-1}g_2)$ and $\dim(h_1^{-1}h/g_1^{-1}g_2,a) \sim \dim(G)$ by Claim 2, we get $\dim C_G(g_1^{-1}g_2) \sim \dim(G)$. This shows that $g_1^{-1}g_2 \in N$, so, as $\dim(g_1^{-1}g_2/a) \sim \dim(G)$ and N is definable over a, we conclude that $\dim(N) \sim \dim(G)$, so G/N is an algebraic [semialgebraic] group by Corollary 6.6.

It is left to show that N is algebraic-by-abelian [semialgebraic-by-abelian]. For any $x \in N$ we have that all fibers of the map $G \to [x,G] := \{[x,y] : y \in G\}$ given by $y \mapsto [x,y]$ are cosets of $C_G(x)$ and hence they have dimension

$\dim(G_G(x)) \sim \dim(G)$. So, by Corollary 4.13(4) we get that $\dim([x,G]) \in \omega$. Thus, for $x_1, x_2 \in N$ the commutator $[x_1,x_2]$ has finite dimension over a, x_1 and over a, x_2, so by Proposition 5.2 $[x_1,x_2] \in (\mathrm{Lin}_K(V(a,x_1)) \cap \mathrm{Lin}_K(V(a,x_2)))^k$. If additionally $\dim(x_1/x_2,a) \sim \dim(x_1/a)$ then, by Corollary 5.3 we get that

$$\mathrm{Lin}_K(V(a,x_1)) \cap \mathrm{Lin}_K(V(a,x_2)) = \mathrm{Lin}_K(V(a)) =: A,$$

so $[x_1,x_2] \in A$.

Now, for arbitrary $y_1, y_2 \in N$, as $\dim(y_2 \cdot C_G(y_1)) = \dim(C_G(y_1)) \sim \dim(G)$, we can find $y_2' \in y_2 \cdot C_G(y_1)$ with $\dim(y_2'/y_1,a) \sim \dim(G)$, so $[y_1,y_2] = [y_1,y_2'] \in A$ by the above paragraph. This shows that $N_0 := \{[y_1,y_2] : y_1,y_2 \in N\} \subseteq A$. Put $N_1 := N \cap A$. As A is definably closed by Fact 2.10(2), we get that N_1 is a definable subgroup of N. So, as $N_1 \supseteq N_0$, we conclude that $N_1 \supseteq N'$. Finally, put

$$N_2 := \bigcap_{g \in N} (N_1)^g$$

and note that N_2 is a definable normal subgroup of N and $N' \leqslant N_2$, so N/N_2 is abelian. Also, $N_2 \leqslant N_1 \subseteq A$. But, as a is finite, we have that A is finite-dimensional, so N_2 is algebraic [semialgebraic] over $K(\mathfrak{C})$ by Corollary 6.6. Hence N is [semi]algebraic-by-abelian, and G is ([semi]algebraic-by-abelian)-by-[semi]algebraic. □

Remark 7.4. — Examples 7.1(1) and (2) show that the conclusion in Theorem 7.3 cannot be strengthened to "G is [semi] algebraic-by-abelian", nor to "G is abelian-by-[semi] algebraic". Indeed, in Example 7.1(1), taking $H := K^*$ acting naturally on $(V,+)$, we get that the commutator group $[G,G] = V \times \{1\}$ is infinite-dimensional, so G is not [semi] algebraic-by-abelian. On the other hand, the Heisenberg group $(G,\cdot) = (V \times V \times K, \cdot)$ in Example 7.1(2) is not abelian-by-[semi] algebraic. Indeed, working for example in ${}_ST_\infty^{\mathrm{ACF}_p}$, if $N \triangleleft G$ is a normal definable subgroup such that G/N is [semi] algebraic, then, by Corollary 6.6 we get that $\dim(N) \sim \dim(G) = [2n+1]$. Hence, if $M = \bigcup_r^a N_r$ and $R \in \omega$ are such that G and N are definable over N_R, then we can find $(v,w,a) \in N(M)$ with $v \notin N_R$ and $w \notin \mathrm{Lin}_{K(M)}(N_R,v)$. Let $v_0,v_1,w_0,w_1 \in V(M)$ be such that $v = v_0 + v_1$, $w = w_0 + w_1$, $v_0,w_0 \in V(N_R)$ and $v_1,w_1 \perp V(N_R)$. As $[\cdot,\cdot]$ is nondegenerate and $v_1 \notin V(N_R)$, there is $z \in V(M)$ with $[v_1,z] \neq 0$ and $z \perp V(N_R)$. Now we can choose $v',w'' \in V(M)$ with $v',w'' \perp v_1,w_1,z,V(N_R)$, $[v',v'] = [v_1,v_1]$, and $[v',w''] = [v_1,w_1]$. Take any $e_0 \in V(M)$ with $e_0 \perp v_1,w_1,z,v',w'',V(N_R)$ and $[e_0,e_0] = 1$. Let $\alpha \in K(M)$ be such that $[\alpha e_0 + w'' + z, \alpha e_0 + w'' + z] = [w_1,w_1]$.

Then putting $w' := \alpha e_0 + w'' + z$ we have $[w', w'] = [w_1, w_1]$, $[v', v'] = [v_1, v_1]$, $[v', w'] = \alpha[v', e_0] + [v', w''] + [v', z] = [v', w''] = [v_1, w_1]$, and $v_1, w_1, v', w' \perp V(N_R)$. Thus $\mathrm{tp}(v', w'/N_R) = \mathrm{tp}(v_1, w_1/N_R)$ and hence $\mathrm{tp}(v_0 + v', w_1 + w'/N_R) = \mathrm{tp}(v, w/N_R)$ so we can choose $b \in K$ with $\mathrm{tp}(v_0 + v', w_1 + w', b/N_R) = \mathrm{tp}(v, w, a/N_R)$. As N is definable over N_R and $(v, w, a) \in N$, we must have $(v_0 + v', w_0 + w', b) \in N$ as well. Now, the commutator $[(v, w, a), (v_0 + v', w_0 + w', b)]$ equals $(0, 0, [v, w_0 + w'] - [v_0 + v', w]) = (0, 0, [v_0, w_0] + [v_1, w'] - ([v_0, w_0] + [v', w_1])) = (0, 0, [v_0, w_0] + [v_1, \alpha e_0] + [v_1, w''] + [v_1, z] - [v_0, w_0]) = (0, 0, [v_1, z]) \neq (0, 0, 0)$. Hence N is not abelian, and G is not abelian-by-[semi] algebraic.

Now we conclude from (the proof of) Theorem 7.3 that all fields definable in T^*_∞ have finite dimension.

Theorem 7.5. — *Every field definable in T^*_∞ is finite-dimensional, and hence definable in the field of scalars K. In particular, there is no definable field structure on V^k for any $k < \omega$.*

Proof. — Suppose F is an infinite-dimensional field definable in T^*_∞. Put $s := \dim(F)$.

Let $G = (F^*, \cdot) \ltimes (F, +)$ be the affine group of F, that is, G consists of pairs (a, b) where $a \in F^*$ and $b \in F$ with multiplication given by:

$$(a, b)(c, d) = (ac, b + ad).$$

Notice that for any $(a, b)(c, d) \in G$ the commutator

$$[(a, b), (c, d)] = (a, b)(c, d)(a, b)^{-1}(c, d)^{-1}$$

is equal to $(1, (a - 1)d + (1 - c)b)$. Hence, if $a \neq 1$ and $(c, d) \in C_G((a, b))$ then $d = \frac{c-1}{a-1}b$, so $\dim(C_G((a, b))) \leqslant s$, whereas $\dim(G) = 2s$ by Corollary 4.13(4) applied to projection on either of the coordinates of the Cartesian product $F^* \times F$. Hence, if we put $N := \{g \in G : \dim(G) - \dim(C_G(g)) \in \omega\}$, we get that $N \subseteq \{1\} \times F$. This implies that the set $\{(a, 0) : a \in F^*\}$ embeds definably in G/N, so $\dim(N) \ll \dim(G)$ which contradicts the proof of Theorem 7.3. $\square$

By Fact 1.1(2) we conclude:

Corollary 7.6. — *Every infinite field definable in T_∞ is definably isomorphic to the field of scalars K.*

By Fact 1.2 we also get:

Corollary 7.7. — *Every infinite field definable in T^{RCF}_∞ is either algebraically closed or real closed.*

8. Independence relations and generics

In this section we relate our notion of dimension in T_∞ to two independence relations, $\mathop{\smile\!\!\!\!\!\!\mid}^{\Gamma}$ introduced in [11, Definition 12.2.1] for T_∞, and Kim-independence (denoted $\mathop{\smile\!\!\!\!\!\!\mid}^{K}$) defined for any theory in [14], and having good properties over models in NSOP_1 theories, and over arbitrary sets in NSOP_1 theories satisfying existence.

We will work in a monster model $\mathfrak{C} \models T_\infty$, that is, a $\overline{\kappa}$-saturated, $\overline{\kappa}$-strongly homogeneous model of T_∞ for some sufficiently large $\overline{\kappa}$. All parameter sets considered will be *small*, that is, of size less than $\overline{\kappa}$.

We say that a set A is an *extension base* if no formula (or equivalently, type) over A forks over A. We say that a theory T satisfies the *existence axiom* (or simply *existence*) if every set of parameters is an extension base. It was asked in [5, Question 6.6] whether any NSOP_1 theory satisfies existence, and a list of positive examples was given in [5, Fact 2.14]. Here we show that T_∞ also satisfies it:

Proposition 8.1. — *T_∞ satisfies existence.*

Proof. — Let $\phi(x,a)$ be a formula over A. Let $p(x)$ be a global generic type in $\phi(x,a)$. As any conjugate of $p(x)$ over A is also a generic type in $\phi(x,a)$, we get by Proposition 6.7(1) that there are only finitely many conjugates of $p(x)$ over A. As $p(x)$ is definable by Proposition 6.7(2), this implies that it is definable over $\mathrm{acl}^{eq}(A)$; in particular, $p(x)$ does not fork over $\mathrm{acl}^{eq}(A)$, so it does not fork over A, so $\phi(x,a)$ does not fork over A. $\square$

Fact 8.2 ([11, Theorem 12.2.2]). — *Let $M \models T_\infty$. Then the relation $\mathop{\smile\!\!\!\!\!\!\mid}^{\Gamma}$ on subsets of M given by Γ-forking is automorphism invariant, symmetric, transitive, satisfies the finite character and extension axioms, and types over models are stationary.*

Below, if $p(x) \in S(B)$ is a complete type in T_∞ and $B \subseteq N_R \models T_R$, then we say that $p(x)$ forks in N_R over some $A \subseteq B$ if its quantifier-free part in the language L_θ^F (which is equivalent to $p(x)$ in T_∞) forks in N_R over A. Likewise, $\mathrm{RM}_{N_R}(p(x))$ means Morley rank of the quantifier-free part of $p(x)$ in the sense of N_R.

Fact 8.3 ([11, Proposition 12.2.3]). — *Let $M \models T_\infty$, let $A \subseteq B \subseteq M$ and let $p(x) \in S(B)$. Let $(N_r : r \in \omega)$ be some approximating sequence for M. Then the following are equivalent:*

(1) *$p(x)$ does not Γ-fork over A;*
(2) *Given any formula $\phi = \phi(x,b) \in p(x)$ there is $R_\phi \in \omega$ such that $\phi(x,b)$ does not fork over $A \cap N_r$ in the structure N_r for all $r \geqslant R_\phi$;*

(3) *For each finite $b \subseteq B$ there is $R_b \in \omega$ such that $p(x)|_{N_r \cap Ab}$ does not fork over $A \cap N_r$ in N_r for all $r \geqslant R_b$.*

Corollary 8.4. — *Let $M \models T_\infty$, let $A \subseteq B \subseteq M$ with A finite, and let $p(x) \in S_{<\omega}(B)$. Then $p(x)$ does not Γ-fork over A if and only if $\dim(p(x)) = \dim(p|_A(x))$.*

Proof. — Assume $\dim(p(x)) = \dim(p|_A(x))$. We will verify that the condition (3) in Fact 8.3 holds for $p(x)$ and A. Consider any finite $b \subseteq B$ and $M = \bigcup_r^a N_r$ containing Ab. Let $R \geqslant \alpha(l(x), l(Ab))$ be such that $Ab \subseteq N_R$. Consider any $r > R$. Note that $\dim(\phi(x)) \in D_{l(x),l(Ab)}$ for any formula $\phi(x)$ with parameters in Ab. Hence, as $\dim(p(x)) = \dim(p|_A(x))$, we have by Lemma 4.7 that

$$\mathrm{RM}_{N_r}(p|_{Ab}(x)) = \mathrm{RM}_{N_r}(p|_A(x)),$$

so $p|_{Ab}(x)$ does not fork over A in N_r, and by Fact 8.3 we get that $p|_{Ab}(x)$ does not Γ-fork over A, as required.

Similarly, if $\dim(p(x)) < \dim(p|_A(x))$ then there is a finite b such that $\dim(p|_{Ab}(x)) < \dim(p|_A(x))$, and if $M = \bigcup_r^a N_r$ and $R \geqslant \alpha(l(x), l(Ab))$ are such that $Ab \subseteq N_R$, then we have by Lemma 4.7 that $\mathrm{RM}_{N_r}(p|_{Ab}(x)) < \mathrm{RM}_{N_r}(p|_A(x))$, so, by Fact 8.3, $p|_{Ab}(x)$ Γ-forks over A. $\square$

Definition 8.5 ([9, Definition 1.11]). — *Let $A \subseteq B \subseteq \mathfrak{C} \models T$ for some theory T, and let G be a group definable in $\mathfrak{C}$ over parameters A. Suppose $\mathop{\smile\!\!\!\!\!\!|}^*$ is an invariant ternary relation between small subsets of $\mathfrak{C}$. We call an element $g \in G$ a (left)* generic *over B, if*

$$\text{for every } h \in G \text{ such that } g \mathop{\smile\!\!\!\!\!\!|}^*_B h \text{ we have } h \cdot g \mathop{\smile\!\!\!\!\!\!|}^*_A B, h.$$

We call a type $p(x) \in S_G(B)$ (left) generic in G if every (equivalently, some) its realisation is a generic in G over B.

This notion of a generic was first studied in groups definable in stable (e.g. [20]), and more generally, simple theories (e.g. [19]), with $\mathop{\smile\!\!\!\!\!\!|}^*$ being the forking independence. In [9] it was studied in rosy theories mainly with $\mathop{\smile\!\!\!\!\!\!|}^*$ being thorn-independence. In the (non-first order) setting of Polish group structures a useful notion of a generic is obtained by taking $\mathop{\smile\!\!\!\!\!\!|}^*$ to be the non-meagre independence ([15]). Below we examine this notion in T_∞ for $\mathop{\smile\!\!\!\!\!\!|}^* = \mathop{\smile\!\!\!\!\!\!|}^\Gamma$ and for $\mathop{\smile\!\!\!\!\!\!|}^* = \mathop{\smile\!\!\!\!\!\!|}^K$.

To avoid confusion, we will say "dim-generic" to mean generic in the sense of Definition 4.18.

Proposition 8.6. — *Suppose $A \subseteq B \subseteq \mathfrak{C} \models T_\infty$, where A is finite, and G is a group definable over A. Then for any $p(x) \in S(B)$ we have that $p(x)$ is $\mathop{\smile\!\!\!\!\!\!|}^{\Gamma}$-generic in G if and only if $p(x)$ is* dim*-generic in G.*

Proof. — Suppose first $p(x)$ is dim-generic in G (i.e. $\dim(p(x)) = \dim(G)$) and fix any $g \models p$ and $h \in G$ such that $g \mathop{\smile\!\!\!\!\!\!|}^{\Gamma}_{B} h$. Then by Corollary 8.4 $\dim(\mathrm{tp}(g/B,h)) = \dim(\mathrm{tp}(g/B)) = \dim(p(x)) = \dim(G)$. As dim is preserved by definable bijections and every formula in $q := \mathrm{tp}(h \cdot g/B,h)$ is a translate of a formula in $\mathrm{tp}(g/B,h)$, we conclude that $\dim(q(x)) = \dim(G)$. On the other hand, $q|_A \vdash G(x)$, so $\dim(q|_A(x)) \leqslant \dim(G)$, so we must have $\dim(q(x)) = \dim(q|_A(x))$. By Corollary 8.4 again, this gives that $q(x)$ does not Γ-fork over A, i.e. $h \cdot g \mathop{\smile\!\!\!\!\!\!|}^{\Gamma}_{A} B, h$.

Now suppose $p = \mathrm{tp}(g/B)$ is a $\mathop{\smile\!\!\!\!\!\!|}^{\Gamma}$-generic in G. By Proposition 4.16 we can find $h \in G$ with $\dim(h/B,g) = \dim(G)$. In particular, $g \mathop{\smile\!\!\!\!\!\!|}^{\Gamma}_{B} h$. As g is generic in G over B, we have $h \cdot g \mathop{\smile\!\!\!\!\!\!|}^{\Gamma}_{A} B, h$ so $h \cdot g \mathop{\smile\!\!\!\!\!\!|}^{\Gamma}_{B} h$. Using this together with Corollary 8.4 in the second equality below, we get:

$$\begin{aligned}\dim(g/B) &\geqslant \dim(g/B,h) = \dim(h \cdot g/B,h) = \dim(h \cdot g/B)\\ &\geqslant \dim(h \cdot g/B,g) = \dim(h/B,g) = \dim(h/B) = \dim(G).\end{aligned}$$

Clearly $\dim(g/B) \geqslant \dim(G)$ implies that

$$\dim(g/B) = \dim(G),$$

as $\mathrm{tp}(g/B) \vdash G(x)$. □

Corollary 8.7. — *For any group G definable in T_∞ over a finite set A and any $B \supseteq A$ there exists a $\mathop{\smile\!\!\!\!\!\!|}^{\Gamma}$-generic over B element in G, and being $\mathop{\smile\!\!\!\!\!\!|}^{\Gamma}$-generic does not depend on the choice of the finite set A over which G is definable.*

Kim-independence was introduced and studied extensively in [14] over models in NSOP_1 theories. It was proved there, among other results, that $\mathop{\smile\!\!\!\!\!\!|}^{K}$ is symmetric and satisfies the independence theorem over models, which was later extended in [5] to arbitrary sets in NSOP_1 theories satisfying existence.

Definition 8.8 ([5, Definition 2.10]).

(1) *We say a formula $\varphi(x,a_0)$* Kim-divides *over A if for some Morley sequence $\langle a_i : i < \omega\rangle$ in tp(a_0/A), $\{\varphi(x,a_i) |\ i < \omega\}$ is inconsistent.*
(2) *A formula $\varphi(x;a)$* Kim-forks *over A if $\varphi(x;a) \vdash \bigvee_{i<k} \psi_i(x;b_i)$ where $\psi_i(x;b_i)$ Kim-divides over A for all $i < k$.*

(3) *Likewise we say a type $p(x)$ Kim-forks or Kim-divides over A if it implies a formula that Kim-forks or Kim-divides over A, respectively.*
(4) *We write $a \mathop{\smile\!\!\!\!\mid}^K_A b$ to denote the assertion that $tp(a/Ab)$ does not Kim-fork over A.*

Fact 8.9 ([5]). — *Suppose T is NSOP$_1$ and satisfies existence. Then:*

(1) *Kim's Lemma holds for $\mathop{\smile\!\!\!\!\mid}^K$, that is, if a formula $\phi(x, a_0)$ Kim-divides over A then for every Morley sequence $\langle a_i : i < \omega\rangle$ in $tp(a_0/A)$, $\{\varphi(x, a_i)| \ i < \omega\}$ is inconsistent*
(2) *A formula Kim-forks over A if and only if it Kim-divides over A*
(3) *$\mathop{\smile\!\!\!\!\mid}^K$ is symmetric*
(4) *The independence theorem for Lascar types for $\mathop{\smile\!\!\!\!\mid}^K$ holds over any set.*

The following folklore fact follows as in [14, Corollary 5.17], using the fact that, for any set C, a sequence is Morley over C iff it is a Morley sequence over $\mathrm{acl}(C)$.

Fact 8.10. — *Suppose T is an NSOP$_1$ theory satisfying existence, and let A, B, and C be any sets. Then $A \mathop{\smile\!\!\!\!\mid}^K_C B \iff \mathrm{acl}(A) \mathop{\smile\!\!\!\!\mid}^K_{\mathrm{acl}(C)} \mathrm{acl}(B)$. Also, it follows from the definition of $\mathop{\smile\!\!\!\!\mid}^K$ that $A \mathop{\smile\!\!\!\!\mid}^K_C B$ implies $A \mathop{\smile\!\!\!\!\mid}^K_C BC$.*

We will now give a description of Kim-independence in T_∞ over arbitrary sets. The proof of it will be essentially the same as in the description of Kim-independence over models given in [14, Proposition 9.37], but the statement there requires two corrections (even when working over a model), which we now explain. If $A \subseteq \mathfrak{C} \models T_\infty$, put $\langle A\rangle := \mathrm{Lin}_{K(\mathfrak{C})}(V(A))$ and let $A_K := A \cap K(\mathfrak{C})$. By $\mathrm{acl}(A)_K$ we mean $(\mathrm{acl}(A))_K$, where acl is the model-theoretic algebraic closure.

By quantifier elimination the structure on the sort K induced from T_∞ is just the pure field structure, so the relation $\mathop{\smile\!\!\!\!\mid}^K$ restricted to the sort K coincides with forking independence $\mathop{\smile\!\!\!\!\mid}^{\mathrm{ACF}}$ in the algebraically closed field K, that is, with algebraic independence in the sense of field theory. As for algebraically closed $A, B \supseteq M \models T_\infty$ the condition $A \cap B = M$ does not imply that $K(A) \mathop{\smile\!\!\!\!\mid}^{\mathrm{ACF}}_{K(M)} K(B)$, the latter is a missing condition in [14, Proposition 9.37].

Also, the following example shows that the condition $A \cap B = M$ for algebraically closed sets A, B and a model M, does not imply that $\langle A\rangle \cap \langle B\rangle = \langle M\rangle$ (even if $K_A = K_B = K_M$), which is also implicitly used in the proof of [14, Proposition 9.37], and which clearly follows from $A \mathop{\smile\!\!\!\!\!\!\;|}^{K}_{M} B$ (see the proof of Proposition 8.12 below).

Example 8.11. — Let $M = (V_0, K_0) \models {}_S T_\infty$ and choose an orthonormal pair (e_0, e_1) with $e_0, e_1 \in M^\perp$. Put $A := (\mathrm{Lin}_{K_0}(M, e_0, e_1), K_0)$. Clearly $A = \mathrm{acl}(A)$. Let $t \in K(\mathfrak{C}) \backslash K_0$ and let t' be such that $t^2 + t'^2 = 1$. Put $f := te_0 + t'e_1$ and $B := (\mathrm{Lin}_{K_0}(M, f), K_0)$. As $[f, f] = 1$ and $f \in M^\perp$, we have $B = \mathrm{acl}(B)$. Clearly $\langle A \rangle \cap \langle B \rangle = \langle B \rangle \neq M$, but $A \cap B = M$: any element of $A \cap B$ is of the form $ae_0 + be_1 + m_0 = cf + m_1$ for some $a, b, c \in K_0$ and $m_0, m_1 \in V(M)$. Then, as $\langle e_0, e_1, f \rangle \subseteq M^\perp$, we have

$$ae_0 + be_1 = cf = cte_0 + ct'e_1$$

so $a = ct$ and $b = ct'$. As $t, t' \notin K_0$, this implies that $a = b = 0$, so $ae_0 + be_1 + m_0 = m_0 \in M$. Hence $A \cap B = M$.

Proposition 8.12. — *Let $A, B, E \subseteq \mathfrak{C} \models T_\infty$ be small algebraically closed sets with $E \subseteq A, B$. Then $A \mathop{\smile\!\!\!\!\!\!\;|}^{K}_{E} B$ if and only if $\langle A \rangle \cap \langle B \rangle = \langle E \rangle$ and $K(A) \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{ACF}}_{K(E)} K(B)$.*

Proof. — Suppose first that $A \mathop{\smile\!\!\!\!\!\!\;|}^{K}_{E} B$. As already pointed out above, this implies that $K(A) \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{ACF}}_{K(E)} K(B)$ and it is left to show that $\langle A \rangle \cap \langle B \rangle = \langle E \rangle$. Suppose this is not the case, so there are vectors $a_1, \ldots, a_m \in A$ and $b_1, \ldots, b_k \in B$ such that $\langle a_1, \ldots, a_m \rangle \cap \langle b_1, \ldots, b_k \rangle$ is not contained in $\langle E \rangle$. By the assumption and [5, Proposition 4.5] there is an A-indiscernible Morley sequence $(\overline{d_i})_{i \in \omega}$ in $\mathrm{tp}(b_1, \ldots, b_k / E)$. In particular, the subspaces $\langle \overline{d_i} \rangle$, $i < \omega$ are linearly independent over $\langle E \rangle$, so $\langle a_1, \ldots, a_m \rangle$ can intersect at most m-many of them outside of $\langle E \rangle$, which contradicts indiscernibility. Hence the implication from left to right is proved.

Let us now assume that $\langle A \rangle \cap \langle B \rangle = \langle E \rangle$ and $K(A) \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{ACF}}_{K(E)} K(B)$. There are only two problems with the proof of Proposition 9.37 in [14] (with $E = M$ a model). First, as shown by Example 8.11, the assumptions there do not imply that $\langle A \rangle \cap \langle B \rangle = \langle M \rangle$, which is used in the construction of the structure N in that proof. This is, however, assumed here. Secondly, in the last paragraph of the proof in [14], the map $\sigma^i : B_0 \to B_i$ need not be elementary over $K(A') = K(A)$ on the sort K. However, assuming that $K(A) \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{ACF}}_{K(M)} K(B)$ we have that $K(B_i) \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{ACF}}_{K(M)} K(A')$ and $K(B_0) \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{ACF}}_{K(M)} K(A')$, so the map $\mathrm{id}_{K(A')} \cup \sigma^i|_{K(B_0)} : K(A') \cup K(B_0) \to K(A') \cup K(B_i)$ is elementary by stationarity of $\mathrm{tp}(K(B_0)/K(M))$. Thus $\mathrm{id}_{K(A')} \cup \sigma^i|_{K(B_0)}$ extends to an isomorphism $\rho : \widetilde{K} \to L$ onto some algebraically closed field L, hence, by the construction of A', the map $\mathrm{id}_{A'} \cup \sigma^i$

extends to an isomorphism between $\mathrm{Lin}_{\tilde{K}}(A' \cup B_0)$ and $\mathrm{Lin}_L(A' \cup B_i)$. By quantifier elimination this isomorphism is an elementary map, so in particular $A'B_0 \equiv A'B_i$, and hence $\mathrm{tp}(A/B)$ does not Kim-divide over M.

When E is an arbitrary algebraically closed set (not necessarily a model), the only difference is that the vector spaces we obtain may be finite-dimensional, which does not cause any problems, as an isomorphism of finite-dimensional vector subspaces of $\mathfrak{C}$ is still elementary in T_∞ by quantifier elimination. Hence the implication from right to left holds. □

By [11, Proposition 9.5.1] $\mathrm{acl}(AC)_K$ is the field-theoretic algebraic closure of $\mathrm{dcl}(AC)_K$ and $\langle \mathrm{acl}(AC)\rangle = \langle AC\rangle$ for any sets A and C, so by Fact 8.10 we conclude:

Corollary 8.13. — *Let $A, B, C \subseteq \mathfrak{C} \models T_\infty$ be any small sets. Then $A \mathop{\smile\!\!\!\!\!\!|}\nolimits^K_C B$ if and only if*

$$\langle AC\rangle \cap \langle BC\rangle = \langle C\rangle \text{ and } \mathrm{dcl}(AC)_K \mathop{\smile\!\!\!\!\!\!|}\limits^{\mathrm{ACF}}_{\mathrm{dcl}(C)_K} \mathrm{dcl}(BC)_K.$$

As $\mathop{\smile\!\!\!\!\!\!|}\nolimits^K$ does not satisfy base monotonicity, it is not obvious whether in the definition of $\mathop{\smile\!\!\!\!\!\!|}\nolimits^K$-genericity over B of an element $g \in G$ with G definable over A it is more reasonable to require that $h \cdot g \mathop{\smile\!\!\!\!\!\!|}\nolimits^K_A B, h$ (as is done in Definition 8.5) or that $h \cdot g \mathop{\smile\!\!\!\!\!\!|}\nolimits^K_B h$, provided that $g \mathop{\smile\!\!\!\!\!\!|}\nolimits^K_B h$. In either case, it turns out that $(V, +)$ does not have any $\mathop{\smile\!\!\!\!\!\!|}\nolimits^K$-generics over any set of parameters. Below we prove it for $\mathop{\smile\!\!\!\!\!\!|}\nolimits^K$-genericity in the sense of Definition 8.5, and exactly the same argument works for the other sense.

Proposition 8.14. — *The $\emptyset$-definable in T_∞ group $(V, +)$ does not have any $\mathop{\smile\!\!\!\!\!\!|}\nolimits^K$-generic type over any set B.*

Proof. — As usually we consider the symmetric case, the alternating case being very similar. By Fact 8.10 being $\mathop{\smile\!\!\!\!\!\!|}\nolimits^K$-generic over B is the same as being $\mathop{\smile\!\!\!\!\!\!|}\nolimits^K$-generic over $\mathrm{acl}(B)$, so we may assume that $B = \mathrm{acl}(B) = (V_0, K_0)$; in particular, K_0 is an algebraically closed field. Consider any $v \in V$ and put $(V_1, K_1) = \mathrm{acl}(B, v)$ and $a = [v, v]$. We will show that v is not a $\mathop{\smile\!\!\!\!\!\!|}\nolimits^K$-generic in $(V, +)$ over B. If $v \in \langle V_0\rangle$ then for any $w \neq v$ with $v \mathop{\smile\!\!\!\!\!\!|}\nolimits^K_B w$ we have that $0 \neq w+v \in \langle w+v\rangle \cap \langle V_0, w\rangle$, so $w+v \not\mathop{\smile\!\!\!\!\!\!|}\nolimits^K B, w$ hence v is not a $\mathop{\smile\!\!\!\!\!\!|}\nolimits^K$-generic in $(V, +)$ over B. So let us assume that $v \notin \langle V_0\rangle$. Let $t \in K(\mathfrak{C}) \setminus K_0$ be such that $K_1 \mathop{\smile\!\!\!\!\!\!|}\nolimits^{\mathrm{ACF}}_{K_0} t$.

Claim 1. — *We may assume there exists $w \in V$ such that $w \perp V_0$, $[w, w] = t$, $[w, v] = -\frac{1}{2}a$ and $\langle V_0, v\rangle \cap \langle V_0, w\rangle = \langle V_0\rangle$.*

Proof of the claim. — As $v \notin \langle V_0\rangle$, by compactness and the Gram–Schmidt process we can easily find some $f \in V$ with $f \perp V_0$ and $[f, v] = -\frac{1}{2}a$. Let $e_1 \in V$ be orthogonal to $\langle V_1, f\rangle$ with $[e_1, e_1] = 1$. Now we can find $\beta \in K(\mathfrak{C})$ such that $[f, f] + \beta^2 = t$. Then putting $w := f + \beta e_1$ we get $[w, w] = [f, f] + \beta^2 = t$ and $[w, v] = [f, v] = -\frac{1}{2}a$. By possibly modifying t we may assume that $\beta \neq 0$, so $\langle V_0, v\rangle \cap \langle V_0, w\rangle = \langle V_0\rangle$. □

Let w be as in the claim. Then $[w+v, w+v] = [w, w] + [v, v] + 2[w, v] = t + a - a = t = [w, w]$, so $w + v \not\downfree^K B, w$. On the other hand, as $w \perp V_0$, we have by [11, Proposition 9.5.1] that $\mathrm{dcl}(B, w)_K = \mathrm{dcl}_{\mathrm{ACF}}(K_0, t)$. As $K_1 = \mathrm{acl}(B, v)_K$, this gives us that $\mathrm{dcl}(B, v)_K \downfree^{\mathrm{ACF}}_{K_0} \mathrm{dcl}(B, w)_K$ by the choice of t. As we also know by the choice of w that $\langle V_0, v\rangle \cap \langle V_0, w\rangle = \langle V_0\rangle$, we conclude that $v \downfree^K_B w$. Hence v is not a $\downfree^K$-generic in $(V, +)$ over B. □

Question 8.15. — *Is there a useful notion of a generic element in a group definable in an NSOP$_1$ theory with existence?*

BIBLIOGRAPHY

[1] E. Bouscaren, "Model theoretic versions of Weil's theorem on pregroups", in *The model theory of groups (Notre Dame, IN, 1985–1987)*, Notre Dame Math. Lectures, vol. 11, Univ. Notre Dame Press, Notre Dame, IN, 1989, p. 177-185.

[2] A. D. Byron & R. Sklinos, "Fields definable in the free group", *Trans. Amer. Math. Soc. Ser. B* **6** (2019), p. 297-345.

[3] A. Chernikov & N. Hempel, "On n-dependent groups and fields II", *Forum Math. Sigma* **9** (2021), article no. e38 (51 pages).

[4] A. Chernikov & N. Ramsey, "On model-theoretic tree properties", *J. Math. Log.* **16** (2016), no. 2, article no. 1650009 (41 pages).

[5] J. Dobrowolski, B. Kim & N. Ramsey, "Independence over arbitrary sets in NSOP$_1$ theories", *Ann. Pure Appl. Logic* **173** (2022), no. 2, article no. 103058 (20 pages).

[6] L. P. D. van den Dries, "Definable groups in characteristic 0 are algebraic groups", *Abstracts Amer. Math. Soc.* (1982), no. 3, p. 142.

[7] ———, "Weil's group chunk theorem: a topological setting", *Illinois J. Math.* **34** (1990), no. 1, p. 127-139.

[8] ———, *Tame topology and o-minimal structures*, London Mathematical Society Lecture Note Series, vol. 248, Cambridge University Press, Cambridge, 1998, x+180 pages.

[9] C. Ealy, K. Krupiński & A. Pillay, "Superrosy dependent groups having finitely satisfiable generics", *Ann. Pure Appl. Logic* **151** (2008), no. 1, p. 1-21.

[10] P. E. Eleftheriou & S. Starchenko, "Groups definable in ordered vector spaces over ordered division rings", *J. Symbolic Logic* **72** (2007), no. 4, p. 1108-1140.

[11] N. Granger, "Stability, simplicity, and the model theory of bilinear forms", PhD Thesis, University of Manchester, 1999.

[12] H. Gross, *Quadratic forms in infinite-dimensional vector spaces*, Progress in Mathematics, vol. 1, Birkhäuser, Boston, Mass., 1979, xii+419 pages.

[13] E. Hrushovski, "Strongly minimal expansions of algebraically closed fields", *Israel J. Math.* **79** (1992), no. 2-3, p. 129-151.

[14] I. Kaplan & N. Ramsey, "On Kim-independence", *J. Eur. Math. Soc. (JEMS)* **22** (2020), no. 5, p. 1423-1474.

[15] K. Krupiński, "Some model theory of Polish structures", *Trans. Amer. Math. Soc.* **362** (2010), no. 7, p. 3499-3533.

[16] A. Macintyre, "On ω_1-categorical theories of fields", *Fund. Math.* **71** (1971), no. 1, p. 1-25. (errata insert).

[17] M. Messmer, "Groups and fields interpretable in separably closed fields", *Trans. Amer. Math. Soc.* **344** (1994), no. 1, p. 361-377.

[18] A. Pillay, "On groups and fields definable in o-minimal structures", *J. Pure Appl. Algebra* **53** (1988), no. 3, p. 239-255.

[19] ———, "Definability and definable groups in simple theories", *J. Symbolic Logic* **63** (1998), no. 3, p. 788-796.

[20] B. Poizat, *Stable groups*, Mathematical Surveys and Monographs, vol. 87, American Mathematical Society, Providence, RI, 2001, Translated from the 1987 French original by Moses Gabriel Klein, xiv+129 pages.

[21] K. Tent & M. Ziegler, *A course in model theory*, Lecture Notes in Logic, vol. 40, Association for Symbolic Logic, La Jolla, CA; Cambridge University Press, Cambridge, 2012, x+248 pages.

[22] A. Weil, "On algebraic groups of transformations", *Amer. J. Math.* **77** (1955), p. 355-391.

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Jan DOBROWOLSKI
Instytut Matematyczny
Uniwersytetu Wrocławskiego
pl. Grunwaldzki 2/4, 50-383 Wrocław (poland)
School of Mathematics
University of Leeds
Leeds LS2 9JT (United Kingdom)
dobrowol@math.uni.wroc.pl