

YNGVE DOMAR

## **A property of Fourier Stieltjes transforms on the discrete group of real numbers**

*Annales de l'institut Fourier*, tome 20, n° 2 (1970), p. 325-334

[http://www.numdam.org/item?id=AIF\\_1970\\_\\_20\\_2\\_325\\_0](http://www.numdam.org/item?id=AIF_1970__20_2_325_0)

© Annales de l'institut Fourier, 1970, tous droits réservés.

L'accès aux archives de la revue « Annales de l'institut Fourier » (<http://annalif.ujf-grenoble.fr/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

## A PROPERTY OF FOURIER-STIELTJES TRANSFORMS ON THE DISCRETE GROUP OF REAL NUMBERS

by Yngve DOMAR

---

1. We denote by  $\hat{G}$  the compact abelian group dual to the discrete group  $R_d$  of real numbers.  $\hat{G}$  is thus the well-known Bohr compactification of the real line. The elements of  $\hat{G}$  are characters on  $R_d$ . A Borel measurable subgroup  $\hat{R}$  of  $\hat{G}$  is formed by the set of characters which are continuous when considered as functions on  $R$  instead of on  $R_d$ . A Fourier-Stieltjes transform on  $R_d$  is by definition a complex-valued function  $\mu$  on  $R_d$  defined for every  $x \in R_d$  by the relation

$$\mu(x) = \int_{\hat{G}} (x, \hat{x}) d\hat{\mu}(\hat{x}),$$

where  $\hat{\mu}$  belongs to the class  $M(\hat{G})$  of all finite regular Borel measures on  $\hat{G}$ . In the following  $K$  denotes the class of Fourier-Stieltjes transforms which we obtain if we restrict ourselves to measures  $\hat{\mu}$  such that  $|\hat{\mu}|$  vanishes on  $\hat{R}$ .

Our main result is the following:

**THEOREM 1.** — *Let  $\mu \in K$  and let  $\lambda$  be an arbitrary real number and  $\varepsilon$  an arbitrary positive number. We consider  $\mu$  as a function on  $R$  instead of on  $R_d$ . Then*

$$\{x \in R \mid \operatorname{Re}(e^{i\lambda} \mu(x)) > \varepsilon\}$$

*has interior Lebesgue measure 0.*

In Theorem 1,  $\varepsilon$  can not be exchanged to 0. This is the immediate consequence of the following theorem.

THEOREM 2. —  $K$  contains a strictly positive function.

In Section 2, 3, and 4 we shall prove these two theorems. In the concluding Section 5 we state an immediate corollary of Theorem 1 and use it to give a new proof of a theorem of Rosenthal [1].

In the discussion we shall use without specific references wellknown facts from harmonic analysis on locally compact abelian groups, in particular on the group  $\hat{G}$ . As a general reference we suggest [2], where the Bohr compactification is discussed on pp. 30-32. The group operation on  $\hat{G}$  is notated as addition.

2. In this section we shall prove a lemma which will be used in Section 3 where Theorem 1 is proved. Let us first observe that there is a bijection from  $R$  to  $\hat{R}$  such that  $t \in R$  corresponds to the character with values  $e^{itx}$  at the points  $x \in R_d$ , and we shall in this sense use a real parameter to denote elements in  $\hat{R}$ .

LEMMA. — Let  $\varepsilon$  be an arbitrary positive number and  $\hat{U}$  an arbitrary open subset of  $\hat{G}$ , containing the subset  $\hat{I} = [-9/\sqrt{\varepsilon}, 9/\sqrt{\varepsilon}]$  of  $\hat{R}$ . Then there exists a non-negative discrete measure  $\nu$  on  $R_d$ , with total mass 1 and support contained in  $[-1/2, 1/2]$ , and such that  $\hat{\nu}$ , defined by

$$\hat{\nu}(\hat{x}) = \int_{R_d} (x, \hat{x}) d\nu(x),$$

$\hat{x} \in \hat{G}$ , satisfies the relation  $|\hat{\nu}(\hat{x})| \leq \varepsilon/4$  on  $\hat{G} \setminus \hat{U}$ .

*Proof of the lemma.* — The above-mentioned bijection from  $R$  to  $\hat{R}$  is continuous, hence  $\hat{I}$  is a compact subset of  $\hat{G}$ . It follows from this and from general properties of topological groups that it is no restriction to assume that  $\hat{U} = \hat{I} + \hat{U}_0$ , where  $\hat{U}_0$  is an open neighborhood of the zero element  $\hat{O}$  in  $\hat{G}$ . By the definition of the topology on  $\hat{G}$  we can furthermore assume that

$$\hat{U}_0 = \prod_{i=1}^n \{\hat{x} \in \hat{G} | (x_i, \hat{x}) - 1| < d\},$$

for some  $d > 0$  and with  $x_i \in R_d$ ,  $i = 1, 2, \dots, n$ . It is

easy to realize that we can restrict ourselves to the case when  $n \geq 2$  and when the elements  $x_i, i = 1, 2, \dots, n$ , are  $\neq 0$  and are not integer multiples of the same real number.

Let  $m$  be a positive number such that  $m \geq \max_{1 \leq i \leq n} |x_i|/\pi$ . For every  $i, 1 \leq i \leq n$ , we denote by  $\hat{f}_{i,m}$  the function on  $\hat{\mathbb{R}}$  with period  $2\pi/|x_i|$  and defined by the relation

$$\hat{f}_{i,m}(t) = \begin{cases} 1 - m|t|, & \text{if } |t| \leq 1/m \\ 0, & \text{if } 1/m < |t| \leq \pi/|x_i|, \end{cases}$$

and we form the non-negative function  $\hat{f}_m$  defined by

$$\hat{f}_m(t) = \frac{m(n+1)}{2} \prod_{i=1}^n \hat{f}_{i,m}(t),$$

for every  $t \in \hat{\mathbb{R}}$ .

The set

$$\hat{E} = \{t \in \hat{\mathbb{R}} | \hat{f}_m(t) > 0\}$$

is a union of disjoint open intervals of length  $\leq 2/m$ . Due to the assumption that at least two of the numbers  $x_i$  have irrational ratio it is obvious that the lower bound of the distances between these intervals tends to  $\infty$ , as  $m \rightarrow \infty$ . One of the intervals, we call it  $\hat{I}_0$ , contains  $\hat{O}$ . Obviously

$$\int_{\hat{I}_0} \hat{f}_m(t) dt = 1,$$

where we integrate with respect to the Lebesgue measure on  $\hat{\mathbb{R}}$ .

If  $\hat{I}_1$  is another arbitrary interval in the above class it is true that

$$\int_{\hat{I}_1} \hat{f}_m(t) dt \leq 1.$$

To see this it is enough to prove that if  $g$  is the function on  $\mathbb{R}$  defined by the relation

$$g = \begin{cases} 1 - |t|, & \text{if } |t| \leq 1 \\ 0, & \text{if } |t| > 1, \end{cases}$$

and if  $t_1, t_2, \dots, t_n$  are arbitrary, then

$$\int_{-\infty}^{\infty} \prod_{i=1}^n g(t + t_i) dt \leq \int_{-\infty}^{\infty} g(t)^n dt.$$

And this is a direct consequence of Hölder's inequality.

Let us now form the function  $\hat{\sigma}$  defined for  $t \in \hat{R}$  by

$$\hat{\sigma}(t) = \int_{-\infty}^{\infty} \frac{16 \sin^2 \frac{s}{4}}{s^2} \hat{f}_m(t-s) ds = \int_{t-s \in \hat{E}} \frac{16 \sin^2 \frac{s}{4}}{s^2} \hat{f}_m(t-s) ds.$$

If  $t \in \hat{I} + \hat{E}$  we have

$$\hat{\sigma}(t) = \int_{s \in \hat{I}} \frac{16 \sin^2 \frac{s}{4}}{s^2} \hat{f}_m(t-s) ds.$$

This relation and the properties of  $\hat{f}_m$  which we obtained earlier show that, if  $m$  is sufficiently large,

$$|\hat{\sigma}(t)| \leq \frac{\varepsilon}{5},$$

on  $\hat{R} \setminus (\hat{I} + \hat{E})$ .

Moreover, if  $m$  is large, the positive number  $\hat{\sigma}(0)$  is close to 1. Hence, if we define  $\hat{\nu}$  by

$$\hat{\nu}(t) = \hat{\sigma}(t)/\hat{\sigma}(0),$$

for  $t \in \hat{R}$ , we have, for large  $m$ , that  $\hat{\nu}$  fulfils the relation  $\hat{\nu}(0) = 1$  and satisfies

$$(1) \quad |\hat{\nu}(t)| \leq \frac{\varepsilon}{4},$$

on  $\hat{R} \setminus (\hat{I} + \hat{E})$ .

$\hat{f}_m$  is a product of positive definite periodic functions, hence it can be expressed as a sum

$$\hat{f}_m(t) = \sum_0^{\infty} a_p e^{i\alpha_p t},$$

where  $a_p \geq 0$  and  $\alpha_p$  are real for every  $p \in \mathbb{N}$ , and where  $\sum_0^{\infty} a_p < \infty$ . The function  $16 \sin^2 \frac{s}{4} / s^2$  is positive definite, too, and its Fourier transform vanishes outside  $[-1/2, 1/2]$ . Hence, since  $\hat{\sigma}$  is defined as a convolution between this function and  $\hat{f}_m$ , there exists a representation

$$\hat{\nu}(t) = \sum_0^{\infty} b_p e^{i\beta_p t},$$

where  $b_p \geq 0$ ,  $|\beta_p| \leq \frac{1}{2}$  for every  $p \in \mathbb{N}$ , and where

$$\sum_0^\infty b_p = \hat{\nu}(0) = 1.$$

Hence, if we define, for every  $\hat{x} \in \hat{G}$ ,

$$(2) \quad \hat{\nu}(\hat{x}) = \sum_0^\infty b_p(\beta_p, \hat{x}),$$

we obtain an extension of  $\hat{\nu}$  to a continuous function on  $\hat{G}$ . But (1) holds on  $\hat{R} \setminus (\hat{I} + \hat{E})$ , and since  $\hat{G} \setminus \text{closure}(\hat{I} + \hat{E})$  is open and  $\hat{R}$  is dense in  $\hat{G}$ , we know that  $|\hat{\nu}(\hat{x})| \leq \varepsilon/4$  on  $\hat{G} \setminus \text{closure}(\hat{I} + \hat{E})$ . Now, if  $m$  is large enough,  $\text{closure}(\hat{I} + \hat{E}) \subseteq \hat{U}$ , and thus

$$|\hat{\nu}(\hat{x})| \leq \varepsilon/4,$$

for every  $\hat{x} \in \hat{G} \setminus \hat{U}$ . (2) can be written

$$\hat{\nu}(\hat{x}) = \int_{\mathbb{R}^d} (x, \hat{x}) \, d\nu(x),$$

where  $\nu$  is a positive discrete measure with total mass 1 and support in  $\left[-\frac{1}{2}, \frac{1}{2}\right]$ , and hence  $\nu$  fulfils all conditions required in the lemma.

3. *Proof of Theorem 1.* — It is obviously enough to prove the theorem when  $\lambda = 0$ , and since  $\mu \in \mathbb{K} \implies \text{Re}(\mu) \in \mathbb{K}$  we can assume that  $\mu$  is real. Furthermore we can of course assume that

$$(3) \quad |\hat{\mu}|(\hat{G}) = 1.$$

We shall give an indirect proof and hence we assume that, for a certain real function  $\mu \in \mathbb{K}$ , satisfying (3), and for a certain  $\varepsilon > 0$ , the relation

$$(4) \quad \mu(x) \geq \varepsilon$$

holds in a set  $E \subset \mathbb{R}$  of positive Lebesgue measure.

By wellknown properties of the Lebesgue measure on  $\mathbb{R}$

there exists an interval  $I \subset \mathbb{R}$  of positive length and such that

$$m(I \cap E) \geq \left(1 - \frac{\varepsilon}{8(1 + \varepsilon)}\right) mI,$$

where  $m(\cdot)$  denotes Lebesgue measure. Using a suitable affine transformation of  $\mu$ , which does not affect the validity of (3), we see that we can assume that  $I = [-1, 1]$ .

The interval  $\hat{I} = [-9/\sqrt{\varepsilon}, 9/\sqrt{\varepsilon}] \subset \hat{\mathbb{R}}$  is, as mentioned before, a compact subset of  $\hat{\mathbb{G}}$ , and by assumption  $|\hat{\mu}|(\hat{I}) = 0$ . Since  $\hat{\mu}$  is a regular measure, there exists in  $\hat{\mathbb{G}}$  an open neighbourhood  $\hat{U}$  of  $\hat{I}$  such that

$$(5) \quad |\hat{\mu}|(\hat{U}) \leq \frac{\varepsilon}{4}.$$

We choose  $\nu$  as in the lemma, with  $\varepsilon$  and  $\hat{U}$  having the same meaning as now, and form, for an arbitrary real  $\lambda$ , the integral

$$(6) \quad J = \int_{\mathbb{R}^d} \mu(x + \lambda) d\nu(x) = \int_{[-1/2, 1/2]} \mu(x + \lambda) d\nu(x).$$

By a simple form of Parseval's relation

$$J = \int_{\hat{\mathbb{G}}} \hat{\nu}(\hat{x})(\lambda, \hat{x}) d\hat{\mu}(\hat{x}).$$

By (3), (5) and the properties of  $\hat{\nu}$ , formulated in the lemma

$$(7) \quad |J| \leq \int_{\hat{U}} |\hat{\nu}(\hat{x})| |d\hat{\mu}(\hat{x})| + \int_{\hat{\mathbb{G}} \setminus \hat{U}} |\hat{\nu}(\hat{x})| |d\hat{\mu}(\hat{x})| \leq \frac{\varepsilon}{2}$$

In order to be able to integrate the relation (6) with respect to the Lebesgue measure we substitute for  $\mu$  the function  $\mu_0$  defined in  $\mathbb{R}$  by

$$\mu_0(x) = \begin{cases} \varepsilon, & \text{if } x \in E \\ -1, & \text{if } x \notin E. \end{cases}$$

By (3) and (4),  $\mu(x) \geq \mu_0(x)$  for every  $x \in \mathbb{R}$  and hence, since  $\nu$  is non-negative, (6) and (7) give

$$\int_{[-1/2, 1/2]} \mu_0(x + \lambda) d\nu(x) \leq \frac{\varepsilon}{2},$$

for every  $\lambda \in \mathbb{R}$ . Integrating this we obtain

$$\int_{[-1/2, 1/2]} \left( \int_{-1/2}^{1/2} \mu_0(x + \lambda) d\lambda \right) d\nu(x) \leq \frac{\varepsilon}{2}.$$

The integration of  $\mu_0$  takes place in an interval  $I_1$  of length 1 included in  $[-1, 1]$ . By assumption the Lebesgue measure of  $[-1, 1] \setminus E$  is  $\leq \frac{\varepsilon}{4(1 + \varepsilon)}$ . Hence the same is true for the Lebesgue measure of  $I_1 \setminus E$  and this gives

$$\int_{-1/2}^{1/2} \mu_0(x + \lambda) d\lambda \geq -\frac{\varepsilon}{4(1 + \varepsilon)} + \varepsilon \left( 1 - \frac{\varepsilon}{4(1 + \varepsilon)} \right) = \frac{3\varepsilon}{4},$$

if  $|x| \leq 1/2$ .

Hence

$$\frac{3\varepsilon}{4} \int_{[-1/2, 1/2]} d\nu(x) \leq \frac{\varepsilon}{2},$$

and this gives a contradiction since  $\varepsilon > 0$  and

$$\nu([-1/2, 1/2]) = \nu(R_d) = 1.$$

4. *Proof of Theorem 2.* — Using transfinite induction we can construct a non-constant character  $\hat{x}$  on  $R_d$  which takes only values of the form  $e^{i\pi r}$ , where  $r$  is rational. Obviously  $\hat{x} \in \hat{G} \setminus \hat{R}$ . We denote by  $H_0$  the subgroup of all  $x \in R_d$  with the property that  $(x, \hat{x}) = 1$ . For every complex  $\lambda$  the subset of  $R_d$  consisting of all  $x$  such that  $(x, \hat{x}) = \lambda$  is either empty or a coset of  $H_0$ . This follows easily from the multiplicativity of the characters. Hence the class of these cosets is denumerable. It is obviously infinite and hence we can order the cosets as an infinite sequence  $(H_n)_0^\infty$ .

For every  $n \geq 0$  we denote by  $\chi_n$  the characteristic function of  $H_n$ . It is well known ([2], 3.1.2) that  $\chi_0$  is the Fourier-Stieltjes transform of the Haar measure  $\hat{\nu}$  of the annihilator subgroup  $\hat{\Gamma}$  of  $H_0$ .  $\hat{\Gamma}$  is a compact subgroup of  $\hat{G}$ , isomorphic to the dual of  $R_d/H_0$ , and it consists of the characters on  $R_d$  which are constant on  $H_0$ . Since  $H_0$  obviously is dense, considered as a subset of  $R$ ,  $\hat{\Gamma}$  does not contain any elements of  $\hat{R}$  except  $\hat{O}$ . Since moreover



$R_d/H_0$  is infinite, its dual  $\hat{\Gamma}$  is infinite and hence  $\hat{\nu}(\{\hat{O}\}) = 0$ . From this we can conclude that

$$(8) \quad |\hat{\nu}|(\hat{R}) = \hat{\nu}(\hat{R}) = 0.$$

Let us now form the function  $\mu$  on  $R_d$ , defined by the relation

$$\mu(x) = \sum_0^{\infty} 2^{-n} \chi_n(x), \quad x \in R_d.$$

For every  $n$ ,  $\chi_n$  is a translate of  $\chi_0$ , that is, there exists an  $x_n \in R_d$  such that

$$\chi_n(x) = \int_{\hat{G}} (x + x_n, \hat{x}) d\hat{\nu}(x), \quad x \in R_d.$$

Hence

$$\mu(x) = \int_{\hat{G}} (x, \hat{x}) \left( \sum_0^{\infty} 2^{-n} (x_n, \hat{x}) \right) d\hat{\nu}(\hat{x}), \quad x \in R_d,$$

and by (8) we see that  $\mu \in K$ . Since  $\mu$  is strictly positive, Theorem 2 is proved.

5. We shall now give a corollary of Theorem 1. Let us first recall a concept from measure theory. We say that a Lebesgue measurable subset  $E$  of  $R$  is of *uniformly positive measure* if for every open set  $U \subset R$  such that  $E \cap U$  is nonempty the Lebesgue measure of  $E \cap U$  is positive. Let us for brevity say that a Fourier-Stieltjes transform is of *continuous type* if it corresponds to a measure on  $\hat{R}$ , that is can be interpreted as a Fourier-Stieltjes transform on  $R$ , of *discontinuous type* if it corresponds to a measure on  $\hat{G} \setminus \hat{R}$ , that is belongs to the class  $K$ . There is obviously a unique decomposition of a Fourier-Stieltjes transform on  $R_d$  as a sum of two transforms of continuous and discontinuous type, respectively.

**COROLLARY.** — *Let  $\mu = \mu_1 + \mu_2$  be a Fourier-Stieltjes transform on  $R_d$ ,  $\mu_1$  and  $\mu_2$  are its components of continuous and discontinuous type, respectively. Then the following holds:*

1° *If  $\mu$ , considered as a function on  $R$ , is Lebesgue measurable on a Lebesgue measurable subset of  $R$ ,  $\mu_2$  vanishes almost everywhere on this set.*

2° If  $\mu$ , considered as a function on  $R$ , is continuous on a subset of  $R$  of uniformly positive measure,  $\mu_2$  vanishes everywhere on this set.

*Proof of the corollary.* — With the assumptions in 1°,  $\mu_2 = \mu - \mu_1$  is measurable on the set, hence by Theorem 1 the measure of the set

$$\{x \in R \mid \operatorname{Re} (e^{i\lambda\mu_2(x)}) > 0\}$$

is 0 for every real  $\lambda$ , and this means that  $\mu_2(x) = 0$  for almost every  $x \in R$ .

Under the assumptions of 2°  $\mu_2$  is continuous on the set. Since the set is of uniform positive measure 1° ascertains that  $\mu_2$  vanishes on the set.

The corollary can be used to prove a theorem of Rosenthal [1], which we formulate in the following way:

**THEOREM.** — Let  $E$  be a Lebesgue measurable subset of  $R$  and let  $\varphi$  be a bounded measurable function defined on  $E$ . We assume that for a certain constant  $A$

$$\left| \sum_{j=1}^n c_j \varphi(x_j) \right| \leq A \|P\|_\infty$$

for all trigonometrical polynomials  $P$  of the form

$$P(y) = \sum_{j=1}^n c_j e^{ix_j y}, \quad y \in R,$$

where  $x_j \in R$  for all  $1 \leq j \leq n$ . Then  $\varphi$  coincides almost everywhere on  $E$  with a Fourier-Stieltjes transform on  $R$ .

*Proof of the theorem.* — As mentioned by Rosenthal, p. 404, Proposition A, the assumptions imply rather easily that there exists a Fourier-Stieltjes transform  $\mu$  on  $R_d$  which coincides with  $\varphi$  everywhere on  $E$ . If we write, as in the corollary,  $\mu = \mu_1 + \mu_2$  and apply 1°, we see that  $\varphi$  coincides almost everywhere on  $E$  with  $\mu_1$ , which proves the theorem.

*Added in proof.* — I. Glicksberg has kindly communicated to the author unpublished generalizations of Rosenthal's theorem by himself and K. de Leeuw, their startingpoint being Theorem 6.4 in their joint paper The decomposition of

certain group representations, *Journal d'Analyse Math.* XV (1965), pp. 135-192. Their methods can be applied as well to prove our Theorem 1.

## BIBLIOGRAPHY

- [1] H. ROSENTHAL, A characterization of restrictions of Fourier-Stieltjes transforms, *Pac. J. Math.* 23 (1967) 403-418.
- [2] W. RUDIN, Fourier analysis on groups. New York 1962.

Manuscrit reçu le 11 octobre 1969.

Yngve DOMAR,  
Department of Mathematics,  
Uppsala University,  
Sysslomangatan 8,  
Uppsala (Suède)

---