Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

Erwan Rousseau & Behrouz Taji

Orbifold Chern classes inequalities and applications

Tome 73, nº 6 (2023), p. 2371-2410.

https://doi.org/10.5802/aif.3571

Les *Annales de l'Institut Fourier* sont membres du Centre Mersenne pour l'édition scientifique ouverte

www.centre-mersenne.org e-ISSN : 1777-5310

Ann. Inst. Fourier, Grenoble
73, 6 (2023) 2371-2410

ORBIFOLD CHERN CLASSES INEQUALITIES AND APPLICATIONS

by Erwan ROUSSEAU & Behrouz TAJI

Abstract. — In this paper we prove that given a pair (X, D) of a threefold X and a boundary divisor D with mild singularities, if $(K_X + D)$ is movable, then the orbifold second Chern class c_2 of (X, D) is pseudoeffective. This generalizes the classical result of Miyaoka on the pseudoeffectivity of c_2 for minimal models. As an application, we give a simple solution to Kawamata's effective non-vanishing conjecture in dimension 3, where we prove that $H^0(X, K_X + H) \neq 0$, whenever $K_X + H$ is nef and H is an ample, effective, reduced Cartier divisor. Furthermore, we study Lang–Vojta's conjecture for codimension one subvarieties and prove that minimal threefolds of general type have only finitely many Fano, Calabi–Yau or Abelian subvarieties of codimension one that are mildly singular and whose numerical classes belong to the movable cone.

Résumé. — Dans cet article nous prouvons que pour une paire (X, D) avec X une variété de dimension 3 et D un diviseur de bord avec peu de singularités, si $(K_X + D)$ est mobile, alors la seconde classe de Chern orbifolde c_2 de (X, D) est pseudoeffective. Cela généralise le résultat classique de Miyaoka sur la pseudoeffectivité de c_2 pour les modèles minimaux. Comme application, nous donnons une solution simple à la conjecture de non-annulation effective de Kawamata en dimension 3, où nous prouvons que $H^0(X, K_X + H) \neq 0$, lorsque $K_X + H$ est nef et H un diviseur de Cartier réduit, ample et effectif. De plus, nous étudions la conjecture de Lang–Vojta pour les sous-variétés de codimension 1 et montrons que les variétés minimales de dimension 3 de type général ont un nombre fini de sous-variétés de codimension 1 Fano, Calabi–Yau ou abéliennes avec peu de singularités et dont les classes numériques appartiennent au cône mobile.

1. Introduction

It is well known that the Chern classes of nef vector bundles over smooth projective varieties satisfy certain inequalities [11]. More generally, a theorem of Miyaoka [30] states that over a normal, projective variety (that

Keywords: Classification theory, Miyaoka–Yau inequality, Movable cone of divisors, Minimal Models, Effective non-vanishing, Lang–Vojta's conjecture.
2020 *Mathematics Subject Classification:* 14E30, 14J70, 14B05.

is smooth in codimension two) any torsion free, coherent sheaf $\mathscr{E}$ that is *semipositive* with respect to the tuple of ample divisors $(H_1, \dots, H_{n-1})$ and whose determinant $\det(\mathscr{E})$ is nef, verifies the inequality

$$c_2(\mathscr{E}) \cdot H_1 \dots H_{n-2} \geqslant 0.$$

On the other hand, thanks to Miyaoka's celebrated *generic semipositivity* result, cf. [30], and the result of Boucksom, Demailly, Păun and Peternell ([4]), when K_X is pseudoeffective, the cotangent bundle Ω^1_X of a smooth projective variety is generically semipositive. As a result, for a smooth projective variety X with K_X nef, the inequality

$$c_2(X) \cdot H_1 \dots H_{n-2} \geqslant 0 \tag{1.1}$$

holds, for any tuple of ample divisors $(H_1, \dots, H_{n-2})$.

Recent works of Campana and Păun ([7, 8]) have generalized some parts of Miyaoka's results, showing in particular that if X is a smooth projective variety with K_X pseudoeffective, then Ω^1_X is semipositive with respect to any *movable* class $\alpha \in \overline{\mathrm{Mov}}_1(X)$ (see Definition 2.3).

Our first result is a natural generalization of the inequality (1.1) to the setting of pairs with movable log-canonical divisors.

Theorem 1.1. — *Let X be a normal projective threefold that is smooth in codimension two and D a reduced effective divisor such that (X, D) has only isolated lc singularities. If $(K_X + D) \in \overline{\mathrm{Mov}}^1(X)$, then for any ample divisor A, the inequality*

$$c_2\big((\Omega^1_X \log(D))^{**}\big) \cdot A \geqslant 0$$

holds.

The second result is another generalization of an inequality established by Miyaoka [30], which is sometimes referred to as the Miyaoka–Yau inequality.

Theorem 1.2. — *Let X be a normal projective threefold that is smooth in codimension two and D a reduced effective divisor such that (X, D) has only isolated lc singularities. If $(K_X + D) \in \overline{\mathrm{Mov}}^1(X)$, then*

$$c_1^2\big((\Omega^1_X \log(D))^{**}\big) \cdot A \leqslant 3c_2\big((\Omega^1_X \log(D))^{**}\big) \cdot A,$$

for any ample divisor A.

Theorem 1.2 will also be established for pairs (X, D) of dimension three with isolated singularities (see Theorem 7.1).

There are two main ingredients in the proof of the above inequalities. The first one is a restriction result for semistable sheaves with respect to certain movable curves. This is described in Section 3. The second component involves the semipositivity of the *orbifold cotangent sheaves* for certain mildly singular pairs and is treated in Section 4.

The rest of the paper is devoted to two applications of Theorems 1.1 and 1.2. The first is concerned with the so-called effective non-vanishing conjecture.

Conjecture 1.3 (Effective non-vanishing conjecture of Kawamata). *Let Y be a normal projective variety and D_Y an effective $\mathbb{R}$-divisor such that (Y, D_Y) is klt. Let H be an ample, or more generally big and nef, divisor such that $(K_Y + D_Y + H)$ is Cartier and nef. Then*

$$H^0(X, K_Y + D_Y + H) \neq 0.$$

Using Theorem 1.1, in Section 6, we obtain a simple proof of the following weak version of Conjecture 1.3 in dimension three.

Theorem 1.4 (Non-vanishing for canonical threefolds). — *Let Y be a normal projective threefold with only canonical singularities. Let H be a very ample divisor. If $(K_Y + H)$ is a nef and Cartier divisor and not numerically trivial, then*

$$H^0(Y, K_Y + H) \neq 0.$$

We note that Theorem 1.4 is stated in [17] under the weaker assumption that H is a nef and big Cartier divisor. The proof relies on an inequality similar to that of Theorem 1.1 but under the weaker assumption that the first Chern class is *nef in codimension one*. It seems that there is a gap in the proof of that inequality, but the author kindly informs us that one can get rid of this assumption and use only the classical result of Miyaoka, where c_1 is assumed to be nef (see the inequality (1.1)).

A second application is given in Section 8 vis-à-vis Lang–Vojta's conjectures on subvarieties of varieties of general type:

Geometric Lang–Vojta conjecture. — *In a projective variety of general type X, subvarieties that are not of general type are contained in a proper algebraic subvariety of X.*

In particular, a variety of general type should have only finitely many codimension one subvarieties that are not of general type. We partially establish this conjecture in the setting of the following theorem.

Theorem 1.5. — *Let X be a normal projective $\mathbb{Q}$-factorial threefold such that $K_X \in \mathrm{Mov}^1(X)$. If X is of general type then X has only a finite number of movable codimension one, normal subvarieties D verifying the following conditions.*

1. *The subvariety D has only canonical singularities.*
2. *The anticanonical divisor $-K_D$ is pseudoeffective.*
3. *The pair (X, D) has only isolated lc singularities.*

In particular, there are only finitely many such Fano, Abelian and Calabi-Yau subvarieties.

Here, by a variety of general type, we mean a normal variety whose resolution has a big canonical bundle.

We remark that—in the smooth setting—a stronger version of Theorems 1.5 and 1.1 has been claimed in [27], where the authors establish these results under the weaker assumption that $(K_X + D)$ is pseudoeffective. Unfortunately the arguments in [27] are not complete. We refer to Remark 8.2 for a detailed discussion of these problems.

Acknowledgements

The authors would like to thank Sébastien Boucksom, Junyan Cao, Paolo Cascini, Andreas Höring, Steven Lu, Mihai Păun for fruitful discussions, and the anonymous referee for all the remarks that greatly improved the reading of this article.

2. Basic definitions and background

2.1. Movable cone

We introduce the *movable* cone of divisors, one of the important cones of divisors that is ubiquitous in birational geometry.

Let X be a normal projective variety and D a $\mathbb{Q}$-divisor on X. The stable base locus of D is defined by

$$\mathbb{B}(D) := \bigcap_m \mathrm{Bs}(|mD|).$$

The restricted base locus is given by

$$\mathbb{B}_-(D) = \bigcup_{A \text{ ample}} \mathbb{B}(D + A).$$

Definition 2.1 (Movable cone of divisors). — *Let* $\mathrm{N}^1(X)_{\mathbb{Q}}$ *be the space of numerical classes of divisors over* $\mathbb{Q}$*, the Neron–Severi space. The movable cone* $\overline{\mathrm{Mov}}^1(X) \subset \mathrm{N}^1(X)_{\mathbb{Q}}$ *is the closure of the convex cone* $\mathrm{Mov}^1(X)$ *generated by the classes of all effective divisors* D *such that* $\mathbb{B}_-(D)$ *has no divisorial components.*

The following inclusions now follow from the definitions.

$$\underbrace{\mathrm{Amp}(X)}_{\text{ample cone}} \subset \underbrace{\mathrm{Nef}(X)}_{\text{nef cone}} \subseteq \overline{\mathrm{Mov}}^1(X) \subseteq \underbrace{\overline{\mathrm{Eff}}(X)}_{\text{pseudoeffective cone}} \subset \mathrm{N}^1(X)_{\mathbb{Q}}.$$

The following proposition gives a more geometric picture of Definition 2.1.

Proposition 2.2 ([3, Prop. 2.3]). — *Given any* α *in the interior of* $\mathrm{Mov}^1(X)$*, there is a birational map* $\phi : Y \to X$ *and an ample divisor* A *on* Y *such that* $[\phi_* A] = \alpha$*.*

2.2. Stability with respect to movable 1-cycles

Now we introduce the notion of movable curves with respect to which a slope stability theory for sheaves can be formulated.

Definition 2.3. — *A class* $\gamma \in \mathrm{N}_1(X)$ *is movable if there is a projective birational morphism* $\pi : \widetilde{X} \to X$ *and a set of ample classes* $H_1, \dots, H_{n-1}$ *in* $\mathrm{N}^1(\widetilde{X})_{\mathbb{Q}}$ *such that* γ *is equal to the class of* $\pi_*(H_1 \cdot \ldots \cdot H_{n-1})$*. We define* $\mathrm{Mov}_1(X)$ *to be the convex cone generated by such* 1*-cycles and denote its closure in* $\mathrm{N}_1(X)_{\mathbb{Q}}$ *by* $\overline{\mathrm{Mov}}_1(X)$*.*

Movable classes form a natural setting for the notion of stability of coherent sheaves (see [9] and [13]). We shall now recall the basic definitions and properties.

Notation 2.4 (determinant sheaves). — Throughout this paper by $\det(\mathscr{E})$ we mean the reflexive hull of the determinant sheaf of $\mathscr{E}$.

Notation 2.5. — Let X be a normal projective variety and $\mathscr{F}$ a coherent sheaf on X of rank r. Let D be a Weil divisor in X such that $\det(\mathscr{F}) \cong \mathscr{O}_X(D)$. When D is $\mathbb{Q}$-Cartier, we set $[\mathscr{F}]$ to denote the numerical class $[D] \in \mathrm{N}^1(X)_{\mathbb{Q}}$ of D.

Definition 2.6. — *Assume that* X *is normal,* $\mathbb{Q}$*-factorial and projective, let* $\gamma \in \mathrm{Mov}_1(X)$*. The slope of a coherent sheaf* $\mathscr{E}$ *of rank* r *with respect to* γ *is defined by*

$$\mu_\gamma(\mathscr{E}) := \frac{1}{r} \cdot [\mathscr{E}] \cdot \gamma \quad \in \mathbb{Q}.$$

Definition 2.7. — *We say that a torsion free sheaf $\mathscr{E}$ is semistable with respect to γ, if $\mu_\gamma(\mathcal{F}) \leqslant \mu_\gamma(\mathscr{E})$ for any coherent subsheaf $0 \subsetneq \mathcal{F} \subset \mathscr{E}$.*

Proposition 2.8 ([13, Cor. 2.27]). — *Let X be a normal, $\mathbb{Q}$-factorial, projective variety, $\gamma \in \mathrm{Mov}_1(X)$ and $\mathscr{E}$ a torsion free sheaf. There exists a unique Harder–Narasimhan (or HN, for short) filtration $(\mathscr{E}, F^{\mathrm{HN}})$, i.e. a filtration $0 = \mathscr{E}_0 \subsetneq \mathscr{E}_1 \subsetneq \cdots \subsetneq \mathscr{E}_r = \mathscr{E}$, where each quotient $\mathcal{Q}_i := \mathscr{E}_i/\mathscr{E}_{i-1}$ is torsion-free, γ-semistable, and where the sequence of slopes $\mu_\gamma(\mathcal{Q}_i)$ is strictly decreasing.*

Remark 2.9. — We note that, by definition, the intersection of $\gamma \in \mathrm{Mov}_1(X)$ with any effective divisor is strictly positive. Therefore, to have a reasonable notion of stability, one works with elements of $\mathrm{Mov}_1(X)$ instead of those of its closure.

Notation 2.10 (Polarization). — Given a projective normal variety X, let $D_1, \ldots, D_k \in \mathrm{N}^1(X)_{\mathbb{Q}}$. By $(D_1, \ldots, D_k)$ we denote the element of $\mathrm{H}^{2k}(X)$ defined by $D_1 \cdot \ldots \cdot D_k$.

2.3. Chern classes for singular spaces

For any $i \in \mathbb{N}$, let X be a quasi-projective variety that is smooth in codimension i. For every coherent sheaf $\mathscr{F}$ on X, by using a finite projective resolution of $\mathscr{F}|_{X_{\mathrm{reg}}}$, we can define the i-th Chern class $c_i(\mathscr{F}|_{X_{\mathrm{reg}}})$ as an element of the Chow ring $\mathrm{A^i(X_{reg})}$, cf. [12]. On the other hand, with $Z := X \backslash X_{\mathrm{reg}}$, there is a natural exact sequence of Abelian groups

$$0 \longrightarrow \mathrm{A}^i(Z) \longrightarrow \mathrm{A}^i(X) \longrightarrow \mathrm{A}^i(X_{\mathrm{reg}}) \longrightarrow 0.$$

Therefore, if $\mathrm{codim}_X(Z) > i$, then we have $\mathrm{A}^i(X) \cong \mathrm{A}^i(X_{\mathrm{reg}})$. In particular, for normal varieties, $c_1(\mathscr{F})$ can be defined as an element of $\mathrm{A}^1(X)$ as the class of the (unique) extension of $c_1(\mathscr{F}|_{\mathrm{reg}})$. Similarly, if X is smooth in codimension two, we can define $c_2(\mathscr{F}) \in \mathrm{A}^2(X)$. Consequently, assuming that X is projective, $c_2(\mathscr{F})$ induces a multilinear form on

$$\underbrace{\mathrm{N}^1(X)_{\mathbb{Q}} \times \cdots \times \mathrm{N}^1(X)_{\mathbb{Q}}}_{(n-2)\text{-times}},$$

where $n = \dim X$.

Remark 2.11 (Non-$\mathbb{Q}$-factorial case). — If for every torsion free subsheaf $\mathscr{F} \subseteq \mathscr{E}$, the class of $\gamma \in \mathrm{Mov}^1(X)$ has a representative by a smooth curve

$C \subset X_{\mathrm{reg}}$ such that $\mathscr{F}|_C$ is locally free, then the $\mathbb{Q}$-factoriality assumption in Definition 2.7 is redundant. In this case we can define γ-slope of $\mathscr{F}$ by

$$\frac{1}{r}c_1(\mathscr{F}) \cdot [C] = \frac{1}{r} \cdot \deg(\mathscr{F}|_C),$$

where $[C] \in A^{n-1}(X)$. One can relax the above – rather stringent – assumptions on C but that would be unnecessary for our purposes in the current article. We note that the $\mathbb{Q}$-factoriality assumption in Proposition 2.8 is redundant if γ is a 1-cycle class of this form.

2.4. $\mathbb{Q}$-twisted sheaves

It will be quite useful in the sequel to work in the more general setting of $\mathbb{Q}$-twisted sheaves as introduced in [30].

Definition 2.12 ($\mathbb{Q}$-twisted sheaves). — *A $\mathbb{Q}$-twisted sheaf is a pair $\mathscr{E}\langle B\rangle$, where $\mathscr{E}$ is a coherent sheaf and B is a $\mathbb{Q}$-Cartier divisor.*

We now recall the usual formulas for Chern classes of $\mathbb{Q}$-twisted locally free sheaves.

Definition 2.13. — *For a $\mathbb{Q}$-twisted locally-free sheaf $\mathscr{E}\langle B\rangle$ of rank r on a normal quasi-projective variety we have*

$$c_1(\mathscr{E}\langle B\rangle) := c_1(\mathscr{E}) + rc_1(B),$$

$$c_2(\mathscr{E}\langle B\rangle) := c_2(\mathscr{E}) + (r-1)c_1(\mathscr{E}) \cdot c_1(B) + \frac{r(r-1)}{2}c_1(B)^2.$$

Notation 2.14. — In the setting of Notation 2.5, for any $\mathbb{Q}$-Cartier divisor A, we set $[\mathscr{F}\langle A\rangle] = [\mathscr{F}] + r \cdot [A]$.

For $\mathbb{Q}$-factorial normal projective varieties we can define a notion of slope stability for $\mathbb{Q}$-twisted sheaves with respect to $\gamma \in \mathrm{Mov}^1(X)$ in the natural way. Moreover, from the definition it follows that $\mathscr{E}$ is γ-semistable (or stable) if and only if $\mathscr{E}\langle B\rangle$ is γ-semistable (resp. stable) as $\mathbb{Q}$-twisted sheaf (see also [26, Rem. 6.4.8]). In particular the following inequality follows from the well-known Bogomolov–Gieseker inequality for smooth projective surfaces [2].

Proposition 2.15 (Bogomolov–Gieseker inequality for semistable $\mathbb{Q}$-twisted sheaves). — *Take S to be a smooth projective surface. Let $\mathscr{E}\langle B\rangle$ be a $\mathbb{Q}$-twisted locally-free sheaf on S of rank r and $A \in \mathrm{Amp}(X)_{\mathbb{Q}}$. If $\mathscr{E}\langle B\rangle$ is semistable with respect to A, then $\mathscr{E}\langle B\rangle$ verifies Bogomolov–Gieseker inequality*

$$2r \cdot c_2(\mathscr{E}\langle B\rangle) - (r-1) \cdot c_1^2(\mathscr{E}\langle B\rangle) \geqslant 0. \tag{2.1}$$

Definition 2.16 (Semipositive sheaves). — *Let X be a normal, $\mathbb{Q}$-factorial, projective variety and $\gamma \in \mathrm{Mov}_1(X)$. A torsion-free sheaf $\mathscr{E}$ is said to be semipositive with respect to γ, if for every torsion-free, quotient sheaf $\mathscr{F}$ of $\mathscr{E}$, we have $[\mathscr{F}] \cdot \gamma \geqslant 0$.*

Remark 2.17. — Similar to the case of stability, if for every torsion free quotient $\mathscr{F}$ of $\mathscr{E}$, the class of γ has a representative by a smooth projective curve C as in Remark 2.11, then the $\mathbb{Q}$-factoriality assumption in Definition 2.16 is not necessary. In this case the semipositivity assumption is given by

$$c_1(\mathscr{F}) \cdot [C] = \deg(\mathscr{F}|_C) \geqslant 0.$$

For a $\mathbb{Q}$-factorial projective variety X the two definitions coincide.

The semipositivity property for sheaves also naturally extends to the setting of $\mathbb{Q}$-twisted sheaves. We say $\mathscr{E}\langle B\rangle$ is semipositive with respect to γ, if $[\mathscr{F}\langle B\rangle] \cdot \gamma \geqslant 0$, for all torsion free quotient sheaves $\mathscr{F}$.

2.5. Orbifold basics

Following the terminology of Campana [5], an *orbifold* is simply a pair (X, D), consisting of a normal quasi-projective variety and a boundary divisor $D = \sum d_i \cdot D_i$, where $d_i = (1 - b_i/a_i) \in [0,1] \cap \mathbb{Q}$. We follow the usual convention that when $d_i = 1$ we have "$a_i = \infty$". Throughout this article all pairs (X, D) will be of this form. As such, we frequently refer to them simply as *pairs*. When X is projective, we refer to (X, D) as above as a *projective pair*. We say (X, D) is log-smooth, if X is smooth and D has simple normal crossing support.

Our main aim is now to define a notion of cotangent sheaf, adapted to a pair. To this end, and since we will not be exclusively working with smooth varieties, we will need a notion of pull-back for Weil divisors (that are not necessarily $\mathbb{Q}$-Cartier). We denote the group of Weil divisors by $\mathrm{WDiv}(X)$ and set $\mathrm{WDiv}(X)_{\mathbb{Q}} := \mathrm{WDiv}(X) \otimes \mathbb{Q}$ to denote the group of $\mathbb{Q}$-Weil divisors.

Definition 2.18 (Pull-back of Weil divisors). — *Let $f : Y \to X$ be a finite morphism between quasi-projective normal varieties X and Y. We define the pull-back $f^*(D)$ of a $\mathbb{Q}$-Weil divisor $D \subset X$ by the Zariski closure of $f^*(D|_{X_{\mathrm{reg}}})$.*

Notation 2.19. — Given a pair (X, D), with $D = \sum d_i \cdot D_i$, we use the following notations:

$$\lfloor D \rfloor := \sum \lfloor d_i \rfloor \cdot D_i,$$
$$\lceil D \rceil := \sum \lceil d_i \rceil \cdot D_i,$$

where $\lfloor d_i \rfloor$ denotes the round-down and $\lceil d_i \rceil$ the round-up.

DEFINITION 2.20 (Adapted and strongly adapted morphisms). — *Let (X, D) be an orbifold. A finite, surjective morphism $f : Y \to X$ is called* adapted *(to D) if, f^*D is an integral Weil divisor and f is unramified at the generic point of $\lfloor D \rfloor$. We say that a given adapted morphism $f : Y \to X$ is* strictly adapted, *if we have $f^*D_i = a_i \cdot D'_i$, for some Weil divisor $D'_i \subset Y$. Furthermore, we call a strictly adapted morphism f,* strongly adapted, *if the branch locus of f only consists of* $\operatorname{supp}\big(D - \lfloor D \rfloor + A\big)$, *where A is a general member of a basepoint free linear system on X.*

Remark 2.21. — For a log-smooth pair (X, D), the existence of a strongly adapted morphism $f : Y \to X$ was established by Kawamata, cf. [26, Prop. 4.1]. A similar strategy can be applied to construct strongly adapted morphisms $f : Y \to X$ when all the irreducible components of D are $\mathbb{Q}$-Cartier; in particular when X is assumed to be $\mathbb{Q}$-factorial. Alternatively, one can use the following more general statement, which follows from Kawamata's original result.

PROPOSITION 2.22. — *Let $D \subset X$ be a prime divisor on a normal quasi-projective variety. For every $m \in \mathbb{N}$ there is a normal variety Y, a finite morphism $f : Y \to X$ and a Weil divisor $D_Y \subset Y$ such that*

(1) *$f^*D = m \cdot D_Y$, and that*
(2) *the branched locus of f consists of D and a general member of a basepoint free linear system.*

Proof. — Let $\pi : (\widetilde{X}, \widetilde{D}) \to (X, D)$ be a log-resolution. By [26, Prop. 4.1] we know that there is a morphism $\widetilde{f} : \widetilde{Y} \to \widetilde{X}$ of smooth quasi-projective varieties such that $\widetilde{f}^*\widetilde{D} = mD_{\widetilde{Y}}$, for some Weil divisor $D_{\widetilde{Y}} \subset \widetilde{Y}$. Define $g := \pi \circ \widetilde{f}$. Now, let $\mu : \widetilde{Y} \to Y$ be the birational morphism and $f : Y \to X$ the finite map arising from Stein factorization of g. After normalization, if necessary, the morphism $f : Y \to X$ satisfies the required properties[1]. □

[1] For an effective, π-exceptional divisor E and ample divisor $A \subset X$, the ample divisor in Kawamata's construction should be taken to be of the form $(\pi^*A - a \cdot E)$, with $a \in \mathbb{Q}^+$ sufficiently small, so that $(\pi^*A - a \cdot E)$ is ample. This guarantees that Property (2) is satisfied.

The following lemma is useful in the course of the arguments in Section 4. Its proof follows directly from the construction of adapted morphisms. Nevertheless, for the reader's convenience, we include a brief argument.

LEMMA 2.23. — *Let (X, D) be a pair with $\lfloor D \rfloor = 0$, and $g : Y \to X$ any finite morphism of normal varieties. There is an adapted morphism $h : Z \to (X, D)$ factoring through g and a morphism $r : Z \to Y$.*

Proof. — Setting $D = \sum(1 - b_i/a_i) \cdot D_i$, for every i, let $n_j \in \mathbb{N}$ be the integer for which we have

$$g^* D_i = \sum_{j(i)=1}^{k_i} n_j \cdot D_{ij},$$

for some $k_i \in \mathbb{N}$. Define $m_i := \mathrm{lcm(n_1, \ldots, n_{k_i})}$. By Proposition 2.22, for each $D_{ij} \subset Y$ we can construct a morphism $r_{ij} : Z_{ij} \to Y$ such that

$$r_{ij}^*(D_{ij}) = \left(\frac{m_i}{n_j} \cdot a_i\right) \cdot B_{ij},$$

for some $B_{ij} \subset Z_{ij}$. Let $r : Z \to Y$ denote the composition of all such maps. Then, by construction we have

$$\begin{aligned} r^*(g^* D_i) &= \sum_{j=1}^{k_i} n_j \cdot r^*(D_{ij}) \\ &= (a_i \cdot m_i) \sum_{j=1}^{k_i} B_{ij}, \end{aligned}$$

as required. □

Notation 2.24. — Let $f : Y \to X$ be a morphism adapted to D, where $D = \sum d_i \cdot D_i$, $d_i = 1 - \frac{b_i}{a_i} \in (0, 1] \cap \mathbb{Q}$. For every irreducible component D_i of $(D - \lfloor D \rfloor)$, let $\{D_{ij}\}_{j(i)}$ be the collection of prime divisors that appear in $f^*(D_i)$. We define new divisors in Y by

$$D_Y^{ij} := b_i \cdot D_{ij} \tag{2.2}$$

$$D_f := f^*(\lfloor D \rfloor). \tag{2.3}$$

Now, let us explain how to define the cotangent sheaf of an orbifold (or a pair).

DEFINITION 2.25 (Orbifold cotangent sheaf). — *In the situation of Notation 2.24, denote Y° to be the log-smooth locus of the pair $(Y, \sum D_{ij} +$*

D_f) and define $D_Y^{ij\circ} := D_Y^{ij}|_{Y^\circ}$. Set $\Omega^1_{(Y^\circ,f,D)}$ to be the kernel of the sheaf morphism

$$(f|_{Y^\circ})^*\big(\Omega_X \log(\lceil D\rceil)\big) \longrightarrow \bigoplus_{i,j(i)} \mathscr{O}_{D_Y^{ij\circ}}$$

induced by the natural residue map. We define the orbifold cotangent sheaf $\Omega^{[1]}_{(Y,f,D)}$ by the coherent extension $(i_{Y^\circ})_*(\Omega^1_{(Y^\circ,f,D)})$, where i_{Y° is the natural inclusion. We define the orbifold tangent sheaf *$\mathscr{T}_{(Y,f,D)}$ by $(\Omega^{[1]}_{(Y,f,D)})^*$.*

3. Restriction results for semistable sheaves

Let $h = (H_1, \ldots, H_{n-1})$ be a tuple of ample divisors on a normal projective variety X of dimension n and $\mathscr{E}$ a torsion free sheaf. A theorem of Mehta–Ramanathan [29] states that if m is large enough and $Y \in |mH_{n-1}|$ is a generic hypersurface, then the maximal destabilizing subsheaf of $\mathscr{E}|_Y$ is the restriction of the maximal destabilizing subsheaf of $\mathscr{E}$.

It is natural to try to extend this restriction theorem to movable polarization. Unfortunately, in general, such results are not valid for movable curve. For example, when X is a projective $K3$ surface then its cotangent bundle Ω^1_X is not pseudoeffective, which gives rise to the existence of movable curves for which the restriction theorem does not hold (cf. [4, Sect. 7]).

In this section, we will prove a restriction theorem for some movable curves (see Proposition 3.9 below). The following lemmas will serve as key technical ingredients in the proof of this result.

Set-up 3.1. — *Let $\pi : \widetilde{S} \to S$ be a birational morphism of smooth projective surfaces $\widetilde{S}$ and S. Let $\widetilde{A}_{\widetilde{S}} \subset \widetilde{S}$ be an ample divisor and define $P_S := [\pi_*(\widetilde{A}_{\widetilde{S}})] \in \mathrm{N}^1(S)_{\mathbb{Q}}$.*

Lemma 3.2 (Induced destabilizing subsheaves of small rank on higher birational models. I). — *In the setting of Set-up 3.1, let $\mathscr{G}_S$ be a locally free sheaf on S of rank two. Assume that $\mathscr{F}_S \subset \mathscr{G}_S$ is a saturated and properly destabilizing subsheaf of $\mathscr{G}_S$. If $\widetilde{\mathscr{B}} \subset \pi^*\mathscr{G}_S$ is the maximal destabilizing subsheaf of $\pi^*\mathscr{G}_S$, then $\big(\pi_*(\widetilde{\mathscr{B}})\big)^{**} \cong \mathscr{F}_S$.*

Proof. — The proof is a direct consequence of the assumptions made on the slopes of $\widetilde{\mathscr{B}}$ and $\mathscr{F}_S$. More precisely, if we consider the exact sequence

$$0 \longrightarrow \overline{\pi^*\mathscr{F}_{\mathscr{S}}} \longrightarrow \pi^*\mathscr{G}_S \longrightarrow \widetilde{\mathscr{Q}} \longrightarrow 0,$$

where $\overline{\pi^*\mathscr{F}_{\mathscr{S}}}$ is the saturation of $\pi^*\mathscr{F}_S$ in $\pi^*\mathscr{G}_S$, we have the slope inequality $\mu_{\widetilde{A}_{\widetilde{S}}}(\widetilde{\mathscr{Q}}) < \mu_{\widetilde{A}_{\widetilde{S}}}(\pi^*\mathscr{G}_S)$. This implies that the induced map from $\widetilde{\mathscr{B}}$ to

$\widetilde{\mathscr{Q}}$ is zero, otherwise we get $\mu_{\widetilde{A}_{\widetilde{S}}}(\widetilde{\mathscr{B}}) < \mu_{\widetilde{A}_{\widetilde{S}}}(\widetilde{\mathscr{Q}})$, which is absurd. It thus follows that there is an injection

$$\widetilde{\mathscr{B}} \hookrightarrow \overline{\pi^*\mathscr{F}_{\mathscr{S}}}$$

But the slope of $\widetilde{\mathscr{B}}$ is maximal. Therefore $\widetilde{\mathscr{B}} \cong \overline{\pi^*\mathscr{F}_{\mathscr{S}}}$ and this proves the claim. □

Lemma 3.3. — *In the setting of Set-up 3.1 let $\mathscr{E}_S$ be a locally free sheaf of rank three on S. Define $\widetilde{\mathscr{G}}_{\widetilde{S}} \subset \pi^*\mathscr{E}_S$ to be the maximal destabilizing subsheaf with respect to $\widetilde{A}_{\widetilde{S}}$. If $\operatorname{rank}(\widetilde{\mathscr{G}}_{\widetilde{S}}) = 2$, then for every saturated, properly destabilizing subsheaf $\mathscr{F}_S \subset \mathscr{E}_S$ of rank one with respect to P_S, there is an injective morphism*

$$\mathscr{F}_S \hookrightarrow \mathscr{G}_S,$$

where $\mathscr{G}_S := \left(\pi_*\widetilde{\mathscr{G}}_{\widetilde{S}}\right)^{**}$.

Proof. — Let $\overline{\pi^*\mathscr{F}_{\mathscr{S}}}$ denote the saturation of $\pi^*\mathscr{F}_S$ in $\pi^*\mathscr{E}_S$ and set $\widetilde{\mathscr{Q}}$ to be the torsion free quotient $\pi^*\mathscr{E}_S/\overline{\pi^*\mathscr{F}_{\mathscr{S}}}$, whose slopes satisfies the inequality

$$\mu(\widetilde{\mathscr{Q}}) < \mu(\pi^*\mathscr{E}_S). \tag{3.1}$$

As $\operatorname{rank}(\mathscr{F}_S) = 1$, there is a nontrivial morphism $\sigma : \widetilde{\mathscr{G}}_{\widetilde{S}} \to \widetilde{\mathscr{Q}}$.

Now, if σ is injective, then $\mu(\widetilde{\mathscr{G}}_{\widetilde{S}}) \leqslant \mu(\widetilde{\mathscr{Q}})$. It then follows from the inequality (3.1) that

$$\mu(\widetilde{\mathscr{G}}_{\widetilde{S}}) < \mu(\pi^*\mathscr{E}_S),$$

a contradiction. Therefore $\widetilde{\mathscr{K}} := \operatorname{Im}(\sigma) \subset \widetilde{\mathscr{Q}}$ is a rank one subsheaf, giving rise to the commutative diagram of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \overline{\pi^*\mathscr{F}_{\mathscr{S}}} \arrow[r] & \pi^*\mathscr{E}_S \arrow[r] & \widetilde{\mathscr{Q}} \arrow[r] & 0 \\
0 \arrow[r] & \widetilde{\mathscr{M}} \arrow[r] \arrow[u] & \widetilde{\mathscr{G}}_{\widetilde{S}} \arrow[r, "\sigma"] \arrow[u] & \widetilde{\mathscr{K}} \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

Claim 3.4. — $\mu_{\widetilde{A}_{\widetilde{S}}}(\widetilde{\mathscr{K}}) > \mu_{\widetilde{A}_{\widetilde{S}}}(\widetilde{\mathscr{Q}})$.

Proof of Claim 3.4. Aiming for a contradiction, assume that $\mu(\widetilde{\mathscr{K}}) \leqslant \mu(\widetilde{\mathscr{Q}})$. Now, as $\widetilde{\mathscr{G}}_{\widetilde{S}}$ is $\widetilde{A}_{\widetilde{S}}$-semistable, the inequality

$$\mu(\widetilde{\mathscr{G}}_{\widetilde{S}}) \leqslant \mu(\widetilde{\mathscr{K}}) \tag{3.2}$$

holds. On the other hand, we have

$$\mu(\widetilde{\mathscr{K}}) \leqslant \mu(\widetilde{\mathscr{Q}}) < \mu(\overline{\pi^*\mathscr{F}_{\mathscr{S}}}), \tag{3.3}$$

where the last inequality follows from (3.1); that is, the fact that $\overline{\pi^*\mathscr{F}_{\mathscr{S}}} \subset \pi^*\mathscr{E}_S$ is properly destabilizing. But (3.2) and (3.3) lead to the inequality

$$\mu(\widetilde{\mathscr{G}}_{\widetilde{S}}) < \mu(\overline{\pi^*\mathscr{F}_{\mathscr{S}}}),$$

which contradicts the assumption on $\widetilde{\mathscr{G}}_{\widetilde{S}}$ having the maximal slope. This finishes the proof of the claim.

We now consider the saturation of $\widetilde{\mathscr{K}}$, which we denote by $\widetilde{\mathscr{K}_1}$, as a properly destabilizing subsheaf of $\widetilde{\mathscr{Q}}$ with the resulting exact sequence of sheaves

$$0 \longrightarrow \widetilde{\mathscr{K}_1} \xrightarrow{\tau} \widetilde{\mathscr{Q}} \longrightarrow \widetilde{\mathscr{A}} \longrightarrow 0$$

that are locally free in codimension one. Let $\widetilde{\mathscr{B}}$ be the kernel of the induced surjection $\gamma : \pi^*\mathscr{E}_S \longrightarrow \widetilde{\mathscr{A}}$ and

$$0 \longrightarrow \widetilde{\mathscr{B}} \longrightarrow \pi^*\mathscr{E}_S \xrightarrow{\gamma} \widetilde{\mathscr{A}} \longrightarrow 0,$$

the corresponding exact sequence.

As $\widetilde{\mathscr{K}}$ generically coincides with $\ker(\tau) = \widetilde{\mathscr{K}_1}$, the induced map $\widetilde{\mathscr{G}}_{\widetilde{S}} \longrightarrow \widetilde{\mathscr{A}}$ is zero. Therefore there is an injection

$$\widetilde{\mathscr{G}}_{\widetilde{S}} \hookrightarrow \widetilde{\mathscr{B}}.$$

Now, since $\widetilde{\mathscr{G}}_{\widetilde{S}}$ is the maximal destabilizing subsheaf (with respect to the ample divisor $\widetilde{A}_{\widetilde{S}}$), it follows that $\widetilde{\mathscr{G}}_{\widetilde{S}}$ and $\widetilde{\mathscr{B}}$ are isomorphic in codimension one.

On the other hand, from the sequence

$$\begin{tikzcd} 0 \arrow[r] & \overline{\pi^*\mathscr{F}_{\mathscr{S}}} \arrow[r] & \pi^*\mathscr{E}_S \arrow[r, "g"] \arrow[rr, bend left, "\gamma"] & \widetilde{\mathscr{Q}} \arrow[r] & \widetilde{\mathscr{A}}, \end{tikzcd}$$

we can see that there is an injection $\overline{\pi^*\mathscr{F}_{\mathscr{S}}} = \ker(g) \hookrightarrow \ker(\gamma) = \widetilde{\mathscr{B}}$. Noting that $\mathscr{F}_S \subset \mathscr{E}_S$ is saturated and thus reflexive, this implies that $\mathscr{F}_S$ injects into $(\pi_*\widetilde{\mathscr{G}}_{\widetilde{S}})^{**}$. $\square$

Lemma 3.5 (Induced destabilizing subsheaves of small rank on higher birational models. II). — *In the situation of Set-up 3.1, let $\mathscr{E}_S$ be a P_S-unstable locally free sheaf of rank three on S. Assume that $\mathscr{E}_S$ contains a saturated properly destabilizing subsheaf $\mathscr{F}_S$ of rank one. Let $\widetilde{\mathscr{G}}_{\widetilde{S}} \subset \pi^*\mathscr{E}_S$ be the maximal $\widetilde{A}_{\widetilde{S}}$-destabilizing subsheaf of $\pi^*\mathscr{E}_S$ and define $\mathscr{G}_S := (\pi_*\widetilde{\mathscr{G}}_{\widetilde{S}})^{**}$. Then,*

(1) *either $\mathscr{G}_S$ is a properly destabilizing subsheaf of $\mathscr{E}_S$,*

(2) *or there is a nontrivial morphism*

$$\pi^* \mathscr{F}_S \to \widetilde{\mathscr{G}}_{\widetilde{S}} \tag{$*$}$$

inducing an injection $\mathscr{F}_S \to \mathscr{G}_S$*, whose image is properly destabilizing.*

(3) *Or the Harder–Narasimhan filtration of* $\pi^* \mathscr{E}_S$ *has two steps. With* $\widetilde{\mathscr{D}}_{\widetilde{S}} := F_2^{\mathrm{HN}}(\pi^* \mathscr{E}_S)$ *and* $\mathscr{D}_S := (\pi_* \widetilde{\mathscr{D}}_S)^{**}$*, there is an injection* $\mathscr{F}_S \to \mathscr{D}_S \subset \mathscr{E}_S$*. Moreover, we either have*

$$\mu_{P_S}(\mathscr{D}_S) > \mu_{P_S}(\mathscr{E}_S)$$

or $\mathscr{D}_S$ *is not semistable, with the image of* $\mathscr{F}_S$ *in* $\mathscr{D}_S$ *being a properly destabilizing subsheaf.*

Proof. — We exclude Item (1) by making the assumption that

$$\mu_{P_S}(\mathscr{G}_S) \leqslant \mu_{P_S}(\mathscr{E}_S). \tag{3.4}$$

Assume further that $\operatorname{rank}(\mathscr{G}_S) = 2$. Then, according to Lemma 3.3 we have an injection

$$\mathscr{F}_S \hookrightarrow \mathscr{G}_S. \tag{3.5}$$

Using (3.4) we can see that the image of $\mathscr{F}_S$ under the map (3.5) properly destabilizes $\mathscr{G}_S$.

Now, if $\operatorname{rank}(\mathscr{G}_S) = 1$, consider the exact sequence

$$0 \longrightarrow \widetilde{\mathscr{G}}_{\widetilde{S}} \longrightarrow \pi^* \mathscr{E}_S \xrightarrow{\ \nu\ } \widetilde{\mathscr{C}} \longrightarrow 0.$$

Using (3.4) again, we can see that there is an injection $\pi^* \mathscr{F}_S \hookrightarrow \widetilde{\mathscr{C}}$. Then, the inequalities

$$\begin{aligned} \mu(\pi^* \mathscr{F}_S) &> \mu(\pi^* \mathscr{E}_S), \quad \text{since } \mathscr{F}_S \subset \mathscr{E}_S \text{ is destabilizing} \\ &> \mu(\widetilde{\mathscr{C}}), \quad \text{as } \widetilde{\mathscr{G}}_{\widetilde{S}} \subset \pi^* \mathscr{E}_S \text{ is destabilizing} \end{aligned}$$

imply that the image of $\pi^* \mathscr{F}_S$ under ν destabilizes $\widetilde{\mathscr{C}}$. Therefore, there is a second step $\widetilde{\mathscr{D}}_{\widetilde{S}}$ in the HN-filtration of $\pi^* \mathscr{E}_S$.

Claim 3.6. — *There is an injection* $\pi^* \mathscr{F}_S \hookrightarrow \widetilde{\mathscr{D}}_{\widetilde{S}}$.

Assuming Claim 3.6 for the moment, we proceed to finish the proof of the lemma. We note that once the injection in Claim 3.6 exists, then we have $\mathscr{F}_S \hookrightarrow \mathscr{D}_S$. Now, either $\mu(\mathscr{D}_S) > \mu(\mathscr{E}_S)$ or

$$\mu(\mathscr{D}_S) \leqslant \mu(\mathscr{E}_S).$$

If the latter inequality holds, then (the image of) $\mathscr{F}_S$ properly destabilizes $\mathscr{D}_S$ and this finishes the proof of the lemma.

It now remains to establish Claim 3.6.

Proof of Claim 3.6. — The first observation is that $\widetilde{\mathscr{D}}_{\widetilde{S}} \subset \pi^*\mathscr{E}_S$ is destabilizing. To see this, we consider the two exact sequences

$$0 \longrightarrow \widetilde{\mathscr{G}}_{\widetilde{S}} \longrightarrow \widetilde{\mathscr{D}}_{\widetilde{S}} \longrightarrow \mathscr{Q}' \longrightarrow 0,$$

$$0 \longrightarrow \widetilde{\mathscr{D}}_{\widetilde{S}} \longrightarrow \pi^*\mathscr{E}_S \xrightarrow{j} \widetilde{\mathscr{A}} \longrightarrow 0,$$

with the two sheaves $\mathscr{Q}'$ and $\widetilde{\mathscr{A}}$ being the successive quotients of the HN-filtration. By the definition of HN-filtration, we know that

$$\mu(\mathscr{Q}') > \mu(\widetilde{\mathscr{A}}). \tag{3.6}$$

On the other hand, as $\widetilde{\mathscr{G}}_{\widetilde{S}}$ is the maximal destabilizing subsheaf, from the first sequence we have

$$\mu(\mathscr{Q}') < \mu(\widetilde{\mathscr{D}}_{\widetilde{S}}) < \mu(\widetilde{\mathscr{G}}_{\widetilde{S}}). \tag{3.7}$$

Combining (3.6) and (3.7) we have

$$\mu(\widetilde{\mathscr{A}}) < \mu(\widetilde{\mathscr{D}}_{\widetilde{S}}).$$

From the second sequence it now follows that

$$\mu(\pi^*\mathscr{E}_S) < \mu(\widetilde{\mathscr{D}}_{\widetilde{S}}),$$

i.e. $\widetilde{\mathscr{D}}_{\widetilde{S}}$ destabilizes $\pi^*\mathscr{E}_S$. Consequently $\pi^*\mathscr{F}_S$ projects to zero via the morphism j so that there is a map $\pi^*\mathscr{F}_S \to \widetilde{\mathscr{D}}_{\widetilde{S}}$, as required. □

Lemma 3.7 (Induced destabilizing subsheaves of small rank on higher birational models. III). — *In the setting of Lemma 3.5, let $\mathscr{E}_S$ be a P_S-unstable locally free sheaf of rank 3 on S. Assume that $\mathscr{E}_S$ contains a saturated proper destabilizing subsheaf $\mathscr{F}_S$ of rank 2. Let $\widetilde{\mathscr{N}_{\widetilde{S}}} \subset \pi^* \bigwedge^2 \mathscr{E}_S$ be the maximal destabilizing subsheaf of $\pi^* \bigwedge^2 \mathscr{E}_S$ and $\mathscr{N}_S := (\pi_* \widetilde{\mathscr{N}_{\widetilde{S}}})^{**}$. Then,*

1. *either the subsheaf $\mathscr{N}_S = \big((\pi_*\widetilde{\mathscr{N}_{\widetilde{S}}})^{**}\big)$ destabilizes $\bigwedge^2 \mathscr{E}_S$,*
2. *or we have an injection $\bigwedge^2 \mathscr{F}_S \hookrightarrow \mathscr{N}_S$ with the image of $\bigwedge^2 \mathscr{F}_S$ being properly destabilizing,*
3. *or the second step $\widetilde{\mathscr{A}_{\widetilde{S}}}$ of the HN-fitlration of $\bigwedge^2 \pi^*\mathscr{E}_S$ descends to a destabilizing subsheaf $\mathscr{A}_S$ of $\bigwedge^2 \mathscr{E}_S$,*
4. *or $\bigwedge^2 \mathscr{F}_S$ maps into $\mathscr{A}_S$ inducing a properly destabilizing subsheaf.*

Proof. — By using the fact that

$$\mu\left(\bigwedge^2 \mathscr{E}_S\right) = 2\cdot\mu(\mathscr{E}_S) \;\text{ and }\; \mu\left(\bigwedge^2 \mathscr{F}_S\right) = 2\cdot\mu(\mathscr{F}_S),$$

we can see that $\bigwedge^2 \mathscr{F}_S \subset \bigwedge^2 \mathscr{E}_S$ is a properly destabilizing, saturated subsheaf. Now, as $\operatorname{rank}(\bigwedge^2 \mathscr{F}_S) = 1$ and $\operatorname{rank}(\bigwedge^2 \mathscr{E}_S) = 3$, Lemma 3.5 applies and settles the proof. □

Remark 3.8. — The main objective of the above lemmas is to show that once we have an unstable bundle $\mathscr{E}_S$ of rank at most 3 on S, there is a term of a HN filtration on $\widetilde{S}$ that "descends" to a destabilizing subsheaf of $\mathscr{E}$ (or $\bigwedge^2 \mathscr{E}$). Now, either the sheaf on $\widetilde{S}$ is the first or second step of the HN-filtration of $\pi^*\mathscr{E}$ (or $\widetilde{\mathscr{N}_{\widetilde{S}}} \subset \bigwedge^2 \pi^*\mathscr{E}$), that is

$$\mu_{P_S}(\mathscr{G}_S) > \mu_{P_S}(\mathscr{E}) \qquad \left(\text{resp. } \mu_{P_S}(\mathscr{N}_S) > \mu_{P_S}\left(\bigwedge^2 \mathscr{E}_S\right)\right), \tag{3.8}$$

or it is the maximal destabilizing subsheaf of one of the two steps of the HN filtration for $\pi^*\mathscr{E}_S$. To be more precise, for example when rank of a properly detabilizing subsheaf $\mathscr{F}$ of $\mathscr{E}_S$ is one, if the inequality (3.8) does not hold, then the image of $\mathscr{F}_S$ in $\mathscr{G}_S$ is destabilizing (see (2)). Therefore, according to Lemma 3.2, the maximal destabilizing subsheaf $\widetilde{\mathscr{B}} \subset \pi^*\mathscr{G}_S$ descends to the sheaf $(\pi_*\widetilde{\mathscr{B}})^{**}$ on S that is isomorphic to the saturation of (the image of) $\mathscr{F}_S$ in $\mathscr{G}_S$. In particular we have

$$\mu_{P_S}\big((\pi_*\widetilde{\mathscr{B}})^{**}\big) > \mu_{P_S}(\mathscr{E}_S).$$

One can argue similarly for the case of Item (3) or when $\operatorname{rank}(\mathscr{F}_S) = 2$. Uniqueness of such sheaves on $\widetilde{S}$ will play a crucial role in the proof of Proposition 3.9.

The next proposition is the main result in this section, proving a restriction result for semistable sheaves with respect to a particular set of movable classes. As we will see later in Section 5, these classes naturally arise in the context of positivity problems for second Chern classes.

Proposition 3.9 (A restriction result for movable classes). — *Let X be a normal projective threefold that is smooth in codimension two. Let $P \in \overline{\operatorname{Mov}}^1(X)$ and $H_1, H_2 \in \operatorname{Amp}(X)_{\mathbb{Q}}$. Let $\mathscr{E}$ be a torsion free sheaf on X of rank 3. There exists a positive integer M_1 such that for all sufficiently divisible integers $m_1 \geqslant M_1$, there is a Zariski open subset $V_{m_1} \subset |m_1 \cdot H_1|$ for which the following properties holds.*

(1) *Every member $S \in V_{m_1}$ is smooth, irreducible and is contained in X_{reg}.*
(2) *The restriction $\mathscr{E}|_S$ is torsion free.*
(3) *The divisor $P|_S$ is nef.*

(4) *For every such S, there exists $M_2 \in \mathbb{N}^+$ such that every sufficiently divisible integer $m_2 \geqslant M_2$ gives rise to a Zariski open subset $V_{m_2} \subset |m_2 \cdot (P + H_2)|_S|$, where every $\gamma \in V_{m_2}$ is a smooth, irreducible curve in S such that $\mathscr{E}|_\gamma$ is locally free and verifies the following property:*

$(*)$ *The formation of the* HN*-filtration of $\mathscr{E}$ with respect to $(H_1, P+H_2)$ commutes with restriction to γ, i.e.* $\mathrm{HN}_\bullet(\mathscr{E})|_\gamma = \mathrm{HN}_\bullet(\mathscr{E}|_\gamma)$.

Proof. — Let $\pi : \widetilde{X} \to X$ be the birational morphism and $\widetilde{X}$ the smooth projective variety with ample divisor $\widetilde{A} \subset \widetilde{X}$ associated to the Fujita approximation of the big movable divisor $P + H_2$ in the interior of $\mathrm{Mov}^1(X)$, i.e.

$$\pi_* \widetilde{A} = [(P + H_2)]$$

(cf. Proposition 2.2, [26, Sect. 11.4] and [31, Chapt. III]).

Now, let $N_1 \in \mathbb{N}^+$ be a sufficiently large and divisible integer such that for every $n_1 \geqslant N_1$, there are open subsets $U_{n_1} \subset |n_1 \cdot \pi^* H_1|$ and $\widetilde{U}_{n_1} \subset |n_1 \cdot \widetilde{A}|$, where for every subscheme $\widetilde{S} := D_{n_1}$ and $\widetilde{\gamma} := \widetilde{D}_{n_1} \cap D_{n_1}$, with $D_{n_1} \in U_{n_1}$ and $\widetilde{D}_{n_1} \in \widetilde{U}_{n_1}$, we have:

(1) Both $\widetilde{S}$ and $\widetilde{\gamma}$ are smooth and irreducible.
(2) The restrictions $(\pi^{[*]}\mathscr{E})|_{\widetilde{S}}$ and $\left(\bigwedge^{[2]} \pi^{[*]}\mathscr{E}\right)|_{\widetilde{S}}$ are locally free.
(3) The HN-filtration of $\pi^{[*]}\mathscr{E}$ with respect to $(\pi^* H_1, \widetilde{A})$ verifies:

$$\mathrm{HN}_\bullet\big((\pi^{[*]}\mathscr{E})|_{\widetilde{S}}\big) = \mathrm{HN}_\bullet(\pi^{[*]}\mathscr{E})|_{\widetilde{S}}.$$

In addition, the same property holds for $\bigwedge^{[2]} \pi^{[*]}\mathscr{E}$.

The positive integer N_1 exists, thanks to Bertini theorem and Langer's restriction theorem for stable sheaves, cf. [24].

Step. 1. (Reflexivity assumption). — By the Bertini theorem and [15, Thm. 12.2.1], and as $P \in \overline{\mathrm{Mov}}^1(X)$, there exists a positive integer N_2 such that for every sufficiently divisible $n_2 \geqslant N_2$ there exists a Zariski open subset $V_{n_2} \subset |n_2 \cdot H_1|$, where every $S \in V_{n_2}$ satisfies the three Properties (1), (2) and (3)(2). We can also ensure that every $S \in V_{n_2}$ is transversal to the exceptional centre of π. Furthermore, as $P|_S$ is nef, we can find $N_3 \in \mathbb{N}^+$ such that for each sufficiently divisible $n_3 \geqslant N_3$, the general member of $\gamma \in |n_3 \cdot (P + H_2)|_S|$ is smooth and is contained in an open subset of X over which the HN-filtration of $\mathscr{E}$ (with respect to $(H_1, P + H_2)$) is a filtration of $\mathscr{E}$ by locally-free sheaves. Therefore, to prove that Property $(*)$ is verified

(2) Here we are using the fact, which is a consequence of Fujita's approximation for movable divisors, that any codimension one movable class is *nef in codimension one*, that is its restriction to a sufficiently general surface is nef.

by γ, we may assume, without loss of generality, that $\mathscr{E}$ is reflexive and therefore for a suitable choice of S, its restriction $\mathscr{E}|_S$ is locally free.

Step. 2. (Construction of γ). — Let $M_2 \geqslant N_3$ be a sufficiently large and divisible integer such that for every $m_2 \geqslant M_2$ there exists a Zariski open subset $V_{m_2} \subset |m_2(P+H_2)|_S|$, where every curve $\gamma \in V_{m_2}$ is smooth and is contained in the complement of the exceptional center of π. Furthermore, $\mathscr{E}|_\gamma$ is locally free, and if $\mathscr{E}|_\gamma$ is not semistable, then $\mathscr{E}_S := \mathscr{E}|_S$ is not semistable with respect to $(P+H_2)|_S$ and more generally we have $\mathrm{HN}_\bullet(\mathscr{E}_S)|_\gamma = \mathrm{HN}_\bullet(\mathscr{E}|_\gamma)$. The existence of such M_2 is guaranteed by Bertini theorem and Mehta–Ramanathan's restriction theorem, cf. [29].

Summarizing these geometric constructions, by choosing sufficiently large n_i and m_2 and by shrinking V_{n_i} and V_{m_2} if necessary, we have $\gamma \subset S$ and $\widetilde{\gamma} \subset \widetilde{S}$, with surjective morphisms

$$\pi|_{\widetilde{S}} : \widetilde{S} \to S \quad \text{and} \quad \pi|_{\widetilde{\gamma}} : \widetilde{\gamma} \to \gamma,$$

and satisfying Properties (1), (2), (3), and those in the setting of the proposition but excluding (*).

Now, to prove the proposition, it suffices to show that if a reflexive sheaf $\mathscr{E}$ is semistable with respect to $(H_1, P+H_2)$, then so is $\mathscr{E}|_\gamma$. So let us now assume that $\mathscr{E}$ is indeed semistable. The next steps are devoted to proving that $\mathscr{E}|_\gamma$ is also semistable.

Aiming for a contradiction, assume that $\mathscr{E}|_\gamma$ is not semistable. It follows that $(\pi^{[*]}\mathscr{E})|_{\widetilde{\gamma}} = \pi^*(\mathscr{E}|_\gamma)$ is not semistable. By the construction of γ, this also implies that $\mathscr{E}_S$ is unstable. Therefore, $\pi^*\mathscr{E}_S$ is unstable with respect to $\widetilde{\gamma}$ (which is numerically proportional to $\widetilde{A}|_{\widetilde{S}}$). Moreover, thanks to [24], unstability of $(\pi^{[*]}\mathscr{E}|_{\widetilde{\gamma}})$ implies that $\pi^{[*]}\mathscr{E}$ is unstable (with respect to $(\pi^*H_1, \widetilde{A})$).

Claim 3.10. — *$\bigwedge^{[2]} \pi^{[*]}\mathscr{E}$ is not semistable with respect to $(\pi^*H_1, \widetilde{A})$.*

Proof of Claim 3.10. — This follows directly from rank considerations. Suppose $\mathscr{M} \subset \pi^{[*]}\mathscr{E}$ is a saturated destabilizing subsheaf.

If $\operatorname{rank}(\mathscr{M}) = 2$, then as $\mu(\bigwedge^2 \mathscr{M}) = 2 \cdot \mu(\mathscr{M})$ and $\mu(\bigwedge^{[2]} \pi^{[*]}\mathscr{E}) = 2 \cdot \mu(\pi^{[*]}\mathscr{E})$, the subsheaf $\bigwedge^2 \mathscr{M} \subset \bigwedge^{[2]} \pi^{[*]}\mathscr{E}$ is destabilizing.

Now, if $\operatorname{rank}(\mathscr{M}) = 1$, then $\mu(\mathscr{Q}) < \mu(\pi^{[*]}\mathscr{E})$, where $\mathscr{Q}$ is the torsion free quotient $\pi^{[*]}\mathscr{E}/\mathscr{M}$. Again by using the fact that $\mu(\bigwedge^2 \mathscr{Q}) = 2 \cdot \mu(\mathscr{Q})$, we find that the inequality

$$\mu\left(\bigwedge^{[2]} \mathscr{Q}\right) < \mu\left(\bigwedge^{[2]} \pi^{[*]}\mathscr{E}\right)$$

holds, implying that $\bigwedge^{[2]} \pi^{[*]}\mathscr{E}$ is not semistable. This finishes the proof of Claim 3.10. □

Now, let $\widetilde{\mathscr{G}}$ and $\widetilde{\mathscr{N}}$ be the first step of the HN-filtration of $\pi^{[*]}\mathscr{E}$ and $\bigwedge^{[2]} \pi^{[*]}\mathscr{E}$, respectively and define

$$\mathscr{G} := \big(\pi_*(\widetilde{\mathscr{G}})\big)^{**} \quad , \quad \mathscr{N} := \big(\pi_*(\widetilde{\mathscr{N}})\big)^{**}.$$

Assuming that they exist, let $\widetilde{\mathscr{D}}$ and $\widetilde{\mathscr{A}}$ be the second step of the HN-filtration of $\pi^{[*]}\mathscr{E}$ and $\bigwedge^{[2]} \pi^{[*]}\mathscr{E}$ and set

$$\mathscr{D} := \big(\pi_*(\widetilde{\mathscr{D}})\big)^{**} \quad , \quad \mathscr{A} := \big(\pi_*(\widetilde{\mathscr{A}})\big)^{**}.$$

Let $m_1 \in \mathbb{N}$ be a sufficiently divisible integer, verifying the inequality $m_1 \geqslant M_1 := \max\{N_1, N_2\}$, and such that there is an open subset $V_{m_1} \subseteq |m_1 \cdot H_1|$ for which we have the following property. After shrinking V_{m_1}, if necessary, for every $S \in V_{m_1}$ (defined in Steps. 1 and 2), we have

(1) $\widetilde{S} := \pi^*(S) \in U_{m_2}$,
(2) $\widetilde{\mathscr{G}}|_{\widetilde{S}}$, $\widetilde{\mathscr{D}}|_{\widetilde{S}}$, $\widetilde{\mathscr{N}}|_{\widetilde{S}}$ and $\widetilde{\mathscr{A}}|_{\widetilde{S}}$ are locally free,
(3) S does not intersect the singular loci of $\mathscr{E}$, $\mathscr{G}$, $\mathscr{D}$, $\mathscr{N}$ and $\mathscr{A}$, and
(4) we have $(\pi_*(\widetilde{\mathscr{G}}|_{\widetilde{S}}))^{**} \cong (\pi_*\widetilde{\mathscr{G}})^{**}|_S$ and the same holds for $\widetilde{\mathscr{D}}$, $\widetilde{\mathscr{N}}$ and $\widetilde{\mathscr{A}}$.

Step. 3. (Extension of maximal destabilizing subsheaves). — We are now in the setting where we can apply Lemmas 3.2, 3.5 and 3.7. Let $\widetilde{\mathscr{G}}_{\widetilde{S}}$ and $\widetilde{\mathscr{D}}_{\widetilde{S}}$ be the first and second steps of the HN-filtration of $\pi^*\mathscr{E}_S$, assuming that the latter exists. By construction, using Property (3) together with Properties (1) and (3), there are isomorphism

$$\widetilde{\mathscr{G}}_{\widetilde{S}} \cong \widetilde{\mathscr{G}}|_{\widetilde{S}} \ , \ \ \widetilde{\mathscr{D}}_{\widetilde{S}} \cong \widetilde{\mathscr{D}}|_{\widetilde{S}}.$$

Let us first assume that $\mathscr{E}_S$ contains a saturated destabilizing subsheaf $\mathscr{F}_S$ of rank one. According to Lemma 3.5, one of the locally free sheaves $\mathscr{G}_S := \pi_*(\widetilde{\mathscr{G}}_{\widetilde{S}})^{**}$ or $\mathscr{D}_S := \pi_*(\widetilde{\mathscr{D}}_{\widetilde{S}})^{**}$

($*$) either destabilizes $\mathscr{E}_S$,

($**$) or it is not semistable and admits an injection from $\mathscr{F}_S$ with a properly destabilizing image.

We identify $\mathscr{F}_S$ with its image under $\mathscr{F}_S \hookrightarrow \mathscr{G}_S$ (respectively, $\mathscr{F}_S$ with its image under $\mathscr{F}_S \hookrightarrow \mathscr{D}_S$).

Now, if ($*$) holds, then we have our desired contradiction since by (4) the subsheaf $(\pi_*\widetilde{\mathscr{G}})^{**} \subset \mathscr{E}$ or $(\pi_*\widetilde{\mathscr{D}})^{**} \subset \mathscr{E}$ is properly destabilizing.

So assume that ($**$) is true. We observe that by our choice of S (Property (3)) we have $(\pi|_S)^*\mathscr{G}_S = \big(\pi^{[*]}\mathscr{G}\big)|_{\widetilde{S}}$ (and $(\pi|_S)^*\mathscr{D}_S = (\pi^{[*]}\mathscr{D})|_{\widetilde{S}}$). We can now apply Lemma 3.2. More precisely, if $\mathscr{F}_S \subset \mathscr{G}_S$ (or $\mathscr{F}_S \subset \mathscr{D}_S$)

is destabilizing $\mathscr{G}_S$ (respectively, $\mathscr{D}_S$), then, according to Lemma 3.2, the maximal destabilizing subsheaf $\widetilde{\mathscr{L}_{\widetilde{S}}}$ of $\pi^*(\mathscr{G}_S)$ verifies the isomorphism

$$((\pi|_{\widetilde{S}})_*(\widetilde{\mathscr{L}_{\widetilde{S}}}))^{**} \cong \overline{\mathscr{F}_{\mathscr{S}}}, \tag{3.9}$$

where $\overline{\mathscr{F}}_S$ is the saturation of $\mathscr{F}_S$ in $\mathscr{G}_S$, and similarly when $\mathscr{F}_S \subset \mathscr{D}_S$ is destabilizing.

On the other hand, again by the restriction result [24], we have

$$\widetilde{\mathscr{L}}|_{\widetilde{S}} \cong \widetilde{\mathscr{L}_{\widetilde{S}}},$$

where $\widetilde{\mathscr{L}}$ is the maximal destabilizing subsheaf of $\pi^{[*]}\mathscr{G}$ (after adjusting the choice of S and $\widetilde{S}$ if necessary). Therefore, by (3.9) $\widetilde{\mathscr{L}}$ descends to a destabilizing subsheaf of $\mathscr{E}$, i.e. $(\pi_*\mathscr{L})^{**} \subset \mathscr{E}$ is destabilizing, a contradiction. Similarly we can argue that the maximal destabilizing subsheaf of $\widetilde{\mathscr{D}}$ descends to a destabilizing subsheaf of $\mathscr{E}$.

Next, we assume that $\operatorname{rank}(\mathscr{F}_S) = 2$. In this case Lemma 3.7 applies. The same arguments as above (this time for $\mathscr{N}$, $\mathscr{A}$ instead of $\mathscr{G}$ and $\mathscr{D}$) then shows that $\bigwedge^2 \mathscr{E}$ is not semistable. On the other hand, thanks to [13, Thm. 4.2], we know that semistable sheaves with respect to movable classes over normal varieties form a tensor category(3). As a result we again get a contradiction to the semistability assumption on $\mathscr{E}$. $\square$

Remark 3.11 (Restriction of HN*-filtration for* $\mathbb{Q}$*-twisted sheaves).* — We note that the consequences of Proposition 3.9 are still valid for $\mathbb{Q}$-twisted torsion-free sheaves. More precisely, given a $\mathbb{Q}$-twisted, torsion-free sheaf $\mathscr{E}\langle B\rangle$ and $H_1, H_2 \in \operatorname{Amp}(X)_{\mathbb{Q}}$, $P \in \overline{\operatorname{Mov}}^1(X)$, there is a complete intersection surface S and $\gamma \subset S$, as in Proposition 3.9, such that $\operatorname{HN}_\bullet(\mathscr{E}\langle B\rangle)|_\gamma = \operatorname{HN}_\bullet(\mathscr{E}\langle B\rangle|_\gamma)$. To see this we can use the fact that, for every torsion free sheaf $\mathscr{F}$ and Weil $\mathbb{Q}$-divisor B, we have

$$\operatorname{HN}_\bullet\big(\mathscr{E}\langle B\rangle\big) = \big(\operatorname{HN}_\bullet(\mathscr{E})\big)\langle B\rangle,$$

which follows directly from the definitions. The rest now follows from Proposition 3.9.

Remark 3.12 (Restriction result in higher dimensions). — Following the same arguments as those of the proof of Proposition 3.9, we can remove the restriction on the dimension, that is the consequences of Proposition 3.9 are still valid, if X is of dimension $n > 3$ and the polarization is $(H_1, H_2, \dots, (P + H_{n-1}))$, for any $H_1, \dots, H_{n-1} \in \operatorname{Amp}(X)_{\mathbb{Q}}$, as long as $\operatorname{rank}(\mathscr{E}) = 3$.

(3) This result has an additional assumption; $\mathbb{Q}$-factoriality of X. As we pointed out in Remark 2.11, this condition is unnecessary in the context of this proposition.

As an immediate consequence we establish a Bogomolov–Gieseker inequality for ($\mathbb{Q}$-twisted) sheaves of small rank that are semistable with respect to movable classes of the form that appear in Proposition 3.9. Although we do not use this inequality in the rest of the paper, we find it to be of independent interest.

Proposition 3.13 (Bogomolov–Gieseker inequality in higher dimensions). — *Let X be an n-dimensional, normal projective variety that is smooth in codimension two and $\mathscr{E}\langle B\rangle$ a $\mathbb{Q}$-twisted, reflexive sheaf of rank at most equal to 3 on X. If $\mathscr{E}\langle B\rangle$ is semistable with respect to $(H_1, H_2, \ldots, (P + H_{n-1}))$, where $H_i \in \operatorname{Amp}(X)_{\mathbb{Q}}$ and $P \in \overline{\operatorname{Mov}}^1(X)$, then*

$$\big(2r \cdot c_2(\mathscr{E}\langle B\rangle) - (r-1) \cdot c_1^2(\mathscr{E}\langle B\rangle)\big) \cdot H_1 \ldots \cdot H_{n-2} \geqslant 0.$$

Proof. — This is an immediate consequence of the restriction result in Proposition 3.9 and Remark 3.11 together with Proposition 2.15. ☐

4. Semipositivity of adapted sheaf of forms

In [8] Campana and Păun remarkably prove that the orbifold cotangent sheaf of a log-smooth pair (X, D) is semipositive with respect to movable curve classes on X (see Theorem 4.1 below). See Definition 2.16 for the definition of this notion of semipositivity. Currently it is not clear if this result can be easily extended to the case of singular pairs. In the present section we show that, for a special subset of movable classes, the generalization to singular pairs can be achieved by essentially reducing to the smooth case.

Theorem 4.1 (Orbifold semipositivity with respect to movable classes, cf. [8, Thm. 1.2]). — *Given a log-smooth pair (X, D), if $(K_X + D)$ is pseudoeffective, then for any movable class $\gamma \in \operatorname{Mov}_1(X)$ and any adapted morphism $f : Y \to X$, where Y is smooth, the orbifold cotangent sheaf $\Omega^1_{(Y,f,D)}$ is semipositive with respect to $f^*(\gamma)$*(4).

In the next proposition we slightly refine Theorem 4.1 for a class of movable 1-cycles that we call *complete intersection* 1*-cycles*. As we will see later in Section 5, such classes appear naturally in our treatment of the pseudoeffectivity of c_2.

(4) Here we are following the notation of [8] for "pullback" of movable 1-cycles. Since in the current paper we are only concerned with those cycles that are defined by divisors, we have forgone the exact definition of this notion.

Definition 4.2 (Complete intersection movable classes). — *We say that* $\gamma \in \mathrm{Mov}_1(X)$ *is a complete intersection movable* 1*-cycle, if there are classes* $B_1, \dots, B_{n-1} \in \mathrm{N}^1(X)_{\mathbb{Q}}$ *such that* γ *is numerically equivalent to the cycle defined by* $(B_1 \cdot \dots \cdot B_{n-1}) \in \mathrm{N}_1(X)_{\mathbb{Q}}$.

Proposition 4.3 (A refinement of the orbifold semipositivity result). *Let* (X, D) *be a log-smooth pair and* $\gamma \in \mathrm{Mov}_1(X)$ *a complete intersection movable cycle. If* $(K_X + D)$ *is pseudoeffective, then for any strictly adapted morphism* $g : Z \to X$ *(see Definition 2.20),* $\Omega^{[1]}_{(Z,g,D)}$ *is semipositive with respect to* $g^*\gamma$.

Proof. — Assume that Z is not smooth, otherwise the claim follows from the arguments of Campana and Păun, cf. [8]. Let $D = \sum d_i \cdot D_i$, where D_i are prime divisors and $d_i = 1 - (b_i/a_i) \in [0, 1] \cap \mathbb{Q}$. By assumption, for every D_i, we have $g^*(D_i) = a_i \cdot D_{Z,i}$, for some $D_{Z,i} \in \mathrm{WDiv}(Z)$.

Now, set $f : Y \to X$ to be a strongly adapted morphism (Definition 2.20), where, thanks to Kawamata's construction, cf. [26, Prop. 4.1.12], the variety Y is smooth. Let W be the normalization of fibre product $Y \times_X Z$ with the resulting commutative diagram

$$\begin{tikzcd} W \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, "h"] & Z \arrow[d, "g"] \\ Y \arrow[r, "f"] & X. \end{tikzcd}$$

Aiming for a contradiction, assume that $\Omega^{[1]}_{(Z,g,D)}$ is not semipositive with respect to $g^*\gamma$, that is there exists a reflexive subsheaf $\mathscr{G}_Z \subset \Omega^{[1]}_{(Z,g,D)}$ such that

$$\big(\gamma^*(K_X + D) - [\mathscr{G}_Z]\big) \cdot g^*\gamma < 0. \tag{4.1}$$

We consider $v^{[*]}(\mathscr{G}_Z) \subset \Omega^{[1]}_{(W,h,D)}$. As γ is, numerically, a complete intersection cycle, we can use the projection formula to conclude that

$$\big(h^*(K_X + D) - [v^{[*]}\mathscr{G}_Z]\big) \cdot h^*\gamma < 0, \tag{4.2}$$

which implies that $\Omega^{[1]}_{(W,h,D)}$ is not semipostive with respect to $h^*\gamma$. Now, let $\Omega^{[1]}_{(W,h,D)} \twoheadrightarrow \mathscr{F}_W$ be the torsion free quotient with the minimum slope with the kernel $\mathscr{G}_W$:

$$0 \to \mathscr{G}_W \to \Omega^{[1]}_{(W,h,D)} \to \mathscr{F}_W \to 0. \tag{4.3}$$

Assuming that $u: W \to Y$ is Galois, let $G \coloneqq \mathrm{Gal}(W/Y)$. Notice that by the construction of f, we have $\Omega^{[1]}_{(W,h,D)} = u^*(\Omega^1_{(Y,f,d)})$. Now, as the inclusion $\mathscr{G}_W \subset \Omega^1_{(W,h,D)}$ is saturated, and since $\mathscr{G}_W$ is a G-subsheaf (thanks to its uniqueness), according to [18, Thm. 4.2.15] or [14, Prop. 2.16], there exists a reflexive subsheaf $\mathscr{G}_Y \subset \Omega^1_{(Y,f,D)}$ such that $u^{[*]}(\mathscr{G}_Y) \cong \mathscr{G}_W$.

Now, by applying the G-invariant section functor $u_*(\cdot)^G$ to the exact sequence (4.3) we find that

$$0 \to \mathscr{G}_Y \to \Omega^1_{(Y,f,D)} \to \big(u_*(\mathscr{F}_W)\big)^G \to 0. \tag{4.4}$$

But, by the projection formula, it follows that

$$\big(f^*(K_X + D) - [\mathscr{G}_Y]\big) \cdot f^*\gamma \leqslant 0,$$

i.e. $\Omega^1_{(Y,f,D)}$ is not semipositive with respect to $f^*\gamma$, which contradicts Theorem 4.1.

For the case where u is not Galois, we can consider the Galois closure $u' : W' \xrightarrow{\sigma} W \xrightarrow{u} Y$ and repeat the above argument for $\sigma^{[*]}(\Omega^{[1]}_{(W,h,D)})$ instead of $\Omega^{[1]}_{(W,h,D)}$. □

The next proposition is the extension of Theorem 4.1 to a special class of complete intersection, movable 1-cycles on a mildly singular X.

Proposition 4.4 (Semipositivity for mildly singular pairs). — *Given a projective pair (X,D), assume that $(K_X + D)$ is pseudoeffective. Let $H_1 \dots, H_{n-1} \in \mathrm{Amp}(X)_{\mathbb{Q}}$ and $P \in \overline{\mathrm{Mov}}^1(X)$. Then, for any strictly adapted morphism $f: Y \to X$, the orbifold cotangent sheaf $\Omega^{[1]}_{(Y,f,D)}$ is semipositive with respect to $f^*(H_1, \dots, H_{n-2}, P + H_{n-1})$, if (X,D) verifies one of the following assumptions.*

(1) *(X,D) has only klt singularities.*
(2) *D is reduced (i.e. $\lfloor D \rfloor = 0$) and (X,D) has only lc singularities.*

Proof. — Assume that the assumption (1) holds. Let $\pi : (\widetilde{X}, \widetilde{D}) \to (X,D)$ be a log-resolution and $\widetilde{Y}$ the normalization of the fibre product $Y \times_X \widetilde{Y}$ with the commutative diagram

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{\pi}"'] & \widetilde{X} \arrow[d, "\pi"] \\ Y \arrow[r, "f"] & X, \end{tikzcd}$$

where $\widetilde{\pi} : \widetilde{Y} \to Y$ and $\widetilde{f} : \widetilde{Y} \to Y$ are the naturally induced projections.

For simplicity, and as the arguments are identical in higher dimensions, we only deal with the case when $\dim X = 3$. Denote $H_{Y,i} = f^*(H_i)$, for $i \in \{1,2\}$ and $P_Y = f^*(P)$.

Now, aiming for a contradiction, assume that $\Omega^{[1]}_{(Y,f,D)}$ is not semipositive with respect to $(H_{Y,1}, P_Y + H_{Y,2})$. This implies that there exists a saturated subsheaf $\mathscr{G} \subset \mathscr{T}_{(Y,f,D)}$ such that $[\mathscr{G}] \cdot (H_{Y,1}, P_Y + H_{Y,2}) > 0$. Define $\widetilde{\mathscr{H}} := (\widetilde{\pi}^{[*]}\mathscr{H}) \cap \mathscr{T}_{(\widetilde{Y},\widetilde{f},D)}$. Let m be a sufficiently large positive integer such that the 1-cycle $\gamma \in \mathrm{Mov}_1(Y)$, that is numerically equivalent to the cycle defined by $m^2(H_{Y,1}, P_Y + H_{Y,2})$, is away from the exceptional centre of $\widetilde{\pi}$. Existence of such γ in particular guarantees that

$$[\widetilde{\mathscr{H}}] \cdot \widetilde{\pi}^*(H_{Y,1}, P_Y + H_{Y,2}) > 0.$$

In other words there exists a torsion-free quotient sheaf

$$\Omega^{[1]}_{(\widetilde{Y},\widetilde{f},\widetilde{D})} \twoheadrightarrow \widetilde{\mathscr{F}} \tag{4.5}$$

on $\widetilde{Y}$ such that $\deg(\widetilde{\mathscr{F}}|_{\widetilde{\gamma}}) < 0$, where $\widetilde{\gamma} := \widetilde{\pi}^{-1}(\gamma)$.
Now, let us consider the logarithmic ramification formula

$$K_{\widetilde{X}} + \widetilde{D} = \pi^*(K_X + D) + \sum a_i \cdot E_i - \sum b_i \cdot E'_i,$$

where $a_i \in \mathbb{Q}^+$, and, because of the assumptions on the singularities, $b_i \in (0,1) \cap \mathbb{Q}$. Define $\widetilde{G} := \sum b_i \cdot E'_i$ and let $\widetilde{h} : Z \to \widetilde{X}$ be the morphism adapted to $(\widetilde{X}, \widetilde{D} + \widetilde{G})$, factoring through $\widetilde{f} : \widetilde{Y} \to \widetilde{X}$

$$\begin{tikzcd} Z \arrow[r, "r"'] \arrow[rr, bend left, "\widetilde{h}"] & \widetilde{Y} \arrow[r, "\widetilde{f}"'] & \widetilde{X} \end{tikzcd},$$

as in Lemma 2.23. Set $B_Z := \widetilde{h}^*(\pi^*(H_1, P + H_2))$ and $B_{\widetilde{Y}} := \widetilde{f}^*(\pi^*(H_1, P + H_2))$. Now, let $\mathscr{G}_{\widetilde{Y}}$ be the kernel of the sheaf morphism (4.5) so that

$$\left(\widetilde{f}^*(K_{\widetilde{X}} + \widetilde{D}) - [\mathscr{G}_{\widetilde{Y}}]\right) \cdot B_{\widetilde{Y}} < 0. \tag{4.6}$$

As γ is away from the exceptional centre of $\widetilde{\pi}$ and since $\widetilde{G}$ is supported on the exceptional locus of π, we have

$$\begin{aligned} \widetilde{h}^*(K_{\widetilde{X}} + \widetilde{D} + \widetilde{G}) \cdot B_Z &= \widetilde{h}^*(K_{\widetilde{X}} + \widetilde{D}) \cdot B_Z \\ &= r^*(\widetilde{f}^*(K_{\widetilde{X}} + \widetilde{D})) \cdot B_Z. \end{aligned}$$

As a result, for the inclusion $r^{[*]}(\mathscr{G}_{\widetilde{Y}}) \subset \Omega^{[1]}_{(Z,\widetilde{h},\widetilde{D}+\widetilde{G})}$, we find that

$$\begin{aligned}\left(\left[\Omega^{[1]}_{(Z,\widetilde{h},\widetilde{D}+\widetilde{G})}\right] - r^{[*]}\mathscr{G}_{\widetilde{Y}}\right)\cdot B_Z &= \left(r^*\left(\widetilde{f}^*(K_{\widetilde{X}}+\widetilde{D})\right) - r^{[*]}\mathscr{G}_{\widetilde{Y}}\right)\cdot B_Z \\ &= (\deg r)\Big(\widetilde{f}^*(K_{\widetilde{X}}+\widetilde{D}) - [\mathscr{G}_{\widetilde{Y}}]\Big)\cdot B_{\widetilde{Y}} \\ &< 0, \qquad\qquad \text{by Inequality 4.6,}\end{aligned}$$

contradicting Proposition 4.3.

Finally, if the assumption (2) holds, the proof follows from simply considering the ramification formula as above and using Theorem 4.1. ☐

5. Pseudoeffectivity of the orbifold c_2

In [30] Miyaoka famously proved that the second Chern class c_2 of a generically semipositive sheaf with nef determinant is pseudoeffective. Thanks to his result on the semipositivity of cotangent sheaves, Miyaoka then established the pseudoeffectivity of $c_2(X)$ for any minimal model X. Our aim in this section is to generalize this result to the case of pairs (X, D) with movable $(K_X + D)$ (Corollary 5.2).

Proposition 5.1 (Pseudoeffectivity of c_2 for semipositive sheaves). — *Let X be a normal projective, threefold with isolated singularities and $A_1 \in \mathrm{Amp}(X)_{\mathbb{Q}}$. Then, the inequality*

$$c_2(\mathscr{E})\cdot A_1 \geqslant 0$$

holds for any reflexive sheaf $\mathscr{E}$ of rank r verifying the following properties.

1. *$[\mathscr{E}] \in \overline{\mathrm{Mov}}^1(X)$.*
2. *For any $A_2 \in \mathrm{Amp}(X)_{\mathbb{Q}}$, the sheaf $\mathscr{E}$ is semipositive with respect to $(A_1, [\mathscr{E}] + A_2)$.*

Proof. — Let c be any positive integer. Consider the $\mathbb{Q}$-twisted reflexive sheaf $\mathscr{E}\langle\frac{1}{c}\cdot H\rangle$. For the choice of polarization $(A_1, [\mathscr{E}\langle\frac{1}{c}\cdot H\rangle])$, the assumptions of Proposition 3.9 are satisfied, for all c.

Now, let S be the complete intersection surface defined in Proposition 3.9 (see also Remark 3.11) so that, using the assumption (2) with $A_2 := \frac{r}{c}H$, the restriction $\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle := (\mathscr{E}\langle\frac{1}{c}\cdot H\rangle)|_S$ is semipositive with respect to

$$\beta := c_1\left(\mathscr{E}_S\left\langle\frac{1}{c}\cdot H_S\right\rangle\right) = \left([\mathscr{E}] + \frac{r}{c}\cdot[H]\right)\Big|_S.$$

Following the arguments of Miyaoka, we now consider two cases based on the stability of $\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle$.

First, we consider the case where $\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle$ is semistable with respect to β. Here, the semipositivity of c_2 follows from Bogomolov–Gieseker inequality for $\mathbb{Q}$-twisted locally-free sheaves (Proposition 2.15).

Now, we assume that $\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle$ is not semistable with respect to β. Let

$$0 \neq \mathscr{E}_S^1\left\langle\frac{1}{m}\cdot H_S\right\rangle \subset \cdots \subset \mathscr{E}_S^t\left\langle\frac{1}{c}\cdot H_S\right\rangle = \mathscr{E}_S\left\langle\frac{1}{c}\cdot H_S\right\rangle \tag{5.1}$$

be the $\mathbb{Q}$-twisted HN-filtration of $\mathscr{E}_S\langle\frac{1}{c}H_S\rangle$. Denote the semistable, torsion-free, $\mathbb{Q}$-twisted sheaves

$$\mathscr{E}_S^i\left\langle\frac{1}{c}\cdot H_S\right\rangle/\mathscr{E}_S^{i-1}\left\langle\frac{1}{c}\cdot H_S\right\rangle$$

of rank r_i by $\mathscr{Q}_S^i\langle\frac{1}{c}\cdot H_S\rangle$ and let $\overline{\mathscr{Q}}_S^i\langle\frac{1}{c}\cdot H_S\rangle$ denote its reflexivization. As the second Chern character $ch_2(\cdot)$ is additive, we have

$$\begin{aligned} 2\cdot c_2\left(\mathscr{E}_S\left\langle\frac{1}{c}\cdot H_S\right\rangle\right) &- c_1^2\left(\mathscr{E}_S\left\langle\frac{1}{c}\cdot H_S\right\rangle\right) \\ &= \sum\left(2\cdot c_2\left(\mathscr{Q}_S^i\left\langle\frac{1}{c}\cdot H_S\right\rangle\right) - c_1^2\left(\mathscr{Q}_S^i\left\langle\frac{1}{c}\cdot H_S\right\rangle\right)\right) \\ &\geqslant \sum\left(2\cdot c_2\left(\overline{\mathscr{Q}}_S^i\left\langle\frac{1}{c}\cdot H_S\right\rangle\right) - c_1^2\left(\mathscr{Q}_S^i\left\langle\frac{1}{c}\cdot H_S\right\rangle\right)\right), \end{aligned} \tag{5.2}$$

where the last inequality follows from the fact that $c_2(\mathscr{Q}_S^i) \geqslant c_2(\overline{\mathscr{Q}}_S^i)$. Now, by applying the Bogomolov inequality (2.15) to each semistable, $\mathbb{Q}$-twisted sheaf $\overline{\mathscr{Q}}_S^i\langle\frac{1}{c}\cdot H_S\rangle$, we find that each term in the right-hand side of the inequality (5.2) verifies the inequality

$$2\cdot c_2\left(\overline{\mathscr{Q}}_S^i\left\langle\frac{1}{c}\cdot H_S\right\rangle\right) - c_1^2\left(\mathscr{Q}_S^i\left\langle\frac{1}{c}\cdot H_S\right\rangle\right) \geqslant \frac{-1}{r_i}\cdot c_1^2\left(\mathscr{Q}_S^i\left\langle\frac{1}{c}\cdot H_S\right\rangle\right).$$

Therefore we have

$$\begin{aligned} 2\cdot c_2\left(\mathscr{E}_S\left\langle\frac{1}{c}\cdot H_S\right\rangle\right) &- c_1^2\left(\mathscr{E}_S\left\langle\frac{1}{c}\cdot H_S\right\rangle\right) \\ &\geqslant \sum \frac{-1}{r_i}\cdot c_1^2\left(\mathscr{Q}_S^i\left\langle\frac{1}{c}\cdot H_S\right\rangle\right). \end{aligned} \tag{5.3}$$

Next, we define the rational number $\alpha_i \in \mathbb{Q}$ by the equality

$$r_i\cdot\alpha_i = \frac{c_1(\mathscr{Q}_S^i\langle\frac{1}{c}\cdot H_S\rangle)\cdot\beta}{c_1^2(\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle)} = \frac{c_1(\mathscr{Q}_S^i\langle\frac{1}{c}\cdot H_S\rangle)\cdot\beta}{\beta^2}. \tag{5.4}$$

It follows that

$$\sum r_i \cdot \alpha_i = 1. \tag{5.5}$$

Furthermore, according to the definition of α_i, and by using the fact that the slopes of the quotients of the HN-filtration (5.1) are strictly decreasing, we know that

$$\alpha_1 > \alpha_2 > \cdots > \alpha_t \geqslant 0, \tag{5.6}$$

where the last inequality follows from the semipositivity of $\mathscr{E}_S\langle \frac{1}{c} H_S\rangle$.

Now, as $\alpha_i \geqslant 0$, for each i, the equality (5.5) implies that $\alpha_i \leqslant 1$. On the other hand, according to the Hodge index theorem we have

$$-c_1^2\left(\mathscr{Q}_S^i\left\langle \frac{1}{c}\cdot H_S\right\rangle\right) \geqslant \frac{\left(c_1(\mathscr{Q}_S^i\langle \frac{1}{c}\cdot H_S\rangle)\cdot\beta\right)^2}{\beta^2},$$

so that

$$-c_1^2\left(\mathscr{Q}_S^i\left\langle \frac{1}{c}\cdot H\right\rangle\right) \geqslant \beta^2(r_i\cdot\alpha_i)^2.$$

Going back to the inequality (5.3) we now find that

$$\begin{aligned} 2\cdot c_2\left(\mathscr{E}_S\left\langle\frac{1}{c}\cdot H_S\right\rangle\right) &\geqslant \beta^2\left(1-\sum r_i\cdot\alpha_i^2\right) \\ &\geqslant \beta^2\left(1-\alpha_1\sum r_i\cdot\alpha_i\right) && \text{by (5.6)} \\ &= \beta^2(1-\alpha_1) && \text{by (5.5)} \\ &\geqslant 0 && \text{as } \alpha_1\leqslant 1. \end{aligned}$$

The inequality $c_2(\mathscr{E}_S) \geqslant 0$ now follows by taking the limit $c\to\infty$. □

As an immediate consequence, we can now prove the pseudoeffectivity of c_2 for the orbifold cotangent sheaves of pairs (X,D) in dimension 3 with only mild isolated singularities and whose $K_X + D$ is movable.

Corollary 5.2 (Positivity of c_2 of orbifold cotangent sheaves). — *Let (X,D) be a projective pair of dimension 3 and with only isolated singularities. Assume that either D is reduced and (X,D) has only lc singularities or (X,D) is klt. If $(K_X+D)\in\overline{\mathrm{Mov}}^1(X)$, then for any ample divisor $A\subset X$ and strongly adapted morphism $f:Y\to X$, the inequality*

$$c_2\left(\Omega^{[1]}_{(Y,f,D)}\right)\cdot f^*(A)\geqslant 0$$

holds.

Proof. — As $[\Omega^{[1]}_{(Y,f,D)}] = f^*(K_X+D)$, the corollary is a direct consequence of Proposition 5.1 together with Proposition 4.4. □

We would like to point out that once we assume that $(K_X + D)$ is nef, then an easy adaptation of the original results of Miyaoka to the case of orbifold Chern classes, together with the semipositivity result of [6] (see also [10, Thm. 5.3]) leads to the following theorem.

Theorem 5.3 (Positivity of orbifold c_2 for log-minimal models). — *Let (X, D) be a projective lc pair of dimension n that is log-smooth in codimension two. If $(K_X + D)$ is nef, then for any strongly adapted morphism $f : Y \to X$, we have*

$$c_2\left(\Omega^{[1]}_{(Y,f,D)}\right) \cdot f^*(A^{n-2}) \geqslant 0,$$

where $A \subset X$ is any ample divisor.

Remark 5.4. — In the above results the assumption that (X, D) is log-smooth in codimension two can be dropped if one is willing to work woth the so-called $\mathcal{Q}$-Chern classes. But in this setting (X, D) would have to be klt and additional assumptions would be needed to guarantee that the covering Y has quotient singularities in codimension two.

6. An effective non-vanishing result for threefolds

The goal of this section is to prove Theorem 1.4. The main point of the strategy is to devise an effective lower bound for $\chi(K_Y + H)$, when Y is terminal (and (Y, H) is lc). First we recall the well-known fact that Hizerbruch–Riemann–Roch Theorem holds for locally free sheaves over projective threefolds or surfaces with only mild singularities and, for the reader's convenience, include a short proof.

Proposition 6.1. — *Let X be a projective variety and L a Cartier divisor on X.*

- *If X is a terminal threefold, then we have*

$$\chi(X, L) = \frac{1}{12} \cdot L \cdot (L - K_X) \cdot (2L - K_X) + \frac{1}{12} \cdot c_2(X) \cdot L + \chi(X, \mathscr{O}_X). \tag{6.1}$$

- *If X is of dimension two and with only rational singularities, then we have*

$$\chi(L) = \frac{1}{2}L^2 - \frac{1}{2}L \cdot K_X + \chi(X, \mathscr{O}_X). \tag{6.2}$$

Proof. — We consider the threefold case first. Let $\pi : \widetilde{X} \to X$ be a resolution such that $\pi^{-1}|_{X_{\mathrm{reg}}}$ is an isomorphism. Remember that, as its singularities are only terminal, X has only rational singularities (in this case since X has only quotient singularities [23, Cor. 4.39], the fact that X has rational singularities follows from the definition). Consequently it follows that

$$\chi(\widetilde{X}, \pi^* L) = \chi(X, L)$$

and in particular we have $\chi(\mathscr{O}_{\widetilde{X}}) = \chi(\mathscr{O}_X)$.

On the other hand, by Hizerbruch–Riemann–Roch theorem for smooth projective threefolds (see [16, Ex. 6.7, App. A]) we have

$$\begin{aligned} \chi(\widetilde{X}, \pi^* L) = \frac{1}{12} \cdot \pi^* L \cdot (\pi^* L - K_{\widetilde{X}}) \cdot (2\pi^* L - K_{\widetilde{X}}) \\ + \frac{1}{12} \cdot c_2(\widetilde{X}) \cdot \pi^* L + \chi(\widetilde{X}, \mathscr{O}_{\widetilde{X}}). \end{aligned} \tag{6.3}$$

Using that fact that X is smooth in codimension two, we now find that the right-hand sides of (6.1) and (6.3) are equal and therefore Equality (6.1) is established.

A similar argument can now be used to show that the equality (6.2) also holds. □

Proposition 6.2 (Lower bounds for the Euler characteristic of adjoint bundles). — *For a terminal projective threefold X the inequality*

$$\chi(X, K_X + A) \geqslant \frac{1}{24} \cdot A \cdot (K_X + A) \cdot (K_X + 2A) \tag{6.4}$$

holds, for any Weil divisor A satisfying the following conditions.

1. *A is irreducible.*
2. *The pair (X, A) is lc and is log-smooth in codimension two.*
3. *The divisors A and $(K_X + A)$ are Cartier and nef.*

Proof. — According to (6.1), with L being replaced by $(K_X + A)$, we have

$$\begin{aligned} \chi(X, K_X + D + A) = \frac{1}{12} \cdot (K_X + A) \cdot A \cdot \big(2(K_X + A) - K_X\big) \\ + \frac{1}{12} \cdot c_2(X) \cdot (K_X + A) + \chi(X, \mathscr{O}_X). \end{aligned} \tag{6.5}$$

Standard Chern class calculations show that we have the equality

$$c_2(X) = c_2 \left(\Omega_X^{[1]} \log(A)\right) - (K_X + A) \cdot A - A^2, \tag{6.6}$$

as linear forms on $\mathrm{N}^1(X)_{\mathbb{Q}}$. After substituting back into Equality 6.5, we find that the equality

$$\begin{aligned}\chi(X, K_X + D + A) = \frac{1}{12} \cdot (K_X + A) \cdot \Big\{ A \cdot (K_X + 2A) \\ + \cdot c_2(\Omega_X^{[1]} \log(A)) - (K_X + A) \cdot A - A^2 \Big\} \\ + \chi(X, \mathscr{O}_X)\end{aligned} \tag{6.7}$$

holds, which then simplifies to

$$\chi(X, K_X + A) = \frac{1}{12}(K_X + A) \cdot \Big\{ A^2 + c_2(\Omega_X^{[1]} \log(A)) \Big\} + \chi(X, \mathscr{O}_X). \tag{6.8}$$

On the other hand, as X is terminal, we know, thanks to [19, Lems. 2.2 and 2.3], that

$$\chi(X, \mathscr{O}_X) \geqslant \frac{-1}{24} K_X \cdot c_2(X). \tag{6.9}$$

Bu using the equality (6.6) we can rewrite this inequality as

$$\begin{aligned}24 \cdot \chi(X, \mathscr{O}_X) &\geqslant \big(A - (K_X + A)\big) \cdot c_2(\Omega_X^{[1]} \log(A)) + K_X \cdot (K_X + A) \cdot A \\ &\geqslant (K_X + A) \cdot \Big\{ K_X \cdot A - c_2(\Omega_X^{[1]} \log(A)) \Big\},\end{aligned}$$

where for the latter inequality we have used the assumption that A is nef and the pseudoeffectivity of c_2 (Theorem 5.3). Now, going back to Equation 6.8, we get

$$24\chi(X, K_X + A) \geqslant (K_X + A) \Big\{ 2A^2 + c_2(\Omega_X^{[1]} \log(A)) + K_X \cdot A \Big\}. \tag{6.10}$$

Again, by using Corollary 5.2 and the nefness assumptions on $(K_X + A)$, we find that

$$\begin{aligned}24 \cdot \chi(X, K_X + A) &\geqslant (K_X + A) \cdot (2A^2) + (K_X + A) \cdot A \\ &= (K_X + A) \cdot A \cdot (K_X + 2A),\end{aligned} \tag{6.11}$$

as required. □

6.1. Proof of Theorem 1.4

Thanks to Kawamata–Viehweg vanishing [28, Thm. 5.2.7], it suffices to prove that $\chi(Y, K_Y + H) \neq 0$. For a general choice of $H' \in |H|$, the pair (Y, H') satisfies Assumption (2). Therefore the assumptions of Proposition 6.2 are satisfied except for the terminal assumption for the singularities.

Now, let $\pi : X \to Y$ be a terminalization of Y, cf. [23, Sect. 6.3]. Set $A := \pi^*(H')$. Since π is small, the adjoint divisor $(K_X + A)$ is also nef. We may now conclude using the strict positivity of the right-hand side of the inequality (6.4) by the following argument. According to the basepoint freeness theorem for log-canonical threefolds, cf. [20], the divisor $K_X + A$ is semi-ample. Therefore, for sufficiently large integer m, we can find an irreducible surface $S \in |m\cdot(K_X+2A)|$ such that $(A|_S)$ is big. On the other hand, the divisor $(K_X+A)|_S$ is nef. It thus follows that $(K_X+A)|_S\cdot A|_S > 0$, thanks to Kleiman's ampleness criterion ([25, Thm. 1.4.29]). □

7. A Miyaoka–Yau inequality in higher dimensions

In [30], Miyaoka generalized the famous inequality $c_1^2 \leqslant 3c_2$ from surfaces with pseudoeffective canonical divisor to higher dimensional varieties with nef canonical divisor. We extend this result to the case of movable canonical divisor.

Theorem 7.1. — *In the setting of Corollary 5.2, we have the inequality*

$$c_1^2\left(\Omega^{[1]}_{(Y,f,D)}\right)\cdot f^*A \leqslant 3c_2\left(\Omega^{[1]}_{(Y,f,D)}\right)\cdot f^*A.$$

Proof. — Let $\widetilde{H} \in \mathrm{Amp}(X)_{\mathbb{Q}}$, $H := f^*\widetilde{H}$ and $\mathscr{E} := \Omega^{[1]}_{(Y,f,D)}$. Let c be any positive integer. Consider the $\mathbb{Q}$-twisted reflexive sheaf $\mathscr{E}\langle\frac{1}{c}\cdot H\rangle$. For the choice of polarization $(f^*A, [\mathscr{E}\langle\frac{1}{c}\cdot H\rangle])$, the assumptions of Proposition 3.9 are satisfied, for all c.

Now, let S be the complete intersection surface defined in Proposition 3.9 (see also Remark 3.11) so that the restriction $\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle := (\mathscr{E}\langle\frac{1}{c}\cdot H\rangle)|_S$ is semipositive with respect to

$$\beta := \left([\mathscr{E}] + \frac{r}{c}\cdot H\right)\Big|_S.$$

Let

$$0 \neq \mathscr{E}^1_S\left\langle\frac{1}{c}\cdot H_S\right\rangle \subset \cdots \subset \mathscr{E}^s_S\left\langle\frac{1}{c}\cdot H_S\right\rangle = \mathscr{E}_S\left\langle\frac{1}{c}\cdot H_S\right\rangle \tag{7.1}$$

be the $\mathbb{Q}$-twisted HN-filtration of $\mathscr{E}_S\langle\frac{1}{c}H_S\rangle$.

The same arguments as those in the proof of Proposition 5.1 show that

$$\left(2c_2\left(\mathscr{E}_S\left\langle\frac{1}{c}\cdot H_S\right\rangle\right) - c_1^2\left(\mathscr{E}_S\left\langle\frac{1}{c}\cdot H_S\right\rangle\right)\right) \geqslant \left(\sum\frac{-1}{r_i}c_1^2(\mathscr{Q}^i_S)\right),$$

where $\mathscr{Q}^i_S\langle\frac{1}{c}\cdot H_S\rangle$ is the torsion free, $\mathbb{Q}$-twisted quotient sheaf of rank r_i of the filtration (7.1).

Again, as in the proof of Proposition 5.1, for each i, we define α_i by the equation

$$r_i \cdot \alpha_i = \frac{c_1(\mathscr{Q}^i_S\langle \frac{1}{c} \cdot H_S\rangle) \cdot \beta}{\beta^2}.$$

From the definition of α_i, it follows that $\sum r_i \cdot \alpha_i = 1$. Moreover, we have $\alpha_1 > \cdots > \alpha_s \geqslant 0$, where the last inequality is due to the semipositivity of $\mathscr{E}_S\langle \frac{1}{c} \cdot H_S\rangle$.

We now deduce

$$\begin{aligned}
&\left(6c_2\left(\mathscr{E}_S\left\langle \frac{1}{c} \cdot H_S\right\rangle\right) - 2c_1^2\left(\mathscr{E}_S\left\langle \frac{1}{c} \cdot H_S\right\rangle\right)\right)\\
&\qquad \geqslant \left(3\left(\sum_{i>1} \frac{-1}{r_i} c_1^2(\mathcal{G}_i)\right) + 6c_2\left(\mathscr{E}^1_S\left\langle \frac{1}{c} \cdot H_S\right\rangle\right)\right.\\
&\qquad\qquad \left. - 3c_1^2\left(\mathscr{E}^1_S\left\langle \frac{1}{c} \cdot H_S\right\rangle\right) + c_1^2\left(\mathscr{E}_S\left\langle \frac{1}{c} \cdot H_S\right\rangle\right)\right).
\end{aligned}$$

And finally,

$$\begin{aligned}
&\left(6c_2\left(\mathscr{E}_S\left\langle \frac{1}{c} \cdot H_S\right\rangle\right) - 2c_1^2\left(\mathscr{E}_S\left\langle \frac{1}{c} \cdot H_S\right\rangle\right)\right)\\
&\qquad \geqslant \left(\left(1 - 3\sum_{i>1} r_i \alpha_i^2\right) \cdot c_1\left(\mathscr{E}_S\left\langle \frac{1}{c} \cdot H_S\right\rangle\right)\right.\\
&\qquad\qquad \left. + 6c_2\left(\mathscr{E}^1_S\left\langle \frac{1}{c} \cdot H_S\right\rangle\right) - 3c_1^2\left(\mathscr{E}^1_S\left\langle \frac{1}{c} \cdot H_S\right\rangle\right)\right).
\end{aligned} \tag{7.2}$$

There are three possibilities: $r_1 \geqslant 3$, $r_1 = 2$ and $r_1 = 1$.

If $r_1 \geqslant 3$, using Bogomolov–Gieseker inequality and the Hodge index theorem, we obtain

$$\begin{aligned}
&\left(6c_2\left(\mathscr{E}_S\left\langle \frac{1}{c} \cdot H_S\right\rangle\right) - 2c_1^2\left(\mathscr{E}_S\left\langle \frac{1}{c} \cdot H_S\right\rangle\right)\right)\\
&\qquad \geqslant \left(\left(1 - 3\sum_{i>1} r_i \alpha_i^2\right) \cdot c_1^2\left(\mathscr{E}_S\left\langle \frac{1}{c} \cdot H_S\right\rangle\right) - 3\frac{1}{r_1} c_1^2(\mathscr{E}_1)\right)\\
&\qquad \geqslant \left(1 - 3\sum_{i} r_i \alpha_i^2\right) \cdot c_1^2\left(\mathscr{E}_S\left\langle \frac{1}{c} \cdot H_S\right\rangle\right)\\
&\qquad \geqslant (1 - 3\alpha_1) \cdot c_1^2\left(\mathscr{E}_S\left\langle \frac{1}{c} \cdot H_S\right\rangle\right) \geqslant 0.
\end{aligned}$$

since $3\alpha_1 \leqslant r_1\alpha_1 \leqslant \sum_i r_i\alpha_i = 1$.

If $r_1 = 2$, we choose S general enough so that $\mathscr{E}_S^1$ injects into

$$\Omega_S^1(\log(f^{-1}\lceil D\rceil_{|S})).$$

Using the Bogomolov–Miyaoka–Yau inequality, we have either

$$\kappa(S, c_1(\mathscr{E}_S^1)) \leqslant 0 \quad \text{or} \quad c_1^2(\mathscr{E}_S^1) \leqslant 3c_2(\mathscr{E}_S^1).$$

In the case $\kappa(S, c_1(\mathscr{E}_S^1)) \leqslant 0$, since $c_1(\mathscr{E}_S^1).\beta > 0$, we have

$$c_1^2(\mathscr{E}_S^1) \leqslant 0.$$

Applying Bogomolov–Gieseker inequality to 7.2:

$$\begin{aligned}
&\left(6c_2\left(\mathscr{E}_S\left\langle\frac{1}{c}\cdot H_S\right\rangle\right) - 2c_1^2\left(\mathscr{E}_S\left\langle\frac{1}{c}\cdot H_S\right\rangle\right)\right)\\
&\quad\geqslant \left(\left(1 - 3\sum_{i>1} r_i\alpha_i^2\right)\cdot c_1^2\left(\mathscr{E}_S\left\langle\frac{1}{c}\cdot H_S\right\rangle\right) - \frac{3}{2}c_1^2\left(\mathscr{E}_S^1\left\langle\frac{1}{c}\cdot H_S\right\rangle\right)\right)\\
&\quad\geqslant \left(1 - 3\sum_{i>1} r_i\alpha_i^2\right)\cdot c_1^2\left(\mathscr{E}_S\left\langle\frac{1}{c}\cdot H_S\right\rangle\right) - \frac{3}{2}c_1^2\left(\mathscr{E}_S^1\left\langle\frac{1}{c}\cdot H_S\right\rangle\right)\\
&\quad\geqslant \left(1 - 3\alpha_2\sum_{i>1} r_i\alpha_i\right)\cdot c_1^2\left(\mathscr{E}_S\left\langle\frac{1}{c}\cdot H_S\right\rangle\right) - \frac{3}{2}c_1^2\left(\mathscr{E}_S^1\left\langle\frac{1}{c}\cdot H_S\right\rangle\right)\\
&\quad= \left(1 - 3\alpha_2\left(1 - 2\alpha_1\right)\right)\cdot c_1^2\left(\mathscr{E}_S\left\langle\frac{1}{c}\cdot H_S\right\rangle\right) - \frac{3}{2}c_1^2\left(\mathscr{E}_S^1\left\langle\frac{1}{c}\cdot H_S\right\rangle\right)\\
&\quad\geqslant \left(1 - 3\alpha_1\left(1 - 2\alpha_1\right)\right)\cdot c_1^2\left(\mathscr{E}_S\left\langle\frac{1}{c}\cdot H_S\right\rangle\right) - \frac{3}{2}c_1^2\left(\mathscr{E}_S^1\left\langle\frac{1}{c}\cdot H_S\right\rangle\right)\\
&\quad= \left(6\left(\alpha_1 - \frac{1}{4}\right)^2 + \frac{5}{8}\right)\cdot c_1^2\left(\mathscr{E}_S\left\langle\frac{1}{c}\cdot H_S\right\rangle\right) - \frac{3}{2}c_1^2\left(\mathscr{E}_S^1\left\langle\frac{1}{c}\cdot H_S\right\rangle\right)\\
&\quad\geqslant -\frac{3}{2}c_1^2\left(\mathscr{E}_S^1\left\langle\frac{1}{c}\cdot H_S\right\rangle\right).
\end{aligned}$$

Finally, we obtain

$$(3c_2(\mathscr{E}_S) - c_1^2(\mathscr{E}_S)) \geqslant 0.$$

In the case $c_1^2(\mathscr{E}_S^1) \leqslant 3c_2(\mathscr{E}_S^1)$ we have from 7.2:

$$\begin{aligned}
&\left(6c_2\left(\mathscr{E}_S\left\langle\frac{1}{c}H_S\right\rangle\right)-2c_1^2\left(\mathscr{E}_S\left\langle\frac{1}{c}H_S\right\rangle\right)\right)\\
&\quad\geqslant\left(1-3\sum_{i>1}r_i\alpha_i^2\right)c_1^2\left(\mathscr{E}_S\left\langle\frac{1}{c}H_S\right\rangle\right)-c_1^2\left(\mathscr{E}_S^1\left\langle\frac{1}{c}H_S\right\rangle\right)\\
&\qquad\qquad+6c_2\left(\mathscr{E}_S\left\langle\frac{1}{c}H_S\right\rangle\right)-2c_1^2\left(\mathscr{E}_S^1\left\langle\frac{1}{c}H_S\right\rangle\right)\\
&\quad\geqslant\left(\left(1-4\alpha_1^2-3\sum_{i>1}r_i\alpha_i^2\right)c_1^2\left(\mathscr{E}_S\left\langle\frac{1}{c}H_S\right\rangle\right)\right)\\
&\qquad\qquad+6c_2\left(\mathscr{E}_S\left\langle\frac{1}{c}H_S\right\rangle\right)-2c_1^2\left(\mathscr{E}_S^1\left\langle\frac{1}{c}H_S\right\rangle\right)\\
&\quad\geqslant\left(\left(1-4\alpha_1^2-3\alpha_2\sum_{i>1}r_i\alpha_i\right)c_1^2\left(\mathscr{E}_S\left\langle\frac{1}{c}\dot{H}_S\right\rangle\right)\right)\\
&\qquad\qquad+6c_2\left(\mathscr{E}_S\left\langle\frac{1}{c}H_S\right\rangle\right)-2c_1^2\left(\mathscr{E}_S^1\left\langle\frac{1}{c}H_S\right\rangle\right)\\
&\quad=\left(1-4\alpha_1^2-3\alpha_2\left(1-2\alpha_1\right)\right)c_1^2\left(\mathscr{E}_S\left\langle\frac{1}{c}H_S\right\rangle\right)\\
&\qquad\qquad+\left(6c_2\left(\mathscr{E}_S\left\langle\frac{1}{c}H_S\right\rangle\right)-2c_1^2\left(\mathscr{E}_S^1\left\langle\frac{1}{c}\cdot H_S\right\rangle\right)\right)\\
&\quad=(1-2\alpha_1)(1+2\alpha_1-3\alpha_2)\cdot c_1^2\left(\mathscr{E}_S\left\langle\frac{1}{c}H_S\right\rangle)\right)\\
&\qquad\qquad+6c_2\left(\mathscr{E}_S\left\langle\frac{1}{c}H_S\right\rangle\right)-2c_1^2\left(\mathscr{E}_S^1\left\langle\frac{1}{c}H_S\right\rangle\right).
\end{aligned}$$

As $3\alpha_2 < r_1\alpha_1 + r_2\alpha_2 \leqslant 1$, we have

$$(6c_2(\mathscr{E}_S) - 2c_1^2(\mathscr{E}_S)) \geqslant 0.$$

Finally, if $r_1 = 1$, a classical result of Bogomolov and Sommese (the Bogomolov–Sommese vanishing) implies that

$$\mathscr{E}_S^1 \subset \Omega_S^1(\log(f^{-1}\lceil\Delta\rceil_{|S}))$$

has Kodaira dimension at most one. Therefore $c_1^2(\mathscr{E}_S^1) \leqslant 0$. From 7.2, one obtains:

$$\begin{aligned}
&\left(6c_2\left(\mathscr{E}_S\left\langle\frac{1}{c}\cdot H_S\right\rangle\right) - 2c_1^2\left(\mathscr{E}_S\left\langle\frac{1}{c}\cdot H_S\right\rangle\right)\right)\\
&\quad \geqslant \left(\left(1 - 3\sum_{i>1} r_i\alpha_i^2\right)\cdot c_1^2\left(\mathscr{E}_S\left\langle\frac{1}{c}\cdot H_S\right\rangle\right)\right) - 3c_1^2\left(\mathscr{E}_S^1\left\langle\frac{1}{c}\cdot H_S\right\rangle\right)\\
&\quad \geqslant \left(\left(1 - 3\alpha_1\sum_{i>1} r_i\alpha_i\right)\cdot c_1^2\left(\mathscr{E}_S\left\langle\frac{1}{c}\cdot H_S\right\rangle\right)\right) - 3c_1^2\left(\mathscr{E}_S^1\left\langle\frac{1}{c}\cdot H_S\right\rangle\right)\\
&\quad = \left((1 - 3\alpha_1(1-\alpha_1))\cdot c_1^2\left(\mathscr{E}_S\left\langle\frac{1}{c}\cdot H_S\right\rangle\right)\right) - 3c_1^2\left(\mathscr{E}_S^1\left\langle\frac{1}{c}\cdot H_S\right\rangle\right)\\
&\quad \geqslant \left(1 - \frac{3}{2}\left(1-\frac{1}{2}\right)\right)\cdot c_1^2\left(\mathscr{E}_S\left\langle\frac{1}{c}\cdot H_S\right\rangle\right) - 3c_1^2\left(\mathscr{E}_S^1\left\langle\frac{1}{c}\cdot H_S\right\rangle\right)\\
&\quad = \frac{1}{4}c_1^2\left(\mathscr{E}_S\left\langle\frac{1}{c}\cdot H_S\right\rangle\right) - 3c_1^2\left(\mathscr{E}_S^1\left\langle\frac{1}{c}\cdot H_S\right\rangle\right)\\
&\quad \geqslant -3c_1^2\left(\mathscr{E}_S^1\left\langle\frac{1}{c}\cdot H_S\right\rangle\right).
\end{aligned}$$

Therefore, we have

$$(6c_2(\mathscr{E}_S) - 2c_1^2(\mathscr{E}_S)) \geqslant 0. \qquad \square$$

We finish this section by pointing out that when $(K_X + D)$ is nef, the original result of Miyaoka can be adapted to the case of orbifold Chern classes. This can then be combined with the semipositivity result of [6] to conclude the following result.

Theorem 7.2. — *Let (X, D) be an n-dimensional lc pair that is smooth in codimension two. If $K_X + D$ is nef, then for any strongly adapted morphism $f : Y \to X$ and any ample divisor A in X, we have*

$$c_1^2\left(\Omega^{[1]}_{(Y,f,D)}\right)\cdot f^*A^{n-2} \leqslant 3c_2\left(\Omega^{[1]}_{(Y,f,D)}\right)\cdot f^*A^{n-2}. \tag{7.3}$$

8. Remarks on Lang–Vojta's conjecture in codimension one

A classical conjecture of Lang predicts that a variety of general type X, admits a proper algebraic subvariety that contains all subvarieties of X that are *not* of general type. In this section, we will prove a particular case of this conjecture for codimension one subvarieties satisfying certain conditions: the codimension one subvariety will be assumed to be movable and with only canonical singularities.

First, an immediate application of the inequality (7.1) gives the following theorem.

Theorem 8.1. — *Let X be a normal, projective and $\mathbb{Q}$-factorial threefold such that $K_X \in \overline{\mathrm{Mov}}^1(X)$. Let H be a nef divisor, D a reduced, irreducible, normal divisor such that (X, D) has only isolated lc singularities. Assume that $[D] \in \overline{\mathrm{Mov}}^1(X)$. If $-K_D$ is pseudoeffective, then*

$$K_X \cdot D \cdot H \leqslant (3c_2 - c_1^2) \cdot H, \tag{8.1}$$

where $c_i(X) \coloneqq c_i(\Omega_X^{[1]})$.

Proof. — As $(K_X + D) \in \overline{\mathrm{Mov}}^1(X)$, from the inequality (7.1), we have $c_1^2(\Omega_X^{[1]}(\log D)) \cdot H \leqslant 3c_2(\Omega_X^{[1]}(\log D)) \cdot H$. Therefore, we have

$$(K_X + D)^2 \cdot H \leqslant 3(c_2 + (K_X + D) \cdot D) \cdot H.$$

It follows that

$$2K_X \cdot D \cdot H \leqslant (3c_2 - c_1^2) \cdot H + 3(K_X + D) \cdot D \cdot H - D^2 \cdot H.$$

Finally, thanks to the adjunction formula, cf. [22, Prop. 16.4], we get

$$K_X \cdot D \cdot H \leqslant (3c_2 - c_1^2) \cdot H + 2K_D \cdot H|_D.$$

The inequality (8.1) now follows from the assumption that $-K_D$ is pseudoeffective. □

Proof of Theorem 1.5. — Let H be an ample divisor in X. The divisor K_X is big so we can find a positive integer m such that $(m \cdot K_X - H)$ is linearly equivalent to an effective divisor E.

Let us first prove that the family of polarized varieties $(D, H|_D)$ is bounded. We note that as each D has only rational singularities, using (6.2), we see that the coefficients of the Hilbert polynomial corresponding to $H|_D$ are determined by Riemann–Roch formula. Therefore, the theorem of Kollár and Matsusaka [21] applies, that is to bound the family $(D, H|_D)$, it suffices to bound the intersection numbers

$$H^2 \cdot D \qquad \text{and} \qquad H \cdot K_D = H \cdot (K_X + D) \cdot D.$$

For $H^2 \cdot D$, we note that, as long as D is not a component of E we can use the inequality (8.1), to get

$$0 \leqslant H^2 \cdot D \leqslant mH \cdot (3c_2 - c_1^2).$$

For the second term $K_D \cdot H$, we use Theorem 1.2 to find

$$\begin{aligned}0 &\leqslant 3c_2(\Omega_X^{[1]}(\log D)) \cdot H - c_1^2(\Omega_X^{[1]}(\log D)) \cdot H \\ &= (3c_2 - c_1^2) \cdot H + 2(K_X + D) \cdot D \cdot H - K_X \cdot D \cdot H.\end{aligned}$$

We immediately deduce that

$$-\frac{1}{2}(3c_2 - c_1^2) \cdot H \leqslant H \cdot (K_X + D) \cdot D = H \cdot K_D \leqslant 0.$$

Therefore, the family of polarized varieties $(D, H|_D)$ is bounded.

It now remains to show the finiteness of the polarized varieties $(D, H|_D)$ with canonical singularities and $-K_D$ pseudoeffective. Let d be the Hilbert polynomial of one of this object D and Hilb_X^d the corresponding Hilbert scheme. All other surfaces with canonical singularities in Hilb_X^d are deformations of D. From simultaneous resolution of families of surfaces with canonical singularities [23], one can assume that the deformation is smooth. Then from the deformation invariance of the Kodaira dimension [32], we obtain that such deformations of D are not of general type. If Hilb_X^d is not finite then X is covered by a family of varieties which are not of general type. This is impossible by the easy additivity of Kodaira dimensions and the fact that X if of general type. The boundedness above gives that polarized varieties $(D, H|_D)$ with canonical singularities and $-K_D$ pseudoeffective are contained in finitely many such Hilbert schemes. This concludes the proof. □

Remark 8.2. — In [27, Thm. 4], in the setting where X is non-uniruled and smooth and D is reduced, the Miyaoka–Yau inequality 7.2 is claimed to be valid. As a consequence a stronger version of Theorem 1.5 is obtained. Unfortunately we have been unable to verify the details of the proof of [27, Thm. 4]. The main point of difficulty is that within the proof of this theorem, in [27, Subsect. 3.1], the authors claim that given a smooth projective, threefold X of general type with an ample divisor H, for sufficiently large m, there is a general member $S \in |m \cdot H|$ for which the following conditions hold.

(1) The restriction $(\Omega_X^1 \log(D))|_S$ is semipositive with respect to $(P_\sigma(K_X + D))|_S$, where P_σ is the positive part of the divisorial Zariski decomposition of $K_X + D$, cf. [31, Chapt. III].
(2) The restriction $(P_\sigma(K_X + D))|_S$ of the positive part of $K_X + D$ verifies the equality $P_\sigma(K_X + D)|_S \cdot N((K_X + D)|_S) = 0$, where $N(K_X + D|_S)$ is the negative part of the Zariski decomposition of the pseudoeffective divisor $(K_X + D)|_S$.

Although Item (1) in the conditions above can most likely be recovered by [7, Thm. 2.1] and the arguments in Sections 3 and 4 in the current paper, the second condition (2) is more problematic as the underlying assumption is that Zariski decomposition is functorial; a condition that in general does not hold.

Remark 8.3. — Starting with a general type variety X and a divisor D such that (X, D) is dlt, thanks to [1], it is certainly possible to establish a Miyaoka–Yau inequality using a minimal model of (X, D). More precisely, let $\pi : (X, D) \dashrightarrow (X', D')$ be a LMMP map resulting in the log-minimal model (X', D'). Let $\widetilde{\pi} : \widetilde{X} \to X'$ be a desingularization of π factoring through $\mu : \widetilde{X} \to X$. Now, as we pointed out prior to Theorem 7.2, thanks to [6], one can use the original arguments of Miyaoka, together with those of Megyesi (and his use of $\mathcal{Q}$-Chern classes), to show that the inequality

$$\big(3c_2(\Omega^{[1]}_{X'} \log(D') - (K'_X + D')^2)\big) \cdot H^{n-2} \geqslant 0$$

holds for any ample divisor $H \subset X'$. Furthermore, we can use known results on the behaviour of Chern classes under birational morphisms to show that

$$\big(3c_2(\Omega^1_{\widetilde{X}} \log(\widetilde{D})) - (K_{\widetilde{X}} + \widetilde{D})^2)\big) \cdot \widetilde{\pi}^*(H)^{n-2} \geqslant 0. \tag{8.2}$$

But the inequality (8.2) is hardly independent of the divisor D. In fact in the inequality (8.2) even the polarization $(\pi^* H)$ depends on D. Therefore, the inequality (8.2) is far from being useful in the context of Lang–Vojta's conjecture.

BIBLIOGRAPHY

[1] C. Birkar, P. Cascini, C. D. Hacon & J. McKernan, "Existence of minimal models for varieties of log general type", *J. Amer. Math. Soc.* **23** (2010), no. 2, p. 405-468.

[2] F. A. Bogomolov, "Holomorphic tensors and vector bundles on projective varieties", *Math. USSR, Izv.* **13** (1979), p. 499-555.

[3] S. Boucksom, "Divisorial Zariski decompositions on compact complex manifolds", *Ann. Sci. École Norm. Sup. (4)* **37** (2004), no. 1, p. 45-76.

[4] S. Boucksom, J.-P. Demailly, M. Păun & T. Peternell, "The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension", *J. Algebraic Geom.* **22** (2013), no. 2, p. 201-248.

[5] F. Campana, "Orbifolds, special varieties and classification theory", *Ann. Inst. Fourier* **54** (2004), no. 3, p. 499-630.

[6] F. Campana & M. Păun, "Positivity properties of the bundle of logarithmic tensors on compact Kähler manifolds", `http://arxiv.org/abs/1407.3431`, 2014.

[7] F. Campana & M. Păun, "Orbifold generic semi-positivity: an application to families of canonically polarized manifolds", *Ann. Inst. Fourier* **65** (2015), no. 2, p. 835-861.

[8] F. Campana & M. Păun, "Foliations with positive slopes and birational stability of orbifold cotangent bundles", `https://arxiv.org/abs/1508.02456`, 2016.

[9] F. Campana & T. Peternell, "Geometric stability of the cotangent bundle and the universal cover of a projective manifold", *Bull. Soc. Math. France* **139** (2011), no. 1, p. 41-74, With an appendix by Matei Toma.

[10] B. Claudon, S. Kebekus & B. Taji, "Generic positivity and applications to hyperbolicity of moduli spaces", `http://arxiv.org/abs/1610.09832`, 2016.

[11] J.-P. Demailly, T. Peternell & M. Schneider, "Compact complex manifolds with numerically effective tangent bundles", *J. Algebraic Geom.* **3** (1994), no. 2, p. 295-345.

[12] W. Fulton, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998, xiv+470 pages.

[13] D. Greb, S. Kebekus & T. Peternell, "Movable curves and semistable sheaves", To appear in *Int Math Res Notices*, `http://arxiv.org/abs/1408.4308`, 2015.

[14] D. Greb, S. Kebekus, T. Peternell & B. Taji, "The Miyaoka–Yau inequality and uniformisation of canonical models", `http://arxiv.org/abs/1511.08822`, 2015.

[15] A. Grothendieck, "Éléments de géométrie algébrique IV, Étude locale des schémas et des morphismes de schémas (Seconde Partie)", *Inst. Hautes Études Sci. Publ. Math.* (1965), no. 24, p. 231.

[16] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, New York-Heidelberg, 1977, xvi+496 pages.

[17] A. Höring, "On a conjecture of Beltrametti and Sommese", *J. Algebraic Geom.* **21** (2012), no. 4, p. 721-751.

[18] D. Huybrechts & M. Lehn, *The geometry of moduli spaces of sheaves*, second ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2010, xviii+325 pages.

[19] Y. Kawamata, "On the plurigenera of minimal algebraic 3-folds with $K \equiv 0$", *Math. Ann.* **275** (1986), no. 4, p. 539-546.

[20] S. Keel, K. Matsuki & J. McKernan, "Log abundance theorem for threefolds", *Duke Math. J.* **75** (1994), no. 1, p. 99-119.

[21] J. Kollár & T. Matsusaka, "Riemann–Roch type inequalities", *Amer. J. Math.* **105** (1983), no. 1, p. 229-252.

[22] J. Kollár (ed.), *Flips and abundance for algebraic threefolds*, Astérisque, vol. 211, Société Mathématique de France, Paris, 1992, papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, 1-258 pages.

[23] J. Kollár & S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, viii+254 pages.

[24] A. Langer, "Semistable sheaves in positive characteristic", *Ann. Math. (2)* **159** (2004), no. 1, p. 251-276.

[25] R. Lazarsfeld, *Positivity in algebraic geometry. I. Classical setting: line bundles and linear series*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 48, Springer-Verlag, Berlin, 2004, xviii+387 pages.

[26] ———, *Positivity in algebraic geometry. II. Positivity for vector bundles, and multiplier ideals*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas.

3rd Series. A Series of Modern Surveys in Mathematics], vol. 49, Springer-Verlag, Berlin, 2004, xviii+385 pages.

[27] S. S.-Y. Lu & Y. Miyaoka, "Bounding codimension-one subvarieties and a general inequality between Chern numbers", *Amer. J. Math.* **119** (1997), no. 3, p. 487-502.

[28] K. Matsuki, *Introduction to the Mori program*, Universitext, Springer-Verlag, New York, 2002, xxiv+478 pages.

[29] V. B. Mehta & A. Ramanathan, "Semistable sheaves on projective varieties and their restriction to curves", *Math. Ann.* **258** (1981/82), no. 3, p. 213-224.

[30] Y. Miyaoka, "The Chern classes and Kodaira dimension of a minimal variety", in *Algebraic geometry, Sendai, 1985*, Adv. Stud. Pure Math., vol. 10, North-Holland, Amsterdam, 1987, p. 449-476.

[31] N. Nakayama, *Zariski-decomposition and abundance*, MSJ Memoirs, vol. 14, Mathematical Society of Japan, Tokyo, 2004, xiv+277 pages.

[32] Y.-T. Siu, "Invariance of plurigenera", *Invent. Math.* **134** (1998), no. 3, p. 661-673.

Manuscrit reçu le 1er octobre 2019,
révisé le 23 novembre 2021,
accepté le 1er août 2022.

Erwan ROUSSEAU
Université de Brest
CNRS UMR 6205
Laboratoire de Mathématiques de Bretagne Atlantique
F-29200 Brest (France)
erwan.rousseau@univ-brest.fr
http://eroussea.perso.math.cnrs.fr/

Behrouz TAJI
School of Mathematics and Statistics – Red Centre
The University of New South Wales
NSW 2052 (Australia)
b.taji@unsw.edu.au
https://web.maths.unsw.edu.au/~btaji/