Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

Yong Li, David Sauzin & Shanzhong Sun

On the Moyal Star Product of Resurgent Series

Tome 73, nº 5 (2023), p. 1987-2027.

https://doi.org/10.5802/aif.3565

Les *Annales de l'Institut Fourier* sont membres du
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org e-ISSN : 1777-5310

Ann. Inst. Fourier, Grenoble
73, 5 (2023) 1987-2027

ON THE MOYAL STAR PRODUCT OF RESURGENT SERIES

by Yong LI, David SAUZIN & Shanzhong SUN (*)

Abstract. — We analyze the Moyal star product in deformation quantization from the resurgence theory perspective. By putting algebraic conditions on Borel transforms, one can define the space of "algebro-resurgent series" (a subspace of 1-Gevrey formal series in $i\hbar$ with coefficients in $\mathbb{C}\{x_1,\dots,x_d\}$), which we show is stable under Moyal star product.

Résumé. — Nous analysons le star produit de Moyal de la quantification par déformation sous l'angle de la théorie de la résurgence. En imposant des conditions algébriques sur les transformées de Borel, on peut définir l'espace des « séries algébro-résurgentes » (un sous-espace des séries formelles Gevrey-1 en l'indéterminée $i\hbar$ à coefficients dans $\mathbb{C}\{x_1,\dots,x_d\}$), dont nous montrons qu'il est stable par star-produit de Moyal.

1. Introduction

1.1. Main results

Given a Poisson structure with constant coefficients in d dimensions

$$\pi = \sum_{1\leqslant i<j\leqslant d} \pi^{i,j}\partial_{x_i}\wedge\partial_{x_j}, \qquad \pi^{j,i} = -\pi^{i,j}\in\mathbb{R}, \tag{1.1}$$

Keywords: Deformation quantization, Moyal product, Resurgence theory, Algebro-resurgent series, Hadamard product.
2020 *Mathematics Subject Classification:* 53D55, 32Dxx.
(*) The first author aknowledges support from NSFC (No.11771303) and thanks Capital Normal University for their hospitality during the period September 2019 – May 2021. The second author thanks Capital Normal University for their hospitality during the period September 2019–February 2020.
The third author is partially supported by National Key R&D Program of China (2020YFA0713300), NSFC (No.s 11771303, 11911530092, 11871045).
This paper is partly a result of the ERC-SyG project, Recursive and Exact New Quantum Theory (ReNewQuantum) which received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under grant agreement No 810573.

the corresponding Poisson bracket can be written

$$\{f,g\} = \mu\circ P(f\otimes g), \quad P := \sum_{1\leqslant i,j\leqslant d} \pi^{i,j}\partial_{x_i}\otimes\partial_{x_j}, \quad \mu := \text{multiplication.} \tag{1.2}$$

The corresponding Moyal star product is then

$$\begin{aligned}\tilde f \star_M \tilde g &= \mu\circ\exp\left(\frac{tP}{2}\right)(\tilde f\otimes\tilde g)\\ &= \tilde f\tilde g + \frac{t}{2}\sum_{i,j}\pi^{i,j}\,\partial_{x_i}(\tilde f)\,\partial_{x_j}(\tilde g)\\ &\qquad + \frac{1}{2!}\left(\frac{t}{2}\right)^2\sum_{i,j,k,\ell}\pi^{i,j}\pi^{k,\ell}\,\partial_{x_i}\partial_{x_k}(\tilde f)\,\partial_{x_j}\partial_{x_\ell}(\tilde g) + \cdots\end{aligned} \tag{1.3}$$

for any formal series $\tilde f, \tilde g$ in $t, x_1, \dots, x_d$ with complex coefficients. One gets a non-commutative associative algebra $\big(\mathbb{C}[\![t, x_1, \dots, x_d]\!], \star_M\big)$, the unit of which is the constant series 1 and the product of which can be viewed as a non-commutative deformation of ordinary multiplication in the direction of π, in the sense that

$$\tilde f \star_M \tilde g = \tilde f\,\tilde g + O(t), \qquad \tilde f \star_M \tilde g - \tilde g \star_M \tilde f = t\{\tilde f, \tilde g\} + O(t^2). \tag{1.4}$$

The main result of this paper is

Theorem 1.1. — *If $\tilde f$ and $\tilde g$ are algebro-resurgent series in the $1+d$ variables $t, x_1, \dots, x_d$, then so is their Moyal star product $\tilde f \star_M \tilde g$.*

Here, "algebro-resurgence" is a property of a formal series $\tilde f(t, x_1, \dots, x_d)$ defined in terms of its formal Borel transform $\hat f(\xi, x_1, \dots, x_d)$ with respect to the first variable, which is required to be convergent, i.e. $\hat f(\xi, x_1, \dots, x_d) \in \mathbb{C}\{\xi, x_1, \dots, x_d\}$, and to admit analytic continuation along all the paths which start close enough to the origin of $\mathbb{C}^{1+d}$ and avoid a proper algebraic subvariety (which depends on $\tilde f$). More details will be given in due time.

This variant of Écalle's definition of resurgence [8, 9] was introduced by M. Garay, A. de Goursac and D. van Straten in their pioneering work on resurgent deformation quantization [12], where they state the

Theorem 1.2. — *If $\tilde f$ and $\tilde g$ are algebro-resurgent series in the $1+2N$ variables $t, q_1, \dots, q_N, p_1, \dots, p_N$, then so is their standard star product*

$$\tilde f \star_S \tilde g := \sum_{k_1,\dots,k_N\geqslant 0} \frac{t^{k_1+\cdots+k_N}}{k_1!\cdots k_N!}\Big(\partial_{p_1}^{k_1}\cdots\partial_{p_N}^{k_N}\tilde f\Big)\Big(\partial_{q_1}^{k_1}\cdots\partial_{q_N}^{k_N}\tilde g\Big). \tag{1.5}$$

However, their proof of Theorem 1.2 is not valid, due to a flaw in one of the key formulas presented in [12]. In this paper, we will give the correct formula and develop somewhat different arguments that lead to a proof of Theorem 1.2.

We will go from Theorem 1.2 to Theorem 1.1 by means of a linear change of variables and a further result on the stability of algebro-resurgence under the "transition operator" introduced in (1.7) infra. Indeed, since the Poisson structure π we started with has constant coefficients, we can pass from the initial coordinates $x_1, \dots, x_d$ to canonical coordinates: setting $2N :=$ rank of the antisymmetric matrix $[\pi^{i,j}]$ and $s := d - 2N \geqslant 0$, a linear change leads us to coordinates $q_1, \dots, q_N, p_1, \dots, p_N, y_1, \dots, y_s$ in which π is the standard Poisson structure in N degrees of freedom,

$$\pi = \sum_{1 \leqslant i < j \leqslant N} \partial_{p_i} \wedge \partial_{q_i}. \tag{1.6}$$

Forgetting about the "inert" variables $y_1, \dots, y_s$, we are now dealing with the Poisson bracket $\{\cdot\,,\cdot\}$ associated with the standard symplectic structure $\mathrm{d}p_1 \wedge \mathrm{d}q_1 + \cdots + \mathrm{d}p_N \wedge \mathrm{d}q_N$, and the corresponding Moyal star product $\star_M$ is known to be the image of the standard star product $\star_S$ by the transition operator

$$T := \exp\left(-\frac{t}{2} \sum_{1 \leqslant j \leqslant N} \partial_{q_j} \partial_{p_j}\right) \tag{1.7}$$

in the sense that $T(\widetilde{f} \star_S \widetilde{g}) = (T\widetilde{f}) \star_M (T\widetilde{g})$. We will establish

Theorem 1.3. — *If $\widetilde{f}$ is an algebro-resurgent series in the $1 + 2N$ variables $t, q_1, \dots, q_N, p_1, \dots, p_N$, then so are $T\widetilde{f}$ and $T^{-1}\widetilde{f}$.*

Theorem 1.1 will follow from Theorems 1.2 and 1.3.

1.2. Background and motivation

We now give some background and motivation.

Ever since quantum mechanics started in the 1920s, quantization and semiclassical limit have become a central theme among a variety of areas in mathematics such as functional analysis, geometry and topology, representation theory, pseudo-differential operators and microlocal analysis and symplectic geometry, to name a few.

Conventional quantum mechanics is formulated in terms of linear operators on Hilbert space that realize the fundamental Canonical Commutation Relations, or of Feynman's path integrals as conceived by Dirac and developed by Feynman to make the quantum picture more compatible with the classical one. Built upon Wigner, Weyl and Groenewold's insights and pioneered by Moyal, deformation quantization is a third formulation, in full phase space, which evolved gradually into an autonomous theory with its own internal logic, that is conceptually very appealing.

The idea of deformation quantization is to achieve Heisenberg's Canonical Commutation Relations by deforming the commutative algebra of functions on the phase space (classical observables) to a non-commutative associative algebra.

In [18], in the case of the standard Poisson structure, Moyal introduced his star product $\star_M$ in relation with statistical properties of quantum mechanics. For one degree of freedom, the standard Poisson structure for functions $f(q,p)$ and $g(q,p)$ in $\mathbb{R}^2$ being

$$\{f,g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} = \mu \circ P(f \otimes g), \quad \text{with } P := \frac{\partial}{\partial p} \otimes \frac{\partial}{\partial q} - \frac{\partial}{\partial q} \otimes \frac{\partial}{\partial p},$$

where μ is the usual pointwise product of functions (or more accurately the restriction to the diagonal when viewing the tensor product of two functions as a function on $\mathbb{R}^2 \times \mathbb{R}^2$), the Moyal star product of two classical observables is the formal series in t obtained as

$$f \star_M g = \mu \circ \exp\left(\frac{tP}{2}\right)(f \otimes g) = \mu \circ \left(\mathrm{Id} + \sum_{k \geqslant 1} \frac{t^k}{2^k k!} P^k\right)(f \otimes g) \tag{1.8}$$

with

$$\begin{aligned} P^k &= \sum_{n=0}^{k} (-1)^{k-n} \binom{k}{n} (\partial_p \otimes \partial_q)^n (\partial_q \otimes \partial_p)^{k-n} \\ &= \sum_{n=0}^{k} (-1)^{k-n} \binom{k}{n} \partial_p^n \, \partial_q^{k-n} \otimes \partial_p^{k-n} \, \partial_q^n, \end{aligned}$$

i.e.

$$f \star_M g = fg + \sum_{k \geqslant 1} \frac{t^k}{2^k k!} \sum_{n=0}^{k} (-1)^{k-n} \binom{k}{n} \left(\partial_p^n \, \partial_q^{k-n} f\right)\left(\partial_p^{k-n} \, \partial_q^n g\right). \tag{1.9}$$

Here, t is the deformation parameter, taken to be $i\hbar$ in quantum mechanics. The extension to $N \geqslant 1$ degrees of fredom is obtained by replacing P by

$$P = \sum_{j=1}^{N} \left(\frac{\partial}{\partial p_j} \otimes \frac{\partial}{\partial q_j} - \frac{\partial}{\partial q_j} \otimes \frac{\partial}{\partial p_j}\right) \tag{1.10}$$

in (2.1). When extended to $C^\infty(\mathbb{R}^{2N})[\![t]\!]$, the Moyal star product is a noncommutative associative product. We have $f \star_M g = fg + \frac{t}{2}\{f,g\} + O(t^2)$, hence

$$[f,g]_M \coloneqq \frac{1}{t}(f \star_M g - g \star_M f) = \{f,g\} + O(t), \tag{1.11}$$

we thus recover the Poisson algebra structure of the classical observables in the limit $\hbar = t/i \to 0$. Moreover, the Canonical Commutation Relations are realized:

$$[p_j, q_j]_M = 1 \text{ and } [p_j, q_k]_M = [p_j, p_k]_M = [q_j, q_k]_M = 0 \text{ for } j \neq k. \tag{1.12}$$

The Moyal star product can be viewed as a non-commutative associative deformation of the usual product of functions in the direction of the Poisson structure. The idea to view Quantum Mechanics as a deformation of Classical Mechanics was promoted by Bayen–Flato–Fronsdal–Lichnerowicz–Sternheimer [2, 3] in the 1970s and led to what is now called Deformation Quantization Theory.

For general symplectic manifolds, the existence of a star product that satisfies the analogue of (1.11) was proved in [5] and [10, 11]. In particular, Fedosov recursively constructed a star product through a canonical flat connection on the Weyl bundle.

For the deformation quantization of an arbitrary Poisson structure π in $\mathbb{R}^d$, Kontsevich constructed in 1997 an intriguing explicit formula for a star product that satisfies the analogue of (1.11) ([14]):

$$f \star_K g = fg + \sum_{k \geqslant 1} t^k \sum_{\Gamma \in G_k} c_\Gamma B_{\Gamma,\pi}(f,g), \tag{1.13}$$

where each G_k is a suitable collection of graphs, the c_Γ's are universal coefficients, and the $B_{\Gamma,\pi}$'s are polydifferential operators depending on the graph Γ and the Poisson structure π. Recently, a deep connection between these universal coefficients c_Γ and multiple zeta values(1) has been brought to light [1].

For a general Poisson manifold (M,π), the existence of a star product that satisfies the analogue of (1.11) is a consequence of the formality theorem which establishes an L_∞ quasi-isomorphism between two differential graded Lie algebras (DGLAs): the Hochschild complex of the associative algebra $A = C^\infty(M)$ and its cohomology.

(1) Interestingly, multiple zeta values are themselves deeply related to Resurgence Theory – see e.g. [23].

1.3. Resurgence theory

It is known after Dyson [6] and others that, in quantum field theory, almost all the series in $\hbar$ describing physical quantities are divergent and must be interpreted as giving asymptotic information. In quantum mechanics, this can even be traced back to as early as Birkhoff. When Voros developed the exact WKB theory [21] to study the spectrum of Sturm–Liouville operators, he already conjectured the resurgent character of these series. Resurgence Theory was then a new perspective, initiated by Écalle [7, 9], to deal with asymptotic series. Écalle immediately clarified and confirmed Voros' conjecture in [8] and [9]. Pham and his collaborators devoted a lot of energy to make the whole picture complete in [19] and a series of papers in the 1990s, culminating in the proof of the conjectural formula proposed by Zinn–Justin [24] on multi-instanton expansions in quantum mechanics (however they had to rely on a resurgence conjecture stated in [8], the proof of which has not yet been given in detail).

In a nutshell, in Resurgence Theory, one considers formal series $\widetilde{\varphi}(t) = \sum_{n\geqslant 0} a_n t^n$ (in applications to physics, the coefficients a_n may be functions on the configuration space or the phase space) and their formal Borel transforms $\widehat{\varphi}(\xi)$ defined by[(2)]

$$\text{(1.14)} \quad \text{formal Borel transform } \beta\colon\ \widetilde{\varphi}(t) = \sum_{n\geqslant 0} a_n t^n \mapsto \widehat{\varphi}(\xi) = \sum_{n\geqslant 0} a_n \frac{\xi^n}{n!},$$

and one imposes the convergence of $\widehat{\varphi}(\xi)$ for $|\xi|$ small enough and suitable conditions on its analytic continuation, so as to be able to analyse the various "Borel–Laplace sums" $\mathscr{S}^\theta\widetilde{\varphi}(t) = \int_0^{e^{i\theta}\infty} e^{-\xi/t}\widehat{\varphi}(\xi)\mathrm{d}t/t$ for all non-singular directions θ: the functions $\mathscr{S}^\theta\widetilde{\varphi}$ all are asymptotic to $\widetilde{\varphi}(t)$ as $|t| \to 0$ but differ by exponentially small quantities. The analytic continuation of the convergent germ $\widehat{\varphi}(\xi)$ is required to have at worse isolated singularities. More precisely (in order of increasing generality): to be "Ω-continuable" with a certain prescribed set Ω of potentially singular points [7, 20]; or "endlessly continuable" [4]; or "continuable without a cut" [9].

(2) In this article, we depart from the usual convention of Resurgence Theory, which is to define the formal Borel transform as $\mathcal{B}\big(\sum_{n\geqslant 1} a_n t^n\big) = \sum_{n\geqslant 1} a_n \frac{\xi^{n-1}}{(n-1)!}$ and to handle the constant term a_0 separately by setting $\mathcal{B}(1) = \delta$ (a symbol that can be identified with the Dirac mass at 0). Obviously, formula (1.14) yields $\beta(\widetilde{\varphi}) = \mathcal{B}(t\widetilde{\varphi})$. The advantage of $\mathcal{B}$ over β is that it gives rise to slightly simpler formula for convolution and Laplace transform. However, β has other advantages, e.g. not to force us to deal separately the t-independent term. Note that the choice of β as formal Borel transform is also the one favored by Voros.

Another variant of this property of continuability in the Borel plane was introduced in [12] under the name “algebro-resurgence”. It was designed for situations where the coefficients a_n depend analytically on affine variables $x = (x_1, \ldots, x_d) \in \mathbb{C}^d$: the singular locus of $\widehat{\varphi}(\xi, x)$ is required to be a proper algebraic subvariety of $\mathbb{C}^{1+d}$, the germ $\widehat{\varphi}(\xi, x)$ should have analytic continuation along all the paths that avoid it; in particular, for fixed x, only finitely many singular points can exist in the Borel plane. It is with this version of resurgence that we will work throughout this article.

The present article does not require any familiarity with Resurgence Theory on the part of the reader. But let us mention that, from the viewpoint of Resurgence Theory, stability properties like those indicated in our main results are reminiscent of the stability of the space of resurgent series under multiplication and other nonlinear operations [7, 4, 20]. The possibility of doing nonlinear analysis with resurgent series has always been emphasized by J. Écalle and, ultimately, is responsible for the success of Resurgence Theory in nonlinear dynamics, its initial area of application in mathematics. The success of the theory in WKB analysis also has recently aroused renewed interest for its applicability to mathematical physics, as testified e.g. by the resurgence conjecture of the recent article [15].

1.4. Algebro-resurgent Moyal product

Our initial motivation was to understand Deformation Quantization and the explicit construction (1.13) of Kontsevich from the viewpoint of Resurgence Theory. But already at the level of the Moyal star product (1.3), even with $N = 1$ and analytic classical observables that do not depend on t, one can see that the star product is generically divergent as a series in t, but with at most factorial growth due to the Cauchy inequalities—see the examples in Section 2.2. It is thus natural to consider the Moyal star product of two elements of $\widetilde{f}$ and $\widetilde{g}$ of $\mathbb{C}\{q_1, \ldots, q_N, p_1, \ldots, p_N\}[\![t]\!]$ and to enquire on $\beta(\widetilde{f} \star_M \widetilde{g})$ in terms of $\beta\widetilde{f}$ and $\beta\widetilde{g}$, i.e. to investigate the Borel counterpart of the Moyal star product:

$$\widehat{f} *_M \widehat{g} \coloneqq \beta\big(\beta^{-1}\widehat{f} \star_M \beta^{-1}\widehat{g}\big). \tag{1.15}$$

This is what Garay, de Goursac and van Straten did in [12] with the “standard star product” $\star_S$ defined by (1.5), which is a star product equivalent to the Moyal one via the transition operator T. They considered

$$\widehat{f} *_S \widehat{g} \coloneqq \beta\big(\beta^{-1}\widehat{f} \star_S \beta^{-1}\widehat{g}\big) \tag{1.16}$$

with a view to proving Theorem 1.2: supposing that $\widetilde{f}$ and $\widetilde{g}$ are algebro-resurgent series, i.e. that $\widehat{f}$ and $\widehat{g}$ are algebro-resurgent germs, is it true that $\widehat{f} *_S \widehat{g}$ is an algebro-resurgent germ (and hence that $\widetilde{f} \star_S \widetilde{g}$ is an algebro-resurgent series)?

However, the analysis in [12] relies on an integral representation of $*_S$ that is flawed (Proposition 3.3 of that article), thus invalidating the purported proof of Theorem 1.2. In Section 2, we will give another integral representation of $*_S$, formula (2.30). The correct formula is more intricate than that of [12]; therefore, following the analytic continuation of $\widehat{f} *_S \widehat{g}$ (where both factors are supposed to be algebro-resurgent) requires considerably more work.

For the sake of clarity, we will begin in Section 2 with the case of one degree of freedom and give in Lemma 2.6 the formula for $*_S$ for that case. It is a mixture of convolution(3) and Hadamard product (which has the classical integral representation given as (6.2) infra); more specifically, the formula involves the Hadamard product with respect to ζ of the Taylor expansions $\widehat{f}(\xi_1, q, p+\zeta) \odot \widehat{g}(\xi_2, q+\zeta, p)$ and then a convolution-like integration with respect to ζ, ξ_1 and ξ_2.

Analytic continuation of convolution is a classical topic in Resurgence Theory [7, 4, 20, 17]. We will adapt these techniques to our more intricate situation in Section 5. The analytic continuation of the Hadamard product of two Ω-continuable germs has been treated in [16], with a possibly infinite singular locus Ω; our situation is simpler inasmuch as it involves only finite singular loci in the Borel plane, as we will see in Section 6 devoted to the Hadamard part of the formula for $*_S$.

The technique for following the analytic continuation of $\widehat{f} *_S \widehat{g}$ in the case of N degrees of freedom is indicated in Section 7. This will lead us to a proof of Theorem 1.2 that follows a path rather different than that of [12]. Then, using the concrete form of the equivalence T between the Moyal and standard star products $\star_M$ and $\star_S$, we will be able to relate $*_M$ and $*_S$ by an integral transform $\widehat{T}$, prove that

$$\widehat{f} \text{ algebro-resurgent germ } \Rightarrow \widehat{T}\widehat{f} \text{ and } \widehat{T}^{-1}\widehat{f} \text{ algebro-resurgent germs}$$

(3) "Convolution" is the operation that corresponds to the multiplication of formal series via formal Borel transform. Beware that Resurgence Theory usually makes use of the formula corresponding to $\mathcal{B}$, rather than β, in accordance with Footnote 2. The image by $\mathcal{B}$ of the product $(\mathcal{B}^{-1}\widehat{\varphi})(\mathcal{B}^{-1}\widehat{\psi})$ is the function $\int_0^\xi \widehat{\varphi}(\xi_1)\widehat{\psi}(\xi-\xi_1)\,\mathrm{d}\xi_1$, whereas the image by β of $(\beta^{-1}\widehat{\varphi})(\beta^{-1}\widehat{\psi})$ is $\frac{\partial}{\partial\xi}\Big(\int_0^\xi \widehat{\varphi}(\xi_1)\widehat{\psi}(\xi-\xi_1)\,\mathrm{d}\xi_1\Big)$.

(which is equivalent to Theorem 1.3) and deduce that

$$\widehat{f} \text{ and } \widehat{g} \text{ algebro-resurgent germs } \Rightarrow \widehat{f} *_M \widehat{g} \text{ algebro-resurgent germ}$$

(which is equivalent to Theorem 1.1).

Hence, algebro-resurgent series form a subalgebra of the associative algebras $(\mathbb{C}\{q_1,\dots,q_N,p_1,\dots,p_N\}[\![t]\!],\star_S)$ or $(\mathbb{C}\{q_1,\dots,q_N,p_1,\dots,p_N\}[\![t]\!],\star_M)$ or, in the case of a general constant-coefficient Poisson structure π, $(\mathbb{C}\{x_1,\dots,x_d\}[\![t]\!],\star_M)$.

1.5. Fundamental open problem in deformation quantization

The formal parameter $\hbar = t/i$ corresponds to the Planck constant, which is a nonzero fundamental constant of nature, and hence can hardly be treated as formal and dimensionless in applications to physics. A question thus naturally arises as to the "convergence of formal deformation quantization", in the sense of giving an analytic meaning to formal star products (recall that all the corresponding power series in t are expected to be generically divergent, so it cannot be "convergence" in the usual sense).

In fact, this question is considered to be one of the fundamental remaining open problems e.g. in [22] and there still is no general theory to answer it. The approach to Deformation Quantization taken in this paper may give hope to reach an answer, at least in some cases, by means of a Borel–Laplace summation of some sort (the usual Borel–Laplace summation $\mathscr{S}^\theta$ or one of the "Borel–Laplace averages" conceived by J. Écalle as a tool to be used when $\mathscr{S}^\theta\widetilde{\varphi}$ is ill-defined due to the presence of singularities of $\widehat{\varphi}(\xi)$ on the integration ray $\mathrm{e}^{\mathrm{i}\theta}\mathbb{R}_{>0}$). We have not pursued the question of Borel summability in this article – see however Remark 2.5.

1.6. Organization of the paper

The paper is organized as follows.

- Section 2 deals with definitions, examples and elementary properties for the Moyal and standard star products, $\star_M$ and $\star_S$, and their Borel counterparts $*_M$ and $*_S$. It also contains the integral representation formulas for $*_S$, $\widehat{T}$ and $\widehat{T}^{-1}$ that will be used in the rest of the article.
- Section 3 deals with the definition of algebro-resurgent series and algebro-resurgent germs, and states three lemmas that are instrumental in our proof of Theorems 1.2 and 1.3.

- Section 4 introduces the notion of a multivariate polynomial that is "simple with respect to one of its variables", as an algebraic preparation to handle more conveniently the algebraic varieties which appear in the singular loci in the Borel plane.
- Section 5 deals with the "convolution part" of our formula for $*_S$.
- Section 6 deals with the "Hadamard part" of the formula.
- Section 7 explains how to adapt the proof from $N = 1$ to N arbitrary.

2. Borel counterparts of the Moyal and standard star products

2.1. Moyal and standard star products

As explained at the end of § 1.1, we can assume without loss of generality that the Poisson structure π is the standard one (1.6), with $2N$ variables $q_1, \dots, q_N, p_1, \dots, p_N$. The formula (1.3) for the star product $\star_M$ is thus

$$\widetilde{f} \star_M \widetilde{g} = \mu \circ \exp\left(\frac{t}{2} \sum_{j=1}^{N} \left(\frac{\partial}{\partial p_j} \otimes \frac{\partial}{\partial q_j} - \frac{\partial}{\partial q_j} \otimes \frac{\partial}{\partial p_j}\right)\right) (\widetilde{f} \otimes \widetilde{g}). \tag{2.1}$$

Formula (2.1) (which boils down to (1.9) when $N = 1$) makes sense in $\mathbb{C}\{q_1, \dots, q_N, p_1, \dots, p_N\}[\![t]\!]$ as well as in

$$\widetilde{\mathcal{Q}}_{2N+1} := \mathbb{C}[\![q_1, \dots, q_N, p_1, \dots, p_N]\!][\![t]\!] = \mathbb{C}[\![t, q_1, \dots, q_N, p_1, \dots, p_N]\!].$$

The same is true for $\star_S$, which is defined by the formula (1.5) or, equivalently

$$\widetilde{f} \star_S \widetilde{g} = \mu \circ \exp\left(t \sum_{j=1}^{N} \left(\frac{\partial}{\partial p_j} \otimes \frac{\partial}{\partial q_j}\right)\right) (\widetilde{f} \otimes \widetilde{g}). \tag{2.2}$$

Recall that the formal deformation parameter is $t = i\hbar$.

It is well-known that $\star_S$ and $\star_M$ are equivalent under the transition operator T defined by (1.7): $T(\widetilde{f} \star_S \widetilde{g}) = (T\widetilde{f}) \star_M (T\widetilde{g})$ with

$$T = \exp\left(-\frac{t}{2} \sum \partial_{q_j} \partial_{p_j}\right) \;\text{ and }\; T^{-1} = \exp\left(\frac{t}{2} \sum \partial_{q_j} \partial_{p_j}\right).$$

In other words, we have

$$T\colon\; \big(\widetilde{\mathcal{Q}}_{2N+1}, \star_S\big) \to \big(\widetilde{\mathcal{Q}}_{2N+1}, \star_M\big) \;\text{ isomorphism of associative algebras.}$$

It is with $\star_S$ that we will work most of the time, because the formulas are simpler with it than with $\star_M$, hence we use abbreviations:

Notation 2.1. — From now on, we set $\star = \star_S$ for the standard star product, and $* = *_S$ for its Borel counterpart (1.16). We will call $*$ the "Borel-star product".

Example 2.2. — With one degree of freedom, $N = 1$, the definition (1.5) boils down to

$$\widetilde{f} \star_S \widetilde{g} = \widetilde{f} \star \widetilde{g} = \sum_{k\geqslant 0} \frac{t^k}{k!} \left(\partial_p^k \widetilde{f}\right) \left(\partial_q^k \widetilde{g}\right). \tag{2.3}$$

Here is a simple example in that case:

$$\begin{aligned} (tp) \star (tq) &= t^2pq + t^3, \qquad & (tq) \star (tp) &= t^2pq, \\ (tp) \star_M (tq) &= t^2pq + \frac{t^3}{2}, \qquad & (tq) \star_M (tp) &= t^2pq - \frac{t^3}{2}. \end{aligned} \tag{2.4}$$

Note that $T(t^2pq) = t^2pq - \frac{t^3}{2}$.

2.2. Stability of 1-Gevrey series

We are mostly interested in the subspace $\mathbb{C}\{q_1,\dots,q_N,p_1,\dots,p_N\}[\![t]\!]$ of $\widetilde{\mathcal{Q}}_{2N+1}$. However, it is important to realize that we cannot restrict ourselves to the too narrow subspace(4) $\mathbb{C}\{t,q_1,\dots,q_N,p_1,\dots,p_N\}$ consisting of formal series which converge in a neighbourhood of the origin in $\mathbb{C}^{2N+1}$, because even if f and g do not depend on t, their star product or their images by T may be divergent. Here is a simple example taken from [12], and a variant:

Example 2.3. — Take $N = 1$ as in (2.3). The geometric series $(1-p)^{-1}$ and $(1-q)^{-1}$ give rise to a divergent series

$$(1-p)^{-1} \star (1-q)^{-1} = \sum_{k\geqslant 0} k!\, t^k \big((1-p)(1-q)\big)^{-k-1}. \tag{2.5}$$

The logarithm series $\log(1-p)$ and $\log(1-q)$ give rise to a divergent series

$$\begin{aligned} &\log(1-p) \star \log(1-q) \\ &\qquad = \log(1-p)\log(1-q) + \sum_{k\geqslant 1} \frac{(k-1)!}{k} t^k \big((1-p)(1-q)\big)^{-k}. \end{aligned} \tag{2.6}$$

(4) However, the even smaller subspace of polynomials $\mathbb{C}[t,q_1,\dots,q_N,p_1,\dots,p_N]$ is stable under $\star_M$ and $\star$.

The transition operator T and its inverse map $(1-p-q)^{-1}$ to the divergent series

$$T^{\pm 1}\big((1-p-q)^{-1}\big) = \sum_{k\geqslant 0} \frac{(2k)!}{k!}\Big(\mp\frac{t}{2}\Big)^k (1-p-q)^{-2k-1}. \tag{2.7}$$

Note however the 1-Gevrey character with respect to t of these examples: the coefficient of t^k essentially has at most factorial growth, hence convergence is restored when t^k is replaced by $\xi^k/k!$, i.e. their image by the formal Borel transform (1.14) belongs to the space of convergent series $\mathbb{C}\{\xi,q,p\}$. This is a general phenomenon. Let us extend the definition of the formal Borel transform by the formula

$$\begin{aligned}\beta\colon\ \widetilde{\varphi} = \sum_{n\geqslant 0} a_n(z_1,\dots,z_r)t^n \in \mathbb{C}[\![t,z_1,\dots,z_r]\!] \\ \mapsto \widehat{\varphi} = \sum_{n\geqslant 0} a_n(z_1,\dots,z_r)\frac{\xi^n}{n!} \in \mathbb{C}[\![\xi,z_1,\dots,z_r]\!],\end{aligned} \tag{2.8}$$

and call 1-Gevrey formal series with respect to t the elements of $\widetilde{\mathcal{Q}}^G_{r+1}$, where

$$\widetilde{\mathcal{Q}}^G_{r+1} \coloneqq \beta^{-1}\big(\widehat{\mathcal{Q}}_{r+1}\big) \subset \mathbb{C}[\![t,z_1,\dots,z_r]\!],\quad \widehat{\mathcal{Q}}_{r+1} \coloneqq \mathbb{C}\{\xi,z_1,\dots,z_r\}. \tag{2.9}$$

We then have, as noted in [12] in the case of the standard star product,

Theorem 2.4. — *The subspace $\widetilde{\mathcal{Q}}^G_{2N+1}$ is stable under the Moyal star product $\star_M$ and the standard product $\star = \star_S$, as well as under the transition operators T and T^{-1}.*

The proof is a consequence of (2.33) in Lemma 2.11.

Examples (2.5)–(2.7) are thus elements of $\widetilde{\mathcal{Q}}^G_3$; their images by β are the following elements of $\widehat{\mathcal{Q}}_3$:

$$\begin{aligned}&\beta\big((1-p)^{-1}\star(1-q)^{-1}\big)\\ &\qquad = (1-p)^{-1}(1-q)^{-1}\Big(1-\xi\big((1-p)(1-q)\big)^{-1}\Big)^{-1},\end{aligned} \tag{2.10}$$

$$\begin{aligned}&\beta\big(\log(1-p)\star\log(1-q)\big)\\ &\qquad = \log(1-p)\log(1-q) + \mathrm{Li}_2\Big(\xi\big((1-p)(1-q)\big)^{-1}\Big),\end{aligned} \tag{2.11}$$

where

$$\mathrm{Li}_2(z) = \sum_{k\geqslant 1}\frac{z^k}{k^2} = -\int_0^z \frac{\log(1-\zeta)}{\zeta}\,\mathrm{d}\zeta \tag{2.12}$$

(the famous dilogarithm function, which plays an ubiquitous role in mathematical physics along with its quantum variant), and

$$\beta \circ T^{\pm 1}\big((1-p-q)^{-1}\big) = (1-p-q)^{-1}\big(1 \pm 2\xi(1-p-q)^{-2}\big)^{-1/2}. \tag{2.13}$$

In fact, the divergent series (2.5) is essentially the famous Euler series, a paradigmatic example of 1-Gevrey series, and the most elementary example of resurgent series (see e.g. [17]).

Remark 2.5. — The above examples can be further generalized: for any $\widetilde{g} = \widetilde{g}(t,q,p) \in \widetilde{\mathcal{Q}}_3^G$, one has $(1-p)^{-1} \star \widetilde{g} = \sum_{n \geqslant 0} (1-p)^{-n-1}\, t^n\, \partial_q^n \widetilde{g}(t,q,p)$, hence $\widehat{g} = \beta\, \widetilde{g} \in \widehat{\mathcal{Q}}_3$ satisfies

$$\begin{aligned} (1-p)^{-1} * \widehat{g} &= \beta\big((1-p)^{-1} \star \widetilde{g}\big) \\ &= (1-p)^{-1} \frac{\partial}{\partial \xi}\left(\int_0^\xi \widehat{g}\big(\xi-\zeta, q+\zeta(1-p)^{-1}, p\big)\,\mathrm{d}\zeta\right), \end{aligned} \tag{2.14}$$

and there is a similar formula for $(1-q)^{-1} \star \widetilde{g}$. Note that if $\theta \in \mathbb{R}$ and U is a subset of $\{(q,p) \in \mathbb{C} \times \mathbb{C} \mid |p-1| \geqslant \varepsilon\}$ (for some $\varepsilon > 0$) such that $\widetilde{g}$ is Borel-summable in the direction θ when $(q,p) \in U$, i.e. $\widehat{g}(\xi,q,p)$ extends analytically in $S_{\theta,\varepsilon'} \times U$ with at most exponential growth at infinity,

$$|\widehat{g}(\xi,q,p)| \leqslant C\, \exp\big(A_1|\xi| + A_2|q| + A_3|p|\big) \tag{2.15}$$

(for some $C, A_1, A_2, A_3 \geqslant 0$), where $S_{\theta,\varepsilon'}$ is a half-strip $\{\mathrm{dist}(\xi, \mathbb{R}_{\geqslant 0}\, \mathrm{e}^{\mathrm{i}\theta}) < \varepsilon'\}$ (for some $\varepsilon' > 0$), then $(1-p)^{-1} \star \widetilde{g}$ is Borel-summable in the direction θ when $(q,p) \in U'$, where

$$U' := \big\{(q,p) \in U \mid \big(q + \xi(1-p)^{-1}, p\big) \in U \text{ for all } \xi \in S_{\theta,\varepsilon'}\big\}. \tag{2.16}$$

2.3. Key formula for the Borel-star product

Recall that we have defined the Borel counterpart $*_M$ of the Moyal star product $\star_M$ by (1.15), and the Borel-star product $* = *_S$, counterpart of the standard star product $\star = \star_S$, by (1.16). We will now give general integral representations for them.

For the sake of clarity, we begin with the case of one degree of freedom, $N = 1$.

Lemma 2.6. — *For any $\widehat{f}, \widehat{g} \in \mathbb{C}[\![\xi,q,p]\!]$,*

$$\begin{aligned} \widehat{f} * \widehat{g}(\xi,q,p) = \frac{\mathrm{d}^3}{\mathrm{d}\xi^3} \int_0^\xi \mathrm{d}\xi_1 \int_0^{\xi-\xi_1} \mathrm{d}\xi_2 \int_0^{\xi-\xi_1-\xi_2} \mathrm{d}\xi_3 \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \\ \times \widehat{f}(\xi_1, q, p+\sqrt{\xi_3}\mathrm{e}^{-\mathrm{i}\theta})\widehat{g}(\xi_2, q+\sqrt{\xi_3}\mathrm{e}^{\mathrm{i}\theta}, p), \end{aligned} \tag{2.17}$$

where the integrand is considered as element of $\mathbb{C}[e^{\pm i\theta}][\![q,p,\xi_1,\xi_2,\sqrt{\xi_3}]\!]$ *and integration in* θ *is performed termwise.*

Moreover, if both factors are convergent, then so is their Borel-star product:

$$\widehat{f},\widehat{g}\in\widehat{\mathcal{Q}}_3 \quad\Rightarrow\quad \widehat{f}*\widehat{g}\in\widehat{\mathcal{Q}}_3. \tag{2.18}$$

Proof. — We expand $\widehat{f}$ and $\widehat{g}$ in powers of ξ as $\widehat{f}=\sum f_m(q,p)\frac{\xi^m}{m!}$, $\widehat{g}=\sum g_n(q,p)\frac{\xi^n}{n!}$, so that (2.3) allows us to compute $\widehat{f}*\widehat{g}(\xi,q,p)$ as the image by β of $\sum_{m,n,k\geqslant 0}\frac{1}{k!}(\partial_p^k f_m)(\partial_q^k g_n)t^{m+n+k}$:

$$\widehat{f}*\widehat{g}(\xi,q,p)=\sum_{m,n,k\geqslant 0}\frac{1}{k!}(\partial_p^k f_m)(\partial_q^k g_n)\frac{\xi^{m+n+k}}{(n+m+k)!}.$$

The identity $\frac{\xi^{m+n+k+3}}{(n+m+k+3)!}=\int_0^\xi d\xi_1\int_0^{\xi-\xi_1}d\xi_2\int_0^{\xi-\xi_1-\xi_2}d\xi_3\frac{\xi_1^m}{m!}\frac{\xi_2^n}{n!}\frac{\xi_3^k}{k!}$ differentiated three times yields

$$\widehat{f}*\widehat{g}(\xi,q,p)=\frac{d^3}{d\xi^3}\int_0^\xi d\xi_1\int_0^{\xi-\xi_1}d\xi_2\int_0^{\xi-\xi_1-\xi_2}d\xi_3 \times\sum_{n,m,k\geqslant 0}(\partial_p^k f_m)(\partial_q^k g_n)\frac{\xi_1^m}{m!}\frac{\xi_2^n}{n!}\frac{\xi_3^k}{k!^2}. \tag{2.19}$$

For each (m,n), with the notations $a_k=\frac{1}{k!}\partial_p^k f_m$, $b_k=\frac{1}{k!}\partial_q^k g_m$, the series $\sum a_k b_k\xi_3^k$ can be interpreted as the evaluation at ξ_3 of the Hadamard product $\phi\odot\psi$ of

$$\phi(\xi)=\sum a_k\xi^k=f_m(q,p+\xi)\ \text{ and }\ \psi(\xi)=\sum b_k\xi^k=g_n(q+\xi,p).$$

According to Section 6.1, $\phi\odot\psi(\xi)=\sum a_k b_k\xi^k$ can be rewritten as

$$\int_0^{2\pi}\phi\left(\sqrt{\xi}e^{-i\theta}\right)\psi\left(\sqrt{\xi}e^{i\theta}\right)\frac{d\theta}{2\pi}$$

with termwise integration in $\mathbb{C}[e^{\pm i\theta}][\![\sqrt{\xi}]\!]$ (see (6.3) infra), thus

$$\begin{aligned}\widehat{f}*\widehat{g}(\xi,q,p)&=\frac{d^3}{d\xi^3}\int_0^\xi d\xi_1\int_0^{\xi-\xi_1}d\xi_2\int_0^{\xi-\xi_1-\xi_2}d\xi_3\\&\qquad\sum_{m,n\geqslant 0}\int_0^{2\pi}\frac{d\theta}{2\pi}f_m\left(q,p+\sqrt{\xi_3}e^{-i\theta}\right)g_n\left(q+\sqrt{\xi_3}e^{i\theta},p\right)\frac{\xi_1^m}{m!}\frac{\xi_2^n}{n!}\\&=\frac{d^3}{d\xi^3}\int_0^\xi d\xi_1\int_0^{\xi-\xi_1}d\xi_2\int_0^{\xi-\xi_1-\xi_2}d\xi_3\\&\qquad\int_0^{2\pi}\frac{d\theta}{2\pi}\widehat{f}\left(\xi_1,q,p+\sqrt{\xi_3}e^{-i\theta}\right)\widehat{g}\left(\xi_2,q+\sqrt{\xi_3}e^{i\theta},p\right).\end{aligned}$$

Now, suppose $\widehat{f},\widehat{g}\in\widehat{\mathcal{Q}}_3=\mathbb{C}\{\xi,q,p\}$. The right hand side of (2.17) involves a function $G(\xi_1,\xi_2,s,q,p)\coloneqq\int_0^{2\pi}\widehat{f}(\xi_1,q,p+s\mathrm{e}^{-\mathrm{i}\theta})\widehat{g}(\xi_2,q+s\mathrm{e}^{\mathrm{i}\theta},p)\frac{\mathrm{d}\theta}{2\pi}$, which clearly belongs to $\mathbb{C}\{\xi_1,\xi_2,s,q,p\}$ and is even in s. Consequently, $F(\xi_1,\xi_2,\xi_3,q,p)\coloneqq G(\xi_1,\xi_2,\pm\sqrt{\xi_3},q,p)$ is a well-defined germ that belongs to $\mathbb{C}\{\xi_1,\xi_2,\xi_3,q,p\}$. The right-hand side of (2.17) can be written as

$$\frac{\mathrm{d}^3}{\mathrm{d}\xi^3}\int_0^{\xi}\mathrm{d}\xi_1\int_0^{\xi-\xi_1}\mathrm{d}\xi_2\int_0^{\xi-\xi_1-\xi_2}\mathrm{d}\xi_3\,F(\xi_1,\xi_2,\xi_3,q,p), \tag{2.20}$$

hence it defines a holomorphic germ in $\mathbb{C}\{\xi,q,p\}$. □

Remark 2.7. — The integral formula (2.17) differs from the one given in Proposition 3.3 of [12], which is not correct. Take for instance $\widehat{f}=\xi p$ and $\widehat{g}=\xi q$: we know by the first equation in (2.4) that we must find

$$(\xi p)*(\xi q)=pq\frac{\xi^2}{2!}+\frac{\xi^3}{3!}, \tag{2.21}$$

and the reader may check that our formula produces the right outcome, but not the formula from [12], which yields a term $\frac{\xi^3}{2!}$ instead of $\frac{\xi^3}{3!}$.

Remark 2.8. — Instead of writing the Hadamard product $\phi\odot\psi(\xi)$ as we did in our proof, we could have used the integration variable $\zeta=\sqrt{\xi}\mathrm{e}^{\mathrm{i}\theta}$ and then the Cauchy theorem, which yields

$$\phi\odot\psi(\xi)=\frac{1}{2\pi\mathrm{i}}\oint_C\phi\left(\frac{\xi}{z}\right)\psi(z)\frac{\mathrm{d}z}{z} \tag{2.22}$$
$$\text{with any circle } C\colon\ \theta\mapsto c\,\mathrm{e}^{\mathrm{i}\theta}\ \text{ of radius } c\in\left(\frac{|\xi|}{R_\phi},R_\psi\right),$$

where R_ϕ and R_ψ are the radii of convergence of ϕ and ψ, and $|\xi|<R_\phi R_\psi$ (see (6.2) infra). Correspondingly, Formula (2.17) can be rewritten

$$\begin{aligned}\widehat{f}*\widehat{g}(\xi,q,p)=\frac{\mathrm{d}^3}{\mathrm{d}\xi^3}\int_0^{\xi}\mathrm{d}\xi_1\int_0^{\xi-\xi_1}\mathrm{d}\xi_2\int_0^{\xi-\xi_1-\xi_2}\mathrm{d}\xi_3\oint_C\frac{\mathrm{d}z}{2\pi\mathrm{i}z}&\\ \times\widehat{f}\Big(\xi_1,q,p+\frac{\xi_3}{z}\Big)\widehat{g}(\xi_2,q+z,p),&\end{aligned} \tag{2.23}$$

where C is an appropriate circle: supposing $\widehat{f}$ and $\widehat{g}$ holomorphic in $\mathbb{D}^3_\tau$ with notation (3.1) and taking $\varepsilon\in(0,\tau)$, $\varepsilon'\in(0,\varepsilon^2)$ and $c\in(\frac{\varepsilon'}{\varepsilon},\varepsilon)$, formula (2.23) holds for $(\xi,q,p)\in\mathbb{D}_{\varepsilon'}\times\mathbb{D}_{\tau-\varepsilon}\times\mathbb{D}_{\tau-\varepsilon}$.

Lemma 2.9. — *If $\widehat{f},\widehat{g}\in\mathbb{C}[\![\xi,q,p]\!]$, then*

$$\widehat{f}*_M\widehat{g}(\xi,q,p)=\frac{\mathrm{d}^4}{\mathrm{d}\xi^4}\int_0^{\xi}\mathrm{d}\xi_1\int_0^{\xi-\xi_1}\mathrm{d}\xi_2\int_0^{\xi-\xi_1-\xi_2}\mathrm{d}\xi_3 \times\int_0^{\xi-\xi_1-\xi_2-\xi_3}\mathrm{d}\xi_4\oint_{C_1}\frac{\mathrm{d}z_1}{2\pi\mathrm{i}z_1}\oint_{C_2}\frac{\mathrm{d}z_2}{2\pi\mathrm{i}z_2} \times\widehat{f}(\xi_1,q+z_1,p+z_2)\,\widehat{g}\left(\xi_2,q+\frac{\xi_4}{2z_2},p-\frac{\xi_3}{2z_1}\right) \tag{2.24}$$

with integration on appropriate circles C_1 and C_2: supposing $\widehat{f}$ and $\widehat{g}$ holomorphic in $\mathbb{D}^3_\tau$ with notation (3.1) *and taking $\varepsilon\in(0,\tau)$, $\varepsilon'\in(0,\varepsilon^2)$, formula* (2.24) *holds for $(\xi,q,p)\in\mathbb{D}_{\varepsilon'}\times\mathbb{D}_{\tau-\varepsilon}\times\mathbb{D}_{\tau-\varepsilon}$ provided C_1 and C_2 are any anticlockwise circles centred at* 0 *with radii in $(\frac{\varepsilon'}{\varepsilon},\varepsilon)$.*

Moreover, if both factors are convergent, i.e. $\widehat{f},\widehat{g}\in\widehat{\mathcal{Q}}_3$, then so is $\widehat{f}*_M\widehat{g}$.

Proof. — Using the same kind of argument as in the proof of Lemma 2.6, we compare the right-hand side and the left-hand side of (2.24):

$$\begin{aligned}
\mathrm{RHS}&=\frac{\mathrm{d}^4}{\mathrm{d}\xi^4}\int_0^{\xi}\mathrm{d}\xi_1\int_0^{\xi-\xi_1}\mathrm{d}\xi_2\int_0^{\xi-\xi_1-\xi_2}\mathrm{d}\xi_3\int_0^{\xi-\xi_1-\xi_2-\xi_3}\mathrm{d}\xi_4\\
&\qquad\times\sum_{n,m}\Big(\partial_p^n\partial_q^m\widehat{f}(\xi_1,q,p)\Big)\Big(\partial_p^m\partial_q^n\widehat{g}(\xi_2,q,p)\Big)\frac{(-1)^m\,\xi_3^m\,\xi_4^n}{2^{m+n}\,m!^2\,n!^2}\\
&=\frac{\mathrm{d}^4}{\mathrm{d}\xi^4}\int_0^{\xi}\mathrm{d}\xi_1\int_0^{\xi-\xi_1}\mathrm{d}\xi_2\int_0^{\xi-\xi_1-\xi_2}\mathrm{d}\xi_3\int_0^{\xi-\xi_1-\xi_2-\xi_3}\mathrm{d}\xi_4\\
&\qquad\times\sum_{n,m,\alpha,\beta}\big(\partial_p^n\partial_q^m f_\alpha(q,p)\big)\big(\partial_p^m\partial_q^n g_\beta(q,p)\big)\frac{(-1)^m\,\xi_1^\alpha\,\xi_2^\beta\,\xi_3^m\,\xi_4^n}{2^{m+n}\,\alpha!\,\beta!\,m!^2\,n!^2}\\
&=\sum_{n,m,\alpha,\beta}\big(\partial_p^n\partial_q^m f_\alpha(q,p)\big)\big(\partial_p^m\partial_q^n g_\beta(q,p)\big)\frac{(-1)^m\,\xi^{\alpha+\beta+m+n}}{2^{m+n}\,m!\,n!\,(\alpha+\beta+m+n)!}\\
&=\mathrm{LHS},
\end{aligned}$$

because the LHS is the Borel image of the Moyal star product of $\widetilde{f}=\sum_\alpha f_\alpha(q,p)t^\alpha$ and $\widetilde{g}=\sum_\beta g_\beta(q,p)t^\beta$, which, according to (1.9), can be written

$$\begin{aligned}
\widetilde{f}\star_M\widetilde{g}&=\sum_{n,m\geqslant 0}\frac{(-1)^m\,t^{n+m}}{2^{n+m}\,n!\,m!}\big(\partial_p^n\,\partial_q^m f\big)\big(\partial_p^m\,\partial_q^n g\big)\\
&=\sum_{n,m,\alpha,\beta}\frac{(-1)^m\,t^{\alpha+\beta+n+m}}{2^{n+m}\,n!\,m!}\big(\partial_p^n\,\partial_q^m f_\alpha\big)\big(\partial_p^m\,\partial_q^n g_\beta\big).
\end{aligned}\tag{2.25}$$

The discussion of convergence is the same as in the proof of Lemma 2.6. □

We could also have derived Lemma 2.9 from Lemma 2.6 and the following integral representations of the Borel counterparts of T and T^{-1},

$$\widehat{T}\widehat{f} := \beta T \beta^{-1}\widehat{f}, \qquad \widehat{T}^{-1}\widehat{f} = \beta T^{-1}\beta^{-1}\widehat{f}. \tag{2.26}$$

Lemma 2.10. — *For any $\widehat{f} \in \mathbb{C}[\![\xi, q, p]\!]$,*

$$\widehat{T}\widehat{f} = \frac{\mathrm{d}}{\mathrm{d}\xi}\int_0^{\xi} \mathrm{d}\xi_1 \oint_C \frac{\mathrm{d}z}{2\pi \mathrm{i} z}\widehat{f}\left(\xi-\xi_1, q+z, p-\frac{\xi_1}{2z}\right), \tag{2.27}$$

$$\widehat{T}^{-1}\widehat{f} = \frac{\mathrm{d}}{\mathrm{d}\xi}\int_0^{\xi} \mathrm{d}\xi_1 \oint_C \frac{\mathrm{d}z}{2\pi \mathrm{i} z}\widehat{f}\left(\xi-\xi_1, q+z, p+\frac{\xi_1}{2z}\right), \tag{2.28}$$

with integration on appropriate circle C.

Moreover, if $\widehat{f}$ is convergent, i.e. $\widehat{f} \in \widehat{\mathcal{Q}}_3$, then so is $\widehat{T}\widehat{f}$.

Proof. — Let us compare the right-hand side and the left-hand side of (2.27):

$$\begin{aligned}
\mathrm{RHS} &= \frac{\mathrm{d}}{\mathrm{d}\xi}\int_0^{\xi} \mathrm{d}\xi_1 \oint_C \frac{\mathrm{d}z}{2\pi \mathrm{i}} \sum_{n,m} \frac{\partial_q^n \partial_p^m \widehat{f}(\xi-\xi_1, q, p)}{n!m!}\left(-\frac{\xi_1}{2}\right)^m z^{n-m-1}\\
&= \frac{\mathrm{d}}{\mathrm{d}\xi}\int_0^{\xi} \mathrm{d}\xi_1 \sum_{n} \frac{\partial_q^n \partial_p^n \widehat{f}(\xi-\xi_1, q, p)}{n!^2}\left(-\frac{\xi_1}{2}\right)^n\\
&= \beta\left(\sum_n \frac{\partial_q^n \partial_p^n \tilde{f}(t, q, p)}{n!}\cdot\left(-\frac{t}{2}\right)^n\right) = \mathrm{LHS}.
\end{aligned}$$

The proof of (2.28) is analogous. □

2.4. More degrees of freedom

We now consider the case of an abitrary number of degrees of freedom, say N. We set $q = (q_1, \dots, q_N)$ and $p = (p_1, \dots, p_N)$. If $\widehat{f}(\xi, q, p) = \sum_{m=0}^{\infty} f_m(q,p)\frac{\xi^m}{m!}$, $\widehat{g}(\xi, q, p) = \sum_{n=0}^{\infty} g_n(q,p)\frac{\xi^n}{n!}$, then it follows from (1.5) that

$$\begin{aligned}
\widehat{f} * \widehat{g}(\xi, q, p) = &\sum_{m,n,k_1,\dots,k_N \geqslant 0} \frac{1}{k_1!\cdots k_N!}\\
&\times (\partial_{p_1}^{k_1}\cdots\partial_{p_N}^{k_N} f_m)(\partial_{q_1}^{k_1}\cdots\partial_{q_N}^{k_N} g_n)\frac{\xi^{k_1+\cdots+k_N+n+m}}{(k_1+\cdots+k_N+n+m)!}.
\end{aligned} \tag{2.29}$$

Lemma 2.11. — *There are integral representation formulas in N degrees of freedom analogous to those of Lemma 2.6/Remark 2.8, Lemma 2.9 and Lemma 2.10. Specifically, for $\widehat{f}, \widehat{g} \in \mathbb{C}[\![\xi, q_1, \dots, q_N, p_1, \dots, p_N]\!]$, a formula generalising* (2.23) *is*

$$\begin{aligned}\widehat{f} * \widehat{g}(\xi, q, p) = {} & \frac{\mathrm{d}^{N+2}}{\mathrm{d}\xi^{N+2}} \int_0^{\xi} \mathrm{d}\xi_1 \int_0^{\xi-\xi_1} \mathrm{d}\xi_2 \cdots \\ & \times \int_0^{\xi-\xi_1-\cdots-\xi_{N+1}} \mathrm{d}\xi_{N+2} \oint_{C_1} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}z_1} \cdots \oint_{C_N} \frac{\mathrm{d}z_N}{2\pi\mathrm{i}z_N} \\ & \times \widehat{f}\left(\xi_{N+1}, q_1, \dots, q_N, p_1 + \frac{\xi_1}{z_1}, \dots, p_N + \frac{\xi_N}{z_N}\right) \\ & \times \widehat{g}(\xi_{N+2}, q_1 + z_1, \dots, q_N + z_N, p_1, \dots, p_N),\end{aligned} \tag{2.30}$$

a formula generalising (2.24) *is*

$$\begin{aligned}\widehat{f} *_M \widehat{g}(\xi, q, p) = {} & \frac{\mathrm{d}^{2N+2}}{\mathrm{d}\xi^{2N+2}} \int_0^{\xi} \mathrm{d}\xi_1 \int_0^{\xi-\xi_1} \mathrm{d}\xi_2 \cdots \\ & \times \int_0^{\xi-\xi_1-\cdots-\xi_{2N+1}} \mathrm{d}\xi_{2N+2} \oint_{C_1} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}z_1} \cdots \oint_{C_{2N}} \frac{\mathrm{d}z_{2N}}{2\pi\mathrm{i}z_{2N}} \\ & \times \widehat{f}(\xi_{2N+1}, q_1 + z_1, \dots, q_N + z_N, p_1 + z_{N+1}, \dots, p_N + z_{2N}) \\ & \times \widehat{g}\left(\xi_{2N+2}, q_1 + \frac{\xi_{N+1}}{2z_{N+1}}, \dots, q_N + \frac{\xi_{2N}}{2z_{2N}}, p_1 - \frac{\xi_1}{2z_1}, \dots, p_N - \frac{\xi_N}{2z_N}\right),\end{aligned} \tag{2.31}$$

and a formula generalising (2.27)–(2.28) *is*

$$\begin{aligned}\widehat{T}^{\pm 1} \widehat{f}(\xi, q, p) = {} & \frac{\mathrm{d}^{N+1}}{\mathrm{d}\xi^{N+1}} \int_0^{\xi} \mathrm{d}\xi_1 \int_0^{\xi-\xi_1} \mathrm{d}\xi_2 \cdots \\ & \times \int_0^{\xi-\xi_1-\cdots-\xi_N} \mathrm{d}\xi_{N+1} \oint_{C_1} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}z_1} \cdots \oint_{C_N} \frac{\mathrm{d}z_N}{2\pi\mathrm{i}z_N} \\ & \times \widehat{f}\left(\xi_{N+1}, q_1 + z_1, \dots, q_N + z_N, p_1 \mp \frac{\xi_1}{2z_1}, \dots, p_N \mp \frac{\xi_N}{2z_{2N}}\right).\end{aligned} \tag{2.32}$$

In each of these formulas, supposing $\widehat{f}$ and $\widehat{g}$ holomorphic in $\mathbb{D}_\tau^{2N+1}$ with notation (3.1) *and taking $\varepsilon \in (0, \tau)$, $\varepsilon' \in (0, \varepsilon^2)$, it is understood that $(\xi, q, p) \in \mathbb{D}_{\varepsilon'} \times \mathbb{D}_{\tau-\varepsilon}^N \times \mathbb{D}_{\tau-\varepsilon}^N$ and the C_j's are any anticlockwise circles centred at* 0 *with radii in $(\frac{\varepsilon'}{\varepsilon}, \varepsilon)$—alternatively, C_j can be taken to be the parametrized circle $\theta_j \in [0, 2\pi] \mapsto \sqrt{\xi_j}\, \mathrm{e}^{\mathrm{i}\theta_j}$, where $\sqrt{\xi_j}$ is any square root of ξ_j.*

These formulas entail that

$$\widehat{f},\widehat{g}\in\widehat{\mathcal{Q}}_{2N+1} \quad\Rightarrow\quad \widehat{f}*\widehat{g},\;\; \widehat{f}*_M\widehat{g},\;\; \widehat{T}\widehat{f},\;\; \widehat{T}^{-1}\widehat{f}\in\widehat{\mathcal{Q}}_{2N+1}. \tag{2.33}$$

We leave it to the reader to work out the details of the proof of Lemma 2.11.

As already mentioned, Theorem 2.4 is a mere consequence of (2.33).

3. Algebro-resurgent germs

We shall use the notation

$$\mathbb{D}_\tau := \{z\in\mathbb{C} \mid |z|<\tau\}. \tag{3.1}$$

We know that if $f\in\mathbb{C}\{z_1,\dots,z_n\}$, then there exists $\tau>0$, such that f is the germ of a function holomorphic in the polydisc $\mathbb{D}_\tau^n\subset\mathbb{C}^n$. Following [12], we set

Definition 3.1. — *For any $n\geqslant 0$, we define the set of "algebro-resurgent germs in $n+1$ variables" by*

$$\widehat{\mathcal{Q}}^{\mathcal{A}}_{n+1} := \left\{ f\in\mathbb{C}\{\xi,z_1,\dots,z_n\} \;\middle|\; \begin{array}{l} \exists \textit{ proper algebraic subvariety} \\ V\subset\mathbb{C}^{n+1}, \textit{ such that } f \textit{ admits analytic} \\ \textit{continuation along any } C^1 \textit{ path } \gamma \\ \textit{contained in } \mathbb{C}^{n+1}-V \textit{ and having} \\ \textit{initial point } \gamma(0) \textit{close enough to } 0 \end{array} \right\}. \tag{3.2}$$

Here "$\gamma(0)$ close enough to 0" means that $\gamma(0)\in\mathbb{D}_\tau^{n+1}$ where $\mathbb{D}_\tau^{n+1}$ is a polydisc where f induces a holomorphic function. We then called "avoidant set" for f any proper algebraic subvariety V satisfying the property indicated above.

Example 3.2. — Formulas (2.10), (2.11) and (2.13) define algebro-resurgent germs in 3 variables, with avoidant sets $\{(1-q)(1-p) = 0 \text{ or } \xi\}$, resp. $\{(1-q)(1-p)=0 \text{ or } \xi\}\cup\{\xi=0\}$, resp. $\{(1-p-q)^2 = 0 \text{ or } 2\,\xi \text{ or } -2\,\xi\}$.

Recall that we have defined $\beta\colon\ \mathbb{C}[\![t,z_1,\dots,z_r]\!]\to\mathbb{C}[\![\xi,z_1,\dots,z_r]\!]$ as the formal Borel transform with respect to the first variable – cf. (2.8).

Definition 3.3. — *We define* $\widetilde{\mathcal{Q}}^{\mathcal{A}}_{n+1}$ *to be the preimage by* β *of* $\widehat{\mathcal{Q}}^{\mathcal{A}}_{n+1}$*:*

$$\widetilde{\mathcal{Q}}^{\mathcal{A}}_{n+1} := \beta^{-1}(\widehat{\mathcal{Q}}^{\mathcal{A}}_{n+1}) \subset \mathbb{C}[\![t, z_1, \dots, z_n]\!].$$

This is the set of all "algebro-resurgent series in $n+1$ *variables".*

Obviously, since $\widehat{\mathcal{Q}}^{\mathcal{A}}_{n+1} \subset \mathbb{C}\{\xi, z_1, \dots, z_n\} = \widehat{\mathcal{Q}}_{n+1}$, we have $\widetilde{\mathcal{Q}}^{\mathcal{A}}_{n+1} \subset \widetilde{\mathcal{Q}}^{G}_{n+1}$: algebro-resurgent series are 1-Gevrey with respect to t. We may also consider the disjoint union of the spaces of algebro-resurgent germs or series in any number of variables:

$$\widehat{\mathcal{Q}}^{\mathcal{A}} := \bigsqcup_{n\geqslant 0} \widehat{\mathcal{Q}}^{\mathcal{A}}_{n+1}, \qquad \widetilde{\mathcal{Q}}^{\mathcal{A}} := \bigsqcup_{n\geqslant 0} \widetilde{\mathcal{Q}}^{\mathcal{A}}_{n+1}.$$

The 1-degree-of-freedom version of Theorem 1.2 is

Theorem 3.4. — *If* $\widetilde{f}(t,q,p),\ \widetilde{g}(t,q,p) \in \widetilde{\mathcal{Q}}^{\mathcal{A}}_3$*, then* $\widetilde{f} \star \widetilde{g}(t,q,p) \in \widetilde{\mathcal{Q}}^{\mathcal{A}}_3$*. Equivalently,*

$$\widehat{f}(t,q,p),\ \widehat{g}(t,q,p) \in \widehat{\mathcal{Q}}^{\mathcal{A}}_3 \quad \Rightarrow \quad \widehat{f} * \widehat{g}(t,q,p) \in \widehat{\mathcal{Q}}^{\mathcal{A}}_3. \tag{3.3}$$

Sections 4–6 are devoted to the proof of Theorem 3.4. (Then Section 7 will show how to prove Theorem 1.2, and also Theorems 1.3 and 1.1.) Using formula (2.17), the proof will be divided into the following three lemmas.

Lemma 3.5. — *If* $\widehat{f}(\xi,q,p), \widehat{g}(\xi,q,p) \in \widehat{\mathcal{Q}}^{\mathcal{A}}_3$*, then*

$$\begin{aligned} F(\xi_1,\xi_2,\xi_3,q,p) \\ := \frac{1}{2\pi}\int_0^{2\pi} \widehat{f}(\xi_1, q, p+\sqrt{\xi_3}\mathrm{e}^{-\mathrm{i}\theta})\widehat{g}(\xi_2, q+\sqrt{\xi_3}\mathrm{e}^{\mathrm{i}\theta}, p)\mathrm{d}\theta \end{aligned} \tag{3.4}$$

is an algebro-resurgent germ in 5 *variables.*

The proof will be found in Section 6, which is treated as the "Hadamard product part" of formula (2.17).

Lemma 3.6. — *If* $F(z_1,\dots,z_n) \in \widehat{\mathcal{Q}}^{\mathcal{A}}_n$ *and* $\overline{P}$ *is a polynomial in* n *variables vanishing at* $(0,\dots,0)$*, then*

$$f(z, z_2, \dots, z_n) := \int_0^{\overline{P}(z,z_2,\dots,z_n)} F(z_1, z_2, \dots, z_n)\mathrm{d}z_1 \tag{3.5}$$

is an algebro-resurgent germ in n *variables.*

The proof will be found in Section 5, which is treated as the "convolution product part" of formula (2.17).

Lemma 3.7. — *If* $F(z_1,\dots,z_n) \in \widehat{\mathcal{Q}}_n^{\mathcal{A}}$ *and* $\overline{P}$ *is a polynomial in* $n-1$ *variables vanishing at* $(0,\dots,0)$*, then*

$$f(z_2,\dots,z_n) := \int_0^{\overline{P}(z_2,\dots,z_n)} F(z_1,z_2,\dots,z_n)\,\mathrm{d}z_1 \tag{3.6}$$

is an algebro-resurgent germ in $n-1$ *variables.*

Lemma 3.7 follows almost directly from Lemma 3.6, as will be shown at the end of Section 5.

4. Simple polynomials with respect to a variable

In this section, we shall work in $\mathbb{C}^n$ with variables $z_1,\dots,z_n$ and give the definition of z_1-simple polynomial. The proposition 4.2 is very useful in the following sections and we will prove it carefully. The reason we use the definition "z_1-simple polynomial" is that we want the set (4.2) to be non-trivial.

Any non-zero polynomial $P \in \mathbb{C}[z_1,\dots,z_n]$ can be written in a unique way as

$$P(z_1,\dots,z_n) = \sum_{i=0}^{M} b_i(z_2,\dots,z_n)\, z_1^i \in \mathbb{C}[z_1,\dots,z_n] = \mathbb{C}[z_2,\dots,z_n][z_1],$$

with $M \geqslant 0$, $b_1,\dots,b_M \in \mathbb{C}[z_2,\dots,z_n]$ and $b_M \neq 0$. We denote by $\mathbb{F}$ the fraction field of $\mathbb{C}[z_2,\dots,z_n]$ and $\overline{\mathbb{F}}$ the algebraic closure of $\mathbb{F}$. Thus, $P(z_1,\dots,z_n)$ can be written as

$$b_M(z_2,\dots,z_n)\prod_{\alpha=1}^{M}\big(z_1-\omega_\alpha(z_2,\dots,z_n)\big) \tag{4.1}$$

with $\omega_\alpha(z_2,\dots,z_n) \in \overline{\mathbb{F}}$.

Definition 4.1. — *A non-zero polynomial* $F(z_1,\dots,z_n) \in \mathbb{C}[z_1,\dots,z_n]$ *is called a* z_1*-simple polynomial if, for any* α_1,α_2, $1 \leqslant \alpha_1 < \alpha_2 \leqslant M$*, we have* $\omega_{\alpha_1}(z_2,\dots,z_n) \neq \omega_{\alpha_2}(z_2,\dots,z_n)$ *in the representation* (4.1) *of* F*. In particular, if the degree* M *of* F *in* z_1 *is zero, then* F *is a* z_1*-simple polynomial.*

Proposition 4.2. — *Any proper algebraic subvariety* V *of* $\mathbb{C}^n$ *can be written as* $V = \bigcap_{J=1}^{K} P_J^{-1}(0)$*, where* K *is a positive integer and* $P_1,\dots,P_K$ *are* z_1*-simple polynomials.*

Proof. — Hilbert's basis theorem states that every algebraic variety can be described as a common zero locus of finitely many polynomials. Thus we assume

$$V = \bigcap_{J=1}^{K} V^J, \quad V^J \coloneqq \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid Q^J(z_1, \dots, z_n) = 0\},$$

where $Q^J, J = 1, \dots, K$, are non-zero polynomials over $\mathbb{C}^n$. We want to prove that, for each Q^J, there exists a z_1-simple polynomial P^J s.t.

$$Q^{-1}(0) = P^{-1}(0).$$

From now on we use the abridged notations Q or P. Suppose $Q(z_1, \dots, z_n) = \sum_{i=0}^{M} b_i(z_2, \dots, z_n) z_1^i$ with $b_0, b_1, \dots, b_M$ polynomials of variables $z_2, \dots, z_n$ and b_M non-zero, then it has the following factorization in $\overline{\mathbb{F}}[z_1]$:

$$b_M(z_2, \dots, z_n) \prod_{\alpha=1}^{N} \big(z_1 - \omega_\alpha(z_2, \dots, z_n)\big)^{s_\alpha},$$

where $\omega_\alpha \in \overline{\mathbb{F}}$, $\omega_{\alpha_1}(z_2, \dots, z_n) \neq \omega_{\alpha_2}(z_2, \dots, z_n)$ for $1 \leqslant \alpha_1 < \alpha_2 \leqslant N$, integer multiplicities $s_\alpha \geqslant 1$ and $\sum_{\alpha=1}^{N} s_\alpha = M$. Let us suppose that for some α, $s_\alpha > 1$ (if not, the proof is trivial). We shall use the following notation:

$$R(z_1, \dots, z_n) \coloneqq \frac{Q(z_1, \dots, z_n)}{b_M(z_2, \dots, z_n)} = \prod_{\alpha=1}^{N} \big(z_1 - \omega_\alpha(z_2, \dots, z_n)\big)^{s_\alpha},$$

$$\widetilde{R}(z_1, \dots, z_n) \coloneqq \prod_{\alpha=1}^{N} \big(z_1 - \omega_\alpha(z_2, \dots, z_n)\big).$$

First, we shall prove $\widetilde{R}(z_1, \dots, z_n) \in \mathbb{F}[z_1]$. In fact, $R(z_1, \dots, z_n)$ is reducible in $\mathbb{F}[z_1]$ (irreducible polynomials are separable polynomials). If we consider the minimal polynomial of each root $\omega_i(y)$, with Abel's irreducibility theorem, then we get:

$$\widetilde{R} = R_1 \cdots R_m, \qquad R = (R_1)^{\sigma_1} \cdots (R_m)^{\sigma_m}$$

with $R_1, \dots, R_m \in \mathbb{F}[z_1]$ and σ_i's are chosen from $\{s_1, \dots, s_N\}$. The idea would be to construct inductively R_1 as the minimal polynomial in $\mathbb{F}[z_1]$ of $\omega_1 \in \overline{\mathbb{F}}$, then $\sigma_1 = s_1$ and R_1 is a product of some of the factors $z_1 - \omega_i(z_2, \dots, z_n)$ including $i = 1$, and we go on with R_2 minimal polynomial of one of the ω_i^J's which has not been included in R_1^J, etc.

Up to now, we have $\widetilde{R} \in \mathbb{F}[z_1]$ as announced, and we have a decomposition of Q in $\mathbb{F}[z_1]$:

$$Q(z_1, \dots, z_n) = b_M(z_2, \dots, z_n) R_1(z_1, \dots, z_n)^{\sigma_1} \cdots R_m(z_1, \dots, z_n)^{\sigma_m}.$$

Each factor R_j $(j=1,\dots,m)$ can be written as

$$R_j(z_1,\dots,z_n)=\frac{1}{L_j(z_2,\dots,z_n)}\widehat{R}_j(z_1,\dots,z_n)$$

taking for L_j the l.c.m. of the denominators of the coefficients of R_j in $\mathbb{F}$, and $\widehat{R}_j(z_1,\dots,z_n)$ is a primitive polynomial in $\mathbb{C}[z_2,\dots,z_n][z_1]$. Gauss's lemma implies that the coefficients of $\widehat{R}_1^{\sigma_1}\cdots\widehat{R}_m^{\sigma_m}$ are relatively prime in $\mathbb{C}[z_2,\dots,z_n]$. Hence the coefficients of Q are also in $\mathbb{C}[z_2,\dots,z_n]$, which implies that $\frac{b_M}{(L_1)^{\sigma_1}\cdots(L_m)^{\sigma_m}}\in\mathbb{C}[z_2,\dots,z_n]$. We define

$$P=\frac{b_M}{L_1^{\sigma_1-1}\cdots L_m^{\sigma_m-1}}R_1\cdots R_m=\frac{b_M}{L_1^{\sigma_1}\cdots L_m^{\sigma_m}}\widehat{R}_1\cdots\widehat{R}_m$$

which is the desired z_1-simple polynomial since $R_1\cdots R_m$ have distinct root in z_1 by the construction. Finally, $P^{-1}(0)=Q^{-1}(0)$ is obvious because both P and Q have common factors $\frac{b_M}{L_1^{\sigma_1}\cdots L_m^{\sigma_m}},\widehat{R}_1,\cdots,\widehat{R}_m$ which are all polynomials in $\mathbb{C}[z_2,\dots,z_n][z_1]$. □

LEMMA 4.3. — *Let $F(z_1,\dots,z_n)\in\mathbb{C}[z_1,\dots,z_n]$ be a z_1-simple polynomial, and denote by M its highest power of z_1. Then*

- $G(p,z,z_2,\dots,z_n):=F(p+z,z_2,\dots,z_n)\in\mathbb{C}[p,z,z_2,\dots,z_n]$ *is both p-simple and z-simple;*
- $G(\xi,z,z_2,\dots,z_n):=z^MF(\frac{\xi}{z},z_2,\dots,z_n)\in\mathbb{C}[\xi,z,z_2,\dots,z_n]$ *is both ξ-simple and z-simple.*

Proof. — The proof is standard and left to the reader. □

LEMMA 4.4. — *If $F(z_1,\dots,z_n)$ is a z_1-simple polynomial, which means that*

$$F(z_1,\dots,z_n)=b_M(z_2,\dots,z_n)\prod_{\alpha=1}^{M}\big(z_1-\omega_\alpha(z_2,\dots,z_n)\big)$$

with $b_M\neq 0$ and $\omega_{\alpha_1}\neq\omega_{\alpha_2}$ for $1\leqslant\alpha_1<\alpha_2\leqslant M$, then

$$\left\{(z_2,\dots,z_n)\;\middle|\;\begin{array}{l}\omega_{\alpha_1}(z_2,\dots,z_n)=\omega_{\alpha_2}(z_2,\dots,z_n)\\ \text{for some } 1\leqslant\alpha_1<\alpha_2\leqslant M\end{array}\right\}\tag{4.2}$$

is an algebraic variety.

Proof. — The set (4.2) is actually $\{(z_2,\dots,z_n)\mid \mathrm{Syl}(F,\partial_{z_1}F)=0\}$, where $\mathrm{Syl}(\cdot,\cdot)$ means the Sylvester matrix, considering F and $\partial_{z_1}F$ as polynomials in z_1 with coefficients in $\mathbb{C}[z_2,\dots,z_n]$. The set (4.2) is thus an algebraic variety, since every element in the Sylvester matrix is a polynomial of $z_2,\dots,z_n$. See [13] for details. □

5. Convolution Product

In this section, our goal is to prove Lemma 3.6. Let $F(z_1,\dots,z_n)\in\widehat{\mathcal{Q}}^n_{\mathcal{A}}$, which means it is holomorphic at origin and there exists an algebraic variety $V_F\subset\mathbb{C}^n$ such that F admits analytic continuation along any path which starts near origin and avoids V_F. From the definition of $f(z,z_2,\dots,z_n)$ in formula (3.5), it is obvious that $f(z,z_2,\dots,z_n)\in\mathbb{C}\{z,z_2,\dots,z_n\}$ since $F(z_1,\dots,z_n)\in\mathbb{C}\{z_1,\dots,z_n\}$ and $\overline{P}$ is a polynomial which vanishes at origin. The remaining part will be proved by constructing an algebraic variety V_f which can be taken as an avoidant set for f.

By Proposition 4.2, we assume

$$V_F=\bigcap_{J=1}^{K}V_F^J,\quad V_F^J\coloneqq\{(z_1,\dots,z_n)\in\mathbb{C}^n\mid P_F^J(z_1,\dots,z_n)=0\},$$

where P_F^J, $J=1,\dots,K$, are z_1-simple polynomials over $\mathbb{C}^n$.

We shall construct algebraic varieties V_f^J, $J=1,\dots,K$, s.t. if F admits analytic continuation along any path which avoids the set $(P_F^J)^{-1}(0)$, then f admits analytic continuation along any path which avoids the set V_f^J. Thus finally, the avoidant set of f is an algebraic variety

$$V_f\coloneqq\bigcap_{J=1}^{K}V_f^J.$$

Let us write

$$P_F^J(z_1,\dots,z_n)=b_M(z_2,\dots,z_n)\prod_{\alpha=1}^{M}\big(z_1-\omega_\alpha(z_2,\dots,z_n)\big)\tag{5.1}$$

with $b_M\neq0$, $\omega_1,\dots,\omega_M\in\overline{\mathbb{F}}$ and $\omega_{\alpha_1}\neq\omega_{\alpha_2}$ if $1\leqslant\alpha_1<\alpha_2\leqslant M$. One may keep in mind that M, b_M and the ω_α's actually depend on J.

Definition 5.1. — *We define V_f^J as follows:*

Case 1. — *If*

$$P_F^J(\overline{P}(z,z_2,\dots,z_n),z_2,\dots,z_n)\neq0\tag{5.2}$$

for some $(z,z_2,\dots,z_n)$, then

$$V_f^J\coloneqq\mathbb{C}^n-\left\{(z,z_2,\dots,z_n)\in\mathbb{C}^n\,\middle|\,\begin{array}{l}(P_F^J)_{(z_2,\dots,z_n)}(z_1)\text{ has }M\text{ distinct}\\ \text{non-zero roots and (5.2) holds}\end{array}\right\},\tag{5.3}$$

where $(P_F^J)_{(z_2,\dots,z_n)}(z_1)\coloneqq P_F^J(z_1,\dots,z_n)$ is treated as a polynomial in z_1 with coefficients in $\mathbb{C}[z_2,\dots,z_n]$.

CASE 2. — *If*

$$P_F^J(\overline{P}(z,z_2,\ldots,z_n),z_2,\ldots,z_n)=0 \quad \textit{for all } z,z_2,\ldots,z_n, \tag{5.4}$$

then

$$V_f^J := \mathbb{C}^n - \left\{ (z,z_2,\ldots,z_n)\in\mathbb{C}^n \,\middle|\, \begin{array}{l} (P_F^J)_{(z_2,\ldots,z_n)}(z_1) \textit{ has} \\ M \textit{ distinct non-zero roots} \end{array} \right\}. \tag{5.5}$$

More precisely, the definition (5.3) is equivalent to

$$V_f^J = \left\{ (z,z_2,\ldots,z_n)\in\mathbb{C}^n \,\middle|\, \begin{array}{ll} b_M(z_2,\ldots,z_n)=0 & \text{or} \\ P_F^J(0,z_2,\ldots,z_n)=0 & \text{or} \\ (P_F^J)_{(z_2,\ldots,z_n)}(z_1) \text{ has a multiple root} & \text{or} \\ P_F^J(\overline{P},z_2,\ldots,z_n)=0 & \end{array} \right\}, \tag{5.6}$$

and the definition (5.5) is equivalent to

$$V_f^J = \left\{ (z,z_2,\ldots,z_n)\in\mathbb{C}^n \,\middle|\, \begin{array}{ll} b_M(z_2,\ldots,z_n)=0 & \text{or} \\ P_F^J(0,z_2,\ldots,z_n)=0 & \text{or} \\ (P_F^J)_{(z_2,\ldots,z_n)}(z_1) \text{ has a multiple root} & \end{array} \right\}. \tag{5.7}$$

In both cases, one may observe that V_f^J is an algebraic variety, by assumption (5.1) and Lemma 4.4.

From the discussion above, to prove Lemma 3.6, we just need the following

LEMMA 5.2. — *Suppose that F is holomorphic at the origin and admits analytic continuation along any path which avoids the algebraic variety $V_F^J = \{(z_1,\ldots,z_n)\in\mathbb{C}^n \mid P_F^J(z_1,\ldots,z_n)=0\}$. Then $f(z,z_2,\ldots,z_n) := \int_0^{\overline{P}(z,z_2,\ldots,z_n)} F(z_1,\ldots,z_n)\,\mathrm{d}z_1$ is holomorphic at the origin and admits analytic continuation along any path γ which avoids V_f^J defined above.*

We now focus on Case 1. Case 2 will be discussed at the end of this section. We begin with the definition of a γ-homotopy.

DEFINITION 5.3. — *For a path $\gamma(t) := (\gamma_z(t),\gamma_{z_2}(t),\ldots,\gamma_{z_n}(t))\in\mathbb{C}^n$, a continuous map $H:[0,1]\times[0,1]\to\mathbb{C}, (t,s)\mapsto H_t(s) := H(t,s)$ is called* *a γ-homotopy if, for any $s,t\in[0,1]$,*

$$H_t(0)=0,\quad H_0(s)=s\cdot\overline{P}(\gamma(0)),\quad H_t(1)=\overline{P}(\gamma(t)),\\ P_F^J(H_t(s),\gamma_{z_2}(t),\dots,\gamma_{z_n}(t))\neq 0. \tag{5.8}$$

To prove Lemma 5.2 in Case 1, it is sufficient to prove the following two claims.

Claim 5.4. — *Let $\gamma:[0,1]\to\mathbb{C}^n-V_f^J$ be a path such that $\gamma(0)$ is near the origin. If there exists a γ-homotopy, then f can be analytically continued along γ.*

Claim 5.5. — *For any path $\gamma:[0,1]\to\mathbb{C}^n-V_f^J$ such that $\gamma(0)$ is near the origin, there exists a γ-homotopy.*

Notation 5.6. — For each $t\in[0,1]$, we denote by $\gamma|_t$ the truncated path defined as follows:

$$\gamma|_t\colon\ \tau\in[0,t]\mapsto\gamma(\tau)\in\mathbb{C}^n.$$

We denote by $\mathrm{cont}_{\gamma|_t} f$ the holomorphic germ at $\gamma(t)$ obtained from f by analytic continuation along $\gamma|_t$.

Proof of Claim 5.4. — One can check that, if there exists γ-homotopy, then the analytic germ at $\gamma(t)$ of f is

$$\begin{aligned}(\mathrm{cont}_{\gamma|_t} f\big)(z,z_2,\dots,z_n)&\\ =\int_{H_t}&\big(\mathrm{cont}_{(H|_t(s),\gamma_{z_2}|_t,\dots,\gamma_{z_n}|_t)} F\big)(z_1,\dots,z_n)\,\mathrm{d}z_1\\ &+\int_{\overline{P}(\gamma(t))}^{\overline{P}(z,z_2,\dots,z_n)}\big(\mathrm{cont}_{(H|_t(1),\gamma_{z_2}|_t,\dots,\gamma_{z_n}|_t)} F\big)(z_1,\dots,z_n)\,\mathrm{d}z_1,\end{aligned}\tag{5.9}$$

where $H|_t(s)$ is the truncated path of $t\mapsto H_t(s)$ for fixed $s\in[0,1]$. Note that $(\mathrm{cont}_{\gamma|_{t_1}} f)(z,z_2,\dots,z_n)=(\mathrm{cont}_{\gamma|_{t_2}} f)(z,z_2,\dots,z_n)$ for t_1 and t_2 close enough to one another, by using Cauchy integral. See [17] for details. □

Proof of Claim 5.5. — To continue analytically f along a path γ that starts near the origin and avoids V_f^J, we need a γ-homotopy $H_t(s)$ that avoids some moving points, so that the germ of F at $(H_t(s),\gamma_{z_2}(t),\dots,\gamma_{z_n}(t))$ should be well-defined (see formula (5.9)). From the last condition of (5.8) and the form of P_F^J (see formula (5.1)), we know these moving points are

$$\omega_i(t)\coloneqq\omega_i(\gamma_{z_2}(t),\dots,\gamma_{z_n}(t))\in\mathbb{C},\ \text{for } i=1,\dots,M. \tag{5.10}$$

In the set (5.6), the first and third conditions mean that there are always M distinguished moving points $\omega_i(t)$. The second and fourth conditions mean that these $\omega_i(t)$'s will not touch the starting point of the homotopy $H_t(0) = 0$ nor its ending point $H_t(1) = \overline{P}(\gamma_z(t), \gamma_{z_2}(t), \ldots, \gamma_{z_n}(t))$.

Now we want to find the γ-homotopy $H_t(s)$. The idea is to find a family of maps $(\Psi_t)_{t\in[0,1]} : \mathbb{C}\times\mathbb{R} \to \mathbb{C}\times\mathbb{R}$ such that, for any s, $H_t(s) := \Pi_{\mathbb{C}}(\Psi_t(H_0(s)))$ yield the desired homotopy, where $\Pi_{\mathbb{C}}$ is the projection from $\mathbb{C}\times\mathbb{R}$ to $\mathbb{C}$. Let $\omega_0(t) := \overline{P}(\gamma(t))$. If $\gamma(t)$ avoids V_f^J, thanks to our assumption (5.2), we have

$$\omega_i(t) \neq \omega_0(t), \quad \omega_i(t) \neq 0, \quad \omega_i(t) \neq \omega_j(t) \tag{5.11}$$

for all $t \in [0,1]$ and $1 \leqslant i \neq j \leqslant M$. See Figure 5.1.

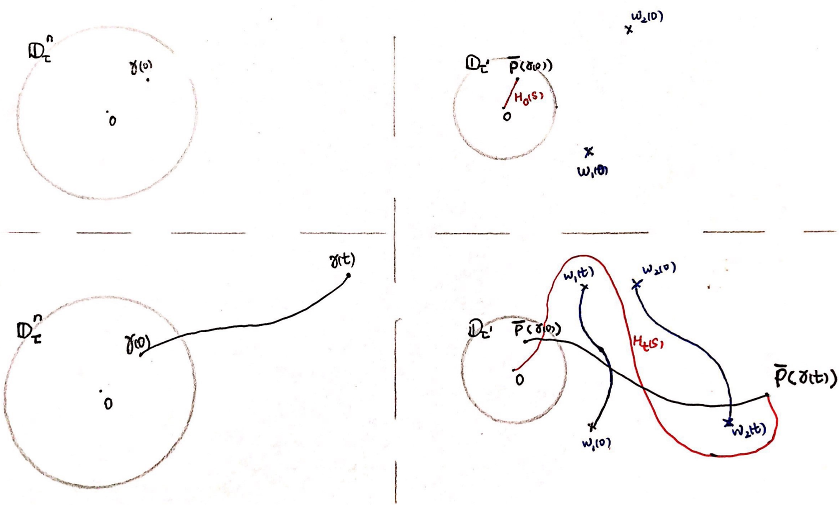

Figure 5.1. Upper-left: f is holomorphic in $\mathbb{D}_\tau^n$ which contains $\gamma(0)$. Upper-right: F is integrated over a line segment $H_0(s)$ contained in $\mathbb{D}_{\tau'}$. When f is continued analytically along $\gamma|_t$ in the lower-left picture, the corresponding homotopy $H_t(s)$ (red curve in lower-right) always exists thanks to the conditions (5.11).

To find the γ-homotopy, it suffices to find injective maps $\Psi_t : \mathbb{C}\times\mathbb{R} \to \mathbb{C}\times\mathbb{R}$ satisfying the following conditions:

$$\begin{aligned}
&(1^\circ) : \Psi_0 = \mathrm{id},\\
&(2^\circ) : \Psi_t(0,0) = (0,0),\\
&(3^\circ) : \Psi_t(\omega_0(0),0) = (\omega_0(t), \mathfrak{L}_{\omega_0(t)}),\ \text{where } \mathfrak{L}_{\omega_0(t)} := \int_0^t |\omega_0'(s)|\,\mathrm{d}s\\
&(4^\circ) : \Psi_t(\omega_i(0),\lambda) = (\omega_i(t), \lambda + \mathfrak{L}_{\omega_i(t)}),\ \text{for } i = 1,\dots,M.
\end{aligned}\tag{5.12}$$

Such maps Ψ_t can be generated by the flow of a non-autonomous vector field $X(\xi,\lambda,t)$ of $\mathbb{C}\times\mathbb{R}$ defined as follows:

$$X(\xi,\lambda,t) := \sum_{i=0}^{N} \frac{N_i(\xi,\lambda,t)}{N_i(\xi,\lambda,t)+\eta_i(\xi,\lambda,t)} \big(\omega_i'(t), |\omega_i'(t)|\big),$$

where, for $i = 1,\dots,N$,

$$\begin{aligned}
N_0(\xi,\lambda,t) &:= \mathrm{dist}((\xi,\lambda),(S(t),\mathbb{R})\cup\{(0,0)\}),\\
N_i(\xi,\lambda,t) &:= \mathrm{dist}((\xi,\lambda),(S_i(t),\mathbb{R})\cup\{(0,0)\}\cup\{(\omega_0(t),\mathfrak{L}_{\omega_0(t)})\}),\\
\eta_0(\xi,\lambda,t) &:= \mathrm{dist}((\xi,\lambda),(\omega_0(t),\mathfrak{L}_{\omega_0(t)})),\\
\eta_i(\xi,\lambda,t) &:= |\xi-\omega_i(t)|,
\end{aligned}$$

with $S(t) := \{\omega_1(t),\dots,\omega_n(t)\}$ and $S_i(t) := S(t) - \{\omega_i(t)\}$.

One can check that $N_i + \eta_i \neq 0$ for $i = 1,\dots,N$. The Cauchy–Lipschitz theorem on the existence and uniqueness of solutions to differential equations applies to $\frac{\mathrm{d}}{\mathrm{d}t}(\xi,\lambda) = X(\xi,\lambda,t)$: for every $(\xi,\lambda)\in\mathbb{C}\times\mathbb{R}$ and $t_0\in[0,1]$ there is a unique solution $t\mapsto\Psi^{t_0,t}(\xi,\lambda)$ such that $\Psi^{t_0,t_0} = \mathrm{id}$.

Let us set $t_0 = 0$ and $\Psi_t := \Psi^{0,t}$ for $t\in[0,1]$. It is easy to see that this family of maps are injective and satisfy the conditions (5.12) of Ψ. This concludes the proof of Lemma 5.2 in Case 1.

Here are two simple examples in Case 1.

Example 5.7. — $f(z,z_2) := \int_0^z \frac{1}{z_2z_1+1}\,\mathrm{d}z_1 = \frac{1}{z_2}\log(z_2z+1)$. One can find a path in $\mathbb{C}^2$ to prove that the singular set of f is $\{(z,z_2) \mid z_2(z_2z+1) = 0\}$. Actually, $\{z_2 = 0\}$ is the first condition in (5.6), $\{z_2z+1 = 0\}$ is the fourth condition.

Example 5.8. — $f(z,z_2) := \int_0^z \frac{1}{(z_1+1)(z_1+z_2+1)}\,\mathrm{d}z_1 = \frac{1}{z_2}\big(\log(z+1) - \log(z+z_2+1) + \log(z_2+1)\big)$. The singular set of f is $\{(z,z_2) \mid z_2(z+1)(z+z_2+1)(z_2+1) = 0\}$. Actually, $\{z_2 = 0\}$ is the third condition in (5.6), $\{(z+1)(z+z_2+1) = 0\}$ is the fourth condition, and $\{z_2+1 = 0\}$ is the second condition.

Remark 5.9. — Although the definition V_f^J gives the possibly singular set, which means that maybe a subset of V_f^J is regular, from these two simple examples, one can observe that all the conditions in (5.6) make sense.

Now we discuss Lemma 5.2 in Case 2. The following example helps us to understand how Case 2 happens.

Example 5.10. — Consider

$$f(z, z_2) := \int_0^{z_2} \frac{1}{z_2 - z_1} \log\big(1 - (z_2 - z_1)\big) \mathrm{d} z_1.$$

We know that $F(z_1, z_2) := \frac{1}{z_2 - z_1} \log\big(1 - (z_2 - z_1)\big)$ is holomorphic at $(0, 0)$ and has singular set $V_f = \{(z_1, z_2) \mid (z_1 - z_2)\big(z_1 - (z_2 - 1)\big) = 0\}$. We thus have $\omega_1(z_2) = z_2 = \overline{P}(z_2)$ and $\omega_2(z_2) = z_2 - 1$. The change of variable $u = z_2 - z_1$ yields

$$f(z, z_2) = \int_0^{z_2} \frac{1}{u} \log(1 - u)\, \mathrm{d}u = -\operatorname{Li}_2(z_2). \tag{5.13}$$

It is obvious that the singular set of f is $\{z_2 = 0 \text{ or } 1\}$ (compute the partial derivative $\frac{\partial f}{\partial z_2}$).

In Case 2, we will use the following

Definition 5.11. — *For a path $\gamma(t) := (\gamma_z(t), \gamma_{z_2}(t), \dots, \gamma_{z_n}(t)) \in \mathbb{C}^n$, a continuous map $H : [0, 1] \times [0, 1] \to \mathbb{C}, (t, s) \mapsto H_t(s) := H(t, s)$ is called a γ'-homotopy if, for any $s, t \in [0, 1]$,*

$$\begin{aligned} H_t(0) = 0, \quad H_0(s) = s \cdot \overline{P}(\gamma(0)), \quad H_t(1) = \overline{P}(\gamma(t)), \\ s \neq 1 \;\Rightarrow\; P_F^J(H_t(s), \gamma_{z_2}(t), \dots, \gamma_{z_n}(t)) \neq 0. \end{aligned} \tag{5.14}$$

In order to prove Lemma 5.2, we will use the same procedure as in Case 1. We shall use the same formula (5.9) to write down the analytic continuation of f along $\gamma(t)$. The only difference between γ'-homotopy and γ-homotopy is the ending points (when $s = 1$) of the fourth condition in (5.8) and (5.14). Thus we shall prove that the germs

$$\left(\operatorname{cont}_{(H|_t(1), \gamma_{z_2}|_t, \dots, \gamma_{z_n}|_t)} F\right)(z_1, \dots, z_n)$$

inside integral representation (5.9) are well-defined.

Let $\omega_i(t) := \omega_i(\gamma_{z_2}(t), \dots, \gamma_{z_n}(t)) \in \mathbb{C},$ for $i = 1, \dots, M$ and $\overline{P}(\gamma(t)) = \omega_1(t)$.

Actually, $\omega_1(t)$ is not a singular point for γ'-homotopy because the variable $(z_1 - \overline{P})$ always lies in the principle sheet when we move along γ. It will be clear after we change the variable:

$$f(z, z_2, \dots, z_n) = \overline{P}(z, z_2, \dots, z_n) \int_{-1}^{0} G(\zeta, z_2, \dots, z_n)\, \mathrm{d}\zeta$$

with $G(\zeta, z_2, \dots, z_n) \coloneqq F\left(\overline{P}(z, z_2, \dots, z_n)(1+\zeta), z_2, \dots, z_n\right)$. We can find sufficient small $R > 0$ s.t.

$$G(\zeta, z_2, \dots, z_n) = \frac{1}{2\pi\mathrm{i}} \oint_{\partial\mathbb{D}_R} \frac{G(\xi, z_2, \dots, z_n)}{\xi - \zeta}\, \mathrm{d}\xi.$$

Indeed, the set (5.5) which $\gamma(t)$ should avoid implies that the moving singular points of G

$$\eta_i(t) \coloneqq \frac{\omega_i(t)}{\omega_1(t)} - 1, \quad \text{for } i = 2, \dots, M$$

never touch 0. This allows us to choose sufficient small R s.t. $\eta_i(t)$ always lie outside $\mathbb{D}_R$. One can prove that such G is always holomorphic at $(0, \gamma_{z_2}(t), \dots, \gamma_{z_n}(t))$. This concludes the proof of Lemma 5.2 in Case 2. □

Proof of Lemma 3.7. — Given $\overline{P}(z_2, \dots, z_n) \in \mathbb{C}[z_2, \dots, z_n]$, we may apply lemma 3.6 treating $\overline{P}$ as an element of $\mathbb{C}[z_1, \dots, z_n]$: $f(z_2, \dots, z_n) = g(z, z_2, \dots, z_n)$ with $g \in \widehat{\mathcal{Q}}_n^{\mathcal{A}}$. We observe that formulas (5.6) and (5.7) yield $V_g = \mathbb{C} \times V_f$, hence $f \in \widehat{\mathcal{Q}}_{n-1}^{\mathcal{A}}$. □

6. Hadamard Product

6.1. Introduction to Hadamard product on $\mathbb{C}$

In this section, we study the analytic continuation of the Hadamard product.

Definition 6.1. — *Let $f(\xi) = \sum_{n=0}^{\infty} a_n \xi^n$ and $g(\xi) = \sum_{m=0}^{\infty} b_m \xi^m$ be two formal series in $\mathbb{C}[\![\xi]\!]$. Their Hadamard product is defined to be the formal series*

$$f \odot g(\xi) = \sum_{n=0}^{\infty} a_n b_n \xi^n. \tag{6.1}$$

If $f, g \in \mathbb{C}\{\xi\}$, then, denoting by R_f and R_g their positive radii of convergence, we have $f \odot g \in \mathbb{C}\{\xi\}$ with radius of convergence $R_{f \odot g} \geqslant R_f R_g$.

Indeed, for any positive $c < R_f$, we have the integral representation (using notation (3.1))

$$\xi \in \mathbb{D}_{cR_g} \;\Rightarrow\; f \odot g(\xi) = \oint_C f(z) g\left(\frac{\xi}{z}\right) \frac{\mathrm{d}z}{2\pi \mathrm{i} z} = \oint_C f\left(\frac{\xi}{z}\right) g(z) \frac{\mathrm{d}z}{2\pi \mathrm{i} z}, \tag{6.2}$$

where C is the parametrized circle $s \in [0, 2\pi] \mapsto c\,\mathrm{e}^{\mathrm{i}s}$.

Note that if $|\xi| < \min\{R_f^2, R_g^2\}$ then one can use $C =$ parametrized circle $\theta \in [0, 2\pi] \mapsto \sqrt{\xi}\,\mathrm{e}^{\mathrm{i}\theta}$, where $\sqrt{\xi}$ is any square root of ξ, which yields

$$\begin{aligned} f \odot g(\xi) &= \int_0^{2\pi} f\left(\sqrt{\xi}\mathrm{e}^{\mathrm{i}\theta}\right) g\left(\sqrt{\xi}\mathrm{e}^{-\mathrm{i}\theta}\right) \frac{\mathrm{d}\theta}{2\pi} \\ &= \int_0^{2\pi} f\left(\sqrt{\xi}\mathrm{e}^{-\mathrm{i}\theta}\right) g\left(\sqrt{\xi}\mathrm{e}^{\mathrm{i}\theta}\right) \frac{\mathrm{d}\theta}{2\pi}. \end{aligned} \tag{6.3}$$

The following theorem is related to the classical "Hadamard multiplication theorem", and is in fact a weaker version of a theorem proved in [16].

Theorem 6.2. — *If $f, g \in \widehat{\mathcal{Q}}_1^{\mathcal{A}}$, which means, $f, g \in \mathbb{C}\{\xi\}$ and they admit analytic continuation along any path which avoids finite sets S_f and S_g respectively, then $f \odot g \in \widehat{\mathcal{Q}}_1^{\mathcal{A}}$ and it admits analytic continuation along any path which starts near origin and avoids $\{0\} \cup S_f \cdot S_g$.*

Example 6.3. — If $f(\xi) = \log(1 - \xi)$, then one can compute

$$\frac{\mathrm{d}}{\mathrm{d}\xi}(f \odot f)(\xi) = -\frac{1}{\xi} \log(1 - \xi),$$

which means the singular points of $f \odot f$ are 0 and 1. In fact $f \odot f = \mathrm{Li}_2$ as in (2.12).

Although a more general statement is proved in [16], for the sake of completeness, let us mention here how Theorem 6.2 can be proved. Let us introduce the following

Definition 6.4. — *A γ^C-homotopy for the one-dimensional case is a continuous map $H\colon\ (t, s) \in [0, 1] \times [0, 2\pi] \mapsto H_t(s) \in \mathbb{C}$ such that:*

$$H_0(s) = c\,\mathrm{e}^{\mathrm{i}s}, \quad H_t(s) \neq 0, \quad H_t(s) \cap S_f = \emptyset, \quad \frac{\gamma(t)}{H_t(s)} \cap S_g = \emptyset.$$

Claim 6.5. — *If there exists a γ^C-homotopy, then we can do analytic continuation in the following way:*

$$(\mathrm{cont}_{\gamma|_t} f \odot g)(\xi) = \frac{1}{2\pi \mathrm{i}} \oint_{H_t} (\mathrm{cont}_{H_{|t}(s)} f)(z) (\mathrm{cont}_{\frac{\gamma|_t}{H_{|t}(s)}} g)\left(\frac{\xi}{z}\right) \frac{\mathrm{d}z}{z}. \tag{6.4}$$

Proof. — Similar to the proof of claim 5.4. ∎

Claim 6.6. — *If γ starts near origin and avoids $\{0\}\cup S_f\cdot S_g$, then there exists a γ^C-homotopy.*

Proof. — Suppose $S_f=\{f_1,\dots,f_s\}$, $S_g=\{g_1,\dots,g_r\}$. Without loss of generality, we assume $g_i\neq 0$ for all $i=1,\dots,r$. Indeed, if $g_1=0$, it has no influence on the γ^C-homotopy because $\gamma(t)$ never touches 0. Let

$$\omega_i(t)=\frac{\gamma(t)}{g_i},\quad i=1,\dots,r,$$

be the moving singular points of the homotopy to be found. By the assumption, $\gamma(t)\neq 0$ implies that $\omega_i(t)\neq\omega_j(t)$ for $i\neq j$ and $\omega_i(t)\neq 0$, $\gamma(t)\notin S_f\cdot S_g$ implies that $\omega_j(t)\neq f_i$.

To find the homotopy, it is sufficient to find a flow $\Psi_t:\mathbb{C}\to\mathbb{C}$ that satisfies:

$$\begin{aligned}\Psi_0&=\mathrm{id},\\ \Psi_t(0)&=0,\\ \Psi_t(f_i)&=f_i &&\text{for } i=1,\dots,s,\\ \Psi_t(\omega_j(0))&=\omega_j(t) &&\text{for } j=1,\dots,r.\end{aligned}$$

(See Figure 6.1.)

To that end, we use the non-autonomous vector field

$$X(\xi,t):=\sum_{i=1}^{r}\frac{\eta_i(\xi,t)}{\eta_i(\xi,t)+\tau_i(\xi,t)}\omega_i'(t),$$

with

$$\eta_i(\xi,t):=\mathrm{dist}\left(\xi,\{0\}\cup S_f\cup\bigcup_{i\neq j}\omega_j(t)\right),\qquad \tau_i(\xi,t):=\mathrm{dist}(\xi,\omega_i(t)).$$

The Cauchy–Lipschitz theorem on the existence and uniqueness of solutions to differential equations applies to $\frac{\mathrm{d}\xi}{\mathrm{d}t}=X(\xi,t)$: for every $\xi\in\mathbb{C}$ and $t_0\in[0,1]$ there is a unique solution $t\to\Phi^{t_0,t}(\xi)$ such that $\Phi^{t_0,t_0}(\xi)=\xi$. Then we define our flow as $\Psi_t:=\Phi^{0,t}$ for $t\in[0,1]$ and the sought γ^C-homotopy as $H_t(s):=\Psi_t(H_0(s))$. □

6.2. Proof of lemma 3.5

Suppose $\widehat{f}(\xi,q,p),\widehat{g}(\xi,q,p)\in\widehat{\mathcal{Q}}_3^{\mathcal{A}}$. We thus may assume that $\widehat{f}$ and $\widehat{g}$ are holomorphic on $\mathbb{D}^3_\tau$ for some $\tau>0$ and that there are avoidant algebraic

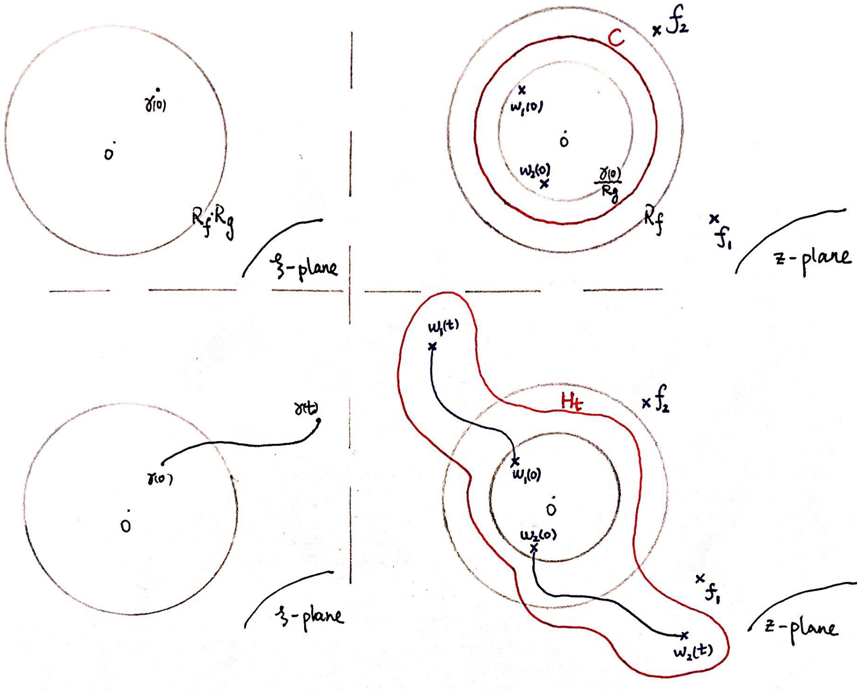

Figure 6.1. Upper-left: $f \odot g$ is holomorphic in $\mathbb{D}_{R_f \cdot R_g}$ which contains $\gamma(0)$. Upper-right: Using C to be the integral curve when $t = 0$, we notice that the singular points of f are outside $\mathbb{D}_{R_f \cdot R_g}$ and the "moving singular points" of the homotopy are inside $\mathbb{D}_{R_f \cdot R_g}$. When $f \odot g$ is continued analytically along $\gamma|_t$ in the lower-left picture, the corresponding homotopy $H_t(s)$(red curve in lower-right) always exists thanks to the conditions on the $\omega(t)$'s.

sets of the form

$$V_f := \bigcap_{J=1}^{R} V_f^J, \quad V_f^J := \{(\xi, q, p) \in \mathbb{C}^3 \mid P_f^J(\xi, q, p) = 0\}, \text{ for } J = 1, \dots, R,$$

$$V_g := \bigcap_{K=1}^{S} V_g^K, \quad V_g^K := \{(\xi, q, p) \in \mathbb{C}^3 \mid Q_g^K(\xi, q, p) = 0\}, \text{ for } K = 1, \dots, S,$$

where the P_f^J's are p-simple polynomials and the Q_g^K's are q-simple polynomials.

For each $J = 1, \dots, R, K = 1, \dots, S$, we shall construct algebraic variety V_F^{JK}, s.t. if f avoids the set $\{x \in \mathbb{C}^n \mid x \in (P_f^J)^{-1}(0)\}$, and g avoids the set $\{x \in \mathbb{C}^n \mid x \in (Q_g^K)^{-1}(0)\}$, then F admits analytic continuation along any γ which avoids the set V_F^{JK}. Thus finally, the avoidant set of F is an algebraic variety

$$V_F = \bigcap_{J,K} V_F^{JK}.$$

To simplify the notation, let

$$\begin{aligned} P_f^J(\xi, q, p) &= a_M(\xi, q) \prod_{\alpha=1}^{M} \big(p - \widetilde{\omega}_\alpha(\xi, q)\big), \\ Q_g^K(\xi, q, p) &= b_N(\xi, p) \prod_{\beta=1}^{N} \big(q - \widetilde{\Omega}_\beta(\xi, p)\big), \end{aligned} \tag{6.5}$$

with $a_M, b_N \neq 0$, $\widetilde{\omega}_{\alpha_1} \neq \widetilde{\omega}_{\alpha_2}$ if $1 \leqslant \alpha_1 < \alpha_2 \leqslant M$, $\widetilde{\Omega}_{\beta_1} \neq \widetilde{\Omega}_{\beta_2}$ if $1 \leqslant \beta_1 < \beta_2 \leqslant N$, just like what we have done in Section 5. We should keep in mind that a_M, M, and the $\widetilde{\omega}_\alpha$'s depend on J, and b_N, N, and the $\widetilde{\Omega}_\beta$'s depend on K.

By the formula (3.4), one can easily prove that F is holomorphic inside $\mathbb{D}(\tau) \times \mathbb{D}(\tau) \times \mathbb{D}(\frac{\tau^2}{4}) \times \mathbb{D}(\frac{\tau}{2}) \times \mathbb{D}(\frac{\tau}{2})$. We shall choose a point $(\xi_1, \xi_2, \xi_3, q, p)$ in this polydisc, then there exists $c > 0$, s.t.

$$\frac{|\xi_3|}{\frac{\tau}{2}} < c < \frac{\tau}{2}.$$

The formula (3.4) is equivalent to

$$F(\xi_1, \xi_2, \xi_3, q, p) = \frac{1}{2\pi \mathrm{i}} \oint_C f(\xi_1, q, p + z) g\left(\xi_2, q + \frac{\xi_3}{z}, p\right) \frac{\mathrm{d}z}{z} \tag{6.6}$$

where C is the circle of radius c with center at origin.

Let us consider the polynomials

$$\begin{aligned} P_f^J(\xi_1, q, p + z) &\coloneqq a_M(\xi_1, q) \prod_{\alpha=1}^{M} (z - \omega_\alpha(\xi_1, q, p)) \in \mathbb{C}[\xi_1, q, p, z], \\ Q_g^K(\xi_2, q + z, p) &\coloneqq b_N(\xi_2, p) \prod_{\beta=1}^{N} (z - \Omega_\beta(\xi_2, q, p)) \in \mathbb{C}[\xi_2, q, p, z]. \end{aligned} \tag{6.7}$$

From the lemma 4.3, we know that these two polynomials are both z-simple polynomials, which means that we have

$$\begin{aligned} a_M \neq 0, &\quad \omega_{\alpha_1} \neq \omega_{\alpha_2} \text{ if } 1 \leqslant \alpha_1 < \alpha_2 \leqslant M, \\ b_N \neq 0, &\quad \Omega_{\beta_1} \neq \Omega_{\beta_2} \text{ if } 1 \leqslant \beta_1 < \beta_2 \leqslant N. \end{aligned} \tag{6.8}$$

We shall define the avoidant set of F in $\mathbb{C}^5$ by using the notations above. One may notice that it is a “symmetry” condition:

Definition 6.7. — *Let*

$$V_F^{JK} := \mathbb{C}^5 - \left\{ (\xi_1,\xi_2,\xi_3,q,p)\in\mathbb{C}^5 \;\middle|\; \begin{array}{l} \mathfrak{P}_{\xi_1,q,p}(z) \text{ has } M \text{ distinct non-zero roots,} \\ \mathfrak{Q}_{\xi_2,q,p}(z) \text{ has } N \text{ distinct non-zero roots,} \\ \xi_3 \notin \{\omega_\alpha\Omega_\beta\}\cup\{0\} \end{array} \right\},$$

where $\mathfrak{P}_{\xi_1,q,p}(z) := P_f^J(\xi_1,q,p+z)$ is treated as a polynomial in z with coefficients in $\mathbb{C}[\xi_1,q,p]$ and $\mathfrak{Q}_{\xi_2,q,p}(z) := Q_g^K(\xi_2,q+z,p)$ is treated as a polynomial in z with the coefficients in $\mathbb{C}[\xi_2,q,p]$.

Remark 6.8. — In fact, V_F^{JK} can be simplified to one sentence:

$$V_F^{JK} = \mathbb{C}^5 - \left\{ (\xi_1,\xi_2,\xi_3,q,p)\in\mathbb{C}^5 \;\middle|\; \begin{array}{l} z^N\mathfrak{P}_{\xi_1,q,p}(z)\mathfrak{Q}_{\xi_2,q,p}\left(\dfrac{\xi_3}{z}\right) \\ \text{has } M+N \text{ distinct non-zero roots} \end{array} \right\}. \tag{6.9}$$

By Lemma 4.4, we know that V_F^{JK} is an algebraic variety.

From the discussion above, to prove Lemma 3.5 it is sufficient to prove the following claim:

Claim 6.9. — *If f and g admit analytic continuation along any path that avoids V_f^J and V_g^K respectively, then F defined by formula (6.6) admits analytic continuation along any path γ that avoids V_F^{JK} defined above.*

By using the following eight conditions, we describe V_F^{JK} more precisely:

$$V_F^{JK} = \left\{ (\xi_1,\xi_2,\xi_3,q,p)\in\mathbb{C}^5 \;\middle|\; \begin{array}{ll} a_M(\xi_1,q)=0 \text{ or } b_N(\xi_2,p)=0 & \text{or} \\ \omega_{\alpha_1}(\xi_1,q,p)=\omega_{\alpha_2}(\xi_1,q,p) \text{ for some } \alpha_1\neq\alpha_2 & \text{or} \\ \Omega_{\beta_1}(\xi_2,q,p)=\Omega_{\beta_2}(\xi_2,q,p) \text{ for some } \beta_1\neq\beta_2 & \text{or} \\ \omega_\alpha(\xi_1,q,p)=\dfrac{\xi_3}{\Omega_\beta(\xi_2,q,p)} \text{ for some } \alpha,\beta & \text{or} \\ \omega_\alpha(\xi_1,q,p)=0 \text{ for some } \alpha & \text{or} \\ \Omega_\beta(\xi_2,q,p)=0 \text{ for some } \beta & \text{or} \\ \xi_3=0 & \end{array} \right\} \tag{6.10}$$

where $\alpha,\alpha_1,\alpha_2\in\{1,\dots,M\}$ and $\beta,\beta_1,\beta_2\in\{1,\dots,N\}$.

Assume that $\gamma\colon\ t\in[0,1]\mapsto\gamma(t)=\big(\gamma_{\xi_1}(t),\gamma_{\xi_2}(t),\gamma_{\xi_3}(t),\gamma_q(t),\gamma_p(t)\big)\in\mathbb{C}^5$ has its starting point $\gamma(0)$ close to 0. To perform analytic continuation, we shall adapt Definition 6.4 to this situation:

Definition 6.10. — *A γ^C-homotopy is a continuous map $H:(t,s)\in[0,1]\times[0,2\pi]\to H_t(s)\in\mathbb{C}$ s.t. for any t,s,α,β:*

$$H_0(s)=c\,\mathrm{e}^{\mathrm{i}s},\qquad\qquad H_t(s)\neq 0,$$

$$H_t(s)\neq\omega_\alpha\big(\gamma_{\xi_1}(t),\gamma_q(t),\gamma_p(t)\big),\qquad H_t(s)\neq\frac{\gamma_{\xi_3}(t)}{\Omega_\beta\big(\gamma_{\xi_2}(t),\gamma_q(t),\gamma_p(t)\big)}.$$

We shall prove that, if there exists such a γ^C-homotopy, then

$$\begin{aligned}(\operatorname{cont}_{\gamma|_t}F)(\xi_1,\xi_2,\xi_3,q,p)&\\ =\frac{1}{2\pi\mathrm{i}}\oint_{H_t}&\Big(\operatorname{cont}_{(\gamma_\xi|_t,\gamma_q|_t,\gamma_p|_t+H|_t(s))}\widehat{f}\Big)(\xi_1,q,p+z)\\ &\cdot\bigg(\operatorname{cont}_{\big(\gamma_{\xi_2}|_t,\gamma_q|_t+\frac{\gamma_{\xi_3}|_t}{H|_t(s)},\gamma_p|_t\big)}\widehat{g}\bigg)\Big(\xi_2,q+\frac{\xi_3}{z},p\Big)\frac{\mathrm{d}z}{z},\end{aligned}$$

which means that F admits analytic continuation along γ (see the proof of Claim 5.4 for details).

Let us now assume that γ avoids V_F^{JK}. To find the desired homotopy, it is sufficient to find a flow $\Psi_t:\mathbb{C}\to\mathbb{C}$ that satisfies

$$\begin{aligned}\Psi_0&=\mathrm{id},\\ \Psi_t(0)&=0,\\ \Psi_t(\omega_i(0))&=\omega_i(t)\quad\text{for } i=1,\dots,N+M,\end{aligned}$$

where $\omega_{N+j}(t)\coloneqq\frac{\gamma_{\xi_3}(t)}{\Omega_j(t)}$, $j=1,\dots,M$. Here we use $\omega_\alpha(t)$ and $\Omega_\beta(t)$ to simplify the notation $\omega_\alpha(\gamma_{\xi_1}(t),\gamma_q(t),\gamma_p(t))$ and $\Omega_\beta(\gamma_{\xi_2}(t),\gamma_q(t),\gamma_p(t))$ respectively.

The conditions in (6.10) ensure that, if γ avoids V_F^{JK}, then $\omega_i(t)\neq\omega_j(t)$ and $\omega_i(t)\neq 0$ for $1\leqslant i\neq j\leqslant M+N$. Thus we can use the non-autonomous vector field:

$$X(\xi,t)\coloneqq\sum_{i=1}^{r}\frac{\eta_i(\xi,t)}{\eta_i(\xi,t)+\tau_i(\xi,t)}\omega_i'(t)$$

where $\eta_i(\xi,t)\coloneqq\operatorname{dist}(\xi,\{0\}\cup\bigcup_{j\neq i}\omega_j(t))$ and $\tau_i(\xi,t)\coloneqq\operatorname{dist}(\xi,\omega_i(t))$. The Cauchy–Lipschitz theorem on the existence and uniqueness of solutions to differential equations applies to $\frac{\mathrm{d}\xi}{\mathrm{d}t}=X(\xi,t)$: for every $\xi\in\mathbb{C}$ and $t_0\in[0,1]$ there is a unique solution $t\to\Phi^{t_0,t}(\xi)$ such that $\Phi^{t_0,t_0}(\xi)=\xi$. Then we define our flow as $\Psi_t=\Phi^{0,t}$ for $t\in[0,1]$ and the desired γ^C-homotopy as $H_t(s)\coloneqq\Psi_t(c\,\mathrm{e}^{\mathrm{i}s})$. This concludes the proof of Claim 6.9.

Example 6.11. — The functions $f(\xi,q,p) \coloneqq \log(3-\xi-q-p)$ and $g(\xi,q,p) \coloneqq \frac{1}{3-\xi-q-p}\cdot\frac{1}{4-\xi-2q-p}$ are holomorphic in $\mathbb{D}(1)\times\mathbb{D}(1)\times\mathbb{D}(1)$. From the discussion above, we know that the corresponding F is holomorphic in $\mathbb{D}(1)\times\mathbb{D}(1)\times\mathbb{D}(\frac{1}{4})\times\mathbb{D}(\frac{1}{2})\times\mathbb{D}(\frac{1}{2})$. We choose $\gamma_{\xi_3}(0)=\frac{1}{8}$, then there exists $c>0$ (we may choose $c=\frac{3}{8}$) s.t. $\frac{\frac{1}{8}}{\frac{1}{2}}<c<\frac{1}{2}$. From formula (6.6), we compute

$$\begin{aligned} F(\xi_1,\xi_2,\xi_3,q,p) &= \oint_C \log(\omega-z)\cdot\frac{1}{A-\frac{\xi_3}{z}}\cdot\frac{1}{B-\frac{2\xi_3}{z}}\cdot\frac{\mathrm{d}z}{2\pi\mathrm{i}z} \\ &= \frac{1}{2\pi\mathrm{i}AB}\oint_C \log(\omega-z)\cdot\frac{z\,\mathrm{d}z}{(z-\frac{\xi_3}{A})(z-\frac{2\xi_3}{B})} \end{aligned}$$

with $\omega \coloneqq 3-\xi_1-q-p$, $A \coloneqq 3-\xi_2-q-p$ and $B \coloneqq 4-\xi_2-2q-p$, and a residue computation yields

$$F(\xi_1,\xi_2,\xi_3,q,p) = \frac{1}{B-2A}\cdot\left[\log\left(\omega-\frac{\xi_3}{A}\right)\cdot\frac{1}{A}-\log\left(\omega-\frac{2\xi_3}{B}\right)\cdot\frac{2}{B}\right],$$

with $B-2A=\xi_2+p-2$. Thus, we directly see that F is an algebro-resurgent germ, with avoidant set

$$V \coloneqq \{\xi_2+p-2=0\}\cup\{A=0\}\cup\{B=0\}\cup\{\xi_3=\omega A\}\cup\{\xi_3=\omega B/2\}.$$

Here we have $P_f(\xi_1,q,p)=3-\xi_1-q-p=\omega$, thus $\omega_1(\xi_1,q,p)=\omega$, and $P_g(\xi_2,q,p)=AB$, thus $\Omega_1=A$, $\Omega_2=B/2$, so formula (6.10) leads to $V_F^{11}\cup V_F^{12}=V\cup\{\omega_1=0\}\cup\{\xi_3=0\}$. Notice that $\{\Omega_1=\Omega_2\}=\{\xi_2+p-2=0\}$ is not singular for the principal branch of F, but it is for some branches of its analytic continuation.

Remark 6.12. — The above computation gives an example where the fourth, fifth and seventh conditions in Definition (6.10) are necessary. Examples can be found for the other conditions too.

7. Conclusion of the proof for an arbitrary number of degrees of freedom

We now assume $N\geqslant 1$, $q=(q_1,\dots,q_N)$ and $p=(p_1,\dots,p_N)$, and want to prove Theorem 1.2, i.e. that

$$\widehat{f},\widehat{g}\in\widehat{\mathcal{Q}}^{\mathcal{A}}_{2N+1} \quad\Rightarrow\quad \widehat{f}*\widehat{g}\in\widehat{\mathcal{Q}}^{\mathcal{A}}_{2N+1}.$$

Recall the integral formula (2.30):

$$\begin{aligned}(7.1)\quad \widehat{f} * \widehat{g}(\xi, q, p) = \frac{\mathrm{d}^{N+2}}{\mathrm{d}\xi^{N+2}} \int_0^{\xi} \mathrm{d}\xi_1 \int_0^{\xi-\xi_1} \mathrm{d}\xi_2 \cdots \\ \times \int_0^{\xi-\xi_1-\cdots-\xi_{N+1}} \mathrm{d}\xi_{N+2} \int_0^{2\pi} \frac{\mathrm{d}\theta_1}{2\pi} \cdots \int_0^{2\pi} \frac{\mathrm{d}\theta_N}{2\pi} \\ \times \widehat{f}\big(\xi_{N+1}, q_1, \ldots, q_N, p_1 + \sqrt{\xi_1}\mathrm{e}^{-\mathrm{i}\theta_1}, \ldots, p_N + \sqrt{\xi_N}\mathrm{e}^{-\mathrm{i}\theta_N}\big) \\ \times\ \widehat{g}\big(\xi_{N+2}, q_1 + \sqrt{\xi_1}\mathrm{e}^{\mathrm{i}\theta_1}, \ldots, q_N + \sqrt{\xi_N}\mathrm{e}^{\mathrm{i}\theta_N}, p_1, \cdots, p_N\big)\end{aligned}$$

(we have chosen C_j = parametrized circle $\theta_j \in [0, 2\pi] \mapsto \sqrt{\xi_j}\,\mathrm{e}^{\mathrm{i}\theta_j}$, where $\sqrt{\xi_j}$ is any square root of ξ_j). We will make use of the following

Definition 7.1. — *Suppose $1 \leqslant i < j \leqslant n$. We define the Hadamard–Taylor operator on coordinates (i, j) as the linear operator $\odot^n_{i,j}$:*

$$F(z_1, \ldots, z_n) \in \mathbb{C}\{z_1, \ldots, z_n\} \mapsto \odot^n_{i,j} F(\xi, z_1, \ldots, z_n) \in \mathbb{C}\{\xi, z_1, \ldots, z_n\},$$

with

$$\begin{aligned}\odot^n_{i,j}\, F(\xi, z_1, \ldots, z_n) \coloneqq \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \\ \times F\Big(z_1, \ldots, z_{i-1}, z_i + \sqrt{\xi}\mathrm{e}^{-\mathrm{i}\theta}, z_{i+1}, \ldots, z_{j-1}, z_j + \sqrt{\xi}\mathrm{e}^{\mathrm{i}\theta}, z_{j+1}, \ldots, z_n\Big).\end{aligned}$$

Notice that the number of variables is not the same in the source and target spaces: a new variable is inserted in first position. In particular, starting with the following function of $4N + 2$ variables,

$$\begin{aligned}(7.2)\quad F(\xi_{N+1}, z_1, \ldots, z_{2N}, \xi_{N+2}, z_{2N+1}, \ldots, z_{4N}) \\ \coloneqq \widehat{f}(\xi_{N+1}, z_1, \ldots, z_{2N})\, \widehat{g}(\xi_{N+2}, z_{2N+1}, \ldots, z_{4N}),\end{aligned}$$

we find that $G_1 \coloneqq \odot^{4N+2}_{2N+1,3N+2} F$ is a function of $4N + 3$ variables such that, when evaluating G_1 on $(\xi_N, \xi_{N+1}, z_1, \ldots, z_{2N}, \xi_{N+2}, z_{2N+1}, \ldots, z_{4N})$ with

$$(7.3)\qquad (z_1, \ldots, z_{4N}) \coloneqq (q, p, q, p),$$

we get the integrand of (7.1) with just the θ_N-integration performed.

Similarly, $G_2 \coloneqq \odot^{4N+3}_{2N+1,3N+2} G_1$ is a function of $4N + 4$ variables; if we evaluate it on $(\xi_{N-1}, \xi_N, \xi_{N+1}, z_1, \ldots, z_{2N}, \xi_{N+2}, z_{2N+1}, \ldots, z_{4N})$ with the z_j's as in (7.3), then we get the integrand of (7.1) with the θ_N-integration *and the θ_{N-1}-integration* performed. Note that, although we use $(i, j) = (2N + 1, 3N + 2)$ in both Hadamard–Taylor operators, the effect of the first one is on a pair (q_N, p_N), while the effect of the second one is on a pair

(q_{N-1}, p_{N-1}) because the insertion of a new variable in first position shifts the old ones by one unit.

In fact, we can rewrite the formula (7.1) as

$$\widehat{f} * \widehat{g}(\xi, q, p) = \frac{\mathrm{d}^{N+2}}{\mathrm{d}\xi^{N+2}} \int_0^{\xi} \mathrm{d}\xi_1 \int_0^{\xi-\xi_1} \mathrm{d}\xi_2 \cdots \times \int_0^{\xi-\xi_1-\cdots-\xi_{N+1}} \mathrm{d}\xi_{N+2}\, \mathcal{G}(\xi_1, \ldots, \xi_{N+2}, q, p), \tag{7.4}$$

where $\mathcal{G}(\xi_1, \ldots, \xi_{N+2}, q, p)$ is the evaluation at

$$(\xi_1, \ldots, \xi_N, \xi_{N+1}, z_1, \ldots, z_{2N}, \xi_{N+2}, z_{2N+1}, \ldots, z_{4N}) \text{ with } z \text{ as in (7.3)}$$

of the function

$$G_N \coloneqq \odot_{i,j}^{n+N-1} \cdots \odot_{i,j}^{n+1} \odot_{i,j}^{n} F$$

with $n \coloneqq 4N+2$ and $(i,j) = (2N+1, 3N+2)$.

In view of Lemmas 3.6 and 3.7, to prove that $\widehat{f} * \widehat{g} \in \mathcal{Q}_{2N+1}^{\mathcal{A}}$, it is thus sufficient to prove the following lemma:

Lemma 7.2. — *If $F(z_1, \ldots, z_n) \in \mathcal{Q}_n^{\mathcal{A}}$, then $\odot_{i,j}^n F(\xi, z_1, \ldots, z_n) \in \mathcal{Q}_{n+1}^{\mathcal{A}}$.*

Proof. — We assume $\{(z_1, \ldots, z_n) \mid P_F(z_1, \ldots, z_n) = 0\}$ is the avoidant set of F. Consider

$$Q(z, \xi, z_1, \ldots, z_n) \coloneqq z^N P_F\left(z_1, \ldots, z_i + z, \ldots, z_j + \frac{\xi}{z}, \ldots, z_n\right),$$

where N is the smallest number such that Q is a polynomial. Treating $Q_{\xi, z_1, \ldots, z_n}(z) \coloneqq Q(z, \xi, z_1, \ldots, z_n)$ as a polynomial in one variable z, of order M, with coefficients in $\mathbb{C}[\xi, z_1, \ldots, z_n]$, one can prove that $\odot_{i,j}^n F$ admits analytic continuation along any path contained in

$$\left\{(\xi, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \,\middle|\, Q_{\xi, z_1, \ldots, z_n}(z) \text{ has } M \text{ distinct non-zero roots}\right\}.$$

The details are left to the reader. □

At this stage, Theorem 1.2 is proved. In view of formula (2.32), exactly the same kind of argument, when applied to (2.32), yields Theorem 1.3, i.e.

$$\widehat{f} \in \widehat{\mathcal{Q}}_{2N+1}^{\mathcal{A}} \quad \Rightarrow \quad \widehat{T}^{\pm 1} \widehat{f} \in \widehat{\mathcal{Q}}_{2N+1}^{\mathcal{A}}. \tag{7.5}$$

Theorem 1.1 then directly follows from Theorems 1.2 and 1.3.

Remark 7.3. — Theorem 1.1 can also be proved more directly by rewriting (2.31) in the same way we have rewritten (2.30) as (7.4), and using Lemmas 7.2, 3.6 and 3.7.

BIBLIOGRAPHY

[1] P. Banks, E. Panzer & B. Pym, "Multiple zeta values in deformation quantization", *Invent. Math.* **222** (2020), no. 1, p. 79-159.

[2] F. Bayen, M. Flato, C. Frønsdal, A. Lichnerowicz & D. Sternheimer, "Deformation theory and quantization. I: Deformations of symplectic structures", *Ann. Phys.* **111** (1978), p. 61-110.

[3] ———, "Deformation theory and quantization. II: Physical applications", *Ann. Phys.* **111** (1978), p. 111-151.

[4] B. Candelpergher, J. C. Nosmas & F. Pham, *Approche de la résurgence*, Hermann, Paris, 1993 (French).

[5] M. De Wilde & P. B. A. Lecomte, "Existence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds", *Lett. Math. Phys.* **7** (1983), p. 487-496.

[6] F. J. Dyson, "Divergence of perturbation theory in quantum electrodynamics", *Phys. Rev., II. Ser.* **85** (1952), p. 631-632.

[7] J. Écalle, *Les fonctions résurgentes. Tome I. Les algèbres de fonctions résurgentes*, Publications Mathématiques d'Orsay, vol. 81-05, Université de Paris-Sud, Département de Mathématiques, Orsay, 1981, 247 pages.

[8] ———, "Cinq applications des fonctions résurgentes", Publ. Math. Orsay 84-62, 109 pages, 1984.

[9] ———, *Les fonctions résurgentes. Tome III. L'équation du pont et la classification analytique des objects locaux*, Publications Mathématiques d'Orsay, vol. 85-05, Université de Paris-Sud, Département de Mathématiques, Orsay, 1985, 587 pages.

[10] B. V. Fedosov, "Formal quantization", in *Some problems in modern mathematics and their applications to problems in mathematical physics*, Moskov. Fiz.-Tekhn. Inst., Moscow, 1985, p. 129–136 (Russian).

[11] ———, "A simple geometrical construction of deformation quantization", *J. Differ. Geom.* **40** (1994), no. 2, p. 213-238.

[12] M. Garay, A. de Goursac & D. van Straten, "Resurgent deformation quantisation", *Ann. Phys.* **342** (2014), p. 83-102.

[13] I. M. Gelfand, M. M. Kapranov & A. V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Birkhäuser Boston, Inc., Boston, MA, 1994.

[14] M. Kontsevich, "Deformation quantization of Poisson manifolds", *Lett. Math. Phys.* **66** (2003), no. 3, p. 157-216.

[15] M. Kontsevich & Y. Soibelman, "Analyticity and resurgence in wall-crossing formulas", *Lett. Math. Phys.* **112** (2022), no. 2, article no. 32 (56 pages).

[16] Y. Li, D. Sauzin & S. Sun, "Hadamard Product and Resurgence Theory", `https://arxiv.org/abs/2012.14175`, 2020.

[17] C. Mitschi & D. Sauzin, *Divergent series, summability and resurgence I. Monodromy and resurgence*, Lect. Notes Math., vol. 2153, Cham: Springer, 2016.

[18] J. E. Moyal, "Quantum mechanics as a statistical theory", *Proc. Camb. Philos. Soc.* **45** (1949), p. 99-124.

[19] F. Pham, "Resurgence, quantized canonical transformations, and multi-instanton expansions", in *Algebraic analysis, Pap. Dedicated to Prof. Mikio Sato on the Occas. of his Sixtieth Birthday. Vol. II*, Academic Press, Boston, MA, 1988, p. 699-726.

[20] D. Sauzin, "On the stability under convolution of resurgent functions", *Funkc. Ekvacioj, Ser. Int.* **56** (2013), no. 3, p. 397-413.

[21] A. Voros, "The return of the quartic oscillator. The complex WKB method", *Ann. Inst. Henri Poincaré, Nouv. Sér., Sect. A* **39** (1983), p. 211-338.

[22] S. Waldmann, "Convergence of star products: from examples to a general framework", *EMS Surv. Math. Sci.* **6** (2019), no. 1-2, p. 1-31.

[23] M. Waldschmidt, "Multiple zeta values: an introduction", *J. Théor. Nombres Bordx.* **12** (2000), no. 2, p. 581-595.

[24] J. Zinn-Justin, "Multi-instanton contributions in quantum mechanics. II", *Nuclear Phys. B* **218** (1983), no. 2, p. 333-348.

Manuscrit reçu le 21 juillet 2021,
révisé le 14 février 2022,
accepté le 12 mai 2022.

Yong LI
Chern Institute of Mathematics and LPMC
Nankai University
Tianjin 300071 (China)
liyong_cnu@163.com

David SAUZIN
CNRS – Observatoire de Paris
PSL Research University
75014 Paris (France)
David.Sauzin@obspm.fr

Shanzhong SUN
Department of Mathematics and Academy for
Multidisciplinary Studies
Capital Normal University
Beijing 100048 (China)
sunsz@cnu.edu.cn