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LIMITS OF INVERSE SYSTEMS OF MEASURES

by Donald J. MALLORY and Maurice SION

Introduction.

Inverse (or projective) systems of measure spaces (see definition 1.1. Sec. I) are used in many areas of mathematics, for example in problems connected with stochastic processes, martingales, etc. One of the first (implicit) uses was made by Kolmogoroff [9], to obtain probabilities on infinite Cartesian product spaces. The concept was later studied explicitly by Bochner, who called such systems stochastic families (see [3]). Since then, inverse systems of measure spaces have been the subject of a number of inbestigations (see e.g. Choksi [5], Metivier [12], Meyer [13], Raoult [16], Scheffer [17], [18]).

The fundamental problem in all of these investigations is that of finding a "limit" for an inverse system of measure spaces (X, p, μ, I). All previous workers in this field have concentrated on getting an appropriate 'limit' measure on the inverse limit set L.

Such an approach presents some serious difficulties, e.g. L may be empty. In this paper, we avoid dependence on L, and hence many of these difficulties, by constructing on the Cartesian product $\widetilde{X}$ of the X_i's, a 'limit' measure $\widetilde{\mu}$ which retains essential features of the usual limit measure. As a result, we are able to get existence theorems with considerably fewer conditions on the systems. (in a recent paper [18], C.L. Scheffer also gets away from dependence on L by working on an abstract representation space. His methods, however, seem to be very different from ours).

Since $L \subset \widetilde{X}$, we investigate the more standard inverse limit measure from the point of view of restricting $\widetilde{\mu}$ to L. This enables us not only to extend known results but also to give a better indication of the reasons for some of the difficulties connected with the standard inverse limit measures. For example, when the index set I is uncountable, L is never $\widetilde{\mu}$-measurable (theorem 2.7, Sec. III).

The topological properties of the limit measures are not discussed in detail (see however 3. Sec. II). They will be examined in a later paper.

Sec. 0 consists of preliminaries in set theory and measure theory. In Sec. I we present our concepts of inverse limits, and in Sec. II we construct our basic limit measure $\widetilde{\mu}$, and develop some of its properties. In Sec. III we apply our results in Sec. II to the standard limit measure problems and show that substantial results can be obtained by using this approach.

0. Preliminaries.

In this section we give such definitions and notation as are of a routine nature.

1. *Set Theory.*

Most of our set theoretic work will use standard notation : $\emptyset$ will represent the empty set, ω the set of non-negative integers, $A \sim B$ and $A \Delta B$ the difference and symmetric differences of the sets A and B. Card (A) denotes the cardinality of the set A. For a family of sets $\mathcal{H}$, $\mathcal{H}_\sigma$ and $\mathcal{H}_\delta$ will represent the families generated from $\mathcal{H}$ by taking countable unions and countable intersections respectively.

For a system of spaces $\{X_i\}_{i \in I}$, $\prod_{i \in I} X_i$ will denote the Cartesian product and π_i will be used for projection onto the space X_i. If, for such a system of spaces, we are given for every $i \in I$ a family $\mathcal{H}_i$ of subsets of X_i, we will say

1.1. $\mathcal{H}$ is a system of set families for $\{X_i\}_{i \in I}$.

For such a system of set families $\mathcal{H}$, we will wish to deal with the "rectangles" in $\prod_{i \in I} X_i$, with a finite number of bases from the families $\mathcal{H}_i$; we thus define.

1.2. $\mathrm{Rect}(\mathcal{H}) = \left\{\alpha : \alpha = \prod_{i\in I} A_i,\ A_j = X_j \text{ for all but a finite number of } j \in I, \text{ and } A_j \in \mathcal{H}_j \text{ otherwise}\right\}$.

Also, for any $\alpha \subset \prod_{i\in I} X_i$ we define.

1.3. $J_\alpha = \{j \in I : \pi_j(\alpha) \neq X_j\}$.

(If α is a "rectangle" J_α is the set of indices of its base).

2. *Measure Theory.*

We use Carathéodory measures throughout and will use the following definitions (See e.g. Sion [19], [20]).

2.1. μ is a Carathéodory measure on X iff μ is a real valued function on the family of subsets of X such that $\mu(\emptyset) = 0$ and

$$0 \leqslant \mu(A) \leqslant \sum_{n\in\omega} \mu(B_n) \leqslant \infty$$

whenever $A \subset \bigcup_{n\in\omega} B_n$.

For such a Caratheodory measure μ on X :

2.2. $A \subset X$ is μ-measurable iff for every $T \subset X$,

$$\mu(T) = \mu(T \cap A) + \mu(T \sim A) .$$

2.3. $\mathfrak{M}_\mu = \{A \; ; \; A \text{ is } \mu\text{-measurable}\}$.

2.4. μ is carried by $A \subset X$ iff $\mu(X \sim A) = 0$.

2.5. μ is pseudo-carried by $A \subset X$ iff $\mu(B) = 0$ whenever $B \in \mathfrak{M}_\mu$ and $B \subset X \sim A$.

2.6. The restriction of μ to $A \subset X$, $\mu|A$, is the measure ν on X defined by $\nu(B) = \mu(B \cap A)$ for all $B \subset X$.

2.7. μ is an outer measure on X iff for every $A \subset X$ there exists $B \subset X$ such that $B \in \mathfrak{M}_\mu$, $A \subset B$ and $\mu(A) = \mu(B)$.

2.8. μ is a semifinite outer measure on X iff μ is an outer measure on X and for every $A \subset X$

$$\mu(A) = \sup\,\{\mu(B) : B \subset A\,,\ \mu(B) < \infty\}\,.$$

2.9. μ is the set function on X generated by g and $\mathcal{H}$ iff $\mathcal{H}$ is a family of subsets of X, $g(H) \geqslant 0$ for every $H \in \mathcal{H}$, and for every $A \subset X$,

$$\mu(A) = \text{int}\left\{ \sum_{H\in\mathcal{H}'} g(H) : \mathcal{H}' \subset \mathcal{H}, \text{Card}(\mathcal{H}') \leqslant \aleph_0, \text{ and } A \subset \bigcup_{H\in\mathcal{H}'} H' \right\}$$

(We will show later that this set function is always a Carathéodory measure).

I. Inverse systems of measures.

In this section we give our definitions of inverse (or projective) systems of measures and define several limits of such systems. We also discuss briefly the reasons for the definitions and the relationships between the limits.

1. *Principal Definitions.*

1.1. (X, p, μ, I) is an *inverse system of outer measures* (a system) iff

1.1.1. I is a directed set (by $\prec$).

1.1.2. $X = \{X_i\}_{i\in I}$, where for each $i \in I$, X_i is a space.

1.1.3. $\mu = \{\mu_i\}_{i\in I}$, where for each $i \in I$, μ_i is an outer measure on X_i. (The μ_i-measurable sets will be denoted by $\mathfrak{M}_i$ instead of $\mathfrak{M}_{\mu_i}$).

1.1.4. p_{ij} is a measurable function from X_j to X_i (i.e. for every $A \in \mathfrak{M}_i$, $p_{ij}^{-1}[A] \in \mathfrak{M}_j$) whenever $i, j \in I$ and $i \prec j$.

1.1.5. for $i, j \in I$ with $i \prec j$, and $A \in \mathfrak{M}_i$,

$$\mu_j(p_{ij}^{-1}[A]) = \mu_i(A)$$

and if $i \prec j \prec k$,

$$\text{(a)} \qquad \mu_k(p_{ik}^{-1}[A] \,\Delta\, (p_{ij} \circ p_{jk})^{-1}[A]) = 0\ .$$

We will usually refer to any (X, p, μ, I) satisfying 1.1 as a *system.*

1.2. ν is an *inverse limit outer measure* for (X, p, μ, I) iff (X, p, μ, I) is a system and

1.2.1. ν is an outer measure on $\prod_{i\in I} X_i$.

1.2.2. ν is carried by

$$L = \{x \in \Pi X_i : p_{ij}(x_j) = x_i \quad \text{for all} \quad i, j \in I, i \prec j\}$$

(the "inverse limit set").

1.2.3. $\pi_i^{-1}[A] \in \mathfrak{M}_\nu$ for every $i \in I$ and $A \in \mathfrak{M}_i$.

1.2.4. $\nu(\pi_i^{-1}[A]) = \mu_i(A)$.

for every $i \in I$ and $A \in \mathfrak{M}_i$.

1.3. ν is a *π-limit outer measure* for (X, p, μ, I) iff (X, p, μ, I) is a measure system and

1.3.1. ν is an outer measure on $\prod_{i \in I} X_i$:

1.3.2. $\pi_i^{-1}[A] \in \mathfrak{M}_\nu$ for every $i \in I$ and $A \in \mathfrak{M}_i$.

1.3.3. $\nu(\pi_i^{-1}[A]) = \mu_i(A)$

and

(b) $$\nu(\Pi_i^{-1}[A] \,\Delta\, (p_{ij} \circ \Pi_j)^{-1}[A]) = 0$$

whenever $i, j \in I$, $i \prec j$ and $A \in \mathfrak{M}_i$.

2. *Remarks.*

2.1. (The definitions) The usual definition of an inverse system of measures (see e.g.) Choksi [5], Metivier [12]) has in place of condition (a) in 1.1.5,

(c) for $i \prec j \prec k$,

$$p_{ik} = p_{ij} \circ p_{ik} \cdot$$

One advantage of using (c) and the "inverse limit set", is that if functions on different spaces X_i of the system are transferred to the inverse limit set L, and if an inverse limit measure exists on L, it is relatively a straightforward task to deal with the limits, integrals etc. of the functions. However further strong conditions on the system are required to assure that L is large enough for this to be successful.

In this paper we are able to avoid strong requirements on the system by placing a "limit" measure on the entire product space $\prod_{i \in I} X_i$ rather than on L. This creates some difficulty in dealing with functions defined on different spaces x_i since the straightforward way of transferring them to $\prod_{i \in I} X_i$ (by using projection mappings) results in values which may be inappropriate. Specifically, even if for $i \prec j$ we have

$$f_j = f_i \circ p_{ij} ,$$

the functions induced on $\prod_{i \in I} X_i$ by

$$\widetilde{f_k} = f_k \circ \pi_k$$

are quite different. Condition (b) is sufficient to assure us that the set of points on which any two such functions differ will have π-limit measure zero.

Since we are weakening the sense in which we obtain a "limit" measure (i.e. π-limit measure) we also weaken the requirements on the system by using condition (a) instead of (c) in our definition.

2.2. (Relations between the limits). It is clear from the definitions that an inverse limit outer measure is a π-limit outer measure. However there exist systems for which π-limit outer measures exist but no inverse limit outer measure exists (Mallory [10]).

π-limit outer measures are not in general closely related to the inverse limit set L. A special type of π-limit outer measure which is more closely related to L and inverse limit measures is the following([1]).

ν is a *generalized inverse limit outer measure* iff ν is a π-limit outer measure and if for every i, $i \in I$ with $i \prec j$, ν is carried by

$$\left\{ x \in \prod_{i \in I} X_i : p_{ij}(x_j) = x_i \right\} .$$

It is not difficult to show that for systems satisfying sequential maximality the existence of a generalized inverse limit outer measure implies the existence of an inverse limit outer measure. The example referred to above satisfies sequential maximality (see definition 1.3 Sec. III) thus a system may have a π-limit measure without having a generalized inverse limit outer measure.

([1]) We are indebted to the referee for suggesting this possibility.

II. Construction of the largest Π-limit outer measure, $\widetilde{\mu}$

In this section we first construct, for a given system (X, p, μ, I), a set function g on the "measurable" rectangles of $\prod_{i \in I} X_i$ in such a way that g has, on these sets, the values that would be required of a π-limit outer measure. From g we then generate the outer measure $\widetilde{\mu}$ which is our "candidate" for a π-limit outer measure and show that if a π-limit outer measure for (X, p, μ, I) exists, then $\widetilde{\mu}$ is the largest such measure. We then prove that it is sufficient to have only a weak "compactness" condition on the measures μ_i in order to guarantee that $\widetilde{\mu}$ s a π-limit outer measure, and we show that $\widetilde{\mu}$ then has substantial approximation properties.

1. *Generation of the measure* $\widetilde{\mu}$.

We begin with the following lemma concerning the values taken by π-limit outer measures on measurable rectangles.

1.1. LEMMA. – *Let* (X, p, μ, I) *be a system and let* ν *be an outer measure on* $\prod_{i \in I} X_i$. *Then* ν *satisfies condition* 1.3.3 *Sec.* I *iff for every* $\alpha \in Rect(\mathfrak{M})$ *and* $j \in I$ *with* $i \prec j$ *for every* $i \in J_a$, *we have :*

$$(*) \qquad \nu(\alpha) = \mu_j\Big(\bigcap_{i \in J_a} p_{ij}^{-1}[\pi_i[\alpha]]\Big) .$$

Proof of 1.1. – Let $A_i = \mu_i[\alpha]$, $J = J_a$ so that

$$\alpha = \bigcap_{i \in J} \pi_i^{-1}[A_i] .$$

Choose $j \in I$ so that $j \succ i$ for all $i \in J$ and let

$$B = \bigcap_{i \in J} p_{ij}^{-1}[A_i] .$$

Then

$$\nu(\pi_j^{-1}[B]) = \mu_j(B) ,$$

$$\pi_j^{-1}[B] \subset \alpha \cup \bigcup_{i \in J} ((p_{ij} \circ \pi_j)^{-1}[A_i] \, \Delta \pi_i^{-1}[A_i]) ,$$

$$\alpha \subset \pi_j^{-1}[B] \cup \bigcup_{i \in J} (\pi_i^{-1}[A_i] \sim (p_{ij} \circ \pi_j)^{-1}[A_i]) .$$

Since for each $i \in \mathrm{J}$,

$$\nu(\pi_i^{-1}[\mathrm{A}_i]\,\Delta\,(p_{ij}\circ\pi_j)^{-1}[\mathrm{A}_i]) = 0\ ,$$

we conclude

$$\nu(\alpha) = \nu(\pi_j^{-1}[\mathrm{B}]) = \mu_j(\mathrm{B})\ .$$

Conversely suppose for each $\alpha \in \mathrm{Rect}(\mathfrak{M})$, $\nu(\alpha) = \mu_j(\mathrm{B})$ where j and B are as above. Then for any $i \in \mathrm{I}$ and $\mathrm{A} \in \mathfrak{M}_i$, letting $\alpha = \pi_i^{-1}[\mathrm{A}]$ and $j = i$, we have $\mathrm{B} = \mathrm{A}$ and therefore

$$\nu(\pi_i^{-1}[\mathrm{A}]) = \mu_i(\mathrm{A})\ .$$

Moreover, for any $j \succ i$, letting $\mathrm{A}_i = \mathrm{A}$,

$$\mathrm{A}_j = \mathrm{X}_j \sim p_{ij}^{-1}[\mathrm{A}]\ ,$$

and

$$\begin{aligned}\alpha &= \pi_i^{-1}[\mathrm{A}_i] \cap \pi_j^{-1}[\mathrm{A}_j]\\ &= \pi_i^{-1}[\mathrm{A}] \sim (p_{ij}\circ\pi_j)^{-1}[\mathrm{A}]\ ,\end{aligned}$$

we get

$$\mathrm{B} = \mathrm{A}_j \cap p_{ij}^{-1}[\mathrm{A}_i] = \emptyset$$

and therefore

$$\nu(\pi_i[\mathrm{A}] \sim (p_{ij}\circ\pi_j)^{-1}[\mathrm{A}]) = \mu_j(\mathrm{B}) = 0\ .$$

Similarly, letting $\mathrm{A}_i = \mathrm{X}_i \sim \mathrm{A}$ and $\mathrm{A}_j = p_{ij}^{-1}[\mathrm{A}]$ we get

$$\nu((p_{ij}\circ\pi_j)^{-1}[\mathrm{A}] \sim \pi_i[\mathrm{A}]) = 0\ .$$

Thus

$$\nu(\pi_i^{-1}[\mathrm{A}]\,\Delta\,(p_{ik}\circ\pi_k)^{-1}[\mathrm{A}]) = 0$$

whenever

$$k \succ i \quad \text{and} \quad \mathrm{A} \in \mathfrak{M}_i\ .$$

We use condition (*) to define a set function g on $\mathrm{Rect}(\mathfrak{M})$ with which we generate a candidate, $\widetilde{\mu}$, for a π-limit outer measure.

The process we use to generate $\widetilde{\mu}$ is motivated by the following theorem due to Carathéodory. We will also use it to establish properties of $\widetilde{\mu}$.

1.2. Theorem. – *For any non-negative set function h and family $\mathcal{H}$ of subsets of a space* X, *the set function μ, generated by h and $\mathcal{H}$ has the following properties.*

1.2.1. *μ is a Carathéodory measure,* 1.2.2. *if* $A \subset A'$ *for some* $A' \in \mathcal{H}_\sigma$ *(in particular if* $\mu(A) < \infty$*), then for every* $\varepsilon > 0$ *there exists* $B \in \mathcal{H}_\sigma$ *such that* $A \subset B$ *and*

$$\mu(B) \leqslant \mu(A) + \varepsilon$$

and hence there exists $B' \in \mathcal{H}_{\sigma\delta}$ *such that* $A \subset B'$ *and* $\mu(A) = \mu(B')$.

1.2.3. *if* $\mathcal{H}$ *is a ring and* g *is finitely additive on* $\mathcal{H}$ *then* $\mathcal{H} \subset \mathfrak{M}_\mu$. *(Hence in view of* 1.2.2, μ *is an outer measure).*

1.2.4. *if* g *is countably subadditive on* $\mathcal{H}$, *then* $\mu(A) = g(A)$ *for every* $A \in \mathcal{H}$.

Proof of 1.2. – See Sion [20] for 1.2.1, 1.2.3, 1.2.3.

Since we always have $\mu(H) \leqslant g(H)$ for $H \in \mathcal{H}$; 1.2.4 follows immediately.

1.3. DEFINITIONS. – Let (X, p, μ, I) be a system.

1.3.1. For $\alpha \in \mathrm{Rect}(\mathfrak{M})$,

$$g(\alpha) = \mu_j \Big(\bigcap_{i \in J_\alpha} p_{ij}^{-1} [\pi_i[\alpha]] \Big)$$

where j is any element of I such that $j \succ i$ for every $i \in J_\alpha$. (Note that J_α is finite and that in view of condition 1.1.5 (a) Sec. I of the definition of a system, g is independent of the choice of j).

1.3.2. $\tilde{\mu}$ is the Carathéodory measure on $\prod_{i \in I} X_i$ generated by g and $\mathrm{Rect}(\mathfrak{M})$.

With no further conditions we have the following lemma.

1.4. LEMMA. – *For any system* (X, p, μ, I) :

1.4.1. $g(\pi_i^{-1}[A]) = \mu_i(A)$ *for every* $i \in I$ *and* $A \in \mathfrak{M}_i$,

1.4.2. *If* J *is a finite directed subset of* I, *and* $\alpha \in \mathrm{Rect}(\mathfrak{M})$; *then* $g(\alpha) = 0$ *whenever*

$$\alpha \cap \Big\{ x \in \prod_{i \in I} X_i \ : p_{ij}(x_j) = x_i \quad \textit{for} \quad i, j \in J, i \prec j \Big\} = \emptyset .$$

In particular,

$$g(\pi_i^{-1}[A] \cap (p_{ij} \circ \pi_j)^{-1}[X_i \sim A]) = 0 \quad \text{whenever}$$
$$i \prec j \quad \text{and} \quad A \in \mathfrak{M}_i .$$

1.4.3. g is finitely additive on $\mathrm{Rect}(\mathfrak{M})$.

1.4.4. $\tilde{\mu}$ is an outer measure on $\tilde{X}$.

1.4.5. $\mathrm{Rect}(\mathfrak{M}) \subset \mathfrak{M}_{\tilde{\mu}}$

Proof of 1.4.1, 1.4.2. – Similar to proof of 1.1.

To show that g is finitely additive we will use the following lemma.

LEMMA A. – *Let* (X, p, μ, I) *be a system, and let* α, β *be disjoint elements of* $\mathrm{Rect}(\mathfrak{M})$. *Then for any* k *such that* $k \succ i$ *for all* $i \in J_\alpha \cup J_\beta$,

$$A = \bigcap_{i \in J_\alpha} p_{ik}^{-1}[\pi_i[\alpha]] \cap \bigcap_{i \in J_\beta} p_{ik}^{-1}[\pi_i[\beta]] = \emptyset .$$

Proof. – Since $\alpha \cap \beta = \emptyset$, there exists $j \in J_\alpha \cap J_\beta$ such that $\pi_j[\alpha] \cap \pi_j[\beta] = \emptyset$. If, however, there exists $x \in A$, then

$$p_{jk}(x) \in \pi_j[\alpha] \cap \pi_j[\mathcal{B}] = \emptyset .$$

Hence no such x exists.

Proof of 1.4.3. – Let $\alpha \in \mathrm{Rect}(\mathfrak{M})$ and let $\beta \subset \mathrm{Rect}(\mathfrak{M})$ be a finite disjoint family such that $\alpha = \bigcup_{\beta \in \mathcal{B}} B$. Let $K = \bigcup_{\beta \in \mathcal{B}} J_B$ and let $j \in I$ be such that $j \succ i$ for every $i \in K$. Then $J_\alpha \subset K$ so that $j \succ i$ for every $i \in J_\alpha$.

Also let

$$A = \bigcap_{i \in J_\alpha} p_{ij}^{-1}[\pi_i[\alpha]]$$

and for every $B \in \mathcal{B}$,

$$A_B = \bigcap_{i \in J_B} p_{ij}^{-1}[\pi_i[B]] .$$

Then $\{A_B : B \in \mathcal{B}\}$ is a finite disjoint subfamily of $\mathfrak{M}_j$ (from lemma A). Since for every $i \in J_\alpha$ and $B \in \mathcal{B}$, $\pi_i[B] \subset \pi_i[\alpha]$ and $J_\alpha \subset J_B$, then $A_B \subset A$, hence $\bigcup_{B \in \mathcal{B}} A_B \subset A$.

On the other hand, for every $x \in A$, choosing $y \in \prod_{i \in I} X_i$ so that $y_i = p_{ij}(x)$ for every $i \in K$, we see that $y \in \alpha$ and thus $y \in B$ for some $B \in \mathcal{B}$. Then

$$\{x\} \subset \bigcap_{i \in J_B} p_{ij}^{-1}[\pi_i(y)] \subset A_B ,$$

hence $x \in A_B$ and

$$A \subset \bigcup_{B \in \mathcal{B}} A_B .$$

Thus

$$g(\alpha) = \mu_j(A) = \sum_{B \in \mathcal{B}} \mu_j(A_B) = \sum_{B \in \mathcal{B}} g(B) .$$

Proof of 1.4.4, 1.4.5. – We first note that we can extend g to a finitely additive function g^* on the ring $\mathcal{R}$ generated by (Rect($\mathfrak{M}$)) (see e.g. J. Kingman and S. Taylor [8] p. 65). Now, the Carathéodory measure $\widetilde{\mu}$ generated by g and Rect($\mathfrak{M}$) is the same as that generated by g^* and $\mathcal{R}$, hence by theorem 1.2, $\widetilde{\mu}$ is an outer measure and $\mathcal{R} \subset \mathfrak{M}_{\widetilde{\mu}'}$.

1.5. *Remarks.* – From the definitions involved and the above results we have the following facts :

1.5.1. If $\widetilde{\mu}$ agrees with g on Rect($\mathfrak{M}$), then $\widetilde{\mu}$ is a π-limit outer measure (note that for $i \in I$ and $A \in \mathfrak{M}_i$,

$\pi_i^{-1}[A] \,\Delta\, (p_{ij} \circ \pi_j)^{-1}[A] =$

$(\pi_i^{-1}[A] \cap (p_{ij} \circ \pi_j)^{-1}[X_i \sim A]) \cup (\pi_i^{-1}[X_i \sim A] \cap (p_{ij} \circ \pi_j)^{-1}[A])$.

1.5.2. If $\widetilde{\mu}$ does not agree with g on Rect($\mathfrak{M}$), then no π-limit outer measure can exist, since g could not be countably subadditive on Rect($\mathfrak{M}$) and hence no outer measure could agree with g.

1.5.3. If ν is a π-limit outer measure then for any

$$A \subset \prod_{i \in I} X_i , \ \nu(A) \leqslant \widetilde{\mu}(A) ,$$

for otherwise there would exist a countable subfamily $\mathcal{F}$ of Rect($\mathfrak{M}$) which covers A and $\sum_{F \in \mathcal{F}} \nu(F) < \nu(A)$, which is impossible. Thus if π-limit outer measures exist, $\widetilde{\mu}$ *is the largest one.*

2. *Conditions for $\widetilde{\mu}$ to be a π-limit outer measure.*

The problem of showing that π-limit outer measures exist is now reduced to that of assuring the countable subadditivity of g^*

on the ring $\mathcal{R}$ generated by Rect($\mathfrak{M}$). A condition that will be shown to be sufficient for this purpose is that the measures μ_i can be approximated from below by "$\aleph_0$-compact" families. These families were first used for similar purposes by Marczewski [11] and have been used by many authors since (Choksi [5], Metivier [12], Meyer [13], Neveu [14], etc.).

2.1. DEFINITION. – *For any family of sets* $\mathcal{C}$, $\mathcal{C}$ *is* $\aleph_0$*-compact if* $\emptyset \in \mathcal{C}$ *and for every countable* $\mathcal{C}' \subset \mathcal{C}$ *with* $\bigcap_{C \in \mathcal{C}'} C = \emptyset$ *there exists a finite sub-family* $\mathcal{T} \subset \mathcal{C}'$ *such that* $\bigcap_{T \in \mathcal{T}} T = \emptyset$.

We will need certain properties of $\aleph_0$-compact families which are given in the following lemmas.

2.2. LEMMA. – *If* $\mathcal{C}$ *is* $\aleph_0$*-compact then :*

2.2.1. $\mathcal{C}_\delta$ *is* $\aleph_0$*-compact.*

2.2.2. *every subfamily of* $\mathcal{C}$ *is* $\aleph_0$*-compact.*

2.2.3. *The family of finite unions of elements of* $\mathcal{C}$ *is* $\aleph_0$*-compact.*

Proof of 2.2.1, 2.2.2. – Immediate from the definitions.

Proof of 2.2.3. – See Meyer [13],

2.3. LEMMA. – *Let* $\{X_i\}_{i \in I}$ *be a system of spaces, and let* $\mathcal{C}$ *be a system of families of sets for* $\{X_i\}_{i \in I}$ *such that for each* $i \in I$, $\mathcal{C}_i$ *is* $\aleph_0$*-compact. Then Rect*$(\mathcal{C})$ *is* $\aleph_0$*-compact.*

Proof of 2.3. – See Neveu [14] p. 85.

We now make precise the sense in which we refer to approximation from below.

2.4. DEFINITION. – *Let* h *be a non-negative set function on a family* $\mathcal{H}$ *of subsets of a space* X, *and* μ *an outer measure on* X.

2.4.1. $\mathcal{C}$ *is an inner family for* h *on* $\mathcal{H}$ *iff* $\mathcal{C} \subset \mathcal{H}$ *and when* $H \in \mathcal{H}$,

$$h(H) = \sup\{h(C) : C \in \mathcal{C}, C \subset H\}.$$

2.4.2. C *is an inner family for* μ *iff* $\mathcal{C}$ *is an inner family for* μ *on* $\mathfrak{M}_\mu$.

The next theorem shows the role played by our $\aleph_0$-compact families.

2.5. THEOREM. — *Let* g *be a non-negative finitely additive set function on a ring* $\mathcal{H}$ *of subsets of a space* X.

If there exists an $\aleph_0$*-compact subfamily* $\mathcal{C} \in \mathcal{H}$ *which is an inner family for* g *on* $\mathcal{H}$, *and if* $g(\mathrm{C}) < \infty$ *for every* $\mathrm{C} \in \mathcal{C}$, *then the outer measure* μ *on* X *generated by* g *and* $\mathcal{H}$ *has the following properties :*

2.5.1. μ *is an extension of* g, i.e. *for all* $\mathrm{H} \in \mathcal{H}$, $\mu(\mathrm{H}) = g(\mathrm{H})$;

2.5.2. $\mathcal{H} \in \mathfrak{M}_\mu$,

2.5.3. *for every* $\mathrm{A} \in \mathfrak{M}_\mu$ *with* $\mu(\mathrm{A}) < \infty$,

$$\mu(\mathrm{A}) = \sup\{\mu(\mathrm{C}) : \mathrm{C} \in \mathcal{C}_\delta , \mathrm{C} \subset \mathrm{A}\} .$$

Proof of 2.5. — Let $\mathrm{H} \in \mathcal{H}$ be such that $g(\mathrm{H}) < \infty$, and let

$$\mathcal{H}' = \{\mathrm{G} \in \mathcal{H} : \mathrm{G} \subset \mathrm{H}\} .$$

Then by a result of Marczewski [11]

i) g can be extended to a countably additive set function ν on the σ-ring $\mathcal{S}$ generated by $\mathcal{H}'$.

ii) for every $\mathrm{G} \in \mathcal{S}$,

$$\nu(\mathrm{G}) = \sup\{\mu(\mathrm{C}) : \mathrm{C} \subset \mathrm{G}, \mathrm{C} \in \mathcal{C}_\delta\}$$

From i) we see that for every $\mathrm{C} \in \mathcal{C}$ and sequence H_n, $n \in \omega$ in $\mathcal{H}$ with $\mathrm{C} \subset \bigcup_{n \in \omega} \mathrm{H}_n$,

$$g(\mathrm{C}) \leqslant \sum_{n \in \omega} g(\mathrm{C} \cap \mathrm{H}_n) \leqslant \sum_{n \in \omega} g(\mathrm{H}_n)$$

and since $\mathcal{C}$ is an inner family for $\mathcal{H}$,

$$\text{if } \mathrm{H} \in \mathcal{H}, \mathrm{H}_n \in \mathcal{H}, \text{ for all } n \in \omega, \text{ and } \mathrm{H} \subset \bigcup_{n \in \omega} \mathrm{H}_n$$

then

$$g(\mathrm{H}) = \sup\{g(\mathrm{C}) : \mathrm{C} \subset \mathrm{H}, \mathrm{C} \in \mathcal{C}\} \leqslant \sum_{n \in \omega} \mathrm{H}_n$$

i.e. g is countably subadditive on $\mathcal{H}$. 2.5.1, and 2.5.2 now follow from Theorem 1.2. From ii) above it is clear that 2.5.3 holds if A is in the σ-ring $\mathcal{T}$ generated by $\mathcal{H}$.

If $B \in \mathfrak{M}_\mu$, $\mu(B) < \infty$ then as a straightforward consequence of 1.2.2

$$\mu(B) = \sup\{\mu(T) : T \in \mathcal{T}\},$$

so that 2.5.3 holds for B also.

The following condition on our systems will allow us to apply the above extension theorem to the set function g^*.

2.6. Definition. – *A system* (X, p, μ, I) *is inner regular relative to* $\mathcal{C}$ *iff* $\mathcal{C}$ *is a system of families of sets for* $\{X_i\}_{i\in I}$ *such that :*

2.6.1. *for every* $i \in I$, $\mathcal{C}_i$ *is an* $\aleph_0$*-compact family of sets which is an inner family for* μ_i,

2.6.2. *for every* $j \in I$ *and* $C \in \mathcal{C}_j$, *and every* $i \prec j$, μ_i *is* σ*-finite on* $p_{ij}[C]$.

It is easily shown that if the above conditions are satisfied then they are also satisfied by the system of families $\mathcal{C}'$, where

$$\mathcal{C}'_i = \{C \in \mathcal{C}_i : \mu_i(C) < \infty\}.$$

Whenever we require the condition in definition 2.7 we shall assume the families $\mathcal{C}_i$ consists of sets of finite measure, and that they are also closed with respect to taking finite unions.

2.7. Theorem. – *If* (X, p, μ, I) *is a system which is inner regular relative to* $\mathcal{C}$ *for some* $\mathcal{C}$, *then* $\tilde{\mu}$ *is a* π*-limit outer measure for* (X, p, μ, I) *on* $\prod_{i\in I} X_i$.

Proof of 2.7. – We first check some approximation properties of g when (X, p, μ, I) is inner regular relative to some system of families of sets $\mathcal{C}$.

Lemma B. – *If* (X, p, μ, I) *is a system which is inner regular relative to* $\mathcal{C}$, *then for any* $\alpha \in \mathrm{Rect}(\mathfrak{M})$, $t < g(a)$ *and* $i \in I$, *there exists* $C \in \mathcal{C}_i$ *such that* $C \subset \pi_i[\alpha]$ *and*

$$g(\alpha \cap \pi_i^{-1}[C]) > t\,.$$

Proof. – Let $k \in I$ with $k \succ j$ for all $j \in J_\alpha \cup \{i\}$, and let

$$A = \bigcap_{j \in J_\alpha} p_{jk}^{-1}[\pi_j[\alpha]] .$$

Then $\mu_k(A) = g(\alpha)$ and there exists $C' \in \mathcal{C}_k$, $C' \subset A$ with $\mu_k(C') > t$. Now choose $B \in \mathcal{M}_i$, $B \subset \pi_i[\alpha]$ such that $p_{ik}[C'] \subset B$ and μ_i is σ-finite on B. Then there exist $C_0, C_1, \ldots \in \mathcal{C}_i$ such that for such $n \in \omega$, $C_n \subset C_{n+1}$, $C_n \subset B$ and

$$\mu_k\Big(B \sim \bigcup_{n \in \omega} C_n\Big) = 0$$

hence

$$\mu_k\Big(C' \sim \bigcup_{n \in \omega} p_{ik}^{-1}[C_n]\Big) = 0$$

so that for some $m \in \omega$

$$\mu_k(C' \cap p_{ik}^{-1}[C_m]) > t .$$

Letting $C = C_m$, we see that $C \subset \pi_i[\alpha]$ and that

$$g(\alpha \cap \pi_i^{-1}[C]) = \mu_k\Big(\bigcap_{j \in J_\alpha} p_{jk}^{-1}[\pi_j[\alpha]] \cap p_{ik}^{-1}[C]\Big)$$

$$\geqslant \mu_k(C' \cap p_{ik}^{-1}[C]) > t .$$

LEMMA C. – *If* (X, p, μ, I) *is a system which is inner regular relative to* $\mathcal{C}$, *then* $\mathrm{Rect}(\mathcal{C})$ *is an inner family for* g *on* $\mathrm{Rect}(\mathcal{M})$.

Proof. – Let $\alpha \in \mathrm{Rect}(\mathcal{M})$ and $J_\alpha = \{i_0, i_1, \ldots, i_n\}$. Given $t < g(\alpha)$,

choose $C_0 \in \mathcal{C}_{i_0}$, $C_0 \subset \pi_{i_0}[\alpha]$ such that

$$g(\alpha \cap \pi_{i_0}^{-1}[C_0]) > t ,$$

and by recursion on m, $C_m \in \mathcal{C}_{i_m}$, $C_m \subset \pi_{i_m}[\alpha]$ such that

$$g\Big(\alpha \bigcap_{l=0}^{m} \pi_{i_1}^{-1}[C_l]\Big) > t .$$

If

$$C = \bigcap_{l=0}^{n} \pi_{i_1}^{-1}[C_1] ,$$

then $C \in \mathrm{Rect}(\mathcal{C})$, $C \subset \alpha$ and $g(C) > t$.

LEMMA D. — *Let g^* be the extension of g to a finitely additive function on the ring $\mathcal{R}$ generated by* Rect($\mathfrak{M}$), *and let* $\mathcal{K} = \{$*finite unions of elements of* Rect($\mathcal{C}$)$\}$. *If* (X, p, μ, I) *is inner regular relative to* $\mathcal{C}$, *then* $\mathcal{K}$ *is an inner family for g^* on* $\mathcal{R}$.

Proof. — Let $\alpha \in \mathcal{R}$. Then there exists a finite disjoint family $\mathcal{B} \subset \mathrm{Rect}(\mathfrak{M})$ such that $\alpha = \bigcup_{B \in \mathcal{B}} B$. If $t < g^*(\alpha)$, choose for each $B \in \mathcal{B}$, $C_B \in \mathrm{Rect}(\mathcal{C})$ so that $C_B \subset B$ and $t < \sum_{B \in \mathcal{B}} g(C_B)$. Then

$$g^*\Big(\bigcup_{B \in \mathcal{B}} C_B\Big) = \sum_{B \in \mathcal{B}} g^*(C_B) = \sum_{B \in \mathcal{B}} g(C_B) > t\,.$$

Theorem 2.7 now follows from theorem 2.5, since $\mathcal{K}$ is $\aleph_0$-compact (by lemmas 2.2 and 2.3).

Without the $\aleph_0$-compactness condition a π-limit outer measure may fail to exist. An example is easily constructed (Mallory [10] from one used by Halmos ([6], p. 214).

3. *Approximation Properties of π-limit Outer Measures.*

We now assume (X, p, μ, I) to be a system which is inner regular relative to some fixed family $\mathcal{C}$ and indicate some approximation properties which $\tilde{\mu}$ then possesses. These properties will be useful when we consider the restriction of $\tilde{\mu}$ to L.

3.1. DEFINITIONS.

3.1.1. $\mathcal{R}$ = *the ring generated by Rect*($\mathfrak{M}$).

3.1.2. $\mathcal{C}' = \{$*finite unions of elements of Rect*($\mathcal{C}$)$\}$.

3.2. THEOREM. — *If* $A \in \mathfrak{M}_{\tilde{\mu}}$, *and* $\tilde{\mu}(A) < \infty$, *then*

$$\mu(A) = \sup\{\tilde{\mu}(C) : C \in \mathcal{C}'_\delta, \subset A\}\,.$$

Proof of 3.2. — Let g^* be the finitely additive extension of g to $\mathcal{R}$. Then $\mathcal{C}'$ is an inner family for g^* on $\mathcal{R}$ (from lemma D in the proof of theorem 2.7), and is $\aleph_0$-compact by lemmas 2.2.3 and 2.3. The result then follows from theorem 2.5, by choosing $\mathcal{C}'$ for the family '$\mathcal{C}$' in theorem 2.5.

Clearly the condition that $\tilde{\mu}(A)$ be finite can be replaced by :

a) $\quad \widetilde{\mu}(\mathrm{A}) = \sup\{\widetilde{\mu}(\mathrm{B}) : \mathrm{B} \subset \mathrm{A}, \widetilde{\mu}(\mathrm{B}) < \infty\}$.

Thus the theorem fails to hold essentially only in the pathological case in which all subsets of A have either infinite measure or measure zero. The following propositions show that under our assumptions such cases are limited and that they do not occur in the sets in which we are primarily interested.

3.3. PROPOSITION. – *For any* $\mathrm{A} \in \mathfrak{M}_{\widetilde{\mu}}$, *at least one of* A, $\widetilde{\mathrm{X}} \sim \mathrm{A}$ *satisfies conditions a) above.*

Proof of 3.3. – If $\widetilde{\mu}(\widetilde{\mathrm{X}}) < \infty$, the result follows from theorem 3.2. Otherwise, for any $i \in \mathrm{I}$, there exists a sequence

$$\mathrm{C}_0, \mathrm{C}_1 \ldots \in \mathcal{C}_i$$

such that

$$\mu_i(\mathrm{X}_i) = \sup\{\mu_i(\mathrm{C}_n) : n \in \omega\} = \infty .$$

Clearly μ_i is σ-finite on $\bigcup_{n \in \omega} \mathrm{C}_n$, so that $\widetilde{\mu}$ is infinite but σ-finite on $\pi_i^{-1}\left[\bigcup_{n \in \omega} \mathrm{C}_n\right]$ $(\in \mathfrak{M}_{\widetilde{\mu}})$. Thus, $\widetilde{\mu}$ is infinite and σ-finite on

$$\mathrm{A} \cap \pi_i^{-1}\left[\bigcup_{n \in \omega} \mathrm{C}_n\right] \quad \text{or on} \quad (\mathrm{X} \sim \mathrm{A}) \cap \pi_i^{-1}\left[\bigcup_{n \in \omega} \mathrm{C}_n\right] ,$$

and hence the result follows easily from theorem 3.2.

3.4. PROPOSITION. – *Every* $\mathrm{A} \in \mathcal{R}$ *satisfies condition* a) *above.*

Proof of 3.4. – By lemma C in the proof of theorem 2.7 for every $\mathrm{A} \in \mathrm{Rect}(\mathfrak{M})$

$$g(\mathrm{A}) = \sup\{g(\mathrm{C}) : \mathrm{C} \subset \mathrm{A}, \mathrm{C} \in \mathrm{Rect}(\mathcal{C})\} ,$$

and since $\mathrm{Rect}(\mathcal{C}) \subset \mathrm{Rect}(\mathfrak{M})$ and $\widetilde{\mu}(\mathrm{A}) = g(\mathrm{A})$ for every

$$\mathrm{A} \in \mathrm{Rect}(\mathfrak{M}) ,$$

then

$$\widetilde{\mu}(\mathrm{A}) = \sup\{\widetilde{\mu}(\mathrm{C}) : \mathrm{C} \in \mathrm{Rect}(\mathcal{C}), \mathrm{C} \subset \mathrm{A}\}$$

and since such sets C have finite measure (from definition 2.6), 3.4 follows easily.

In view of proposition 3.4 we can adjust our measure so as to eliminate the pathological cases.

3.5. THEOREM. – *Let, for every* $A \subset \widetilde{X}$,

$$\widetilde{\mu}'(A) = \sup\{\widetilde{\mu}(B) : B \subset A \quad \textit{and} \quad \widetilde{\mu}(B) < \infty\}\,;$$

then

3.5.1. $\widetilde{\mu}'$ *is a semifinite* π*-limit outer measure,*

3.5.2. $\mathfrak{M}_{\widetilde{\mu}'} = \mathfrak{M}_{\widetilde{\mu}}$.

3.5.3. *for every* $A \in \mathfrak{M}_{\widetilde{\mu}}$,

$$\widetilde{\mu}'(A) = \sup\{\widetilde{\mu}'(C) : C \in \mathcal{C}'_\delta, C \subset A\}.$$

Proof of 3.5. – It is easily seen that $\widetilde{\mu}'$ is an outer measure and that $\mathfrak{M}_{\widetilde{\mu}'} = \mathfrak{M}_{\widetilde{\mu}}$ (see e.g. Berberian [2]). Furthermore it follows from 3.4 that $\widetilde{\mu}'(B) = \widetilde{\mu}(B)$ for every $B \in \mathcal{R}$, so that $\widetilde{\mu}'$ is a π-limit outer measure, 3.5.3 then follows from the fact that every $A \in \mathfrak{M}_{\widetilde{\mu}}$ satisfies condition a) above.

We note that if the spaces X_i were topological, and the measures μ_i Radón (not restricted to locally compact spaces), then we would take for $\mathcal{C}_i$ the closed compact sets. If we then replaced the original topology $\mathcal{G}_i$ on X_i by the topology $\mathcal{G}'_i$ of complements of closed compact sets, μ_i would remain Radón, and the class of sets $\mathcal{C}'_\delta$ (Def. 3.1) would be closed and compact in the resulting product topology. Theorem 3.2 above could then be used to show that under very weak conditions a Radón π-limit outer measure exists on $\prod_{i \in I} X_i$ with the new product topology (Mallory [10]).

III. Limit measures on the inverse limit set.

In this section we try to answer the following question. When does a π-limit outer measure exist on the inverse limit set, L ?

Our approach is from the point of view of restricting $\widetilde{\mu}$ to L.

The proofs of all lemmas and theorems in this section will be given in subsection 3 at the end.

1. *Definitions and Notation.*

Here we collect definitions and notation used in the sequel.

1.1. *Basic Assumptions.* – Throughout this section we assume (X, p, μ, I) is a system and

$$p_{ik} = p_{ij} \circ p_{jk}$$

whenever $i \prec j \prec k$, so that (X, p, I) is actually an inverse system of spaces. We also assume that the inverse limit set, L_I, is such that for every $i \in I$, $\pi_i[L_I] = X_i$ (simple maximality).

1.2. *Definitions (Subsystems).* – For any directed subset J of I,

1.2.1. (X, p, μ, J) will denote the subsystem obtained by restricting X and μ to J and p to

$$\{(i, j) : i \prec j \quad \text{and} \quad i, j \in J\}.$$

Clearly (X, p, μ, J) is also a system.

1.2.2. $\widetilde{X}_J = \prod_{i \in J} X_i$.

(In case $J = I$ we may write $\widetilde{X}$ for $\widetilde{X}_I$).

1.2.3. $L_J = \{x \in \widetilde{X}_J \ : \ \pi_i(x) = p_{ij}(\pi_j(x))$ whenever $i \prec j$ and $i, j \in J\}$.

Thus L_J is the inverse limit set of (X, p, μ, J).

1.2.4. r_J is the function on $\widetilde{X}$ to $\widetilde{X}_J$ such that for every

$$x \in \widetilde{X}, \ r_J(x) = x \mid J .$$

In the next two definitions we introduce properties of the system and of measures which we will use in the theorems to follow.

1.3. DEFINITION. – (X, p, μ, I) *satisfies sequential maximality iff, for every countable directed subset* J *of* I, *the range of* $r_J \mid L_I$ *is all of* L_J, i.e. : *for every sequence* $i_0, i_1, \ldots$ *in* I *with* $i_n \prec i_{n+1}$ *and sequence* y *with* $y_n \in X_{i_n}$ *and* $p_{i_n i_{n+1}}(y_{n+1}) = y_n$ *for every* $n \in \omega$, *there exists* $x \in L_I$ *such that* $x_{i_n} = y_n$ *for every* $n \in \omega$.

1.4. DEFINITION. – *An outer measure* φ *on a space* S *is almost separable iff there exists a countable family* $\mathcal{B} \subset \mathfrak{M}_\varphi$, *and a set* $T \subset S$ *such that* $\varphi(T) = 0$ *and for every* $x, y \in S \sim T$ *with* $x \neq y$ *there exists* $B \in \mathcal{B}$ *with* $x \in B$ *and* $y \notin B$.

The extension process we are using to obtain $\widetilde{\mu}$ is such that $\widetilde{\mu}$

may not be semifinite even if the measures μ_i are semifinite. To simplify some results we will refer to the semifinite part using the following definition. (See also theorem 3.5 Sec. II).

1.5. DEFINITION. – *If ν is an outer measure on a space* S, *ν' is the semifinite outer measure on* S *derived from ν by taking*

$$\nu'(A) = \sup \{\nu(B) : B \subset A \quad \textit{and} \quad \nu(B) < \infty\},$$

for every $A \subset S$.

2. *Existence of an Inverse Limit Measure.*

In this section we consider the problem of the existence of an inverse limit outer measure. We begin by indicating the relation between such a measure and the measure $\tilde{\mu}$ introduced in 1.3 Sec. II.

2.1. LEMMA. – *An inverse limit outer measure exists iff $\tilde{\mu} \mid L_I$ is an inverse limit outer measure.*

2.2. LEMMA. – *If for each $i \in I$, μ_i is a semifinite outer measure, then $\tilde{\mu} \mid L_I$ is an inverse limit outer measure* (i.e. *such a measure exists*) *iff $\tilde{\mu}$ is a π-limit outer measure such that the semifinite outer measure $\tilde{\mu}'$ derived from $\tilde{\mu}$ is pseudo-carried by* L_I.

In view of the above lemmas we devote the rest of this section to determining conditions under which $\tilde{\mu}$ or $\tilde{\mu}'$ is carried or pseudo-carried by L_I. We have two types of conditions under which this occurs and we discuss them separately. First we consider "separability" conditions.

2.3. LEMMA. – *Suppose that for every $i \in I$, μ_i is almost separable.*

2.3.1. *If* I *is countable, then $\tilde{\mu}$ is carried by* L_I.

2.3.2. *If sequential maximality is satisfied and for each $i \in I$, μ_i is semifinite, then $\tilde{\mu}'$ is pseudo-carried by* L_I.

2.4. *Remark.* – 2.3.1 remains true even if (X, p, μ, I) is a system for which (X, p, I) is not necessarily an inverse system of spaces. In effect the rest of the hypotheses force L_I to be large enough to carry $\tilde{\mu}$. By combining lemma 2.3 with the fundamental existence theorem 2.7 of Sec. II we obtain the following theorem.

2.5. THEOREM. – *If* (X, p, μ, I) *is inner relative to* $\mathcal{C}$ *for some* $\mathcal{C}$ *(definition 2.5* Sec. II) *and if for each* $i \in I$, μ_i *is almost separable, then an inverse limit outer measure exists whenever one of the following conditions holds.*

2.5.1. I *is countable,*

2.5.2. (X, p, μ, I) *satisfies sequential maximality.*

2.6. *Remarks.* – Previously known existence theorems require further conditions on the images and inverse images of the functions p_{ij} than are used in theorem 2.5 (see e.g. Choksi [5], Metivier [12]).

In view of lemma 2.3.1 we can conclude that in 2.5.1 $\tilde{\mu}$ is a π-limit outer measure which is carried by L_I and not just pseudo-carried. The following theorem shows that this is not the case for any nontrivial system when I is uncountable.

2.7. THEOREM. – *If* I *is uncountable and* X_i *contains at least two points for uncountably many* $i \in I$, *then for every* $A \in \mathrm{Rect}(\mathfrak{M})$,

$$\tilde{\mu}(A) = \tilde{\mu}(A \sim L_I) ,$$

hence $\tilde{\mu}$ *is not carried by* L_I *whenever* $\tilde{\mu} \neq 0$.

From the above theorem we see that in many significant cases where an inverse limit outer measure does exist, L_I is not $\tilde{\mu}$-measurable. This may explain many of the difficulties encountered by inverse limit measures. For example, even if $\tilde{\mu}$ is Radón, its restriction to L_I may not be.

We now examine another type of conditions under which an inverse limit outer measure exists. Here we establish a "topological" relationship between L_I and inner families $\mathcal{C}_i$ for the measures μ_i. Conditions similar to ours have been used by previous workers (e.g. Bochner [3], Choksi [5], Metivier [12]) who worked only with L_I (not considering its relation with $\tilde{X}$).

The following theorem is a basic existence theorem from this point of view.

2.8. THEOREM. — *Let* (X, p, μ, I) *be inner regular relative to* $\mathcal{C}$. *If for every sequence* $i \in I$ *with* $i_n \prec i_{n+1}$ *for all* $n \in \omega$, *the family*

$$\{\pi_{i_n}^{-1}[C] \cap L_I : C \in \mathcal{C}_{i_n} \text{ for some } n \in \omega\}$$

is $\aleph_0$*-compact, then* $\widetilde{\mu}'$ *is pseudo-carried by* L_I, *hence* $\widetilde{\mu}' \mid L_I$ *is an inverse limit outer measure.*

2.9. *Remarks.* — The hypotheses of theorem 2.8 are obviously satisfied if the spaces C_i are compact Hausdorff, the measures μ_i Radón, and the functions p_{ij} continuous. In this case L_I is compact so that

$$\{\pi_i^{-1}[C] \cap L_I : C \in \mathcal{C}_i \quad \text{for some} \quad i \in I\}$$

consists of sets which are compact in the product topology, and thus are certainly $\aleph_0$-compact.

Since in other cases it may be difficult to check the hypotheses of theorem 2.8 directly, we give in the following theorem a condition on countable subsystems which, if sequential maximality holds, will ensure the existence of an inverse limit outer measure. We should note that the following theorem is essentially that of Metivier [12], though we include the semifinite case.

2.10. THEOREM. — *Suppose that* (X, p, μ, I) *is inner regular w.r.t.* $\mathcal{C}$, *and that sequential maximality holds. Then* $\widetilde{\mu}'$ *is pseudo-carried by* L_I *hence* $\widetilde{\mu}' \mid L_I$ *is an inverse limit outer measure, whenever the following conditions hold : if* i *is a sequence in* I *with* $i_n \prec i_{n+1}$ *for every* $n \in \omega$, *and*

$$\mathcal{K}_m = \{p_{i_m i_n}[C] : C \in \mathcal{C}_{i_n} \text{ for some } n \in \omega \text{ with } m \leqslant n\},$$

2.10.1. *then* $\mathcal{K}_m$ *is* $\aleph_0$*-compact for every* $m \in \omega$, *and*

2.10.2. $\{p_{i_l i_m}^{-1}[x] \cap K : K \in \mathcal{K}_m\}$ *is* $\aleph_0$*-compact for every* $l, m \in \omega$ with $l < m$, and $x \in X_{i_l}$.

We conclude this section by indicating how one can transfer an inverse limit outer measure for a system to one for a subsystem, and vice-versa.

2.11. THEOREM. — *Suppose* ν *is an inverse limit outer measure. Then for any directed subset* J *of* I, *the set function* Ψ *generated by the family.*

$$\mathfrak{A} = \{A \subset X_J : r_J^{-1}[A] \in \mathfrak{M}_\nu\}$$

and the set function h on $\mathfrak{A}$, *defined by*

$$h(A) = \nu(r_J^{-1}[A]) \quad \textit{for all} \quad A \in \mathfrak{A} \ ,$$

is an inverse limit outer measure for (X, p, μ, J).

2.12. THEOREM. – *Let* J *be a confinal subset of* I. *Then :*

2.12.1. $r_J \mid L_I$ *is one-to-one and onto* L_J.

2.12.2. *if* ν *is an inverse limit outer measure for* (X, p, μ, J), *the set function* Ψ *defined by*

$$\Psi(A) = \nu(r_J[A \cap L_I])$$

for every $A \subset \widetilde{X}_I$, *is an inverse limit outer measure for* (X, p, μ, I).

From the above theorems we see that a system has an inverse limit outer measure if it can be imbedded in a system which does have one, and that theorems 2.5, 2.8 and 2.10 can be somewhat extended by requiring that their hypotheses be satisfied only for a cofinal subsystem.

3. *Proofs.*

Proof of 2.1. – Clearly we need only to show that if $\widetilde{\mu} \mid L_I$ is not an inverse limit outer measure then no such measure exists. To do this we first establish the following lemma.

LEMMA A. – *Let* $\alpha \in \mathrm{Rect}(\mathfrak{M})$, *and* $k \in I$ *be such that* $k \succ j$ *for all* $j \in J_\alpha$. *If we let*

$$B = \bigcap_{j \in J_\alpha} p_{jk}^{-1}[\pi_j[\alpha]] \ ,$$

then,

3.1. $B \in \mathfrak{M}_k$,

3.2. $\pi_k^{-1}[B] \cap L_I = \alpha \cap L_I$,

3.3. $g(\alpha) = \mu_k(B)$.

Proof. – Immediate from the definitions.

Suppose that $\widetilde{\mu} \mid L_I$ is not an inverse limit outer measure. We know from lemma 1.4.5 Sec. II that $\mathrm{Rect}(\mathfrak{M}) \subset \mathfrak{M}_{\widetilde{\mu}}$ hence for every $i \in I$ and $A \in \mathfrak{M}_i$,

$$\pi_i^{-1}[A] \in \mathfrak{M}_{\tilde{\mu}} \mid L_I .$$

Thus it must be that for some $j \in I$ and $B \in \mathfrak{M}_j$

$$\tilde{\mu} \mid L_I(\pi_j^{-1}[B]) \neq g(\pi_j^{-1}[B]) = \mu_j(B) .$$

Since

$$\tilde{\mu}(\pi_j^{-1}[B]) \leqslant g(\pi_j^{-1}[B])$$

we must have

$$\tilde{\mu} \mid L_I(\pi_j^{-1}[B]) = \tilde{\mu}(\pi_j^{-1}[B] \cap L_I) < \mu_j(B) .$$

Then by the definition of $\tilde{\mu}$, there exists a countable family $\mathcal{D} \subset \mathrm{Rect}(\mathfrak{M})$ such that

$$\pi_j^{-1}[B] \cap L_I \subset \bigcup_{D \in \mathcal{D}} D$$

and

$$\sum_{D \in \mathcal{D}} g(D) < g(\pi_j^{-1}[B]) .$$

Let $\mathcal{D} = \{D_0, D_1, \ldots\}$ and for each $n \in \omega$ let $i_n \in I$ and $B_n \in \mathfrak{M}_{i_n}$ be such that

$$D_n \cap L_I = \pi_{i_n}^{-1}[B_n] \cap L_I$$

and $g(D_n) = \mu_{i_n}(B_n)$ (this is possible by lemma A).

Then

$$\pi_j^{-1}[B] \cap L_I \subset \bigcup_{n \in \omega} \pi_i^{-1}[B_n] \cap L_I$$

and

$$\sum_{n \in \omega} \mu_{i_n}(B_n) < \mu_k[B] .$$

Hence there cannot exist an outer measure ν carried by L_I for which

$$\nu(\pi_i^{-1}[B]) = \mu_i(B)$$

and

$$\nu(\pi_{i_n}^{-1}[B_n]) = \mu_{i_n}(B_n)$$

for every $n \in \omega$, i.e. there cannot exist an inverse limit outer measure.

Proof of 2.2. – Suppose first that $\tilde{\mu}$ is a π-limit outer measure and that $\tilde{\mu}'$ is pseudo-carried by L_I.

Let $A \in \mathfrak{M}_i$. Then

$$\begin{aligned}\tilde{\mu}(\pi_i^{-1}[A]) &= \mu_i(A)\\ &= \sup\{\mu_i(B) : B \subset A,\ B \in \mathfrak{M}_i,\ \mu_i(B) < \infty\}\\ &= \sup\{\tilde{\mu}(\pi_i^{-1}[B]) : B \subset A,\ B \in \mathfrak{M}_i,\ \mu_i(B) < \infty\}\end{aligned}$$

Thus

$$\begin{aligned}\mu_i(A) &= \tilde{\mu}'(\pi_i^{-1}[A]) = \tilde{\mu}'(\pi_i^{-1}[A] \cap L_I)\\ &= \tilde{\mu}(\pi_i^{-1}[A] \cap L_I) = \tilde{\mu} \mid L_I(\pi_i^{-1}[A])\ ,\end{aligned}$$

and thus $\tilde{\mu} \mid L_I$ is an inverse limit outer measure.

Now suppose that $\tilde{\mu} \mid L_I$ is an inverse limit outer measure. Then since an inverse limit outer measure is also a π-limit outer measure it follows from remark 1.5.3 Sec. II that $\tilde{\mu}$ is a π-limit outer measure.

Suppose also that there exists $A \in \mathfrak{M}_{\tilde{\mu}'}\ (= \mathfrak{M}_{\tilde{\mu}})$ such that

$$A \subset \tilde{X} \sim L_I \quad \text{and} \quad 0 < \tilde{\mu}'(A)\ .$$

Then from the definition of $\tilde{\mu}'$ there exists a set $B \subset A$, $B \in \mathfrak{M}_{\tilde{\mu}}$ (since $A \in \mathfrak{M}_{\tilde{\mu}}$) with $0 < \tilde{\mu}(B) < \infty$. Then by definition of $\tilde{\mu}$ there exists for $0 < \varepsilon < \dfrac{\tilde{\mu}(B)}{2}$ a finite family $\mathcal{D} \subset \mathrm{Rect}(\mathfrak{M})$ with

$$\tilde{\mu}\Big(\bigcup_{D \in \mathcal{D}} (D \sim B)\Big) < \varepsilon$$

and

$$\varepsilon < \tilde{\mu}(B) - \varepsilon < \tilde{\mu}\Big(\bigcup_{D \in \mathcal{D}} D\Big)$$

Furthermore we can choose $\mathcal{D}$ to be a disjoint family. Then

$$\sum_{D \in \mathcal{D}} \tilde{\mu}(D \cap L_I) \leqslant \sum_{D \in \mathcal{D}} \tilde{\mu}(D \sim B) < \varepsilon .$$

But, since $\tilde{\mu} \mid L_I$ is an inverse limit outer measure, lemma A shows that $\tilde{\mu}(D \cap L_I) = \tilde{\mu}(D)$ for every $D \in \mathcal{D}$, hence

$$\sum_{D \in \mathcal{D}} \tilde{\mu}(D \cap L_I) = \sum_{D \in \mathcal{D}} \mu(D) = \mu\Big(\bigcup_{D \in} D\Big) > \varepsilon\ .$$

Hence no such sets A, B exist, so $\tilde{\mu}'$ is pseudo-carried by L_I.

Proof of 2.3. – We first establish the following lemma.

LEMMA B. – *Let for each* $i \in I$, μ_i *be almost separable, and let* I_0 *be a countable directed subset of* I. *Then*

$$\widetilde{\mu}(\widetilde{X} \sim r_{I_0}^{-1}[L_{I_0}]) = 0 .$$

Proof. – For each $i \in I_0$ let $T_i \subset X_i$ and $\mathcal{B}_i \subset \mathfrak{M}_i$ be such that $\mu_i(T_i) = 0$, $\mathcal{B}_i$ is countable, and for every $x, y \in X_i \sim T_i$ with $x \neq y$, there exists $B \in \mathcal{B}_i$ such that $x \in B$, $y \notin B$. For each $i, j \in I_0$ with $i \prec j$ and $B \in \mathcal{B}_i$ let

$$B_{ij} = \pi_i^{-1}[B] \cap \pi_j^{-1}[p_{ij}^{-1}[X_i \sim B]] .$$

Then,

$$\widetilde{\mu}(B_{ij}) = \mu_j(p_{ij}^{-1}[B] \cap p_{ij}^{-1}[X_i \sim B]) = g(\emptyset) = 0$$

for every such i, j. Let

$$\mathcal{B}' = \{B_{ij} : i, j \in I_0, i \prec j \quad \text{and} \quad B \in \mathcal{B}_i\} ,$$

then

$$\widetilde{X} \sim r_{I_0}^{-1}[L_I] \subset \bigcup_{B' \in \mathcal{B}'} B'$$

hence

$$\widetilde{\mu}(\widetilde{X} \sim r_{I_0}^{-1}[L_{I_0}]) = 0 .$$

Lemma 2.3.1 follows immediately from lemma B.

To prove 2.3.2 we will use the following lemma.

LEMMA C. – *Let μ_i be almost separable for every $i \in I$ and let sequential maximality hold. Then for every $\alpha \in \mathrm{Rect}(\mathfrak{M})$,*

$$\widetilde{\mu}(\alpha) = \widetilde{\mu}(\alpha \cap L_I) .$$

Proof. – For $\varepsilon > 0$ let $\mathcal{H} \subset \mathrm{Rect}(\mathfrak{M})$ be a countable cover of $\alpha \cap L_I$ such that

$$\sum_{H \in \mathcal{H}} g(H) \leqslant \widetilde{\mu}(\alpha \cap L_I) + \varepsilon ,$$

and let $T = \bigcup_{H \in \mathcal{H}} J_H \cup J_\alpha$ and let $K \subset I$ be a countable directed set with $T \subset K$. By sequential maximality, for each $x \in \alpha \cap r_k^{-1}[L_k]$ there exists $x' \in L_I$ such that for every $k \in K$, $x'_k = x_k$. Then $\pi_j(x') \in \pi_j[\alpha]$ for every $j \in J_\alpha$, hence

$$x' \in \alpha \cap L_I \subset \bigcup_{H \in \mathcal{H}} H ,$$

so that

$$\bigcap_{k \in K} \pi_k^{-1}[\{x'_k\}] \subset \bigcup_{H \in \mathcal{H}} H ,$$

hence $x \in \bigcup_{H \in \mathcal{H}} H$ and

$$\alpha \cap r_K^{-1}[L_K] \subset \bigcup_{H \in \mathcal{H}} H .$$

Thus, using lemma B,

$$\tilde{\mu}(\alpha) \leqslant \tilde{\mu}\Big(\bigcup_{H \in \mathcal{H}} H\Big) + \tilde{\mu}(\tilde{X} \sim r_K^{-1}[L_K])$$
$$\leqslant \tilde{\mu}(\alpha \cap L_I) + \varepsilon + 0 .$$

Since ε is arbitrary, $\tilde{\mu}(\alpha) = \tilde{\mu}(\alpha \cap L_I)$.

Turning now to the proof of 2.3.2, suppose that $A \in \mathfrak{M}_{\tilde{\mu}'}(=\mathfrak{M}_{\tilde{\mu}})$ is such that $A \subset \tilde{X} \sim L_I$ and $\tilde{\mu}'(A) > 0$. Then as in the proof of lemma 2.2 there exists $B \subset A$, $B \in \mathfrak{M}_{\tilde{\mu}}$ with $0 < \tilde{\mu}(B) < \infty$, and, for

$$0 < \varepsilon < \frac{\tilde{\mu}(B)}{2}$$

a finite disjoint $\mathcal{D} \subset \mathrm{Rect}(\mathfrak{M})$ with

$$\tilde{\mu}\Big(\bigcup_{D \in \mathcal{D}} D \sim B\Big) < \varepsilon$$

and

$$\tilde{\mu}(B) - \varepsilon \leqslant \tilde{\mu}\Big(\bigcup_{D \in} D\Big) .$$

Thus we have again

$$\sum_{D \in \mathcal{D}} \tilde{\mu}(D \cap L_I) \leqslant \sum_{D \in \sigma} \tilde{\mu}(D \sim B) < \varepsilon ,$$

and from lemma C,

$$\sum_{D \in \mathcal{D}} \tilde{\mu}(D \cap L_I) = \sum_{D \in \mathcal{D}} \tilde{\mu}(D) > \tilde{\mu}(B) - \varepsilon > \varepsilon ,$$

which is a contradiction. Hence no such A exists and $\tilde{\mu}'$ is pseudo-carried by L_I.

Proof of 2.5. – By theorem 2.7 Sec. II, $\tilde{\mu}$ is a π-limit outer measure, and by lemma 2.3, $\tilde{\mu}'$ is pseudo-carried by L_I under conditions 2.5.1 or 2.5.2. Then lemma 2.2 shows that $\tilde{\mu} \mid L_I$ is an inverse limit outer measure.

Proof of 2.7. – Let $A \in \mathrm{Rect}(\mathfrak{M})$, $\mathcal{H} \subset \mathrm{Rect}(\mathfrak{M})$ be a countable cover of $A \sim L_I$, and let $T = \bigcup_{H \in \mathcal{H}} J_H \cup J_A$ and $x \in A \cap L_I$. Then, since I is uncountable let $i \in \mathrm{I} \sim \mathrm{T}$ and $y \in \mathrm{X}_i$ be such that $y \neq x_i$. If we define $x' \in \widetilde{\mathrm{X}}$ by letting $x'_j = x_j$ for $j \neq i$ and $x'_i = y$, then $x' \in \mathrm{A} \sim \mathrm{L_I}$. Hence $x' \in \bigcup_{\mathrm{H} \in \mathcal{H}} \mathrm{H}$. Thus

$$x \in \bigcap_{j \in \mathrm{T}} \pi_j^{-1}[\{x_j\}] = \bigcap_{j \in \mathrm{T}} \pi_j^{-1}[\{x'_j\}] \subset \bigcup_{\mathrm{H} \in \mathcal{H}} \mathrm{H}$$

Hence $\mathrm{A} \cap \mathrm{L_I} \subset \bigcup_{\mathrm{H} \in \mathcal{H}} \mathrm{H}$ so that $\mathrm{A} \subset \bigcup_{\mathrm{H} \in \mathcal{H}} \mathrm{H}$ and therefore from the definition of $\widetilde{\mu}$,

$$\widetilde{\mu}(\mathrm{A} \sim \mathrm{L_I}) = \widetilde{\mu}(\mathrm{A}) .$$

Proof of 2.8. – By theorem 3.5 Sec. II, $\widetilde{\mu}'$ is a π-limit outer measure, and for every $\mathrm{A} \in \mathfrak{M}_{\widetilde{\mu}'}$

$$\widetilde{\mu}'(\mathrm{A}) = \sup \{\widetilde{\mu}(\mathrm{C}) : \mathrm{C} \subset \mathrm{A}, \mathrm{C} \in \mathcal{C}'_\delta\},$$

where $\mathcal{C}'$ consists of finite unions of elements of $\mathrm{Rect}(\mathcal{C})$ (see definition 3.1 Sec. II).

Suppose $\mathrm{B} \in \mathfrak{M}_{\widetilde{\mu}'}$, $\widetilde{\mu}'(\mathrm{B}) > 0$ and $\mathrm{B} \subset \widetilde{\mathrm{X}} \sim \mathrm{L_I}$. Then there exists for $t < \widetilde{\mu}'(\mathrm{B})$ a sequence $\mathrm{C}_0, \mathrm{C}_1, \ldots$ in $\mathcal{C}'$ such that $\mathrm{C}_{n+1} \subset \mathrm{C}_n$ for each $n \in \omega$ (since $\mathrm{Rect}(\mathcal{C})$ is closed under finite intersection), $\bigcap_{n \in \omega} \mathrm{C}_n \subset \mathrm{B}$, and $\widetilde{\mu}\left(\bigcap_{n \in \omega} \mathrm{C}_n\right) > t$. For each $n \in \omega$ there exists a finite disjoint family $\mathcal{B}_n \subset \mathrm{Rect}(\mathfrak{M})$ such that $\mathrm{C}_n = \bigcup_{\mathrm{B} \in \mathcal{B}_n} \mathrm{B}$. Furthermore we can choose the families $\mathcal{B}$, so that if $m < n$, every $\mathrm{B} \in \mathcal{B}_n$ is a subset of some element of $\mathcal{B}_m$.

Let i_0 be such that $i_0 \succ j$ for every $j \in \bigcup_{\mathrm{B} \in \mathcal{B}_0} \mathrm{J_B}$ and choose by recursion $i_{n+1} \in \mathrm{I}$ so that $i_{n+1} \succ i_n$ and $i_{n+1} \succ j$ for all $j \in \bigcup_{\mathrm{B} \in \mathcal{B}_{n+1}} \mathrm{J_B}$. For each $n \in \omega$ let

$$\mathrm{D}_n = \bigcup_{\mathrm{B} \in \mathcal{B}_n} \bigcap_{j \in \mathrm{J_B}} p_{j i_n}^{-1}[\pi_j[\mathrm{B}]] .$$

Then for each $m, n \in \omega$ with $m < n$

$$\mathrm{D}_n \subset p_{i_m i_n}^{-1}[\mathrm{D}_m]$$

and

$$\mu_{i_m}(\mathrm{D}_m) = \widetilde{\mu}(\mathrm{C}_m) > t$$

since we have

$$\tilde{\mu}(C_m) = \sum_{B \in \mathcal{B}_m} \tilde{\mu}(B) = \sum_{B \in \mathcal{B}_m} g(B)$$

$$= \sum_{B \in \mathcal{B}_m} \mu_{i_m}\left(\bigcap_{j \in J_B} p_{ji_n}^{-1}[\pi_j[B]]\right) = \mu_{i_m}(D_m) \ .$$

Let

$0 < \varepsilon < t/2$ and for each $n \in \omega$ choose $K_n \subset D_n$, $K_n \in \mathcal{C}_{i_n}$ such that

$$\mu_{i_n}(D_n \sim K_n) < \frac{\varepsilon}{2^{n+1}} \ .$$

For each $n \in \omega$ let

$$E_n = \bigcap_{m=0}^{n} p_{i_m i_n}^{-1}[K_m] \ .$$

Then

$$E_n \supset D_n \sim \bigcup_{m=0}^{n} p_{i_m i_n}^{-1}[D_m \sim K_m]$$

hence,

$$\mu_{i_n}(E_n) \geqslant t - \sum_{m=0}^{n} \frac{\varepsilon}{2^{n+1}} > \frac{t}{2} \ .$$

Thus, from simple maximality

$$\pi_i^{-1}[E_n] \cap L_I \neq \emptyset \ .$$

Also from lemma A it is clear that

$$\pi_{i_n}^{-1}[E_n] \cap L_I = \bigcap_{m=0}^{n} \pi_{i_m}^{-1}[K_m] \cap L_I$$

so that

$$\bigcap_{m=0}^{n} \pi_{i_m}^{-1}[K_m] \cap L_I \neq \emptyset \ .$$

for any $n \in \omega$, hence

$$\bigcap_{m \in \omega} \pi_{i_m}^{-1}[K_m] \cap L_I \neq \emptyset \ .$$

But, from lemma A, for every $n \in \omega$

$$\pi_{i_m}^{-1}[K_m] \cap L_I \subset C_m \cap L_I$$

hence

$$\emptyset \neq \bigcap_{n \in \omega} C_n \cap L_I \subset B \cap L_I$$

.

contradicting the fact that $B \subset X \sim L_I$. Hence no such B exists and $\widetilde{\mu}'$ is pseudo-carried by L_I.

Proof of 2.10. – We shall check that the hypothesis of theorem 2.8 is satisfied. Let i be a sequence in I with $i_n \prec i_{n+1}$ for every $n \in \omega$,

$$\mathcal{F} = \{\pi_i^{-1}[C] \cap L_I : C \in \mathcal{C}_{i_n}, n \in \omega\},$$

and F be a sequence in $\mathcal{F}$ with $\bigcap_{m=0}^{n} F_m \neq \emptyset$ for every $n \in \omega$. We have to show that $\bigcap_{m \in \omega} F_m \neq \emptyset$.

For each $m \in \omega$, let $j(m)$ be the smallest integer k with

$$F_m = \pi_{i_k}^{-1}[C] \cap L_I$$

for some $C \in \mathcal{C}_{i_k}$, and let $C_m \in \mathcal{C}_{i_{j(m)}}$ be such that

$$F_m = \pi_{i_k}^{-1}[C_m] \cap L_I .$$

Let for each $m \in \omega$,

$$\mathcal{A}_m = \{p_{i_m i_{j(n)}}[C_n] : n \in \omega \quad \text{and} \quad j(n) > m\}.$$

and

$$K_m = \bigcap_{A \in \mathcal{A}_m} A .$$

Since, for each $m \in \omega$, the family

$$\mathcal{K}_m = \{p_{i_m i_n}[C] : C \in \mathcal{C}_{i_n} \quad \text{and} \quad n \in \omega, n \geqslant m\}$$

is $\aleph_0$-compact and, for each $n \in \omega$

$$\bigcap_{l=0}^{n} p_{i_0 i_{j(l)}}[C_l] = \pi_{i_0}\left[\bigcap_{l=0}^{n} \pi_{i_{j(l)}}^{-1}[C_l] \cap L_I\right] \neq \emptyset ,$$

we see that $K_0 \neq \emptyset$. Similar considerations show that for any $n \in \omega$, $K_n \neq \emptyset$. Let $x_0 \in K_0$. Then

$$p_{i_0 i_1}^{-1}[\{x_0\}] \cap K_1 \neq \emptyset$$

otherwise, by condition 2.10.1, there would exist m with

$$p_{i_0 i_1}^{-1}[\{x_0\}] \cap \bigcap_{l=0}^{m} p_{i_l i_{j(l)}}[C_l] = \emptyset ,$$

hence,

$$x_0 \notin \bigcap_{l=0}^{m} p_{i_0 i_{j(l)}}[C_l] ,$$

so that $x_0 \notin K_0$, contradicting the choice of x_0. Thus there exists $x_1 \in K_1$ with $p_{i_0 i_1}(x_1) = x_0$, and by similar arguments we can choose, by recursion, for each $n \in \omega$, $x_n \in K_n$ such that

$$p_{i_m i_n}(x_n) = x_m$$

whenever $m \leqslant n$. Let (by sequential maximality) $y \in L_I$ be such that $y_{i_n} = x_n$ for every $n \in \omega$. Clearly for every $n \in \omega$

$$y \in \pi_{i_n}^{-1}[C_n] \cap L_I$$

thus

$$y \in \bigcap_{n \in \omega} (\pi_{i_n}^{-1}[C_n] \cap L_I)$$

so that

$$\{\pi_{i_n}^{-1}[C] \cap L_I : C \in \mathcal{C}_n, n \in \omega\}$$

is $\aleph_0$-compact.

Thus the conditions of theorem 2.8 are satisfied, hence $\widetilde{\mu} \mid L_I$ is an inverse limit outer measure.

Proof of 2.11. – It is clear that h is countably additive on $\mathfrak{A}$ and that $\mathfrak{A}$ is a ring, hence Ψ is an outer measure on $\widetilde{X}_J$, and $\mathfrak{A} \subset \mathfrak{M}_\Psi$. Ψ is supported by L_J since

$$r_J^{-1}(X_J \sim L_J) \subset \widetilde{X}_I \sim L_I ,$$

and since (using $\widetilde{\pi}$ for projection in $\widetilde{X}_J$)

$$\Psi(\pi_j^{-1}[B]) = h(\widetilde{\pi}_j^{-1}[B]) = \nu(\pi_j^{-1}[B]) = \mu_j(B)$$

if $j \in J$ and $B \in \mathfrak{M}_j$, Ψ is an inverse limit outer measure for (X, p, μ, J).

Proof of 2.12.1. – Immediate from the definitions.

Proof of 2.12.2. – Let $\widetilde{\pi}_j$ denote projection onto the j^{th} coordinate from $\widetilde{X}_J$. Then for every $j \in J$, $i \prec j$ and $B \in \mathfrak{M}_i$,

$$\Psi(\pi_i^{-1}[B]) = \Psi(\pi_j^{-1}[p_{ij}^{-1}[B]]) = \nu(\widetilde{\pi}_j^{-1}[p_{ij}^{-1}[B]]) ,$$

hence

$$\Psi(\pi_i^{-1}[B]) = \mu_j(p_{ij}^{-1}[B]) = \mu_i(B) .$$

Since r_I is 1 : 1 on L_I and $\Psi(\widetilde{X} \sim L_I) = 0$, Ψ is an outer measure and with i, j, B as above, $\pi_i^{-1}[B] \in \mathfrak{M}_\Psi$ since $\Psi(\pi_i^{-1}[B] \sim L_I) = 0$ and

$$r_J[\pi_i^{-1}[B] \cap L_I] = \widetilde{\pi}_j^{-1}[p_{ij}^{-1}[B]] \cap L_J$$

which is in $\mathfrak{M}_\nu$. Hence Ψ is an inverse limit outer measure.

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