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# COMBINATORIAL STRUCTURE OF STURMIAN WORDS AND CONTINUED FRACTION EXPANSION OF STURMIAN NUMBERS

**by Yann BUGEAUD & Michel LAURENT**

Abstract. — Let $\theta = [0; a_1, a_2, \dots]$ be the continued fraction expansion of an irrational real number $\theta \in (0,1)$. It is well-known that the characteristic Sturmian word of slope $\theta$ is the limit of a sequence of finite words $(M_k)_{k \geqslant 0}$, with $M_k$ of length $q_k$ (the denominator of the $k$-th convergent to $\theta$) being a suitable concatenation of $a_k$ copies of $M_{k-1}$ and one copy of $M_{k-2}$. Our first result extends this to any Sturmian word $\mathbf{s}$. Let $b \geqslant 2$ be an integer. Our second result gives the continued fraction expansion of any real number $\xi$ whose $b$-ary expansion is a Sturmian word $\mathbf{s}$ over the alphabet $\{0, b-1\}$. This extends a classical result of Böhmer who considered only the case where $\mathbf{s}$ is characteristic. As a consequence, we obtain a formula for the irrationality exponent of $\xi$ in terms of the slope and the intercept of $\mathbf{s}$.

Résumé. — Soit $\theta = [0; a_1, a_2, \dots]$ le développement en fraction continue d'un nombre irrationnel $\theta \in (0,1)$ et soit $q_k$ le dénominateur de la $k$-ième réduite de $\theta$. On sait que les préfixes $M_k$ de longueur $q_k$ du mot sturmien caractéristique de pente $\theta$ vérifient la relation de récurrence $M_k = M_{k-1}^{a_k} M_{k-2}$ pour tout $k \geqslant 2$. Nous établissons une relation de concaténation analogue pour les préfixes d'un mot sturmien quelconque $\mathbf{s}$. Soit $b$ un entier $\geqslant 2$. Nous obtenons en deuxième lieu une formule explicite pour le développement en fraction continue de tout nombre réel $\xi \in (0,1)$ dont la suite des chiffres en base $b$ forme une suite sturmienne $\mathbf{s}$ sur l'alphabet $\{0, b-1\}$. On généralise ainsi un résultat classique de Böhmer qui traitait le cas particulier où $\mathbf{s}$ est une suite sturmienne caractéristique. Nous en déduisons une formule donnant l'exposant d'irrationalité de $\xi$ en fonction de la pente et de l'intercept de $\mathbf{s}$.

## 1. Introduction

For a positive integer $n$, a factor of length $n$ of an infinite word $w_1 w_2 \dots$ is a finite word $w_j w_{j+1} \dots w_{j+n-1}$ composed of $n$ letters. Sturmian words

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are infinite words over a two-letter alphabet that have exactly $n+1$ distinct factors of length $n$ for every $n \geqslant 1$. They are the non-ultimately periodic words which are closest to ultimately periodic words. They admit several equivalent definitions and appear in many different areas of mathematics, including combinatorics, number theory, and dynamical systems; good references include [26, Chapter 2], [6], and [7]. The arithmetic description of Sturmian words is as follows. Throughout this paper, we let $\lfloor x \rfloor$ (resp., $\lceil x \rceil$) denote the largest (resp., smallest) integer less than or equal (resp., greater than or equal) to the real number $x$.

Let $\theta$ and $\rho$ be real numbers with $0 \leqslant \theta, \rho < 1$ and $\theta$ irrational. For $n \geqslant 1$, set

$$\begin{aligned} s_n &:= s_n(\theta, \rho) = \lfloor n\theta + \rho \rfloor - \lfloor (n-1)\theta + \rho \rfloor, \\ s'_n &:= s'_n(\theta, \rho) = \lceil n\theta + \rho \rceil - \lceil (n-1)\theta + \rho \rceil. \end{aligned}$$

Then, the infinite words

$$\mathbf{s}_{\theta,\rho} := s_1 s_2 s_3 \ldots, \quad \mathbf{s}'_{\theta,\rho} := s'_1 s'_2 s'_3 \ldots$$

are, respectively, the lower and upper Sturmian words of slope $\theta$ and intercept $\rho$, written over the alphabet $\{0,1\}$. Observe that $\mathbf{s}_{\theta,0}$ and $\mathbf{s}'_{\theta,0}$ differ only by their first letter, thus, there exists an infinite word $\mathbf{c}_\theta$, called the characteristic Sturmian word of slope $\theta$, such that

$$\mathbf{s}_{\theta,0} = 0\mathbf{c}_\theta, \quad \mathbf{s}'_{\theta,0} = 1\mathbf{c}_\theta.$$

Explicitly, we have

$$\mathbf{c}_\theta = \mathbf{s}_{\theta,\theta} = \mathbf{s}'_{\theta,\theta} = c_1 c_2 c_3 \ldots,$$

with

$$c_n = \lfloor (n+1)\theta \rfloor - \lfloor n\theta \rfloor = \lceil (n+1)\theta \rceil - \lceil n\theta \rceil, \qquad \text{for } n \geqslant 1.$$

Alternatively, the characteristic word $\mathbf{c}_\theta = \mathbf{s}_{\theta,\theta} = \mathbf{s}'_{\theta,\theta}$ can be defined as follows. Let $[0; a_1, a_2, \ldots]$ denote the continued fraction expansion of the slope $\theta$, with partial quotients $a_1, a_2, \ldots$ and convergents $p_k/q_k = [0; a_1, \ldots, a_k]$ for $k \geqslant 1$. Let $(M_k)_{k \geqslant 0}$ be the sequence of finite words over the alphabet $\{\mathbf{a}, \mathbf{b}\}$ associated with $(a_j)_{j \geqslant 1}$ defined by

$$M_0 = \mathbf{a}, \quad M_1 = \mathbf{a}^{a_1 - 1}\mathbf{b}, \quad M_k = (M_{k-1})^{a_k} M_{k-2}, \quad \text{for } k \geqslant 2.$$

Then, the limit $\lim_{k \to +\infty} M_k$ exists: it is the characteristic Sturmian word of slope $\theta$ over $\{\mathbf{a}, \mathbf{b}\}$. Replacing $\mathbf{a}$ by 0 and $\mathbf{b}$ by 1, we get

$$\mathbf{c}_\theta = \lim_{k \to +\infty} M_k. \tag{1.1}$$

Furthermore, the length (that is, the number of letters) of $M_k$ is equal to $q_k$ for $k \geqslant 1$.

Our first result, stated as Theorem 2.1, extends (1.1) by showing how an arbitrary Sturmian word of slope $\theta$ and intercept $\rho$ can be expressed as the limit of a sequence of finite words $(V_k)_{k\geqslant 0}$, with $V_k$ (of length $q_k$) being a suitable concatenation of $a_k$ copies of $V_{k-1}$ and one copy of $V_{k-2}$, defined in terms of the $\theta$-Ostrowski expansion of the intercept $\rho$.

Then, we will consider some Diophantine properties of the real numbers whose sequence of digits in some given integer base $b$ form a Sturmian word. Such real numbers are called $b$-Sturmian numbers, or shortly Sturmian numbers, when we do not need to refer to the base. The transcendence of characteristic Sturmian numbers was established by Böhmer [11] in 1927, assuming that the sequence of partial quotients $(a_k)_{k\geqslant 1}$ is unbounded. He also gave explicitly their continued fraction expansion; see Theorem 2.2 below. This has been rediscovered by Danilov [19], Davison [20], and by Adams and Davison [5] (see also [2], [13, Theorem 7.22], and [6, Section 9.3] for a special case). Ferenczi and Mauduit [21] used combinatorial properties of Sturmian words and a deep result from Diophantine approximation (Ridout's theorem, which is a $p$-adic extension of Roth's theorem) to establish that Sturmian numbers are transcendental. Specifically, they proved that every Sturmian word contains, for some positive $\varepsilon$, infinitely many $(2+\varepsilon)$-powers of blocks (that is, a block followed by itself and by a prefix of it of relative length at least $\varepsilon$) occurring not too far from its beginning.

Subsequently, Berthé, Holton and Zamboni [10] established that any Sturmian word, whose slope has a bounded continued fraction expansion, has infinitely many prefixes which are $(2+\varepsilon)$-powers of blocks, for some positive real number $\varepsilon$ depending only on the word. This implies that the associated Sturmian number $\xi$ is rather close to rational numbers whose $b$-ary expansion is purely periodic and gives that the irrationality exponent of $\xi$ is at least equal to $2+\varepsilon$.

Definition 1.1. — *The irrationality exponent $\mu(\zeta)$ of an irrational real number $\zeta$ is the supremum of the real numbers $\mu$ such that the inequality*

$$\left|\zeta - \frac{p}{q}\right| < \frac{1}{q^{\mu}}$$

*has infinitely many solutions in rational numbers $\frac{p}{q}$. If $\mu(\zeta)$ is infinite, then $\zeta$ is called a Liouville number.*

Recall that the irrationality exponent of an irrational number $\zeta$ is always at least equal to 2, with equality for almost all $\zeta$, in the sense of the Lebesgue measure. As observed in [1] (see also [13, Section 8.5]), it follows from the results of [10] and [4] that the irrationality exponent of any Sturmian number exceeds 2. Further progress has been made recently in [14], where it is proved that the irrationality exponent of a $b$-Sturmian number can be read on its $b$-ary expansion. This is equivalent to saying that, among the very good rational approximants to a $b$-Sturmian number, infinitely many of them can be constructed by cutting its $b$-ary expansion and completing by periodicity.

Furthermore, [14, Theorem 4.3] asserts that the irrationality exponent of a Sturmian number is at least equal to $\frac{5}{3} + \frac{4\sqrt{10}}{15} = 2.5099\ldots$, and that equality occurs in some cases. This result is obtained by means of a careful analysis of the repetitions occurring near the beginning of a given Sturmian word.

Our second main result, stated as Theorem 2.3, extends Böhmer's result and gives explicitly the continued fraction expansion of any $b$-Sturmian number over the alphabet $\{0, b-1\}$. From this we deduce in Theorem 2.4 an exact formula giving its irrationality exponent. Our approach also allows us to improve the best known transcendence measures for Sturmian numbers, see Theorem 2.7.

## 2. Results

Before stating our first result, we briefly recall the definition of the Ostrowski numeration system; see e.g. [9, Proposition 2]. We keep the notation from Section 1. Set $q_0 = 1$, $p_0 = 0$ and $\theta_k = q_k\theta - p_k$ for $k \geqslant 0$. Note that $\theta_k < 0$ if and only if $k$ is odd. Let $\sigma$ be an arbitrary number in the interval $[-\theta, 1-\theta]$. Then $\sigma$ can be written as

$$\sigma = \sum_{k \geqslant 1} b_k \theta_{k-1},$$

where $0 \leqslant b_1 \leqslant a_1 - 1$, $0 \leqslant b_k \leqslant a_k$ for $k \geqslant 2$, and $b_k = 0$ if $b_{k+1} = a_{k+1}$ (these are the so-called Ostrowski numeration rules). Assume that $\sigma$ does not belong to $\mathbb{Z}\theta + \mathbb{Z}$, or that $\sigma$ belongs to $\mathbb{Z}_{\geqslant 0}\theta + \mathbb{Z}$. Then, we can moreover ensure that there are infinitely many odd (resp., even) integers $k$ such that $b_k < a_k$. The latter condition guarantees the uniqueness of the representation which is called the *Ostrowski expansion of* $\sigma$. When $\sigma$ belongs to $\mathbb{Z}_{\geqslant 0}\theta + \mathbb{Z}$, the digits $b_k$ vanish for large $k$.

Theorem 2.1. — *Let $\theta$ and $\rho$ be real numbers with $0 \leqslant \theta, \rho < 1$ and $\theta$ irrational. Assume that $\rho$ does not belong to $\mathbb{Z}\theta + \mathbb{Z}$, or that $\rho$ belongs to*

$\mathbb{Z}_{\geqslant 1}\theta + \mathbb{Z}$. *Then* $\mathbf{s}_{\theta,\rho} = \mathbf{s}'_{\theta,\rho}$. *Let*

$$\rho - \theta = \sum_{h \geqslant 1} b_h \theta_{h-1}$$

*be the Ostrowski expansion of* $\rho - \theta$ *in base* $\theta$. *Define the words* $V_0, V_1, \ldots$ *by* $V_0 = 0$, $V_1 = 0^{a_1 - b_1 - 1} 1 0^{b_1}$, *and*

$$V_{k+1} = V_k^{a_{k+1} - b_{k+1}} V_{k-1} V_k^{b_{k+1}}, \quad k \geqslant 1.$$

*Then, the sequence* $(V_k)_{k \geqslant 0}$ *converges and*

$$\mathbf{s}_{\theta,\rho} = \mathbf{s}'_{\theta,\rho} = \lim_{k \to +\infty} V_k.$$

*Furthermore, setting*

$$t_k = b_1 + b_2 q_1 + \cdots + b_k q_{k-1} \quad \text{and} \quad r_k = q_k - t_k,$$

*and denoting by* $T_k$ *(resp.,* $R_k$*) the prefix (resp., suffix) of length* $t_k$ *(resp.,* $r_k$*) of* $M_k$ *for* $k \geqslant 1$, *we have*

$$V_k = R_k T_k \quad \text{and} \quad M_k = T_k R_k, \quad k \geqslant 1.$$

A similar result holds in the remaining case where $\rho - \theta = -m\theta + p$ for integers $m \geqslant 1$ and $p$. This case corresponds to the sequences which are ultimately equal to the characteristic word $\mathbf{c}_\theta$. Some technical difficulties occur, due to the fact that the choice of the lower/upper integral part does matter; see Section 3 for a precise statement and its proof. Previously, Arnoux, Ferenczi, and Hubert [8] have linked Sturmian sequences and Ostrowski expansions, but their result is different from ours; see also a work of Chuan [17] on $\alpha$-words.

Theorem 2.1 is a key tool for our extension of the following result of Böhmer [11].

Theorem 2.2 (Böhmer). — *For a positive real irrational number* $\theta = [0; a_1, a_2, \ldots]$ *in* $(0,1)$ *and an integer* $b \geqslant 2$, *set*

$$\xi_b(\theta) = (b-1) \sum_{j=1}^{+\infty} \frac{1}{b^{\lfloor j/\theta \rfloor}}.$$

*For* $k \geqslant 1$, *let* $p_k/q_k$ *denote the* $k$*-th convergent to* $\theta$ *and set*

$$A_k := \frac{b^{q_k} - b^{q_{k-2}}}{b^{q_{k-1}} - 1},$$

*where* $q_{-1} = 0$ *and* $q_0 = 1$. *Then, we have*

$$\xi_b(\theta) = [0; A_1, A_2, A_3, \ldots]$$

*and the irrationality exponent of* $\xi_b(\theta)$ *is given by*

$$\mu(\xi_b(\theta)) = 1 + \limsup_{k\to+\infty} \frac{q_k}{q_{k-1}}.$$

Note that $A_k$ is an integer multiple of $b^{q_{k-2}}$ since $q_k - q_{k-2}$ is an integer multiple of $q_{k-1}$.

The last assertion of the theorem follows from the well-known fact that the irrationality exponent of an irrational real number $\zeta = [A_0; A_1, A_2, \ldots]$ is given by

$$\mu(\zeta) = 1 + \limsup_{j\to+\infty} \frac{\log Q_{j+1}}{\log Q_j},$$

where $[A_0; A_1, A_2, \ldots, A_j] = P_j/Q_j$, for $j \geqslant 1$. Indeed, the sequence of convergents $(P_j/Q_j)_{j\geqslant 1}$ comprises all the best rational approximations to $\zeta$ and we have

$$\frac{1}{2Q_{j+1}Q_j} < \left|\zeta - \frac{P_j}{Q_j}\right| < \frac{1}{Q_{j+1}Q_j}.$$

Theorem 2.2 describes the first known class of real numbers having the property that both their $b$-ary expansion (for some integer $b \geqslant 2$) and their continued fraction expansion are explicitly determined. There are only few such classes; see [13, Section 7.6] for other examples, see also [28].

Our second main result extends Böhmer's theorem to an arbitrary $b$-Sturmian number with digits in $\{0, b-1\}$. Define

$$\xi_b(\theta,\rho) = (b-1)\sum_{n=1}^{+\infty} \frac{s_n(\theta,\rho)}{b^n}, \quad \xi'_b(\theta,\rho) = (b-1)\sum_{n=1}^{+\infty} \frac{s'_n(\theta,\rho)}{b^n}.$$

Let $\xi$ denote one of these numbers. Let $(b_k)_{k\geqslant 1}$ and $(t_k)_{k\geqslant 1}$ be the sequences of integers defined in Theorem 2.1 (or in Theorem 4.2 if $\rho$ is of the form $-m\theta+p$, with $m,p$ nonnegative integers) applied to the Sturmian sequence defining $\xi$. Put $t_0 = 0$ and $r_0 = 1$. For $k \geqslant 0$, set

$$c_k = b^{r_k+q_{k-1}} \frac{b^{(a_{k+1}-b_{k+1}-1)q_k} - 1}{b^{q_k} - 1}, \quad d_k = b^{t_k} - 1,$$
$$e_k = b^{r_k} - 1, \qquad f_k = b^{t_k} \frac{b^{b_{k+1}q_k} - 1}{b^{q_k} - 1}.$$

We point out that some elements of these four sequences may not be positive integers. For example, $f_k$ is equal to 0 when $b_{k+1} = 0$ and $c_{k+1}$ is equal to 0 when $a_{k+2} = b_{k+2} + 1$. More intriguing is the case where $a_{k+2} = b_{k+2}$. Then, we have $b_{k+1} = 0$, thus $r_k + q_{k+1} = r_{k+1} + q_k$ and

$$c_{k+1} = b^{r_{k+1}+q_k} \frac{b^{-q_{k+1}} - 1}{b^{q_{k+1}} - 1} = \frac{b^{r_k} - b^{r_k+q_{k+1}}}{b^{q_{k+1}} - 1} = -b^{r_k} = -e_k - 1 \tag{2.1}$$

is a negative integer. Keeping this in mind, and with some abuse of language, the next theorem asserts that

$$[0; c_0, d_0, 1, e_0, f_0, c_1, d_1, 1, e_1, f_1, c_2, \ldots]$$

is an (improper) continued fraction expansion of $\xi$. The precise statement is as follows.

THEOREM 2.3. — *Let $\xi$ be as above and keep the notation introduced above. If $a_k - b_k \geqslant 2$ and $b_k \geqslant 1$ for every $k \geqslant 1$, then the continued fraction expansion of $\xi$ is given by*

$$\xi_b(\theta, \rho) = [0; c_0 + 1, e_0, f_0, c_1, d_1, 1, e_1, f_1, c_2, \ldots].$$

*Otherwise, let $A_1, A_2, A_3, \ldots$ be the sequence of positive integers obtained from the sequence $c_0, d_0, 1, e_0, f_0, c_1, d_1, 1, e_1, f_1, c_2, \ldots$ after the application of the following rules:*

(i) *For every $k$ such that $c_{k+1} < 0$, replace the nine integers $c_k, d_k, 1, e_k, f_k, c_{k+1}, d_{k+1}, 1, e_{k+1}$ by the positive integer $c_k + 1 + e_{k+1}$;*
(ii) *Replace any three consecutive elements of this new sequence of the form $x, 0, y$ by the integer $x + y$.*

*Then, the continued fraction expansion of $\xi$ is given by*

$$\xi_b(\theta, \rho) = [0; A_1, A_2, A_3, \ldots].$$

Observe that the sequence $(A_j)_{j \geqslant 1}$ is well-defined. Indeed, $c_k$ and $c_{k+1}$ cannot be both negative, since we cannot have simultaneously $a_{k+2} = b_{k+2}$ and $a_{k+1} = b_{k+1}$.

Let us briefly show that Theorem 2.3 includes Böhmer's result. First, note that $\xi_b(\theta) = \xi_b(\theta, \theta)$, since, for a positive integer $j$, we have $\lfloor j/\theta \rfloor$ equals the integer $\ell$ if and only if $\ell < j/\theta < \ell + 1$, that is, if and only if, $\lfloor (\ell+1)\theta \rfloor - \lfloor \ell\theta \rfloor = 1$. Then, observe that the Ostrowski expansion of $\theta - \theta = 0$ in base $\theta$ is given by the constant sequence equal to 0. Consequently, the sequences defined in Theorem 2.3 are equal to

$$d_k = f_k = 0, \quad e_k = b^{q_k} - 1, \quad c_k = b^{q_k + q_{k-1}} \frac{b^{(a_{k+1}-1)q_k} - 1}{b^{q_k} - 1}, \quad k \geqslant 0.$$

It then follows from Theorem 2.3 that

$$\begin{aligned}
\xi_b(\theta) = &\Big[0; b^{q_0+0}\,\frac{b^{(a_1-1)q_0}-1}{b^{q_0}-1}, 0, 1, b^{q_0}-1, 0,\\
&\qquad\qquad b^{q_1+q_0}\,\frac{b^{(a_2-1)q_1}-1}{b^{q_1}-1}, 0, 1, b^{q_1}-1, 0, c_2, \dots\Big]\\
&= \Big[0; b^{q_0+0}\,\frac{b^{(a_1-1)q_0}-1}{b^{q_0}-1}+1,\\
&\qquad\qquad b^{q_0}-1+b^{q_1+q_0}\,\frac{b^{(a_2-1)q_1}-1}{b^{q_1}-1}+1, b^{q_1}-1, 0, c_2, \dots\Big]\\
&= \Big[0; \frac{b^{a_1q_0}-1}{b^{q_0}-1}, \frac{b^{q_2}-b^{q_0}}{b^{q_1}-1}, b^{q_1}-1, 0, c_2, \dots\Big]\\
&= \Big[0; \frac{b^{q_1}-1}{b^{q_0}-1}, \frac{b^{q_2}-b^{q_0}}{b^{q_1}-1}, \frac{b^{q_3}-b^{q_1}}{b^{q_2}-1}, \dots\Big].
\end{aligned}$$

We get the sequence of partial quotients $c_0+1, e_0+c_1+1, e_1+c_2+1, \dots$ and we recover Theorem 2.2.

Theorem 2.3 is proved in Section 7, where we give additional informations on the shape of the convergents to $\xi$ and its partial quotients; see Proposition 7.2.

As a consequence of Theorem 2.3, we obtain an expression for the irrationality exponent of any Sturmian number in terms of its slope and its intercept.

Keep our notation and define

$$\nu_k(1) = 2 + \frac{t_k}{r_{k+1}}, \qquad \nu_k(2) = 2 + \frac{r_k}{r_{k+1}+t_k},$$
$$\nu_k(3) = 1 + \frac{q_{k+1}}{r_{k+1}+q_k}, \qquad \nu_k(4) = 1 + \frac{r_{k+2}}{q_{k+1}}.$$

Put

$$\nu(1) = \limsup_{k\to+\infty}\,\{\nu_k(1) \,:\, a_{k+1}-b_{k+1} \geqslant 1 \text{ and } a_{k+2}-b_{k+2} \geqslant 1\},$$
$$\nu(2) = \limsup_{k\to+\infty}\,\{\nu_k(2) \,:\, a_{k+2}-b_{k+2} \geqslant 1\},$$

and, for $j = 3, 4$,

$$\nu(j) = \limsup_{k\to+\infty} \nu_k(j).$$

Theorem 2.4. — *Let $\xi$ be as above. Then, its irrationality exponent is equal to*

$$\max\{\nu(1), \nu(2), \nu(3), \nu(4)\}.$$

We recover, for the initial repetitions, the formulas found in [10] for the critical initial exponent, namely the contributions of $\nu(2)$ and $\nu(4)$. Theorem 2.4 is established at the end of Section 6; see Theorem 6.4.

Furthermore, we derive easily a necessary and sufficient condition under which a Sturmian number is a Liouville number, thereby reproving the first part of [4, Théorème 3.1] (see also [23, 27]).

Corollary 2.5. — *A Sturmian number is a Liouville number if and only if its slope has unbounded partial quotients in its continued fraction expansion.*

Theorem 2.4 allows us to study in depth the irrationality exponents of Sturmian numbers. For instance, we can fix a slope $\theta$ and consider the spectrum $\mathcal{L}(\theta)$ consisting of the set the irrationality exponents of Sturmian numbers of slope $\theta$. Recall that [14, Theorem 4.3] asserts that the irrationality exponent of a Sturmian number is at least equal to $\frac{5}{3} + \frac{4\sqrt{10}}{15}$.

Theorem 2.6. — *Let $\theta$ be an irrational number in $(0,1)$ with bounded partial quotients. Then,*

$$\mathcal{L}(\theta) \subset \left[\frac{5}{3} + \frac{4\sqrt{10}}{15}, 1 + \mu(\xi_b(\theta))\right]$$

*and there exists an intercept $\rho(\theta)$ such that*

$$\mu(\xi_b(\theta, \rho(\theta))) = 1 + \mu(\xi_b(\theta)).$$

A detailed study of the sets $\mathcal{L}(\theta)$ will be the purpose of a forthcoming work.

Theorem 2.3 allows us also to improve the best known transcendence measures for Sturmian numbers. Let $\zeta$ be a transcendental real number. Following Koksma [22], for any integer $d \geqslant 1$, we let $w_d^*(\zeta)$ denote the supremum of the exponents $w$ for which

$$0 < |\zeta - \alpha| < H(\alpha)^{-w-1}$$

has infinitely many solutions in real algebraic numbers $\alpha$ of degree at most $d$. Here, $H(\alpha)$ stands for the naïve height of the minimal defining polynomial of $\alpha$ over $\mathbb{Z}$. Clearly, the functions $\mu - 1$ and $w_1^*$ are equal and the functions $w_d^*$ are invariant by rational translation and by multiplication by a nonzero rational number, for $d \geqslant 1$. We direct the reader to [12] for classical results on the functions $w_d^*$ and on Mahler's and Koksma's classifications of real numbers. As a particular case of [4, Théorème 1.1], we know that, for any Sturmian number $\xi$ which is not a Liouville number, there exists a positive real number $c$, depending only on $\xi$, such that

$$w_d^*(\xi) \leqslant (2d)^{c(\log 3d)(\log\log 3d)}, \quad d \geqslant 1.$$

This can be improved as follows.

Theorem 2.7. — *Let $\xi$ be a Sturmian number. Assume that the partial quotients of its slope are ultimately bounded from above by $M$. Then, there exists a positive real number $\kappa$, depending only on $M$, such that*

$$w_d^*(\xi) \leqslant (2d)^{\kappa(\log\log 3d)}, \quad d \geqslant 1.$$

We point out that the transcendence measure obtained in Theorem 2.7 does not depend on the intercept of the Sturmian number.

We believe that Theorem 2.1 will have many applications. We use it in a follow-up work [16] devoted to the transcendence of Hecke–Mahler series evaluated at algebraic points. We refer to [15, 24, 25] for various applications of Sturmian numbers to the dynamics of piecewise affine maps.

The present paper is organized as follows. We show in Section 3 that any Sturmian word $\mathbf{s}$ of slope $\theta$ and intercept $\rho$ can be expressed in a way similar to (1.1) and we define its formal intercept. The link between the formal intercept and the expansion of the intercept $\rho$ in the $\theta$-Ostrowski numeration system is established in Section 4, thereby proving Theorem 2.1. In Section 5, we apply Theorem 2.1 to give a precise description of the repetitions occurring near the beginning of $\mathbf{s}$. From this, in the next section, we deduce four one-parametric families of rational numbers which approximate very well the Sturmian number $\xi$ associated to $\mathbf{s}$, the exact rate of approximation to $\xi$ by these rational numbers being given in Proposition 6.1. We derive the continued fraction expansion of $\xi$ in Section 7, thereby proving Theorem 2.4 and Corollary 2.5, since we see that all the very good approximants to $\xi$ belong to one of the four families defined in Section 6. The final Section is devoted to the proofs of the other results stated in Section 2.

## 3. The formal intercept of a Sturmian word

We keep the notation of Section 1 with the alphabet $\{0,1\}$. Let $\mathbf{s}$ be an arbitrary Sturmian word of slope $\theta$. The goal of this section is to establish that any Sturmian word can be expressed as in (1.1), that is, as the limit of a suitable sequence $(V_k)_{k\geqslant 1}$ of binary words $V_k$ of length $q_k$ constructed inductively.

Throughout, the length $|W|$ of a finite word $W$, that is, the number of letters composing $W$, is denoted by $|W|$. If $W$ has at least one letter (resp., at least two letters), then we let $W^-$ (resp., $W^{--}$) denote the work $W$ deprived of its last letter (resp., its last two letters).

Definition 3.1. — *A word $V$ is a* conjugate *of $M_k$ if there exist words $T$ and $R$ such that*

$$V = RT \quad \text{and} \quad M_k = TR,$$

*with $0 \leqslant t := |T| < q_k$. Then, $R$ is the non-empty suffix of $M_k$ of length $q_k - t$.*

Observe that the $q_k$ conjugates $V$ of the word $M_k$ are distinct. We label these translated words $V$ by the length $t, 0 \leqslant t < q_k$ of the (possibly empty) prefix $T$ in the decomposition $M_k = TR, V = RT$. The whole set of conjugates $V$ of $M_k$ is clearly obtained as the set of factors of length $q_k$ in the word $M_k M_k^-$. Each such factor $V$ is determined by its $q_k - 1$ first letters which form the $q_k$ distinct factors of length $q_k - 1$ contained in the word $M_k M_k^{--}$.

As an example, for $k = 1$, we have $M_1 = 0^{a_1-1}1$. Any conjugate $V$ of $M_1$ can be written in the form

$$V = 0^{a_1-1-b_1}10^{b_1} = RT, \quad M_1 = TR, \quad \text{with} \quad T = 0^{b_1}, \quad R = 0^{a_1-1-b_1}1,$$

for some integer $b_1$ with $0 \leqslant b_1 \leqslant a_1 - 1$. Thus, in this case, we have $t = b_1$.

Definition 3.2. — *For each $k \geqslant 1$, let $V_k$ be the conjugate of $M_k$ whose first $q_k - 1$ letters coincide with those of $\mathbf{s}$. Let $T_k$ and $R_k$ be the words such that*

$$V_k = R_k T_k \quad \text{and} \quad M_k = T_k R_k,$$

*with $R_k$ non-empty. Let $t_k$ denote the length of $T_k$. Put $R_0 = 0$, and let $T_0$ be the empty word.*

Then, the following recursion formulae hold. The notion of formal intercept was first introduced by Wojcik [29, 30], but our presentation is different.

Lemma 3.3 (formal intercept). — *Put $t_1 = b_1^*$. For any $k \geqslant 1$, there exists an integer $b_{k+1}^*$ such that $0 \leqslant b_{k+1}^* \leqslant a_{k+1}$ and*

$$t_{k+1} = t_k + b_{k+1}^* q_k.$$

*When $b_{k+1}^* = a_{k+1}$, we necessarily have $t_k < q_{k-1}$, so that $b_k^* = 0$ and $t_k = t_{k-1}$ in this case. Moreover, the sequences of words $(T_k)_{k\geqslant 0}$ and $(R_k)_{k\geqslant 0}$ satisfy the recursion formulae*

$$T_{k+1} = M_k^{b_{k+1}^*} T_k = T_k V_k^{b_{k+1}^*}$$

*and we have*

$$R_1 = 0^{a_1-b_1-1}1 \quad \textit{and} \quad R_{k+1} = \begin{cases} R_k M_k^{a_{k+1}-b^*_{k+1}-1} M_{k-1} & \textit{if } b^*_{k+1} < a_{k+1}, \\ R_{k-1} & \textit{if } b^*_{k+1} = a_{k+1}, \end{cases}$$

*for $k \geqslant 1$. The sequence $(b^*_k)_{k\geqslant 1}$ is called the formal intercept of* $\mathbf{s}$.

*Proof.* — The word $V_{k+1}$ is a factor of the word $M_{k+1}M^-_{k+1}$ beginning somewhere on the first factor $M_{k+1}$. Assume first that $V_{k+1}$ begins on the prefix $M_k^{a_{k+1}}$ of $M_{k+1} = M_k^{a_{k+1}} M_{k-1}$ and let $P$ be the prefix of length $q_k$ of $V_{k+1}$. Thus, for some integer $0 \leqslant b^*_{k+1} < a_{k+1}$, the prefix $P$ begins on the second factor $M_k$ in the product $M_k^{a_{k+1}} = M_k^{b^*_{k+1}} M_k M_k^{a_{k+1}-b^*_{k+1}-1}$. Then, $P$ is a factor of

$$M_k M_k^{a_{k+1}-b^*_{k+1}-1} M^-_{k+1} = M_k M_k^{2a_{k+1}-b^*_{k+1}-1} M^-_{k-1}$$

beginning on the first factor $M_k$. Since $2a_{k+1} - b^*_{k+1} - 1 \geqslant 1$, we see that $P$ is located over the product $M_k M_k^-$, where $M_k^-$ is the prefix of $M_k^{2a_{k+1}-b^*_{k+1}-1}$ of length $q_k - 1$. As the first $q_k - 1$ letters of $P$ coincide with those of $\mathbf{s}$, we deduce that $P = V_k = R_k T_k$, and next that

$$T_{k+1} = M_k^{b^*_{k+1}} T_k \quad \text{and} \quad R_{k+1} = R_k M_k^{a_{k+1}-b^*_{k+1}-1} M_{k-1}.$$

Note finally that

$$M_k^{b^*_{k+1}} T_k = (T_k R_k)^{b^*_{k+1}} T_k = T_k (R_k T_k)^{b^*_{k+1}} = T_k V_k^{b^*_{k+1}}.$$

Suppose now that $V_{k+1}$ begins on the second factor $M_{k-1}$ in

$$M_{k+1} M^-_{k+1} = M_k^{a_{k+1}} M_{k-1} M^-_{k+1} = M_k^{a_{k+1}} T_{k-1} R_{k-1} M^-_{k+1}$$

and put $b^*_{k+1} = a_{k+1}$. Then,

$$T_{k+1} = M_k^{a_{k+1}} T_{k-1} \quad \text{and} \quad R_{k+1} = R_{k-1},$$

observing that $V_{k-1} = R_{k-1} T_{k-1}$ equals the prefix of $V_{k+1}$ of length $q_{k-1}$. Notice now that $M_k$ is a prefix of $M_{k-1} M^-_{k+1}$. Writing

$$M_{k-1} M^-_{k+1} = M_k \cdots = T_{k-1} R_{k-1} M_{k-1}^{a_k - 1} M_{k-2} \cdots$$

we see that $T_k = T_{k-1}$ and $R_k = R_{k-1} M_{k-1}^{a_k-1} M_{k-2}$. Thus $b^*_k = 0$ by the preceding case applied to the level $k - 1$. □

We now deal with binary recursions expressing $V_{k+1}$ in terms of $V_k$ and $V_{k-1}$ extending the classical formulae $M_{k+1} = M_k^{a_{k+1}} M_{k-1}$. Set $V_0 = R_0 T_0 = 0$.

Lemma 3.4 (binary recursion). — *We have the relation*

$$V_1 = 0^{a_1-1-b_1^*}10^{b_1^*}$$

*and, for any $k \geqslant 1$, we have*

$$V_{k+1} = V_k^{a_{k+1}-b_{k+1}^*}V_{k-1}V_k^{b_{k+1}^*}.$$

*Proof.* — The formula $V_1 = 0^{a_1-1-b_1^*}10^{b_1^*}$ has already been verified. For $k \geqslant 1$, we distinguish two cases, either $b_{k+1}^* < a_{k+1}$ or $b_{k+1}^* = a_{k+1}$. Assume first that $b_{k+1}^* < a_{k+1}$. According to Lemma 3.3, we write $V_{k+1} = R_{k+1}T_{k+1}$ with

$$\begin{aligned} R_{k+1} &= R_kM_k^{a_{k+1}-b_{k+1}^*-1}M_{k-1} = R_k(T_kR_k)^{a_{k+1}-b_{k+1}^*-1}T_{k-1}R_{k-1} \\ &= (R_kT_k)^{a_{k+1}-b_{k+1}^*-1}R_kT_{k-1}R_{k-1} = V_k^{a_{k+1}-b_{k+1}^*-1}R_kT_{k-1}R_{k-1} \end{aligned}$$

and

$$T_{k+1} = T_kV_k^{b_{k+1}^*}.$$

Thus

$$V_{k+1} = V_k^{a_{k+1}-b_{k+1}^*-1}R_kT_{k-1}R_{k-1}T_kV_k^{b_{k+1}^*}.$$

Since

$$\begin{aligned} R_kT_{k-1}R_{k-1}T_k &= R_kT_{k-1}R_{k-1}T_{k-1}(R_{k-1}T_{k-1})^{b_k^*} \\ &= R_kT_kR_{k-1}T_{k-1} = V_kV_{k-1}, \end{aligned}$$

we get

$$V_{k+1} = V_k^{a_{k+1}-b_{k+1}^*}V_{k-1}V_k^{b_{k+1}^*}.$$

Assume now that $b_{k+1}^* = a_{k+1}$. Then $b_k^* = 0$. From Lemma 3.3, we know that $T_k = T_{k-1}$ and that

$$T_{k+1} = T_kV_k^{b_{k+1}^*} = T_{k-1}V_k^{b_{k+1}^*} \quad \text{and} \quad R_{k+1} = R_{k-1}.$$

Thus

$$V_{k+1} = R_{k+1}T_{k+1} = R_{k-1}T_{k-1}V_k^{b_{k+1}^*} = V_{k-1}V_k^{b_{k+1}^*},$$

as asserted. □

We conclude this section with a corollary, which shows how any prefix of $M_{n+1}$ can be expressed in terms of $M_0, \ldots, M_n$.

Recall that the Ostrowski numeration system in base $\theta$ is defined as follows: every positive integer $N$ can be uniquely written in the form

$$N = d_1 + d_2q_1 + \cdots + d_{r+1}q_r,$$

where $0 \leqslant d_j \leqslant a_j$ for $j = 1, \ldots, r+1$, $d_{r+1} > 0$, $d_1 < a_1$ and $d_j = 0$ if $d_{j+1} = a_{j+1}$.

Corollary 3.5 (product formula for prefixes). — *Let $T$ be the prefix of $M_{n+1}$ of length $t < q_{n+1}$. Write*

$$t = d_1 + d_2 q_1 + \cdots + d_{n+1} q_n$$

*where $d_1, \ldots, d_{n+1}$ are the digits of the integer $t$ in the Ostrowski numeration system in base $\theta$. Then, we have the product formula*

$$T = M_n^{d_{n+1}} M_{n-1}^{d_n} \cdots M_0^{d_1} = V_0^{d_1} V_1^{d_2} \cdots V_n^{d_{n+1}},$$

*where the words $V_0, \ldots, V_n$ are defined recursively by the formulae*

$$V_0 = 1,\ V_1 = 0^{a_1 - d_1 - 1} 1 0^{d_1},\ V_{k+1} = V_k^{a_{k+1} - d_{k+1}} V_{k-1} V_k^{d_k},\ 1 \leqslant k < n.$$

*Proof.* — By Lemma 3.3, we have $T = T_{n+1}$ and $t = t_{n+1}$. The recurrence relations

$$T_{k+1} = M_k^{d_{k+1}} T_k = T_k V_k^{d_{k+1}}$$

yield inductively the product formula

$$T = T_{n+1} = M_n^{d_{n+1}} M_{n-1}^{d_n} \cdots M_0^{d_1} = V_0^{d_1} V_1^{d_2} \cdots V_n^{d_{n+1}}.$$

This establishes the corollary. □

## 4. Linking formal intercept and Ostrowski numeration

We link the formal intercept, that is the sequence $(b_k^*)_{k \geqslant 1}$ such that

$$t_k = b_1^* + b_2^* q_1 + \cdots + b_k^* q_{k-1}, \quad k \geqslant 1,$$

to the intercept $\rho$ thanks to the

Proposition 4.1. — *Let $0 < \rho < 1$ be a real number either not belonging to $\mathbb{Z}\theta + \mathbb{Z}$, or of the form $\mathbb{Z}_{\geqslant 1}\theta + \mathbb{Z}$. Let*

$$\rho - \theta = \sum_{h \geqslant 1} b_h \theta_{h-1}$$

*be the Ostrowski expansion of $\rho - \theta$ in base $\theta$. For every $k \geqslant 1$, put*

$$t_k = b_1 + b_2 q_1 + \cdots + b_k q_{k-1}.$$

*Then, $t_k$ is the length of the word $T_k$ associated to the Sturmian word $\mathbf{s}_{\theta,\rho} = \mathbf{s}'_{\theta,\rho}$. In other words, we have $b_k = b_k^*$ for $k \geqslant 1$, meaning that the formal intercept of this Sturmian word coincides with the sequence of digits of the number $\rho - \theta$ in its Ostrowski expansion in base $\theta$.*

*Proof.* — By definition, we have

$$s_n = \lfloor n\theta + \rho \rfloor - \lfloor (n-1)\theta + \rho \rfloor, \quad n \geqslant 1,$$

and

$$s'_n = \lceil n\theta + \rho \rceil - \lceil (n-1)\theta + \rho \rceil, \quad n \geqslant 1,$$

while the $n$-th letter of $\mathbf{c}_\theta$ is

$$c_n = \lfloor (n+1)\theta \rfloor - \lfloor n\theta \rfloor = \lceil (n+1)\theta \rceil - \lceil n\theta \rceil, \quad n \geqslant 1.$$

Thus

$$s_n = \lfloor (n+1)\theta + \rho - \theta \rfloor - \lfloor n\theta + \rho - \theta \rfloor = \lfloor (n+1+t_k)\theta + \sigma_k \rfloor - \lfloor (n+t_k)\theta + \sigma_k \rfloor,$$

and

$$s'_n = \lceil (n+1)\theta + \rho - \theta \rceil - \lceil n\theta + \rho - \theta \rceil = \lceil (n+1+t_k)\theta + \sigma_k \rceil - \lceil (n+t_k)\theta + \sigma_k \rceil,$$

where we have set

$$\sigma_k = \sum_{h \geqslant k} b_{h+1}\theta_h.$$

We claim that

$$\lfloor q\theta + \sigma_k \rfloor = \lfloor q\theta \rfloor \quad \text{and} \quad \lceil q\theta + \sigma_k \rceil = \lceil q\theta \rceil$$

for every integer $q$ with $1 \leqslant q \leqslant q_k + t_k$. This yields that

$$s_n = \lfloor (n+1+t_k)\theta + \sigma_k \rfloor - \lfloor (n+t_k)\theta + \sigma_k \rfloor = \lfloor (n+1+t_k)\theta \rfloor - \lfloor (n+t_k)\theta \rfloor = c_{n+t_k}$$

and

$$s'_n = \lceil (n+1+t_k)\theta + \sigma_k \rceil - \lceil (n+t_k)\theta + \sigma_k \rceil = \lceil (n+1+t_k)\theta \rceil - \lceil (n+t_k)\theta \rceil = c_{n+t_k}$$

for every $1 \leqslant n \leqslant q_k - 1$, and will establish the proposition, noting that $M_k M_k$ is a prefix of $\mathbf{c}_\theta$.

To that purpose, we bound $|\sigma_k|$. Observe that $\theta_k$ is positive when $k$ is even and negative when $k$ is odd. Moreover $b_{h+1} \leqslant a_{h+1}$ for any $h \geqslant 1$, while $b_1 \leqslant a_1 - 1$. Thus,

$$\begin{aligned} |\sigma_k| &< \max(|a_{k+1}\theta_k + a_{k+3}\theta_{k+2} + \cdots|, |a_{k+2}\theta_{k+1} + a_{k+4}\theta_{k+3} + \cdots|) \\ &= \max(|\theta_{k-1}|, |\theta_k|) = |\theta_{k-1}|, \end{aligned}$$

noting that

$$\begin{aligned}
&a_{k+1}\theta_k + a_{k+3}\theta_{k+2} + \cdots \\
&\qquad = \lim_{n\to\infty} \left( \left( \sum_{h=0}^{n} a_{k+2h+1} q_{k+2h} \right) \theta - \left( \sum_{h=0}^{n} a_{k+2h+1} p_{k+2h} \right) \right) \\
&\qquad = \lim_{n\to\infty} \left( \sum_{h=0}^{n} (q_{k+2h+1} - q_{k+2h-1})\theta - \sum_{h=0}^{n} (p_{k+2h+1} - p_{k+2h-1}) \right) \\
&\qquad = \lim_{n\to\infty} \Big( (q_{k+2n+1} - q_{k-1})\theta - (p_{k+2n+1} - p_{k-1}) \Big) = -\theta_{k-1}.
\end{aligned}$$

The inequality $|\sigma_k| < |\theta_{k-1}|$ is strict because either $\rho-\theta$ does not belong to $\mathbb{Z}\theta+\mathbb{Z}$, or $\rho-\theta$ belong to $\mathbb{Z}_{\geqslant 0}\theta+\mathbb{Z}$, so that the sequence of digits $(b_h)_{h\geqslant 1}$ cannot be ultimately of the form $a_{k+1}, 0, a_{k+3}, 0 \ldots$.

Observe now that

$$|\sigma_k - b_{k+1}\theta_k| < a_{k+2}|\theta_{k+1}| + a_{k+4}|\theta_{k+3}| + \cdots = |\theta_k|.$$

It follows that $\sigma_k$ and $\theta_k$ share the same sign when $b_{k+1} \geqslant 1$ and that $|\sigma_k| < |\theta_k|$ when $b_{k+1} = 0$. In particular, the stronger inequality $|\sigma_k| < |\theta_k|$ holds when $\theta_k$ and $\sigma_k$ have opposite signs.

The upper bound $|\sigma_k| < |\theta_{k-1}|$ can also be sharpened when $t_k \geqslant q_{k-1}$. Indeed in this case we have $b_k \geqslant 1$ and thus $b_{k+1}$ cannot be equal to $a_{k+1}$ by Ostrowski's numeration rules. We now bound $b_{k+1} \leqslant a_{k+1} - 1$ to obtain

$$|\sigma_k| < |\theta_{k-1}| - |\theta_k|. \tag{4.1}$$

Let $\|x\|$ denote the distance from the real number $x$ to the closest integer. We now show that $\|q\theta\|$ is larger than $|\sigma_k|$ when $q$ differs from $q_k$, so that $q\theta$ and $q\theta + \sigma_k$ belong to the same integer open interval of length 1 and have thus the same upper and lower integer parts. We distinguish three cases. If $q < q_k$, then

$$\|q\theta\| \geqslant |\theta_{k-1}| > |\sigma_k|,$$

as required. Assume secondly that $t_k < q_{k-1}$ and $q = q_k + v$ for some $1 \leqslant v \leqslant t_k$. Then

$$\|q\theta\| = \|v\theta+\theta_k\| \geqslant \|v\theta\| - |\theta_k| \geqslant |\theta_{k-2}| - |\theta_k| = |\theta_{k-2} - \theta_k| \geqslant |\theta_{k-1}| > |\sigma_k|.$$

Thirdly, assume $t_k \geqslant q_{k-1}$ and $q = q_k + v$ with $1 \leqslant v \leqslant q_k - 1$. Then,

$$\|q\theta\| \geqslant \|v\theta\| - |\theta_k| \geqslant |\theta_{k-1}| - |\theta_k| > |\sigma_k|,$$

by (4.1).

These three cases cover all the values of $q$ with $1 \leqslant q \leqslant q_k + t_k$, except $q = q_k$, which we consider now. We have $\|q_k\theta\| = |\theta_k|$. When $\theta_k$ and $\sigma_k$ share the same sign, we have

$$|\theta_k| < |\theta_k + \sigma_k| = |\theta_k| + |\sigma_k| \leqslant |\theta_k| + |\theta_{k-1}| < 1.$$

Thus, $q_k\theta$ and $q_k\theta + \sigma_k$ both belong either to $(p_k, p_k+1)$ or to $(p_k - 1, p_k)$. When $\theta_k$ and $\sigma_k$ have opposite signs, we know that $|\sigma_k| < |\theta_k|$, so that $\theta_k$ and $\theta_k + \sigma_k$ have the same sign and both have absolute value less than 1. The claim is proved, which yields the proposition. □

A similar result holds in the remaining case where $\rho - \theta = -m\theta + p$ for integers $m \geqslant 1$ and $p$. Assume first that $\rho$ is positive, that is to say $m \geqslant 2$. Let $l \geqslant 0$ be defined by the inequalities $q_l < m \leqslant q_{l+1}$ and let

$$q_{l+1} - m = b_1 q_0 + \cdots + b_{l+1} q_l$$

be the Ostrowski expansion of the integer $q_{l+1} - m$ (see the definition at the end of Section 3). Observe that

$$b_{l+1} \leqslant a_{l+1} - 1 \quad \text{and} \quad \text{that} \quad b_l = 0 \quad \text{when} \quad b_{l+1} = a_{l+1} - 1.$$

Then $\rho - \theta \in (-\theta, 1-\theta)$ has two Ostrowski expansions of the form

$$\rho - \theta = b_1\theta_0 + \cdots + b_{l+1}\theta_l + \sum_{k \geqslant 1} a_{l+2k+1}\theta_{l+2k}$$

and

$$\rho - \theta = b_1\theta_0 + \cdots + b_l\theta_{l-1} + (b_{l+1}+1)\theta_l + (a_{l+2}-1)\theta_{l+1} + \sum_{k\geqslant 2} a_{l+2k}\theta_{l+2k-1},$$

when $l \geqslant 1$, or

$$\rho - \theta = (b_1+1)\theta_0 + (a_2 - 1)\theta_1 + \sum_{k \geqslant 2} a_{2k}\theta_{2k-1},$$

when $l = 0$. Set

$$b_{l+2} = 0, b_{l+3} = a_{l+3}, b_{l+4} = 0, b_{l+5} = a_{l+5}, \ldots$$

and

$$b'_1 = b_1, \ldots, b'_l = b_l, b'_{l+1} = b_{l+1}+1, b'_{l+2} = a_{l+2}-1, b'_{l+3} = 0, b'_{l+4} = a_{l+4}, \ldots$$

when $l \geqslant 1$, or

$$b'_1 = b_1 + 1, b'_2 = a_2 - 1, b'_3 = 0, b'_4 = a_4, \ldots$$

when $l = 0$, so that $(b_k)_{k\geqslant 1}$ and $(b'_k)_{k \geqslant 1}$ are the sequences of digits appearing in the two above expansions of $\rho - \theta$. Notice that both sequences satisfy the Ostrowski numeration rules for digits in base $\theta$. When $\rho = 0$, we use the two proper expansions

$$1 - \theta = (a_1 - 1)\theta_0 + \sum_{k \geqslant 1} a_{2k+1}\theta_{2k},$$

and

$$-\theta = \sum_{k \geqslant 1} a_{2k}\theta_{2k-1},$$

to define respectively the sequences of digits $(b_k)_{k\geqslant 1}$ and $(b'_k)_{k\geqslant 1}$. Then, we have the following analogue of Theorem 2.1.

Theorem 4.2. — *Assume that $\rho - \theta = -m\theta + p$ where $m \geqslant 1$ and $p$ are integers. When $m \geqslant 2$, let $l \geqslant 0$ be defined by the inequalities $q_l < m \leqslant q_{l+1}$. When $m = 1$, set $l = 0$. Let $(V_k)_{k\geqslant 0}$ and $(V'_k)_{k\geqslant 0}$ be the two sequences of words recursively defined as in Theorem 2.1, with respect to the two sequences of digits $(b_k)_{k\geqslant 1}$ and $(b'_k)_{k\geqslant 1}$ defined above. When $l$ is odd, we have*

$$\mathbf{s}_{\theta,\rho} = \lim_{k\to+\infty} V_k \quad \text{and} \quad \mathbf{s}'_{\theta,\rho} = \lim_{k\to+\infty} V'_k.$$

*When $l$ is even, we have*

$$\mathbf{s}_{\theta,\rho} = \lim_{k\to+\infty} V'_k \quad \text{and} \quad \mathbf{s}'_{\theta,\rho} = \lim_{k\to+\infty} V_k.$$

*Moreover, the analogous decompositions $V_k = R_k T_k$ and $V'_k = R'_k T'_k$, as in Theorem 2.1, hold true with*

$$t_k = b_1 + \cdots + b_k q_{k-1} \quad \text{and} \quad t'_k = b'_1 + \cdots + b'_k q_{k-1}.$$

*Proof.* — We only give a complete proof for the sequence of digits

$$(b_k)_{k\geqslant 1} = \Big\{ b_1, \ldots, b_{l+1}, 0, a_{l+3}, 0, a_{l+5}, \ldots \Big\}.$$

Assume that $m \geqslant 2$ and $l$ is odd, and recall the notations

$$t_k = b_1 + \cdots + b_k q_{k-1} \quad \text{and} \quad \sigma_k = b_{k+1}\theta_k + b_{k+2}\theta_{k+2} + \cdots$$

The argumentation is similar to the proof of Proposition 4.1. It suffices to show that

$$\lfloor q\theta + \sigma_k \rfloor = \lfloor q\theta \rfloor \tag{4.2}$$

for every integer $q$ with $1 \leqslant q \leqslant q_k + t_k$. If $k \leqslant l$, we compute

$$\sigma_k = b_{k+1}\theta_k + \cdots + b_{l+1}\theta_{l+1} - \theta_{l+1}.$$

Since the tail $b_{k+1}, b_{k+2}\dots$ of the sequence $(b_k)_{k\geqslant 1}$ contains the subsequence $\dots b_{l+1}, 0, a_{l+3}, \dots$ and that $b_{l+1} \leqslant a_{l+1}-1$, observe that this tail is neither of the form $a_j, 0, a_{j+1}, 0, \dots$ nor $0, a_j, 0, a_{j+1} \dots$. Then, (4.2) holds true by taking again the proof of Proposition 4.1. When $k \geqslant l+1$, we have

$$t_k = \begin{cases} -m + q_{k-1} & \text{if } k = l+2j, \quad (j \geqslant 1), \\ -m + q_k & \text{if } k = l+2j+1, \quad (j \geqslant 0), \end{cases}$$

and

$$\sigma_k = \begin{cases} -\theta_{k-1} & \text{if } k = l+2j, \quad (j \geqslant 1), \\ -\theta_k & \text{if } k = l+2j+1, \quad (j \geqslant 0). \end{cases}$$

Assume first that $k$ has the same parity as $l$, namely $k = l+2j$ for some $j \geqslant 1$. Then $\sigma_k = -\theta_{k-1}$ and $t_k = -m + q_{k-1}$. In order to check (4.2), we distinguish three subcases. Assume first $q \leqslant q_k - 1$. Then $\|q\theta\| \geqslant |\theta_{k-1}|$ with equality only when $q = q_{k-1}$. If $q \neq q_{k-1}$, then we have

$$\|q\theta\| > |\theta_{k-1}|,$$

so that $q\theta$ and $q\theta - \theta_{k-1}$ are located in the same open interval of length one, so that (4.2) holds true. If $q = q_{k-1}$, then we have

$$q_{k-1}\theta - \theta_{k-1} = p_{k-1} \quad \text{and} \quad q_{k-1}\theta = p_{k-1} + \theta_{k-1},$$

so that (4.2) holds, since $\theta_{k-1}$ is positive, noting that $k-1 = l-1+2j$ is even. Assume secondly that $q = q_k$. Then,

$$q_k\theta - \theta_{k-1} = p_k + \theta_k - \theta_{k-1} \quad \text{and} \quad q_k\theta = p_k + \theta_k.$$

This shows that (4.2) holds, since both numbers $\theta_k$ and $\theta_k - \theta_{k-1}$ are negative with absolute value less than 1. Assume thirdly that $q = q_k + v$ for some integer $v$ with $1 \leqslant v \leqslant t_k = -m + q_{k-1}$. Then,

$$q\theta - \theta_{k-1} = p_k + \theta_k - \theta_{k-1} + v\theta \quad \text{and} \quad q\theta = p_k + \theta_k + v\theta.$$

Notice now that $\|v\theta\| \geqslant |\theta_{k-2}|$ with equality only when $v = q_{k-2}$. If $v \neq q_{k-2}$, then we have

$$\|v\theta\| > |\theta_{k-2}|.$$

Then, $q\theta$ and $q\theta - \theta_{k-1}$ are located in the same open interval of length one, since

$$|\theta_{k-2}| \geqslant |\theta_k| + |\theta_{k-1}|,$$

so that (4.2) holds true. If $v = q_{k-2}$, then we have

$$q\theta - \theta_{k-1} = p_k + p_{k-2} + \theta_k - \theta_{k-1} + \theta_{k-2} \quad \text{and} \quad q\theta = p_k + p_{k-2} + \theta_k + \theta_{k-2},$$

so that (4.2) holds, since $\theta_k - \theta_{k-1} + \theta_{k-2}$ and $\theta_k + \theta_{k-2}$ are both negative with absolute value less than 1.

We assume now that $k = l + 2j + 1$ for some $j \geqslant 0$. Then $\sigma_k = -\theta_k$ and $t_k = -m + q_k$. We distinguish again three subcases. Assume first that $q \leqslant q_k - 1$. Then, $\|q\theta\| \geqslant |\theta_{k-1}|$, so that $q\theta$ and $q\theta - \theta_k$ are located in the same open interval of length one. It folllows that (4.2) holds. Assume secondly that $q = q_k$. Then,

$$q_k\theta - \theta_k = p_k \quad \text{and} \quad q_k\theta = p_k + \theta_k.$$

This shows that (4.2) holds, since $\theta_k$ is positive because $k = l + 2j + 1$ is even. Assume thirdly that $q = q_k + v$ for some integer $v$ with $1 \leqslant v \leqslant t_k = -m + q_k$. Then,

$$q\theta - \theta_k = p_k + v\theta \quad \text{and} \quad q\theta = p_k + \theta_k + v\theta.$$

Notice now that $\|v\theta\| \geqslant |\theta_{k-1}| > |\theta_k|$. Thus (4.2) holds. All cases have been checked.

When $l$ is even, the numbers $\theta_{l+2j}$ (resp. $\theta_{l+2j+1}$) turn to be positive (resp. negative), and the above argumentation remains valid provided that we replace the usual integer part $\lfloor\cdot\rfloor$ by the upper integer part $\lceil\cdot\rceil$. □

To illustrate this statement, take $\rho = 0$, $a_1 = 5$, $a_2 = 3$, $a_3 = 2$; then

$$V_0 = 0, \quad V_1 = 10^4, \quad V_2 = 10^4 10^4 10^4 0,$$
$$V_3 = 10^4 10^4 10^4 10^4 0 10^4 10^4 10^4 0 = 10^4 10^4 10^4 10^5 10^4 10^4 10^5,$$

$$V_0' = 0, \quad V_1' = 0^4 1, \quad V_2' = 0 0^4 1 0^4 1 0^4 1,$$
$$V_3' = 0 0^4 1 0^4 1 0^4 1 0 0^4 1 0^4 1 0^4 1 0^4 1 = 0^5 1 0^4 1 0^4 1 0^5 1 0^4 1 0^4 1 0^4 1.$$

Note also that

$$M_0 = 0, \quad M_1 = 0^4 1, \quad M_2 = 0^4 1 0^4 1 0^4 1 0,$$
$$M_3 = 0^4 1 0^4 1 0^4 1 0 0^4 1 0^4 1 0^4 1 0 0^4 1 = 0^4 1 0^4 1 0^4 1 0^5 1 0^4 1 0^4 1 0^5 1.$$

By induction, we check that $V_n$ is the mirror image of $V_n'$. We know that $M_n^{--}$ is a palindrome. We also have that

$$^-V_n^- = {}^-(V_n')^- = M_n^{--},$$

where $^-W$ means the word $W$ deprived of its first letter. In other words, for $n \geqslant 1$, the words $V_n$ and $V_n'$ deprived of their first and last letters are equal to the palindrome $M_n^{--}$.

## 5. Repetitions in a Sturmian word

We keep our notation. Recall that $\mathbf{s}$ denotes an arbitrary Sturmian word of slope $\theta$.

We show that Proposition 1 of [15] can be deduced from the recursion formulae for the words $V_k$ and we give further informations on the occurrence of the various cases. Proposition 5.1 will be used in the next section to compare $\mathbf{s}$ with four families of (shifted for two of them) periodic words, depending on a parameter $k$, constructing thus families of strong rational approximations to the associated Sturmian number.

Proposition 5.1. — *Let $k$ be an integer with $k \geqslant 2$. Then, there exist a uniquely determined non-empty suffix $U_k$ of $M_k M_{k+1} = (M_k)^{a_{k+1}+1} M_{k-1}$ and an integer $\widetilde{a}_{k+1}$ such that*

$$\widetilde{a}_{k+1} \in \{a_{k+1}, a_{k+1}+1\}$$

*and*

$$\mathbf{s} = U_k (M_k)^{\widetilde{a}_{k+1}} M_{k-1} M_k^- \ldots$$

*More precisely, when $a_{k+2} - b_{k+2} \geqslant 2$, we have*

$$U_k = R_{k+1} \quad \text{and} \quad \widetilde{a}_{k+1} = a_{k+1}.$$

*When $a_{k+2} - b_{k+2} = 1$, we have*

$$U_k = R_{k+1} \quad \text{and} \quad \widetilde{a}_{k+1} = \begin{cases} a_{k+1} + 1 & \text{if } \ b_{k+3} < a_{k+3}, \\ a_{k+1} & \text{if } \ b_{k+3} = a_{k+3}. \end{cases}$$

*When $a_{k+2} = b_{k+2}$, we have $U_k = R_k M_{k+1}$. Moreover $\widetilde{a}_{k+1} = a_{k+1}$, unless*

$$a_{k+2} = 1,\ a_{k+3} - b_{k+3} \geqslant 2,$$

*or*

$$a_{k+2} = 1,\ a_{k+3} = 1,\ b_{k+3} = 0,\ a_{k+4} = b_{k+4},$$

*in which cases $\widetilde{a}_{k+1} = a_{k+1} + 1$.*

*Remark 5.2.* — For $k \geqslant 3$, the fact that $\mathbf{s} = U_k (M_k)^{\widetilde{a}_{k+1}} M_{k-1} M_k^- \ldots$ means that after the prefix of length $|U_k|$, we have exactly $\widetilde{a}_{k+1} + 1$ copies of $M_k$, followed by the prefix of $M_k$ of length $q_{k-1} - 2$, since $M_{k-1} M_k^-$ and $M_k M_{k-1}^-$ differ only by their last letter. In addition, we observe that when $a_{k+2} = b_{k+2}$ we have $b_{k+1} = 0$ and we take $U_{k-1} = R_k$.

*Proof.* — The idea of the proof is to show that the prefix of $\mathbf{s}$ of length $2q_{k+1} + q_k - 1$ coincides with one of the three words $V_{k+1}^2 V_k^-$ or $V_{k+1} V_k V_{k+1}^-$ or $V_k V_{k+1} V_{k+1}^-$.

Assume first that $a_{k+2} - b_{k+2} \geqslant 2$. Then

$$V_{k+2} = V_{k+1}^{a_{k+2}-b_{k+2}} V_k V_{k+1}^{b_{k+2}} = V_{k+1}^2 V_k^- \dots,$$

observing that $V_k^-$ is a prefix of $V_{k+1}$ (this follows from Definition 3.2). But

$$\begin{aligned} V_{k+1}V_{k+1}V_k &= R_{k+1}T_{k+1}R_{k+1}T_{k+1}R_kT_k \\ &= R_{k+1}M_{k+1}T_{k+1}R_kT_k \\ &= R_{k+1}M_{k+1}M_k^{b_{k+1}}T_kR_kT_k \\ &= R_{k+1}M_{k+1}M_k^{b_{k+1}+1}T_k = R_{k+1}M_k^{a_{k+1}}M_{k-1}M_k \dots \end{aligned}$$

Assume secondly that $a_{k+2} - b_{k+2} = 1$ and that $a_{k+3} - b_{k+3} \geqslant 1$. Then

$$\begin{aligned} V_{k+3} = V_{k+2}^{a_{k+3}-b_{k+3}} V_{k+1} V_{k+2}^{b_{k+3}} &= V_{k+2}V_{k+1}^- \dots \\ &= (V_{k+1}V_kV_{k+1}^{b_{k+2}+1})^- \dots = V_{k+1}V_kV_{k+1}^- \dots \end{aligned}$$

Actually, we can be more precise and claim that $V_{k+1}V_kV_{k+1}V_k^-$ is a prefix of $\mathbf{s}$. This is obvious unless $b_{k+2} = 0$ (then $a_{k+2} = 1$ and $V_{k+2} = V_{k+1}V_k$) and $a_{k+3} - b_{k+3} = 1$ and $b_{k+3} = 0$ (then $V_{k+3} = V_{k+2}V_{k+1} = V_{k+1}V_kV_{k+1}$). Assume that these three equalities hold. If $a_{k+4} > b_{k+4}$, we have

$$V_{k+4} = V_{k+3}V_{k+2}^- \dots = V_{k+1}V_kV_{k+1}V_{k+2}^- \dots = V_{k+1}V_kV_{k+1}V_{k+1}V_k^- \dots,$$

then $V_{k+1}V_kV_{k+1}V_k^-$ is indeed a prefix of $\mathbf{s}$. Otherwise, we have

$$V_{k+4} = V_{k+2}V_{k+3} \dots = V_{k+1}V_kV_{k+1}V_kV_{k+1} \dots = V_{k+1}V_kV_{k+1}V_k \dots,$$

and the same conclusion holds.

We claim that

$$V_{k+1}V_kV_{k+1}V_k = R_{k+1}M_k^{a_{k+1}+1}M_{k-1}M_k^{b_{k+1}+1}T_k,$$

which yields that $U_k = R_{k+1}$ and $\widetilde{a}_{k+1} = a_{k+1} + 1$. For the proof, we distinguish two cases, either $a_{k+1} > b_{k+1}$, or $a_{k+1} = b_{k+1}$. In the first case, we have

$$T_{k+1} = M_k^{b_{k+1}}T_k \quad \text{and} \quad R_{k+1} = R_kM_k^{a_{k+1}-b_{k+1}-1}M_{k-1},$$

so that we compute

$$\begin{aligned} V_{k+1}V_kV_{k+1}V_k &= R_{k+1}T_{k+1}R_kT_kR_{k+1}T_{k+1}R_kT_k \\ &= R_{k+1}M_k^{b_{k+1}}T_kR_kT_kR_kM_k^{a_{k+1}-b_{k+1}-1}M_{k-1}M_k^{b_{k+1}}T_kR_kT_k \\ &= R_{k+1}M_k^{a_{k+1}+1}M_{k-1}M_k^{b_{k+1}+1}T_k. \end{aligned}$$

For the latter case, we have

$$T_k = T_{k-1}, \quad R_k = R_{k-1} M_{k-1}^{a_k - 1} M_{k-2},$$

and

$$T_{k+1} = M_k^{a_{k+1}} T_{k-1} = M_k^{a_{k+1}} T_k, \quad R_{k+1} = R_{k-1}.$$

Thus

$$\begin{aligned} V_{k+1} V_k V_{k+1} V_k & \\ &= R_{k+1} T_{k+1} R_k T_k R_{k+1} T_{k+1} R_k T_k \\ &= R_{k+1} M_k^{a_{k+1}} T_{k-1} R_{k-1} M_{k-1}^{a_k - 1} M_{k-2} T_{k-1} R_{k-1} M_k^{a_{k+1}} T_k R_k T_k \\ &= R_{k+1} M_k^{a_{k+1}+1} M_{k-1} M_k^{a_{k+1}+1} T_k. \end{aligned}$$

The claim is established.

Assume thirdly that $a_{k+2} - b_{k+2} = 1$ and that $a_{k+3} = b_{k+3}$. Then, $b_{k+2} = 0$ and $a_{k+2} = 1$. We find

$$V_{k+3} = V_{k+1} V_{k+2}^{a_{k+3}} = V_{k+1} (V_{k+1} V_k)^{a_{k+3}} = V_{k+1}^2 V_k \dots.$$

The first case shows that $U_k = R_{k+1}$ and $\widetilde{a}_{k+1} = a_{k+1}$, as asserted.

Suppose finally that $a_{k+2} = b_{k+2}$. Then $b_{k+1} = 0$ and $a_{k+3} > b_{k+3}$, since $a_{k+3} = b_{k+3}$ should yield $a_{k+2} = b_{k+2} = 0$. Thus,

$$\begin{aligned} V_{k+3} = V_{k+2}^{a_{k+3} - b_{k+3}} V_{k+1} V_{k+2}^{b_{k+3}} &= V_{k+2} V_{k+1}^- \cdots \\ &= V_k V_{k+1}^{a_{k+2}} V_{k+1}^- \cdots = V_k V_{k+1} V_{k+1}^- \cdots \end{aligned}$$

Here, again, we can be more precise and show that $\mathbf{s}$ is either of the form

$$\mathbf{s} = V_k V_{k+1} V_k V_{k+1} V_k^- \dots, \tag{5.1}$$

or of the form

$$\mathbf{s} = V_k V_{k+1} V_{k+1} V_k^- \dots. \tag{5.2}$$

If $a_{k+2} \geqslant 2$, we have

$$V_{k+3} = V_{k+2} V_{k+1}^- \cdots = V_k V_{k+1}^{a_{k+2}} V_k^- \cdots = V_k V_{k+1}^2 V_k^- \cdots$$

Thus (5.2) holds. When $a_{k+2} = 1$ and $a_{k+3} - b_{k+3} \geqslant 2$, we have $V_{k+2} = V_k V_{k+1}$ and

$$V_{k+3} = V_{k+2}^2 V_{k+1}^- \cdots = V_k V_{k+1} V_k V_{k+1} V_k^- \cdots$$

Thus (5.1) holds. When $a_{k+2} = 1$, $a_{k+3} - b_{k+3} = 1$ and $b_{k+3} \geqslant 1$, we have

$$V_{k+3} = V_{k+2} V_{k+1} V_{k+2}^{b_{k+3}} = V_k V_{k+1}^2 V_{k+2}^{b_{k+3}},$$

so that (5.2) holds true. When $a_{k+2}=1$, $a_{k+3}=1$ and $b_{k+3}=0$, we have $V_{k+3}=V_kV_{k+1}^2$. If $a_{k+4}-b_{k+4}\geqslant 1$, we have

$$V_{k+4}=V_{k+3}V_{k+2}^-\cdots=V_kV_{k+1}^2V_k^-\ldots$$

so that (5.2) holds, while

$$V_{k+4}=V_{k+2}V_{k+3}\cdots=V_kV_{k+1}V_kV_{k+1}^2\cdots$$

if $a_{k+4}=b_{k+4}$. Then (5.1) holds. Now, we compute

$$\begin{aligned}V_kV_{k+1}V_{k+1}V_k&=R_kT_kR_{k+1}T_{k+1}R_{k+1}T_{k+1}R_kT_k\\&=R_kT_k(R_kM_k^{a_{k+1}-1}M_{k-1})M_{k+1}T_kR_kT_k\\&=R_kM_k^{a_{k+1}}M_{k-1}M_k^{a_{k+1}}M_{k-1}M_kT_k\\&=R_kM_{k+1}M_k^{a_{k+1}}M_{k-1}T_k,\end{aligned}$$

and

$$\begin{aligned}&V_kV_{k+1}V_kV_{k+1}V_k\\&\qquad=R_kT_kR_{k+1}T_{k+1}R_kT_kR_{k+1}T_{k+1}R_kT_k\\&\qquad=R_kT_kR_kM_k^{a_{k+1}-1}M_{k-1}T_kR_kT_kR_kM_k^{a_{k+1}-1}M_{k-1}T_kR_kT_k\\&\qquad=R_kM_k^{a_{k+1}}M_{k-1}M_k^{a_{k+1}+1}M_{k-1}M_kT_k\\&\qquad=R_kM_{k+1}M_k^{a_{k+1}+1}M_{k-1}M_kT_k.\end{aligned}$$

Thus $U_k=R_kM_{k+1}$ in both cases. We have $\widetilde{a}_{k+1}=a_{k+1}$ when (5.2) holds, while $\widetilde{a}_{k+1}=a_{k+1}+1$ whenever (5.1) is satisfied. □

We have used at several places the obvious property that $V_k^-$ is a prefix of $V_{k+1}$, which holds since by definition $V_k$, $V_{k+1}$ and $\mathbf{s}$ share the same prefix of length $q_k-1$. A question which arises naturally is to know when $V_k$ is a prefix of $V_{k+1}$.

Proposition 5.3. — *For any $k\geqslant 0$, the word $V_k$ is a prefix of $V_{k+1}$ if and only if the sequence $b_1,\ldots,b_{k+1}$ differs from $0,a_2,0,a_4,\ldots,a_{k+1}$ when $k$ is odd, or differs from $a_1-1,0,a_3,0,\ldots,a_{k+1}$ when $k$ is even.*

For $k\geqslant 0$, let $W_k$ denote the longest common prefix of $V_{k+1}V_k$ and $V_kV_{k+1}$.

Lemma 5.4. — *We have $W_0=0^{a_1-1-b_1}$ and $W_{k+1}=V_{k+1}^{a_{k+2}-b_{k+2}}W_k$ for $k\geqslant 0$. Consequently, the length $w_k$ of $W_k$ is given by*

$$w_k=a_1-1-b_1+\sum_{j=1}^{k}(a_{j+1}-b_{j+1})q_j=q_{k+1}+q_k-t_{k+1}-2,\quad k\geqslant 0.$$

*Proof.* — Recall that $V_0 = 0$, and $V_1 = 0^{a_1-1-b_1}10^{b_1}$. This implies that $V_0V_1 = 0^{a_1-b_1}10^{b_1}$, thus

$$W_0 = V_0^{a_1-1-b_1}, \quad w_0 = a_1 - 1 - b_1.$$

We proceed by induction. Let $k \geqslant 0$ be an integer.

Assume first that $a_{k+2} - b_{k+2} \geqslant 1$.

Since $V_{k+2} = V_{k+1}^{a_{k+2}-b_{k+2}} V_k V_{k+1}^{b_{k+2}}$, we get

$$V_{k+2}V_{k+1} = V_{k+1}^{a_{k+2}-b_{k+2}} V_k V_{k+1}^{b_{k+2}+1}$$

and

$$V_{k+1}V_{k+2} = V_{k+1}^{a_{k+2}-b_{k+2}+1} V_k V_{k+1}^{b_{k+2}},$$

thus

$$W_{k+1} = V_{k+1}^{a_{k+2}-b_{k+2}} W_k.$$

Assume now that $a_{k+2} = b_{k+2}$. In that case, we know that $b_{k+1} = 0$. Then, assuming moreover that $k \geqslant 1$, we have

$$V_{k+2}V_{k+1} = V_k V_{k+1}^{a_{k+2}+1} = V_k V_k^{a_{k+1}} V_{k-1} V_k^{a_{k+1}} V_{k-1} V_{k+1}^{a_{k+2}-1}$$

and

$$V_{k+1}V_{k+2} = V_k^{a_{k+1}} V_{k-1} V_k V_{k+1}^{a_{k+2}},$$

thus

$$W_{k+1} = V_k^{a_{k+1}} W_{k-1} = W_k = V_{k+1}^{a_{k+2}-b_{k+2}} W_k.$$

In the remaining case $k = 0$, we have

$$V_1 = 0^{a_1-1}1 \quad \text{and} \quad V_2 = 0V_1^{a_2},$$

so that $W_1 = 0^{a_1-1} = W_0$, as required. □

*Proof of Proposition 5.3.* — Since $q_k$ is the length of $V_k$, the word $V_k$ is a prefix of $V_{k+1}$ exactly when $w_k \geqslant q_k$. Lemma 5.4 tells us that $w_k \geqslant q_k$ if and only if $t_{k+1} \leqslant q_{k+1} - 2$. Observe finally that $t_{k+1} \leqslant q_{k+1} - 1$ with equality if and only if

$$b_{k+1} = a_{k+1},\ b_k = 0,\ b_{k-1} = a_{k-1},\ b_{k-2} = 0, \ldots.$$

This completes the proof. □

## 6. The sequence of convergents contributing to the exponent of irrationality

In this section and the next one, $b \geqslant 2$ is an integer and $\xi$ denotes one of the numbers $\xi_b(\theta, \rho)$ or $\xi'_b(\theta, \rho)$. We analyze the convergents which contribute to the exponent of irrationality of $\xi$, which we call 'strong convergents'. According to [14], all of them are obtained by truncating the $b$-ary expansion of $\xi$ and completing by periodicity. Thus, their denominators are either of the form $b^s - 1$ (purely periodic case) or $b^r(b^s - 1)$ (existence of a preperiod).

We adopt the following conventions of writing. Any finite word $Y = y_1 \dots y_r$ with letters in $\{0, \dots, b-1\}$ is as well viewed as the natural integer

$$Y = y_1 b^{r-1} + \cdots + y_r,$$

whose sequence of $b$-ary digits is given by $Y$. Then, for any words $Y = y_1 \dots y_r$ and $Z = z_1 \dots z_s$, we have the $b$-ary expansions

$$\frac{Z}{b^s - 1} = 0.Z^\infty,$$

and

$$\frac{YZ - Y}{b^r(b^s - 1)} = 0.YZ^\infty,$$

where $0.z_1 z_2 \cdots = \frac{z_1}{b} + \frac{z_2}{b^2} + \cdots$ and $YZ$ stands for the number whose $b$-ary sequence of digits is the concatenation of the words $Y$ and $Z$, that is, $YZ = y_1 \dots y_r\, z_1 \dots z_s$.

Let $u$ and $v$ be two positive quantities depending upon a parameter $k$. As usual, we write $u \asymp v$ when there exist positive constants $c_1$ and $c_2$, independent of $k$, such that $c_1 u \leqslant v \leqslant c_2 u$.

The candidates for the sequence of strong convergents belong to four types. We label them by the index $k \geqslant 0$. The sequences of finite words $(R_k)_{k \geqslant 0}, (T_k)_{k \geqslant 0}, (V_k)_{k \geqslant 0}$ are given by Theorem 2.1 (or Theorem 4.2), but we now replace the alphabet $\{0, 1\}$ by $\{0, b-1\}$. We recall that $R_0 = 0$ and $T_0$ is the empty word. Below, the height means the logarithmic height $\log H / \log b$, that is, roughly speaking, the largest exponent of $b$ appearing in the denominator.

The first possible convergent is

$$(1)_k = \frac{R_{k+1} - R_k}{b^{r_k}(b^{r_{k+1} - r_k} - 1)},$$

with height $\asymp r_{k+1}$ and $b$-ary expansion (below, we set $M_{-1} = b-1$)

$$(1)_0 = 0 \cdot R_0 (M_0^{a_1-b_1-2} M_{-1})^\infty,$$
$$(1)_k = 0 \cdot R_k (M_k^{a_{k+1}-b_{k+1}-1} M_{k-1})^\infty, \quad k \geqslant 1.$$

Of course, $(1)_k$ is meaningful only when $r_{k+1} > r_k$, that is to say when $a_{k+1} - b_{k+1} \geqslant 1$ when $k \geqslant 1$, or $a_1 - b_1 \geqslant 2$ for $k = 0$. The second candidate is

$$(2)_k = \frac{R_{k+1} T_k}{b^{r_{k+1}+t_k} - 1},$$

with height $\asymp r_{k+1} + t_k$, associated to the periodic word $(R_{k+1}T_k)^\infty$. The third is

$$(3)_k = \frac{R_{k+1} M_k - R_{k+1}}{b^{r_{k+1}} (b^{q_k} - 1)},$$

with height $\asymp r_{k+1} + q_k$, associated to the word $R_{k+1} M_k^\infty$. The fourth is

$$(4)_k = \frac{V_{k+1}}{b^{q_{k+1}} - 1},$$

with height $\asymp q_{k+1} = r_{k+1} + t_{k+1}$, associated to the periodic word $V_{k+1}^\infty = (R_{k+1}T_{k+1})^\infty$.

We say that a rational $x$ precedes another one $y$, and we write $x \prec y$, when the height of $x$ is less than the height of $y$. Clearly

$$(1)_k \prec (2)_k \preceq (4)_k.$$

We have $(2)_k = (4)_k$ exactly when $t_k = t_{k+1}$, that is to say when $b_{k+1} = 0$. Then,

$$(1)_k \prec (2)_k = (4)_k \prec (3)_k.$$

If $b_{k+1} \geqslant 1$, we have

$$t_{k+1} \geqslant t_k + q_k,$$

so that

$$(1)_k \prec (2)_k \prec (3)_k \prec (4)_k,$$

in this case. When $a_{k+1} = b_{k+1}$, obviously $b_{k+1} \geqslant 1$, so that the above inequality

$$(2)_k \prec (3)_k \prec (4)_k$$

hold with $(1)_k$ being omitted.

An important observation is that we have the following coincidences between levels $k-2$, $k-1$ and $k$, where $k \geqslant 2$ is an integer.

If $a_{k+1} - b_{k+1} = 1$, then we have

$$(1)_k = (3)_{k-1},$$

since

$$R_k (M_k^{a_{k+1}-b_{k+1}-1} M_{k-1})^\infty = R_k M_{k-1}^\infty.$$

If $a_{k+1} = b_{k+1}$, then we have

$$(2)_k = (4)_{k-2},$$

since we have $R_{k+1} = R_{k-1}$ and $T_k = T_{k-1}$ (because $b_k = 0$), so that

$$R_{k+1}T_k = R_{k-1}T_{k-1} = V_{k-1}.$$

If, in addition, $b_{k-1} = 0$, then $T_{k-1} = T_{k-2}$ and $(2)_{k-2} = (2)_k = (4)_{k-2}$.

Observe also that if $a_{k+1} - b_{k+1} = 1$, then we have

$$(2)_k = \frac{V_k V_{k-1}}{b^{q_k+q_{k-1}} - 1},$$

since

$$R_{k+1}T_k = R_k M_{k-1} M_{k-1}^{b_k} T_{k-1} = R_k T_k R_{k-1} T_{k-1} = V_k V_{k-1},$$

noting that

$$\begin{aligned} T_k R_{k-1} = T_{k-1} V_{k-1}^{b_k} R_{k-1} &= T_{k-1}(R_{k-1}T_{k-1})^{b_k} R_{k-1} \\ &= (T_{k-1}R_{k-1})^{b_k+1} = M_{k-1}^{b_k+1}. \end{aligned}$$

To go further for linking consecutive blocks (with indices $k-1$ and $k$), we need to know when the rationals $(1)_k, (2)_k, (3)_k, (4)_k$ are indeed convergents. We indicate as well in the next proposition the value of the exponential rate of approximation $\mu_k(j)$ such

$$|\xi - (j)_k| \asymp \frac{1}{H((j)_k)^{\mu_k(j)}} = \frac{1}{b^{h((j)_k)\mu_k(j)}},$$

for all large $k$ and $1 \leqslant j \leqslant 4$, where $h((j)_k)$ is the base-$b$ logarithm of the height $H((j)_k)$ of $(j)_k$. We determine in which cases the exponent $\mu_k(j)$ is bigger than 2, thanks to Proposition 5.1. To that purpose, let us introduce the following quantities

$$\nu_k(1) = 1 + \frac{r_{k+1} + t_k}{r_{k+1}}, \qquad \nu_k(2) = 1 + \frac{r_{k+1} + q_k}{r_{k+1} + t_k},$$
$$\nu_k(3) = 1 + \frac{q_{k+1}}{r_{k+1} + q_k}, \qquad \nu_k(4) = 1 + \frac{r_{k+2}}{q_{k+1}}.$$

They are equal to one plus the ratio of the height of two consecutive points in the sequence $\dots (1)_k, (2)_k, (3)_k, (4)_k, (1)_{k+1}, \dots$. Then, we can state the following criterion.

Proposition 6.1. — *Let $k \geqslant 2$ be an integer such that $t_{k-1}$ is positive. The rational*

$$(1)_k = \frac{R_{k+1} - R_k}{b^{r_k}(b^{r_{k+1}-r_k} - 1)}$$

*is a convergent to $\xi$ if and only if*

$$a_{k+1}-b_{k+1}\geqslant 1,\, a_{k+2}-b_{k+2}\geqslant 1 \quad \textit{and then} \quad \mu_k(1)=\nu_k(1),$$

*or*

$$b_k\geqslant 1,\, a_{k+1}=1,\, b_{k+1}=0,\, a_{k+2}=b_{k+2} \quad \textit{and then} \quad \mu_k(1)=\nu_{k-1}(3).$$

*The rational*

$$(2)_k=\frac{R_{k+1}T_k}{b^{r_{k+1}+t_k}-1}$$

*is a convergent to $\xi$ if and only if $a_{k+2}-b_{k+2}\geqslant 1$ and then*

$$\mu_k(2)=\begin{cases}\nu_k(2) & \textit{if } b_{k+1}\geqslant 1,\\ \nu_k(4) & \textit{if } b_{k+1}=0,\, a_{k+3}-b_{k+3}\geqslant 1,\\ \nu_{k+2}(2) & \textit{if } b_{k+1}=0,\, a_{k+3}=b_{k+3}.\end{cases}$$

*The rational*

$$(3)_k=\frac{R_{k+1}M_k-R_{k+1}}{b^{r_{k+1}}(b^{q_k}-1)}$$

*is a convergent to $\xi$ if and only if*

$$b_{k+1}\geqslant 1,\, a_{k+2}-b_{k+2}\geqslant 2 \quad \textit{and then} \quad \mu_k(3)=\nu_k(3),$$

*or*

$$a_{k+2}-b_{k+2}=1,\, a_{k+3}-b_{k+3}\geqslant 1 \quad \textit{and then} \quad \mu_k(3)=\nu_{k+1}(1),$$

*or*

$$b_{k+1}\geqslant 1,\, a_{k+2}=1,\, b_{k+2}=0,\, a_{k+3}=b_{k+3} \quad \textit{and then} \quad \mu_k(3)=\nu_k(3),$$

*The rational*

$$(4)_k=\frac{V_{k+1}}{b^{q_{k+1}}-1}$$

*is a convergent to $\xi$ if and only if*

$$a_{k+2}-b_{k+2}\geqslant 2,\, a_{k+3}-b_{k+3}\geqslant 1 \quad \textit{and then} \quad \mu_k(4)=\nu_k(4),$$

*or*

$$b_{k+1}=0,\, a_{k+2}-b_{k+2}=1,\, a_{k+3}-b_{k+3}\geqslant 1 \quad \textit{and then} \quad \mu_k(4)=\nu_k(2)=\nu_k(4),$$

*or*

$$a_{k+3}=b_{k+3} \quad \textit{and then} \quad \mu_k(4)=\nu_{k+2}(2)=1+\nu_k(4).$$

*Proof.* — We only prove Proposition 6.1 assuming that $t_{k-1}$ is large enough. In fact, crude estimates of the constants involved in the symbols $\asymp$ show that the lower bound $b^{t_{k-1}} \geqslant 4$ is sufficient for our purpose. Relaxing the assumption to $t_{k-1} \geqslant 1$ follows from an alternative argumentation which will be given in the next Section 7. Our present approach is based on Legendre's theorem asserting that $P/Q$ is a convergent to $\xi$ when $|\xi - P/Q| < 1/(2Q^2)$.

Let $\mathbf{s}$ be the Sturmian word composed of the $b$-ary digits of $\xi$.

For $(1)_k$, the relevant assumption is $a_{k+1} - b_{k+1} \geqslant 1$. Assume first that $a_{k+2} - b_{k+2} \geqslant 1$. Then, Proposition 5.1 gives

$$\begin{aligned}
\mathbf{s} &= R_{k+1} M_k^{\widetilde{a_{k+1}}} M_{k-1} M_k^{-} \dots \\
&= R_k M_k^{a_{k+1}-b_{k+1}-1} M_{k-1} M_k^{\widetilde{a_{k+1}}} M_{k-1} \dots \\
&= R_k M_k^{a_{k+1}-b_{k+1}-1} M_{k-1} M_k^{a_{k+1}-b_{k+1}-1} M_k^{\widetilde{a_{k+1}}-a_{k+1}+b_{k+1}+1} M_{k-1} \dots \\
&= R_k M_k^{a_{k+1}-b_{k+1}-1} M_{k-1} M_k^{a_{k+1}-b_{k+1}-1} M_k M_{k-1} \dots,
\end{aligned}$$

to be compared with the word $R_k(M_k^{a_{k+1}-b_{k+1}-1} M_{k-1})^\infty$. When $a_{k+1} - b_{k+1} \geqslant 2$, we can write

$$\begin{aligned}
&R_k(M_k^{a_{k+1}-b_{k+1}-1} M_{k-1})^\infty \\
&\qquad\qquad = R_k M_k^{a_{k+1}-b_{k+1}-1} M_{k-1} M_k^{a_{k+1}-b_{k+1}-1} M_{k-1} M_k \dots
\end{aligned}$$

to obtain the estimate

$$|\xi - (1)_k| \asymp \frac{1}{b^{r_k + 2(a_{k+1}-b_{k+1}-1)q_k + 2q_{k-1} + q_k}} = \frac{1}{b^{2r_{k+1}+t_k}}.$$

When $a_{k+1} - b_{k+1} = 1$ the same estimate holds, since then

$$\mathbf{x} = R_k M_{k-1} M_k M_{k-1} \cdots = R_k M_{k-1}^{a_k+1} M_{k-2} M_{k-1} \dots,$$

while

$$R_k(M_k^{a_{k+1}-b_{k+1}-1} M_{k-1})^\infty = R_k M_{k-1}^\infty = R_k M_{k-1}^{a_k+1} M_{k-1} M_{k-2} \dots.$$

Thus $(1)_k$ is a convergent to $\xi$ and

$$\mu_k(1) = \frac{2r_{k+1} + t_k}{r_{k+1}} = 1 + \frac{r_{k+1} + t_k}{r_{k+1}} = \nu_k(1).$$

When $a_{k+2} = b_{k+2}$, we have $b_{k+1} = 0, U_k = R_k M_{k+1}$, and Proposition 5.1 gives

$$\mathbf{s} = U_k \dots = R_k M_k^{a_{k+1}-1} M_k M_{k-1} \dots$$

We distinguish two subcases. If $a_{k+1} \geqslant 2$, we write

$$R_k(M_k^{a_{k+1}-1} M_{k-1})^\infty = R_k M_k^{a_{k+1}-1} M_{k-1} M_k M_k^{a_{k+1}-2} M_{k-1} \dots.$$

Thus,

$$|\xi - (1)_k| \asymp \frac{1}{b^{r_k + a_{k+1} q_k + q_{k-1}}} = \frac{1}{b^{r_{k+1} + q_k}},$$

so that

$$(r_{k+1}+q_k)-2r_{k+1} = q_k - r_{k+1} = q_k - (r_k + (a_{k+1}-1)q_k + q_{k-1}) = t_k - q_{k+1} + q_k$$

is negative, since $q_{k+1} > 2q_k$. Therefore $(1)_k$ is not a convergent in this subcase. When $a_{k+1} = 1$, write

$$\mathbf{s} = R_k M_k M_{k-1} \cdots = R_k M_{k-1}^{a_k} M_{k-2} M_{k-1} \ldots,$$

while

$$R_k M_{k-1}^{\infty} = R_k M_{k-1}^{a_k} M_{k-1} M_{k-2} \ldots$$

Thus,

$$|\xi - (1)_k| \asymp \frac{1}{b^{r_k + (a_k+1) q_{k-1} + q_{k-2}}} = \frac{1}{b^{r_k + q_k + q_{k-1}}} = \frac{1}{b^{r_{k+1} + q_k}},$$

so that

$$\begin{aligned}(r_{k+1} + q_k) - 2r_{k+1} = -r_{k+1} + q_k &= -(r_k + q_{k-1}) + q_k \\ &= t_k - q_{k-1} = t_{k-1} + (b_k - 1) q_{k-1}.\end{aligned}$$

We conclude by noticing that $t_{k-1} + (b_k - 1)q_{k-1}$ is positive if $b_k \geqslant 1$ and negative when $b_k = 0$. Thus,

$$\mu_k(1) = \frac{r_{k+1} + q_k}{r_{k+1}} = 1 + \frac{q_k}{r_{k+1}} = 1 + \frac{q_k}{r_k + q_{k-1}} = \nu_{k-1}(3).$$

Observe that, in this case, we have the ordering

$$(1)_k = (3)_{k-1} \prec (2)_{k+1} = (4)_{k-1},$$

while $(2)_k, (3)_k, (4)_k$ and $(1)_{k+1}$ are not convergents to $\xi$.

We now deal with $(2)_k$. Assume first that $a_{k+2} - b_{k+2} \geqslant 1$.

In the subcase $a_{k+1} - b_{k+1} \geqslant 1$ and $b_{k+1} \geqslant 1$, we have

$$R_{k+1} = R_k M_k^{a_{k+1} - b_{k+1} - 1} M_{k-1}$$

and Proposition 5.1 gives

$$\begin{aligned}\mathbf{s} &= R_{k+1} M_k^{\widetilde{a_{k+1}}} M_{k-1} M_k^{-} \ldots \\ &= R_k M_k^{a_{k+1} - b_{k+1} - 1} M_{k-1} M_k^{\widetilde{a_{k+1}}} M_{k-1} M_k^{-} \ldots \\ &= R_k M_k^{a_{k+1} - b_{k+1} - 1} M_{k-1} M_k^{a_{k+1} - b_{k+1}} M_k^{\widetilde{a_{k+1}} - a_{k+1} + b_{k+1}} M_{k-1} M_k^{-} \ldots \\ &= R_k M_k^{a_{k+1} - b_{k+1} - 1} M_{k-1} M_k^{a_{k+1} - b_{k+1}} M_k M_{k-1} \ldots\end{aligned}$$

since $\widetilde{a}_{k+1} - a_{k+1} + b_{k+1} \geqslant 1$. Comparing with the word

$$(R_k M_k^{a_{k+1}-b_{k+1}-1} M_{k-1} T_k)^\infty \\ = R_k M_k^{a_{k+1}-b_{k+1}-1} M_{k-1} M_k^{a_{k+1}-b_{k+1}} M_{k-1} M_k \ldots,$$

we obtain

$$|\xi - (2)_k| \asymp \frac{1}{b^{r_k+2(a_{k+1}-b_{k+1})q_k+2q_{k-1}}} = \frac{1}{b^{2(r_{k+1}+t_k)+r_k}}.$$

Thus, $(2)_k$ is a convergent of $\xi$ and

$$\mu_k(2) = \frac{2(r_{k+1}+t_k)+r_k}{r_{k+1}+t_k} = 2 + \frac{r_k}{r_{k+1}+t_k} = 1 + \frac{r_{k+1}+q_k}{r_{k+1}+t_k} = \nu_k(2).$$

In the subcase $a_{k+1} = b_{k+1}$ (and thus $b_{k+1} \geqslant 1$), we have $(2)_k = (4)_{k-2}$. Assuming temporarily that Proposition 6.1 has been checked for $(4)_{k-2}$, it yields that $(2)_k$ is again a convergent to $\xi$ with exponent $\mu_k(2) = \mu_{k-2}(4) = \nu_k(2)$ as asserted.

Consider finally the subcase $b_{k+1} = 0$. Then $(2)_k = (4)_k$ and Proposition 6.1 for $(4)_k$, tells us that $(2)_k$ is indeed a convergent to $\xi$ with exponent $\mu_k(2) = \mu_k(4)$ which will be computed below.

Assume now that $a_{k+2} = b_{k+2}$. Then, Proposition 5.1 gives

$$\mathbf{s} = R_k M_{k+1} \cdots = R_k M_k^{a_{k+1}} M_{k-1} \cdots = R_k M_k^{a_{k+1}-1} M_k M_{k-1} \ldots,$$

while

$$(R_{k+1}T_k)^\infty = R_k M_k^{a_{k+1}-1} M_{k-1} T_k R_k \cdots = R_k M_k^{a_{k+1}-1} M_{k-1} M_k \ldots$$

since $b_{k+1} = 0$. It follows that

$$|\xi - (2)_k| \asymp \frac{1}{b^{r_k+a_{k+1}q_k+q_{k-1}}} = \frac{1}{b^{r_{k+1}+q_k}} = \frac{1}{b^{2(r_{k+1}+t_k)-(q_{k+1}-r_k)}}.$$

Then $(2)_k$ is not a convergent to $\xi$.

We now deal with $(3)_k$. Assume first that $a_{k+2}-b_{k+2} \geqslant 1$. Proposition 5.1 gives

$$\mathbf{s} = R_{k+1} M_k^{\widetilde{a}_{k+1}} M_{k-1} M_k^- \ldots$$

Since

$$R_{k+1} M_k^\infty = R_{k+1} M_k^{\widetilde{a}_{k+1}} M_k M_{k-1} \ldots$$

we obtain the estimate

$$|\xi - (3)_k| \asymp \frac{1}{b^{r_{k+1}+(\widetilde{a}_{k+1}+1)q_k+q_{k-1}}}.$$

Write

$$\begin{aligned}(r_{k+1}+(\widetilde{a}_{k+1}+1)q_k+q_{k-1})-2(r_{k+1}+q_k) &= -r_{k+1}+(\widetilde{a}_{k+1}-1)q_k+q_{k-1}\\ &= t_{k+1}+(\widetilde{a}_{k+1}-a_{k+1}-1)q_k.\end{aligned}$$

If $a_{k+2}-b_{k+2}=1$ and $a_{k+3}-b_{k+3}\geqslant 1$, we know that $\widetilde{a}_{k+1}=a_{k+1}+1$, so that $t_{k+1}+(\widetilde{a}_{k+1}-a_{k+1}-1)q_k>0$. If $a_{k+2}-b_{k+2}\geqslant 2$, or if $a_{k+2}=1, b_{k+2}=0, a_{k+3}=b_{k+3}$, we know that $\widetilde{a}_{k+1}=a_{k+1}$, so that

$$t_{k+1}+(\widetilde{a}_{k+1}-a_{k+1}-1)q_k=t_k+(b_{k+1}-1)q_k.$$

Now, $t_k+(b_{k+1}-1)q_k$ is positive when $b_{k+1}\geqslant 1$ and negative when $b_{k+1}=0$. We get the three cases announced. Concerning the exponent $\mu_k(3)$, we find

$$\mu_k(3)=\frac{r_{k+1}+(\widetilde{a}_{k+1}+1)q_k+q_{k-1}}{r_{k+1}+q_k}.$$

When $\widetilde{a}_{k+1}=a_{k+1}$, we get

$$\mu_k(3)=\frac{r_{k+1}+q_{k+1}+q_k}{r_{k+1}+q_k}=1+\frac{q_{k+1}}{r_{k+1}+q_k}=\nu_k(3),$$

while, in the case $\widetilde{a}_{k+1}=a_{k+1}+1$, we have

$$\mu_k(3)=\frac{r_{k+1}+q_{k+1}+2q_k}{r_{k+1}+q_k}=1+\frac{q_{k+1}+q_k}{r_{k+1}+q_k}=1+\frac{r_{k+2}+t_{k+1}}{r_{k+2}}=\nu_{k+1}(1),$$

since $r_{k+2}=r_{k+1}+q_k$ when $a_{k+2}-b_{k+2}=1$. It remains for us to prove that $(3)_k$ is not a convergent when $a_{k+2}=b_{k+2}$. Then, $b_{k+1}=0$ and $r_{k+1}=r_k+(a_{k+1}-1)q_k+q_{k-1}$. In this case, Proposition 5.1 gives

$$\mathbf{s}=R_kM_{k+1}\cdots=R_kM_k^{a_{k+1}-1}M_kM_{k-1}\ldots$$

while

$$R_{k+1}M_k^{\infty}=R_kM_k^{a_{k+1}-1}M_{k-1}M_k\ldots.$$

Thus

$$|\xi-(3)_k|\asymp\frac{1}{b^{r_k+a_{k+1}q_k+q_{k-1}}}\asymp\frac{1}{b^{r_{k+1}+q_k}},$$

and $(3)_k$ is not a convergent to $\xi$.

For the last rational

$$(4)_k=\frac{V_{k+1}}{b^{q_{k+1}}-1}$$

Proposition 5.1 tells us that $\mathbf{s}=V_{k+1}^2V_k^-\ldots$ whenever

$$a_{k+2}-b_{k+2}\geqslant 2$$

or

$$a_{k+2}=1 \quad\text{and}\quad b_{k+2}=0 \quad\text{and}\quad a_{k+3}=b_{k+3}.$$

Then, the initial exponent of repetition of $V_{k+1}$ is clearly larger than 2, so that $(4)_k$ is a convergent to $\xi$. When $a_{k+2} - b_{k+2} = 1$ and $a_{k+3} - b_{k+3} \geqslant 1$, we have

$$\mathbf{s} = V_{k+1} V_k V_{k+1}^{-} \dots$$

By Lemma 5.4, the common prefix $W_k$ to $V_k V_{k+1}$ and $V_{k+1} V_k$ has length

$$w_k = a_1 - 1 - b_1 + \sum_{j=1}^{k} (a_{j+1} - b_{j+1}) q_j = q_{k+1} + q_k - t_{k+1} - 2.$$

Noting that

$$t_{k+1} = \sum_{j=0}^{k} b_{j+1} q_j$$

is larger or smaller than $q_k$ when $b_{k+1} \geqslant 1$ or $b_{k+1} = 0$, we deduce that $(4)_k$ is then a convergent to $\xi$ when $b_{k+1} = 0$ and is not when $b_{k+1} \geqslant 1$. This yields the case

$$b_{k+1} = 0 \quad \text{and} \quad a_{k+2} - b_{k+2} = 1 \quad \text{and} \quad a_{k+3} - b_{k+3} \geqslant 1.$$

When $a_{k+3} - b_{k+3} \geqslant 1$ and $a_{k+2} - b_{k+2} \geqslant 1$, Proposition 5.1, with $k$ replaced by $k+1$, tells us that

$$\mathbf{s} = R_{k+2} M_{k+1} \dots = R_{k+1} M_{k+1}^{a_{k+2} - b_{k+2} - 1} M_k M_{k+1} \dots,$$

while

$$V_{k+1}^{\infty} = R_{k+1} M_{k+1}^{\infty} = R_{k+1} M_{k+1}^{a_{k+2} - b_{k+2} - 1} M_{k+1} M_k \dots.$$

It follows that

$$|\xi - (4)_k| \asymp \frac{1}{b^{r_{k+1} + (a_{k+2} - b_{k+2}) q_{k+1} + q_k}} = \frac{1}{b^{r_{k+2} + q_{k+1}}}.$$

Thus,

$$\mu_k(4) = \frac{r_{k+2} + q_{k+1}}{q_{k+1}} = 1 + \frac{r_{k+2}}{q_{k+1}} = \nu_k(4).$$

Notice that $\nu_k(4) = \nu_k(2)$ in the case $a_{k+2} - b_{k+2} = 1$ and $b_{k+1} = 0$, since $(1)_{k+1} = (3)_k$ and $(4)_k = (2)_k$.

When $a_{k+3} = b_{k+3}$, Proposition 5.1 with $k$ replaced by $k+1$, gives

$$\mathbf{s} = R_{k+1} M_{k+2} \widetilde{M}_{k+1}^{a_{k+2}} \dots = R_{k+1} M_{k+1}^{a_{k+2}} M_k M_{k+1} \dots$$

It follows that

$$|\xi - (4)_k| \asymp \frac{1}{b^{r_{k+1} + (a_{k+2} + 1) q_{k+1} + q_k}} = \frac{1}{b^{r_{k+2} + 2q_{k+1}}},$$

since $r_{k+2} = r_{k+1} + (a_{k+2} - 1) q_{k+1} + q_k$. Thus,

$$\mu_k(4) = \frac{r_{k+2} + 2q_{k+1}}{q_{k+1}} = 2 + \frac{r_{k+2}}{q_{k+1}} = 1 + \nu_k(4) = \nu_{k+2}(2),$$

noting that

$$r_{k+2} + q_{k+1} = r_{k+1} + q_{k+2} = r_{k+3} + q_{k+2},$$

and

$$q_{k+1} = r_{k+1} + t_{k+1} = r_{k+3} + t_{k+2},$$

since $b_{k+2} = 0$.

When $a_{k+2} = b_{k+2}$, the word $\mathbf{s}$ has a prefix of the form

$$\mathbf{s} = V_{k+2} \cdots = V_k V_{k+1}^{b_{k+2}} \ldots$$

and the common prefix of $V_{k+1}^{\infty}$ and $\mathbf{s}$ has length at most

$$w_k \leqslant q_{k+1} + q_k - 2 < 2q_{k+1}.$$

Thus, $(4)_k$ cannot be a convergent to $\xi$. □

The next proposition describes a tail of the sequence of strong convergents ordered by increasing height. We start with the cyclic sequence $\mathcal{S}$

$$(1)_0, (2)_0, (3)_0, (4)_0, (1)_1, \ldots, (4)_{k-1}, (1)_k, (2)_k, (3)_k, (4)_k, (1)_{k+1}, \ldots$$

built with the $(j)_k$. As already observed, some elements of $\mathcal{S}$ may coincide and the height function is not necessarily increasing along $\mathcal{S}$. Assume that $\mathbf{s}$ differs from $\mathbf{c}_\theta$, so that $t_k$ is positive for any $k \geqslant h$ and some $h \geqslant 1$. Then, let $\mathcal{S}^+$ be the tail of $\mathcal{S}$ formed by the elements $(j)_k$ with $k \geqslant h+1$. Assuming moreover that $1 \leqslant b_k \leqslant a_k - 2$ for every $k \geqslant h+1$, Proposition 6.1 tells us that the sequence of strong convergents $(j)_k$, restricted to the indices $k \geqslant h+1$, coincides with $\mathcal{S}^+$. Otherwise, the following modifications are needed.

Proposition 6.2. — *A tail of the ordered sequence of strong convergents to $\xi$ is obtained by applying to $\mathcal{S}^+$ the following replacement rules.*

(i) *Assume $a_{k+2} = b_{k+2}$. When $b_k \geqslant 1$, we replace the string of seven elements $(4)_{k-1}, \ldots, (2)_{k+1}$ by the single element $(4)_{k-1} = (2)_{k+1}$. When $b_k = 0$, we replace the string of nine elements $(2)_{k-1}, \ldots, (2)_{k+1}$ by the single element $(2)_{k-1} = (4)_{k-1} = (2)_{k+1}$.*

(ii) *Assume $a_{k+2} - b_{k+2} = 1$ and $a_{k+3} - b_{k+3} \geqslant 1$. When $b_{k+1} \geqslant 1$, we replace the three elements $(3)_k, (4)_k, (1)_{k+1}$ by the single element $(3)_k = (1)_{k+1}$, and the four elements $(2)_k, (3)_k, (4)_k, (1)_{k+1}$ by the pair*

$$(2)_k = (4)_k \prec (3)_k = (1)_{k+1},$$

*when $b_{k+1} = 0$.*

(iii) *Assume that $a_{k+2} - b_{k+2} \geqslant 2$ and $a_{k+3} - b_{k+3} \geqslant 1$. When $b_{k+1} = 0$, we replace the three elements $(2)_k, (3)_k, (4)_k$ by the single element $(2)_k = (4)_k$.*

*Remark 6.3.* — Observe that there is no overlap for the above replacement rules, since the case (i) cannot appear for two consecutive indices $k$ by Ostrowski's numeration rules.

*Proof.* — We check in each case (i), (ii) and (iii) that the elements $(j)_k$ in $\mathcal{S}$ which are erased do not belong to the list provided by Proposition 6.1, while the remaining ones belong indeed to the list.

For instance, in the case (i) with $b_k = 0$, Proposition 6.1 tells us that $\mu_{k-1}(2) = \nu_{k+1}(2)$. Moreover,

$$(2)_{k-1} = (4)_{k-1} = (2)_{k+1} \prec (3)_{k+1}$$

are convergents to $\xi$, while the intermediate rationals $(3)_{k-1}$, $(1)_k$, $(2)_k$, $(3)_k$, $(4)_k$, $(1)_{k+1}$ are not, as can be verified by reading the necessary and sufficient conditions displayed in Proposition 6.1 for each element involved. $\square$

It will be proved in Proposition 7.2 that the subset of convergents to $\xi$ given by Proposition 6.1 provides all the convergents contributing to the irrationality exponent of $\xi$. We thus obtain the

Theorem 6.4. — *The irrationality exponent of $\xi$ is equal to*

$$\limsup_{k\to+\infty} \max\{\mu_k(1), \mu_k(2), \mu_k(3), \mu_k(4)\} = \max\{\nu(1), \nu(2), \nu(3), \nu(4)\},$$

*where*

$$\begin{aligned}
\nu(1) &= \limsup_{k\to+\infty}\{\nu_k(1) \,:\, a_{k+1} - b_{k+1} \geqslant 1 \quad \text{and} \quad a_{k+2} - b_{k+2} \geqslant 1\},\\
\nu(2) &= \limsup_{k\to+\infty}\{\nu_k(2) \,:\, a_{k+2} - b_{k+2} \geqslant 1\},\\
\nu(3) &= \limsup_{k\to+\infty}\{\nu_k(3)\},\\
\nu(4) &= \limsup_{k\to+\infty}\{\nu_k(4)\}.
\end{aligned}$$

*Proof.* — For any convergent $(j)_k$ to $\xi$, we have expressed $\mu_k(j)$ as some value $\nu_{k'}(j')$, thanks to Proposition 6.1. Conversely, for any given $\nu_k(j)$, we analyze under which conditions it contributes to the exponent of irrationality of $\xi$. For instance, Proposition 6.1 tells us that $\nu_k(1)$ occurs exactly when $a_{k+1} - b_{k+1} \geqslant 1$ and $a_{k+2} - b_{k+2} \geqslant 1$, leading to the definition of $\nu(1)$. Similarly, $\nu_k(2)$ appears in Proposition 6.1 if and only if

$$b_{k+1} \geqslant 1, \quad a_{k+2} - b_{k+2} \geqslant 1,$$

or

$$b_{k+1} = 0, \quad a_{k+2} - b_{k+2} = 1, \quad a_{k+3} - b_{k+3} \geqslant 1,$$

or

$$a_{k+1} = b_{k+1}.$$

Remark first that the third case is included in the first case, because $a_{k+1} = b_{k+1}$ implies $a_{k+2} - b_{k+2} \geqslant 1$ by Ostrowski's rules. Recall that $(2)_k = (4)_k$ when $b_{k+1} = 0$. Observe now that the assumptions $b_{k+1} = 0$ and $a_{k+2} - b_{k+2} \geqslant 1$ yield the inequality $\nu_k(2) \leqslant \nu_k(4)$, with equality if and only if $a_{k+2} - b_{k+2} = 1$, since $t_{k+1} = t_k$ and

$$\begin{aligned}\nu_k(2) = 1 + \frac{r_{k+1} + q_k}{r_{k+1} + t_k} &= 1 + \frac{r_{k+2} - (a_{k+2} - b_{k+2} - 1)q_{k+1}}{q_{k+1}} \\ &\leqslant 1 + \frac{r_{k+2}}{q_{k+1}} = \nu_k(4).\end{aligned}$$

We may thus remove the condition $b_{k+1} \geqslant 1$ in the first case, since the additional contributions are taken into account by $\nu(4)$. Finally, the single constraint $a_{k+2} - b_{k+2} \geqslant 1$ remains. We are thus led to introduce the quantity $\nu(2)$.

We now deal with the contribution of $\nu_k(4)$. It occurs in Proposition 6.1 exactly when

$$a_{k+2} - b_{k+2} \geqslant 2, \quad a_{k+3} - b_{k+3} \geqslant 1,$$

or

$$b_{k+1} = 0, \quad a_{k+2} - b_{k+2} = 1, \quad a_{k+3} - b_{k+3} \geqslant 1.$$

Observe that $\nu_k(4) = 1 + \frac{r_{k+2}}{q_{k+1}}$ is at most equal to 2 when $b_{k+1} \geqslant 1$ and $a_{k+2} - b_{k+2} = 1$, since then

$$r_{k+2} = r_{k+1} + q_k = \begin{cases} r_k + q_{k+1} - b_{k+1}q_k \leqslant q_{k+1} - t_k & \text{if } a_{k+1} - b_{k+1} \geqslant 1, \\ r_{k-1} + q_k \leqslant q_{k+1} & \text{if } a_{k+1} = b_{k+1}. \end{cases}$$

We may thus forget the condition $b_{k+1} = 0$ in the second case above. Observe also that $\nu_k(4) < 2$ when $a_{k+2} = b_{k+2}$. It remains the constraint $a_{k+3} - b_{k+3} \geqslant 1$. Notice however that we may remove this last constraint as asserted. Indeed, when $a_{k+3} = b_{k+3}$, Proposition 6.1 tells us that $(4)_k = (2)_{k+2}$ is a convergent to $\xi$ with approximation exponent $\nu_{k+2}(2) = 1 + \nu_k(4)$. Since $a_{k+4} - b_{k+4} \geqslant 1$ by Ostrowski's rules, the number $\nu_{k+2}(2) > \nu_k(4)$ is taken into account by $\nu(2)$. We may thus define $\nu(4)$ unconditionally as above.

We finally deal with the contribution of $\nu_k(3)$. It appears when

$$b_{k+1} \geqslant 1, \quad a_{k+2} - b_{k+2} \geqslant 2,$$

or

$$b_{k+1} \geqslant 1, \quad a_{k+2} = 1, \quad b_{k+2} = 0, \quad a_{k+3} = b_{k+3}.$$

We may relax the constraints as follows. We first forget the assumption $b_{k+1} \geqslant 1$, since when $b_{k+1} = 0$, we have $r_{k+1} = r_k + q_{k+1} - q_k$, so that

$$\nu_k(3) = 1 + \frac{q_{k+1}}{r_{k+1} + q_k} = 1 + \frac{q_{k+1}}{r_k + q_{k+1}} < 2.$$

We may also relax the assumptions $a_{k+2} = 1, b_{k+2} = 0, a_{k+3} = b_{k+3}$ in the second case above to $a_{k+2} - b_{k+2} = 1$, since when $a_{k+2} - b_{k+2} = 1$ and $a_{k+3} - b_{k+3} \geqslant 1$, we have $(3)_k = (1)_{k+1}$, while

$$\begin{aligned}\nu_k(3) = 1 + \frac{q_{k+1}}{r_{k+1} + q_k} &= 1 + \frac{q_{k+1}}{r_{k+2}} \\ &< 1 + \frac{q_{k+1} + q_k}{r_{k+2}} = 1 + \frac{r_{k+2} + t_{k+1}}{r_{k+2}} = \nu_{k+1}(1).\end{aligned}$$

The additional contributions are then covered by $\nu(1)$. It remains the constraint $a_{k+2} - b_{k+2} \geqslant 1$. But when $a_{k+2} = b_{k+2}$, we have $b_{k+1} = 0$, so that $\nu_k(3) \leqslant 2$, as already observed. □

## 7. The partial quotients

We keep the notation of the previous section.

For $k \geqslant 0$, recall that we have set

$$c_k = b^{r_k + q_{k-1}} \frac{b^{(a_{k+1} - b_{k+1} - 1)q_k} - 1}{b^{q_k} - 1}, \qquad d_k = b^{t_k} - 1,$$

$$e_k = b^{r_k} - 1, \qquad f_k = b^{t_k} \frac{b^{b_{k+1} q_k} - 1}{b^{q_k} - 1}.$$

The integers $c_k, d_k, e_k, f_k$ are positive, unless $t_k = 0$ (and then $d_k = 0$) or $b_{k+1} = 0$ (and then $f_k = 0$) or $a_{k+1} - b_{k+1} \leqslant 1$ (and then $c_k = 0$ if $a_{k+1} - b_{k+1} = 1$, while otherwise $c_k = -b^{r_{k-1}} = -e_{k-1} - 1$, by (2.1).

Recall that we have defined the possible convergents by

$$(1)_k = \frac{R_{k+1} - R_k}{b^{r_k}(b^{r_{k+1} - r_k} - 1)}, \qquad (2)_k = \frac{R_{k+1} T_k}{b^{r_{k+1} + t_k} - 1}, \qquad k \geqslant 0,$$

$$(3)_k = \frac{R_{k+1} M_k - R_{k+1}}{b^{r_{k+1}}(b^{q_k} - 1)}, \qquad (4)_k = \frac{V_{k+1}}{b^{q_{k+1}} - 1}, \qquad k \geqslant 0.$$

Put also

$$(4)_{-1} = \frac{V_0}{b^{q_0} - 1} = \frac{0}{b - 1}.$$

From a Diophantine point of view, $(1)_k$ is meaningful only when $r_{k+1} > r_k$, that is to say when $a_{k+1} - b_{k+1} \geqslant 1$. Nevertheless, it can be formally defined as well when $r_{k+1} < r_k$, in which case numerator and denominator are negative integers.

We use the notation $\frac{P}{Q} = c \cdot \frac{P'}{Q'} \dot{+} \frac{P''}{Q''}$ between fractions to mean that both relations $P = cP' + P''$ and $Q = cQ' + Q''$ hold true. Similarly, $(2)_k \dot{-} (1)_k$ stands below for the fraction whose numerator (resp. denominator) is the difference between the numerators (resp. denominators) of $(2)_k$ and $(1)_k$. Then, we have the

Lemma 7.1. — *For $k \geqslant 0$, we have the following relations:*

$$\begin{gathered}
(1)_k = c_k \cdot (4)_{k-1} \dot{+} (3)_{k-1}, \quad (k \neq 0), \\
(2)_k \dot{-} (1)_k = d_k \cdot (1)_k \dot{+} (4)_{k-1}, \\
(2)_k = 1 \cdot ((2)_k \dot{-} (1)_k) \dot{+} (1)_k, \\
(3)_k = e_k \cdot (2)_k \dot{+} ((2)_k \dot{-} (1)_k), \\
(4)_k = f_k \cdot (3)_k \dot{+} (2)_k.
\end{gathered}$$

*Proof.* — Let us begin with the first equality. If $a_{k+1} - b_{k+1} \geqslant 1$, then

$$\begin{aligned}
&c_k(b^{q_k} - 1) + b^{r_k}(b^{q_{k-1}} - 1) \\
&\qquad = b^{r_k + q_{k-1}}\left(b^{(a_{k+1} - b_{k+1} - 1)q_k} - 1\right) + b^{r_k}(b^{q_{k-1}} - 1) \\
&\qquad = b^{r_k + (a_{k+1} - b_{k+1} - 1)q_k + q_{k-1}} - b^{r_k} \\
&\qquad = b^{r_{k+1}} - b^{r_k},
\end{aligned}$$

which is the denominator of $(1)_k$. Likewise, we have

$$\begin{aligned}
&V_k \times b^{r_k + q_{k-1}} \frac{b^{(a_{k+1} - b_{k+1} - 1)q_k} - 1}{b^{q_k} - 1} \\
&\quad = R_k T_k \times \left(b^{r_k + q_{k-1}} + b^{r_k + q_{k-1} + q_k} + \ldots + b^{r_k + q_{k-1} + (a_{k+1} - b_{k+1} - 2)q_k}\right) \\
&\quad = (R_k T_k)^{a_{k+1} - b_{k+1} - 1} b^{r_k + q_{k-1}} \\
&\quad = (R_k T_k)^{a_{k+1} - b_{k+1} - 1} R_k M_{k-1} - R_k M_{k-1} \\
&\quad = R_{k+1} - R_k M_{k-1} = (R_{k+1} - R_k) - (R_k M_{k-1} - R_k),
\end{aligned}$$

if $a_{k+1} - b_{k+1} \geqslant 2$, while

$$V_k \times b^{r_k + q_{k-1}} \frac{b^{(a_{k+1} - b_{k+1} - 1)q_k} - 1}{b^{q_k} - 1} = 0 = (R_{k+1} - R_k) - (R_k M_{k-1} - R_k),$$

if $a_{k+1} - b_{k+1} = 1$, because we then have $R_{k+1} = R_k M_{k-1}$. In both cases we end up with the numerator of $(1)_k$ minus the numerator of $(4)_{k-1}$.

Now, assume that $a_{k+1} = b_{k+1}$. Then, $c_k = -b^{r_{k-1}}$ and we check that

$$(-b^{r_{k-1}})(b^{q_k} - 1) + b^{r_k}(b^{q_{k-1}} - 1) = b^{r_{k-1}} - b^{r_k} = b^{r_{k+1}} - b^{r_k},$$

since $r_{k-1} + q_k = r_k + q_{k-1}$ and $r_{k-1} = r_{k+1}$. As for the numerators, we have

$$\begin{aligned} V_k \times (-b^{r_{k-1}}) &= -V_k R_{k-1} + R_{k-1} \\ &= -R_k T_k R_{k-1} + R_{k-1} \\ &= -R_k M_{k-1} + R_{k+1} = (R_{k+1} - R_k) - (R_k M_{k-1} - R_k), \end{aligned}$$

which confirms our claim.

For the second equality, observe that

$$b^{t_k}\left(b^{r_k}(b^{r_{k+1}-r_k} - 1)\right) + (b^{q_k} - 1) = b^{q_k+r_{k+1}} - b^{t_k+r_k} + b^{q_k} - 1 = b^{q_k+r_{k+1}} - 1$$

is the denominator of $(2)_k$. Note also that

$$b^{t_k}(R_{k+1} - R_k) = R_{k+1}T_k - R_k T_k = R_{k+1}T_k - V_k$$

is the numerator of $(2)_k$ minus the numerator of $(4)_k$. This completes the proof of the second equality. The third one is a tautology. The remaining two equalities are proved in a way similar to the proof of the second one. We omit the details. $\square$

Define two sequences $(P_j)_{j \geqslant -1}$ and $(Q_j)_{j \geqslant -1}$ of integers by setting

$$P_{-1} = b - 1, \quad Q_{-1} = 0, \quad P_0 = 0, \quad Q_0 = b - 1,$$

and, denoting by $(\alpha_j)_{j \geqslant 1}$ the sequence of integers $c_0, d_0, 1, e_0, f_0, c_1, \ldots$,

$$P_{j+2} = \alpha_{j+2} P_{j+1} + P_j, \quad Q_{j+2} = \alpha_{j+2} Q_{j+1} + Q_j, \quad j \geqslant -1.$$

Since

$$c_0 = \frac{b^{a_1 - b_1} - b}{b - 1}, \quad d_0 = 0, \quad e_0 = b - 1,$$

we get

$$\begin{gathered} P_1 = b - 1, \quad Q_1 = b^{a_1 - b_1} - b, \quad P_2 = P_0, \quad Q_2 = Q_0, \\ P_3 = b - 1, \quad Q_3 = b^{a_1 - b_1} - 1, \quad P_4 = (b - 1)^2, \quad Q_4 = b^{a_1 - b_1}(b - 1), \ldots. \end{gathered}$$

Thus

$$\frac{P_1}{Q_1} = (1)_0, \quad \frac{P_2}{Q_2} = (2)_0 \dot{-} (1)_0, \quad \frac{P_3}{Q_3} = (2)_0, \quad \frac{P_4}{Q_4} = (3)_0, \quad \ldots .$$

Using Lemma 7.1, we check by induction on $j \geqslant 1$ that the greatest prime divisor of the integers $P_j$ and $Q_j$ is equal to $b-1$ and that $P_j$ and $Q_j$ are the numerator and denominator of a fraction of one of the five types $(1)_k$, $(2)_k \dot{-} (1)_k$, $(2)_k$, $(3)_k$, $(4)_k$, more precisely, they correspond to

- the fraction $(1)_k$ if $\alpha_j = c_k$;
- the fraction $(2)_k \dot{-} (1)_k$ if $\alpha_j = d_k$;
- the fraction $(2)_k$ if $\alpha_j = 1$;
- the fraction $(3)_k$ if $\alpha_j = e_k$;
- the fraction $(4)_k$ if $\alpha_j = f_k$.

We explain below how to derive the sequence of partial quotients of $\xi$ from the sequence $(\alpha_j)_{j \geqslant 1}$.

To do this, we work with matrices and recall that

$$\begin{aligned}\begin{pmatrix} P_{j+1} & P_j \\ Q_{j+1} & Q_j \end{pmatrix} &= \begin{pmatrix} P_j & P_{j-1} \\ Q_j & Q_{j-1} \end{pmatrix} \cdot \begin{pmatrix} \alpha_{j+1} & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & b-1 \\ b-1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} \alpha_{j+1} & 1 \\ 1 & 0 \end{pmatrix}, \quad j \geqslant 0.\end{aligned}$$

So we have a product of elementary integer 2 by 2 matrices $\left(\begin{smallmatrix} \alpha_j & 1 \\ 1 & 0 \end{smallmatrix}\right)$, exactly as in the continued fraction algorithm. Here, however, some coefficients $\alpha_j$ may be 0 or negative. The point is that it is possible to transform this formal infinite product into a product of elementary integer 2 by 2 matrices $\left(\begin{smallmatrix} \alpha''_j & 1 \\ 1 & 0 \end{smallmatrix}\right)$ where all the $\alpha''_j$'s are positive. This defines a regular continued fraction and we show that this is precisely the continued fraction expansion of $\xi$. A general study of the multiplicative relations between elementary matrices may be found in [18].

Simple calculations show that for nonnegative integers $x$ and $y$ we have

$$\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} -x-1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \tag{7.1}$$

and

$$\begin{pmatrix} y & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} y & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \tag{7.2}$$

If for some integer $j \geqslant 1$ the integer $\alpha_{j+5} = c_{k+1}$ is negative, then $c_{k+1} = -e_k - 1$ and, as $b_{k+1} = 0$, we get $d_{k+1} = d_k$, $f_k = 0$, and the septuple $(\alpha_{j+1}, \ldots, \alpha_{j+7})$ is equal to $(d_k, 1, e_k, 0, -e_k - 1, d_k, 1)$. Consequently, by (7.1) and (7.2), we have

$$\begin{pmatrix} \alpha_{j+1} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} \alpha_{j+7} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

We derive that

$$\begin{aligned} \begin{pmatrix} P_{j+8} & P_{j+7} \\ Q_{j+8} & Q_{j+7} \end{pmatrix} &= \begin{pmatrix} P_{j-1} & P_{j-2} \\ Q_{j-1} & Q_{j-2} \end{pmatrix} \cdot \begin{pmatrix} \alpha_j & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} \alpha_{j+8} & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} P_{j-1} & P_{j-2} \\ Q_{j-1} & Q_{j-2} \end{pmatrix} \cdot \begin{pmatrix} \alpha_j & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_{j+8} & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} P_{j-1} & P_{j-2} \\ Q_{j-1} & Q_{j-2} \end{pmatrix} \cdot \begin{pmatrix} \alpha_j + \alpha_{j+8} + 1 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} (\alpha_j + \alpha_{j+8} + 1)P_{j-1} + P_{j-2} & P_{j-2} \\ (\alpha_j + \alpha_{j+8} + 1)Q_{j-1} + Q_{j-2} & Q_{j-2} \end{pmatrix} \\ &= \begin{pmatrix} (c_k + e_{k+1} + 1)P_{j-1} + P_{j-2} & P_{j-1} \\ (c_k + e_{k+1} + 1)Q_{j-1} + Q_{j-2} & Q_{j-1} \end{pmatrix}. \end{aligned} \tag{7.3}$$

This shows that $P_{j-1}$ is followed by $P_{j+8} = (c_k + e_{k+1} + 1)P_{j-1} + P_{j-2}$, and similarly for $Q_{j-1}$.

Consider now the sequence $(\alpha'_j)_{j\geqslant 1}$ constructed inductively from $(\alpha_j)_{j\geqslant 1}$ as follows. We put $\alpha'_j = \alpha_j$ for $j < j_0$, where $j_0 \geqslant 1$ is the smallest integer such that $\alpha_{j_0} = c_k$, with $\alpha_{j_0+5} = c_{k+1} < 0$. Then, we put $\alpha'_{j_0} = c_k + e_{k+1} + 1$ and $\alpha'_{j_0+1} = \alpha_{j_0+9} = f_{k+1}$. We continue with $\alpha'_{j_0+2} = c_{k+2}$, unless $c_{k+3} < 0$, in which case we put $\alpha'_{j_0+2} = c_{k+2} + e_{k+3} + 1$. And so on. The sequence $(\alpha'_j)_{j\geqslant 1}$ is well-defined since $c_k$ and $c_{k+1}$ cannot be simultaneously negative.

Said differently, for each index $k$ such that $c_{k+1} < 0$, we replace the 10 consecutive partial quotients $c_k, d_k, \ldots, e_{k+1}, f_{k+1}$ by the 2 partial quotients $c_k + e_{k+1} + 1, f_{k+1}$. Let us add that $f_{k+1}$ is positive since $b_{k+2}$ is positive.

We have constructed from $(\alpha_j)_{j\geqslant 1}$ a sequence of nonnegative integers $(\alpha'_j)_{j\geqslant 1}$. Define

$$P'_{-1} = b - 1, \quad P'_0 = 0, \quad Q'_{-1} = 0, \quad Q'_0 = b - 1,$$

and

$$P'_{j+2} = \alpha'_{j+2} P_{j+1} + P'_j, \quad Q'_{j+2} = \alpha'_{j+2} Q_{j+1} + Q'_j, \quad j \geqslant -1.$$

By construction, the sequence of pairs $((P'_j, Q'_j))_{j\geqslant 0}$ is a subsequence of $((P_j, Q_j))_{j\geqslant 0}$. Furthermore, it follows from (7.3) that $P'_j$ and $Q'_j$ are the numerator and denominator of

- the fraction $(1)_k$ if $\alpha'_j = c_k$;
- the fraction $(2)_k \dot{-} (1)_k$ if $\alpha'_j = d_k$;
- the fraction $(2)_k$ if $\alpha'_j = 1$;
- the fraction $(3)_k$ if $\alpha'_j = e_k$;
- the fraction $(3)_{k+1}$ if $\alpha'_j = c_k + e_{k+1} + 1$;
- the fraction $(4)_k$ if $\alpha'_j = f_k$.

Now, we have to get rid of the 0's in $(\alpha'_j)_{j\geqslant 1}$ and construct a sequence $(\alpha''_j)_{j\geqslant 1}$ of positive integers. Since $e_k$ is positive for $k \geqslant 0$, there are no sequences of more than 3 consecutive 0's in $(\alpha'_j)_{j\geqslant 1}$.

As already observed, for nonnegative integers $x$ and $y$, we have

$$\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} y & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x+y & 1 \\ 1 & 0 \end{pmatrix}$$

and, if $\alpha'_{j+1} = 0$, we get

$$\begin{aligned} \begin{pmatrix} P'_{j+2} & P'_{j+1} \\ Q'_{j+2} & Q'_{j+1} \end{pmatrix} &= \begin{pmatrix} P'_{j-1} & P'_{j-2} \\ Q'_{j-1} & Q'_{j-2} \end{pmatrix} \cdot \begin{pmatrix} \alpha'_j & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha'_{j+1} & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha'_{j+2} & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} P'_{j-1} & P'_{j-2} \\ Q'_{j-1} & Q'_{j-2} \end{pmatrix} \cdot \begin{pmatrix} \alpha'_j + \alpha'_{j+2} & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} (\alpha'_j + \alpha'_{j+2}) P'_{j-1} + P'_{j-2} & P'_{j-1} \\ (\alpha'_j + \alpha'_{j+2}) Q'_{j-1} + Q'_{j-2} & Q'_{j-1} \end{pmatrix}. \end{aligned} \tag{7.4}$$

This shows that $P'_{j-1}$ is followed by $P'_{j+2} = (\alpha'_j + \alpha'_{j+2}) P_{j-1} + P_{j-2}$, and similarly for $Q'_{j-1}$.

By (7.4), if $x, 0, y$ are consecutive elements in $(\alpha'_j)_{j\geqslant 1}$, they have to be replaced by the single element $x + y$ in $(\alpha''_j)_{j\geqslant 1}$ and the pair associated with the partial quotient $x+y$ is the pair associated with $\alpha'_{j+2}$, that is, the pair $(P'_{j+2}, Q'_{j+2})$. Define recursively

$$P''_{-1} = b - 1, \quad P''_0 = 0, \quad Q''_{-1} = 0, \quad Q''_0 = b - 1,$$

and

$$P''_{j+2} = \alpha''_{j+2} P''_{j+1} + P''_j, \quad Q''_{j+2} = \alpha''_{j+2} Q''_{j+1} + Q''_j, \quad j \geqslant -1.$$

By construction, the sequence of pairs $((P''_j, Q''_j))_{j\geqslant 0}$ is a subsequence of $((P'_j, Q'_j))_{j\geqslant 0}$, hence of $((P_j, Q_j))_{j\geqslant 0}$. Let us discuss more in details which are the possible elements of the sequence $(\alpha''_j)_{j\geqslant 1}$. The following cases may occur:

(i) $b_k = 0$ and $a_{k+2} = b_{k+2}$ with $k \geqslant 1$, corresponding to the string

$$1, e_{k-1}, f_{k-1} = 0, c_k + e_{k+1} + 1, f_{k+1},$$

where $e_{k-1} > 0$, $c_k + e_{k+1} + 1 > 0$, $f_{k+1} > 0$. We get the partial quotient $e_{k-1} + c_k + e_{k+1} + 1$ in $(\alpha''_j)_{j\geqslant 1}$ and

$$[0; \alpha''_1, \ldots, 1, e_{k-1} + c_k + e_{k+1} + 1] = (3)_{k+1},$$

the preceding convergent being $(2)_{k-1}$.

(ii) $b_{k+1} = 0$, $t_{k+1} \geqslant 1$, $a_{k+2} \geqslant b_{k+2} + 2$, $a_{k+3} > b_{k+3}$ with $k \geqslant 0$, corresponding to the string

$$1, e_k, f_k = 0, c_{k+1}, d_{k+1}, \quad \text{where} \quad e_k > 0,\ c_{k+1} > 0,\ d_{k+1} > 0.$$

We get the partial quotient $e_k + c_{k+1}$ in $(\alpha''_j)_{j\geqslant 1}$ and

$$[0; \alpha''_1, \ldots, 1, e_k + c_{k+1}] = (1)_{k+1},$$

the preceding convergent being $(2)_k$.

(iii) $b_{k+1} \geqslant 1$, $a_{k+2} = b_{k+2} + 1$, $a_{k+3} > b_{k+3}$ with $k \geqslant 0$, corresponding to the string

$$e_k, f_k, c_{k+1} = 0, d_{k+1}, 1, \quad \text{where} \quad e_k > 0,\ f_k > 0,\ d_{k+1} > 0.$$

We get the partial quotient $f_k + d_{k+1}$ in $(\alpha''_j)_{j\geqslant 1}$ and

$$[0; \alpha''_1, \ldots, e_k, f_k + d_{k+1}] = (2)_{k+1} \dot{-} (1)_{k+1},$$

the preceding convergent being $(3)_k$.

(iv) $b_{k+1} = 0$, $t_{k+1} \geqslant 1$, $a_{k+2} = b_{k+2} + 1$, $a_{k+3} > b_{k+3}$ with $k \geqslant 0$, corresponding to the string

$$1, e_k, f_k = 0, c_{k+1} = 0, d_{k+1}, 1, \quad \text{where} \quad e_k > 0,\ d_{k+1} > 0.$$

In this case, we simply remove the two zeros from the sequence $(\alpha'_j)_{j\geqslant 1}$. We get

$$[0; \alpha''_1, \ldots, e_k, d_{k+1}] = (2)_{k+1} \dot{-} (1)_{k+1},$$

the preceding convergent being $(3)_k$.

(v) $t_{k+1} = 0$, $a_{k+2} \geqslant b_{k+2} + 2$, $a_{k+3} > b_{k+3}$ with $k \geqslant 0$, corresponding to the string

$$1, e_k, f_k = 0, c_{k+1}, d_{k+1} = 0, 1, e_{k+1}, \quad \text{where} \quad e_k > 0,\ c_{k+1} > 0,\ e_{k+1} > 0.$$

We get the partial quotient $e_k + c_{k+1} + 1$ in $(\alpha''_j)_{j\geqslant 1}$ and

$$[0; \alpha''_1, \ldots, 1, e_k + c_{k+1} + 1] = (2)_{k+1},$$

the preceding convergent being $(2)_k$.

(vi) $t_{k+1} = 0$ and $a_{k+2} = b_{k+2} + 1$, $a_{k+3} > b_{k+3}$ with $k \geqslant 0$, corresponding to the string

$$1, e_k, f_k = 0, c_{k+1} = 0, d_{k+1} = 0, 1, e_{k+1}, \quad \text{where} \quad e_k > 0,\ e_{k+1} > 0.$$

We get the partial quotient $e_k + 1$ in $(\alpha''_j)_{j\geqslant 1}$ and

$$[0; \alpha''_1, \dots, 1, e_k + 1] = (2)_{k+1},$$

the preceding convergent being $(2)_k$.

Since $d_0 = 0$ and $e_0 > 0$, we always have the initial reduction

$$[0; c_0, 0, 1, e_0, \dots] = [0; c_0 + 1, e_0, \dots],$$

which is not taken into account by the preceding cases.

The cases (v) and (vi) occur only when $d_{k+1} = 0$, that is, when $b_1 = \cdots = b_{k+1} = 0$. They are not reflected in Proposition 6.2, where it is assumed that $t_k$ is positive. The link with Proposition 6.2 is as follows:

- Case (i) of Proposition 6.2 corresponds to the construction of $(\alpha'_j)_{j\geqslant 1}$ from $(\alpha_j)_{j\geqslant 1}$, with, if in addition $b_k = 0$, Case (i) above.
- Case (ii) of Proposition 6.2 corresponds to Case (iii) above if $b_{k+1}$ is positive, and to Case (iv) above if $b_{k+1} = 0$.
- Case (iii) of Proposition 6.2 corresponds to Case (ii) above.

Since the sequence $(\alpha''_j)_{j\geqslant 1}$ is composed of positive integers, the real number

$$\zeta := [0; \alpha''_1, \alpha''_2, \dots]$$

is well defined by its continued fraction expansion. We have proved that all of its convergents are of the form $P_j/Q_j$ for some index $j$.

It also follows from our discussion that $(2)_{k+1}$ is a convergent to $\zeta$ if $c_{k+1}$ and $c_{k+2}$ are nonnegative. If $c_{k+1} < 0$ and $f_{k-1} > 0$, then $f_{k-1}$ is an element of $(\alpha''_j)_{j\geqslant 1}$, associated with $(4)_{k-1} = (2)_{k+1}$. If $c_{k+1} < 0$ and $f_{k-1} = 0$, then $b_k = 0$ and $1, e_{k-1} + c_k + e_{k+1} + 1$ are consecutive elements of $(\alpha''_j)_{j\geqslant 1}$, with this partial quotient 1 being associated to $(2)_{k-1}$ and we have $(2)_{k-1} = (4)_{k-1} = (2)_{k+1}$. To summarize, we have shown that $(2)_{k+1}$ is a convergent to $\zeta$ unless $c_{k+2}$ is negative, that is, unless $a_{k+3} = b_{k+3}$. However, Proposition 6.1 asserts that $(2)_{k+1}$ is a convergent to $\xi$ if and only if $a_{k+3} \geqslant b_{k+3} + 1$. Since there are infinitely many $h$ such that $a_h \geqslant b_h + 1$, we deduce that $\xi$ and $\zeta$ have infinitely many partial quotients in common, thus $\xi = \zeta$.

The next statement summarizes what we have established. For $j \geqslant 1$, write $P''_j/Q''_j = [0; \alpha''_1, \alpha''_2, \dots, \alpha''_j]$ for the $j$-th convergent to $\xi$.

Proposition 7.2. — *All of the convergents to $\xi$ are of one of the five types $(1)_k$, $(2)_k \dot{-} (1)_k$, $(2)_k$, $(3)_k$, $(4)_k$. All of its partial quotients are of the form*

$$1, c_k, d_k, e_k, f_k,$$

*or belong to the set*

$$\begin{aligned}\{c_0+1\} &\cup \bigcup_{k\geqslant 1}\{e_{k-1}+c_k+e_{k+1}+1\}\\ &\cup \bigcup_{k\geqslant 0}\{c_k+e_{k+1}+1, e_k+c_{k+1}, f_k+d_{k+1}, e_k+c_{k+1}+1, e_k+1\}.\end{aligned}$$

*More precisely, we have $P''_1/Q''_1 = (2)_0$ with $\alpha''_1 = c_0+1$ and*

$$P''_j/Q''_j = \begin{cases} (1)_k & \text{if } \alpha''_j \in \{c_k, e_{k-1}+c_k\},\\ (2)_k \dot{-} (1)_k & \text{if } \alpha''_j \in \{d_k, f_{k-1}+d_k\},\\ (2)_k & \text{if } \alpha''_j \in \{1, e_{k-1}+c_k+1, e_{k-1}+1\},\\ (3)_k & \text{if } \alpha''_j \in \{e_k, c_{k-1}+e_k+1, e_{k-2}+c_{k-1}+e_k+1\},\\ (4)_k & \text{if } \alpha''_j = f_k,\end{cases}$$

*for $j \geqslant 2$.*

## 8. Remaining proofs

*Proof of Corollary 2.5.* — Assume that $\theta$ has unbounded partial quotients (the case of bounded partial quotients is treated in Theorem 2.6). Let $\mathcal{K}$ be an infinite set of positive integers such that the subsequence $(a_k)_{k\in\mathcal{K}}$ is increasing. Assume first that there exists an infinite set $\mathcal{K}' \subset \mathcal{K}$ such that $(a_k - b_k)_{k\in\mathcal{K}'}$ is increasing. For $k \geqslant 3$ in $\mathcal{K}'$ we have

$$\nu_{k-2}(4) = 1 + \frac{r_k}{q_{k-1}} \geqslant 1 + \frac{(a_k - b_k - 1)q_{k-1}}{q_{k-1}},$$

and, since $a_k - b_k$ can be arbitrarily large with $k$ in $\mathcal{K}'$, we deduce that $\nu(4)$ is infinite.

Assume now that there exist an infinite set $\mathcal{K}' \subset \mathcal{K}$ and a nonnegative integer $\delta$ such that $a_k - b_k = \delta$ for $k$ in $\mathcal{K}'$. For $k \geqslant 3$ in $\mathcal{K}'$ we have

$$\nu_{k-1}(3) = 1 + \frac{q_k}{r_k + q_{k-1}} \geqslant \frac{a_k q_{k-1}}{r_{k-1} + \delta q_{k-1} + q_{k-2}} \geqslant \frac{a_k}{\delta + 2}.$$

We deduce that $\nu(3)$ is infinite. Consequently, any Sturmian number whose slope has unbounded partial quotients is a Liouville number. □

*Proof of Theorem 2.6.* — Assume that $\theta$ has bounded partial quotients. Observe that

$$\nu_k(3) = 1 + \frac{q_{k+1}}{r_{k+1} + q_k} \leqslant 1 + \frac{q_{k+1}}{q_k}, \quad \nu_k(4) = 1 + \frac{r_{k+2}}{q_{k+1}} \leqslant 1 + \frac{q_{k+2}}{q_{k+1}}.$$

If $a_{k+1} = b_{k+1}$, then $r_{k+1} = r_{k-1}$ and $t_k = t_{k-1}$, thus

$$\nu_k(2) = 2 + \frac{r_k}{r_{k+1} + t_k} = 2 + \frac{r_k}{r_{k-1} + t_{k-1}} \leqslant 2 + \frac{q_k}{q_{k-1}}.$$

If $a_{k+1} > b_{k+1}$, then $r_{k+1} \geqslant r_k + q_{k-1}$, thus

$$\nu_k(2) = 2 + \frac{r_k}{r_{k+1} + t_k} \leqslant 2 + \frac{r_k}{r_k + t_k} \leqslant 3,$$

and

$$\nu_k(1) = 2 + \frac{t_k}{r_{k+1}} \leqslant 2 + \frac{t_k}{r_k + q_{k-1}} \leqslant 2 + \frac{q_k}{q_{k-1}}.$$

This shows that the irrationality exponent of $\xi_b(\theta, \rho)$ satisfies

$$\mu(\xi_b(\theta, \rho)) \leqslant 2 + \limsup_{k \to +\infty} \frac{q_k}{q_{k-1}} = 1 + \mu(\xi_b(\theta)). \tag{8.1}$$

Let us now show that there exist intercepts $\rho$ for which equality holds. Let $\mathcal{K}$ be an infinite set of positive integers such that

$$\lim_{k \to +\infty, k \in \mathcal{K}} \frac{q_k}{q_{k-1}} = \mu(\xi_b(\theta)) - 1.$$

Take $k_1 \geqslant 3$ in $\mathcal{K}$ and set $b_1 = \cdots = b_{k_1} = 0$. Put $a_{k_1+1} = b_{k_1+1}$ and $b_{k_1+2} = b_{k_1+3} = \cdots = b_{k_2} = 0$, where $k_2 > k_1 + 2$ is in $\mathcal{K}$ and sufficiently large to ensure that $r_{k_2} \geqslant q_{k_2}/2$. Then, put $b_{k_2+1} = a_{k_2+1}$ and $b_{k_2+2} = \cdots = b_{k_3} = 0$, where $k_3 > k_2 + 2$ is in $\mathcal{K}$ and sufficiently large to ensure that $r_{k_3} \geqslant 2q_{k_3}/3$. Proceeding like this, we define inductively an increasing sequence $(k_j)_{j \geqslant 2}$ of integers in $\mathcal{K}$ such that $b_{k_j+1} = a_{k_j+1}$ and $b_k = 0$ for every $k$ not in $(k_j)_{j \geqslant 2}$. In addition, we have $r_{k_j} \geqslant (j-1)q_{k_j}/j$, for $j \geqslant 2$.

Let $\rho$ denote the intercept defined by this sequence $(b_k)_{k \geqslant 1}$ and let us determine the irrationality exponent of $\xi_b(\theta, \rho)$.

Recall that for an index $h$ such that $b_{h+1} = a_{h+1}$ we have $r_{h+1} = r_{h-1}$ and $t_h = t_{h-1}$, thus

$$\nu_h(2) = 2 + \frac{r_h}{r_{h+1} + t_h} = 2 + \frac{r_h}{r_{h-1} + t_{h-1}} = 2 + \frac{r_h}{q_{h-1}}.$$

Consequently, we get

$$\nu_{k_j}(2) \geqslant 2 + \frac{(j-1)q_{k_j}}{j q_{k_j - 1}}, \quad j \geqslant 2,$$

and

$$\mu(\xi_b(\theta, \rho)) \geqslant \nu(2) \geqslant 2 + \limsup_{j \to +\infty} \frac{q_{k_j}}{q_{k_j - 1}} = 2 + \lim_{k \to +\infty, k \in \mathcal{K}} \frac{q_k}{q_{k-1}} = \mu(\xi_b(\theta)) + 1.$$

The reverse inequality follows from (8.1). Consequently, we get

$$\mu(\xi_b(\theta,\rho)) = 1 + \mu(\xi_b(\theta)).$$

This proves the theorem. $\square$

*Proof of Theorem 2.7.* — Assume that not all $b_k$ are 0. Let $k$ be an integer large enough to ensure that $t_k$ is positive and that $a_k, a_{k+1}, \dots$ are all at most equal to $M$. Then, it follows from Proposition 6.2 that there are four (possibly overlapping) cases:

(i) If $a_{k+2} = b_{k+2}$, then $(2)_{k+1}$ is a convergent to $\xi$;
(ii) If $a_{k+3} = b_{k+3}$, then $(2)_{k+2}$ is a convergent to $\xi$;
(iii) If $a_{k+4} = b_{k+4}$, then $(2)_{k+3}$ is a convergent to $\xi$;
(iv) If (i), (ii), and (iii) do not hold, then $(1)_{k+1}$ and $(2)_{k+1}$ are convergents to $\xi$.

In case (i), the rate of approximation of $\xi$ by $(2)_{k+1}$ is at least equal to

$$\nu_{k+1}(2) = 2 + \frac{r_{k+1}}{r_{k+2} + t_{k+1}} = 2 + \frac{r_{k+1}}{q_k} \geqslant 2 + \frac{q_{k-1}}{q_k} \geqslant 2 + \frac{1}{M+1},$$

since $r_{k+2} = r_k$, $t_{k+1} = t_k$, and $r_{k+1} \geqslant q_{k-1}$.

Similarly, in case (ii) (resp., (iii)), the rate of approximation of $\xi$ by $(2)_{k+2}$ (resp., by $(2)_{k+3}$) is at least equal to $2 + 1/(M+1)$.

In case (iv), note that $r_{k+2} + t_{k+1} \leqslant (a_{k+2}+1)q_{k+1}$, thus

$$\nu_{k+1}(1) = 2 + \frac{t_{k+1}}{r_{k+2}} \geqslant 2 + \frac{t_{k+1}}{(M+1)q_{k+1}}$$

and

$$\nu_{k+1}(2) = 2 + \frac{r_{k+1}}{r_{k+2} + t_{k+1}} \geqslant 2 + \frac{r_{k+1}}{(M+1)q_{k+1}}.$$

Recalling that $r_{k+1} + t_{k+1} = q_{k+1}$, we get

$$\max\{\nu_{k+1}(1), \nu_{k+1}(2)\} \geqslant 2 + \frac{1}{2(M+1)}.$$

This shows that, for every sufficiently large $k$, there exists a rational number $P/Q$ with

$$b^{q_k} \leqslant Q \leqslant b^{q_{k+4}} \tag{8.2}$$

such that $|\xi - P/Q| \leqslant Q^{-2-1/(2(M+1))}$. We are then in position to apply Théorème 3.1 of [3] with $\varepsilon = \frac{1}{2(M+1)}$ and $\mathcal{S}$ the empty set. Note that, by (8.2), the number $c$ introduced in [3, (3.2)] can be taken to be $(M+1)^5$. Consequently, the upper bound

$$w_d^*(\xi) \leqslant (2d)^{\kappa(\log\log 3d)}, \quad d \geqslant 1,$$

given by [3, Théorème 3.1] holds with a real number $\kappa$ depending only on $M$. $\square$

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