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## THE GROWTH OF ENTIRE SOLUTIONS OF DIFFERENTIAL EQUATIONS OF FINITE AND INFINITE ORDER

by Lawrence GRUMAN

Let  $f(z)$  be an entire function (of one or several variables) of finite order  $\rho$ . A proximate order  $\rho(r)$  is a function which satisfies the conditions

$$\lim_{r \rightarrow \infty} \rho(r) = \rho \quad \text{and} \quad \lim_{r \rightarrow \infty} r \rho'(r) \ln r = 0. \quad (1)$$

The function  $L(r) = r^{\rho(r) - \rho}$  satisfies

$$\lim_{r \rightarrow \infty} \frac{L(kr)}{L(r)} = 1 \quad \text{uniformly for} \quad 0 < a \leq k \leq b < \infty. \quad (2)$$

We assume in addition that  $\lim_{r \rightarrow \infty} L(r)$  exists (perhaps infinite). For every entire function of order  $\rho$ , there exists a proximate order  $\rho(r)$  with respect to which  $f(z)$  has normal type [5].

For a given proximate order  $\rho(r)$ , we define the functions

$$h_r^*(z) = \overline{\lim}_{z' \rightarrow z} \left[ \overline{\lim}_{r \rightarrow \infty} \frac{\ln |f(rz')|}{r^{\rho(r)}} \right], \quad r > 0$$

(resp.  $h_c^*(z) = \overline{\lim}_{z' \rightarrow z} \left[ \overline{\lim}_{|u| \rightarrow \infty} \frac{\ln |f(uz')|}{|u|^{\rho(r)}} \right], \quad u \in \mathbb{C}$ ).

If  $f(z)$  is of normal type with respect to the proximate order  $\rho(r)$ , it follows from (2) that these functions are pluri-subharmonic and real positive homogeneous (resp. complex homogeneous) of order  $\rho$  [4]. The function  $h_r^*(z)$  (resp.  $h_c^*(z)$ ) is called the radial (resp. circular) indicator of growth function of  $f(z)$ .

A convex homogeneous function  $g(z)$  is one which satisfies  $g(z_1 + z_2) \leq g(z_1) + g(z_2)$  and  $g(tz) = tg(z)$ ,  $t \geq 0$ . To every convex

homogeneous function  $g(z)$ , we associate the compact convex set  $K_g = \{w : \operatorname{Re} \langle w, z \rangle \leq g(z) \forall z \in \mathbb{C}^n\}$ , and to every compact convex set  $K$ , we associate the convex homogeneous function

$$g_K(z) = \sup_{w \in K} \operatorname{Re} \langle w, z \rangle,$$

which is called the support function of  $K$ . If  $\rho \equiv 1$ , we define  $h_K(z)$ , the *convex indicator of growth function* of  $f(z)$ , to be the least convex homogeneous majorant of  $h^*(z)$ . It is evidently the support function of the closed convex hull of the set

$$\{w : \operatorname{Re} \langle w, z \rangle \leq h^*(z) \forall z \in \mathbb{C}^n\}.$$

If the dimension  $n = 1$ , these two functions are the same [5].

In § 1, we investigate for the case  $n = 1$  the relationship between the growth of the function  $f(z)$  and that of solutions  $u(z)$  of the differential equation  $P(D)u = f$  (where  $D = \frac{\partial}{\partial z}$  and  $P(D)$  is a differential polynomial).

Let  $p(z)$  be a complex norm (i.e.  $p(\lambda z) = |\lambda| p(z)$ ,  $\lambda \in \mathbb{C}$ ),  $B_A^\rho$  the space of functions which satisfy a majoration

$$|f(z)| \leq C_A \exp\{(A p(z))^\rho\}$$

and  $E_R^\rho = \bigcap_{\Lambda > R} B_\Lambda^\rho$ . In [8], A. Martineau introduced the notion of a constant coefficient differential operator as a convolution operator on the dual space  $(E_R^\rho)'$  of continuous linear functionals defined on  $E_R^\rho$ . We will take as our definition of such an operator the *transpose*, which is a linear operator on the space  $E_R^\rho$  into itself. This category includes the usual constant coefficient differential operator as a special case. For  $\rho \geq 1$ , Martineau showed that for every such operator  $\hat{\mu}$  on  $E_R^\rho$  and every  $f \in E_R^\rho$ , there exists a solution  $g \in E_R^\rho$  of the equation  $\hat{\mu}(g) = f$ .

In § 2, we extend this notion and this result to the case of  $p(z)$  a pseudo-norm and  $\rho(r)$  a proximate order ( $\rho \neq 1$ ), including the important case of  $\rho < 1$ . In § 3, we extend this notion and result to the case  $\rho = 1$  and  $p(z)$  an arbitrary convex homogeneous function. In § 4, we extend this notion and result to those functions which satisfy a majoration of the type  $\exp\{k(\ln r)^\rho\}$  for  $\rho > 1$ .

*Remark.* — The case of proximate orders for  $\rho = 1$  is rendered much more difficult by the special role played by the exponentials. We do not treat this case.

### 1. Ordinary differential equations.

Let  $f(z)$  be an entire function of a single variable and  $h_r^*(z)$  its indicator function with respect to a proximate order  $\rho(r)$ . We will henceforth in this section use the notation  $k_f(\theta) = h_r^*(e^{i\theta})$ , which is the standard notation for  $n = 1$ . If  $u(z)$  is a solution of the constant coefficient differential equation  $P(D)u = f$ , then it is an easy consequence of Cauchy's theorem that  $k_f(\theta) \leq k_u(\theta)$ . We are interested in seeing if we can choose a solution such that equality holds (at least locally). We will need

LEMMA 1. — *The number of disjoint open intervals on which  $k_f(\theta)$  can be negative is at most  $\sup_{a < 1} [2a\rho]$  (where  $[ \ ]$  means "greatest integer in").*

*Proof.* — For  $\theta_1 < \theta_2 < \theta_3$  and  $\theta_3 - \theta_1 < \pi/\rho$ , we have [5, p. 70]

$$k_f(\theta_1) \sin \rho(\theta_2 - \theta_3) + k_f(\theta_2) \sin \rho(\theta_3 - \theta_1) + k_f(\theta_3) \sin \rho(\theta_1 - \theta_2) \leq 0.$$

Thus, any two disjoint intervals on which  $k_f(\theta)$  is negative are separated by an interval of length at least  $\pi/\rho$  on which  $k_f(\theta)$  is non-negative. Q.E.D.

THEOREM 1. — *Let  $f(z)$  be an entire function with indicator  $k_f(\theta)$  with respect to the proximate order  $\rho(r)$ . Then there exists a solution  $u(z)$  of the differential equation  $P(D)u = f$  such that*

i)  $k_u(\theta) = k_f(\theta)$  for  $\rho \leq 1$ .

ii)  $k_f(\theta) \leq k_u(\theta) \leq k_f^+(\theta) = \max(k_f(\theta), 0)$  for  $\rho > 1$  and for any specific interval  $(\theta_1, \theta_2)$  on which  $k_f(\theta)$  is negative, there exists a unique solution  $u$  with this property such that  $k_u(\theta) = k_f(\theta)$  for  $\theta_1 \leq \theta \leq \theta_2$ .

*Proof.* — It is enough to consider solutions of the equation  $(D - a)u = f$  and then iterate the result. All such solutions are given by

$$u(z) = e^{az} \int_0^z f(\zeta) e^{-a\zeta} d\zeta + Ce^{az}. \quad (3)$$

If for some open interval of  $\theta$ , the function  $f(z) e^{-az}$  has negative indicator (with respect to *any* proximate order), then

$$C = \int_0^\infty f(t\xi) e^{-at\xi} \xi dt, \quad \xi = e^{i\theta},$$

defines a constant for all  $\theta$  in this interval. If there is no such region, we choose  $C = 0$ . By Lemma 1, for  $\rho \leq 1$ , there is at most one such interval, but for  $\rho > 1$  there may be more than one such interval and we may only be able to choose  $C$  to satisfy this relation in one of the intervals. (This explains the difference between i) and ii) above).

From (1), we have that

$$(r^{\rho(r)})' = \rho(r) r^{\rho(r)-1} + r^{\rho(r)} \rho'(r) \ln r \rightarrow \rho(r) r^{\rho(r)-1}. \quad (4)$$

Let us consider the case  $\rho < 1$ . For a given  $\xi = e^{i\theta}$ , let  $b = k_f(\theta)$  and  $s = \operatorname{Re} a\xi$ . Then given  $\varepsilon > 0$ , we have  $|f(t\xi)| \leq K \exp(b + \varepsilon) t^{\rho(t)}$ .

i) If  $s < 0$  and  $b < 0$  and if  $\varepsilon < -\frac{b}{2}$ , then

$$\begin{aligned} |u(r\xi)| &\leq K e^{sr} \int_0^r e^{(b+\varepsilon)t^{\rho(t)} - st} dt + |C| e^{sr} \\ &\leq K'_1 e^{sr} \int_{q_0}^r \left[ (b+\varepsilon) \frac{d}{dt} (t^{\rho(t)}) - s \right] e^{(b+\varepsilon)t^{\rho(t)} - st} dt + |C| e^{sr}, \end{aligned}$$

where  $q_0$  is chosen so large that  $\left[ (b+\varepsilon) \frac{d}{dt} (t^{\rho(t)}) - s \right]$  is bounded below and  $K'_1$  depends on  $q_0$ .

$$\begin{aligned} |u(r\xi)| &\leq K'_1 [e^{(b+\varepsilon)r^{\rho(r)}} - e^{sr} \cdot K_{q_0}] + |C| e^{sr} \\ &\leq K''_1 e^{(b+\varepsilon)r^{\rho(r)}}. \end{aligned}$$

ii) If  $s \geq 0$  and  $b < 0$ , then by the choice of  $C$ , we have

$$\begin{aligned}
 |u(r\xi)| &\leq K e^{sr} \int_r^\infty e^{(b+\varepsilon)t^{\rho(t)}} \cdot e^{-st} dt \\
 &\leq K \int_r^\infty e^{(b+\varepsilon)t^{\rho(t)}} dt \leq K e^{(b+2\varepsilon)r^{\rho(r)}} \int_r^\infty e^{-\varepsilon t^{\rho(t)}} dt \\
 &\leq K'_2 e^{(b+2\varepsilon)r^{\rho(r)}},
 \end{aligned}$$

since by (4),  $r^{\rho(r)}$  is increasing for sufficiently large  $r$ .

iii) If  $s > 0$  and  $b \geq 0$ , then

$$\begin{aligned}
 |u(r\xi)| &\leq K e^{sr} \int_r^\infty e^{(b+\varepsilon)t^{\rho(t)} - st} dt \\
 &\leq K'_3 e^{sr} \int_r^\infty \left[ (b+\varepsilon) \frac{d}{dt} (t^{\rho(t)}) - s \right] e^{(b+\varepsilon)t^{\rho(t)} - st} dt \\
 &\leq K''_3 e^{(b+\varepsilon)r^{\rho(r)}}.
 \end{aligned}$$

iv) If  $s \leq 0$  and  $b \geq 0$ , then

$$\begin{aligned}
 |u(r\xi)| &\leq K e^{sr} \int_0^r e^{(b+\varepsilon)t^{\rho(t)} - st} dt + |C| e^{sr} \\
 &\leq K'_4 r e^{(b+\varepsilon)r^{\rho(r)}}.
 \end{aligned}$$

The case  $\rho \geq 1$  is treated similarly (for  $\rho = 1$ , we must make use of the assumption that  $\lim_{r \rightarrow \infty} r^{\rho(r) - \rho}$  exists). For  $\rho > 1$ , if for some  $\theta$ ,  $k_f(\theta) \neq k_u(\theta)$ , then  $u(z) = w(z) + Ce^{az}$ , where  $k_f(\theta) = k_w(\theta) < 0$ , so  $k_u(\theta) = 0$ . Q.E.D.

*Remark.* — It follows from Theorem 6 below that if  $P(D)$  has a non-zero constant term, then for  $\rho < 1$ , the solution  $u(z)$  in i) is unique.

The following example shows that it is not always possible to find a solution  $u$  of  $P(D)u = f$  with the same indicator as  $f$ . Let  $f(z) = e^{z^2}$  and let  $u$  be a solution of  $Du = f$ . The function  $f(z)$  has two intervals on which its indicator is negative. If we integrate  $f(z)$  along the positive imaginary axis, we obtain a constant different from that which we obtain by integrating along the negative imaginary axis.

There is even a more intimate connection between the growth of the function  $f(z)$  and the solution  $u(z)$  of  $P(D)u = f$ . If  $f(z)$  grows regularly in a given direction, then so will  $u(z)$ . We introduce our criterion for regularity of growth.

Let  $E$  be a measurable set of positive real numbers and let  $E^r = E \cap [0, r]$ . A set is said to have upper relative measure  $U$  if  $\overline{\lim}_{r \rightarrow \infty} \frac{\text{meas}(E^r)}{r} = U$ . If  $U = 0$ ,  $E$  is an  $E^0$ -set.

DEFINITION [5]. — Let  $f(z)$  be an entire function with indicator  $k_f(\theta)$  with respect to a given proximate order  $\rho(r)$ ;  $f(z)$  is said to be of completely regular growth along the ray  $re^{i\theta}$  if

$$\lim_{r \rightarrow \infty} \frac{\ln |f(re^{i\theta})|}{r\rho(r)} = k_f(\theta),$$

where  $r$  takes on all values except perhaps for some  $E^0$ -set.

Remark. — The property of being of completely regular growth is not invariant with respect to a change in proximate orders.

THEOREM 2. — If  $u(z)$  is a solution of  $P(D)u = f$  for an entire function  $f(z)$  and if  $\rho(r)$  is a proximate order with respect to which both  $k_f(\theta)$  and  $k_u(\theta)$  are bounded, then if  $f(z)$  is of completely regular growth along the ray  $re^{i\theta}$ , so is  $u(z)$ .

Proof. — We consider a solution of  $(D - a)u = f$ . By Theorem 1, for given  $\theta$ , there is an interval  $(\theta_1, \theta_2)$  containing  $\theta$  such that  $u = w + Ce^{az}$  and  $w$  has the same indicator as  $f$  in the interval  $(\theta_1, \theta_2)$ . Thus, if  $k_u(\theta) \neq k_f(\theta)$ , we have that  $\lim_{r \rightarrow \infty} \frac{\ln |u(re^{i\theta})|}{r\rho(r)}$  exists with no exceptional set. Hence, in the following, we assume that  $k_u(\theta) = k_f(\theta)$ . We assume without loss of generality that  $\theta = 0$ .

Let  $\varepsilon$  and  $\eta$  be given positive numbers. Then there exists a set  $E_1$  of upper relative measure less than  $\eta/4$  such that if  $r \notin E_1$ , the family of functions  $k_{u,r}(\phi) = \frac{\ln |u(re^{i\phi})|}{r\rho(r)}$  is equicontinuous [5, p. 96].

Thus, there is a  $\delta > 0$  such that for  $|\phi| < \delta$ ,

$$|k_{u,r}(\phi) - k_{u,r}(0)| < \frac{\varepsilon}{4} \text{ and } |k_u(\phi) - k_u(0)| < \frac{\varepsilon}{4} \text{ for } r \notin E_1.$$

Since  $f$  is of completely regular growth along the positive real axis, given  $\gamma > 0$  (depending eventually on  $\eta$  and  $\varepsilon$ ), for  $r$  not in some  $E^0$ -set  $E_2$ ,

$$-\frac{\gamma}{4} + k_f(0) \leq \frac{\ln |f(r)|}{r^{\rho(r)}} \leq k_f(0) + \frac{\gamma}{4} = k_u(0) + \frac{\gamma}{4}. \quad (5)$$

We choose  $r$  so large that  $\text{meas}(E_2^r) < \frac{\eta}{4} r$  and  $\frac{\ln |u(re^{i\phi})|}{r^{\rho(r)}} \leq k_u(\phi) + \frac{\gamma}{4}$

[5, p. 71]. By Cauchy's formula,

$$f(r) e^{-ar} = \frac{1}{2\pi i} \int_{|\xi|=1} \frac{u(\xi+r)}{\xi^2} e^{-a(\xi+r)} d\xi.$$

So by (5) for  $r \notin E_2$  and  $r$  sufficiently large, there exists  $w$  with  $|w-r|=1$  such that, noting  $\phi_w = \arg w$ ,

$$|a| + \ln |u(w)| > \left\{ k_f(0) - \frac{\gamma}{4} \right\} r^{\rho(r)} \geq \left\{ k_f(\phi_w) - \frac{\gamma}{2} \right\} |w|^{\rho(|w|)}$$

Let  $R_m = \left( \frac{1+\eta}{1-\eta} \right)^m$ . Then, as in the proof of Theorem 31 [5, p. 73], we can choose  $\gamma$  so small (depending on  $\varepsilon$  and  $\eta$  but independent of  $w$  since  $k_u(\theta)$  is bounded) such that

$$\frac{\ln |u(r\phi_w)|}{r^{\rho(r)}} > k_u(\phi_w) - \frac{\varepsilon}{4}$$

except perhaps on a set of measure at most  $\frac{\eta^2}{4} R_m$  for

$$(1 - 2\eta) R_m \leq r \leq (1 + 2\eta) R_m$$

(for  $m \geq m_0$  so large that the above inequalities hold). Let

$$E_3 = [0, R_{m_0}] \cup \left( \bigcup_{m \geq m_0} E_m \right).$$

Then

$$\begin{aligned} \frac{\text{meas}(E_3^r)}{r} &\leq \frac{R_{m_0} + \sum_{i=m_0}^m \frac{\eta^2}{4} \frac{(R_{m_0} - R_m)}{1 - (1 + \eta)}}{R_m(1 - \eta)} \\ &\leq 0(1) + \frac{\eta}{2} \left( 1 - \frac{R_{m_0}}{R_m} \right) < \frac{\eta}{4} \end{aligned}$$



for  $m$  sufficiently large. Let  $E_\eta = E_1 \cup E_2 \cup E_3$ . Then

$$\overline{\lim}_{r \rightarrow \infty} \frac{\text{meas}(E'_\eta)}{r} < \eta ,$$

and gathering together our inequalities, we have  $|k_{u,r}(0) - k_u(0)| < \varepsilon$  for  $r \notin E$ . To see that this implies the theorem, we refer the reader to Theorem 1, part 3 [5, p. 141]. Q.E.D.

*Remark.* – The fact that a function is of completely regular growth in an interval has important consequences for the distribution of its zeros. This is fully discussed in [5].

## 2. Differential operators with constant coefficients.

Let  $p_n(z)$  be a decreasing sequence of real valued functions and  $B_n$  the space of entire functions such that  $|f(z) \exp\{-p_n(z)\}|$  goes to zero at infinity. This is a Banach space with norm

$$\|f\|_n = \sup_z |f(z) \exp\{-p_n(z)\}| .$$

We then set

$$E = \bigcap_n B_n , \tag{6}$$

which is a Fréchet space when we equip it with the projective limit topology. If  $B'_n$  is the dual space of  $B_n$ ,  $E'$  that of  $E$ , then  $E' = \bigcup_n B'_n$ .

Let  $p(z)$  be a complex pseudo-norm and  $\rho(r)$  a proximate order. The space  $E_p^{\rho(r)}$  will designate the space we get in (6) by setting  $p_n(z) = \left\{ p(z) + \frac{1}{n} \|z\| \right\}^{\rho(r)}$  (where  $r = \|z\|$ , and we use the Euclidean norm). The space  $E^0$  will be the space we get in (6) by setting  $p_n(z) = \|z\|^{1/n}$  (the space of entire functions of zero order).

For a given proximate order  $\rho(r)$ , we have by (4) that  $r^{\rho(r)}$  is increasing for sufficiently large  $r$ . For a given integer  $q$ , we define  $\phi(q) = r_q$  to be the largest solution of  $q = r^{\rho(r)}$ . Then the type with respect to  $\rho(r)$  of an entire function of one variable with coefficients  $c_q$  (in its Taylor series expansion at the origin) is given by the formula

$$(\rho e)^{1/\rho} = \overline{\lim}_{q \rightarrow \infty} (\phi(q) |c_q|^{1/q}) \quad [5, p. 42] . \tag{7}$$

If  $f \in E_p^{\rho(r)}$ , we expand  $f$  at the origin in homogeneous polynomials  $f(z) = \sum_q P_q(z)$ . Let  $A_q = \left(\frac{\phi(q)^\rho}{e\rho}\right)^{q/\rho}$ . If we set

$$f_t(z) = \sum_q A_q P_q(z) ,$$

then  $f_t(z)$  is a holomorphic function in the open set  $D = \{z : p(z) < 1\}$ , and when we equip the space  $\mathcal{H}(D)$  of holomorphic functions defined on  $D$  with the topology of uniform convergence on compact subsets, the mapping  $f \rightarrow f_t$  becomes an isomorphism of  $E_p^{\rho(r)}$  onto  $\mathcal{H}(D)$  (cf. [8], Prop. 4, p. 116 and [4]).

For  $\mu \in (E_p^{\rho(r)})'$ , we define the linear functional  $\mu_t$  on  $\mathcal{H}(D)$  by  $(f_t, \mu_t) = (f, \mu)$ . This is an isomorphism of  $(E_p^{\rho(r)})'$  onto  $\mathcal{H}'(D)$ , the space of continuous linear functionals on  $\mathcal{H}(D)$ . We say that a linear functional  $\mu_t$  is carried by the compact convex set  $K$  if for every open neighborhood  $\Omega$  of  $K$ , there exists a constant  $C_\Omega$  such that  $|\mu_t(f_t)| \leq C_\Omega \sup_\Omega |f_t|$ . Every  $\mu_t \in \mathcal{H}'(D)$  is carried by one of

$$\text{the sets } K_n = \left\{ z : p(z) + \frac{1}{n} \|z\| \leq 1 \right\} .$$

We define the Fourier-Borel transform of the functional  $\mu_t$  to be the entire function  $\tilde{\mu}_t(u) = \mu_t(\exp \langle z, u \rangle)$ . Then we have [3], [7].

PROPOSITION 1. — *The functional  $\mu_t$  is carried by the compact convex set  $K$  if and only if*

$$\tilde{\mu}_t(u) \leq C_\delta \exp(H_K(u) + \delta \|u\|) \quad \text{for all } \delta > 0 ,$$

where  $H_K(u)$  is the support function of  $K$ .

Let  $p'_n(u) = \sup_{z \in K_n} \text{Re} \langle z, u \rangle$ . Then  $p'_n(u)$  is a family of increasing complex norms, and since each  $\mu_t \in \mathcal{H}'(D)$  is carried by some  $K_n$ , we have

$$\tilde{\mu}_t(u) \leq C_n \exp H_{K_n}(u) \quad \text{for } n \text{ sufficiently large.}$$

Let  $\alpha$  be a multi-index of positive numbers,  $|\alpha| = \sum \alpha_i$  and

$z^\alpha = z^{\alpha_1} \dots z^{\alpha_n}$ . Since the polynomials converge to  $\exp \langle z, u \rangle$  in  $\mathfrak{E}(D)$ , we have

$$\begin{aligned} \mu_t(\exp \langle z, u \rangle) &= \mu_t \sum_q \sum_{|\alpha|=q} z^\alpha \frac{u^\alpha}{\alpha!} = \sum_q \sum_{|\alpha|=q} \mu_t(z^\alpha) \frac{u^\alpha}{\alpha!} \\ &= \sum_q P_q^{\mu_t}(u) \end{aligned}$$

and from (7) and Proposition 1, we have

$$\overline{\lim}_{q \rightarrow \infty} \left\{ \frac{q}{e} |P_q^{\mu_t}(u)|^{1/q} \right\} \leq p'_n(u)$$

for  $n$  sufficiently large. From the relation  $\mu_t(z^\alpha) = \frac{1}{A_{|\alpha|}} \mu(z^\alpha)$ , we see that  $\mu \in (E_p^{\rho(r)})'$  (resp.  $(E^0)'$ ) if and only if

$$\overline{\lim}_{q \rightarrow \infty} \left\{ \frac{q}{e} \left| \frac{1}{A_q} \sum_{|\alpha|=q} \mu(z^\alpha) \frac{u^\alpha}{\alpha!} \right|^{1/q} \right\} \leq p'_n(u) \quad (8)$$

for  $n$  sufficiently large (resp. for  $\rho$  sufficiently small).

For  $\mu \in (E_p^{\rho(r)})'$  (resp.  $(E^0)'$ ), we define its Fourier-Borel transform to be the *formal* power series

$$\tilde{\mu}(u) = \mu(\exp \langle z, u \rangle) = \sum_q \sum_{|\alpha|=q} \mu(z^\alpha) \frac{u^\alpha}{\alpha!} = \sum_q P_q^\mu(u).$$

If  $\rho > 1$ , we assume that the proximate order  $\rho(r)$  satisfies :

- i)  $\rho(r) > 1$  for all  $r$
- ii)  $\frac{d}{dr} (r^{\rho(r)-1}) > 0$  for all  $r$ .

By (1), these properties hold eventually, so this is an inessential assumption. Then the equation  $r = t^{\rho(t)-1}$  has a unique solution for all  $r$ . We define

$$\rho^*(r) = \frac{\rho(t)}{\rho(t) - 1}, \text{ where } t \text{ is this unique solution.}$$

It is an easy calculation to show that  $\rho^*(r)$  satisfies the conditions (1) and so is a proximate order. For  $\rho > 1$ , we designate

$$F_{A\rho'}^{\rho^*(r)} = \bigcup_n E_{A\rho'_n}^{\rho^*(r)},$$

where  $A = \frac{(\rho - 1)^{\frac{\rho-1}{\rho}}}{\rho}$

THEOREM 3. — *The mapping  $\mu \mapsto \tilde{\mu}(u)$  is a one-to-one linear mapping of  $(E_p^{\rho(r)})'$  (resp.  $(E^0)'$ ) onto*

- i)  $F_{A\rho'}^{\rho^*(r)}$  for  $\rho > 1$
- ii) the set  $Q_{\rho'}^{\rho(r)}$  of formal power series at the origin which satisfy (8) for some  $n$  for  $\rho < 1$
- iii) the set  $Q_0$  of formal power series at the origin which satisfy (8) for some  $\rho > 0$  for  $(E^0)'$ .

*Proof.* — We have that (8) holds for some  $n_0$ . Since

$$A_q^{1/q} = \frac{\phi(q)}{(e\rho)^{1/\rho}}, \quad \frac{q}{e} \frac{1}{A_q^{1/q}} = \frac{A r_q^{\rho(r_q)-1}}{(e\rho^*)^{1/\rho^*}}$$

(where  $r_q = \phi(q)$ ). Let  $r'_q = r_q^{\rho(r_q)-1}$ . Then

$$\begin{aligned} (r'_q)^{\rho^*(r'_q)} &= (r_q^{\rho(r_q)-1})^{\rho^*(r_q^{\rho(r_q)-1})} \\ &= (r_q^{\rho(r_q)-1})^{\frac{\rho(r_q)}{\rho(r_q)-1}} = r_q^{\rho(r_q)} = q \end{aligned}$$

so if  $\phi'(q)$  is the unique solution of  $(r'_q)^{\rho^*(r)} = q$ , we have that  $\frac{q}{e} \frac{1}{A_q^{1/q}} = A \frac{\phi'(q)}{(e\rho^*)^{1/\rho^*}}$  so the mapping is into. Since the calculations are all reversible, the mapping is also onto. This proves case i). Cases ii) and iii) follow directly from (8). Q.E.D.

Let  $\mu \in (E_p^{\rho(r)})'$ . Then for any other element  $\nu$ , we define the convolution of  $\nu$  with  $\mu$ ,  $\mu * \nu = \tau$  by  $(f(z), \mu * \nu) = (\mu_w f(z + w), \nu)$ . This is defined at least on the polynomials, which are dense in  $E_p^{\rho(r)}$ . For  $\rho > 1$ , it is also defined on the exponentials [8]. We then have the relationship (for  $\rho \neq 1$ )  $\tilde{\tau}(u) = \tilde{\mu}(u) \cdot \tilde{\nu}(u)$ , which, for the case  $\rho < 1$ , follows from

LEMMA 2. — For  $\tilde{\mu}(u), \tilde{\nu}(u) \in Q_p^{\rho(r)}$  (resp.  $Q_0$ ), we have  $\tilde{\tau}(u) = \tilde{\mu}(u) \tilde{\nu}(u) \in Q_p^{\rho(r)}$  (resp.  $Q_0$ ) for  $\rho < 1$  (i.e. these spaces are algebras).

*Proof.* — We choose  $n_0$  so large so that for  $n \geq n_0$ , (8) holds for both  $\mu$  and  $\nu$ . Consider such an  $n$  and let  $\varepsilon > 0$  be given. Then there exist constants  $C_\varepsilon^\mu$  and  $C_\varepsilon^\nu$  such that

$$|P_q^\mu(u)| \leq C_\varepsilon^\mu [p'_n(u) + \varepsilon \|u\|]^q \left(\frac{\Phi(q)^\rho}{e\rho}\right)^{q/\rho} \left(\frac{e}{q}\right)^q$$

and

$$|P_q^\nu(u)| \leq C_\varepsilon^\nu [p'_n(u) + \varepsilon \|u\|]^q \left(\frac{\Phi(q)^\rho}{e\rho}\right)^{q/\rho} \left(\frac{e}{q}\right)^q.$$

Then

$$\begin{aligned} |P_q^\tau(u)| &= \left| \sum_{m+n=q} P_m^\mu(u) P_n^\nu(u) \right| \\ &\leq C_\varepsilon^\mu C_\varepsilon^\nu [p'(u) + \varepsilon \|u\|]^q \left(\frac{\Phi(q)^\rho}{e\rho}\right)^q \left(\frac{e}{q}\right)^q \\ &\qquad \sum_{m+n=q} \left[ \frac{\Phi(m)^m \Phi(n)^n}{\Phi(m+n)^{m+n}} \right] \frac{(m+n)^{m+n}}{m^m n^n}. \end{aligned}$$

Let  $r_q = \Phi(q)$ . Then  $\frac{q}{\Phi(q)} = r_q^{\rho(r_q)-1}$ , and hence, since by (1),  $r^{\rho(r)-1}$  is decreasing for  $r$  sufficiently large

$$\sum_{m+n=q} \frac{[r_{m+n}^{\rho(r_{m+n})-1}]^{m+n}}{[r_m^{\rho(r_m)-1}]^m [r_n^{\rho(r_n)-1}]^n} \leq K q \text{ for some constant } K.$$

Thus  $|P_q^\tau(u)|$  satisfies (8). For  $Q_0$ , we choose  $\rho_0$  so small that (8) holds for both  $\mu$  and  $\nu$  for  $\rho < \rho_0$ . The result then follows from the above calculations. Q.E.D.

Thus, by Theorem 3, for  $\rho < 1$ , the mapping  $\nu \rightarrow \mu * \nu$  is a map of  $(E_p^{\rho(r)})'$  (resp.  $(E^0)'$ ) into  $(E_p^{\rho(r)})'$  (resp.  $(E_0)'$ ). If  $\rho > 1$ , this is only the case if  $\tilde{\mu}(u)$  is of minimal type with respect to the proximate order  $\rho^*(r)$ . Assuming  $\mu$  to satisfy these conditions, we define  $\check{\mu}$  to be the transpose of  $\mu$ ,  $(\check{\mu}(f), \nu) = (f, \mu * \nu)$ . We are interested in proving that the mapping  $\check{\mu}(E_p^{\rho(r)})$  (resp.  $E^0$ ) is onto (i.e. that there always exists a solution  $g$  such that  $\check{\mu}(g) = f$ ). We will make use of [cf. 9, p. 85].

PROPOSITION 2. — Let  $E, F$  be two Fréchet spaces,  $\alpha$  a continuous linear map of  $E$  into  $F$ . The two following are equivalent

- i)  $\alpha$  is onto
- ii)  ${}^t\alpha : F' \rightarrow E'$  (the transpose map) is one-to-one and its image  ${}^t\alpha(F')$  is weakly closed in  $E'$ .

We shall prove the closure of  $\mu * \nu$  in the equivalent spaces as determined by Theorem 3, but first we must equip these spaces with topologies. For  $\rho > 1$ , we equip the space  $F_{\Delta p}^{\rho*(r)}$  with the topology of pointwise convergence. For  $\rho < 1$ , we equip  $Q_p^{\rho(r)}$  (resp.  $Q_0$ ) with the topology of convergence of Taylor's series coefficients. Each of these topologies is at least as weak as the weak topology.

We define a differential operator with constant coefficients (with respect to a given proximate order  $\rho(r)$ ) to be

i)  $\check{\mu}$  for  $\mu \in (E_p^{\rho(r)})'$  for  $\rho < 1$

ii)  $\check{\mu}$  for  $\mu \in (E^0)'$

iii)  $\check{\mu}$  for  $\mu \in (E_p^{\rho(r)})'$  such that  $\check{\mu}(u)$  is of minimal type with respect to  $\rho^*(r)$  for  $\rho > 1$ .

For  $\rho > 1$ , the mapping  $\check{\nu}(u) \rightarrow \check{\mu}(u) \check{\nu}(u)$  is closed in the topology we have chosen (the proof is carried out in [8] ; the modifications necessary to treat the case of proximate orders are obvious). Thus, we limit ourselves to the case  $\rho < 1$  and  $E^0$ .

LEMMA 3. — Let  $A_n(u) = \frac{B_{n+m}(u)}{C_m(u)}$  be a homogeneous polynomial which is the ratio of two homogeneous polynomials. Furthermore, assume that for some complex norm  $p_0(u)$  that

$$|B_{n+m}(u)| \leq C [p_0(u)]^{n+m} .$$

Then given  $\delta > 0$ , there is a constant  $K_\delta$  (depending only on  $C_m(u)$  and  $\delta$ ) such that  $|A_n(u)| \leq C K_\delta [p_0(u)]^n (1 + \delta)^{n+m}$ .

Proof. — Let  $\Omega = \{u : 1 - \delta \leq p_0(u) \leq 1 + \delta\}$ . For every point  $u$  in  $\Omega$  we find a polydisc (by making a non-singular linear change of variable if necessary)  $\Delta(u ; r^u)$  centered at  $u$  and lying in  $\Omega$  such that  $C_m(u'_1, \dots, u'_{n-1}, \xi_n) \neq 0$  for  $|\xi_n - u_n| = r_n^u$  and

$$|u'_i - u_i| \leq r_i^u, i = 1, \dots, n - 1 [2].$$

Let  $\Omega' = \{u : p_0(u) = 1\}$ . We now consider the polydisc  $\Delta'_u = \Delta\left(u; \frac{r^u}{2}\right)$ . Since  $\Omega'$  is compact, it can be covered by a finite number of  $\Delta'_{u_j}$ ,  $j = 1, \dots, N$ . The function  $\frac{1}{C_m(u)}$  is bounded, say by  $\frac{K_\delta}{2}$ , on the compact set

$$K = \bigcup_j \{u' : u' \in \Delta'_{u_j}, |u'_i - u_i| \leq r_i^{u'_j}, i = 1, \dots, n-1, |u'_n - u_n| \leq r_n^{u'_j}\}.$$

Let the function  $A_n$  take its maximum on  $\Omega'$  at the point  $u^0$ . Then  $u^0 \in \Delta'_{u_j}$  for some  $j$ . By Cauchy's formula

$$|A_n(u^0)| = \left| \frac{1}{2\pi i} \int_{|\xi_n - u_n^0| = r_n^j} \frac{B_{n+m}(u_1^0, \dots, u_{n-1}^0, \xi_n) d\xi_n}{C_m(u_1^0, \dots, u_{n-1}^0, \xi_n) (\xi_n - u_n^0)} \right| = K_\delta C p_0(u) (1 + \delta)^{n+m}. \quad \text{Q.E.D.}$$

**THEOREM 4 (Division Theorem).** — Let  $H(u), F(u) \in Q_p^{\rho(r)}$  for  $\rho < 1$  (resp.  $Q_0$ ) with  $H(u) = F(u) G(u)$ , where  $G(u)$  is a formal power series at the origin. Then  $G(u) \in Q_p^{\rho(r)}$  (resp.  $Q_0$ ).

*Proof.* — Let  $\varepsilon > 0$  be given and let

$$G(u) = \sum_q R_q(u), H(u) = \sum_q P_q(u), \text{ and } F(u) = \sum_q T_q(u),$$

with  $s$  the smallest integer such that  $T_s(u) \neq 0$ . We choose  $n_0$  so large that (8) holds for both  $H(u)$  and  $F(u)$  for  $n \geq n_0$ . Thus, there exist constants  $C_1$  and  $C_2$  such that

$$|P_q(u)| \leq C_1 [p'_n(u) + \varepsilon \|u\|]^q \left(\frac{\phi(q)^\rho}{e\rho}\right)^{q/\rho} \left(\frac{e}{q}\right)^q$$

and

$$|T_q(u)| \leq C_2 [p'_n(u) + \varepsilon \|u\|]^q \left(\frac{\phi(q)^\rho}{e\rho}\right)^{q/\rho} \left(\frac{e}{q}\right)^q.$$

We have

$$P_{q+s}(u) = \sum_{m+k=q} R_m(u) T_{k+s}(u)$$

or

$$R_q(u) = \frac{P_{q+s}(u) - \sum_{\substack{m+k=q \\ m \neq q}} R_m(u) T_{k+s}(u)}{T_s(u)}.$$

We now show by induction that there exist constants  $K_q$  (with  $K_{q-1} \leq K_q$ ) such that

$$|R_q(u)| \leq K_q [p'_n(u) + \varepsilon \|u\|]^q (1 + \delta)^q q \left(\frac{\phi(q+s)^\rho}{e\rho}\right)^{\frac{\rho+s}{\rho}} \left(\frac{e}{q+s}\right)^{q+s},$$

where  $K_q = K_{q-1}$  for  $q$  sufficiently large.

For  $q = 0$ , it follows from Lemma 3. We assume it true for  $q \leq q_0 - 1$ .

$$\begin{aligned} |R_{q_0}(u)| &\leq \frac{|P_{q_0+s}(u)| + \sum_{\substack{m+k=q_0 \\ m \neq q_0}} |R_m(u) T_{k+s}(u)|}{|T_s(u)|} \\ &\leq K_\delta (1 + \delta)^s [p'_n(u) + \varepsilon \|u\|]^{q_0} (1 + \delta)^{q_0} \left(\frac{\phi(q_0+s)^\rho}{e\rho}\right)^{\frac{q_0+s}{\rho}} \left(\frac{e}{q+s}\right)^{q_0+s} \times \\ &\quad \times \left\{ C_1 + \sum_{\substack{m+k=q_0 \\ m \neq q_0}} K_{q-1} C_2 m \left[ \frac{\phi(m)^m \phi(k+s)^{k+s}}{\phi(k+m+s)^{k+m+s}} \right] \frac{(m+k+s)^{m+k+s}}{m^m(k+s)^{k+s}} \right. \\ &\leq \max [K_0(1 + \delta)^s C_1, K_{q-1} C_2] [p'_n(u) + \varepsilon \|u\|]^{q_0} (1 + \delta)^{q_0} \left(\frac{\phi(q+s)^\rho}{e\rho}\right)^{\frac{q_0+s}{\rho}} \left(\frac{e}{q+s}\right)^{q_0+s} \times \\ &\quad \times \left. \left\{ 1 + \sum_{\substack{m+k=q_0 \\ m \neq q_0}} K_\delta (1 + \delta)^s m \left[ \frac{\phi(m)^m \phi(k+s)^{k+s}}{\phi(m+k+s)^{m+k+s}} \right] \frac{(m+k+s)^{m+k+s}}{(k+s)^{k+s} m^m} \right\} \right\}. \end{aligned}$$

We assume that the function  $r^{1-\rho(r)}$  is increasing. By (1), this holds eventually, so this is an inessential assumption



$$\left[ \frac{\phi(m)^m \phi(k+s)^{k+s}}{\phi(q_0+s)^{q_0+s}} \right] \frac{(q_0+s)^{q_0+s}}{m^m(k+s)^{k+s}} = \frac{1}{\left[ \frac{r_{q_0+s}^{1-\rho(r_{q_0+s})}}{r_m^{1-\rho(r_m)}} \right]^m \left[ \frac{r_{q_0+s}^{1-\rho(r_{q_0+s})}}{r_{k+s}^{1-\rho(r_{k+s})}} \right]^{k+s}}$$

Let us assume for the moment that  $k+s \leq \frac{3}{4}(q_0+s)$ . Then

$$\left[ \frac{r_{q_0+s}^{1-\rho(r_{q_0+s})}}{r_{k+s}^{1-\rho(r_{k+s})}} \right]^{k+s} = \left[ \frac{r_{q_0+s}^{\frac{1-\rho(r_{q_0+s})}{1-\rho}} \cdot \frac{\rho}{2}}{r_{k+s}^{\frac{1-\rho(r_{k+s})}{1-\rho}} \cdot \frac{\rho}{2}} \right]^{(k+s) \frac{2}{\rho} (1-\rho)}$$

Let  $\psi(r) = r^{\frac{1-\rho(r)}{1-\rho}} \cdot \frac{\rho}{2}$ . Then

$$\begin{aligned} \psi(r_{q_0+s}) - \psi(r_{k+s}) &= \int_{r_{k+s}}^{r_{q_0+s}} \frac{d}{dr} \psi(r) dr \geq \int_{\frac{3}{4}(r_{q_0+s})}^{r_{q_0+s}} \frac{d}{dr} \psi(r) dr \geq \\ &\geq \int_{\frac{3}{4}(r_{q_0+s})}^{r_{q_0+s}} \frac{d}{dr} r^{\frac{\rho(r_{q_0+s})}{4}} dr \end{aligned}$$

for  $q_0+s$  sufficiently large, by (1). Thus

$$\psi(r_{q_0+s}) - \psi(r_{k+s}) \geq r_{q_0+s}^{\frac{\rho(r_{q_0+s})}{4}} \left[ 1 - \left( \frac{3}{4} \right)^{1/4} \right] = T(q_0+s)^{1/4}.$$

For  $(k+s) \geq 12 \frac{\rho}{2} \frac{1}{1-\rho} = \alpha$ , we have

$$\begin{aligned} &\left[ \frac{r_{q_0+s}^{\frac{1-\rho(r_{q_0+s})}{1-\rho}} \cdot \frac{\rho}{2}}{r_{k+s}^{\frac{1-\rho(r_{k+s})}{1-\rho}} \cdot \frac{\rho}{2}} \right]^{(k+s) \frac{2}{\rho} (1-\rho)} \geq \left[ 1 + \frac{T(q_0+s)^{1/4}}{r_{k+s}^{\frac{1-\rho(r_{k+s})}{1-\rho}} \cdot \frac{\rho}{2}} \right]^{(k+s) \frac{2}{\rho} (1-\rho)} \\ &\geq \left[ 1 + \frac{(k+s) T(q_0+s)^{1/4}}{r_{k+s}^{\frac{1-\rho(r_{k+s})}{1-\rho}} \cdot \frac{\rho}{2}} + \dots + K T^\gamma(q_0+s)^\gamma + \dots \right]^{\frac{2}{\rho} (1-\rho)}, \end{aligned}$$

where  $\gamma \geq 3 \frac{\rho}{2} \frac{1}{1-\rho}$ , (since  $r_{k+s}^{\frac{1-\rho(r_{k+s})}{1-\rho}} \cdot \frac{\rho}{2} = O(k+s)^{1/2}$ )  
 $\geq T'(q_0 + s)^3$ .

For  $(k + s) \leq \alpha + 1$

$$\left[ \frac{r_{q_0+s}^{1-\rho(r_{q_0+s})}}{r_{k+s}^{1-\rho(r_{k+s})}} \right]^{k+s} \geq \left[ \frac{r_{q_0+s}^{1-\rho(r_{q_0+s})}}{\beta} \right]^{k+s} \geq (\alpha + 1)^2 K_\delta (1 + \delta)^s 3,$$

(where  $\beta = \max_{(k+s) \leq \alpha} r_{k+s}^{1-\rho(r_{k+s})}$ ) for  $q_0$  sufficiently large. By symmetry, similar inequalities exist if we replace  $(k + s)$  by  $m$ . We choose  $q_0$  so large that  $\frac{K_\delta (1 + \delta)^s}{T'(q_0 + s)^2} \leq \frac{1}{q_0}$ . Thus

$$\left\{ 1 + \sum_{\substack{m+k=q_0 \\ m \neq q_0}} K_\delta (1 + \delta)^s m \left[ \frac{\phi(m)^m \phi(k+s)^{k+s}}{\phi(m+k+s)^{m+k+s}} \right] \frac{(m+k+s)^{m+k+s}}{m^m (k+s)^{k+s}} \right\} \\ \leq 1 + \frac{(q_0 - 1)}{3} + 2 \leq q_0$$

for  $q_0$  sufficiently large, which establishes the induction.

Furthermore,

$$\left[ \frac{\phi(q+s)}{q+s} \right]^{q+s} = \left[ \frac{r_{q+s}^{1-\rho(r_{q+s})}}{r_{q+s}^{1-\rho(r_{q+s})}} \right]^{q+s} = \left[ \frac{\phi(q)}{q} \right]^{q+s} \left[ \frac{r_{q+s}^{1-\rho(r_{q+s})}}{r_q^{1-\rho(r_q)}} \right]^{q+s} \\ \leq (1 + \delta)^{q+s} \left[ \frac{\phi(q)}{q} \right]^{q+s+1}$$

for arbitrary  $\delta > 0$  when  $q$  is sufficiently large. Thus

$$\overline{\lim}_{q \rightarrow \infty} \left\{ \frac{q}{e} \left| \frac{1}{A_q} R_q(u) \right|^{1/q} \right\} \leq p'_n(u),$$

which proves the theorem.

Q.E.D.

COROLLARY. — Let  $F(u) = \sum_q T_q(u)$ ,  $H(u) = \sum_q P_q(u)$  be in  $Q_p^{\rho(r)}$  (resp.  $Q_0$ ) and assume  $T_0 \neq 0$ . Then there exists a unique  $G(u) \in Q_p^{\rho(r)}$  (resp.  $Q_0$ ) such that  $F(u) G(u) = H(u)$ .

*Proof.* — It is well known that the set of formal power series with non-zero constant term forms a group under multiplication. By Theorem 4,  $G(u) \in Q_p^{\rho(r)}$  (resp.  $Q_0$ ). Q.E.D.

Combining Theorem 4 with Proposition 2, we obtain the following

THEOREM 5. — Let  $\check{\mu}$  be a differential operator with constant coefficients for some space  $E_p^{\rho(r)}$  for a complex pseudo-norm  $p(z)$  and a proximate order  $\rho(r)$  ( $\rho \neq 1$ ) (resp.  $E^0$ ). Then for  $f \in E_p^{\rho(r)}$  (resp.  $E^0$ ), there always exists  $g \in E_p^{\rho(r)}$  (resp.  $E^0$ ) such that  $\check{\mu}(g) = f$ . For  $\rho < 1$  (resp.  $E^0$ ), if  $\check{\mu}(1) \neq 0$ , the solution  $g$  is unique.

*Proof.* — As a result of Theorem 4, the mapping  $\nu \rightarrow \mu * \nu$  is one-to-one and closed. If  $\check{\mu}(u)$  has a non-zero constant term, then by the corollary to Theorem 4, this mapping is also onto, so its transpose  $\check{\mu}$  is one-to-one. Q.E.D.

We now show that for  $\rho < 1$ , the uniqueness of the solution has important consequences for the circular indicator function. Instead of a complex pseudo-norm, we let  $p_0(z)$  be any positive upper semi-continuous complex homogeneous function (i.e.  $p_0(\lambda z) = |\lambda| p_0(z)$ ). We construct the space  $E_{p_0}^{\rho(r)}$  as in (6).

LEMMA 4. — Let  $p_0(z)$  be a positive upper semi-continuous complex homogeneous function,  $\mathfrak{P} = \{p(z) : p(z) \text{ a complex norm, } p(z) \geq p_0(z)\}$ . Then  $p_0(z) = \inf_{p(z) \in \mathfrak{P}} \{p(z)\}$ .

*Proof.* — Let  $D = \{z : p_0(z) < 1\}$ ,  $D_\epsilon = \{z : p_0(z) + \epsilon \|z\| < 1\}$ , which are open. Consider a complex line  $(\lambda z_0)$ ,  $\lambda \in \mathbb{C}$  (which we assume to be  $(\lambda(z_1, 0, \dots, 0))$ , and let

$$D^{z_0} = D \cap (\lambda z_0) , D_\epsilon^{z_0} = D_\epsilon \cap (\lambda z_0) .$$

This determines two concentric circles in the  $(\lambda z_0)$  line. We choose a radius  $r_{z_0} < \infty$  between the radii of these two concentric circles and  $\epsilon_{z_0}$  so small that the convex set

$$K_{z_0} = \{ z : \|z_1\| < r_{z_0}, \sqrt{\sum_{i=2}^n |z_i|^2} < \varepsilon_{z_0} \} \subset D .$$

We define  $p_{z_0}(z) = \inf_{\frac{1}{t} z \in K_{z_0}} t$ , which is a complex norm. Since  $D_\varepsilon$

is a compact set, it can be covered by a finite number of the open sets  $K_{z_j}, j = 1, \dots, N$ . Then  $p_0(z) \leq \inf_j p_{z_j}(z) \leq p_0(t) + \varepsilon \|t\|$ .

Q.E.D.

**THEOREM 6.** — *Let  $\rho < 1$  and let  $f$  have circular indicator  $h_c^*(z)$  with respect to  $\rho(r)$ . Let  $\mu \in \bigcap_{A>0} (E_{A\|z\|}^{\rho(r)})'$  such that  $\mu(1) \neq 0$ . Then there is a unique solution  $g$  of the equation  $\check{\mu}(x) = f$  such that, if  $k_c^*(z)$  is the circular indicator of  $g$  with respect to  $\rho(r)$ ,  $k_c^*(z) \leq h_c^*(z)$ .*

*Proof.* — Let  $p_\alpha(z)$  be a family of norms such that

$$h_c^*(z)^{1/\rho} = \inf_\alpha p_\alpha(z) .$$

Then  $\mu \in (E_{p_\alpha(z)}^{\rho(r)})'$  for every  $\alpha$ , and by Theorem 5, there exists a unique solution  $g$  to the equation  $\check{\mu}(g) = f$ . We clearly have

$$k_c^*(z) \leq h_c^*(z) . \qquad \text{Q.E.D.}$$

In particular, if  $P(D)$  is a differential polynomial with constant coefficients and non-zero constant term, then for  $\rho < 1$ , there is a unique solution  $g$  of the differential equation  $P(D)g = f$  where  $g$  has the same circular indicator as  $f$ .

### 3. The case of $\rho = 1$ and convex functions.

Let  $h_k$  be a convex function,  $K$  the associated convex compact set. We make the space  $E_{h_k}$  of entire functions  $F(u)$  whose convex indicator functions are less than or equal to  $h_k$  into a Frechet space as in (6) by choosing  $p_n(z) = h_k(z) + \frac{1}{n} \|z\|$ ;  $(E_{h_k})'$  is its dual space. We have the following characterization of  $(E_{h_k})'$  [8].

PROPOSITION 3. — *The space  $(E_{h_k})'$  is just the set of measures  $m$  for which there exists an  $\varepsilon > 0$  such that  $m \cdot e^{h_k(z) + \varepsilon \|z\|}$  is a bounded measure.*

We recall some of the basic notions that A. Martineau [8] used in defining the projective Laplace transformation of a function  $f(z)$  of exponential type. Let  $V$  be an  $n$ -dimensional linear vector space,  $V'$  its dual. Let  $P(V)$  be the projective space obtained from  $V$  by adding the points at infinity,  $P(V')$  that obtained from  $V'$  by adding the points at infinity. We write the coordinates of  $P(V)$  as  $(\xi_0, z)$ , those of  $P(V')$  as  $(\xi_0, \xi)$ , and we let  $\bar{\xi}$  be the hyperplane

$$\xi_0 \cdot \xi_0 + \langle z, \xi \rangle = 0 .$$

We introduce the differential forms  $\pi(z) = dz_1 \wedge \dots \wedge dz_n$ ,

$$\theta(\xi) = \sum_{j=1}^n (-1)^j \xi_j d\xi_1 \wedge \dots \wedge \hat{d\xi}_j \wedge \dots \wedge d\xi_n$$

( $d\xi_j$  omitted) and  $\bar{\omega}(\xi, z) = \theta(\xi) \wedge \pi(z)$ , which is defined in  $V \times P(V')$ .

Let  $\Gamma$  be the boundary of a strictly convex open set  $\Omega$  and assume  $\Gamma$  regular and oriented by Stokes' formula  $\int_{\partial\Omega} \pi = \int_{\Omega} d\pi$ . To each point  $z \in \Gamma$ , we have the associated hyperplane  $\bar{\xi}(z)$  through  $z$  tangent to  $\Gamma$ . This defines a manifold  $\Sigma(\Gamma)$  in  $V \times P(V')$ .

For a compact convex set  $K$ , we designate by  $\overset{*}{C}K$  the open subset of  $P(V')$  formed of hyperplanes  $\bar{\xi}$  such that  $\bar{\xi} \cap K = \{\phi\}$ .

PROPOSITION 4 [8]. — *Suppose  $K$  convex and compact. Let  $\psi$  be a function defined in  $\overset{*}{C}K$ , holomorphic there, and zero at the points at infinity ( $\xi_0 = 0$ ). Let  $\bar{f} \in \mathfrak{H}(K)$  (functions holomorphic in a neighborhood of  $K$ ) and  $f$  a representative of  $\bar{f}$  in an open neighborhood  $\Omega$  of  $K$ . Let  $\omega$  be a strictly convex neighborhood of  $K$  with regular boundary included in  $\Omega$ . Posing*

$$T_{\psi}(\bar{f}) = \frac{1}{(2\pi i)^n} \int_{\Sigma(\omega)} f(z) \frac{\partial^{n-1}}{\partial \xi_0^{n-1}} \left( \frac{1}{\xi_0} \psi(\xi) \right) \bar{\omega}(z, \xi) \quad (9)$$

*we define a continuous linear functional on  $\mathfrak{H}(K)$  which is independant of the choice of the representative  $f$  and of  $\omega$ .*

Let  $F(u)$  be an arbitrary element of  $E_{h_k}$ . We define the function

$$\mathcal{P}_F(\bar{\xi}) = \xi_0 \int_0^\infty F(-\xi t) e^{-\xi_0 t} dt .$$

This defines a function in  $\overset{*}{\mathbb{C}}K$  which is zero at the points at infinity  $\xi_0 = 0$ . The function  $\mathcal{P}_F$  is called the projective Fourier-Borel transform of  $F$ . We then have

PROPOSITION 5 [8]. — *Let  $F(u) \in E_{h_k}$ . Then*

$$F(u) = \frac{1}{(2\pi i)^n} \int_{\Sigma(\omega)} \exp \langle z, u \rangle \frac{\partial^{n-1}}{\partial \xi_0^{n-1}} \left( \frac{\mathcal{P}_F(\xi)}{\xi_0} \right) \bar{\omega}(z, \xi), \quad (10)$$

where  $\omega$  is any strictly convex neighborhood of  $K$  with regular boundary.

Let  $\mu \in (E_{h_k})'$ . We define the Fourier-Borel transform of  $\mu$  to be  $f_\mu(z) = \mu(\exp \langle z, u \rangle)$ , which, by Proposition 3, defines a function holomorphic in a neighborhood of  $K$ . For  $\nu \in (E_{h_k})'$ , we define the convolution of  $\mu$  with  $\nu$  as  $(\nu * \mu)(F(u)) = \mu_\nu(\nu, F(u + \nu))$ . We refer the reader again to [8] to see that the convolution is well defined. We then have the relationship that  $f_{\nu * \mu}(z) = f_\nu(z) \cdot f_\mu(z)$  where these functions are defined.

On the other hand, let  $g(z)$  be a function holomorphic in a neighborhood of  $K$ . Then  $g$  defines a continuous linear operator  $S_g$  from  $E_{h_k}$  into  $E_{h_k}$  by

$$S_g(F(u)) = \frac{1}{(2\pi i)^n} \int_{\Sigma(\omega)} g(z) \exp \langle z, u \rangle \frac{\partial^{n-1}}{\partial \xi_0^{n-1}} \left( \frac{\mathcal{P}_F(\xi)}{\xi_0} \right) \bar{\omega}(z, \xi),$$

where  $\omega$  is a suitably small strictly convex regular neighborhood of  $K$ .

LEMMA 5. — *Let  $\psi_{z_0} = \mathcal{P}_{\exp \langle z_0, u \rangle}$  for  $z_0 \in K$ . Then the linear functional on  $\mathfrak{H}(K)$  determined by  $\psi_{z_0}$ ,  $T_{\psi_{z_0}} = \delta(z_0)$ , the Dirac measure.*

*Proof.* — Let  $f$  be a representative of  $\bar{f} \in \mathfrak{H}(K)$  defined in some convex neighborhood  $\omega$  of  $K$ . Since  $\omega$  is a Runge domain,  $f$  can be

uniformly approximated by polynomials in an open neighborhood of  $K$ , and since  $z_i = \lim_{|\lambda| \rightarrow 0} \frac{e^{z_i \lambda} - 1}{\lambda}$ ,  $\lambda \in \mathbb{C}$ ,  $f$  can be uniformly approximated by exponentials. But by (10), we have that  $T_{\psi_{z_0}}$  is just  $f(z_0)$  for the exponentials. It now follows from the uniform convergence in a neighborhood of  $K$  that  $T_{\psi_{z_0}}(f) = f(z_0)$ . Q.E.D.

LEMMA 6. — *Let  $\nu \in (E_{n_k})'$ . If  $f_\nu$  is its Fourier-Borel transform, then the linear operator  $Q_{f_\nu} : E_{n_k} \rightarrow E_{n_k}$  is just the transpose of the convolution  $\nu * \mu$  (i.e.  $(Q_{f_\nu}(F), \mu) = (F, \nu * \mu)$ ).*

*Proof.* — By Proposition 3, we can represent  $\mu$  by a measure  $m_\mu$  such that  $m_\mu e^{h_k(u) + \varepsilon \|u\|}$  is a bounded measure for  $\varepsilon$  sufficiently small. Then

$$\mu(F(u)) = \frac{1}{(2\pi i)^n} \int_{\Sigma(\omega)} \mu(\exp \langle z, u \rangle) \frac{\partial^{n-1}}{\partial \xi^{n-1}} \left( \frac{\rho_F(\xi)}{\xi_0} \right) \bar{\omega}(z, \xi)$$

follows from Fubini's theorem for  $\omega$  a sufficiently small, strictly convex neighborhood of  $K$ . Thus,  $\mu$  is completely determined by its values on a set of exponentials  $\exp \langle z, u \rangle$  defined for  $z$  in a neighborhood of  $K$ . We choose  $\omega$  so small that  $f_\nu$  is defined and bounded in  $\omega$ . Then for  $z_0 \in \omega$ ,

$$\begin{aligned} (Q_\nu(\exp \langle z_0, u \rangle), \mu) &= \\ &= \mu \left( \frac{1}{(2\pi i)^n} \int_{\Sigma(\omega)} \exp \langle z, u \rangle f_\nu(z) \frac{\partial^{n-1}}{\partial \xi^{n-1}} \left( \frac{\psi_{z_0}(\xi)}{\xi_0} \right) \bar{\omega}(z, \xi) \right) = \\ &= f_\nu(z_0) \mu(\exp \langle z_0, u \rangle) = f_\nu(z_0) f_\mu(z_0), \end{aligned}$$

from which the lemma follows. Q.E.D.

For  $\nu \in (E_{n_k})'$ , we define the differential operator with constant coefficients  $\check{\nu}$  on  $E_{n_k}$  to be the transpose of the convolution operation  $\mu \rightarrow \nu * \mu$  on  $(E_{n_k})'$ .

**THEOREM 7.** — *Let  $\check{v}$  be a differential operator with constant coefficients on  $E_{h_k}$ . Then*

(a) *for  $F \in E_{h_k}$ , there always exists  $G \in E_{h_k}$  such that  $\check{v}(G) = F$ ,*

(b) *if  $f_\nu$  has no zeros in  $K$ , then  $G$  is unique*

(c) *the polynomial exponential solutions of  $\check{v}(x) = 0$  are dense in the space of all solutions of this equation.*

*Proof.* — (a) The mapping  $\mu \rightarrow f_\mu$  is a one-to-one linear mapping of  $(E_{h_k})'$  onto  $\mathfrak{H}\mathcal{E}(K)$ . We topologize  $\mathfrak{H}\mathcal{E}(K)$  with the topology of convergence of the Taylor series coefficients at each point of  $K$ . This is at least as weak as the equivalent on  $\mathfrak{H}\mathcal{E}(K)$  of the weak topology on  $(E_{h_k})'$ , since, for a multi-index  $\alpha$ ,

$$\mu(u^a \exp \langle z_0, u \rangle) = \frac{\partial^{|\alpha|} f_\mu(z_0)}{\partial z^\alpha}.$$

If  $f_\nu \cdot f_{\mu_\gamma}$  is a filter converging to  $g \in \mathfrak{H}\mathcal{E}(K)$ , then we must have  $g = f_\gamma \cdot f_g$ , since the Taylor series of  $g$  is divisible by that of  $f_\nu$  at each point of  $K$ . Thus the mapping  $f_\mu \rightarrow f_\nu \cdot f_\mu$  is one-to-one and closed, so  $\mu \rightarrow \nu * \mu$  is also one-to-one and closed. By Proposition 2, its transpose is onto.

(b) If  $f_\nu$  has no zeros in  $K$ , then  $f_\mu \rightarrow f_\nu \cdot f_\mu$  is onto so  $\mu \rightarrow \nu * \mu$  is onto and hence its transpose is one-to-one.

(c) See [8] and [6].

Q.E.D.

The following example, due to C.O. Kiselman, shows that in some sense the results of § 2 and § 3 are sharp. Let  $P(D) = \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}$  and let  $f(z) = \cos \sqrt{z_1 z_2}$ , which is of exponential type. Let  $u$  be a solution of exponential type of  $P(D)u = f$ . Then

$$\begin{aligned} u(0, r) - u(-r, 0) &= \int_0^1 \frac{d}{dt} u(-r(1-t), tr) dt = \\ &= r \int_0^1 \cos r \sqrt{-t(1-t)} dt = \frac{r}{2} \int_0^1 (e^{r\sqrt{t(1-t)}} + e^{-r\sqrt{t(1-t)}}) dt \geq \\ &\geq \frac{r}{2} \int_0^1 e^{r\sqrt{t(1-t)}} dt \geq \frac{r}{2\sqrt{2}} e^{\frac{r}{2\sqrt{2}}}. \end{aligned}$$



But  $h_c^*(z)$  the circular indicator of  $f(z)$ , is zero in both the complex line  $(\lambda(0, z_2))$  and  $(\lambda(z_1, 0))$ , so that the circular indicator (and hence the radial indicator) of  $u$  is strictly greater than that of  $f$ .

#### 4. Functions of slow growth.

In this section, we extend the notion of a differential operator with constant coefficients to entire functions which satisfy a majoration of the form

$$|f(z)| \leq C_k \exp(\ln[p(z)])^k \quad (11)$$

asymptotically for some  $k > 1$  and some norm  $p(z)$ . These functions are known to have very even growth [1].

We define the *logarithmic order*  $\rho$  of such a function to be the infimum of all  $k$  for which (11) holds. We define the *logarithmic type*  $\sigma$  of  $f$  (with respect to a logarithmic order  $\rho$ ) to be the infimum of all  $b$  such that

$$|f(z)| \leq C_b \exp b(\ln p(z))^\rho.$$

These values are clearly independent of the norm used to define them.

**THEOREM 8.** — *Let  $m$  be a multi-index of positive numbers  $m = (m_1, \dots, m_n)$ ,  $|m| = \sum m_i$ . Then the logarithmic order and logarithmic type of a function  $f$  are given by*

$$\begin{aligned} \frac{\rho}{\rho - 1} &= \overline{\lim}_{|m| \rightarrow \infty} \frac{\ln \ln^+ \frac{1}{|c_m|}}{\ln n} \text{ and } \left(\frac{\rho - 1}{\rho}\right) \left[\frac{1}{\sigma\rho}\right]^{\frac{1}{\rho-1}} = \\ &= \overline{\lim}_{|m| \rightarrow \infty} \frac{\ln \frac{1}{|c_m|}}{\frac{\rho}{n^{\rho-1}}} \end{aligned}$$

where  $f(z) = \sum_m c_m z^m$  and  $\ln^+ a = \sup(0, \ln a)$ .

*Remark.* — We interpret this to mean  $\rho = 1$  if the limit in (12) is infinite. In this case, if we have  $\sigma < +\infty$ , we have a polynomial. We do not consider this case but rather assume that if  $\rho = 1$  that  $\sigma = +\infty$ .

*Proof.* — Let  $b > 0$  and  $k > 1$  be numbers such that

$$|f(z)| \leq C \exp b(\ln r)^k .$$

We assume without loss of generality that  $r = \|z\|_1$ , where  $\|z\|_1 = \max_i |z_i|$ . By applying Cauchy's formula to the distinguished boundary of the polydisc of radius  $r$ , we get

$$|c_n| \leq C \exp \{ b(\ln r)^k - |m| \ln r \} .$$

This function takes on its maximum (for  $k > 1$ ) when  $\ln r = \frac{|m|^{\frac{1}{k-1}}}{kb}$

and equals  $\exp \left\{ \left( \frac{1}{kb} \right)^{\frac{1}{k-1}} \left( \frac{1}{k} - 1 \right) |m|^{\frac{k}{k-1}} \right\}$ , which establishes the theorem in one direction.

On the other hand, if  $|c_m| \leq K \exp \left\{ \left( \frac{1}{kb} \right)^{\frac{1}{k-1}} \left( \frac{1}{k} - 1 \right) |m|^{\frac{k}{k-1}} \right\}$ ,

$$|f(z)| \leq \sum_m K |m|^n \exp \left\{ \left( \frac{1}{kb} \right)^{\frac{1}{k-1}} \left( \frac{1}{k} - 1 \right) |m|^{\frac{k}{k-1}} + |m| \ln r \right\}$$

on the distinguished boundary of the polydisc of radius  $r$ . The function

$\left( \frac{1}{kb} \right)^{\frac{1}{k-1}} \left( \frac{1}{k} - 1 \right) x^{\frac{k}{k-1}} + x \ln r$  takes on its maximum for

$$x = \{ (kb)^{\frac{1}{k-1}} \ln r \}^{k-1}$$

and equals  $\exp b(\ln r)^k$ .

Let  $M_0 = \{ \{ (kb)^{\frac{1}{k-1}} \ln r \}^{k-1} \}$  and

$$M'_0 = \left[ \left\{ \frac{1}{2} \frac{k}{(k-1)} (kb)^{\frac{1}{k-1}} \ln r \right\}^{k-1} \right]$$

("greatest integer in"). Then

$$|f(z)| \leq K' (\ln r)^{2n(k-1)} \exp b(\ln r)^k + \sum_{|m|=M'_0+1}^{\infty} r^{|m|} \exp \left\{ \left( \frac{1}{kb} \right)^{\frac{1}{k-1}} \left( \frac{1}{k} - 1 \right) |m|^{\frac{k}{k-1}} \right\} .$$

But

$$\sum_{|m| \leq M'_0 + 1}^{\infty} r^{|m|} \exp \left\{ \left( \frac{1}{kb} \right)^{\frac{1}{k-1}} \left( \frac{1}{k} - 1 \right) |m|^{\frac{k}{k-1}} \right\} \leq$$

$$\leq \sum_{|m| \leq M'_0 + 1}^{\infty} \exp \left\{ \left( \frac{1}{kb} \right)^{\frac{1}{k-1}} \left( \left( \frac{1}{k} - 1 \right) |m|^{\frac{k}{k-1}} + |m| (M_0 + 1)^{\frac{1}{k-1}} \right) \right\}$$

and this last series is bounded independently of  $M'_0$  since

$$\left( \frac{1}{k} - 1 \right) |m|^{\frac{k}{k-1}} + |m| (M_0 + 1)^{\frac{1}{k-1}} =$$

$$= |m| \left( \frac{1}{k} - 1 \right) \left( |m|^{\frac{1}{k-1}} - \frac{(k-1)}{k} (M_0 + 1)^{\frac{1}{k-1}} \right) < |m| \left( \frac{1}{k} - 1 \right) T$$

for some  $T > 0$ .

Q.E.D.

We let  $E_{\sigma, \rho}$  be the Fréchet space that we get by taking

$$p_n = \left( \sigma + \frac{1}{n} \right) (\ln r)^\rho$$

in (6),  $E_1$  that which we get by taking  $p_n = (\ln r)^{(1 + \frac{1}{n})}$ , and we designate their duals by  $(E_{\sigma, \rho})'$  and  $(E_1)'$ .

LEMMA 7. — *A linear functional  $\mu$  on  $E_{\sigma, \rho}$  (resp.  $E_1$ ) is in  $(E_{\sigma, \rho})'$  (resp.  $(E_1)'$ ) if and only if*

$$|\mu(z^m)| \leq K_\epsilon \exp \left[ \frac{1}{(\sigma + \epsilon) \rho} \right]^{\rho-1} \left[ 1 - \frac{1}{\rho} \right] |m|^{\frac{\rho}{\rho-1}} \quad (13)$$

(resp.

$$|\mu(z^m)| \leq K_\epsilon \exp \left[ \frac{1}{1 + \epsilon} \right]^{\frac{1+\epsilon}{\epsilon}} \left[ \frac{\epsilon}{1 + \epsilon} \right] |m|^{\frac{1+\epsilon}{\epsilon}} \quad (14)$$

for some  $\epsilon > 0$ .

*Proof.* — It follows from the proof of Theorem 8 that the Taylor series of an element in  $E_{\sigma, \rho}$  (resp.  $E_1$ ) converges to the function in this space (cf. [8]). Thus, if  $\mu$  is a continuous linear functional, it follows that (13) (resp. (14)) holds.

On the other hand, if (13) (resp. (14)) holds, it follows from the estimates of Theorem 8 that  $\mu$  is a continuous linear functional on  $E_{\sigma, \rho}$  (resp.  $E_1$ ). Q.E.D.

For  $\mu \in (E_{\sigma, \rho})'$  (resp.  $(E_1)'$ ), we define its Fourier-Borel transform  $\tilde{\mu}(u) = \mu(\exp \langle z, u \rangle) = \sum \mu(z^m) \frac{u^m}{m!}$ , in the sense of a formal power series at the origin. We topologize this space with the topology of convergence of coefficients. Let  $Q_{\sigma, \rho}$  (resp.  $Q_1$ ) be the space of formal power series whose coefficients satisfy (13) (resp. (14)) above.

For  $\nu, \mu \in (E_{r, \rho})'$  (resp.  $(E_1)'$ ), we define the convolution of  $\mu$  with  $\nu$ ,  $\nu * \mu$  to be

$$\nu * \mu(f(u)) = \mu(\nu_v(f(u + v))) .$$

A differential operator with constant coefficients on  $E_{\sigma, \rho}$  (resp.  $E_1$ ) is defined as the transpose of this convolution operation. We then have the following

**THEOREM 9.** — *Let  $\check{\nu}$  be a differential operator with constant coefficients on the space  $E_{\sigma, \rho}$  (resp.  $E_1$ ). Then for  $f \in E_{\sigma, \rho}$  (resp.  $E_1$ ) there is always a solution  $g \in E_{\sigma, \rho}$  (resp.  $E_1$ ) of the equation  $\check{\nu}(g) = f$ . If  $\check{\nu}(1) = 0$ , then  $g$  is unique.*

The proof is the same as that of Theorem 6, with some alterations in the calculations of Theorem 5 to prove that the operation of convolution is closed. The details are left to the interested reader.

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