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ANOTHER CHARACTERIZATION OF ABSOLUTE STABILITY

by Roger C. McCANN

It is well known that absolute stability of a compact subset M of a locally compact metric space can be characterized by the presence of a fundamental system of absolutely stable neighborhoods, and also by the existence of a continuous Liapunov function φ defined on some neighborhood of $M = \varphi^{-1}(0)$, [1]. In a more general setting it has been shown that a set M is closed and absolutely stable if and only if $M = \cap \varphi_i^{-1}(0)$ for suitable Liapunov functions φ_i, [2]. This paper presents a more elementary description of absolute stability in terms of positively invariant neighborhoods only.

Throughout this paper R and R^+ will denote the reals and il non-negative reals respectively. A rational number r is called dyadic iff there are integers n and j such that $n \geqslant 0$, $1 \leqslant j < 2^n$, and $r = j/2^n$.

A dynamical system on a topological space X is a mapping π of $X \times R$ into X satisfying the following axioms (where $x\pi t = \pi(x, t)$) :

(1) $x\pi 0 = x$ for $x \in X$.

(2) $(x\pi t)\pi s = x\pi(t + s)$ for $x \in X$ and $t, s \in R$.

(3) π is continuous in the product topology.

If $A \subset X$ and $B \subset R$, then $A\pi B$ will denote the set $\{x\pi t : x \in A, t \in B\}$ A subset A of X is called positively invariant if and only if $A\pi R^+ = A$.

A mapping $\varphi : X \to R^+$ is called a Liapunov function (relative to π) if and only if φ is continuous and $\varphi(x\pi t) \leqslant \varphi(x)$ for all $x \in X$ and $t \in R^+$.

Absolute stability is defined in terms of a prolongation ([1], [2]) and, in [1], is characterized in a special setting by the following theorem.

THEOREM A. — *Let* M *be a compact subset of a locally compact metric space. Then the following are equivalent:*

(*a*) *There is a Liapunov function* φ *with* $\varphi^{-1}(0) = M$.

(*b*) M *possesses a fundamental system of absolutely stable neighborhoods.*

(*c*) M *is absolutely stable.*

In [2], absolutely stable sets, in a more general setting, are characterized by Liapunov functions.

THEOREM B. — *Let* M *be a subset of a space* X *which is Hausdorff paracompact, and locally compact. Then* M *is closed and absolutely stable if and only if* $M = \cap \varphi_i^{-1}(0)$ *for suitable Liapunov functions* $\varphi_i : X \to [0, 1]$.

In order to obtain our result we will need the following result [2, Corollary 18].

THEOREM C. — *In a locally compact metric space* X, *the closed absolutely stable sets are precisely the zero-sets of Liapunov functions mapping* X *into* [0, 1].

THEOREM. — *Let* M *be a closed subset of a locally compact metric space* X. *Then* M *is absolutely stable if and only if* M *possesses a family* $\mathcal{F}$ *of neighborhoods satisfying*

(*i*) *If* $U \in \mathcal{F}$, *then* U *is open and positively invariant.*

(*ii*) $\cap \mathcal{F} = M$.

(*iii*) *If* $U \in \mathcal{F}$, *then there is a* $V \in \mathcal{F}$ *such that* $\overline{V} \subset U$.

(*iv*) *If* $U, V \in \mathcal{F}$ *are such that* $\overline{U} \subset V$, *then there is a* $W \in \mathcal{F}$ *such that* $\overline{U} \subset W \subset \overline{W} \subset V$.

Proof. — *If.* Let $U \in \mathcal{F}$. For each dyadic rational r we construct a set $U(r) \subset U$ such that $U(r) \in \mathcal{F}$ and $\overline{U}(r) \subset U(s)$ if $r < s$. Then we construct a Liapunov function $\varphi_U : X \to [0, 1]$ and show that $M = \cap \{\varphi_U^{-1}(0) : U \in \mathcal{F}\}$. The result will then follow from Theorem B. First obtain from $\mathcal{F}$ a system of neighborhoods $U\left(\frac{1}{2^n}\right)$, n a non-negative integer, such that $U(1) = U$ and $U\left(\frac{1}{2^{n+1}}\right) \subset U\left(\frac{1}{2^n}\right)$. This is clearly possible by (iii). Using (iv) this system of neighborhoods can be extended to one with the desired properties. For example, we choose $U\left(\frac{3}{4}\right)$ to be any member W of $\mathcal{F}$ such that $\overline{U}\left(\frac{1}{2}\right) \subset W \subset \overline{W} \subset U(1)$. Now define $\varphi_U : X \to R^+$ by $\varphi_U(x) = 1$ if $x \notin U = U(1)$ and $\varphi_U(x) = \inf\{\varphi : x \in U(r)\}$ if $x \in U$. If $x \in U(r)$ and $t \in R^+$, then $x\pi t \in U(r)$ since $U(r)$ is positively invariant. Therefore

$$\varphi_U(x) = \inf\{r : x \in U(r)\} \geqslant \inf\{r : x\pi t \in U(r)\} = \varphi_U(x\pi t).$$

The continuity of φ_U is proved as in the proof of Urysohn's lemma. Thus for each $U \in \mathcal{F}$ we have constructed a continuous Liapunov function φ_U such that $M \subset \varphi_U^{-1}(0) \subset U$. By (ii), $\cap \varphi_U^{-1}(0) = M$.

Only if. — Let M be absolutely stable. Then by theorem C, $M = \varphi_U^{-1}(0)$ for some Liapunov function φ. Let $\mathcal{F}$ consist of all sets of the form $\{x : \varphi(x) < r\}$ where $r \in (0, 1)]$. Evidently $\mathcal{F}$ satisfies conditions (i)-(iv).

Remark. — In the « If » part of the proof we only need that X is Hausdorff, paracompact, and locally compact.

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